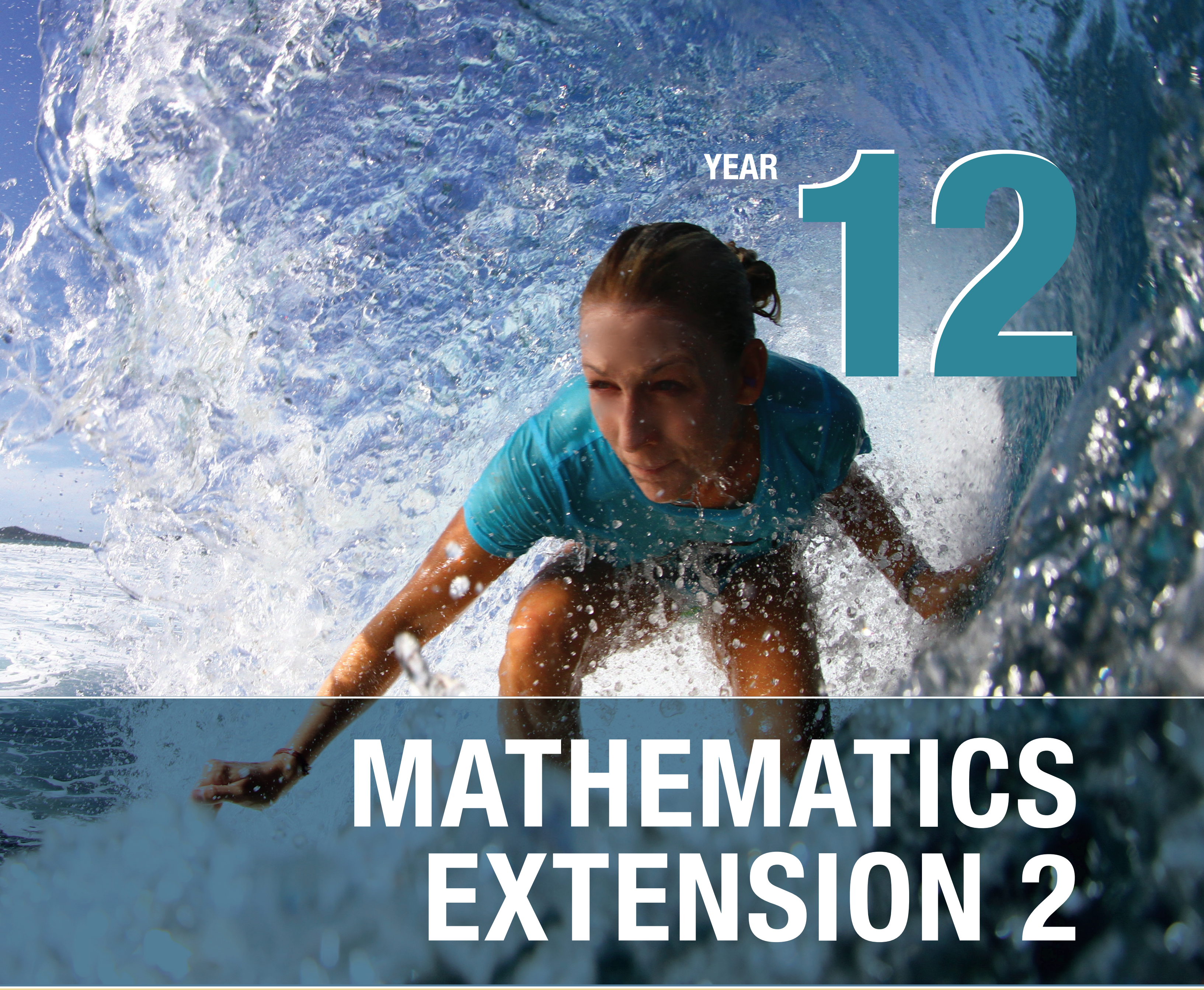

YEAR 12

MATHEMATICS EXTENSION 2

CambridgeMATHS

STAGE 6

DAVID SADLER | DEREK WARD

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Introduction

This resource covers the Mathematics Extension 2 syllabus and is designed to be used in conjunction with *CambridgeMATHS Mathematics Advanced Year 12* and *Extension 1 Year 12*. It is based on the authors' Extension 2 resource developed for the previous syllabus, and retains the same design and structure, but has been re-written to cover the new syllabus implementing in 2020.

The online Curriculum Grid, Scope and Sequence, and Teaching Program are provided to guide you in planning a sound mathematical journey through a complex syllabus.

The Exercises are divided into Foundation, Development and Enrichment to gradually lead you to achieve your highest potential. The Enrichment questions are particularly challenging.

Essential rules, formulae and important concepts are highlighted in numbered boxes for quick reference and revision.

Chapter review exercises are provided for all chapters.

The Interactive Textbook powered by Cambridge HOTmaths offers selected worked solutions (as an option that teachers can choose to enable for student access), workspaces with self-assessment tools, and access to a downloadable PDF textbook for offline use.

The Online Teaching Suite provides a test for each chapter.

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Rationale

The exercises

No-one should try to do all the questions! The exercises are deliberately long so that everyone will find enough questions of a suitable standard — each student will need to select from them, and there should be plenty left for revision. The book provides a great variety of questions, and representatives of all types should be selected.

The **Foundation** section contains routine questions designed to reinforce the elementary concepts and skills that must be mastered before more abstract and sophisticated questions can be attempted.

The **Development** section is graded from reasonably straightforward to diffic ult. The harder questions may be more complicated algebraically or they may require deeper thinking. These questions are designed to provide students with the opportunity to attempt a wide variety of problem types.

The **Enrichment** section is intended to extend and challenge the best students while at the same time matching the standard of the hardest questions in the past Extension 2 HSC papers. We assume that the examinations for the new courses will continue to contain some very demanding questions.

Syllabus coverage of the chapters

Chapter 1: Complex numbers (part 1)

Syllabus References: N1.1
N1.2
N2.1
N2.2

Complex numbers is covered over two chapters because some of the applications of de Moivres theorem are very demanding and need to be delayed until later in the course. The chapter starts with the straightforward arithmetic of complex numbers in Section 1A and then moves on to quadratic equations with complex roots in Section 1B. The Argand diagramis introduced in Section 1C which allows the introduction of the modulus-argument form in Section 1D. Then the vector representation of a complex number is used in Section 1E to solve geometric problems in the complex plane. In Section 1F, both algebraic and geometric approaches are used to sketch curves in the complex plane. Finally, we discuss complex conjugate zeroes of polynomials with real coefficients in Section 1G.

Chapter 2: Proof

Syllabus References: P1
P2

Mathematical proof requires precise logic and clarity of explanation. Section 2A introduces the terminology and symbolic notation required in this chapter. It also emphasises the fact that logic can not only be conveyed symbolically, but also using simple words such as and and or. In Section 2B, proofs involving numbers (mostly positive integers) are treated. Many of these proofs are based on divisibility arguments. Section 2C introduces indirect proof: proof by contradiction and by the contrapositive. In Section 2D, inequalities are proven by algebraic techniques. One of the most important ideas is to

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understand that proving LHS > RHS is equivalent to proving LHS − RHS > 0. Section 2E is proof by mathematical induction. This builds upon and extends the work done in Chapter 2 of the Year 12 Extension 1 book. For instance, many inequalities can be proven by induction. The syllabus doesnt specifically mention it, but a treatment of inequalities would be incomplete without a discussion of inequalities in calculus and geometry. Such problems have occurred frequently in past Extension examination papers. This material is covered in Section 2F. It may be appropriate to delay this section to later in the course, such as at the end of Chapter 4, Integration.

Chapter 3: Complex numbers (part 2)

Syllabus References: N1.3
N2.1
N2.2

In Section 3A, de Moivres theorem is proven by induction and the exercise focuses on powers of complex numbers. Sections 3B and 3C are demanding. De Moivres theorem is used in 3B to prove some complicated trigonometric identities, and it is used again in 3C to find the complex roots of certain polynomial equations: most typically $z^n = \pm 1$. This then leads to the factorisation of various polynomials. Section 3D introduces the exponential form of a complex number via Eulers formula $e^{i\theta} = \cos\theta + i\sin\theta$. Then there are some applications of the exponential form in Section 3E.

Chapter 4: Integration

Syllabus References: C1

The intention in Section 4A is that students become familiar with the standard integrals. There was a more comprehensive and user-friendly list provided in the previous course which would be a useful supplement for students. Section 4B is a short section focusing on some simple algebraic tricks for manipulating the integrand so that the primitive can easily be found. For example, copying the denominator into the numerator of a fraction. Sections 4C to 4F develop the standard methods for integration: substitution, partial fractions, quadratic denominators and integration by parts. Then in Section 4G these methods are applied to trigonometric integrals. Sequences of integrals are introduced in the difficult Section 4H, where reduction formulae are covered. Section 4I is a miscellaneous collection of problems, where the student must determine which method is most appropriate. Often there is more than one way to find a primitive.

Chapter 5: Vectors

Syllabus References: V1.1
V1.2
V1.3

This topic has been delayed because students will need to have completed Chapter 8 of the Extension 1 book on vectors in two dimensions. Section 5A introduces coordinates in three dimensions, along with some basic three-dimensional coordinate geometry. This will promote the transition from two-dimensional thinking to three dimensions. Section 5B then introduces column and component vectors in three dimensions. In Section 5C, the dot (scalar) product is defined for three-dimensional vectors, and then applied to a wide variety of problems in Section 5D. The vector equation of a line in both two and three dimensions is discussed in Section 5E, followed by vector equations of circles and spheres in Section 5F. The vector equation of a plane is also included, as it is mentioned in the support material. The Cartesian form of a line in three dimensions is not discussed.

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Chapter 6: Mechanics

Syllabus References: M1.1
M1.2
M1.3
M1.4

Section 5A examines forces and the resultant acceleration in terms of t, x or v. Constant and non-constant forces are included, as well as concurrent forces. Newtons laws are used to determine equations of motion. Simple harmonic motion is treated in Sections 5B and 5C. The time equations are the focus of 5B, while the displacement equations are dealt with in 5C. Horizontal and vertical resisted motion are discussed in Sections 5D and 5E, while projectile motion is the focus of Section 5F, following on from the work in Chapter 10 of the Extension 1 book. Section 5G is a miscellaneous set of problems, some of which are very challenging. The intention here is to expose students to a wide variety of interesting problems.

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1

Complex Numbers I

Chapter Overview: One of the significant properties of the real numbers is that any of the four arithmetic operations of addition, subtraction, multiplication and division can be applied to any pair of real numbers, with the exception that division by zero is undefined. As a result, every linear equation

$$ax + b = 0 \quad \text{where} \quad a \neq 0$$

can be solved.

The situation is not so satisfactory when quadratic equations are considered. There are some quadratic equations that can be solved, but others, like

$$x^2 + 2x + 3 = 0\,,$$

have no real solution. This apparent inconsistency, that some quadratics have a solution whilst others do not, can be resolved by the introduction of a new type of number, the complex number.

But there is more to complex numbers than just solving quadratic equations. In this chapter the reader is shown an application to geometry and how they can be used in higher degree polynomials. These new numbers have many applications beyond this course, such as in evaluating certain integrals and in solving problems in electrical engineering. Complex numbers also provide links between seemingly unrelated quantities and areas of mathematics. Here is a stunning example. The four most significant real numbers encountered so far are 0, 1, e and π. As will be shown in a later chapter on complex numbers, these four numbers are connected in a remarkably simple equation involving the special complex number i, namely

$$e^{i\pi} + 1 = 0\,.$$

1A The Arithmetic of Complex Numbers

Introducing A New Type of Number: The investigation is begun by examining the roots of various quadratic equations. For convenience in presenting the new work, the method of completing the square is used exclusively.

First consider only those quadratic equations with rational solutions, such as the equation $x^2 - 4x - 12 = 0$. Completing the square:

$$(x-2)^2 = 16$$

so $\quad x - 2 = 4 \text{ or } -4$

which leads to the two roots

$$\alpha = 6 \text{ and } \beta = -2\,.$$

Note that $\alpha + \beta = 4$ and $\alpha\beta = -12$.

Repeating this process for a number of quadratics with rational solutions, it soon becomes evident that if $ax^2 + bx + c = 0$ has solutions α and β then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}\,.$$

Further investigation reveals that there are some quadratic equations which do not have rational solutions, such as $x^2 - 4x - 1 = 0$. Completing the square:

$$(x-2)^2 = 5\,.$$

Herein lies a problem since there is no rational number which when squared equals 5. This problem is overcome by introducing a new type of number, in this case the irrational number $\sqrt{5}$ which has the property that $\left(\sqrt{5}\right)^2 = 5$. Assuming that the normal rules of algebra apply to this new number, it follows that $\left(-\sqrt{5}\right)^2 = \left(\sqrt{5}\right)^2 = 5$, so that 5 has two square roots, namely $\sqrt{5}$ and $-\sqrt{5}$. If the introduction of this new type of number is valid then the solution may proceed. Thus

$$x - 2 = \sqrt{5} \text{ or } -\sqrt{5}$$

which leads to the two roots

$$\alpha = 2 + \sqrt{5} \text{ and } \beta = 2 - \sqrt{5}\,.$$

Note that $\alpha + \beta = 4$ and $\alpha\beta = -1$.

Repeating this process for a number of quadratics with irrational solutions, it soon becomes evident that if $ax^2 + bx + c = 0$ has irrational roots α and β then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}\,.$$

Since this is consistent with the quadratic equations with rational solutions, it seems that the introduction of surds into the number system is valid. Indeed surds have been used since Year 8 and students will be proficient in their use.

Yet further investigation reveals that there are some quadratic equations which have neither rational nor irrational solutions, such as $x^2 - 4x + 5 = 0$. Completing the square yields:

$$(x-2)^2 = -1\,.$$

Again there is a problem since there is no known number which when squared equals -1. Just as before, this problem is overcome by introducing a new type of number. In this case the so called imaginary number i is introduced which has the property that $i^2 = -1$. Assuming that the normal rules of algebra apply to this new number, it follows that $(-i)^2 = i^2 = -1$, so that -1 has two square roots, namely i and $-i$. If the introduction of this new type of number is valid then the solution may proceed. Thus

$$x - 2 = i \text{ or } -i$$

which leads to the two roots

$$\alpha = 2 + i \text{ and } \beta = 2 - i\,.$$

Note that $\alpha + \beta = 4$ and

$$\alpha\beta = (2-i)(2+i)$$

$$= 2^2 - i^2 \quad \text{(difference of two squares)}$$
$$= 4 + 1$$
$$= 5\,.$$

Repeating this process for a number of quadratics with solutions which involve the imaginary number i, it soon becomes evident that if $ax^2 + bx + c = 0$ has solutions α and β then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}\,.$$

Since this is consistent with all previously encountered quadratic equations, it seems reasonable to include the imaginary number i in the number system.

A New Number in Arithmetic: The imaginary number i is now formally included into the system of numbers. It has the special property that $i^2 = -1$. This new number i will be treated as if it were an algebraic pronumeral when it is combined with real numbers using the four arithmetic operations of addition, subtraction, multiplication and division.

1 A NEW NUMBER: The new number i has the special property that

$$i^2 = -1\,.$$

It may be used like a pronumeral with real numbers in addition, subtraction, multiplication and division.

It is instructive to write out the first four positive powers of i. They are:

$$i^1 = i \qquad i^2 = -1 \text{ (by definition)} \qquad \begin{aligned} i^3 &= i^2 \times i \\ &= -1 \times i \\ &= -i \end{aligned} \qquad \begin{aligned} i^4 &= i^3 \times i \\ &= -i \times i \\ &= 1 \end{aligned}$$

Writing out the next four powers of i, it is found that this sequence repeats.

$$\begin{aligned} i^5 &= i^4 \times i \\ &= 1 \times i \\ &= i \end{aligned} \qquad \begin{aligned} i^6 &= i^4 \times i^2 \\ &= 1 \times (-1) \\ &= -1 \end{aligned} \qquad \begin{aligned} i^7 &= i^4 \times i^3 \\ &= 1 \times (-i) \\ &= -i \end{aligned} \qquad \begin{aligned} i^8 &= \left(i^4\right)^2 \\ &= 1 \end{aligned}$$

It should be clear from these calculations that the sequence continues to cycle. In general, the result can be determined from the remainder when the index is divided by 4.

2 POWERS OF THE IMAGINARY NUMBER: A power of i may take only one of four possible values. If k is an integer, then these values are:

$$i^{4k} = 1\,, \quad i^{4k+1} = i\,, \quad i^{4k+2} = -1\,, \quad i^{4k+3} = -i\,.$$

WORKED EXAMPLE 1: Simplify: (a) i^{23} (b) $i^7 + i^9$

SOLUTION:

(a) $$\begin{aligned} i^{23} &= i^{4\times5+3} \\ &= -i \end{aligned}$$

(b) $$\begin{aligned} i^7 + i^9 &= -i + i \\ &= 0 \end{aligned}$$

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Complex Numbers: Since i has been included in the number system and since it is to be treated as a pronumeral, the number system must now include the real numbers plus new quantities like

$$2i\,, \qquad -7i\,, \qquad 5+4i \qquad \text{and} \qquad \sqrt{6}-3i\,.$$

The set which includes all such quantities as well as the real numbers is given the symbol $\mathbf{C}$. Each quantity in $\mathbf{C}$ is called a *complex number*. Thus 5, $2i$ and $\sqrt{6}-3i$ are all examples of complex numbers. In the first case, 5 is also a real number, and the real numbers form a special subset of the complex numbers. The number $2i$ is an example of another special subset of the complex numbers. This set consists of all the real multiples of i, which are called *imaginary numbers*. Thus $-7i$ is another example of an imaginary number.

3

> TWO NEW TYPES OF NUMBERS: Let a and b be real numbers.
> COMPLEX NUMBERS: Numbers of the form $a+ib$ are called *complex numbers*. The set of all complex numbers is given the symbol $\mathbf{C}$.
> IMAGINARY NUMBERS: Numbers of the form ib, that is the complex numbers for which $a=0$, are called *imaginary numbers*.

Again noting that i is treated as a pronumeral, the addition, subtraction and multiplication of complex numbers presents no problem.

$$(2-3i)+(5+7i)=7+4i\,, \qquad (7+2i)-(5-3i)=2+5i\,,$$
$$3(-5+7i)=-15+21i\,, \qquad \sqrt{3}(2+i\sqrt{3})=2\sqrt{3}+3i\,.$$

The following worked examples of multiplication involve binomial expansions and the property that $i^2=-1$.

WORKED EXAMPLE 2: Simplify:

(a) $(2-3i)(5+7i)$ (b) $(3-2i)^2$ (c) $(4+3i)^2$ (d) $(2+5i)(2-5i)$

SOLUTION:

(a) $$\begin{aligned}(2-3i)(5+7i)&=10-i-21i^2\\&=10-i+21\\&=31-i\end{aligned}$$

(c) $$\begin{aligned}(4+3i)^2&=16+24i+9i^2\\&=16+24i-9\\&=7+24i\end{aligned}$$

(b) $$\begin{aligned}(3-2i)^2&=9-12i+4i^2\\&=9-12i-4\\&=5-12i\end{aligned}$$

(d) $$\begin{aligned}(2+5i)(2-5i)&=4-25i^2\\&=4+25\\&=29\end{aligned}$$

The last three examples above demonstrate the expansions of $(x+iy)^2$, $(x-iy)^2$ and $(x+iy)(x-iy)$ for real values of x and y. Note that in the final example, the result is the sum of two squares and is a real number. This will always be the case, regardless of the values of x and y.

4

> THE SUM OF TWO SQUARES: Let x and y be real numbers, then
> $$(x+iy)(x-iy)=x^2+y^2$$
> which is always a real number.

Complex Conjugates: The last result is significant and will be used frequently. Clearly the pair of numbers $x+iy$ and $x-iy$ have a special relationship. They are called *complex conjugates*. Thus the complex conjugate of $3+2i$ is $3-2i$. Similarly the conjugate of $7-5i$ is $7+5i$.

When the conjugate is required, the complex number is written with a bar above it. Thus:

$$\overline{2+i} = 2 - i \qquad \overline{-3i} = 3i$$

$$\overline{-1+4i} = -1 - 4i \qquad \overline{-3-5i} = -3 + 5i$$

5 COMPLEX CONJUGATES: Let x and y be real numbers, then the two complex numbers $x + iy$ and $x - iy$ are called complex conjugates.
A: The conjugate of $x + iy$ is $\overline{x+iy} = x - iy$.
B: The conjugate of $x - iy$ is $\overline{x-iy} = x + iy$.

Division: Just like real numbers, division by zero is undefined. Dividing a complex number by any other real number presents no problem. As with rational numbers, fractions should be simplified wherever possible by cancelling out common factors.

$$\frac{6+8i}{2} = 3 + 4i \qquad \frac{-2-6i}{3} = -\tfrac{2}{3} - 2i$$

$$\frac{\sqrt{2}-2i}{\sqrt{2}} = 1 - i\sqrt{2} \qquad \frac{-12+21i}{15} = \frac{-4+7i}{5} \quad \text{or} \quad -\tfrac{4}{5} + \tfrac{7}{5}\,i$$

There is a potential problem if one complex number is divided by another, such as in $\frac{2+i}{3-i}$. As it stands, it is not clear that this sort of quantity is even allowed in the new number system, since it is not in the standard form $x + iy$.

The problem is resolved by taking a similar approach to that used to deal with surds in the denominator. The process here is called *realising the denominator*. Thus if the divisor is an imaginary number then simply multiply the fraction by i/i, as in the following two examples.

WORKED EXAMPLE 3: Realise the denominators of: (a) $\dfrac{1}{4i}$ (b) $\dfrac{1+2i}{3i}$

SOLUTION:

(a)
$$\begin{aligned}\frac{1}{4i} &= \frac{1}{4i} \times \frac{i}{i} \\ &= \frac{i}{4i^2} \\ &= -\tfrac{1}{4}\,i\end{aligned}$$

(b)
$$\begin{aligned}\frac{1+2i}{3i} &= \frac{1+2i}{3i} \times \frac{i}{i} \\ &= \frac{i+2i^2}{3i^2} \\ &= \frac{2-i}{3}\end{aligned}$$

If on the other hand the denominator is a complex number then the method is to multiply top and bottom by its conjugate, as demonstrated here.

WORKED EXAMPLE 4: Realise the denominators: (a) $\dfrac{5}{2+i}$ (b) $\dfrac{5+2i}{3-4i}$

SOLUTION:

(a)
$$\begin{aligned}\frac{5}{2+i} &= \frac{5}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{5(2-i)}{4+1} \\ &= 2 - i\end{aligned}$$

(b)
$$\begin{aligned}\frac{5+2i}{3-4i} &= \frac{5+2i}{3-4i} \times \frac{3+4i}{3+4i} \\ &= \frac{15+26i-8}{9+16} \\ &= \frac{7+26i}{25}\end{aligned}$$

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> REALISING THE DENOMINATOR: There are two cases.
> A: If the denominator is an imaginary number, multiply top and bottom by i.
> B: If the denominator is complex, multiply top and bottom by its conjugate.

There is now significant evidence that the complex numbers form a valid number system. It has been seen on the previous pages that the four basic arithmetic operations of addition, subtraction, multiplication and division all behave in a sensible way, consistent with real arithmetic.

A Convention for Pronumerals: It is often necessary in developing the theory of complex numbers to perform algebraic manipulations with unknown complex numbers. In order to help distinguish between real and complex variables, the convention that will be used in this text is that the pronumerals x, y, a and b will represent real numbers and the pronumerals z and w will represent complex numbers. Thus in a statement like " Let $z = x+iy$" it is automatically understood that x and y are real whilst z is complex.

Real and Imaginary Parts: Given the complex number $z = x + iy$, the real part of z is the real number x, and the imaginary part of z is the real number y. It is convenient to define two new functions of the complex variable z for these two quantities. Thus

$$\mathrm{Re}(z) = x \qquad \text{and} \qquad \mathrm{Im}(z) = y$$

from which it is clear that

$$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)\,.$$

WORKED EXAMPLE 5: Determine $\mathrm{Re}(z^2 - iz)$ when $z = 3 - i$.

SOLUTION: Expanding the quadratic in z first,

$$\begin{aligned} z^2 - iz &= (3-i)^2 - i(3-i) \\ &= 8 - 6i - 3i - 1 \\ &= 7 - 9i\,, \end{aligned}$$

so $\qquad \mathrm{Re}(z^2 - iz) = 7\,.$

If two complex numbers z and w are equal, by analogy with surds, it is natural to expect that $\mathrm{Re}(z) = \mathrm{Re}(w)$ and $\mathrm{Im}(z) = \mathrm{Im}(w)$. This is in fact the case.

7

> EQUALITY OF COMPLEX NUMBERS: If two complex numbers z and w are equal then
> $$\mathrm{Re}(z) = \mathrm{Re}(w) \qquad \text{and} \qquad \mathrm{Im}(z) = \mathrm{Im}(w)\,.$$

PROOF: Let $z = x + iy$ and $w = a + ib$, and suppose that $z = w$. Then

$$x + iy = a + ib\,.$$

Rearranging $i(y - b) = a - x\,.$ $\qquad (**)$

By way of contradiction, suppose that $y - b \neq 0$, then

$$i = \frac{a-x}{y-b}\,, \text{ which is a real number.}$$

But i is an imaginary number and so there is a contradiction. Thus $y - b = 0$ and hence $y = b$. It follows from equation $(**)$ that $x = a$, and the proof is complete.

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The careful reader will have noticed that the definitions of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ given above are not in terms of the variable z. Both of these functions can be expressed in terms of z by first writing down z and its conjugate.

$$\begin{aligned} z &= x + iy \\ \overline{z} &= x - iy \end{aligned}$$

This pair of simultaneous equations can be solved for x and y to obtain:

$$\mathrm{Re}(z) = \tfrac{1}{2}(z + \overline{z}) \quad \text{and} \quad \mathrm{Im}(z) = \tfrac{1}{2i}(z - \overline{z})\,.$$

8 REAL AND IMAGINARY PARTS: These can be written as functions of z.

$$\mathrm{Re}(z) = \tfrac{1}{2}(z + \overline{z}) \quad \text{and} \quad \mathrm{Im}(z) = \tfrac{1}{2i}(z - \overline{z})\,.$$

The Arithmetic of Conjugates: Since taking the complex conjugate of z simply changes the sign of the imaginary part, when it is applied twice in succession the end result leaves z unchanged. Thus

$$\overline{(\overline{z})} = \overline{\left(\overline{x + iy}\right)} = \overline{x - iy} = x + iy = z\,.$$

Another important property of taking conjugates is that it commutes with the four basic arithmetic operations. For example, with addition,

$$\begin{aligned} \overline{(3+i)+(2-4i)} &= \overline{5-3i} \\ &= 5+3i\,, \\ \text{and} \qquad \overline{3+i} + \overline{2-4i} &= 3 - i + 2 + 4i \\ &= 5 + 3i\,. \\ \text{Thus} \qquad \overline{(3+i)+(2-4i)} &= \overline{3+i} + \overline{2-4i}\,. \end{aligned}$$

Notice that it does not matter whether the addition is done before or after taking the conjugate, the result is the same. Here is an example with multiplication.

$$\begin{aligned} \overline{(3+i)(2-4i)} &= \overline{10-10i} \\ &= 10+10i\,, \\ \text{and} \qquad \overline{3+i} \times \overline{2-4i} &= (3-i)(2+4i) \\ &= 10+10i\,. \\ \text{Thus} \qquad \overline{(3+i)(2-4i)} &= \overline{3+i} \times \overline{2-4i}\,. \end{aligned}$$

Again notice that it does not matter whether the multiplication is done before or after taking the conjugate, the result is the same. This is always the case for addition, subtraction, multiplication and division.

9 THE ARITHMETIC OF CONJUGATES: The taking of complex conjugates is commutative with addition, subtraction, multiplication and division.

(a) $\overline{w+z} = \overline{w} + \overline{z}$ (c) $\overline{wz} = \overline{w} \times \overline{z}$

(b) $\overline{w-z} = \overline{w} - \overline{z}$ (d) $\overline{w \div z} = \overline{w} \div \overline{z}$

The proof of these results is left as a question in the exercise. There are two special cases of these results. To get the conjugate of a negative, put $w = 0$ into (b).

$$\begin{aligned} \overline{(-z)} &= \overline{0 - z} \\ &= \overline{0} - \overline{z} \\ \text{thus} \qquad \overline{(-z)} &= -\overline{z}\,. \end{aligned}$$

For the conjugate of a reciprocal, put $w = 1$ in (d) to get

$$\begin{aligned} \overline{z^{-1}} &= \overline{1 \div z} \\ &= \overline{1} \div \overline{z} \\ &= 1 \div \overline{z} \\ \text{thus} \qquad \overline{z^{-1}} &= (\overline{z})^{-1}\,. \end{aligned}$$

Integer Powers: The careful reader will have noted that several of the examples used above involve powers of a complex number despite the fact that the meaning of z^n has not yet been properly defined. If the index n is a positive integer then the meaning of z^n is analogous to the real number definition. Thus

$$z^n = \underbrace{z \times z \times \ldots \times z}_{n \text{ factors}}$$

or, the recursive definition may be used:

$$\begin{aligned} z^1 &= z\,, \\ z^n &= z \times z^{n-1} \qquad \text{for} \qquad n > 1\,. \end{aligned}$$

Just like the real numbers, if $z = 0$ then z^0 is undefined. For all other complex numbers, $z^0 = 1$. Again continuing the analogy with the real numbers, a negative integer power yields a reciprocal. Thus if n is a positive integer then

$$z^{-n} = \frac{1}{z^n}\,, \qquad z \neq 0\,.$$

As with other division by complex numbers, the denominator is usually realised by multiplying by the conjugate. The case when $n = 1$ occurs frequently and should be learnt.

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}}$$

Indices which are not integers will not be considered in this text.

Exercise **1A**

1. Use the rule given in Box 2 to simplify:

(a) i^2 (b) i^4 (c) i^7 (d) i^{13} (e) i^{29} (f) i^{2010} (g) $i^3 + i^4 + i^5$ (h) $i^7 + i^{16} + i^{21} + i^{22}$

2. Evaluate:

(a) $\overline{2i}$ (b) $\overline{3 + i}$ (c) $\overline{1 - i}$ (d) $\overline{5 - 3i}$ (e) $\overline{-3 + 2i}$

3. Express in the form $a + ib$, where a and b are real.

(a) $(7 + 3i) + (5 - 5i)$ (b) $(-8 + 6i) + (2 - 4i)$ (c) $(4 - 2i) - (3 - 7i)$ (d) $(3 - 5i) - (-4 + 6i)$

4. Express in the form $x + iy$, where x and y are real.

(a) $(4 + 5i)i$ (b) $(1 + 2i)(3 - i)$ (c) $(3 + 2i)(4 - i)$ (d) $(-7 + 5i)(8 - 6i)$ (e) $(5 + i)^2$ (f) $(2 - 3i)^2$ (g) $(2 + i)^3$ (h) $(1 - i)^4$ (i) $(3 - i)^4$

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5. Use the rule for the sum of two squares given in Box 4 to simplify:

(a) $(1+2i)(1-2i)$
(b) $(4+i)(4-i)$
(c) $(5+2i)(5-2i)$
(d) $(-4-7i)(-4+7i)$

6. Express in the form $x+iy$, where x and y are real.

(a) $\dfrac{1}{i}$
(b) $\dfrac{2+i}{i}$
(c) $\dfrac{5-i}{1-i}$
(d) $\dfrac{6-7i}{4+i}$
(e) $\dfrac{-11+13i}{5+2i}$
(f) $\dfrac{(1+i)^2}{3-i}$

7. Let $z=1+2i$ and $w=3-i$. Find, in the form $x+iy$:

(a) $\overline{(iz)}$ (b) $w+\overline{z}$ (c) $2z+iw$ (d) $\operatorname{Im}(5i-z)$ (e) z^2

8. Let $z=8+i$ and $w=2-3i$. Find, in the form $x+iy$:

(a) $\overline{z}-w$ (b) $\operatorname{Im}(3iz+2w)$ (c) zw (d) $65\div z$ (e) $z\div w$

9. Let $z=2-i$ and $w=-5-12i$. Find, in the form $x+iy$:

(a) $-zw$ (b) $(1+i)\overline{z}-w$ (c) $\dfrac{10}{\overline{z}}$ (d) $\dfrac{w}{2-3i}$ (e) $\operatorname{Re}\Big((1+4i)z\Big)$

DEVELOPMENT

10. By equating real and imaginary parts, find the real values of x and y given that:

(a) $(x+yi)(2-3i)=-13i$
(b) $(1+4i)(x+yi)=6+7i$
(c) $(1+i)x+(2-3i)y=10$
(d) $x(1+2i)+y(2-i)=4+5i$
(e) $\dfrac{x}{2+i}+\dfrac{y}{2+3i}=4+i$

11. Express in the form $x+iy$, where x and y are real.

(a) $\dfrac{1}{1+i}+\dfrac{2}{1+2i}$
(b) $\dfrac{1+i\sqrt{3}}{2}+\dfrac{2}{1+i\sqrt{3}}$
(c) $\dfrac{3+2i}{2-5i}+\dfrac{3-2i}{2+5i}$
(d) $\dfrac{-8+5i}{-2-4i}-\dfrac{3+8i}{1+2i}$

12. Given that $z=x+iy$ and $w=a+ib$, where a, b, x and y are real, prove that:

(a) $\overline{z+w}=\overline{z}+\overline{w}$
(b) $\overline{z-w}=\overline{z}-\overline{w}$
(c) $\overline{zw}=\overline{z}\,\overline{w}$
(d) $\overline{z^2}=(\overline{z})^2$
(e) $\overline{\left(\dfrac{1}{z}\right)}=\dfrac{1}{\overline{z}}$, $z\neq 0$
(f) $\overline{\left(\dfrac{z}{w}\right)}=\dfrac{\overline{z}}{\overline{w}}$, $w\neq 0$

13. Let $z=a+ib$, where a and b are real and non-zero. Prove that:

(a) $z+\overline{z}$ is real,
(b) $z-\overline{z}$ is imaginary,
(c) $z^2+(\overline{z})^2$ is real,
(d) $z\overline{z}$ is real and positive.

14. Let $z=a+ib$, where a and b are real. If $\dfrac{z}{z-i}$ is real, show that z is imaginary or 0.

15. Prove that if $z^2=(\overline{z})^2$ then z can only be purely real or purely imaginary.

16. If $z=x+iy$, where x and y are real, express in the form $a+ib$, where a and b are written in terms of x and y.

(a) z^{-1} (b) z^{-2} (c) $\dfrac{z-1}{z+1}$

ENRICHMENT

17. If both $z + w$ and zw are real, prove that either $z = \overline{w}$ or $\mathrm{Im}(z) = \mathrm{Im}(w) = 0$.

18. Given that $z = 2(\cos\theta + i\sin\theta)$, show that $\mathrm{Re}\left(\dfrac{1}{1-z}\right) = \dfrac{1 - 2\cos\theta}{5 - 4\cos\theta}$.

19. Show that $\dfrac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$.

20. If $z = \cos\theta + i\sin\theta$, show that $\dfrac{2}{1+z} = 1 - it$, where $t = \tan\frac{\theta}{2}$.

1B Quadratic Equations

Now that the arithmetic of complex numbers has been satisfactorily developed, it is appropriate to return to the original problem of solving quadratic equations. To reflect the fact that the solutions may be complex, the variable z will be used.

Quadratic Equations with Real Coefficients: The simplest quadratic equations are the perfect square

$$(z - \lambda)^2 = 0$$

for which $z = \lambda$,

and the difference of two squares

$$z^2 - \lambda^2 = 0$$

for which $z = -\lambda$ or λ.

It is now also possible to solve equations involving the sum of two squares, using the result of Box 4 in Section 1A.

Given $z^2 + \lambda^2 = 0$

$$(z + i\lambda)(z - i\lambda) = 0 \qquad \text{(the sum of two squares)}$$

so $z = -i\lambda$ or $i\lambda$.

Thus there are three possible cases for a simple quadratic equation: a perfect square, the difference of two squares, or the sum of two squares.

WORKED EXAMPLE 6: Find the two imaginary solutions of $z^2 + 10 = 0$.

SOLUTION: From the sum of two squares

$$(z + i\sqrt{10})(z - i\sqrt{10}) = 0$$

so $z = -i\sqrt{10}$ or $i\sqrt{10}$

For more general quadratic equations, it is simply a matter of completing the square in z to obtain one of the same three situations: a perfect square, the difference of two squares, or the sum of two squares.

WORKED EXAMPLE 7: Find the complex solutions of $z^2 + 6z + 25 = 0$.

SOLUTION: Completing the square:

$$(z + 3)^2 + 16 = 0$$

so $(z + 3 + 4i)(z + 3 - 4i) = 0$ (sum of two squares)

thus $z = -3 - 4i$ or $-3 + 4i$.

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Notice that the sum of two squares situation always yields two roots which are complex conjugates.

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> QUADRATIC EQUATIONS WITH REAL COEFFICIENTS: Complete the square in z to obtain one of the following situations:
> A. A PERFECT SQUARE: There is only one real root.
> B. THE DIFFERENCE OF TWO SQUARES: There are two real roots.
> C. THE SUM OF TWO SQUARES: There are two conjugate complex roots.

There are several of ways of proving the assertion that complex solutions to quadratic equations with real coefficients must occur as conjugate pairs. The approach presented here will later be extended to encompass all polynomials with real coefficients.

PROOF: Let $Q(z) = az^2 + bz + c$, where a, b and c are real numbers. Suppose that the equation $Q(z) = 0$ has a complex solution $z = w$. It follows that

$$aw^2 + bw + c = 0\,.$$

Take the conjugate of both sides of this equation to get

$$\overline{aw^2 + bw + c} = \overline{0}\,.$$

Now the conjugate of a real number is the same real number. Further, as noted in Box 9, taking a conjugate is commutative with addition and multiplication. Thus the last equation becomes

$$a(\overline{w})^2 + b(\overline{w}) + c = 0\,,$$

that is, $$Q(\overline{w}) = 0\,.$$

Hence $z = \overline{w}$ is also a complex root of $Q(z) = 0$. That is, $Q(z) = 0$ must have two conjugate complex roots, $z = w$ and $z = \overline{w}$, and the proof is complete.

WORKED EXAMPLE 8: Find a quadratic equation with real coefficients given that one of the roots is $w = 5 - i$.

SOLUTION: The coefficients are real so the roots occur in conjugate pairs. Hence the other root is $\overline{w} = 5 + i$. Thus the monic quadratic equation is:

$$\big(z - (5 - i)\big)\big(z - (5 + i)\big) = 0$$

or $$\big((z - 5) + i\big)\big((z - 5) - i\big) = 0$$

thus $$(z - 5)^2 + 1 = 0\,.$$

Finally $$z^2 - 10z + 26 = 0\,.$$

In general, a quadratic equation with real coefficients which has a complex root $z = \alpha$ is

$$z^2 - 2\,\mathrm{Re}(\alpha)\,z + \alpha\,\overline{\alpha} = 0\,,$$

as can be observed in the three worked examples above. The proof is quite straightforward, and is one of the questions in the exercise.

11

> REAL QUADRATIC EQUATIONS WITH COMPLEX ROOTS: A quadratic equation with real coefficients which has a complex root $z = \alpha$ is
> $$z^2 - 2\,\mathrm{Re}(\alpha)\,z + \alpha\,\overline{\alpha} = 0\,.$$

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The Quadratic Method: Many readers will know the quadratic formula as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

There is a problem with this formula when complex numbers are involved. When applied to real numbers, the symbol $\sqrt{\ }$ means the positive square root, but it is unclear what "positive" means when applied to complex numbers. It might be tempting to say that i is positive and $-i$ is negative, but then what is to be said about numbers like $(-1+i)$ or $(1-i)$? In short, it does not make sense to speak of positive and negative complex numbers, and so the positive square root has no meaning. Thus it is not appropriate to blindly use the quadratic formula to solve an equation with complex roots.

Recall that the quadratic formula arose from applying the method of completing the square. Here that process is reviewed.

Given $\qquad az^2 + bz + c = 0\,,$

$\qquad z^2 + \frac{b}{a}z = -\frac{c}{a}$

so $\qquad \left(z + \frac{b}{2a}\right)^2 = \frac{\Delta}{(2a)^2}\,, \qquad$ where $\Delta = b^2 - 4ac\,.$

Now suppose there exists a number λ, possibly complex, such that $\Delta = \lambda^2$.

Then $\qquad \left(z + \frac{b}{2a}\right)^2 - \left(\frac{\lambda}{2a}\right)^2 = 0$

whence $\quad \left(z + \frac{b+\lambda}{2a}\right)\left(z + \frac{b-\lambda}{2a}\right) = 0 \qquad$ (the difference of two squares)

and so $\qquad z = \frac{-b-\lambda}{2a}$ or $\frac{-b+\lambda}{2a}\,.$

Thus if there is a number λ, possibly complex, for which $\lambda^2 = \Delta$, then the solution to the quadratic equation can be written using the last line above. If the quadratic formula is to be applied then this method should always be followed.

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THE QUADRATIC METHOD: Use the following steps to solve $az^2 + bz + c = 0\,.$

1. First find $\Delta = b^2 - 4ac\,.$
2. Next find a number λ, possibly complex, such that $\lambda^2 = \Delta\,.$
3. Finally, the roots are $z = \dfrac{-b-\lambda}{2a}$ or $\dfrac{-b+\lambda}{2a}\,.$

WORKED EXAMPLE 9: Solve $z^2 + 2z + 6 = 0\,.$

SOLUTION:

$$\begin{aligned} \Delta &= 2^2 - 4 \times 1 \times 6 \\ &= -20 \\ &= \left(2i\sqrt{5}\right)^2, \end{aligned}$$

hence

$$\begin{aligned} z &= \frac{-2 - 2i\sqrt{5}}{2} \text{ or } \frac{-2 + 2i\sqrt{5}}{2} \\ &= -1 - i\sqrt{5} \text{ or } -1 + i\sqrt{5}\,. \end{aligned}$$

Complex Square Roots: Before extending the above work to the case of a quadratic equation with complex coefficients, it is necessary to develop methods for finding the square roots of complex numbers.

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The first thing to notice is that, just like real numbers, every complex number has two square roots. The proof is quite straightforward. Suppose that the complex number z is a square root of another complex number w then

$$z^2 = w\,.$$

Further $$(-z)^2 = z^2 = w\,.$$

Hence w has a second square root which is the opposite of the first, namely $(-z)$. Thus for example $-2i$ has two opposite square roots, $(1-i)$ and $(-1+i)$. This is not really very surprising since all real numbers (other than zero) have two opposite square roots. For example, 9 has square roots 3 and -3, whilst -5 has square roots $i\sqrt{5}$ and $-i\sqrt{5}$. The proof that there are no more than two square roots is left as an exercise.

Complex Square Roots and Pythagoras: A simple way to find complex square roots is to equate the real and imaginary parts of $z^2 = w$ in order to obtain a pair of simultaneous equations.

Given $$(x+iy)^2 = a+ib\,, \quad \text{where } x,\ y,\ a \text{ and } b \text{ are real,}$$
$$x^2 - y^2 + 2ixy = a + ib\,.$$

Equating real and imaginary parts yields
$$x^2 - y^2 = a$$
and $$xy = \tfrac{1}{2}\,b\,.$$

In many cases this pair of equations can be easily solved by inspecting the factors of $\frac{1}{2}b$, as in the following example.

WORKED EXAMPLE 10: Find the square roots of $7+24i$.

SOLUTION: Let $$(x+iy)^2 = 7+24i\,, \quad \text{where } x \text{ and } y \text{ are real,}$$
then $$(x^2-y^2) + 2ixy = 7 + 24i\,.$$
Equating real and imaginary parts yields the simultaneous equations
$$x^2 - y^2 = 7$$
and $$xy = 12\,.$$

Inspecting the factors of 12, it is clear that $x = 4$ and $y = 3$, or $x = -4$ and $y = -3$. Hence the square roots of $7 + 24i$ are the opposites

$$4 + 3i \quad \text{and} \quad -4 - 3i\,.$$

Some readers will have noticed in the above example that 7 and 24 are the first two numbers of the Pythagorean triad 7, 24, 25. This is no coincidence. It is often the case that if b is even and the numbers $|a|$, $|b|$ and $\sqrt{a^2+b^2}$ form a Pythagorean triad then the resulting equations for x and y can be simply solved by inspecting the factors of $\frac{1}{2}\,b$.

13 COMPLEX SQUARE ROOTS AND PYTHAGORAS: Given $(x+iy)^2 = a+ib$, equate the real and imaginary parts to get the simultaneous equations

$$\begin{aligned} x^2 - y^2 &= a \\ xy &= \tfrac{1}{2}\,b\,. \end{aligned}$$

If b is even and the numbers $|a|$, $|b|$ and $\sqrt{a^2+b^2}$ form a Pythagorean triad then these equations can often be solved by inspecting the factors of $\frac{1}{2}\,b$.

Quadratic Equations with Complex Coefficients: Simple quadratic equations with complex coefficients can now be solved. All that is needed is to combine the above method for finding the roots of a complex number with either the method of completing the square or the quadratic method in Box 12.

WORKED EXAMPLE 11: Complete the square to solve $z^2-(2+6i)z+(-5+2i) = 0$.

SOLUTION: Rearranging

$$\begin{aligned} z^2 - 2(1+3i)z &= 5-2i \\ \text{so} \quad (z-(1+3i))^2 &= (1+3i)^2 + 5 - 2i \\ &= -8+6i+5-2i\,, \\ \text{thus} \quad (z-(1+3i))^2 &= -3+4i\,. \end{aligned}$$

$$\begin{aligned} \text{Let} \quad (x+iy)^2 &= -3+4i \\ \text{then} \quad x^2-y^2 &= -3 \\ \text{and} \quad xy &= 2 \end{aligned}$$

so by inspection one solution is $x = 1$ and $y = 2$.

$$\begin{aligned} \text{Hence} \quad (z-(1+3i))^2 &= (1+2i)^2 \\ \text{and thus} \quad z &= (1+3i)+(1+2i) \ \text{ or } \ (1+3i)-(1+2i) \\ \text{that is} \quad z &= 2+5i \ \text{ or } \ i\,. \end{aligned}$$

Whilst the focus here is on the quadratic method and completing the square, those two methods should only be used when required. It is always preferable to solve a quadratic equation by factors whenever they can be easily identified.

WORKED EXAMPLE 12: Solve $z^2 + 4i\,z - 3 = 0$.

SOLUTION: Noting that $-3 = i \times 3i$ and $4i = i + 3i$,

$$\begin{aligned} (z+i)(z+3i) &= 0 \\ \text{hence} \quad z &= -i \ \text{ or } \ -3i\,. \end{aligned}$$

Harder Complex Square Roots: Often the simultaneous equations given in Box 13 cannot be solved by inspection. Fortunately there is an identity that can be used to help solve these equations. Recall that if

$$\begin{aligned} (x+iy)^2 &= a+ib \\ \text{then} \quad x^2-y^2 &= a \qquad (1) \\ \text{and} \quad 2xy &= b\,. \qquad (2) \end{aligned}$$

Squaring these and adding:

$$a^2+b^2 = (x^2-y^2)^2 + (2xy)^2$$

$$= (x^2)^2 + 2x^2y^2 + (y^2)^2$$
$$= (x^2 + y^2)^2 .$$

Hence $\quad x^2 + y^2 = \sqrt{a^2 + b^2} \qquad (3)$

Equations (1) and (3) now form a very simple pair of simultaneous equations to solve. Equation (2) is used to determine whether x and y have the same sign, when $b > 0$, or opposite sign, when $b < 0$.

14

> SQUARE ROOTS OF A COMPLEX NUMBER: Given $(x + iy)^2 = a + ib$ then x and y are solutions of the pair of simultaneous equations
>
> $$x^2 - y^2 = a$$
> $$x^2 + y^2 = \sqrt{a^2 + b^2}$$
>
> with the same sign if b is positive, and opposite sign if b is negative.

WORKED EXAMPLE 13: Determine the two square roots of $-4 + 2i$.

SOLUTION: Let $(x + iy)^2 = -4 + 2i$. Since $\mathrm{Im}(-4 + 2i) > 0$, x and y have the same sign. Further, $(-4)^2 + 2^2 = 20$, so solve

$$x^2 - y^2 = -4 \qquad (1)$$

and $\quad x^2 + y^2 = 2\sqrt{5} \qquad (2)$

Adding (1) and (2) yields

$$2x^2 = -4 + 2\sqrt{5}$$

so $\quad x = -\sqrt{-2 + \sqrt{5}}$ or $\sqrt{-2 + \sqrt{5}}$.

Subtracting (1) from (2) yields

$$2y^2 = 4 + 2\sqrt{5}$$

so $\quad y = -\sqrt{2 + \sqrt{5}}$ or $\sqrt{2 + \sqrt{5}}$.

Hence $\quad x + iy = -\sqrt{-2 + \sqrt{5}} - i\sqrt{2 + \sqrt{5}}$ or $\sqrt{-2 + \sqrt{5}} + i\sqrt{2 + \sqrt{5}}$.

In fact, the result in Box 14 can be used to develop a formula for the square roots of any complex number, which is derived in one of the Exercise questions. However that formula is not part of the course and should not be memorised.

Harder Quadratic Equations: Any quadratic equation can now be solved, including those with complex discriminants. Box 14 is used to find the square roots of discriminants that cannot be found by inspection.

WORKED EXAMPLE 14: [A HARD EXAMPLE] Solve $z^2 + (4 - 2i)z + 1 = 0$ by using the quadratic method.

SOLUTION:

$$\begin{aligned} \Delta &= (4 - 2i)^2 - 4 \\ &= 12 - 16i - 4 \\ &= 8 - 16i . \end{aligned}$$

Let $\quad (x + iy)^2 = 8 - 16i.$

Now $\mathrm{Im}(8 - 16i) < 0$ so x and y have opposite sign, with

$$x^2 - y^2 = 8 \qquad (1)$$
and
$$x^2 + y^2 = \sqrt{8^2 + 16^2}$$
or
$$x^2 + y^2 = 8\sqrt{5} \qquad (2)$$
Adding and subtracting equations (1) and (2) yields
$$2x^2 = 8 + 8\sqrt{5} \qquad \text{and} \qquad 2y^2 = -8 + 8\sqrt{5}$$
$$x^2 = 4(1+\sqrt{5}) \qquad\qquad y^2 = 4(-1+\sqrt{5})\,.$$
Thus
$$\Delta = \left(2\sqrt{1+\sqrt{5}} - 2i\sqrt{-1+\sqrt{5}}\right)^2$$
and so
$$z = \tfrac{1}{2}\left(-4 + 2i + 2\sqrt{1+\sqrt{5}} - 2i\sqrt{-1+\sqrt{5}}\right)$$
$$\text{or } \tfrac{1}{2}\left(-4 + 2i - 2\sqrt{1+\sqrt{5}} + 2i\sqrt{-1+\sqrt{5}}\right)$$
that is
$$z = \left(\left(-2 + \sqrt{1+\sqrt{5}}\right) + i\left(1 - \sqrt{-1+\sqrt{5}}\right)\right)$$
$$\text{or } \left(\left(-2 - \sqrt{1+\sqrt{5}}\right) + i\left(1 + \sqrt{-1+\sqrt{5}}\right)\right)$$

Exercise **1B**

1. Solve for z.
(a) $z^2 + 9 = 0$
(b) $(z-2)^2 + 16 = 0$
(c) $z^2 + 2z + 5 = 0$
(d) $z^2 - 6z + 10 = 0$
(e) $16z^2 - 16z + 5 = 0$
(f) $4z^2 + 12z + 25 = 0$

2. Write as a product of two complex linear factors.
(a) $z^2 + 36$
(b) $z^2 + 8$
(c) $z^2 - 2z + 10$
(d) $z^2 + 4z + 5$
(e) $z^2 - 6z + 14$
(f) $z^2 + z + 1$

3. Form a quadratic equation with real coefficients given that one root is:
(a) $i\sqrt{2}$
(b) $1 - i$
(c) $-1 + 2i$
(d) $2 - i\sqrt{3}$

4. In each case, find the two square roots of the given number by the inspection method.
(a) $2i$
(b) $3 + 4i$
(c) $-8 - 6i$
(d) $35 + 12i$
(e) $-5 + 12i$
(f) $24 - 10i$
(g) $-15 - 8i$
(h) $9 - 40i$

DEVELOPMENT

5. (a) Find the two square roots of $-3 - 4i$.
(b) Hence solve $z^2 - 3z + (3 + i) = 0$.

6. (a) Find the two square roots of $-8 + 6i$.
(b) Hence solve $z^2 - (7 - i)z + (14 - 5i) = 0$.

7. Use the method outlined in Box 11 to solve for z.
(a) $z^2 - z + (1 + i) = 0$
(b) $z^2 + 3z + (4 + 6i) = 0$
(c) $z^2 - 6z + (9 - 2i) = 0$
(d) $(1 + i)z^2 + z - 5 = 0$
(e) $z^2 + (2 + i)z - 13(1 - i) = 0$
(f) $iz^2 - 2(1 + i)z + 10 = 0$

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8. (a) Find the value of w if i is a root of the equation $z^2 + wz + (1+i) = 0$.
(b) Find the real numbers a and b given that $3-2i$ is a root of the equation $z^2+az+b=0$.
(c) Given that $1-2i$ is a root of the equation $z^2-(3+i)z+k=0$, find k and the other root of the equation.

9. Find the two complex numbers z satisfying $z\overline{z} = 5$ and $\dfrac{z}{\overline{z}} = \dfrac{1}{5}(3+4i)$.

10. (a) Solve $z^2 - 2z\cos\theta + 1 = 0$ for z by completing the square.
(b) Rearranging the equation in part (a) gives $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$. Confirm this result for each of the solutions to part (a) by substitution.

11. By first factoring the sum or difference of two cubes, solve for z.
(a) $z^3 = -1$ (b) $z^3 + i = 0$

12. Consider the quadratic equation $az^2+bz+c=0$, where a, b and c are real and $b^2-4ac<0$. Suppose that ω is one of the complex roots of the equation.
(a) Explain why $a\omega^2 + b\omega + c = 0$.
(b) By taking the conjugate of both sides of the result in (a), and using the properties of conjugates, show that $a\left(\overline{\omega}\right)^2 + b\,\overline{\omega} + c = 0$.
(c) What have you just proved about the two complex roots of the equation?

13. Suppose that $z = \alpha$ is a complex solution to a quadratic equation with real coefficients.
(a) Which other number is also a solution of this quadratic equation?
(b) Hence prove that one such quadratic equation is $z^2 - 2\,\mathrm{Re}(\alpha)z + \alpha\overline{\alpha} = 0$.

14. Let $(x+iy)^2 = a + ib$, then we have $x^2 - y^2 = a$ and $2xy = b$.
(a) For the moment, assume that both a and b are positive.
(i) Sketch the graphs of these two equations on the same number plane.
(ii) What feature of your sketch indicates that there are two square roots of $a+ib$?
(b) Investigate how the sketch changes when either a or b or both are negative or zero.

15. Use the results of Box 13 to find the two square roots of:
(a) $-i$ (b) $-6+8i$ (c) $2+2i\sqrt{3}$ (d) $10-24i$ (e) $2-4i$

16. Find the discriminant and its square roots, and hence solve:
(a) $z^2 + (4+2i)z + (1+2i) = 0$
(b) $z^2 - 2(1+i)z + (2+6i) = 0$
(c) $z^2 + 2(1-i\sqrt{3})z + 2 + 2i\sqrt{3} = 0$
(d) $z^2 + (1-i)z + (i-1) = 0$

ENRICHMENT

17. Let α and β be the two complex roots of $z^3 = 1$. Show that:
(a) $\beta = \overline{\alpha}$, (b) $\alpha^2 = \beta$ and $\beta^2 = \alpha$, (c) $1 + \alpha + \alpha^2 = 0$,
(d) the sum of the first n terms of the series $1 + \alpha + \alpha^2 + \alpha^3 + \ldots$ is either 0, 1 or $-\alpha^2$, depending on the remainder when n is divided by 3.

18. Let a, b and c be real with $b^2 - 4ac < 0$, and suppose that the quadratic equation $az^2 + bz + c = 0$ has complex solutions $\alpha = x + iy$ and $\beta = u + iv$.
(a) By considering the sum and product of the roots, show that

$$\mathrm{Im}(\alpha+\beta) = 0 \quad \text{and} \quad \mathrm{Im}(\alpha\beta) = 0\,.$$

(b) Hence show that $\alpha = \overline{\beta}$.

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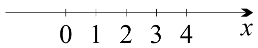
19. Let $(x+iy)^2 = a+ib$, where $b \neq 0$. Use the result of Box 13 to prove the formula:

$$x+iy = \pm\left(\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)} + i\,\frac{b}{|b|}\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}\,\right).$$

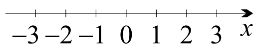
Explain the significance of the term $b/|b|$ in this formula.

1C The Argand Diagram

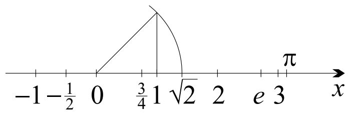
Mathematics requires a knowledge of numbers, and throughout high school that understanding of numbers has been enhanced by being able to plot them on a number line, to visualise their properties and relationships. Initially there were the natural numbers, shown at discrete intervals on the number line.



When negative numbers were included to create the integers, the number line was extended to the left of the origin to show these new numbers.



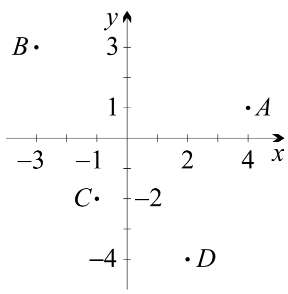
Next came the rationals, the fractions which exist in the spaces between integers. Eventually the irrationals were discovered and included, which fit in the gaps that are somehow left between rationals. Some irrational numbers like $\sqrt{2}$ can be constructed geometrically, but others like e and π can only be approximated to so many decimal places. The construction for $\sqrt{2}$ is shown here along with the positions of $-\frac{1}{2}$, $\frac{3}{4}$, e and π.



The number line is now full, the reals have filled it up, and there is no space left for any new objects like complex numbers. Further, since complex numbers come in two parts, real and imaginary, there is no satisfactory way of representing them on a number line. A two dimensional representation is needed.

The Complex Number Plane: Keeping to things that are familiar, the number plane would seem to be a convenient way to represent complex numbers. More formally, for each complex number $z = x+iy$ there corresponds a point $Z(x,y)$ in the Cartesian plane. Equally, given any point $W(a,b)$ in the real number plane, the associated complex number is $w = a+ib$.

Thus in the diagram on the right, the complex numbers $4+i$ and $-3+3i$ are represented by the points A and B respectively. The points C and D represent the complex numbers $-1-2i$ and $2-4i$. Several different names are used to describe a coordinate plane that is used to represent complex numbers. One name is the *Argand diagram*, after the French mathematician Jean-Robert Argand, born in Geneva in 1768. The terms *complex number plane* or z-plane are also used.



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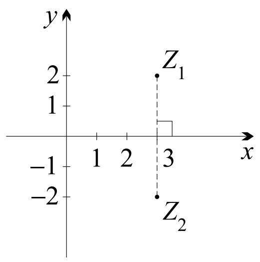
15 THE ARGAND DIAGRAM: The complex number $z = x + iy$ is associated with the point $Z(x, y)$ in the real number plane. A complex number may be represented by a point, and a point may be represented by a complex number.

As a convenient abbreviation, the point $Z(x, y)$ will sometimes be simply referred to as the point z in the Argand diagram. It is important to remember that the complex number plane is just a real number plane which is used to display complex numbers. By the nature of this representation, if two complex numbers are equal then they represent the same point. The converse is also true.

The Real and Imaginary Axes: If $\text{Im}(z) = 0$, that is $z = x + 0i$, then z is a real number and the corresponding point $Z(x, 0)$ in the Argand diagram lies on the horizontal axis. Thus the horizontal axis is called the *real axis*.

Likewise, if $\text{Re}(z) = 0$, that is $z = 0 + iy$, then z is an imaginary number and the corresponding point $Z(0, y)$ in the Argand diagram lies on the vertical axis. Thus the vertical axis is called the *imaginary axis*.

Some Simple Geometry: Now that the complex plane has been introduced, it is immediately possible to observe the geometry of some simple complex number operations. The simplest of these are the geometries of conjugates, opposites, and multiplication by i, which are now investigated.

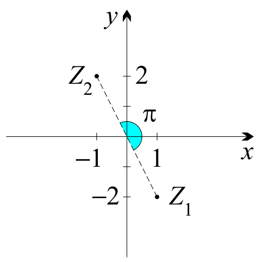


Let $\quad z = x + iy\,,\quad$ then the conjugate is

$$\overline{z} = x - iy\,,$$

that is, y has been replaced by $-y$. This was encountered in the work on graphs and is known to be a reflection in the real axis. This is clearly evident in the example of $z_1 = 3 + 2i$ and $z_2 = 3 - 2i = \overline{z_1}$ shown on the right.

16 THE GEOMETRY OF CONJUGATES: The points z and $\overline{z}$ in the Argand diagram are reflections of each other in the real axis.

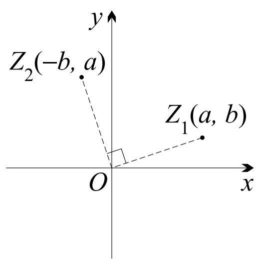


Next, let $\quad z = x + iy\,,\quad$ then the opposite is

$$-z = -x - iy\,.$$

In this case, x and y have been replaced by $-x$ and $-y$ respectively. Thus the result is obtained by reflecting the point in both axes in succession. Alternatively, it is a rotation by π about the origin. The diagram on the right with $z_1 = 1 - 2i$ and $z_2 = -1 + 2i = -z_1$ demonstrates this rotation.

17 THE GEOMETRY OF OPPOSITES: The points z and $-z$ in the Argand diagram are rotations of each other by π about the origin.



Now let $\;z_1 = a + ib\,,\;$ then $z_2 = i\,z_1$ is given by

$$z_2 = -b + ia\,.$$

Consider the corresponding points Z_1 and Z_2 shown in the Argand diagram on the right, where neither a nor b is zero. The product of the gradients of OZ_1 and OZ_2 is

$$\frac{b}{a} \times \frac{a}{-b} = -1\,.$$

Hence OZ_2 is perpendicular to OZ_1 and the conclusion is that multiplication by i is equivalent to an anticlockwise rotation by $\frac{\pi}{2}$ about the origin. The situation is the same even when z_1 is real or imaginary, but not zero, and the proof is left as an exercise.

18 THE GEOMETRY OF MULTIPLICATION BY i: The point iz in the Argand Diagram is the result of rotating the point z by $\frac{\pi}{2}$ anticlockwise about the origin.

Note that multiplication by i twice in succession yields a rotation of $2 \times \frac{\pi}{2} = \pi$. This is consistent with the geometry of opposites, since $i(iz) = i^2 z = -z$.

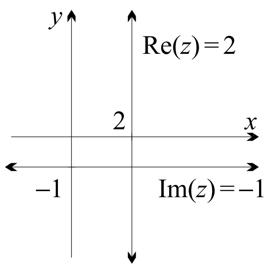
WORKED EXAMPLE 15: Let $z = x+iy$. Determine $i\,\overline{z}$ and hence give a geometric interpretation of the result.

SOLUTION:
$$\begin{aligned} i\,\overline{z} &= i(x - iy) \\ &= y + ix\,. \end{aligned}$$

This is just z with x and y swapped. Thus it is a reflection in the line $y = x$.

Curves in the Argand Diagram: So far attention has been given to individual points in the complex plane. Often an equation in z will correspond to a well known line or curve in the Argand diagram. In the simple cases dealt with here, the equation can be found by putting $z = x + iy$.

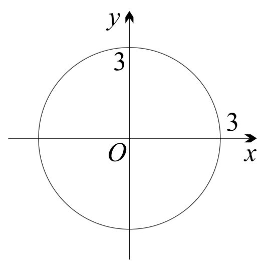
WORKED EXAMPLE 16: Graph the following:
(a) $\text{Re}(z) = 2$, (b) $\text{Im}(z) = -1$.



SOLUTION: The equations give:
(a) the vertical line $x = 2$ and
(b) the horizontal line $y = -1$,
as shown in the diagram on the right.
Note that these two lines intersect at $z = 2 - i$.

19 VERTICAL AND HORIZONTAL LINES: In the Argand diagram:
- the equation $\text{Re}(z) = a$ is the vertical line $x = a$
- the equation $\text{Im}(z) = b$ is the horizontal line $y = b$
- these two lines intersect at $z = a + ib$.

WORKED EXAMPLE 17: Let the point P in the complex plane represent the number $z = x+iy$. Given that $z\,\overline{z} = 9$, find the curve that P is on, and sketch it.



SOLUTION: The given equation becomes

$$(x + iy)(x - iy) = 9$$

so $\quad x^2 + y^2 = 3^2$

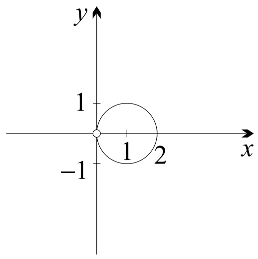
that is, a circle with centre the origin and radius 3.

In some examples it is best to manipulate the given equation in z first, and then substitute $x+iy$. It is also important to note any restrictions on z before starting. Both of these points feature in the following example.

WORKED EXAMPLE 18: Find and describe the curve specified by

$$\frac{1}{z} + \frac{1}{\overline{z}} = 1\,.$$

SOLUTION: Note that in the given equation $z \neq 0$, since the LHS is undefined there. Multiply both sides by the lowest common denominator to get



$$\begin{aligned} &\overline{z} + z = z\,\overline{z} \\ \text{so}\quad &2x = x^2 + y^2 \\ \text{or}\quad &0 = x^2 - 2x + y^2 \\ \text{thus}\quad &1 = (x-1)^2 + y^2 \end{aligned}$$

that is, the circle with radius 1 and centre $(1, 0)$, excluding the origin.

Exercise 1C

1. Write down the coordinates of the point in the complex plane that represents:

(a) 2 (b) i (c) $-3 + 5i$ (d) $\overline{2 + 2i}$ (e) $-5(1 + i)$ (f) $(2 + i)i$

2. Write down the complex number that is represented by the point:

(a) $(-3, 0)$ (b) $(0, 3)$ (c) $(7, -5)$ (d) (a, b)

3. Let $z = 1 + 3i$, and let A, B, C and D be the points representing z, iz, i^2z and i^3z respectively.

(a) Plot the points A, B, C and D in the complex plane.

(b) What type of special quadrilateral is $ABCD$?

(c) What appears to be the geometric effect of multiplying a complex number by i?

4. Let $z = 3 + i$ and $w = 1 + 2i$. Plot the points representing each group of complex numbers on separate Argand diagrams, and describe any geometry you observe.

(a) $z,\ iz,\ -z,\ -iz$ (b) $w,\ iw,\ -w,\ -iw$ (c) $z,\ \overline{z},\ w,\ \overline{w}$ (d) $z,\ w,\ z + w$ (e) $z,\ w,\ z - w$ (f) $z,\ w,\ w - z$

5. Graph the following sets of points in the Argand diagram.

(a) $\text{Re}(z) = -3$ (b) $\text{Im}(z) = 2$ (c) $\text{Im}(z) < 1$ (d) $\text{Re}(z) \geq -2$ (e) $\text{Re}(z) = \text{Im}(z)$ (f) $2\,\text{Re}(z) = \text{Im}(z)$ (g) $\text{Re}(z) \leq 2\,\text{Im}(z)$ (h) $\text{Re}(z) > -\text{Im}(z)$

DEVELOPMENT

6. Let the point P represent the complex number $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$, and let the points Q, R, S and T, represent $\overline{z}$, $-z$, iz and $\frac{1}{z}$ respectively. Plot all these points on an Argand diagram.

7. Show that the point representing $-\overline{z}$ is a reflection of the point representing z in the y-axis.

8. Consider the points represented by the complex numbers z, $\overline{z}$, $-z$ and $-\overline{z}$.
Show that these points form a rectangle by using:

(a) coordinate geometry to show that the diagonals are equal and bisect each other,

(b) the geometry of conjugates and opposites.

9. In the text it was proven that when z is complex, iz is a rotation by $\frac{\pi}{2}$ about the origin. Prove the same result when z is: (a) real, (b) imaginary.

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10. The numbers $z = a + ib$ and $w = iz$ are plotted in the complex plane at A and B respectively.
(a) By considering gradients, show that $OA \perp OB$.
(b) Use the distance formula to show that $OA = OB$.
(c) What type of triangle is $\triangle OAB$?

11. The point P in the complex plane represents the number z. Find and describe the curve that P is on given that

$$\frac{1}{z} - \frac{1}{\overline{z}} = i\,.$$

12. The complex number z is represented by the point C in the Argand diagram. Find and describe the curve that C is on if

$$\operatorname{Re}\left(\frac{z-6}{z}\right) = 0\,.$$

13. Show that $(z-2)\,\overline{(z-2)} = 9$ represents a circle in the Argand diagram.

14. Find and describe the curve in the Argand diagram specified by

$$z\overline{z} = \Big(\operatorname{Re}(z - 1 + 3i)\Big)^2\,.$$

ENRICHMENT

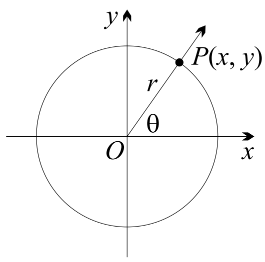
15. Sketch the curve in the complex plane specified by the given equation.
(a) $\operatorname{Im}\left(z^2\right) = 2c^2$ (b) $\operatorname{Re}\left(z^2\right) = c^2$

16. Show that the point representing $-i\overline{z}$ is a reflection of the point representing z in $y = -x$.

17. Show that $\frac{1}{z}$ is a reflection and enlargement of z.

1D Modulus-Argument Form

Recall that in the study of trigonometry it was found that the location of a point P could be expressed either in terms of its horizontal and vertical positions, x and y, or in terms of its distance $OP = r$ from the origin and the angle θ that the ray OP makes with the positive x-axis. The situation is shown in the number plane on the right.



The Modulus and Argument of a Complex Number: In the Argand diagram the distance r is called the *modulus* of z, and owing to its geometric definition as a distance it is written as $|z|$. On squaring:

$$\begin{aligned} |z|^2 &= r^2 \\ &= x^2 + y^2 \\ &= (x+iy)(x-iy) \qquad \text{(sum of two squares)} \end{aligned}$$

hence $\quad |z|^2 = z\overline{z}\,.$

The angle θ is called the *argument* of z, and is written $\theta = \arg(z)$. Just as with trigonometry, θ can take infinitely many values for the same point P. It is often necessary to restrict the angle to just one value called the *principal argument*, which is written $\operatorname{Arg}(z)$. The value of the principal argument is always in the range $-\pi < \operatorname{Arg}(z) \le \pi$. Note the strict inequality on the left hand side, and the use of radian measure.

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There is a problem with measuring θ when $z = 0$ because then P coincides with the origin and there is no angle to measure. For this reason, both $\arg(0)$ and $\operatorname{Arg}(0)$ are undefined. However, it should be clear $|0|$ is defined and that $|0| = 0$.

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MODULUS AND ARGUMENT: Let P represent the complex number $z = x + iy$ in the Argand diagram, with origin O.
- The *modulus* of z is the distance $|z| = r = OP$. Note that $|z|^2 = z\overline{z}$.
- The *argument* of $z \neq 0$ is any angle $\arg(z) = \theta$ that the ray OP can make with the positive real axis. However, $\arg(0)$ is undefined.
- The *principal argument* of $z \neq 0$ is the unique angle $\operatorname{Arg}(z) = \theta$ which is in the range $-\pi < \theta \leq \pi$. However, $\operatorname{Arg}(0)$ is undefined.

From the trigonometric definitions it is clear that

$$x = r\cos\theta \qquad (1)$$

and $$y = r\sin\theta\,, \qquad (2)$$

from which it follows that

$$z = r\cos\theta + ir\sin\theta\,.$$

Notice that the modulus r is a common factor in this last expression and it is more commonly written as

$$z = r(\cos\theta + i\sin\theta)$$

or $$z = r\operatorname{cis}\theta \qquad \text{for short.}$$

In order to contrast the two ways of writing a complex number, $z = x + iy$ is called *real-imaginary* or *Cartesian* form whilst $z = r(\cos\theta + i\sin\theta)$ is called *modulus-argument* form, or *mod-arg* form for short. Another name for mod-arg form is *polar form*, because the radius is measured from the origin which acts as a pole. Equations (1) and (2) above serve to link the two forms.

WORKED EXAMPLE 19: Express each complex number in real-imaginary form.

(a) $z = 4\operatorname{cis}\pi$ (b) $z = 2\operatorname{cis}\frac{\pi}{6}$ (c) $z = \operatorname{cis}\frac{2\pi}{3}$

SOLUTION:

(a) $z = 4\cos\pi + 4i\sin\pi$
$= -4$

(b) $z = 2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6}$
$= \sqrt{3} + i$

(c) $z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$
$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

WORKED EXAMPLE 20: Express each complex number in mod-arg form using the principal argument. In part (c) give $\operatorname{Arg}(z)$ correct to two decimal places.

(a) $z = 5i$ (b) $z = 3 - 3i$ (c) $z = -4 - 3i$

SOLUTION: In each case let $z = r\operatorname{cis}\theta$, with Z the point in the Argand diagram.

(a) In this case Z is on the positive imaginary axis so

$$r = 5$$

and $$\theta = \frac{\pi}{2}\,,$$

hence $$z = 5\operatorname{cis}\frac{\pi}{2}\,.$$

(b) Now Z is in the fourth quadrant with

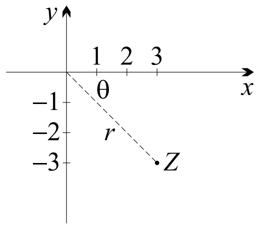
$$r^2 = 3^2 + 3^2$$

or $\quad r = 3\sqrt{2}\,.$

Thus $\quad \cos\theta = \dfrac{1}{\sqrt{2}}$

and $\quad \theta = -\frac{\pi}{4}\,,$

hence $\quad z = 3\sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)\,.$



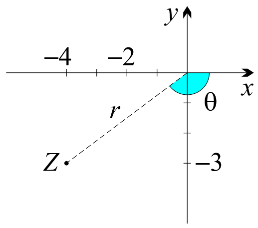
(c) In this case there is a Pythagorian triad so

$$r = 5.$$

Now $\quad \cos\theta = -\frac{4}{5}$ and Z is in the third quadrant

so $\quad \theta = -\pi + \cos^{-1}\frac{4}{5} \doteqdot -2{\cdot}50$ radians,

hence $\quad z \doteqdot 5\,\mathrm{cis}(-2{\cdot}50)\,.$



21 FORMS OF A COMPLEX NUMBER:

- $x + iy$ is called the *real-imaginary* form or *Cartesian* form of z.
- $r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$ is called the *modulus-argument* form of z.
- The equations relating the two forms are:

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

Note that some people prefer to use $\tan\theta = \frac{y}{x}$ to find $\mathrm{Arg}\,z$. This alternative formula should only be used when the quadrant is known for θ, otherwise there is a potential problem. By way of example, suppose that $z = k(1+i)$ for some constant k. The alternative formula would then give

$$\tan\theta = 1\,.$$

It would be tempting at this point to write $\mathrm{Arg}\,z = \frac{\pi}{4}$. Whilst this is correct when $k > 0$, it is wrong when $k \le 0$. In fact when $k < 0$, z lies in the third quadrant and so $\mathrm{Arg}\,z = -\frac{3\pi}{4}$. And, of course, $\mathrm{Arg}\,z$ is undefined when $k = 0$.

Some Simple Algebra: As will be revealed over the remainder of this chapter, the use of mod-arg form is a powerful tool, both in simplifying much algebra and in providing geometric interpretations.

Perhaps the most obvious thing to notice is that $|\,\mathrm{cis}\,\theta| = 1$. The geometry of the situation makes the result obvious since if $z = \cos\theta + i\sin\theta$ then the point $Z(\cos\theta, \sin\theta)$ lies on the unit circle. Hence $|z| = OZ = 1$. Here is an algebraic derivation of the same result.

$$\begin{aligned} |\cos\theta + i\sin\theta|^2 &= \cos^2\theta + \sin^2\theta \\ &= 1\,, \end{aligned}$$

hence $\quad |\,\mathrm{cis}\,\theta| = 1\,.$

This identity has immediate applications in quadratic equations. If $\mathrm{cis}\,\theta$ is one of the roots, then the constant term of the monic quadratic must be 1, as the following worked example demonstrates.

WORKED EXAMPLE 21: Find a quadratic equation with real coefficients given that one root is $z = \mathrm{cis}\,\theta\,.$

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SOLUTION: The coefficients are real so the other root must be $\overline{\operatorname{cis}\theta}$. Thus:

$$(z - \operatorname{cis}\theta)(z - \overline{\operatorname{cis}\theta}) = 0$$

or $$z^2 - (\operatorname{cis}\theta + \overline{\operatorname{cis}\theta})z + \operatorname{cis}\theta \times \overline{\operatorname{cis}\theta} = 0$$

that is $$z^2 - 2\operatorname{Re}(\operatorname{cis}\theta)z + |\operatorname{cis}\theta|^2 = 0$$

thus $$z^2 - 2z\cos\theta + 1 = 0\,.$$

The Product of Two Complex Numbers: The modulus-argument form of the product of two numbers is a particularly important result. Let $w = a\operatorname{cis}\theta$ and $z = b\operatorname{cis}\phi$, with $a \neq 0$ and $b \neq 0$. Then:

$$\begin{aligned} wz &= a(\cos\theta + i\sin\theta) \times b(\cos\phi + i\sin\phi) \\ &= ab\Big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\Big) \\ &= ab\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big)\,, \\ &= ab\operatorname{cis}(\theta+\phi)\,. \end{aligned}$$

Thus $\quad |wz| = ab \quad$ and $\quad \arg(wz) = \theta + \phi\,.$

This yields the following two significant results:

$$|wz| = |w|\,|z|$$

and $$\arg(wz) = \arg(w) + \arg(z)\,.$$

WORKED EXAMPLE 22: Let $w = \sqrt{3} + i$ and $z = 1 + i$.

(a) Evaluate wz in real-imaginary form.

(b) Express w and z in mod-arg form and hence evaluate wz in mod-arg form.

(c) Hence find the exact value of $\cos\frac{5\pi}{12}$.

SOLUTION:

(a) $$\begin{aligned} wz &= (\sqrt{3} + i)(1 + i) \\ &= (\sqrt{3} - 1) + i(\sqrt{3} + 1)\,. \end{aligned}$$

(b) Now $$\begin{aligned} w &= 2\operatorname{cis}\tfrac{\pi}{6} \\ z &= \sqrt{2}\operatorname{cis}\tfrac{\pi}{4}\,, \end{aligned}$$

hence $$\begin{aligned} wz &= 2\sqrt{2}\operatorname{cis}(\tfrac{\pi}{6} + \tfrac{\pi}{4}) \\ &= 2\sqrt{2}\operatorname{cis}\tfrac{5\pi}{12}\,. \end{aligned}$$

(c) Equating the real parts of parts (a) and (b) yields

$$2\sqrt{2}\cos\tfrac{5\pi}{12} = \sqrt{3} - 1$$

hence $$\cos\tfrac{5\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}}\,.$$

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THE PRODUCT OF TWO COMPLEX NUMBERS: Let w and z be two complex numbers.

- The modulus of the product is the product of the moduli, that is:

$$|wz| = |w|\,|z|\,.$$

- The argument of the product is the sum of the arguments, that is:

$$\arg(zw) = \arg(w) + \arg(z) \qquad \text{(provided } w \neq 0 \text{ and } z \neq 0\,.)$$

A question in the exercise deals with the case of division of complex numbers.

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Some Simple Geometry Again: It is instructive to re-examine the geometry of conjugates, opposites and multiplication by i using mod-arg form. Beginning with the conjugate, recall that the result is a reflection in the real axis. Thus the modulus should be unchanged, and the argument should be opposite.

Let $z = r\operatorname{cis}\theta$ then

$$\begin{aligned}\overline{z} &= r\cos\theta - ir\sin\theta\\ &= r\cos(-\theta) + ir\sin(-\theta)\\ &= r\operatorname{cis}(-\theta)\end{aligned}$$

hence $|\overline{z}| = |z|$

and $\arg(\overline{z}) = -\arg(z)$

that is, the modulus is unchanged and the angle is opposite, as expected.

The cases of opposites and multiplication by i are more simply dealt with. Recall that these operations represented rotations in the complex plane by π and $\frac{\pi}{2}$ respectively. Thus, again, the modulus should be the same, and the argument should be increased appropriately. Looking at opposites first:

$$|-z| = |(-1)\times z| = |-1|\times|z| = |z|\,,$$

and $\arg(-z) = \arg(-1\times z) = \arg(-1) + \arg(z) = \pi + \arg(z)\,.$

That is, the moduli of opposites are equal and the arguments differ by π.

Similarly $|iz| = |i|\,|z| = |z|\,,$

and $\arg(iz) = \arg(i) + \arg(z) = \frac{\pi}{2} + \arg(z)\,.$

That is, the moduli of z and iz are equal and the arguments differ by $\frac{\pi}{2}$. In both cases the results are exactly as expected. Also notice that the principal arguments of -1 and i have been used. As an exercise, justify why this is correct.

The Geometry of Multiplication: Aside from the special cases above, the geometry of multiplication is evident in the results of Box 22. The product of the moduli indicates an enlargement with centre the origin, and the sum of the arguments represents an anticlockwise rotation about the origin.

Consider these two transformations individually and let $w = r\operatorname{cis}\theta$. When $\theta = 0$ the product wz reduces to $wz = rz$, which is an enlargement without any rotation. Thus both z and rz lie on the same ray.

When $|w| = r = 1$ the product wz becomes $wz = z\operatorname{cis}\theta$. Using Box 22:

$$\begin{aligned}|wz| &= |w||z|\\ &= |z|\end{aligned}$$

and

$$\begin{aligned}\arg(wz) &= \arg w + \arg z\\ &= \theta + \arg z\,.\end{aligned}$$

This is simply a rotation without any enlargement. Thus z and $z\operatorname{cis}\theta$ both lie on a circle of radius $|z|$. The following example serves to demonstrate the situation.

WORKED EXAMPLE 23: Let $z = 1 + i$.

(a) Find, in Cartesian form, the complex number w such that wz is a rotation of z by $\frac{\pi}{3}$ about the origin.

(b) Evaluate wz in Cartesian form.

(c) Verify that $|wz| = |z|$, then plot z and wz on an Argand diagram.

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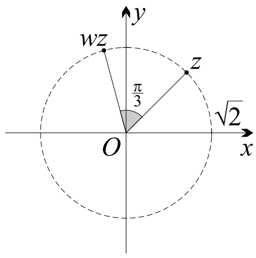
SOLUTION:

(a) Clearly $w = \operatorname{cis}\frac{\pi}{3}$
$= \frac{1}{2}(1 + i\sqrt{3})\,.$

(b) $wz = \frac{1}{2}(1 + i\sqrt{3})(1 + i)$
$= \frac{1}{2}\Big((1 - \sqrt{3}) + i(1 + \sqrt{3})\Big)\,.$

(c) Now $|wz|^2 = \frac{1}{4}\Big((1 - \sqrt{3})^2 + (1 + \sqrt{3})^2\Big)$
$= \frac{1}{4}(1 + 3 + 1 + 3)$
$= 2\,.$

Hence $|wz| = \sqrt{2}$
$= |z|\,.$

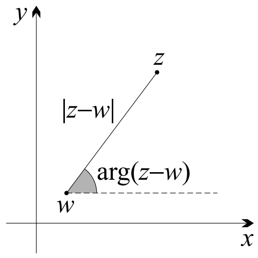


23 **THE GEOMETRY OF MULTIPLICATION:** Let $w = r\operatorname{cis}\theta$. Then the complex number wz is the result of a rotation of z by θ anti-clockwise about the origin and an enlargement of z by factor r with centre the origin.

The corresponding geometry for the division of complex numbers is similar and is dealt with in one of the exercise questions.

Shifting in the Complex Plane: Recall that, for a real number x, the value of $|x|$ is the distance from the origin to x. Shifting this, $|x - a|$ is the distance from a to x. Although it will not be proven here, the results for shifting can also be applied to the Argand diagram. Thus since $|z|$ is the distance from the origin to z, it follows that $|z - w|$ is the distance from w to z. An algebraic proof of this important result is the subject of a question in the exercise.

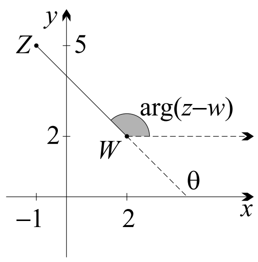
Likewise, since $\arg(z)$ is the angle at the origin between z and the positive real axis, the value of $\arg(z - w)$ is the angle at the vertex w between z and the right half of the horizontal line through w. The situation is shown in the diagram on the right.



WORKED EXAMPLE 24: Let points W and Z represent $w = 2 + 2i$ and $z = -1 + 5i$ respectively. Find:

(a) the length of WZ,

(b) the angle θ that WZ makes with the positive x-axis.



SOLUTION: First note that $z - w = -3 + 3i$.

(a) $WZ = |z - w|$
$= |-3 + 3i|$
$= 3\sqrt{2}$

(b) $\theta = \operatorname{Arg}(z - w)$ (corresponding angles, parallel lines)
$= \operatorname{Arg}(-3 + 3i)$
$= \dfrac{3\pi}{4}\,.$

24 SHIFTING IN THE COMPLEX PLANE: Functions of complex numbers can be shifted in a similar way to functions of real numbers. In particular:
- $|z - w|$ is the distance from w to z.
- $\arg(z - w)$ is the angle at the vertex w between z and the right half of the horizontal line though w.

Exercise **1D**

1. Find $|z|$ given:

(a) $z = 3$ (b) $z = -5i$ (c) $z = 1 - i$ (d) $z = -\sqrt{3} - i$ (e) $z = -3 + 4i$ (f) $z = 15 + 8i$

2. Find $\text{Arg}(z)$ given:

(a) $z = -2$ (b) $z = 4i$ (c) $z = 2 - 2i$ (d) $z = 1 + \sqrt{3}\,i$ (e) $z = -3 + 3i$ (f) $z = -\sqrt{3} - i$

3. Express each complex number in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$.

(a) $2i$ (b) -4 (c) $1 + i$ (d) $\sqrt{3} - i$ (e) $-1 + \sqrt{3}\,i$ (f) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

4. Repeat the previous question for each of these complex numbers, writing θ in radians correct to two decimal places.

(a) $3 + 4i$ (b) $12 - 5i$ (c) $-2 + i$ (d) $-1 - 3i$

5. Express in the form $a + ib$, where a and b are real.

(a) $3\,\text{cis}\,0$ (b) $5\,\text{cis}\left(-\frac{\pi}{2}\right)$ (c) $4\,\text{cis}\,\frac{\pi}{4}$ (d) $6\,\text{cis}\left(-\frac{\pi}{6}\right)$ (e) $2\,\text{cis}\,\frac{3\pi}{4}$ (f) $2\,\text{cis}\left(-\frac{2\pi}{3}\right)$

6. Given that $z = 1 - i$, express in mod-arg form:

(a) z (b) $\overline{z}$ (c) $-z$ (d) iz (e) z^2 (f) $(\overline{z})^{-1}$

7. Simplify each expression, leaving your answer in mod-arg form.

(a) $5\,\text{cis}\,\frac{\pi}{12} \times 2\,\text{cis}\,\frac{\pi}{4}$ (b) $3\,\text{cis}\,\theta \times 3\,\text{cis}\,2\theta$ (c) $6\,\text{cis}\,\frac{\pi}{2} \div 3\,\text{cis}\,\frac{\pi}{6}$ (d) $\dfrac{3\,\text{cis}\,5\alpha}{2\,\text{cis}\,4\alpha}$ (e) $\left(4\,\text{cis}\,\frac{\pi}{5}\right)^2$ (f) $\left(2\,\text{cis}\,\frac{2\pi}{7}\right)^3$

8. Find the distance $|z - w|$ between the following pairs of numbers in the complex plane.

(a) $w = -1 + i$, $z = 1 + 3i$
(b) $w = 4 + 2i$, $z = 1 - i$
(c) $w = 1 + i\sqrt{3}$, $z = 4 - 2i\sqrt{3}$
(d) $w = -3 + i\sqrt{3}$, $z = 3 + 3i\sqrt{3}$
(e) $w = -1 - 3i$, $z = 2 + i$
(f) $w = -1 + i$, $z = -2 - i$

9. Find $\text{Arg}(z - w)$ for each pair of numbers in the previous question. Approximate your answers to 2 decimal places where necessary.

10. Suppose that multiplying a complex number by w produces a rotation of θ radians about the origin. Find w in Cartesian form for the given values of θ.

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{3}$ (d) $\frac{3\pi}{4}$ (e) $\frac{5\pi}{6}$ (f) $-\frac{\pi}{2}$ (g) $-\frac{\pi}{4}$ (h) $-\frac{2\pi}{3}$

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DEVELOPMENT

11. Let z be a non-zero complex number such that $0 < \arg z < \frac{\pi}{2}$. Indicate points A, B, C and D in the complex plane representing the complex numbers z, $-iz$, $(2\operatorname{cis}\frac{\pi}{3})z$ and $\left(\frac{1}{2}\operatorname{cis}(-\frac{\pi}{4})\right)z$.

12. Replace z with $z \div w$ in Box 22 to prove that for $z \neq 0$ and $w \neq 0$:

(a) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ (b) $\arg\left(\dfrac{z}{w}\right) = \arg z - \arg w$

13. Suppose that $z_1 = \sqrt{3} + i$ and $z_2 = 2\sqrt{2} + 2\sqrt{2}\,i$.

(a) Write z_1 and z_2 in mod-arg form. (b) Hence write $z_1 z_2$ and $\dfrac{z_2}{z_1}$ in mod-arg form.

14. Repeat the previous question for $z_1 = -\sqrt{3} + i$ and $z_2 = -1 - i$.

15. (a) Express $\dfrac{1 + i\sqrt{3}}{1+i}$ in real-imaginary form.

(b) Write $1 + i$ and $1 + i\sqrt{3}$ in mod-arg form and hence express $\dfrac{1+i\sqrt{3}}{1+i}$ in mod-arg form.

(c) Hence find $\cos\frac{\pi}{12}$ in surd form.

16. Let $z_1 = 1 + 5i$ and $z_2 = 3 + 2i$, and let $z = \dfrac{z_1}{z_2}$.

(a) Find $|z|$ without finding z.

(b) Find $\tan(\tan^{-1} 5 - \tan^{-1}\frac{2}{3})$, and hence find $\arg z$ without finding z.

(c) Hence write z in the form $x + iy$, where x and y are real.

17. Show that for any non-zero complex number $z = r\operatorname{cis}\theta$:

(a) $z\,\overline{z} = |z|^2$, (b) $\arg(z^2) = 2\arg(z)$, (c) if $|z| = 1$ then $\overline{z} = z^{-1}$.

18. Let z be any non-zero complex number. By considering $\arg(|z|^2)$, use the result in part (a) of the previous question to prove that $\arg\overline{z} = -\arg z$.

19. Let $z = \cos\theta + i\sin\theta$. Determine z^2 in two different ways and hence show that:

(a) $\cos 2\theta = \cos^2\theta - \sin^2\theta$ (b) $\sin 2\theta = 2\sin\theta\cos\theta$

20. The complex number z satisfies the equation $|z - 1| = 1$. Prove that $|z|^2 = 2\operatorname{Re}(z)$ by:

(a) letting $z = x + iy$,

(b) squaring the equation and then using the result $|z|^2 = z\overline{z}$.

21. Given that z is a complex number satisfying $|2z - 1| = |z - 2|$, prove that $|z| = 1$ by:

(a) letting $z = x + iy$,

(b) squaring the equation and then using the result $|z|^2 = z\overline{z}$.

22. Let $z = 1 + \cos\theta + i\sin\theta$, where $-\pi < \theta < \theta$.

(a) Show that $|z| = 2\cos\frac{\theta}{2}$ and $\arg z = \frac{\theta}{2}$. (b) Hence show that $z^{-1} = \frac{1}{2} - \frac{1}{2}i\tan\frac{\theta}{2}$.

ENRICHMENT

23. Let $w = x_1 + iy_1$ and $z = x_2 + iy_2$.

(a) Show that the distance WZ between the points w and z is $|z - w|$.

(b) Show that the angle that the line WZ makes with the positive real axis is $\arg(z - w)$.

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24. Let $z = \operatorname{cis}\theta$ and $w = \operatorname{cis}\phi$, noting that $|z| = |w| = 1$. Evaluate $z+w$ in mod-arg form and hence show that $\arg(z+w) = \frac{1}{2}(\arg z + \arg w)$. [HINT: Use the sum to product identities $\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$ and $\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$.]

25. (a) Prove that $\operatorname{Re}(z) \le |z|$. Under what circumstances are they equal?

(b) Prove that $|z+w| \le |z| + |w|$. Begin by writing $|z+w|^2 = (z+w)\overline{(z+w)}$.

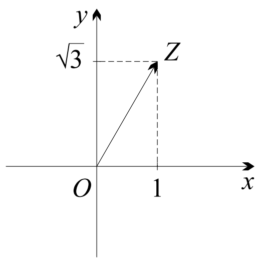
1E Vectors and the Complex Plane

The geometry of multiplication and division became evident with the introduction of the modulus-argument form in the previous section. Since the arguments are added or subtracted, it is clear that a rotation is involved. Since the moduli are multiplied or divided, it is clear that an enlargement is involved.

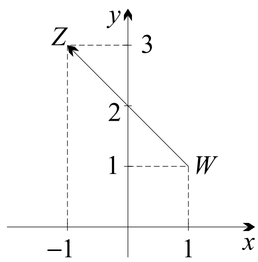
So far, the observed geometry of addition and subtraction has been limited. A better understanding of these two operations is desirable and can be achieved by yet another representation of complex numbers, this time as vectors. In this section, a few vector tools will be introduced to help better understand complex numbers. A more detailed study of vectors is given in the Extension 1 course.

Vectors: In the simple geometric definition used in this section, a vector has two characteristics, a magnitude and a direction. Thus the instruction on a pirate treasure map “walk 40 paces east” is an example of a displacement vector. The magnitude is “40 paces” and the direction is “east”. A train travelling from Sydney to Perth across the Nullarbor at 120 km/h is an example of a velocity vector. The magnitude is 120 km/h and the direction is west.

In the number plane, a vector is represented by an arrow, which is more properly called a directed line segment. The length of the arrow indicates the magnitude of the vector and the direction of the arrow is the direction of the vector. In particular, in the Argand diagram an arrow joining two points will be used to represent the vector from one complex number to another. When naming a vector, the two letter name of the line segment is used with an arrow above it to indicate the direction, as in the following two examples.



$\overrightarrow{OZ}$ is the vector from the origin to $z = 1 + i\sqrt{3}$.



$\overrightarrow{WZ}$ is the vector from $w = 1 + i$ to $z = -1 + 3i$.

Vectors and Complex Numbers: Look further at the examples above. In the first, it is natural to assume that the vector $\overrightarrow{OZ}$ represents the complex number z. But what complex number does the vector $\overrightarrow{WZ}$ represent in the second example?

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The magnitude of a vector is the distance between the end-points of its arrow. That is, the magnitude of $\overrightarrow{WZ}$ is $|z-w|$. The direction of a vector may be specified by the angle its arrow makes with the horizontal. That is, the direction of $\overrightarrow{WZ}$ is given by $\text{Arg}(z-w)$. Since any complex number is completely determined by its modulus and argument, it follows that the $\overrightarrow{WZ}$ must represent the complex number $(z-w)$. This is always the case, regardless of the values of z and w.

25 VECTORS AND COMPLEX NUMBERS: Let the vector from w to z in the Argand diagram be $\overrightarrow{WZ}$. Then the vector $\overrightarrow{WZ}$ represents the complex number $(z-w)$.

Equal Vectors: Suppose that two vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ represent the same complex number. That is, both vectors have the same magnitude and direction. It makes sense to say that these vectors are equal and to write

$$\overrightarrow{AB} = \overrightarrow{PQ}$$

since there is nothing to distinguish between them.

By way of example, let Z_0, Z_1 and Z_2 represent the complex numbers $1+2i$, $2+i$ and $3+3i$ respectively. Then by Box 25,

$$\begin{aligned}\overrightarrow{OZ_0} &= (1+2i) - 0\\ &= 1+2i\end{aligned}$$

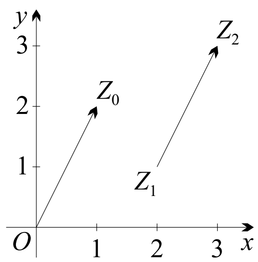
and

$$\begin{aligned}\overrightarrow{Z_1Z_2} &= (3+3i) - (2+i)\\ &= 1+2i.\end{aligned}$$

That is, the same complex number is represented by both vectors, and hence

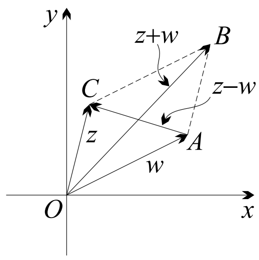
$$\overrightarrow{OZ_0} = \overrightarrow{Z_1Z_2}\,.$$

The diagram to the right shows the situation. Notice that both vectors clearly have the same length and direction.



26 EQUAL VECTORS: Vectors with the same magnitude and direction are equal. Equal vectors represent the same complex number.

Addition and Subtraction: Consider the three points A, B and C which represent the complex numbers w, $w+z$ and z. Now $\overrightarrow{OB} = (z+w)$ and, by Box 25, $\overrightarrow{AC} = (z-w)$. So the diagonals of $OABC$ have special significance. Does this quadrilateral have any other special characteristics?



The magnitude of $\overrightarrow{AB}$ is

$$|(w+z) - w| = |z|\,,$$

thus $AB = OC$. Likewise, the magnitude of $\overrightarrow{CB}$ is

$$|(w+z) - z| = |w|\,,$$

thus $CB = OA$. Now since $OABC$ has opposite sides of equal length, it must be a parallelogram. Consequently the opposite sides are also parallel.

Hence $\overrightarrow{OC} = \overrightarrow{AB}$

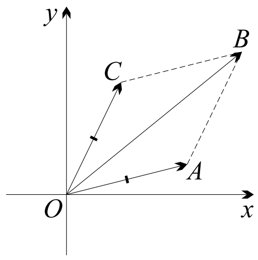
and $\overrightarrow{OA} = \overrightarrow{CB}$.

Thus in order to add or subtract two complex numbers geometrically, simply construct the parallelogram $OABC$ from the vectors $\overrightarrow{OA}$ and $\overrightarrow{OC}$. Then the sum is the vector $\overrightarrow{OB}$ and the difference is $\overrightarrow{AC}$. This result is most useful in solving certain algebraic problems geometrically.

WORKED EXAMPLE 25: Given two non-zero complex numbers w and z with equal moduli and acute arguments, show that $\text{Arg}(w+z) = \frac{1}{2}\big(\text{Arg}(w) + \text{Arg}(z)\big)$.

SOLUTION: Consider the points $OABC$ in the z-plane representing the complex numbers 0, w, $w+z$ and z respectively. Now $OABC$ is a parallelogram. Further $OA = OC$, since $|w| = |z|$. Thus in fact $OABC$ is a rhombus. Since the diagonal OB of the rhombus bisects the angle at the vertex O, it follows that

$$\text{Arg}(w+z) = \tfrac{1}{2}\big(\text{Arg}(w) + \text{Arg}(z)\big).$$



27

> THE GEOMETRY OF ADDITION AND SUBTRACTION: Let the points O, A and C represent the complex numbers 0, w and z. Construct the parallelogram $OABC$. The diagonal vector $\overrightarrow{OB}$ represents the complex number $(z+w)$ and the other diagonal vector $\overrightarrow{AC}$ represents the complex number $(z-w)$.

WORKED EXAMPLE 26: The points $OABC$ represent the complex numbers 0, w, $w+z$ and z. Given that $z - w = i(z+w)$, explain why $OABC$ is a square.

SOLUTION: Firstly $OABC$ is a parallelogram, where $\overrightarrow{OB}$ represents $z+w$ and $\overrightarrow{AC}$ represents $z-w$. Since $z-w = i(z+w)$ it follows that

$$\begin{aligned}\arg(z-w) &= \arg(i(z+w)) \\ &= \arg(i) + \arg(z+w) \\ &= \tfrac{\pi}{2} + \arg(z+w),\end{aligned}$$

and
$$\begin{aligned}|z-w| &= |i(z+w)| \\ &= |i| \times |z+w| \\ &= |z+w|.\end{aligned}$$

Thus the diagonals OB and AC are at right angles to each other and have the same length. Hence $OABC$ is a square.

The Triangle Inequality: An important identity encountered with the absolute value of real numbers is the triangle inequality

$$\Big||x| - |y|\Big| \le |x+y| \le |x| + |y|.$$

Given that the absolute value of a real number is analogous to the modulus of a complex number, it is not surprising that the same result holds for the modulus of complex numbers, that is

$$\Big||z| - |w|\Big| \le |z+w| \le |z| + |w|.$$

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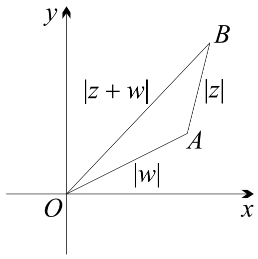
This result can be explained in terms of the geometry of the addition of complex numbers. Consider only the points O, A and B as defined previously and shown in the diagram on the right. Recall that the three vectors $\overrightarrow{OA}$, $\overrightarrow{AB}$ and $\overrightarrow{OB}$ represent the complex numbers w, z and $z + w$ respectively. Hence the three moduli $|w|$, $|z|$ and $|z + w|$ are the lengths of the sides of $\triangle OAB$.



It is a well known result of Euclidean geometry that the length of one side of a triangle must be less than or equal to the sum of the other two, thus

$$|z + w| \le |z| + |w| \,,$$

with equality when O, A and B are collinear. Similarly the length of one side is greater than or equal to the difference of the other two, thus

$$\Big||z| - |w|\Big| \le |z + w| \,,$$

with equality again when the points are collinear. Combining these two yields

$$\Big||z| - |w|\Big| \le |z + w| \le |z| + |w| \,,$$

and replacing w with $-w$ throughout gives

$$\Big||z| - |w|\Big| \le |z - w| \le |z| + |w| \,.$$

These inequalities are called the *triangle inequalities*, after their geometric origins.

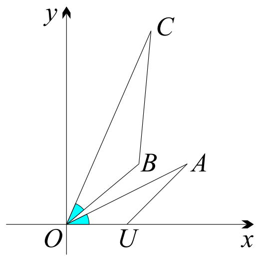
28 THE TRIANGLE INEQUALITIES: For all complex numbers z and w,

- $\Big||z| - |w|\Big| \le |z + w| \le |z| + |w|$
- $\Big||z| - |w|\Big| \le |z - w| \le |z| + |w|$

Multiplication and Division: The geometry of these two operations has already been satisfactorily explained as a rotation and enlargement. This interpretation is further demonstrated by the following example.

The diagram below shows the points O, U, A, B and C which correspond to the complex numbers 0, 1, w, z and wz respectively. In $\triangle UOA$ and $\triangle BOC$,

$$\begin{aligned} \angle BOC &= \arg(wz) - \arg(z) \\ &= \arg(w) + \arg(z) - \arg(z) \\ &= \arg(w) \\ &= \angle UOA \,, \end{aligned}$$



and
$$\begin{aligned} \frac{OC}{OB} &= \frac{|wz|}{|z|} \\ &= |w| \\ &= \frac{OA}{OU} \,. \end{aligned}$$

Hence $\triangle BOC \,|||\, \triangle UOA$ (SAS)

Note that the similarity ratio is $OB : OU = |z| : 1$.

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This provides a novel way of constructing the point C for any given complex numbers w and z. First construct $\triangle UOA$, then use the base OB to construct the similar triangle $\triangle BOC$ by applying the similarity ratio $|z| : 1$.

Other than being an application of similar triangles, this construction method is rarely required. The geometry of the situation should always be remembered as a rotation of w by $\arg(z)$ and an enlargement by factor $|z|$.

Two Special Cases: A vector approach is very helpful in analysing the geometry in two special cases of division. Let z_1, z_2, z_3 and z_4 be four complex numbers corresponding to the points A, B, C and D, and let

$$\frac{z_2 - z_1}{z_3 - z_4} = \lambda\,,$$

so that $z_2 - z_1 = \lambda(z_3 - z_4)\,.$

Suppose that λ is real and non-zero, then one vector is a multiple of the other. Hence both vectors have the same direction if $\lambda > 0$ (but may differ in length) and opposite direction if $\lambda < 0$. In either case the lines AB and CD are parallel.

In the case where λ is imaginary, the two vectors must be perpendicular, since multiplication by i is equivalent to a rotation by $\frac{\pi}{2}$. Hence $AB \perp CD$. The sign of $\text{Im}(\lambda)$ determines whether the rotation is anticlockwise or clockwise.

Worked Example 27: Let z_1 and z_2 be any two complex numbers representing the points A and B in the complex plane. Consider the complex number z given by the equation $\dfrac{z - z_1}{z_2 - z_1} = t$ where t is real. Let the point C represent z.

(a) Show that A, B and C are collinear.

(b) Hence show that $AC : BC = |t| : |1 - t|$.

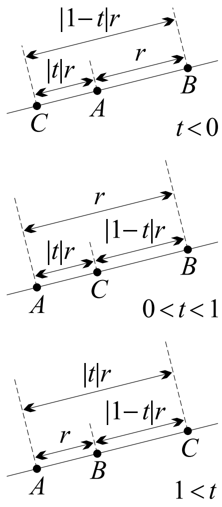
Solution:

(a) First note that $\overrightarrow{AB}$ represents $z_2 - z_1$ and that $\overrightarrow{AC}$ represents $z - z_1$. Since $\dfrac{z - z_1}{z_2 - z_1}$ is real it follows that AB and AC are parallel. Further since A is common to both lines, it follows that A, B and C are collinear.

(b) If $t < 0$ then $\overrightarrow{AC}$ has the opposite direction to $\overrightarrow{AB}$ and the order of the points is CAB.
If $t = 0$ then A and C coincide.
If $0 < t < 1$ then both $\overrightarrow{AC}$ and $\overrightarrow{AB}$ have the same direction and $\overrightarrow{AB}$ has the greater magnitude. Hence the order of the points is ACB.
If $t = 1$ then C and B coincide.
If $t > 1$ then the vectors have the same direction but $\overrightarrow{AC}$ has greater magnitude, so the order is ABC.
In each case let $|z_2 - z_1| = r$. Then:

$$\begin{aligned} AC : AB &= |z - z_1| : |z_2 - z_1| \\ &= |t|r : r \\ &= |t| : 1\,. \end{aligned}$$

Hence in all cases $AC : BC = |t| : |1 - t|$. The three non-zero cases are shown on the right.



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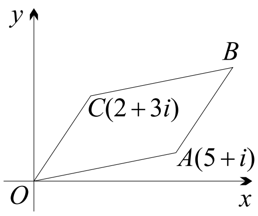
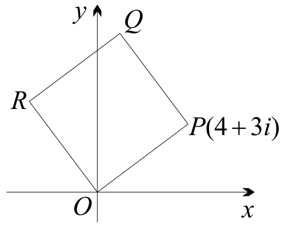
Cambridge Maths Stage 6

Exercise 1E

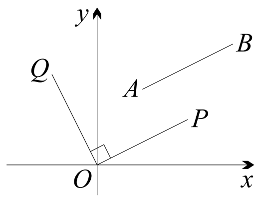
1. In the diagram on the right, $OABC$ is a parallelogram. The points A and C represent $5 + i$ and $2 + 3i$ respectively. Find the complex numbers represented by:
   (a) the vector OB,
   (b) the vector AC,
   (c) the vector CA.



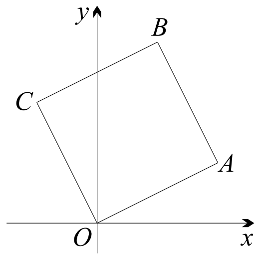
2. In the diagram on the right, $OPQR$ is a square. The point P represents $4 + 3i$. Find the complex numbers represented by:
   (a) the point R,
   (b) the point Q,
   (c) the vector QR,
   (d) the vector PR.



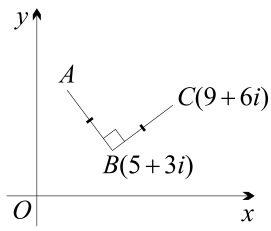
3. In the diagram on the right, intervals AB, OP and OQ are equal in length, OP is parallel to AB and $\angle POQ = \frac{\pi}{2}$. If A and B represent the complex numbers $3+5i$ and $9+8i$ respectively, find the complex number which is represented by Q.



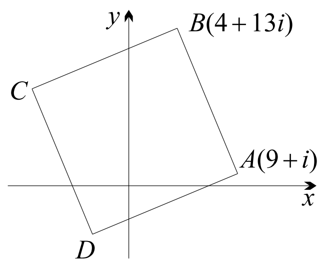
4. In the diagram on the right, $OABC$ is a square. The point A represents the complex number $2 + i$.
   (a) Find the numbers represented by B and C.
   (b) If the square is rotated 45° anticlockwise about O to give $OA'B'C'$, find the number represented by B'.



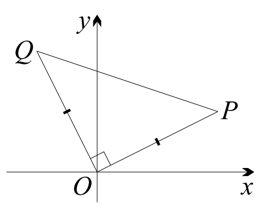
5. In the diagram on the right, $AB = BC$ and $\angle ABC = 90^\circ$. The points B and C represent $5 + 3i$ and $9 + 6i$ respectively. Find the complex numbers represented by:
   (a) the vector BC,
   (b) the vector BA,
   (c) the point A.



6. The diagram on the right shows a square $ABCD$ in the complex plane. The vertices A and B represent the complex numbers $9 + i$ and $4 + 13i$ respectively. Find the complex numbers that correspond to:
   (a) the vector AB,
   (b) the vertex D.

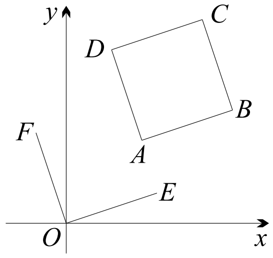


DEVELOPMENT



7. In the diagram on the right, the points P and Q correspond to the complex numbers z and w respectively. The triangle OPQ is isosceles and the angle POQ is a right angle.
   Prove that $z^2 + w^2 = 0$.

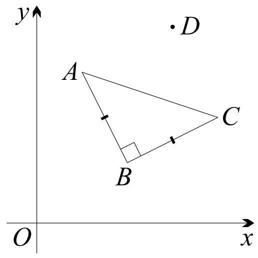
8. In the Argand diagram on the right, $ABCD$ is a square, and OE and OF are parallel and equal in length to AB and AD respectively. The vertices A and B correspond to the complex numbers w_1 and w_2 respectively. What complex numbers correspond to the points E, F, C and D?



9. In the diagram on the right, the vertices of a triangle ABC are represented by the complex numbers z_1, z_2 and z_3 respectively. The triangle is isosceles, and right-angled at B.

(a) Explain why $(z_1 - z_2)^2 + (z_3 - z_2)^2 = 0$.

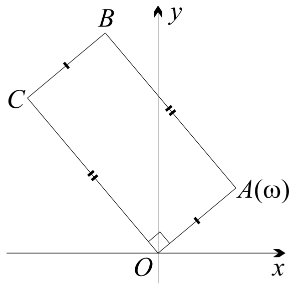
(b) Suppose that D is the point such that $ABCD$ is a square. Find, in terms of z_1, z_2 and z_3, the complex number that the point D represents.



10. In the Argand diagram on the right, $OABC$ is a rectangle, with $OC = 2OA$. The vertex A corresponds to the complex number ω.

(a) What complex number corresponds to the vertex C?

(b) What complex number corresponds to the point of intersection D of the diagonals OB and AC?



11. The vertices of an equilateral triangle are equidistant from the origin. One of its vertices is at $1 + \sqrt{3}i$. Find the complex numbers represented by the other two vertices. [HINT: What is the angle subtended by the vertices at the origin?]

12. Given $z = 3 + 4i$, find the two possible values of w so that the points representing 0, z and w form a right-angled isosceles triangle with right-angle at the point representing:

(a) 0 (b) z (c) w

13. Given that $z_1 = 1 + i$, $z_2 = 2 + 6i$ and $z_3 = -1 + 7i$, find the three possible values of z_4 so that the points representing z_1, z_2, z_3 and z_4 form a parallelogram.

14. Suppose that the complex number z has modulus one, and that $0 < \operatorname{Arg} z < \frac{\pi}{2}$. Prove that $2\operatorname{Arg}(z + 1) = \operatorname{Arg} z$.

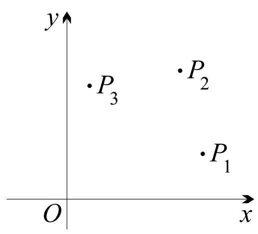
15. The vertices of the quadrilateral $ABCD$ in the complex plane represent the complex numbers z_1, z_2, z_3 and z_4 respectively.

(a) If $z_1 - z_2 = z_4 - z_3$, show that the quadrilateral $ABCD$ is a parallelogram.

(b) If $z_1 - z_2 = z_4 - z_3$ and $z_1 - z_3 = i(z_4 - z_2)$, show that $ABCD$ is a square.

16. A triangle has vertices at points in the Argand diagram which represent the complex numbers z_1, z_2 and z_3.
If $\dfrac{z_2 - z_1}{z_3 - z_1} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$, show that the triangle is equilateral.

17. In the diagram on the right, the points P_1, P_2 and P_3 represent the complex numbers z_1, z_2 and z_3 respectively. If $\dfrac{z_2}{z_1} = \dfrac{z_3}{z_2}$, show that OP_2 bisects $\angle P_1OP_3$.



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18. Let $z_1 = 2i$ and $z_2 = 1 + \sqrt{3}\, i$.

(a) Express z_1 and z_2 in mod-arg form.

(b) Plot in the complex plane the points P, Q, R and S representing z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ respectively.

(c) Find the exact values of: (i) $\arg(z_1 + z_2)$ (ii) $\arg(z_1 - z_2)$

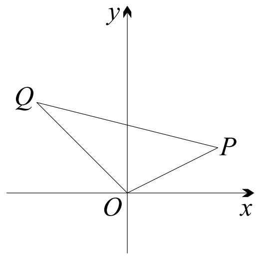
19. (a) Prove that for any complex number z, $|z|^2 = z\overline{z}$.

(b) Hence prove that for any complex numbers z_1 and z_2:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

(c) Explain this result geometrically.

20. In the diagram on the right, the points P and Q represent the complex numbers z and w respectively.



(a) Explain why $|z - w| \le |z| + |w|$.

(b) Indicate on the diagram the point R representing $z + w$.

(c) What type of quadrilateral is $OPRQ$?

(d) If $|z - w| = |z + w|$, what can be said about the complex number $\dfrac{w}{z}$?

21. (a) Prove that the points z_1, z_2 and z_3 are collinear if $\dfrac{z_3 - z_1}{z_2 - z_1}$ is real.

(b) Hence show that the points representing $5 + 8i$, $13 + 20i$ and $19 + 29i$ are collinear.

ENRICHMENT

22. The complex numbers ω_1 and ω_2 have modulus 1, and arguments α_1 and α_2 respectively, where $0 < \alpha_1 < \alpha_2 < \frac{\pi}{2}$.
Show that $\operatorname{Arg}(\omega_1 - \omega_2) = \frac{1}{2}(\alpha_1 + \alpha_2 - \pi)$.

23. [Circle Geometry] It is known that $\arg\left(\dfrac{z_4 - z_1}{z_2 - z_1}\right) + \arg\left(\dfrac{z_2 - z_3}{z_4 - z_3}\right) = \pi$. Explain why the points representing these complex numbers are concyclic.

24. [Circle Geometry] The points representing the complex numbers 0, z_1, z_2 and z_3 are concyclic and in anticlockwise order. Prove that the points representing $\dfrac{1}{z_1}$, $\dfrac{1}{z_2}$ and $\dfrac{1}{z_3}$ are collinear. [Hint: Show that $\dfrac{z_1^{-1} - z_2^{-1}}{z_1^{-1} - z_3^{-1}}$ is real.]

1F Curves and Regions in the Argand Diagram

In many situations a set of equations or conditions on a variable complex number z yields a set of points in the Argand diagram which is a familiar geometric object, such as a line or a circle. The main aim of this section is to provide a geometric description for a curve or region specified algebraically. Therefore the examples in this text have been grouped by the various geometries.

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There are two basic approaches used in this section, algebraic or geometric. The advantage of the algebraic approach is that most readers will already be proficient at manipulating equations in x and y. Unfortunately the geometry of the situation may be obscured by the algebra. The advantage of the geometric approach is that it will often provide a very elegant solution to the problem, but may also require a keen insight. Both methods should be practised, with the aim to become proficient at the geometric approach.

Straight Lines: Some simple straight lines have already been encountered in 1C, such as the vertical line $\text{Re}(z) = a$. Here are some other examples and their geometric interpretations.

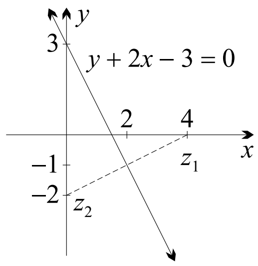
In coordinate geometry, given the coordinates of two points A and B, the task of finding the equation of the perpendicular bisector of AB is a lengthy one. The equivalent complex equation is remarkably simple.

WORKED EXAMPLE 28: Let $z_1 = 4$ and $z_2 = -2i$, and let the variable point z satisfy the equation $|z - z_1| = |z - z_2|$.

(a) Put $z = x + iy$ and hence show that z lies on the straight line $y + 2x - 3 = 0$.

(b) Describe this line geometrically in terms of z_1 and z_2.

SOLUTION:



(a) Substitute the values of z_1 and z_2, then square to get

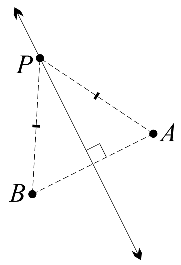
$$(x-4)^2 + y^2 = x^2 + (y+2)^2$$

or $\quad (x-4)^2 - x^2 = (y+2)^2 - y^2$

thus $\quad -4(2x-4) = 2(2y+2)$

so $\quad 4 - 2x = y + 1$

hence $\quad y + 2x - 3 = 0\,.$



(b) Let z, z_1 and z_2 be the points P, A and B in the Argand diagram. Since the modulus is a distance, the given equation yields

$$PA = PB\,.$$

Thus either $\triangle APB$ is always isosceles, or P bisects AB. Hence P is on the perpendicular bisector of AB.

29 THE PERPENDICULAR BISECTOR OF AN INTERVAL: Let z_1 and z_2 be the fixed points A and B in the Argand diagram, and let z be a variable point P. If

$$|z - z_1| = |z - z_2|$$

then P is on the perpendicular bisector of AB.

WORKED EXAMPLE 29: Let $z_0 = a + ib$ be the fixed point T and let $z = x + iy$ be a variable point P in the complex plane. It is known that

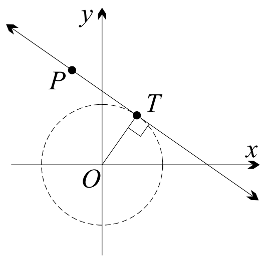
$$z - z_0 = ikz_0\,,$$

where k is a real number. It is also known that P is on a straight line.

(a) Find the equation of that straight line in terms of x and y.

(b) What is the geometry of the situation?

SOLUTION:



(a) The given equation expands to

$$x + iy - (a + ib) = ik(a + ib)\,.$$

Equating real and imaginary parts yields

$$x - a = -kb$$

and $$y - b = ka\,.$$

Eliminating k from this pair of equations gives

$$b(y - b) = -a(x - a)$$

or $$ax + by = a^2 + b^2$$

(b) Some readers will recognise this equation as the tangent to a circle. This geometry is confirmed by examining the given equation more closely.

Since multiplication by i represents a rotation of $\frac{\pi}{2}$, it follows that for $k \neq 0$ the vector $\overrightarrow{TP}$ is perpendicular to $\overrightarrow{OT}$. That is, P lies on a line perpendicular to OT. Further, when $k = 0$, $z = z_0$, so this line passes through T. That is, PT is the tangent to the circle with radius OT, as shown above.

Rays: The horizontal and vertical lines in 1C and the first example above demonstrate some of the geometry of the recently introduced functions $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ and $|z|$. The new function $\mathrm{Arg}(z)$ describes a ray in the z-plane.

WORKED EXAMPLE 30: The complex number z satisfies $\mathrm{Arg}(z) = \frac{\pi}{3}$.

(a) Let $|z| = r$. Write z in modulus-argument form.

(b) Plot z when $r = 1, 2, 3, 4$, and observe that z lies on a ray.

(c) Explain why the origin must be excluded.

(d) Use shifting to sketch $\mathrm{Arg}(z - 2 - i) = \frac{\pi}{3}$.

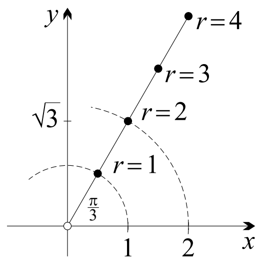
SOLUTION:

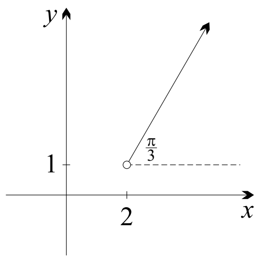
(a) $z = r(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$.

(b) See the first graph on the right.

(c) $\mathrm{Arg}(0)$ is undefined so the origin is not included.

(d) $\mathrm{Arg}(z - 2 - i) = \mathrm{Arg}(z - (2 + i))$ so the ray has been shifted to the point $2 + i$, as shown on the right.





30 **RAYS IN THE ARGAND DIAGRAM:**

- The equation $\mathrm{Arg}(z) = \theta$ represents the ray which makes an angle θ with the positive real axis, omitting the origin.
- The locus $\mathrm{Arg}(z - z_0) = \theta$ is the result of shifting the above ray from the origin to the point z_0.

Circles and Parabolas: The circle can easily be written as an equation of a complex variable. The parabola can also be written as an equation in z, though it is somewhat contrived and is not a significant result. The geometric definitions of each in terms of distances is the key.

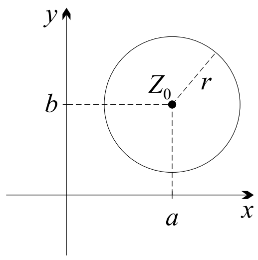
WORKED EXAMPLE 31: Consider the equation $|z-z_0| = r$ for some fixed complex number $z_0 = a + ib$ and positive real number r.

(a) Explain why this represents a circle. State the centre and radius.

(b) Confirm the result by putting $z = x + iy$ and finding the Cartesian equation.

(c) Expand $|z-1|^2$ in terms of z and $\overline{z}$. Hence determine the curve specified by $|z|^2 = z + \overline{z}$.

SOLUTION:



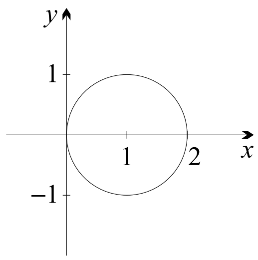
(a) The equation specifies that the distance between z and z_0 is fixed. This is the geometric definition of a circle. The centre is z_0 and the radius is r.

(b) Begin by squaring both sides:

$$|z - z_0|^2 = r^2$$

so $\quad |(x-a) + i(y-b)|^2 = r^2$

thus $\quad (x-a)^2 + (y-b)^2 = r^2\,.$



(c) From $|w|^2 = w\,\overline{w}$ it follows that

$$\begin{aligned} |z-1|^2 &= (z-1)\,\overline{(z-1)} \\ &= (z-1)(\overline{z}-1) \\ &= |z|^2 - (z+\overline{z}) + 1\,. \end{aligned}$$

Since $|z|^2 = z + \overline{z}$, it also follows that

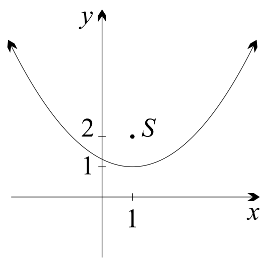
$$|z-1|^2 = 1\,,$$

that is, the circle with centre $z = 1$ and radius 1.

31

> CIRCLES IN THE ARGAND DIAGRAM: Let z_0 be the fixed point C in the Argand diagram, and let z a variable point P. If
>
> $$|z - z_0| = r$$
>
> then the point P is on the circle with centre C and radius r.

WORKED EXAMPLE 32: Let S be the fixed point $1 + 2i$ and z be the variable point P in the Argand diagram. It is known that $|z - (1+2i)| = \mathrm{Im}(z)$. Show algebraically that P lies on a parabola by putting $z = x + iy$.



SOLUTION: Squaring both sides of the given equation:

$$(x-1)^2 + (y-2)^2 = y^2$$

so $\quad (x-1)^2 = y^2 - (y-2)^2 = 4y - 4$

thus $\quad (x-1)^2 = 4(y-1)\,.$

This is the equation of a parabola with vertex at $1 + i$, as shown on the right.

This last worked example may seem a little contrived, but it demonstrates that there are many situations where further insight into a problem may be achieved by considering a corresponding problem in the complex plane.

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Regions: In many instances a curve divides the plane into two or more regions. In simple cases a region is defined by the corresponding inequation. When more intricate regions are required, two or more simple regions may be combined by taking the union or intersection of the corresponding inequations. Some common examples follow.

Worked Example 33: Sketch the following regions in the complex plane.

(a) $1 \le \mathrm{Re}(z) \le 3$

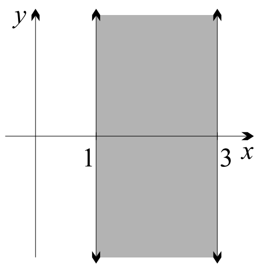
(b) $0 \le \mathrm{Arg}(z) \le \frac{\pi}{4}$

(c) $|z - 2 + i| < 1$

(d) $|z| > |z + 2 - 2i|$

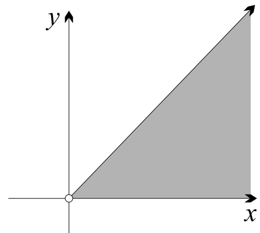
Solution: The first three can be easily explained geometrically.

(a)



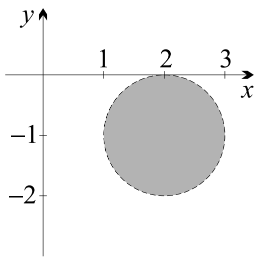
This is $1 \le x \le 3$, the vertical strip between $x = 1$ and $x = 3$.

(b)



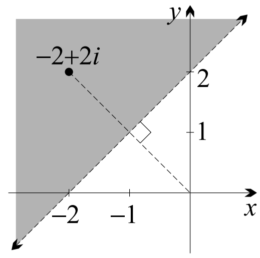
Put $z = r\,\mathrm{cis}\,\theta$ to get $0 \le \theta \le \frac{\pi}{4}$, which defines a wedge excluding the origin, since $\mathrm{Arg}(0)$ is undefined.

(c)



The boundary curve is the circle with radius 1 and centre $2-i$, and is not included. The region includes the centre of the circle since the LHS of the inequality is zero there.

(d)



The perpendicular bisector of the segment from 0 to $-2 + 2i$ is the boundary, and is not included. The region includes the point $-2 + 2i$, since the RHS of the inequality is zero there.

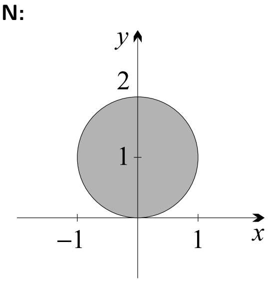
Worked Example 34:

(a) Sketch the regions (i) $|z - i| \le 1$ and (ii) $-\frac{\pi}{6} < \mathrm{Arg}(z + 1 - i) < \frac{\pi}{6}$.

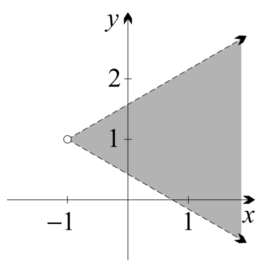
(b) Hence sketch (i) the union and (ii) the intersection of these regions.

Solution:

(a) (i)



(ii)



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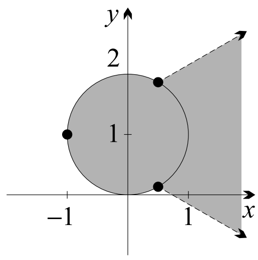
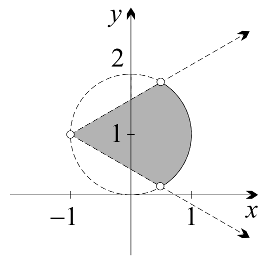
(b) The boundaries intersect at $-1+i$ and, from trigonometry, they intersect again at $\frac{1}{2}+i\,(1+\frac{\sqrt{3}}{2})$ and $\frac{1}{2}+i\,(1-\frac{\sqrt{3}}{2})$. Here are the graphs.

(i)



(ii)



Circle Geometry: Many of the circle geometry theorems encountered in Year 10 may be expressed in terms of a complex number. One significant result is included here, with other examples to be found in the exercise.

WORKED EXAMPLE 35: [A HARD EXAMPLE] Let $z_1 = 3+i$ and $z_2 = 1-i$.

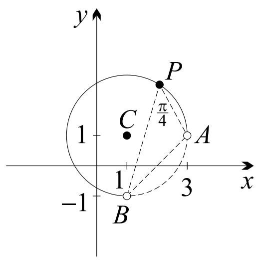
Describe and sketch the set of points z, where $\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \frac{\pi}{4}$.

SOLUTION: Let z_1, z_2 and z represent the points A, B and P respectively. First note that the equation can be written as

$$\arg(z-z_1) - \arg(z-z_2) = \tfrac{\pi}{4}\,.$$

Recall that $z-z_1$ is the vector $\overrightarrow{AP}$, so $\arg(z-z_1)$ is the direction of this vector. Likewise $\arg(z-z_2)$ is the direction of vector $\overrightarrow{BP}$. Thus the difference is the angle between them and is always $\frac{\pi}{4}$. Using the converse of the angles in the same segment theorem, it follows that P must lie on the arc of a circle with chord AB. Further, since

$$\angle APB = \tfrac{\pi}{4} < \tfrac{\pi}{2}$$



it is a major arc. As P moves along the arc from A to B, it moves anticlockwise about the centre, because angles are measured anticlockwise. Lastly, since $\arg(0)$ is undefined, the endpoints of the arc are not included. It simply remains to find the centre and radius of this circle. Let C be the centre, then

$\angle ACB = \frac{\pi}{2}$ (Angles at the centre and circumference)

hence $\angle CAB = \frac{\pi}{4}$ (base angles of isosceles triangle.)

Since $\operatorname{Arg}(z_1 - z_2) = \frac{\pi}{4}$

it follows that AC is horizontal, as the alternate angles are equal. Thus BC is vertical. Hence $C = 1+i$ is the centre of the circle and $AC = 2$ is its radius.

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THE ARC OF A CIRCLE: Let points A and B represent the complex numbers z_1 and z_2. Let the variable point P represent z. The equation

$$\operatorname{Arg}\left(\frac{z-z_1}{z-z_2}\right) = \alpha \qquad \text{where} \qquad 0 < \alpha < \pi\,,$$

is the arc AB of a circle with the endpoints excluded. As P moves along the arc from A to B its motion is anticlockwise about the centre.

Exercise **1F**

1. Sketch each straight line by using the result of Box 28, then find its Cartesian equation.

(a) $|z+3|=|z-5|$ (b) $|z-i|=|z+1|$ (c) $|z+2-2i|=|z|$ (d) $|z-i|=|z-4+i|$

2. Sketch the rays specified by the following equations. Box 29 may be of help.

(a) $\arg(z-4)=\frac{3\pi}{4}$ (b) $\arg(z+1)=\frac{\pi}{4}$ (c) $\arg(z-1-i\sqrt{3})=\frac{\pi}{3}$

3. Use Box 31 to sketch these circles.

(a) $|z+1-i|=1$ (b) $|z-3-2i|=2$ (c) $|z-1+i|=\sqrt{2}$

4. In each case sketch the boundary or boundaries of the region and then shade the region.

(a) $|z-8i|\ge|z-4|$ (d) $0\le\arg(z)\le\frac{3\pi}{4}$ (g) $|z|>2$
(b) $|z-2+i|\le|z-4+i|$ (e) $-\frac{\pi}{3}<\arg(z)<\frac{\pi}{6}$ (h) $|z+2i|\le1$
(c) $|z+1-i|\ge|z-3+i|$ (f) $-\frac{\pi}{4}\le\arg(z+2+i)<\frac{\pi}{4}$ (i) $1<|z-2+i|\le2$

DEVELOPMENT

5. In each case sketch (i) the intersection and (ii) the union of the given pair of regions.

(a) $|z-2+i|\le2$, $\mathrm{Im}(z)\ge0$
(b) $0\le\mathrm{Re}(z)\le2$, $|z-1+i|\le2$
(c) $|z-\overline{z}|<2$, $|z-1|\ge1$
(d) $\mathrm{Re}(z)\le4$, $|z-4+5i|\le3$
(e) $|z-1-i|\le2$, $0\le\arg(z-1-i)\le\frac{\pi}{4}$
(f) $|z|\le1$, $0\le\arg(z+1)\le\frac{\pi}{4}$
(g) $|z+1-2i|\le3$, $-\frac{\pi}{3}\le\arg z\le\frac{\pi}{4}$
(h) $|z-3-i|\le5$, $|z+1|\le|z-1|$

6. Put $z=x+iy$ to help sketch these hyperbolas.

(a) $z^2-(\overline{z})^2=16i$ (b) $z^2-(\overline{z})^2=12i$

7. In each case the given equation represents a parabola. Find the Cartesian equation by putting $z=x+iy$, and hence sketch the parabola.

(a) $|z-3i|=\mathrm{Im}(z)$ (b) $|z+2|=-\mathrm{Re}(z)$ (c) $|z|=\mathrm{Re}(z+2)$ (d) $|z-i|=\mathrm{Im}(z+i)$

8. By putting $z=x+iy$, or otherwise, sketch the graph defined by the equation:

(a) $\mathrm{Im}(z)=|z|$ (b) $\mathrm{Re}\left(1-\dfrac{4}{z}\right)=0$ (c) $\mathrm{Re}\left(z-\dfrac{1}{z}\right)=0$

9. Sketch the arcs of circles specified by the following equations, showing the centre and radius in each case.

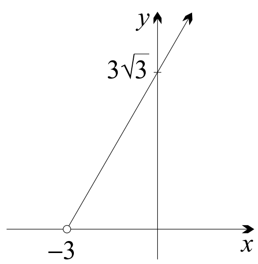
(a) $\arg\left(\dfrac{z-2}{z}\right)=\dfrac{\pi}{2}$ (c) $\arg\left(\dfrac{z-i}{z+i}\right)=\dfrac{\pi}{4}$ (e) $\arg\left(\dfrac{z-2i}{z+2i}\right)=\dfrac{\pi}{6}$
(b) $\arg\left(\dfrac{z-1+i}{z-1-i}\right)=\dfrac{\pi}{2}$ (d) $\arg\left(\dfrac{z+1}{z-3}\right)=\dfrac{\pi}{3}$ (f) $\arg\left(\dfrac{z}{z+4}\right)=\dfrac{3\pi}{4}$

10. A complex number z satisfies $\arg z=\frac{\pi}{3}$.

(a) Use a diagram to show that $|z-2i|\ge1$. (b) For which value of z is $|z-2i|=1$?

11. Consider the graph in the Argand diagram on the right.

(a) Write down an equation for this graph in terms of z.
(b) Find the modulus and argument of z at the point where $|z|$ takes its minimum value.
(c) Hence find z in Cartesian form when $|z|$ takes its least value.



12. (a) A complex number z satisfies $|z-1|=2$. Draw a diagram and hence find the greatest and least possible values of $|z|$.

(b) If z is a complex number such that $\text{Re}(z) \le 2$ and $|z-3|=2$, show with the aid of a diagram that $1 \le |z| \le \sqrt{7}$.

13. (a) A complex number z satisfies $|z-2|=1$.

(i) Sketch the graph of $|z-2|=1$. (ii) Show that $-\frac{\pi}{6} \le \arg z \le \frac{\pi}{6}$.

(b) The complex number z is such that $|z|=1$. Use your answers to part (a) to explain why $-\frac{\pi}{6} \le \arg(z+2) \le \frac{\pi}{6}$.

14. The complex number w satisfies $|w|=10$ and $0 \le \arg w \le \frac{\pi}{2}$, and the complex number z is specified by $z = 3+4i+w$.

(a) Sketch the graph of $z=3+4i+w$..

(b) Use your sketch to determine the maximum value of $|z|$.

(c) What is the value of z for which this maximum occurs?

15. (a) Show that the circle equation $|z-z_0|=r$ is equivalent to

$$z\,\overline{z} - (z\,\overline{z_0} + \overline{z}\,z_0) + z_0\,\overline{z_0} - r^2 = 0\,.$$

[HINT: Square both sides of $|z-z_0|=r$ and use the result $|w|^2 = w\,\overline{w}$.]

(b) Use part (a) to write these equations in the form $|z-z_0|=r$, and hence state the centre and radius of each circle.

(i) $z\,\overline{z}+2(z+\overline{z})=0$ (ii) $z\,\overline{z}-(1+i)\overline{z}-(1-i)z+1=0$ (iii) $\dfrac{1}{z}+\dfrac{1}{\overline{z}}=1$

16. Sketch the graph of the set of points z for which $\dfrac{z-1}{z-i}$ is: (a) real, (b) imaginary.

17. Sketch the graph of: (a) $\arg(z+i) = \arg(z-1)$ (b) $\arg(z+i) = \arg(z-1)+\pi$

18. (a) The variable complex number z satisfies $|z-2-i|=1$. Use a diagram to find the maximum and minimum values of: (i) $|z|$ (ii) $|z-3i|$

(b) A complex number z satisfies $|z|=3$. Use a sketch to find the greatest and least values of $|z+5-i|$.

(c) The variable complex number z satisfies $|z-z_0|=r$. Use a similar approach to parts (a) and (b) to find the maximum and minimum values of: (i) $|z|$ (ii) $|z-z_1|$

(d) Confirm your answers to the previous parts by using the triangle inequality

$$\big||z|-|w|\big| \le |z+w| \le |z|+|w|.$$

ENRICHMENT

19. Describe the graph of $\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \alpha$, where α is constant, if:

(a) $\alpha = 0$ (b) $0 < \alpha < \frac{\pi}{2}$ (c) $\alpha = \frac{\pi}{2}$ (d) $\frac{\pi}{2} < \alpha < \pi$ (e) $\alpha = \pi$

20. [PYTHAGORAS] Describe the graph defined by the equation $|z-z_1|^2+|z-z_2|^2 = |z_1-z_2|^2$.

21. Suppose that $k|z-z_1| = \ell|z-z_2|$, where $k \ne \ell$ and both are positive real numbers.

(a) Show that the graph specified by the equation is a circle with centre $\dfrac{k^2z_1-\ell^2z_2}{k^2-\ell^2}$ and radius $\dfrac{kl|z_2-z_1|}{|k^2-\ell^2|}$,

(i) by letting $z=x+iy$, (ii) by geometric methods.

(b) What happens in the limit as k approaches ℓ?

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1G Polynomials and Complex Numbers

A significant application of complex numbers is in the study of polynomials. This section further develops the work on polynomials already done in the Mathematics Extension 1 course. That work is assumed knowledge though some parts of the theory are repeated here for the sake of convenience. The focus is on polynomials with real coefficients and the relationships with the zeroes, particularly when they are either complex, or real and repeated.

The crux of the work is later in this section where the Fundamental Theorem of Algebra is presented along with some of its consequences. The theorem is left unproven as any proof is beyond the scope of the course.

Polynomials with Integer Coefficients: If a polynomial with integer coefficients has an integer zero $x = k$, then k is a factor of the constant term. This is a significant aid in factorising a polynomial.

Worked Example 36: It is known that the polynomial $P(x) = x^3 - x^2 - 8x - 6$ has only one integer zero. Find it and hence factorise $P(x)$ completely.

Solution: The zero is a factor of 6, so the possible values are: $\pm1, \pm2, \pm3, \pm6$. Testing these one by one:

$$P(1) = -14\,, \qquad P(-1) = 0\,,$$

and there is no need to continue further. By the factor theorem, $(x + 1)$ is a factor of $P(x)$. Performing the long division:

$$\begin{array}{r|rrrr} & x^2 & -2x & -6 & \\ \hline (x+1) & x^3 - x^2 & -8x & -6 & \\ & x^3 + x^2 & & & \\ \hline & -2x^2 & -8x & -6 & \\ & -2x^2 & -2x & & \\ \hline & & -6x & -6 & \\ & & -6x & -6 & \\ \hline & & & 0 & \end{array}$$

Thus
$$\begin{aligned} P(x) &= (x+1)(x^2 - 2x - 6) \\ &= (x+1)\Big((x-1)^2 - 7\Big) && \text{(completing the square)} \\ &= (x+1)(x - 1 - \sqrt{7})(x - 1 + \sqrt{7}) && \text{(difference of two squares.)} \end{aligned}$$

33 **Integer coefficients and zeroes:** If the polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

with integer coefficients a_0, a_1, $a_2, \ldots, a_n$, has an integer zero $x = k$, then k is a factor of the constant term a_0.

Proof: Since $P(k) = 0$, it follows that

$$\begin{aligned} a_0 + a_1k + a_2k^2 + \ldots + a_nk^n &= 0 \\ \text{so} \qquad a_1k + a_2k^2 + \ldots + a_nk^n &= -a_0 \\ \text{thus} \qquad k \times (a_1 + a_2k + \ldots + a_nk^{n-1}) &= -a_0\,. \end{aligned}$$

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Since all the terms in the brackets are integers, it follows that the result is also an integer. Thus the left hand side is the product of two integers. Hence, as asserted, k is a factor of a_0.

Polynomials and Complex Numbers: Consider the general polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \,.$$

Each term in this expression involves an integer power and multiplication by a constant. The terms are then simply added. Since integer powers, multiplication and addition are all natural operations with complex numbers, it follows that the polynomial can be evaluated when x is a complex number. For example if $P(x) = x^2 - 2x + 4$ then at $x = i$ its value is

$$\begin{aligned} P(i) &= i^2 - 2i + 4 \\ &= 3 - 2i \,. \end{aligned}$$

In some examples the polynomial will be written as a function of z in order to emphasise the fact that complex numbers may be substituted. Thus the above example may be written as $P(z) = z^2 - 2z + 4$.

Remainders and Factors: Here is a quick summary of certain important results from the Mathematics Extension 1 course. In the usual notation, let $P(x)$ and $D(x)$ be any pair of polynomials, where $D(x) \neq 0$. There is a unique pair of polynomials $Q(x)$ and $R(x)$, such that

$$P(x) = D(x) \times Q(x) + R(x) \,,$$

and where either

$$\deg(D) > \deg(R) \qquad \text{or} \qquad R(x) = 0 \,.$$

This is known as the division theorem. As a consequence, if $D(x) = (x - \alpha)$ then $R(x)$ must be a constant, either zero or non-zero. Let this constant be r. Re-writing the division theorem:

$$P(x) = (x - \alpha) \times Q(x) + r \,,$$

whence $\quad P(\alpha) = r \,,$

which is known as the remainder theorem.

If $R(x) = 0$ then from the division theorem

$$P(x) = D(x) \times Q(x) \,,$$

so that $P(x)$ is a product of the factors $D(x)$ and $Q(x)$. In particular, $x - \alpha$ is a factor of $P(x)$ if and only if $P(\alpha) = 0$. This is known as the factor theorem.

The division theorem, the remainder theorem and the factor theorem are valid for complex numbers as well as real numbers. Though these claims will not be proven here, the results may be freely applied to solve problems.

WORKED EXAMPLE 37: Let $P(x) = x^3 - 2x^2 - x + k$, where k is real.

(a) Show that $P(i) = (2 + k) - 2i$.

(b) When $P(x)$ is divided by $x^2 + 1$ the remainder is $4 - 2x$. Find the value of k.

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SOLUTION:

(a) $P(i) = i^3 - 2i^2 - i + k$
$= -i + 2 - i + k$
$= (2 + k) - 2i\,.$

(b) By the division theorem,
$$P(x) = (x^2 + 1) \times Q(x) + 4 - 2x\,.$$
Thus $\quad P(i) = 4 - 2i$
hence $\quad (2 + k) - 2i = 4 - 2i\,.$
Equating the real parts gives $k = 2\,.$

Real Coefficients and Remainders: Suppose that the polynomial $P(z)$ has real coefficients. If the remainder when $P(z)$ is divided by $(z - \alpha)$ is β then the remainder when $P(z)$ is divided by $(z - \overline{\alpha})$ is $\overline{\beta}$. Using the remainder theorem, this is equivalent to the statement that if $P(\alpha) = \beta$ then $P(\overline{\alpha}) = \overline{\beta}\,.$

WORKED EXAMPLE 38:

(a) Use the remainder theorem to find the remainder when $P(z) = z^3 - 2z^2 + 3z - 1$ is divided by $(z - i)\,.$

(b) Hence find the remainder when $P(z)$ is divided by $(z + i)\,.$

SOLUTION:

(a) The remainder is:
$$P(i) = i^3 - 2i^2 + 3i - 1 = 1 + 2i\,.$$

(b) It is: $P(-i) = P(\,\overline{i}\,) = \overline{1 + 2i} = 1 - 2i\,.$

34 | REAL COEFFICIENTS AND REMAINDERS: If the polynomial $P(z)$ has real coefficients and if $P(\alpha) = \beta$ then $P(\overline{\alpha}) = \overline{\beta}\,.$

The proof is not too difficult and is dealt with in a question of the exercise.

Real Coefficients and Complex zeroes: Suppose that the polynomial $P(z)$ has real coefficients. If $P(z)$ has a complex zero $z = \alpha$ then it is guaranteed to have a second complex zero $z = \overline{\alpha}$. Further, by the factor theorem, there exists another polynomial $Q(z)$ such that:
$$\begin{aligned} P(z) &= (z - \alpha)(z - \overline{\alpha}) \times Q(z) \\ &= (z^2 - (\alpha + \overline{\alpha})z + \alpha\overline{\alpha}) \times Q(z) \\ &= (z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2) \times Q(z)\,. \end{aligned}$$
Thus $P(z)$ has a quadratic factor with real coefficients: $(z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2)\,.$

WORKED EXAMPLE 39: Consider the polynomial $P(z) = 2z^3 - 3z^2 + 18z + 10\,.$

(a) Given that $1 - 3i$ is a zero of $P(z)$, explain why $1 + 3i$ is another zero.

(b) Find the third zero of the polynomial.

(c) Hence write $P(z)$ as a product of:

(i) linear factors,

(ii) a linear factor and a quadratic factor, both with real coefficients.

.

SOLUTION: (a) Since $P(z)$ has real coefficients, $\overline{(1-3i)} = 1+3i$ is also a zero.

(b) Let the third zero be a, then by the sum of the roots:

$$a + (1-3i) + (1+3i) = \tfrac{3}{2}$$

so $$a + 2 = \tfrac{3}{2}$$

and $$a = -\tfrac{1}{2}\,.$$

(c) (i) By the factor theorem:

$$\begin{aligned} P(z) &= 2(z+\tfrac{1}{2})(z-1+3i)(z-1-3i) \\ &= (2z+1)(z-1+3i)(z-1-3i)\,. \end{aligned}$$

(ii) $P(z) = (2z+1)(z^2-2z+10)\,.$

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> REAL COEFFICIENTS AND ZEROES: If the polynomial $P(z)$ has real coefficients and a complex zero $z = \alpha$ then it is guaranteed to have a second complex zero $z = \overline{\alpha}$.
>
> Consequently $P(z)$ has $(z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2)$ as a factor, which is a quadratic with real coefficients.

PROOF: Suppose that the complex number $z = \alpha$ is a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n\,,$$

where the coefficients $a_0, a_1, \ldots, a_n$ are all real. That is $P(\alpha) = 0$. Then

$$\begin{aligned} P(\overline{\alpha}) &= a_0 + a_1\overline{\alpha} + a_2\overline{\alpha}^{\,2} + \ldots + a_n\overline{\alpha}^{\,n} \\ &= a_0 + a_1\overline{\alpha} + a_2\overline{\alpha^2} + \ldots + a_n\overline{\alpha^n} && (\text{since } \overline{z}^{\,n} = \overline{z^n}) \\ &= \overline{a_0} + \overline{a_1\alpha} + \overline{a_2\alpha^2} + \ldots + \overline{a_n\alpha^n} && (\text{since } c\,\overline{z} = \overline{cz} \text{ for real } c) \\ &= \overline{a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_n\alpha^n} && (\text{since } \overline{w} + \overline{z} = \overline{w+z}) \\ &= \overline{P(\alpha)} \\ &= \overline{0} \\ &= 0\,. \end{aligned}$$

Hence $z = \overline{\alpha}$ is also a zero of the polynomial $P(z)$. Further, as shown above:

$$P(z) = (z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2) \times Q(z)\,.$$

Multiple Zeroes: Recall that if $P(x) = (x-\alpha)^m Q(x)$, where $Q(\alpha) \neq 0$, then the value $x = \alpha$ is called a zero of multiplicity m. It is also the case that $x = \alpha$ is a zero of $P'(x)$ with multiplicity $(m-1)$. In fact this result is also true for polynomials with complex zeroes but the general proof is beyond the scope of this course. However, it is possible to prove the result in the special case of a polynomial with real coefficients and a complex zero with multiplicity 2. This is dealt with in a question of the exercise, and extending this to arbitrary multiplicity may be suitable as a class investigation.

In Extension 1, the derivative result may have been applied a second time to find a triple zero. In fact it can be extended to the general case of a polynomial $P(x)$ with real coefficients which has a real zero $x = \alpha$ of multiplicity m. The value $x = \alpha$ is also a zero of each of the derivatives $P^{(j)}(x)$, for $j = 1, \ldots, (m-1)$.

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36 MULTIPLE ZEROES AND HIGHER DERIVATIVES: Suppose that the polynomial $P(x)$ with real coefficients has a real zero $x = \alpha$ with multiplicity $m > 1$.

Then $x = \alpha$ is a zero of each of the derivatives $P^{(j)}(x)$, for $j = 1, \ldots, (m-1)$.

This result can be proved relatively easily by induction and is left as an exercise for the chapter on proof. It is also true for real polynomials with complex zeroes, and can likewise be proved by induction.

The Fundamental Theorem of Algebra: All the work encountered so far in this section deals with finding the zeroes of various polynomials. Up to this point it has been possible to sidestep an important question: does every polynomial have a zero? For there is no point in searching for one if none exists.

In order to emphasise this point, consider the polynomial $P(x) = x^2 + 1$. Clearly this function has no real zero, and there is no point in searching for one. Yet the polynomial does indeed have two zeroes, both of which happen to be complex numbers: namely i and $-i$. Could it be that there is another polynomial which has neither real nor complex zeroes?

The answer to this question is: every polynomial with degree ≥ 1 has at least one zero, though that zero may be complex. This is such an important and basic fact in the study of mathematics that it is given a title — *The Fundamental Theorem of Algebra*.

37 THE FUNDAMENTAL THEOREM OF ALGEBRA: Every polynomial with degree ≥ 1 has at least one zero, though that zero may be complex.

Several eminent mathematicians worked on this theorem including Leibniz, Euler and Argand. But credit is usually given to Gauss for the first proof, which he presented in his doctoral thesis in 1799. This, or any other proof of the theorem, is beyond the scope of this course. Although the wording given in the box above is imprecise, it is usually sufficient for the problems encountered at this level.

The Degree and the Number of Zeroes: Although the Fundamental Theorem of Algebra cannot be proven here, it is possible to prove two significant consequences of the theorem. The first is that every polynomial of degree $n \geq 1$ with complex coefficients has precisely n zeroes, as counted by their multiplicities.

This is also true for polynomials with real coefficients. To demonstrate the result, the cubic $P(x) = x^3 - 3x^2 + 4 = (x-2)^2(x+1)$ has three zeroes: the simple zero $x = -1$ and the double zero $x = 2$.

38 THE DEGREE AND THE NUMBER OF ZEROES: Every polynomial of degree $n \geq 1$ with complex coefficents has precisely n zeroes, as counted by their multiplicities.

PROOF: This proof uses induction, and may be better left as an exercise for the chapter on proof.

A. Consider the general polynomial of degree one with complex coefficients:

$$P_1(x) = a_0 + a_1 x\,, \qquad \text{where } a_1 \neq 0\,.$$

Clearly this polynomial has one zero $x = \alpha_1$, where

$$\alpha_1 = -\frac{a_0\overline{a_1}}{|a_1|^2}\,.$$

Thus the result is true for $n = 1$.

B. Suppose that the result is true for some integer $k \geq 1$. That is, suppose that every polynomial of degree k with complex coefficients

$$P_k(x) = a_0 + a_1x + \ldots + a_kx^k\,, \qquad \text{where } a_k \neq 0\,,$$

has k zeroes, $x = \alpha_1, \ldots, \alpha_k$, as counted by their multiplicities. $(**)$

The statement is now proven true for $n = k+1$. That is, it is proven that every polynomial of degree $k+1$ with complex coefficents

$$P_{k+1}(x) = a_0 + a_1x + \ldots + a_{k+1}x^{k+1}\,, \qquad \text{where } a_{k+1} \neq 0\,,$$

has $k+1$ zeroes as counted by their multiplicities.

Now for any particular polynomial $P_{k+1}(x)$, that polynomial has at least one zero by the Fundamental Theorem of Algebra. Let this zero be $x = \alpha_{k+1}$. Then, by the factor theorem, it follows that

$$P_{k+1}(x) = (x - \alpha_{k+1})Q_k(x)$$

for some polynomial $Q_k(x)$ of degree k. But by the induction hypothesis above $(**)$, $Q_k(x)$ has k zeroes, all of which are thus inherited by $P_{k+1}(x)$.

Hence $P_{k+1}(x)$ has $k+1$ zeroes, $x = \alpha_1, \ldots, \alpha_k, \alpha_{k+1}$, as counted by their multiplicities. Clearly this follows for each and every polynomial $P_{k+1}(x)$.

C. It follows from parts A and B by mathematical induction that the statement is true for all integers $n \geq 1$.

Real Linear and Quadratic Factors: The second significant consequence of the Fundamental Theorem of Algebra is that every polynomial of degree $n \geq 1$ with real coefficients can be written as a product of factors which are either linear or *irreducible* quadratics, each with real coefficients. In this context the word *irreducible* is used to indicate that the quadratic has no real zero.

In order to demonstrate the result, notice that the polynomial $P(x) = x^3 - 1$ can be written as the product

$$P(x) = (x-1)(x^2 + x + 1)\,.$$

The quadratic factor $(x^2 + x + 1)$ is irreducible since it has no real zero.

39 REAL LINEAR AND QUADRATIC FACTORS: Every polynomial of degree $n \geq 1$ which has real coefficents can be written as a product of factors which are either linear or irreducible quadratics, each with real coefficients.

PROOF: Let $P_n(x) = a_0 + a_1x + \ldots + a_nx^n$ be a polynomial with degree $n \geq 1$ which has real coefficients. By the previous result, this polynomial has n zeroes. Let these zeroes be $x = \alpha_1, \ldots, \alpha_n$. If none of these zeroes is complex then the result is obviously true since then

$$P_n(x) = a_n \prod_{k=1}^{n} (x - \alpha_k)\,.$$

If all of the zeroes are complex then the result is again obviously true. Since $P_n(x)$ has real coefficients, the roots come in conjugate pairs, by which it is known that n is even. Thus

$$\begin{aligned} P_n(x) &= a_n \prod_{k=1}^{n/2} (x-\alpha_k) \times (x-\overline{\alpha_k}) \\ &= a_n \prod_{k=1}^{n/2} \left(x^2 - 2\,\mathrm{Re}(\alpha_k)x + |\alpha_k|^2\right). \end{aligned}$$

Lastly, if some of the zeroes are complex numbers then once again they occur as conjugate pairs, since the coefficients of $P_n(x)$ are real. Let the number of conjugate pairs be j, where $1 < 2j < n$. Now re-order and re-label the zeroes with the conjugate pairs listed first. Thus the first conjugate pair is $x = \alpha_1, \overline{\alpha_1}$, and the last conjugate pair is $x = \alpha_j, \overline{\alpha_j}$.

If there are any other zeroes then they are real. The first of these is $x = \alpha_{2j+1}$ and the last is $x = \alpha_n$. So by the factor theorem, and using product notation:

$$\begin{aligned} P_n(x) &= a_n \times \left(\prod_{k=1}^{j} (x-\alpha_k)(x-\overline{\alpha_k})\right) \times \left(\prod_{\ell=2j+1}^{n} (x-\alpha_\ell)\right) \\ &= a_n \times \left(\prod_{k=1}^{j} \left(x^2 - 2\,\mathrm{Re}(\alpha_k)x + |\alpha_k|^2\right)\right) \times \left(\prod_{\ell=2j+1}^{n} (x-\alpha_\ell)\right). \end{aligned}$$

In each of the three cases the result is a product of factors with real coefficients, which are either linear or irreducible quadratic factors. Put more simply, multiply all the complex factors together in conjugate pairs to get irreducible quadratic factors with real coefficients, and any remaining factors are both linear and real.

Exercise 1G

1. It is known that in each case the given polynomial $P(x)$ has only one integer zero. Find it and hence factorise $P(x)$ completely.

(a) $P(x) = x^3 - 6x + 4$ (b) $P(x) = x^3 + 3x^2 - 2x - 2$ (c) $P(x) = x^3 - 3x^2 - 2x + 4$

2. It is known that $1 + i$ is a zero of the polynomial $P(x) = x^3 - 8x^2 + 14x - 12$.

(a) Why is $1 - i$ also a zero of $P(x)$?

(b) Use the sum of the zeroes to find the third zero of $P(x)$.

3. It is known that $1 - 2i$ is a zero of the polynomial $P(x) = x^3 + x + 10$.

(a) Write down another complex zero of $P(x)$, and give a reason for your answer.

(b) Hence show that $x^2 - 2x + 5$ is a factor of $P(x)$.

(c) Find the third zero, and hence write $P(x)$ as a product of factors with real coefficients.

4. It is known that $-3i$ is a zero of the polynomial $P(z) = 2z^3 + 3z^2 + 18z + 27$.

(a) Write down another complex zero of $P(z)$. Justify your answer.

(b) Hence write down a quadratic factor of $P(z)$ with real coefficients.

(c) Write $P(x)$ as a product of factors with real coefficients.

5. Let $P(z) = 2z^3 - 13z^2 + 26z - 10$.

(a) Show that $P(3 + i) = 0$.

(b) State the value of $P(3 - i)$, and give a reason for your answer.

(c) Hence write $P(z)$ as a product of:

(i) linear factors,

(ii) a linear factor and a quadratic factor, both with real coefficients.

DEVELOPMENT

6. Consider the polynomial $Q(x) = x^4 - 6x^3 + 8x^2 - 24x + 16$.
  (a) It is known that $Q(2i) = 0$. Why does it follow immediately that $Q(-2i) = 0$?
  (b) By using the sum and the product of the zeroes of $Q(x)$, or otherwise, find the other two zeroes of $Q(x)$.
  (c) Hence write $Q(x)$ as a product of:
    (i) four linear factors,
    (ii) three factors with real coefficients,
    (iii) two factors with integer coefficients.

7. (a) Solve the equation $x^4 - 3x^3 + 6x^2 + 2x - 60 = 0$ given that $x = 1 + 3i$ is a root.
  (b) Solve the equation $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$ given that $x = 1 - i$ is a root.

8. Consider the polynomial equation $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$.
  (a) Show that $x = 3$ is a double root of the equation.
  (b) Hence solve the equation.

9. Two of the zeroes of $P(z) = z^4 - 12z^3 + 59z^2 - 138z + 130$ are $a + ib$ and $a + 2ib$, where a and b are real and $b > 0$.
  (a) Find the value of a by considering the sum of the zeroes.
  (b) Use the product of the zeroes to show that $4b^4 + 45b^2 - 49 = 0$, and hence find b.
  (c) Hence express $P(z)$ as the product of quadratic factors with real coefficients.

10. Suppose that $P(x) = x^3 + kx^2 + 6$, where k is real.
  (a) Show that $P(2i) = (6 - 4k) - 8i$.
  (b) When $P(x)$ is divided by $x^2 + 4$ the remainder is $-4x - 6$. Find the value of k.

11. Let $P(x) = x^3 - x^2 + mx + n$, where both m and n are integers.
  (a) Show that $P(-i) = (1 + n) + i(1 - m)$.
  (b) When $P(x)$ is divided by $x^2 + 1$ the remainder is $6x - 3$. Find the values of m and n.

12. Suppose that $P(x) = x^3 + x^2 + 6x - 3$.
  (a) Use the remainder theorem to find the remainder when $P(x)$ is divided by $x + 2i$.
  (b) Hence find the remainder when $P(x)$ is divided by: (i) $x - 2i$, (ii) $x^2 + 4$.

13. Let $P(z) = z^8 - \frac{5}{2}z^4 + 1$. Suppose that w is a root of $P(z) = 0$.
  (a) Show that iw and $\frac{1}{w}$ are also roots of $P(z) = 0$.
  (b) Find one of the roots of $P(z) = 0$ in exact form.
  (c) Hence find all the roots of $P(z) = 0$.

14. Suppose that $P(x) = x^4 + Ax^2 + B$, where A and B are positive real numbers.
  (a) Explain why $P(x)$ has no real zeroes.
  (b) Given that ic and id are zeroes of $P(x)$, where c and d are real and $c \neq -d$, write down the other two zeroes of $P(x)$, and give a reason.
  (c) Prove that $c^4 + d^4 = A^2 - 2B$.

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15. The polynomial $P(x) = x^3 + cx + d$, where c and d are real and non-zero, has a negative real zero k, and two complex zeroes. The graph of $y = P(x)$ has two turning points.

(a) What can be said about the two complex zeroes of $P(x)$, and why?

(b) By considering $P'(x)$, show that $c < 0$.

(c) Sketch the graph of $y = P(x)$.

(d) If $a \pm ib$, where a and b are real, are the complex zeroes of $P(x)$, deduce that $a > 0$.

(e) Prove that $d = 8a^3 + 2ac$.

16. Consider the polynomial function $f(x) = x^3 - 3x + k$, where k is an integer greater than 2.

(a) Show that $f(x)$ has exactly one real zero r, and explain why $r < -1$.

(b) Give a reason why the two complex zeroes of $f(x)$ form a conjugate pair.

(c) If the complex zeroes are $a + ib$ and $a - ib$, use the result for the sum of the roots two at a time to show that $b^2 = 3(a^2 - 1)$.

(d) Find the three zeroes of $f(x)$ given that $k = 2702$, and that a and b are integers.

17. Prove that $P(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$, where $n \geq 2$, has no multiple zeroes.

18. Consider the polynomial $P(z) = z^4 + 4z^3 + 14z^2 + 20z + 25$.

(a) Show that $P(-1 + 2i)$ and $P'(-1 + 2i)$ are both zero.

(b) What can we deduce from (a)?

(c) Explain why $-1 - 2i$ is also a double zero of $P(z)$.

(d) Hence factorise $P(z)$ over the complex numbers and then over the real numbers.

ENRICHMENT

19. In the text it was proven that if $P(z)$ is a polynomial with real coefficients and if $P(\alpha) = 0$ then $P(\overline{\alpha}) = 0$. Use a similar approach to prove that if $P(\alpha) = \beta$ then $P(\overline{\alpha}) = \overline{\beta}$.

20. Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ be a polynomial with integer coefficients. Suppose that $P(x)$ has a rational zero $x = \frac{p}{q}$ where p and q have highest common factor 1. Show that p is a factor of a_0 and that q is a factor of a_n.

21. Use the Fundamental Theorem of Algebra to carefully explain why every polynomial of odd degree with real coefficients has at least one real zero.

22. The polynomial $P(z)$ has real coefficients and a double complex zero $z = \alpha$.

(a) Prove that $z = \overline{\alpha}$ is also a double zero.

(b) Explain why $\left(z^2 - 2\,\mathrm{Re}(\alpha)z + |\alpha|^2\right)^2$ is a factor of $P(z)$.

(c) Hence prove that $P'(\alpha) = 0$.

(d) Try to generalise this result to complex zeros with higher multiplicity.

23. (a) Let u and v be two numbers of the form $u = a + b\sqrt{c}$, where a, b and c are rational numbers, with $\sqrt{c}$ an irrational constant. Let the notation u^* indicate the value of u when the sign of b is reversed. That is, $u^* = a - b\sqrt{c}$.

(i) Show that $u^* + v^* = (u + v)^*$.

(ii) Show that $\lambda u^* = (\lambda u)^*$ whenever λ is a rational number.

(iii) Prove by induction that $(u^n)^* = (u^*)^n$ for positive integers n.

(b) Suppose that $u = a + b\sqrt{c}$ is a zero of a certain polynomial with rational coefficients. Use the results of part (a) to show that $u^* = a - b\sqrt{c}$ is also a zero of this polynomial.

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1H Chapter Review Exercise

Exercise 1H

1. If $z = 3 - i$ and $w = 17 + i$, find:

(a) $6z - \overline{w}$ (b) z^3 (c) $\dfrac{w}{z}$

2. Write as a product of two complex linear factors.

(a) $z^2 + 100$ (b) $z^2 + 10z + 34$

3. Solve each quadratic equation for z.

(a) $z^2 - 8z + 25 = 0$ (b) $16z^2 + 16z + 13 = 0$

4. Find the square roots of:

(a) $5 - 12i$ (b) $7 + 6\sqrt{2}\,i$

5. Solve for z:

(a) $z^2 - 5z + (7 + i) = 0$ (b) $z^2 - (6 + i)z + (14 + 8i) = 0$

6. If $3i$ is a zero of a polynomial $P(z)$ with real coefficients, explain why $z^2 + 9$ is a factor of $P(z)$.

7. It is known that $2 + 5i$ is a zero of the polynomial $P(z) = z^3 - 8z^2 + 45z - 116$.

(a) Why is $2 - 5i$ also a zero of $P(z)$?

(b) Use the sum of the zeroes to find the third zero of $P(z)$.

(c) Hence write $P(z)$ as a product of two factors with real coefficients.

8. Express each complex number in modulus-argument form.

(a) $1 - i$ (b) $-3\sqrt{3} + 3i$

9. Express each complex number in Cartesian form.

(a) $4\operatorname{cis}\frac{\pi}{2}$ (b) $\sqrt{6}\operatorname{cis}(-\frac{3\pi}{4})$

10. Simplify:

(a) $2\operatorname{cis}\frac{\pi}{2} \times 3\operatorname{cis}\frac{\pi}{3}$ (b) $\dfrac{10\operatorname{cis}10\theta}{5\operatorname{cis}5\theta}$ (c) $(3\operatorname{cis}3\alpha)^2$

11. Sketch the graph in the complex plane represented by the equation:

(a) $|z - 2i| = 2$ (c) $\arg(z + 2) = -\frac{\pi}{4}$

(b) $|z| = |z - 2 - 2i|$ (d) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$

12. Shade the region in the complex plane that simultaneously satisfies $|z| \geq 1$, $\text{Re}(z) \leq 2$ and $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$.

13. Suppose that $z = -1 + \sqrt{3}\,i$ and $w = 1 + i$.

(a) Find $\dfrac{z}{w}$ in the form $a + ib$, where a and b are real.

(b) Write z and w in modulus-argument form.

(c) Hence write $\dfrac{z}{w}$ in modulus-argument form.

(d) Deduce that $\cos\frac{5\pi}{12} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$.

14. Sketch the graph specified by the equation:

(a) $z\overline{z} = z + \overline{z}$ (b) $\overline{z} = iz$ (c) $|z + 2| = 2|z - 4|$

15. A triangle PQR in the complex plane is isosceles, with $\angle P = 90°$. The points P and Q represent the complex numbers $4 - 2i$ and $7 + 3i$ respectively. It is also known that the points P, Q and R are in anticlockwise order. Find the complex numbers represented by:

(a) the vector PQ, (b) the vector PR, (c) the point R.

16. If $z_1 = 4 - i$ and $z_2 = 2i$, find in each case the two possible values of z_3 so that the points representing z_1, z_2 and z_3 form an isosceles right-angled triangle with the right-angle at:

(a) z_1 (b) z_2

17. In an Argand diagram, O is the origin and the points P and Q represent the complex numbers z_1 and z_2 respectively.

If triangle OPQ is equilateral, prove that ${z_1}^2 + {z_2}^2 = z_1 z_2$.

18. If $z_1 = 2\operatorname{cis}\frac{\pi}{12}$ and $z_2 = 2i$, find:

(a) $\arg(z_1 + z_2)$ (b) $\arg(z_2 - z_1)$

19. If z_1 and z_2 are complex numbers such that $|z_1| = |z_2|$, prove that

$$\arg(z_1 z_2) = \arg\left((z_1 + z_2)^2\right).$$

20. If $z = \operatorname{cis}\theta$, show that $\dfrac{z^2 - 1}{z^2 + 1} = i\tan\theta$.

21. The points A, B, C and O represent the numbers z, $\dfrac{1}{z}$, 1 and 0 respectively in the complex plane. Given that $0 < \arg z < \frac{\pi}{2}$, prove that $\angle OAC = \angle OCB$.

22. (a) By drawing a suitable diagram, prove the triangle inequality $|z_1 - z_2| \geq |z_1| - |z_2|$.

(b) Hence find the maximum value of $|z|$ given that $\left|z - \dfrac{4}{z}\right| = 2$.

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2

Proof

CHAPTER OVERVIEW: There are many opportunities for presenting or reading proofs in the Extension 1 and Extension 2 Mathematics courses. In this chapter, the basic concepts, terminology and notation are discussed, and a few common methods of proof are presented. Logic plays an important part in proof, but the emphasis here is on clear argument rather than fancy notation.

Section 2A introduces the necessary language. The remaining sections each focus on a common type of proof. Section 2B investigates various simple problems in number theory, that is, the basic structures of the integers. Proof by contradiction is presented in Section 2C. Algebraic inequalities are proven in Section 2D, whilst 2E introduces harder types of induction. The final section considers inequalities in calculus, which some may prefer to postpone until after the Integration chapter.

2A The Language of Proof

This long first section introduces some basic concepts, terminology and notation which are used in the proofs throughout the remainder of the chapter.

Statements: Statements are the basic building blocks of proof. Despite being so fundamental, it is rather difficult to define what a statement is. Two of the definitions given in The Macquarie Dictionary are as follows:

1. something stated
2. a communication or declaration in speech or writing setting forth facts, particulars, etc.

The latter is perhaps more relevant to mathematical proof. Thus a statement can be a simple sentence, such as:

n is a multiple of 3.

It may also be a mathematical declaration, such as:

let p be a prime number.

A statement may take the form of an assertion or definition, such as:

even numbers are divisible by 2,

or it may be a claim to be proven or a deduction which has already been proven, such as:

every multiple of 6 is also a multiple of 3.

Although these four examples do not say precisely what a statement is, they will enable discussions about statements with some agreement about what is meant.

Logical Values: Every statement must take one of two logical values: true or false. Using the first example of a statement above,

n is a multiple of 3

is true if it is known that $n = 6$, but is clearly false if it is known that $n = 7$.

1 | LOGICAL VALUES: A statement must take one of two logical values: true or false.

Notice in this example that the logical value of a statement may change with circumstance, but it cannot be simultaneously both true and false. There is no integer n which is both a multiple of 3 and not a multiple of 3. Likewise a statement cannot be neither true nor false. Given an integer n, it must always fall into one of two categories: a multiple of 3, or not a multiple of 3.

A Proven Statement: Whilst some statements may change logical value due to circumstance, other statements never change their logical value. The statement

34 is a Fibonacci number

is always true, as is determined by simply writing out the first few terms:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

A statement which is shown to be true is said to be *proven*, and the evidence used to establish the truth is called the *proof*. Thus the statement "34 is a Fibonacci number" is proven, and the proof is the listing of the first few terms of the sequence.

2 | A PROVEN STATEMENT: A statement which is shown to be true is said to be proven, and the evidence used to establish the truth is called the proof.

Examples and Counterexamples: Whilst it is rarely feasible, a statement can be shown to be true by example, but care must be taken to examine every possible case. One statement where this is feasible is:

every prime on a regular die is also a Fibonacci number.

The primes on a die are 2, 3 and 5, which are also Fibonacci numbers, so the statement is proven. Clearly this is a contrived situation, but it serves as a reminder that proof by exhaustive examples is sometimes possible.

In contrast, the statement

all primes are odd

is clearly false, since 2 is both prime and even. When a statement is shown to be false by example in this way, that example is called a *counterexample*, and the statement is said to be *disproved*. It is important to note that only one counterexample is needed in order to disprove a statement.

3 | A COUNTEREXAMPLE: A statement may be disproved (shown to be false) by a single example, called a counterexample.

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Negation: The negation of a statement changes its logical value. In English, negation is commonly associated with the word *not*. Thus, whilst the statement

all Fibonacci numbers are odd

is clearly false, its negation

not all Fibonacci numbers are odd

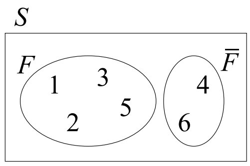
is equally clearly true. In English, this last statement might be considered clumsy, and so may be replaced with

some Fibonacci numbers are even.

Thus, whilst the presence of the word “not” clearly indicates a negation, its presence as an indicator cannot always be relied upon.

It is sometimes convenient to associate negation with complementary sets or events. By way of example:

let x be a Fibonacci number on a die



is true when $x = 1, 2, 3$ or 5 and false when $x = 4$ or 6. In contrast:

let x be a non-Fibonacci number on a die

is false when $x = 1, 2, 3$ or 5 and true when $x = 4$ or 6. Notice that, whilst the “not” has been modified into the prefix “non-”, the logical value of the statement has been changed as expected. All this is clearly evident in the Venn diagram showing the set of Fibonacci numbers F and its complementary set $\overline{F}$.

4 | Negation: The negation of a statement changes its logical value. It is commonly associated with the word *not*, and with complementary sets.

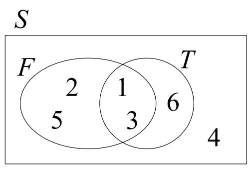
Notation: It is sometimes convenient to present and manipulate statements in an algebraic manner. Pronumerals are used to represent statements, as in:

suppose that p is the statement ‘x is a multiple of 2.’

The letter p is often used as it is the first letter of the word *proposition*, a synonym for a statement. There are two symbols which are commonly used to negate a statement. These are $\neg$ and $\sim$. Thus $\neg p = \sim p$, which is said “not-p” and means:

x is not a multiple of 2.

And, Or and Negation: In the Advanced and Extension 1 work on probability it was established that *and* corresponds to the intersection of sets, whilst *or* corresponds to the union. Thus if F is the set of Fibonacci numbers and T is the set of triangular numbers on a die, then



$$\begin{aligned} F \text{ and } T &= F \cap T \\ &= \{1, 3\}, \end{aligned}$$

whereas

$$\begin{aligned} F \text{ or } T &= F \cup T \\ &= \{1, 2, 3, 5, 6\}. \end{aligned}$$

It makes sense to extend this correspondence to negation and so write:

$$\neg(F \text{ and } T) = \overline{F \cap T},$$

with

$$\neg(F \text{ or } T) = \overline{F \cup T}.$$

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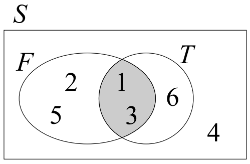
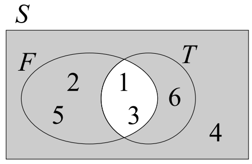
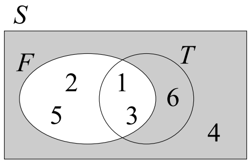
Here is the Venn diagram of $F \cap T$. And here is the Venn diagram of $\overline{F \cap T}$.

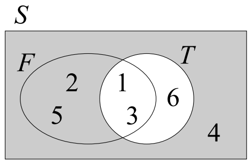




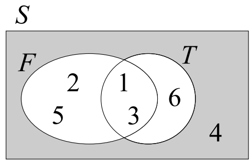
Analysing the latter diagram carefully, it should be clear that the result is



$\overline{F}$



or $\overline{T}$



or both.

That is, $\overline{F \cap T} = \overline{F} \cup \overline{T}$. Hence by analogy $\neg(F \text{ and } T) = \neg F \text{ or } \neg T$.

A practical example may help make it clear. Suppose that a school offers two languages: French and Japanese. A student who does not study both French and Japanese, either does not study French, or does not study Japanese, or does not study any language.

Similar analysis gives $\neg(F \text{ or } T) = \neg F \text{ and } \neg T$, which is left as an exercise.

5 AND, OR AND NEGATION: The rules are analogous to the complements of intersections of sets and unions of sets.

- The negation of *and* is *or*, so that $\neg(F \text{ and } T) = \neg F \text{ or } \neg T$.
- The negation of *or* is *and*, so that $\neg(F \text{ or } T) = \neg F \text{ and } \neg T$.

Implication: An implication is the relationship between two statements by which one is a logical consequence of the other. By way of example, if

n is a multiple of 3

then it logically follows that

$2n$ is also a multiple of 3.

In English, this is often written as a single *if … then …* statement. Thus:

if n is a multiple of 3 then $2n$ is also a multiple of 3.

Other words used in English for such logical deductions include: hence, thus, so, so that, consequently, and therefore. These words will have been regularly seen in the proofs of various theorems given in Year 11 mathematics.

Sometimes the order of the two statements is reversed in English, so care must be taken to determine which statement is the logical conclusion of the other. For example, the logical order of the deduction

a^2 is odd because a is odd

is made far clearer when it is re-written as:

if a is odd then a^2 is odd.

It is good practice when writing mathematics to put the conclusion second in this way. This order should be followed whenever it is practical to do so.

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Two other words commonly associated with implications are *each* and *all*. Here is a similar example written all three ways.

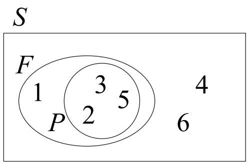
All odd squares are one more than a multiple of 4.

Each odd square is one more than a multiple of 4.

If a number is odd then its square is one more than a multiple of 4.

The careful reader will have also realised from the examples given so far that implication is commonly associated with subsets. Thus the odd squares form a subset of those numbers which are 1 more than a multiple of 4.

This is perhaps more easily seen in a simpler example. Consider the prime numbers P and Fibonacci numbers F on a die, as shown in the Venn diagram. Quite clearly:



all primes on a die are Fibonacci numbers

and it is equally clear that $P \subseteq F$.

6	IMPLICATION: An implication is the relationship between two statements by which one is a logical consequence of the other. In English, it is often written as an *if … then …* statement. Implication is also often associated with subsets.

The mathematical notation used for implication is a double tailed arrow $\Rightarrow$ which points towards the conclusion. Thus, returning to the example of odd squares:

a is odd $\Rightarrow a^2$ is 1 more than a multiple of 4.

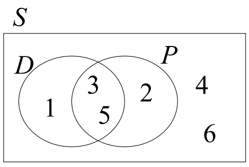
Whilst this notation is a useful tool, it should be used sparingly.

Quantifiers: The words *all* and *some* have been used in various examples above. These words, and their synonyms, are called *quantifiers* because they indicate a quantity. Thus, from above,

all of the primes on a die are Fibonacci numbers,

whereas, in contrast,

some of the primes on a die are odd.



Whilst *all* often indicates subsets, the word *some* is often associated with intersections of sets. This is made clear in the Venn diagram of the second example, where D is the set of odds and P is the set of primes.

7	QUANTIFIERS: The words *all* and *some*, and their synonyms, are called quantifiers. • *All* is associated with subsets. • *Some* is associated with intersections.

An important synonym for *some* is *there exists*. Thus, continuing with the odds and primes on a die, it is also true to state that:

there exists a prime on a die that is odd,

there exists a prime on a die that is not odd.

It was mentioned above that, whilst logic notation is a useful tool, it should be used sparingly. Indeed it is often clearer to write a statement in words than in symbols. To make this point, some more useful symbols will now be introduced.

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Recall that:

all odd squares are one more than a multiple of 4.

The symbol $\forall$ means *for all*. Thus the above implication might be written as:

$\forall\ a$ odd, a^2 is 1 more than a multiple of 4.

Next observe that the value of a can always be written as $a = 2m + 1$, where $m \in \mathbf{Z}$. That is, m is an integer. So

$\forall\ m \in \mathbf{Z}$, $(2m + 1)^2 = (\text{multiple of } 4) + 1$.

Lastly, the multiple of 4 can always be found. For example, $5^2 = 6 \times 4 + 1$. To put it another way, there exists an integer k which is the multiple of 4. The symbol for *there exists* is $\exists$. So finally the statement becomes:

$\forall\ m \in \mathbf{Z}, \exists\ k \in \mathbf{Z}$ such that $(2m + 1)^2 = 4k + 1$.

Compare this last jumble of symbols with the clear prose given in the statement at the top of the page. The symbols have got in the way and hindered an easy reading of the implication. Here is another pair of statements to show that the use of symbols can be a distraction.

Not all primes are odd.

$\exists\ x \in \{\text{primes}\}$ such that $x \neq 2m + 1, \forall\ m \in \mathbf{Z}$.

Of course, these symbols will be needed in certain problems, and students will be expected to understand them when they are used in questions. Nevertheless, symbols used in a proof or logical argument in this course should be the exception rather than the rule.

Sufficient and Necessary: Two other terms are strongly associated with implication and are often used in mathematical proofs or discussions about proofs. It is easiest to explain these terms when the statement is written in *if ... then ...* form. So, here is the statement about primes on a die written in that way.

If a number on a die is prime then it is a Fibonacci number.

The first part of an *if ... then ...* statement is called a *sufficient* condition. That is, it is sufficient to know that a number on a die is a prime to guarantee that it is a Fibonacci number. The second part of an *if ... then ...* statement is called a *necessary* condition. That is, it is necessary to know that a number on a die is a Fibonacci number in order for it to be prime. However, it does not guarantee that it is a prime. It could be the number 1.

The situation is once again made clear by the fact that on the die, the primes form a subset of the Fibonacci numbers. The subset corresponds to the sufficient condition, and the superset corresponds to the necessary condition. A practical example should help make it clear.

If I travel by bus then I use public transport.

It is sufficient that I travel by bus in order to use public transport, but it is necessary that I use public transport in order to travel by bus.

8

SUFFICIENT AND NECESSARY: In the statement *if A then B*,
- A is a sufficient condition for B,
- B is a necessary condition for A.

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Converse Statements: Consider the following two statements.

If x is a multiple of 4 then x is even.

If x is even then x is a multiple of 4.

When the two parts of an *if … then …* statement are swapped in this way, the result is called the *converse*. In this case, clearly the first statement is true, and the converse is false. This is clear because the multiples of 4 form a strict subset of the evens, so the converse cannot possibly be true. In other words, the converse of a statement is not the same as the original statement.

In contrast, here are two converse statements from Year 9 geometry.

If a quadrilateral is a rhombus then it has four equal sides.

If a quadrilateral has four equal sides then it is a rhombus.

In this case both statements are true, but it is important to realise that the converse statement is still not the same as the original. In the first statement, it is given that the quadrilateral is a rhombus. This is called the *premise*, and the conclusion is that it has four equal sides. The second statement differs because the premise is now that the quadrilateral has four equal sides, and the conclusion is that it is a rhombus.

9 CONVERSE STATEMENTS:
The converse of the statement *if A then B* is the statement *if B then A*.
A statement and its converse may have different logical values.

Equivalent Statements: Two statements are called *equivalent* if each is a logical consequence of the other. When solving an equation, the separate steps were called equivalent equations, which are examples of equivalent statements. Thus

$$2x + 3 = 11 \qquad \text{and} \qquad 2x = 8$$

are equivalent. The latter can be obtained by subtracting 3 from both sides of the first. The former can be obtained by adding 3 to both sides of the second. So each is a logical consequence of the other. This symmetry means that the two statements can be written as an *if … then …* statement and its converse, both of which are true.

If $2x + 3 = 11$ then $2x = 8$.

If $2x = 8$ then $2x + 3 = 11$.

10 EQUIVALENT STATEMENTS: Two statements are equivalent if each is a consequence of the other. This symmetry means that the two statements can be written as an *if … then …* statement and its converse, both with the same logical value.

Clearly equivalent statements are important and so they have special terminology and notation. The two implications are often abbreviated into one statement using the words *if and only if*. Thus, from the geometry example above:

a quadrilateral is a rhombus if and only if it has four equal sides.

When written, the words *if and only if* may be abbreviated to *iff*, or the symbol $\Leftrightarrow$ may be used. Here is the same statement written in those two ways.

A quadrilateral is a rhombus iff it has four equal sides.

A quadrilateral is a rhombus $\Leftrightarrow$ a quadrilateral has four equal sides.

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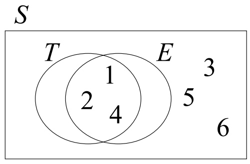
Whichever of the new terminology or symbol is used, it is important to remember that the statement is always an abbreviation of two implications. Thus

n is divisible by 3 iff the sum of its digits is divisible by 3

is a mathematical abbreviation for the two implications

if n is divisible by 3 then the sum of its digits is divisible by 3,

if the sum of the digits of n is divisible by 3 then n is divisible by 3.



Since equivalent statements can be written as a pair of implication statements, it follows that each corresponds to a subset of the other. Consider the the powers of 2 on a die, which are 1, 2 and 4, denoted by set T. These are also the proper factors of 8, denoted by set E. Thus:

x is a power of 2 on a die iff it is a proper factor of 8,

$x \in T \Leftrightarrow x \in E$.

Since each set is a subset of the other, it follows that both sets are equal. That is $T = E$, as shown in the Venn diagram above.

11	EQUIVALENT STATEMENTS AND SETS: The sets corresponding to equivalent statements are equal.

Recall that an implication is associated with the words *sufficient* and *necessary*. Writing the last example of equivalence as an implication and its converse:

if x is a power of 2 on a die then it is a proper factor of 8,

if x is a proper factor of 8 on a die then it is a power of 2.

From the first of these, it is sufficient that x is a power of 2 on a die for it to be a proper factor of 8. From the second, it is necessary that x is a power of 2 on a die for it to be a proper factor of 8. In other words, x is a power of 2 on a die is both a *necessary and sufficient* condition for it to be a proper factor of 8. A similar situation will hold for any pair of equivalent statements.

12	NECESSARY AND SUFFICIENT: When two statements are equivalent, each is both a necessary and sufficient condition for the other to be true.

The Contrapositive: There are many examples of pairs of equivalent statements, but one particular situation plays an important part in many proofs. By way of example, recall that

if x is a prime on a die then it is a Fibonacci number.

This is logically equivalent to the contrapositive statement

if x is not a Fibonacci number on a die then it is not a prime.

Notice that the two parts of the *if … then …* statement have been negated and the order swapped. This is always the case with the contrapositive statement. It is also always the case that an implication and its contrapositive are equivalent.

A practical example may help cement this new concept. Thus

if I own a dog then I have a pet

is logically equivalent to its contrapositive

if I do not have a pet then I do not own a dog.

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This means that an implication and its contrapositive can be written in the form of an *if and only if* statement:

A implies B if and only if not B implies not A,

or, writing this entirely in symbols:

$(A \Rightarrow B) \Leftrightarrow (\sim B \Rightarrow \sim A)$.

13 THE CONTRAPOSITIVE: An implication, if A then B, is logically equivalent to its contrapositive, if not B then not A.

The situation is clear from the Venn diagram of the primes and Fibonacci numbers on a die. In that case $P \subseteq F$, and consequently $\overline{F} \subseteq \overline{P}$. That is, if x is not a Fibonacci number on a die then it is not prime. Here are Venn diagrams with $\overline{F}$ and $\overline{P}$ shaded to demonstrate that $\overline{F} \subseteq \overline{P}$.

It is further confirmed that $\overline{F} \subseteq \overline{P}$ by listing the elements of each set. Thus $\overline{F} = \{4, 6\}$ whilst $\overline{P} = \{1, 4, 6\}$.

Exercise **2A**

1. Identify each of these symbols.

(a) $=$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\forall$ (e) $\exists$

2. Write down the converse of each statement, and state whether the converse is true or false.

(a) If a triangle has two equal sides, then it has two equal angles.
(b) If a number is odd, then its square is odd.
(c) If I am a horse, then I have four legs.
(d) If a number ends with the digit 6, then it is even.
(e) Every square is a rhombus.
(f) If $\sqrt{n} \in \mathbf{R}$, then $n \geq 0$.

3. Indicate whether each statement is true or false.

(a) Having four legs is a necessary condition for being a cat.
(b) Having four legs is a sufficient condition for being a cat.
(c) Owning a car is a necessary condition for holding a driver's licence.
(d) Owning a car is a sufficient condition for holding a driver's licence.
(e) If two statements are equivalent, then each is a necessary condition for the other to be true.
(f) If two statements are equivalent, then each is a sufficient condition for the other to be true.

4. Write down the negation of each statement.

(a) All cars are red.
(b) $a > b$
(c) Hillary likes steak and pizza.
(d) Bill is correct or Dave is correct.
(e) If I live in Tasmania, then I live in Australia.
(f) If Nikhil doesn't study, then he will fail.
(g) $-3 \leq x \leq 8$
(h) $x < -5$ or $x \geq 0$

5. Write down the contrapositive of each statement.

(a) If I water my plants, then they will grow.

(b) If you do not live in Australia, then you do not live in Melbourne.

(c) If a triangle has three equal sides, then it has three equal angles.

(d) If I like cycling, then I do not like motorists.

(e) If a number is even, then the next number is odd.

(f) If a and b are both positive and $a > b$, then $\frac{1}{a} < \frac{1}{b}$.

6. Write down (in 'if... then' form) the two converse statements equivalent to each 'if and only if' statement.

(a) A number is divisible by 15 if and only if it is divisible by both 3 and 5.

(b) A triangle has two equal angles if and only if it has two equal sides.

(c) An integer n greater than one is prime if and only if its only divisors are 1 and n.

(d) A quadrilateral is a parallelogram if and only if a pair of opposite sides are equal and parallel.

DEVELOPMENT

7. State whether each statement is true $\forall x \in \mathbf{R}$. If it is false, provide a counterexample.

(a) $x - 3 < x$

(b) $3x \geq x$

(c) $10^x > 0$

(d) $x \leq x^2$

(e) $|-x| = x$

(f) $|x| = \sqrt{x^2}$

8. State whether each statement is true $\forall a, b \in \mathbf{R}$. If it is false, provide a counterexample.

(a) If $a > b$, then $a^2 > b^2$

(b) If $a^2 > b^2$, then $a > b$

(c) If $a > b$, then $a^3 > b^3$

(d) If $a, b < 0$ and $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

(e) $|a + b| \geq |a| + |b|$

(f) $|a - b| \geq \big||a| - |b|\big|$

9. Assuming that all variables are *real* variables, insert the correct symbol $\Rightarrow$ or $\Leftrightarrow$ in each statement below.

(a) it is raining ... there are clouds in the sky

(b) $3a = 6 \ldots 5a = 10$

(c) $a > b \ldots -b > -a$

(d) $x = 5 \ldots x^2 = 25$

(e) $x = 5 \ldots x^3 = 125$

(f) a is an integer ... a^2 is an integer

10. Answer true or false. Assume that $a, b, c \neq 0$.

(a) $\theta = \frac{\pi}{6} \Leftrightarrow \sin\theta = \frac{1}{2}$

(b) $\sin\theta = -\frac{1}{\sqrt{2}} \Leftrightarrow \tan\theta = \pm 1$

(c) $x^2 + y^2 < 1 \Leftrightarrow (x, y)$ is a point inside the circle $x^2 + y^2 = 1$

(d) α and β are the roots of $ax^2 + bx + c = 0 \Leftrightarrow \frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of $cx^2 + bx + a = 0$.

11. Suppose that p is the statement 'Jack does Extension 2 Mathematics' and q is the statement 'Jack is crazy'. Write each of the following as English sentences.

(a) $p \Rightarrow q$

(b) $\sim (p \Rightarrow q)$

(c) $\sim p \Rightarrow q$

(d) $\sim q \Rightarrow p$

(e) $\sim p \Rightarrow \sim q$

(f) $p \not\Rightarrow q$

12. Write each statement as an English sentence, without any use of symbols.

(a) $\forall n \in \mathbf{Z}\ \exists m \in \mathbf{Z}$ such that $m > n$

(b) $a \in \mathbf{R}$ and $a > 0 \Rightarrow a + \dfrac{1}{a} \geq 2$

13. An implication is false if the premise is true but the conclusion is false. Otherwise it is true. State whether each statement is true or false.

(a) $1+1=3 \Rightarrow 2+2=4$

(b) $1+1=2 \Rightarrow 2+2=3$

(c) $1+1=3 \Rightarrow 2+2=3$

(d) $1+1=2 \Rightarrow 2+2=4$

14. (a) Explain why "$1 < 0$" and "1 is a negative number" are equivalent statements.

(b) Combine them into an *if ... then ...* statement. Is the result true?

15. Consider the following statements:

(1) If I do not do my homework then I will fail.
(2) If I study hard then I will pass.

If statements (1) and (2) are both true and I passed, then:

(a) Did I do my homework?

(b) Did I study hard?

ENRICHMENT

16. Consider this statement:

'If either Anna or Bryan passed the exam, then either Anna and Chris both passed or Bryan and Chris both passed.'

If the statement is **false**, determine whether Chris passed or failed.

17. On a train, Pender, Sadler and Ward are the the fireman, guard and driver, but NOT respectively. Also aboard the train are three passengers who have the same names: Dr Pender, Mr Sadler and Mr Ward.

1. Mr Sadler lives in Sydney.
2. The guard lives exactly half way between Melbourne and Sydney.
3. Mr Ward earns exactly \$100 000 per year.
4. The guard's nearest neighbour, one of the passengers, earns exactly three times as much as the guard.
5. Pender beats the fireman at pool.
6. The passenger whose name is the same as the guard's lives in Melbourne.

Who is the driver? Clearly explain your reasoning.

2B Number Proofs

Now that the basic concepts, terminology and notation of proof are understood, they can be put together to write proofs of simple results. The geometry studied in Years 7 to 10 involved many such proofs, and it would be worth reviewing some of those in light of this new understanding. In this section and in 2C, however, attention is focused on some basic results in *number theory*, that is, the study of the structures and number patterns in:

$\mathbf{Z}$	the integers	$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$
$\mathbf{Z}^+$	the positive integers	$1, 2, 3, \ldots$
$\mathbf{N}$	the natural numbers	$0, 1, 2, 3, \ldots$

A proper investigation of number theory would occupy an entire book, so only a few simple results in divisibility are proven here.

Divisibility: It is important to begin with a clear definition of what divisibility means. Let $a, b, m \in \mathbf{Z}$ and suppose that $b = am$. The numbers a and m are called factors of b. Furthermore b is said to be divisible by a. (Equally, b is divisible by m.) To put it another way, if b and a are integers and b is divisible by a then there exists an integer m such that $b = am$.

14 | DIVISIBILITY: If $a, b \in \mathbf{Z}$ and b is divisible by a then $\exists\, m \in \mathbf{Z}$ such that $b = am$.

This definition will be essential in the following worked examples, and in the exercise questions. Although the wording of the definition may seem obscure, in practice it is quite obvious. For example, 12 is divisible by 4 because $12 = 4 \times 3$.

WORKED EXAMPLE 1: Let a and b be two integers divisible by 3.

(a) Prove that $(a + b)$ is divisible by 3.

(b) Prove that $(ax + by)$ is divisible by 3, for all $x, y \in \mathbf{Z}$.

SOLUTION:

(a) By the definition of divisibility, there exist $m, n \in \mathbf{Z}$ such that

$$a = 3m$$

and $$b = 3n$$

Hence $$(a + b) = (3m + 3n) = 3(m + n)$$

That is, $(a + b)$ is divisible by 3.

(b) Likewise, for all $x, y \in Z$,

$$(ax + by) = (3mx + 3ny) = 3(mx + ny)$$

That is, $(ax + by)$ is divisible by 3.

WORKED EXAMPLE 2: Prove that $a^2 - a$ is even for all $a \in \mathbf{Z}$.

SOLUTION: Factoring, $a^2 - a = a(a - 1)$.
Now, if a is odd, then $(a - 1)$ is even,
and if a is even, then $(a - 1)$ is odd.
In either case, the product of an odd and even is even.
Hence $a^2 - a$ is even.

Note: this is essentially proof by example, with every case examined.

WORKED EXAMPLE 3: A student claims that if an integer n is divisible by both 4 and 6 then the number is divisible by $4 \times 6 = 24$.

(a) Disprove this claim by finding a counterexample.

(b) Explain what has gone wrong, and determine the correct conclusion.

SOLUTION:

(a) Clearly $12 = 4 \times 3$ and $12 = 6 \times 2$,
hence 12 is divisible by both 4 and 6,
but it is not divisible by 24.

(b) The problem is that 4 and 6 have common factor 2.
Since n is divisible by 4 and 6, there exist integers k and ℓ such that

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$$\begin{aligned} && n &= 4k \\ \text{and} && n &= 6\ell \\ \text{so} && 4k &= 6\ell \\ \text{or} && 2k &= 3\ell\,. \end{aligned}$$

Since 2 and 3 are primes, k is divisible by 3 (and ℓ is even),
so there exists an integer m such that $k = 3m$.

$$\begin{aligned} \text{Hence} \quad n &= 4 \times 3m \\ &= 12m \end{aligned}$$

That is, if n is divisible by both 4 and 6 then it is divisible by 12.

WORKED EXAMPLE 4: Let $n = 10x+y$, where $n, x, y \in \mathbf{Z}^+$, the positive integers.

(a) Prove that if n is divisible by 7 then $(x - 2y)$ is also divisible by 7.
(b) Further, prove that the converse is true.
(c) Write the result as an iff statement.
(d) Hence determine whether or not 3871 is divisible by 7.

SOLUTION:

(a) Since n is divisible by 7, $\exists\, m \in \mathbf{Z}^+$ such that

$$10x + y = 7m$$

The key to the proof is that y has been doubled:

$$\begin{aligned} \text{so} && 20x + 2y &= 7 \times (2m) \\ \text{thus} && -x + 2y &= 7(2m - 3x) && \text{(subtracting } 21x \text{ from both sides)} \\ \text{or} && x - 2y &= 7(3x - 2m) \end{aligned}$$

Hence $(x - 2y)$ is divisible by 7.

(b) Since the equations in part (a) are equivalent, it follows that

$$\Big[10x + y = 7m\Big] \Leftrightarrow \Big[x - 2y = 7(3x - 2m)\Big]\,.$$

Hence if $(x - 2y)$ is divisible by 7 then $n = 10x + y$ is also divisible by 7.

(c) That is, combining parts (a) and (b),

$n = 10x + y$ is divisible by 7 if and only if $(x - 2y)$ is divisible by 7.

(d) Since the result is an equivalence relation, the divisibility test can be applied recursively. Thus:

3871 is divisible by 7 iff $387 - 2 \times 1 = 385$ is divisible by 7.
385 is divisible by 7 iff $38 - 2 \times 5 = 28$ is divisible by 7.
Now $28 = 7 \times 4$ is clearly divisible by 7.
Hence 3871 is divisible by 7.

Exercise **2B**

NOTE: In this exercise you may assume that all pronumerals represent integers.

1. (a) If a and b are even, prove that $a + b$ is even.
(b) If a and b are odd, prove that $a + b$ is even.
(c) If a is even and b is odd, prove that $a + b$ is odd.

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2. (a) If a and b are even, prove that ab is even.
(b) If a and b are odd, prove that ab is odd.
(c) If a is even and b is odd, prove that ab is even.

3. Prove that:
(a) the square of an even number is even,
(b) the square of an odd number is odd.

4. Prove that:
(a) if b and $b + c$ are both divisible by a, then c is divisible by a,
(b) if b and $b - c$ are both divisible by a, then c is divisible by a.

5. If b and c are both divisible by a, prove that $bx + cy$ is divisible by a for any $x, y \in \mathbf{Z}$.

6. If a, b, c and d are consecutive integers, prove that:
(a) $a + d = b + c$ (b) $ad = bc - 2$ (c) $a^2 + d^2 = b^2 + c^2 + 4$

DEVELOPMENT

7. Prove that if $a - b$ is even, then $a^2 - b^2$ is divisible by 4.

8. Suppose that $2a + b$ and $3a + 2b$ are both divisible by n. Prove that a and b are both divisible by n.

9. Suppose that $a^2 + a$ and $a^2 - a$ are both divisible by 4. Prove that a is even.

10. Prove that $a^3 - a$ is divisible by $6 \ \forall \ a \in \mathbf{Z}$.

11. If a is even, prove that $a^3 + 2a^2$ is divisible by 8.

12. Prove that a number is divisible by 6 if and only if it is divisible by both 2 and 3. (Remember that to prove $A \Leftrightarrow B$, you must prove $B \Rightarrow A$ and $A \Rightarrow B$.)

13. Prove that an integer is the sum of 7 consecutive integers if and only if it is divisible by 7.

14. (a) If n is odd, prove that the sum of n consecutive numbers is divisible by n.
(b) If n is even, is the sum of n consecutive numbers divisible by n? Explain your answer.

15. Prove that a 4-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

16. Let $n = 10x + y$, where $n, x, y \in \mathbf{Z}^+$.
(a) Prove that if n is divisible by 13, then $x + 4y$ is also divisible by 13.
(b) Prove that the converse of part (a) is true.
(c) Combine (a) and (b) into an 'if and only if' statement using mathematical symbols.
(d) Use part (a) recursively to show that 8112 is divisible by 13.

17. (a) Show that $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x^2 + x + 1)$.
(b) Hence prove that:
(i) $7^n - 1$ is divisible by $6 \ \forall \ n \in \mathbf{Z}^+$,
(ii) if $a^n - 1$ is prime, then $a = 2$

18. Suppose that $n = p^a q^b$, where p and q are primes and $p \neq q$.
(a) Use combinatorics (a counting argument) to explain why n has $(a+1)(b+1)$ factors.
(b) Hence determine the number of factors of 80 000.

19. Prove the statement: $a - c$ is a divisor of $ab + cd \Rightarrow a - c$ is a divisor of $ad + bc$.

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ENRICHMENT

20. (a) Factorise $a^4 + 4b^4$ by adding and subtracting $4a^2b^2$.
(b) Hence prove that $545^4 + 4^{545}$ is not a prime number.

21. (a) Prove that the square of an even number is divisible by 4.
(b) Prove that the remainder is 1 when the square of an odd number is divided by 8.
(c) Hence prove that if a and b are both odd, then $a^2 + b^2$ is not a square.

22. Suppose that p is a prime number greater than 30. Prove that when p is divided by 30, the remainder is either 1 or prime.

23. Numbers such as 6 and 28 are known as *perfect* numbers because they are equal to the sum of their factors, excluding the number itself.
(a) Confirm that 6 and 28 are perfect numbers.
(b) Prove that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is a perfect number.
[HINT: Separate the factors of $2^{n-1}(2^n - 1)$ into two groups: those that are strictly powers of 2 and those that have the factor $2^n - 1$.]

24. Suppose that I choose six of the first ten positive integers. Prove that I must have chosen two numbers such that one is a divisor of the other. [HINT: Write each of the 10 numbers as a power of 2 multiplied by an odd number, then use the pigeonhole principle.]

2C Proof by Contraposition and by Contradiction

Two methods of proof commonly used in mathematics involve negation. These are called *proof by contraposition* and *proof by contradiction*. The latter is also given the Latin name *reductio ad absurdum*, which literally means *reduce to absurdity*. The reason for this will be explained later.

Proof by Contraposition: This style of proof takes advantage of the fact that an implication is equivalent to its contrapositive. Thus when an implication is not easy to prove directly, it may be suitable to use proof by contraposition instead. It is important to clearly state what is being done at the outset.

WORKED EXAMPLE 5: Prove that if n^2 is even then n is even.

SOLUTION: The contrapositive is proven instead. That is:

if n is not even then n^2 is not even,

or, more naturally:

if n is odd then n^2 is odd.

Let $n = 2m + 1$, then n is odd and

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \end{aligned}$$

which is one more than a multiple of 2, and is thus odd.
Hence, by the contrapositive, if n^2 is even then n is even.

15 PROOF BY CONTRAPOSITION: An implication and its contrapositive are equivalent. Hence an implication may be proven by proving its contrapositive instead.

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Proof by Contradiction: This style of proof is based on the fact that a statement can only have one of two values, true or false, and the negated statement must have the opposite value. That is, if a statement is true then its negation must be false. Hence proving a statement is equivalent to showing that its negation is false. Thus when a statement is not easy to prove directly, it may be suitable to show instead that the negated statement is false.

The first step is to write down the negation as an assumption. It is then shown that this leads to an absurd statement, like $-1 > 2$, or it leads to a contradiction of the initial assumption. This is why the method is also called *reductio ad absurdum*. Since the negation is equivalent to a false statement, the negation is also false. It must therefore hold that the proposition is true.

Worked Example 6: Prove that $\sqrt{2}$ is irrational.

Solution: By way of contradiction, assume that $\sqrt{2}$ is rational.
That is, assume there exist $m, n \in \mathbf{N}$ (natural numbers) such that

$$\sqrt{2} = \frac{m}{n}$$

where $n \geq 1$ and the HCF (highest common factor) of m and n is 1.
That is, the fraction has been reduced to lowest terms.
Squaring and re-arranging gives

$$2n^2 = m^2\,.$$

Thus m^2 is divisible by 2.
Now if m were not divisible by 2 then m^2 would not be divisible by 2.
Hence m is also divisible by 2. So let $m = 2p$ and write

$$2n^2 = 4p^2$$

or $$n^2 = 2p^2\,.$$

Thus n^2 is divisible by 2.
Now if n were not divisible by 2 then n^2 would not be divisible by 2.
Hence n is also divisible by 2.
That is, 2 is a common factor of m and n.
But the HCF is 1, so there is a contradiction.
Hence $\sqrt{2}$ is irrational.

16 | Proof by contradiction: Proving a statement is equivalent to showing that its negation is false. Begin by writing down the negation and show this leads to a contradiction.

Notice the careful use of two contrapositive statements in the above proof. Each is a restatement of what was proven in Worked Example 5. The proof that $\sqrt{3}$ is irrational requires similar steps. That is:

if m were not divisible by 3 then m^2 would not be divisible by 3.

A problem arises at this point because that claim has not been proven. Likewise, the proof that $\sqrt{7}$ is irrational leads to the claim that

if m were not divisible by 7 then m^2 would not be divisible by 7

which also has not been proven. In this course, all such claims will be taken to be intuitively obvious unless a proof is specifically required by the question.

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The next worked example follows a slightly different argument to show that $\log_2 3$ is irrational. This result is extremely important in music. Musicians will know that when a fifth is played on a piano, say from A to E, it is not a pure fifth but slightly short. A pure fifth has frequencies in the ratio 2 : 3. Thus using A440 should give E660, but on a piano it is approximately E659. This is done so that the cycle of fifths works, and it comes about because $\log_2 3$ is irrational.

Worked Example 7: Prove that $\log_2 3$ is irrational.

Solution: By way of contradiction, assume that $\log_2 3$ is rational.
That is, assume there exist $m, n \in \mathbf{N}$ such that

$$\log_2 3 = \frac{m}{n}$$

where $n \geq 1$ and the HCF of m and n is 1.
That is, the fraction has been reduced to lowest terms.
From the definition of logs, this can be re-written as

$$3 = 2^{\frac{m}{n}}\,.$$

Take the n^{th} power of both sides to get

$$3^n = 2^m$$

Now since $n \geq 1$ it follows that 3 is a factor of the LHS.
But clearly 3 is not a factor of the RHS, so there is a contradiction.
Hence $\log_2 3$ is irrational.

The Fundamental Theorem of Arithmetic: In the last proof it was assumed that a positive power of 3 cannot equal a positive power of 2. Intuition and experience certainly seem to suggest this is true, but it has not been proven. The result is a specific case of a more general theorem. The theorem effectively states that there is only one way to write a number in prime factored form. This is such an important result in number theory that it is given the name *the fundamental theorem of arithmetic*. Although any proof of this result is beyond the scope of this course, it is such an important theorem that a proof has been included in the appendix to this chapter.

Exercise **2C**

1. Prove by contradiction that $\log_7 13$ is irrational.
   Start by assuming that $\log_7 13$ is rational, so $\log_7 13 = \frac{m}{n}$, where $m, n \in \mathbf{Z}$ and m, n have no common factors other than 1.

2. Prove by contradiction that $\sqrt{5}$ is irrational.
   Start by assuming that $\sqrt{5}$ is rational, so $\sqrt{5} = \frac{m}{n}$, where $m, n \in \mathbf{Z}$ and m, n have no common factors other than 1.

3. Consider this statement for $a \in \mathbf{N}$: 'If a^2 is odd then a is odd.'
   (a) Write down the contrapositive of the statement.
   (b) Prove the statement by proving its contrapositive.

4. By proving the contrapositive, prove that if $m^2 + 4m + 7$ is even, then m is odd.

5. Suppose that $a, b \in \mathbf{Z}^+$. By proving the contrapositive, prove that if ab is even, then a is even or b is even.

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DEVELOPMENT

6. (a) Prove that $\log_3 5$ is irrational.
(b) Hence prove that $\log_3 15$ is irrational.

7. (a) Prove that $\sqrt{11}$ is irrational.
(b) Hence prove that $\sqrt{44}$ is irrational.

8. Suppose that the number n is composite and has two prime factors p_1 and p_2. Use contradiction to prove that at least one of p_1 and p_2 is less than $\sqrt{n}$.

9. Suppose that $\exists\, n \in \mathbf{N}$ such that $n^2 + 2$ is divisible by 4.
(a) Deduce that n is even.
(b) Hence prove by contradiction that no such n exists.

10. (a) Explain why every odd number is one more or one less than a multiple of 4.
(b) Prove that the product of any two positive integers of the form $4n + 1$, where n is a positive integer, is also of the form $4n + 1$.
(c) Hence prove by contradiction that any composite number of the form $4n - 1$ must have at least one prime factor of the form $4n - 1$.

11. Prove by contradiction that if $n \in \mathbf{Z}^+$, then $\sqrt{4n-2}$ is irrational.

12. Prove by contradiction that $\sqrt{3} + 1$ is irrational.

13. (a) Prove that $\sqrt{6}$ is irrational.
(b) Hence prove that $\sqrt{3} + \sqrt{2}$ is irrational.

14. Prove that if $2^n - 1$ is prime then n is prime by proving the contrapositive.

15. Prove by contradiction that there are infinitely many prime numbers.
[Hint: Assume that p is the largest prime number and consider the number $p! + 1$, which is not divisible by any number from 2 to p.]

ENRICHMENT

16. Consider the following theorem about prime numbers.

$$\text{If } \big(\forall\, a, b \in \mathbf{Z}^+,\ p|ab \Rightarrow p|a \text{ or } p|b\big) \text{ then } p \text{ is prime.}$$

(a) State the contrapositive of the theorem.
(b) Prove the theorem by proving the contrapositive.
[Hint: Put $a = p_1$, where p_1 is a prime divisor of p, and $b = \frac{p}{p_1}$.]

17. Suppose that a, b, c and d are positive integers and c is not a square.
(a) Prove that if $\dfrac{a}{b + \sqrt{c}} + \dfrac{d}{\sqrt{c}}$ is rational, then $b^2 d = c(a + d)$.
(b) Hence prove by contradiction that $\dfrac{a}{1 + \sqrt{c}} + \dfrac{d}{\sqrt{c}}$ is irrational.

18. Suppose that p is a prime number greater than 3 and that for some $n \in \mathbf{N}$, p^n is a 20-digit number. Prove that among these 20 digits, there are at least three that are equal.
[Hint: Use proof by contradiction and the test for divisibility by 3.]

19. Prove by contradiction that $\forall\, a, b \in \mathbf{Z}^+$, $(36a + b)(36b + a)$ is not a power of 2.
[Hint: Assume that 2^k is the smallest power of 2 equal to $(36a + b)(36b + a)$.]

2D Algebraic Inequalities

Two fundamental assumptions about inequalities will be used throughout this course. They should be intuitively obvious from previous work on equations and inequations. Those assumptions are, for real numbers a, b and c:

if $a > b$ then $a + c > b + c$,

if $a > b$ and $c > 0$ then $ac > bc$,

with similar results for $a < b$. The first statement means that an inequality is unchanged when the same amount is added to both sides. In the second case the inequality is unchanged when both sides are multiplied by the same positive amount. A particular case of the first statement is when $c = -b$, which gives

if $a > b$ then $a - b > 0$.

That is, if $a > b$ then $(a - b)$ is positive. The converse is also assumed true.

There is an amazing number of types of algebraic inequalities, but most can be assigned to one of four broad categories for proving. Those are:

- put everything on one side,
- squares of reals cannot be negative,
- combinations of inequalities,
- begin with a known result.

The first category takes advantage of the last statement above.

Put Everything on One Side: In a few instances the inequality can easily be proved by moving all terms to one side and considering the sign of that expression.

WORKED EXAMPLE 8: Let $a < b$ be real numbers. Prove that the average of the squares of a and b is greater than the square of the average.

SOLUTION: The corresponding inequality to prove is

$$\frac{a^2 + b^2}{2} > \left(\frac{a+b}{2}\right)^2 .$$

Now

$$\begin{aligned} \text{LHS} - \text{RHS} &= \frac{a^2 + b^2}{2} - \frac{a^2 + 2ab + b^2}{4} \\ &= \frac{a^2 - 2ab + b^2}{4} \\ &= \frac{(a-b)^2}{4} \\ &> 0 \qquad \text{(since squares cannot be negative.)} \end{aligned}$$

Hence $\dfrac{a^2 + b^2}{2} - \left(\dfrac{a+b}{2}\right)^2 > 0$

and so

$$\frac{a^2 + b^2}{2} > \left(\frac{a+b}{2}\right)^2 .$$

Squares Cannot be Negative: The crucial step in the previous worked example was that the square of a real number cannot be negative. That fact can be deduced from the assumptions at the start of this section, and the proof is left as an exercise. This important result can be used to solve numerous other inequalities.

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Arithmetic and Geometric Means: Recall from sequences and series that if a and b are positive real numbers then the sequence a, x, b will be arithmetic if

$$x = \frac{a+b}{2}.$$

The value of x is called the *arithmetic mean* of a and b.

Likewise, the sequence a, y, b will be geometric if

$$y = \sqrt{ab}.$$

The value of y is called the *geometric mean* of a and b.

17 | ARITHMETIC AND GEOMETRIC MEANS: Suppose that a and b are positive.
- The arithmetic mean of a and b is $\dfrac{a+b}{2}$.
- The geometric mean of a and b is $\sqrt{ab}$.

The AM/GM Inequality: One very important result is that the arithmetic mean is at least as large as the geometric mean. This is sometimes called the *AM/GM inequality*. It is proved by noting that squares of real numbers cannot be negative.

WORKED EXAMPLE 9: Prove the AM/GM inequality.

SOLUTION: Let a and b be two positive real numbers.
Since squares cannot be negative,

$$\left(\sqrt{a} - \sqrt{b}\right)^2 \ge 0.$$

Expanding $\quad a - 2\sqrt{ab} + b \ge 0$

so $\quad a + b \ge 2\sqrt{ab}$

or $\quad \dfrac{a+b}{2} \ge \sqrt{ab}.$

18 | THE AM/GM INEQUALITY: The arithmetic mean and geometric mean of two positive numbers a and b are related by the inequality

$$\frac{a+b}{2} \ge \sqrt{ab}.$$

The AM/GM inequality can also be proven using circle geometry and that is done in a worked example in Section 2E.

Combinations of Inequalities: Having established an inequality, it can be restated using other variables and the results combined to form a new inequality.

WORKED EXAMPLE 10:

(a) Suppose that $1 \le k \le n$.
  (i) Show that $n \le k(n-k+1)$.
  (ii) Explain why $\sqrt{n} \le \sqrt{k(n-k+1)} \le \dfrac{n+1}{2}$.

(b) Hence prove that, for all positive integers n,

$$\sqrt{n^n} \le n! \le \left(\frac{n+1}{2}\right)^n.$$

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Solution:

(a) (i) The right hand side is a quadratic in k with negative leading coefficient:

$$k(n+1) - k^2 .$$

Hence its graph is concave down, and any minimum will occur at an end point of the domain. Direct substitution of the two end points gives

$$1 \times n = n \times 1 = n.$$

(ii) The left hand inequality follows directly from part (i).
By the AM/GM inequality,

$$\begin{aligned}\sqrt{k(n-k+1)} &\le \frac{k + (n-k+1)}{2} \\ &\le \frac{n+1}{2} .\end{aligned}$$

(b) By part (a) it follows that

$$\begin{aligned}
&\sqrt{n} \le \sqrt{1 \times n} \le \frac{n+1}{2} && \text{(k = 1)} \\
&\sqrt{n} \le \sqrt{2(n-1)} \le \frac{n+1}{2} && \text{(k = 2)} \\
&\sqrt{n} \le \sqrt{3(n-2)} \le \frac{n+1}{2} && \text{(k = 3)} \\
&\quad \vdots \\
&\sqrt{n} \le \sqrt{n \times 1} \le \frac{n+1}{2} && \text{(k = n)}
\end{aligned}$$

Now multiply all these results together to get

$$\left(\sqrt{n}\right)^n \le \sqrt{\Big(1 \times 2 \times \ldots \times n\Big) \times \Big(n \times (n-1) \times \ldots \times 1\Big)} \le \left(\frac{n+1}{2}\right)^n$$

hence $\quad \sqrt{n^n} \le \sqrt{n! \times n!} \le \left(\frac{n+1}{2}\right)^n$

or $\quad \sqrt{n^n} \le n! \le \left(\frac{n+1}{2}\right)^n$

This example happened to use multiplication of inequalities. Other examples may require addition, subtraction or division. Multiplication and division should be avoided unless it is guaranteed that the quantities involved do not change sign. Otherwise it is not known whether the direction of the inequality is affected.

Begin with a Known Result: Many problems begin with a known result from which another inequality is to be obtained. An important example of this is the relation

$$|x| + |y| \ge |x+y| \ge \Big||x| - |y|\Big|$$

which is known as the triangle inequality. This was proven in the chapter on complex numbers by using geometry. In the following Worked Example the left hand inequality is proven algebraically.

Worked Example 11: Use the fact that $|a| \ge a$ to prove that $|x|+|y| \ge |x+y|$.

Solution: Square the left hand side to get

$$\begin{aligned}(|x|+|y|)^2 &= x^2 + 2|x||y| + y^2 \\ &\ge x^2 + 2xy + y^2 \qquad (\text{since } |a| \ge a)\end{aligned}$$

so $\quad (|x|+|y|)^2 \ge (x+y)^2\,.$

Now take the square root of both sides to get

$$|x|+|y| \ge |x+y| \qquad (\text{since } \sqrt{a^2} = |a| \text{ on the RHS.})$$

The proof of the other part of the triangle inequality is left as an exercise.

In other problems restrictions are placed on the variables, such as $a+b+c=1$, from which it follows that $ab+ac+bc < \frac{1}{3}$. The usual approach is to begin manipulating the expression and at crucial steps apply the given restrictions.

In some instances the solution also involves changing the value of a fraction by altering the numerator or denominator. For example, decreasing the numerator or increasing the denominator will reduce the value of the fraction. This is clearly evident in the following numerical example.

$$\frac{3}{5} > \frac{2}{5} > \frac{2}{7}$$

The strategy is used twice in the next worked example, along with a restriction.

Worked Example 12: Let a, b and c be three positive real numbers. It is known that $a+b \ge c$.

(a) Show that $\dfrac{a+b}{1+a+b} \ge \dfrac{c}{1+c}\,.$

(b) Hence show that $\dfrac{a}{1+a} + \dfrac{b}{1+b} - \dfrac{c}{1+c} \ge 0\,.$

Solution:

(a)
$$\begin{aligned}\text{LHS} - \text{RHS} &= \frac{(a+b)(1+c) - c(1+a+b)}{(1+a+b)(1+c)} \\ &= \frac{(a+b)-c}{(1+a+b)(1+c)} \\ &\ge \frac{c-c}{(1+a+b)(1+c)} \qquad (\text{since } a+b \ge c)\end{aligned}$$

Thus $\text{LHS} - \text{RHS} \ge 0$, and hence $\text{LHS} \ge \text{RHS}\,.$

(b)
$$\begin{aligned}\frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} &\ge \frac{a}{1+a} + \frac{b}{1+b} - \frac{a+b}{1+a+b} \quad (\text{by part (a)}) \\ &= \frac{a}{1+a} + \frac{b}{1+b} - \frac{a}{1+a+b} - \frac{b}{1+a+b} \\ &\ge \frac{a}{1+a} + \frac{b}{1+b} - \frac{a}{1+a} - \frac{b}{1+b}\,,\end{aligned}$$

since decreasing the denominators increases the subtracted fractions.

Hence $\dfrac{a}{1+a} + \dfrac{b}{1+b} - \dfrac{c}{1+c} \ge 0\,.$

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Exercise **2D**

1. Given that $a > 1$, prove that $a^2 > 1$ by proving that LHS − RHS > 0.

2. Prove these inequalities for $a, b \in \mathbf{R}$. [HINT: Begin with LHS − RHS.]

(a) $a^2 + b^2 \geq 2ab$ (b) $\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2} \geq 2$ (c) $\dfrac{a^2 + b^2}{2} \geq \left(\dfrac{a+b}{2}\right)^2$

3. Prove that $a + \dfrac{1}{a} > 2$ for $a > 0$.

4. Prove, for $a, b > 0$, that:

(a) $\frac{1}{2}(a+b) \geq \sqrt{ab}$ (b) $\frac{1}{3}a + \frac{3}{4}b \geq \sqrt{ab}$

5. If $a > b > 0$, prove that:

(a) $a^2 - b > b^2 - a$ (b) $a^3 - b^3 > a^2b - ab^2$

6. (a) Given that x and y are non-negative, prove that $x + y \geq 2\sqrt{xy}$.

(b) Hence prove that $(x+y)(x+z)(y+z) \geq 8xyz$, where z is also non-negative.

DEVELOPMENT

7. Suppose that p, q and r are real and distinct.

(a) Prove that $p^2 + q^2 > 2pq$.

(b) Use part (a) three times to prove that $p^2 + q^2 + r^2 > pq + qr + rp$.

(c) Given that $p + q + r = 1$, prove that $pq + qr + rp < \frac{1}{3}$.
[HINT: Begin with $(p + q + r)^2 = 1$ and use part (b).]

8. Suppose that a, b and c are real numbers.

(a) Prove that $a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2$.

(b) Hence show that $a^2b^2 + a^2c^2 + b^2c^2 \geq a^2bc + b^2ac + c^2ab$.

(c) Deduce that if $a + b + c = d$, then $a^4 + b^4 + c^4 \geq abcd$.

9. Suppose that a, b and c are positive.

(a) Prove that $a^2 + b^2 \geq 2ab$.

(b) Hence prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

(c) Given that $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$,
prove that $a^3 + b^3 + c^3 \geq 3abc$.

(d) If x, y and z are positive, show that $x + y + z \geq 3\,(xyz)^{\frac{1}{3}}$.

10. (a) Prove that $1 + x \geq 2\sqrt{x}$ for $x > 0$.

(b) Suppose that $x, y, z > 0$ and $(1+x)(1+y)(1+z) = 8$. Prove that $xyz \leq 1$.

11. (a) Expand $\left(\dfrac{a}{b} - \dfrac{b}{a}\right)^4$.

(b) Hence prove that $\dfrac{a^4}{b^4} + \dfrac{b^4}{a^4} + 6 \geq 4\left(\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2}\right)$ for $a, b \in \mathbf{R}$.

(c) Deduce that $\dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} + 6 \geq 4\left(\dfrac{x}{y} + \dfrac{y}{x}\right)$ for $x, y > 0$.

12. (a) Suppose that a, b, c and d are positive.
Use the fact that $a^2 + b^2 \geq 2ab$ to show that $\dfrac{a^2 + b^2 + c^2 + d^2}{4} \geq \sqrt{abcd}$.

(b) Hence show that $\dfrac{w + x + y + z}{4} \geq \sqrt[4]{wxyz}$ for $w, x, y, z > 0$.

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13. (a) Show that $a^2 + b^2 \geq 2ab$ for all real numbers a and b.
(b) Hence show that $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$.
(c) Deduce that for all positive real numbers a, b and c:
$$(a + b + c)^2 \geq 3(ab + bc + ca)$$
(d) Suppose that a, b and c are the side lengths of a triangle.
(i) Explain why $(b - c)^2 \leq a^2$.
(ii) Deduce that $(a + b + c)^2 \leq 4(ab + bc + ca)$.

14. Suppose that a, b and c are positive.
(a) Prove that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$.
(b) Hence show that $(a + b + c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) \geq 9$.
(c) (i) Prove that $a^3 + b^3 \geq \left(\dfrac{a}{c} + \dfrac{b}{c}\right) abc$, and write down similar inequalities for $b^3 + c^3$ and $c^3 + a^3$.
(ii) Hence prove that $a^3 + b^3 + c^3 \geq 3abc$.
(iii) Deduce that $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} \geq 3$.

15. In the previous question we proved that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$ for $a, b > 0$. Use this result to prove that $ab(a + b) + bc(b + c) + ca(c + a) \geq 6abc$ for $a, b, c > 0$.

16. Suppose that x and y are positive.
(a) Prove that $\dfrac{1}{x} + \dfrac{1}{y} \geq \dfrac{4}{x + y}$.
(b) (i) Prove that $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{2}{xy}$.
(ii) Use part (i) and the square of part (a) to prove that $\dfrac{1}{x^2} + \dfrac{1}{y^2} \geq \dfrac{8}{(x + y)^2}$.

17. It is known that $|a| \geq a$ for any real number a.
Let x and y be any two real numbers. Use the above result to prove the triangle inequality
$$|x - y| \geq \big||x| - |y|\big|.$$
[Hint: Begin with $\text{LHS}^2 - \text{RHS}^2$.]

18. [Triangle inequality with complex numbers]
(a) Let $z = x + iy$ be a complex number. Prove algebraically that $\text{Re}(z) \leq |z|$.
(b) Let z and w be two complex numbers. Prove that $|z + w| \leq |z| + |w|$.
Begin by writing $|z + w|^2 = (z + w)\overline{(z + w)}$.
(c) Under what circumstances is $|z + w| = |z| + |w|$?

ENRICHMENT

19. Given $a, b, c > 0$ and $abc = 1$, use the AM/GM inequality with three terms to prove that
$$\frac{1 + ab}{1 + a} + \frac{1 + bc}{1 + b} + \frac{1 + ca}{1 + c} \geq 3.$$

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20. (a) Suppose that $a, b, c, a+b-c, a+c-b$ and $b+c-a$ are all positive.

(i) Prove that $(a+b-c)(a+c-b) \le a^2$.

(ii) Hence prove that $(a+b-c)(a+c-b)(b+c-a) \le abc$.

(b) Now consider a triangle ABC with side lengths a, b and c.

(i) Prove that $\sin^2 \frac{1}{2}A = \dfrac{(a+b-c)(a+c-b)}{4bc}$.

(ii) Deduce that $\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C \le \dfrac{1}{8}$.

21. If $a, b, c > 0$, prove that $\dfrac{a}{b+c} + \dfrac{b}{a+c} + \dfrac{c}{a+b} \ge \dfrac{3}{2}$.
[HINT: Begin by adding one to each fraction on the LHS.]

2E Induction

Induction is an important topic in the Mathematics Extension 1 course, and candidates in Extension 2 are expected to be proficient at the two main styles already met, sums and divisibility. The Exercise questions include problems which review this work. Induction will further be applied to inequalities and verifying formulae for sequences specified recursively.

Review of Induction: Recall that there are three main parts to an induction proof. First, the statement is verified for any initial terms. The second step proves the implication that if the statement is true for some integer k then it is true for the next integer $(k+1)$. Each proof then concludes with an appeal to the principle of mathematical induction.

WORKED EXAMPLE 13: Prove by mathematical induction that

$$\sum_{j=1}^{n} (-1)^j \, j^2 = \tfrac{1}{2}(-1)^n \, n(n+1)$$

for all positive integers n.

SOLUTION: This is just the n^{th} partial sum of the sequence $T_j = (-1)^j \, j^2$,

so let
$$\begin{aligned} S_n &= \sum_{j=1}^{n} T_j \\ &= -1^2 + 2^2 - 3^2 + \ldots + (-1)^n \, n^2 \,. \end{aligned}$$

Also recall that for the partial sums of any sequence

$$S_{n+1} = S_n + T_{n+1} \,.$$

A. When $n = 1$,
$$\begin{aligned} \text{LHS} &= (-1)^1 \times 1^2 \\ &= -1 \end{aligned}$$
and
$$\begin{aligned} \text{RHS} &= \tfrac{1}{2}(-1)^1 \times 1 \times 2 \\ &= -1 \\ &= \text{LHS} \end{aligned}$$
so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.
That is, assume that $S_k = \frac{1}{2}(-1)^k\, k(k+1)$ (†)
Now prove the statement for $n = k+1$. That is, prove that
$$S_{k+1} = \tfrac{1}{2}(-1)^{k+1}\,(k+1)(k+2).$$
But LHS $= S_k + T_{k+1}$
so by the induction hypothesis (†):
$$\begin{aligned} \text{LHS} &= \tfrac{1}{2}(-1)^k\, k(k+1) + (-1)^{k+1}(k+1)^2 \\ &= \tfrac{1}{2}(-1)^{k+1}(k+1)\Big(-k + 2(k+1)\Big) \\ &= \tfrac{1}{2}(-1)^{k+1}\,(k+1)(k+2) \\ &= \text{RHS}. \end{aligned}$$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \ge 1$.

Harder questions may involve divisibility problems. In other cases the statement may only be true for values of n in some specified sequence. The easiest way to manage this is to replace n with the formula for the m^{th} term of the sequence. The next Worked Example demonstrates both situations.

Worked Example 14: Prove by mathematical induction that $3^n - 2^n$ is divisible by 5 when n is a positive even integer.

Solution: Since n is even, put $n = 2m$.
Now show that $3^{2m} - 2^{2m}$ is divisible by 5 for $m \in \mathbf{Z}^+$.

A. When $m = 1$, $3^{2m} - 2^{2m} = 9 - 4$
$$= 5 \times 1$$
so the result is true for $m = 1$.

B. Assume the statement is true for the positive integer $m = k$.
That is, assume that $3^{2k} - 2^{2k} = 5p$, for some integer p. (†)
Now prove the statement for $m = k+1$.
That is, prove that $3^{2(k+1)} - 2^{2(k+1)}$ is divisible by 5.
Now $3^{2(k+1)} - 2^{2(k+1)} = 3^{2k} \times 9 - 2^{2k} \times 4$
so by the induction hypothesis (†):
$$\begin{aligned} 3^{2(k+1)} - 2^{2(k+1)} &= (5p + 2^{2k}) \times 9 - 2^{2k} \times 4 \\ &= 45p + 2^{2k} \times 5 \\ &= 5(9p + 2^{2k}) \end{aligned}$$
which is divisible by 5.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $m \ge 1$, and hence is true for all positive even integers n.

Proving Inequalities: Often it is necessary to prove an inequality by induction, thus combining the techniques of this section with those presented in Section 2D. In many cases, the best approach for the inequality is to put everything on one side.

WORKED EXAMPLE 15: Prove by mathematical induction that $2^n > n^3$ for all integers $n \geq 10$.

SOLUTION:

A. When $n = 10$,
$$\begin{aligned} \text{LHS} &= 2^{10} \\ &= 1024 \end{aligned}$$
and
$$\begin{aligned} \text{RHS} &= 10^3 \\ &= 1000 \end{aligned}$$
so the result is true for $n = 10$.

B. Assume the statement is true for the positive integer $n = k$.
That is, assume that $2^k > k^3$ (†)
Now prove the statement for $n = k + 1$. That is, prove that
$$2^{k+1} > (k+1)^3 .$$
This will be done by proving LHS − RHS > 0.
$$\begin{aligned} \text{LHS} - \text{RHS} &= 2^{k+1} - (k+1)^3 \\ &= 2 \times 2^k - (k^3 + 3k^2 + 3k + 1)\,, \end{aligned}$$
so by the induction hypothesis (†):
$$\begin{aligned} \text{LHS} - \text{RHS} &> 2k^3 - (k^3 + 3k^2 + 3k + 1) \\ &= k^3 - (3k^2 + 3k + 1) \\ &> k^3 - (3k^2 + 3k^2 + 3k^2) \quad \text{(since } k > 1) \\ &= k^2(k - 9) \\ &> 0 \qquad \text{(since } k \geq 10.) \end{aligned}$$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 10$.

Proving Recursive Formulae: Recall that sequences can be defined using an initial value and a recursive formula. For example, consider the sequence defined by

$$T_n = T_{n-1} + 2n\,, \quad \text{for } n > 1, \qquad \text{where} \quad T_1 = 2\,.$$

The first few terms of that sequence are $2, 6, 12, 20, 30, 42, \ldots$ It appears that a simpler formula for this sequence is

$$T_n = n^2 + n$$

and this can be proved by mathematical induction.

WORKED EXAMPLE 16: For the sequence defined recursively by

$$T_n = T_{n-1} + 2n\,, \quad \text{for } n > 1, \qquad \text{where} \quad T_1 = 2\,,$$

prove by mathematical induction that

$$T_n = n^2 + n\,, \quad \text{for all } n \geq 1.$$

SOLUTION:

A. When $n = 1$,
$$\begin{aligned} \text{RHS} &= 1^2 + 1 \\ &= 2 \end{aligned}$$
so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.
That is, assume that $T_k = k^2 + k$ (†)
Now prove the statement for $n = k + 1$. That is, prove that

$$T_{k+1} = (k+1)^2 + (k+1)$$
$$= k^2 + 3k + 2\,.$$

From the given recursion formula:

$$\text{LHS} = T_k + 2(k+1)$$

so by the induction hypothesis (†):

$$\text{LHS} = k^2 + k + 2(k+1)$$
$$= k^2 + 3k + 2$$
$$= \text{RHS}\,.$$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \ge 1$.

Notice how easy this last induction proof was. It is generally the case that recursive formulae are easy to prove by induction, but there are exceptions.

Exercise 2E

1. Prove by induction that for all positive integer values of n:

(a) $\displaystyle\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$

(b) $\displaystyle\sum_{r=1}^{n} r(r+1) = \tfrac{1}{3}n(n+1)(n+2)$

(c) $\displaystyle\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$

(d) $\displaystyle\sum_{r=1}^{n}(2r-1)^2 = \tfrac{1}{3}n(2n-1)(2n+1)$

(e) $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

(f) $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$

2. Prove by induction that for all positive integer values of n:

(a) $5^n + 3$ is divisible by 4

(b) $2^{3n} + 6$ is divisible by 7

(c) $5^n + 2^{n+1}$ is divisible by 3

(d) $9^{n+2} - 4^n$ is divisible by 5

(e) $6^n - 5n + 4$ is divisible by 5

(f) $4^n + 6n - 1$ is divisible by 9

DEVELOPMENT

3. Prove by induction that for all positive integer values of n:

(a) $1 + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+n) = \tfrac{1}{6}n(n+1)(n+2)$

(b) $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2$
[HINT: Use question 1 part (a).]

4. Prove by mathematical induction that for even values of n:

(a) $n^2 + 2n$ is a multiple of 8

(b) $n^3 + 2n$ is a multiple of 12

5. Prove by mathematical induction that for odd values of n:

(a) $7^n + 2^n$ is divisible by 9

(b) $7^n + 13^n + 19^n$ is divisible by 13

6. Prove these inequalities by mathematical induction:

(a) $n^2 \ge 3n - 2$ for $n \ge 1$

(b) $2^n \ge 1 + 3n$ for $n \ge 4$

7. (a) Prove by induction that $(1+c)^n > 1+cn$ for all integers $n \geq 2$, where c is a non-zero constant greater than -1.

(b) Hence show that $\left(1 - \dfrac{1}{2n}\right)^n > \dfrac{1}{2}$ for all integers $n \geq 2$.

8. (a) Solve the inequation $x^2 > 2x + 1$.

(b) Hence prove by induction that $2^n > n^2$ for all integers $n \geq 5$.

9. In each case a sequence has been defined recursively and then a formula given for the nth term. Use mathematical induction to prove each formula.

(a) If $T_1 = 1$ and $T_n = T_{n-1} + n$ for $n \geq 2$, then $T_n = \frac{1}{2}n(n-1)$ for $n \geq 1$.

(b) If $T_1 = 1$ and $T_n = 2T_{n-1} + 1$ for $n \geq 2$, then $T_n = 2^n - 1$ for $n \geq 1$.

(c) If $T_1 = 5$ and $T_n = 2T_{n-1} + 1$ for $n \geq 2$, then $T_n = 6 \times 2^{n-1} - 1$ for $n \geq 1$.

(d) If $T_1 = 1$ and $T_n = \dfrac{3T_{n-1} - 1}{4T_{n-1} - 1}$ for $n \geq 2$, then $T_n = \dfrac{n}{2n-1}$ for $n \geq 1$.

10. (a) By differentiating from first principles, show that $\dfrac{d}{dx}(x) = 1$.

(b) Use mathematical induction and the product rule to show that $\dfrac{d}{dx}\big(x^n\big) = nx^{n-1}$ for all positive integer values of n.

11. Prove by induction that the interior angle sum of a polygon with n sides is $(n-2) \times 180°$. [HINT: Dissect the $(k+1)$-gon into a k-gon and a triangle.]

12. Prove by induction that a polygon with n sides has $\frac{1}{2}n(n-3)$ diagonals.

13. Prove by induction that n lines in the plane, no two being parallel and no three concurrent, divide the plane into $\frac{1}{2}(n^2+n+2)$ regions. [HINT: The $(k+1)$th line will cross the other k lines in k distinct points, and so will add $k+1$ regions.]

14. Prove by mathematical induction that every set with n members has 2^n subsets. [HINT: When a new member is added to a k-member set, then every subset of the resulting $(k+1)$-member set either contains or does not contain the new member.]

15. Prove by induction that for all positive integer values of n:

(a) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} \leq 2 - \dfrac{1}{n}$

(b) $\dfrac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times 2n} \geq \dfrac{1}{2n}$

16. Prove by mathematical induction that $n^3 - n$ is divisible by 24 for odd values of n.

17. [Formulae for APs and GPs] Use mathematical induction to prove each result.

(a) If $T_1 = a$ and $T_n = T_{n-1} + d$ for $n \geq 2$, then $T_n = a + (n-1)d$ for $n \geq 1$.

(b) If $T_1 = a$ and $T_n = r\,T_{n-1}$ for $n \geq 2$, then $T_n = ar^{n-1}$ for $n \geq 1$.

(c) If $S_1 = a$ and $S_n - S_{n-1} = a + (n-1)d$ for $n \geq 2$, then

$$S_n = \tfrac{1}{2}n\big(2a + (n-1)d\big) \quad \text{for } n \geq 1.$$

(d) If $S_1 = a$ and $S_n - S_{n-1} = ar^{n-1}$ for $n \geq 2$, then

$$S_n = \frac{a\,(r^n - 1)}{r-1} \quad \text{for } n \geq 1.$$

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18. Suppose that $a, b > 0$ and n is a positive integer.

(a) Prove the inequality $a^{n+1} + b^{n+1} \ge a^n b + b^n a$.

(b) Hence prove by induction that $\left(\dfrac{a+b}{2}\right)^n \le \dfrac{a^n + b^n}{2}$ for all positive integers n.

19. (a) By rationalising the numerator, prove that $\sqrt{n+1} - \sqrt{n} > \dfrac{1}{2\sqrt{n+1}}$.

(b) Hence prove by induction that $1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \sqrt{n}$, for $n \ge 7$.

20. (a) Prove the identity $2\cos A \sin B = \sin(A+B) - \sin(A-B)$.

(b) Hence prove by induction that for all positive integers n,

$$\tfrac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta = \frac{\sin(n - \frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}.$$

21. (a) Prove that for positive values of x and y,

$$\frac{x}{y} + \frac{y}{x} \ge 2.$$

(b) Hence prove by induction that for positive values of $a_1, a_2, \ldots, a_n$,

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge n^2.$$

(c) Deduce that $\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta \ge 9\cos^2\theta$.

ENRICHMENT

22. In each case a sequence has been defined recursively and then a formula given for the nth term. Use a stronger form of induction to prove each formula.

(a) If $T_1 = 3$, $T_2 = 6$, and $T_n = 3\,T_{n-1} - 2\,T_{n-2} - 1$ for $n \ge 3$, then $T_n = n + 2^n$ for $n \ge 1$.

(b) If $T_1 = 8$, $T_2 = 34$, and $T_n = 8\,T_{n-1} - 15\,T_{n-2}$ for $n \ge 3$, then $T_n = 5^n + 3^n$ for $n \ge 1$.

(c) If $T_1 = 12$, $T_2 = 30$, and $T_n = 5\,T_{n-1} - 6\,T_{n-2}$ for $n \ge 3$, then

$$T_n = 3 \times 2^n + 2 \times 3^n \quad \text{for } n \ge 1.$$

(d) If $T_0 = T_1 = 2$, and $T_n = 2\,T_{n-1} + T_{n-2}$ for $n \ge 2$, then

$$T_n = \left(1 + \sqrt{2}\right)^n + \left(1 - \sqrt{2}\right)^n \quad \text{for } n \ge 0.$$

23. (a) Prove the inequality

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) > \frac{k}{(2k+1)(2k+2)}.$$

(b) Hence prove by induction that for all $n \ge 2$,

$$n\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right).$$

24. Define S_n as the sum of the products of all the pairs of distinct integers that can be formed from the first n positive integers.

So, for example, $S_3 = 1 \times 2 + 1 \times 3 + 2 \times 3$.

Prove by mathematical induction that $S_n = \frac{1}{24}(n-1)(n)(n+1)(3n+2)$ for $n \ge 2$.

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25. A squad of n footballers put their training tops out to wash. When the washing has finished drying, each player takes a training top, but it is found that no-one has their own. This situation is called a derangement. Let D_n be the number of derangements.

(a) In some derangements, Ben and another player have each other's top. Explain why the number of these derangements is $(n-1)D_{n-2}$, for $n > 2$.

(b) Find a similar formula for the remaining derangements and hence show that

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}\,, \quad \text{for } n > 2\,.$$

(c) Use the above result to show that

$$D_n - nD_{n-1} = (-1) \times \Big(D_{n-1} - (n-1)D_{n-2}\Big)\,, \quad \text{for } n > 2\,.$$

(d) Find D_1 and D_2, then prove by induction that $D_n - nD_{n-1} = (-1)^n$, for $n > 1$.

(e) Hence prove by induction that $D_n = n!\displaystyle\sum_{r=0}^{n} \frac{(-1)^r}{r!}$ for all $n \in \mathbf{Z}^+$.

2F Inequalities in Geometry and Calculus

The questions in this section often require the calculus of integration or curve sketching. Consequently it may be appropriate to delay this section to later in the course, such as at the end of the Extension 2 Integration chapter.

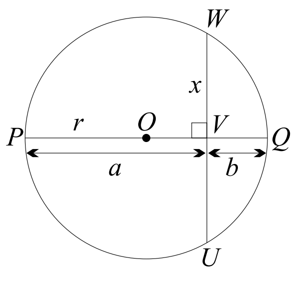
An Inequality Using Geometry: The AM/GM inequality was proven algebraically in Section 2D. Here is a geometric proof of this important result. It has the advantage of allowing readers to see how the inequality works, and identify the special case of equality.

WORKED EXAMPLE 17: Prove the AM/GM inequality for positive real numbers a and b using circle geometry. That is, prove that

$$\frac{a+b}{2} \geq \sqrt{ab}$$

and identify when the two sides are equal.

SOLUTION:



Construct line segment PQ with $|PQ| = a + b$.
Construct the circle with diameter $|PQ|$ and centre O.
The radius of this circle is clearly

$$r = \frac{a+b}{2} \qquad \text{(the arithmetic mean.)}$$

Let V be the point that divides PQ in the ratio $a : b$.
Construct chord UW perpendicular to PQ at V.
Hence V bisects UW, so let $UV = VW = x$.

Now $\quad x^2 = ab \quad$ (product of intercepts of intersecting chords)

so $\quad x = \sqrt{ab} \quad$ (the geometric mean.)

Finally, the longest chord in a circle is a diameter, hence

$$r \geq x$$

that is $\quad \dfrac{a+b}{2} \geq \sqrt{ab}$

with equality when UV is also a diameter. That is, $a = b$.

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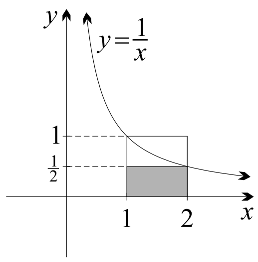
An Inequality Using Calculus:



Many inequalities arise through the study of calculus and geometry. As a very simple example, using an upper and a lower rectangle to approximate the area between the hyperbola $y = \frac{1}{x}$ and the x-axis for $1 < x < 2$ gives

$$\tfrac{1}{2} < \log 2 < 1\,.$$

Often knowledge of other topics is also required. The following worked example makes use of calculus and a geometric series to find an approximation for $\log \frac{3}{2}$ which is accurate to two decimal places.

Worked Example 18: Consider the geometric series

$$S_{2n} = 1 - h + h^2 - \ldots + h^{2n},$$

where $0 < h < 1$.

(a) Show that $S_{2n-1} < \dfrac{1}{1+h} < S_{2n}$.

(b) Integrate the previous result between $h = 0$ and $h = x$, where $0 < x < 1$, and hence write down a polynomial inequality for $\log(1 + x)$.

(c) Use $n = 3$ to estimate the value of $\log \frac{3}{2}$.

Solution:

(a) Now

$$\begin{aligned} S_{2n-1} &= 1 - h + h^2 - h^3 + \ldots - h^{2n-1} \\ &= \frac{1-(-h)^{2n}}{1-(-h)} \qquad \text{(by GP theory)} \\ &= \frac{1-h^{2n}}{1+h} \\ &< \frac{1}{1+h} \qquad \text{(increase numerator)} \end{aligned}$$

and

$$\begin{aligned} S_{2n} &= 1 - h + h^2 - \ldots + h^{2n} \\ &= \frac{1-(-h)^{2n+1}}{1-(-h)} \qquad \text{(by GP theory)} \\ &= \frac{1+h^{2n+1}}{1+h} \\ &> \frac{1}{1+h} \qquad \text{(decrease numerator)} \end{aligned}$$

hence $S_{2n-1} < \dfrac{1}{1+h} < S_{2n}$.

(b)

$$\int_0^x S_{2n-1}\,dh < \int_0^x \frac{1}{1+h}\,dh < \int_0^x S_{2n}\,dh$$

or

$$\left[h - \frac{h^2}{2} + \ldots - \frac{h^{2n}}{2n}\right]_0^x < \Big[\log(1+h)\Big]_0^x < \left[h - \frac{h^2}{2} + \ldots + \frac{h^{2n+1}}{2n+1}\right]_0^x$$

so

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots - \frac{x^{2n}}{2n} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{x^{2n+1}}{2n+1}$$

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(c) Put $n = 3$ and $x = \frac{1}{2}$ to get

$$\tfrac{1}{2} - \tfrac{1}{8} + \tfrac{1}{24} - \tfrac{1}{64} + \tfrac{1}{160} - \tfrac{1}{384} < \log \tfrac{3}{2} < \tfrac{1}{2} - \tfrac{1}{8} + \tfrac{1}{24} - \tfrac{1}{64} + \tfrac{1}{160} - \tfrac{1}{384} + \tfrac{1}{896}$$

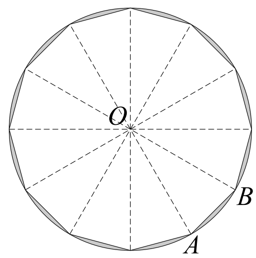
so $$\tfrac{259}{640} < \log \tfrac{3}{2} < \tfrac{909}{2240}$$

or $$0{\cdot}4047 < \log \tfrac{3}{2} < 0{\cdot}4058$$

These calculations suggest that $\log \frac{3}{2} \doteqdot 0{\cdot}405$, correct to three decimal places. The actual value is $0{\cdot}40547$, correct to five decimal places.

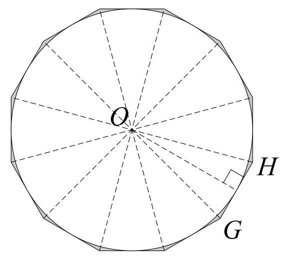
Exercise **2F**

1. (a) A regular dodecagon is drawn inside a circle of radius 1 cm and centre O so that its vertices lie on the circumference, as shown in the first diagram. Determine the area of $\triangle OAB$, and hence find the exact area of the inscribed dodecagon.



(b) (i) Use the formula for $\tan 2\theta$ to show that $\tan 15^\circ = 2 - \sqrt{3}$.

(ii) Another regular dodecagon is drawn with centre O, so that each side is tangent to the circle, as shown in the second diagram. Find the area of $\triangle OGH$ and hence find the exact area of the circumscribed dodecagon.



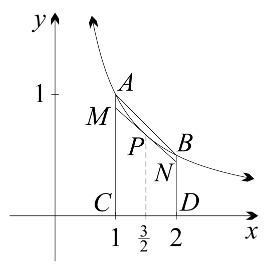
(c) By considering the results in parts (a) and (b), show that

$$3 < \pi < 12(2 - \sqrt{3}) \doteqdot 3{\cdot}24\,.$$

2. (a) Use Simpson's rule with three function values to approximate the area under $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{3}$.

(b) Hence show that $\pi \doteqdot \frac{18}{13}(4 - \sqrt{3})$, which is accurate to two decimal places.

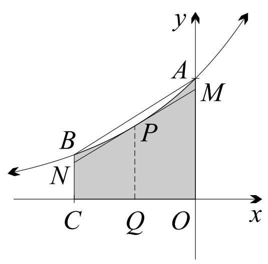
3. The points A, P and B on the curve $y = \dfrac{1}{x}$ have x-coordinates 1, $1\frac{1}{2}$ and 2 respectively. The points C and D are the feet of the perpendiculars drawn from A and B to the x-axis. The tangent to the curve at P cuts AC and BD at M and N respectively.



(a) Find the areas of trapezia $ABDC$ and $MNDC$.

(b) Hence show that $\frac{2}{3} < \ln 2 < \frac{3}{4}$.

4. The diagram shows the points $A(0, 1)$ and $B(-1, e^{-1})$ on the curve $y = e^x$, and the point $C(-1, 0)$ on the x-axis. The tangent at $P(-\frac{1}{2}, e^{-\frac{1}{2}})$ intersects OA at M and BC at N.



(a) Determine the exact area of the region bounded by the curve, BC, CO and OA.

(b) Find the area of trapezium $OABC$.

(c) Show that the area of $OMNC$ is equal to $CO \times PQ$, and hence find the area of the trapezium.

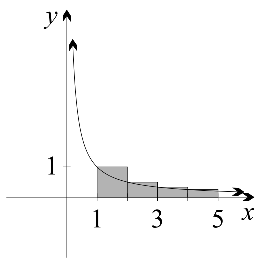
(d) Hence show that $\frac{1}{2}(3 + \sqrt{5}) < e < 3$.

5. Suppose that $m \le f(x) \le M$ in the interval $a \le x \le b$. Use a diagram to help prove that

$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a)\,.$$

DEVELOPMENT

6. The diagram shows upper rectangles for the graph of $y = \dfrac{1}{x}$.



(a) By considering appropriate areas, show that

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \geq \log(n+1)\,.$$

(b) What do you conclude about the infinite series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots ?$$

7. (a) Show, using calculus, that the graph of $y = \ln x$ is concave down throughout its domain.

(b) Sketch the graph of $y = \ln x$, and mark two points $A(a, \ln a)$ and $B(b, \ln b)$ on the curve, where $0 < a < b$.

(c) Find the coordinates of the point P that divides the interval AB in the ratio $2 : 1$.

(d) Using parts (b) and (c), deduce that $\frac{1}{3}\ln a + \frac{2}{3}\ln b < \ln(\frac{1}{3}a + \frac{2}{3}b)$.

8. Let $f(x) = x^n\, e^{-x}$, where $n > 1$.

(a) Show that $f'(x) = x^{n-1}e^{-x}(n - x)$.

(b) Show that $(n, n^n e^{-n})$ is a maximum turning point of the graph of $f(x)$, and hence sketch the graph for $x \geq 0$. (Don't attempt to find points of inflexion.)

(c) Explain why $x^n e^{-x} < n^n e^{-n}$ for $x > n$. Begin by considering the graph of $f(x)$ for $x > n$.

(d) Deduce from part (c) that $(1 + \frac{1}{n})^n < e$.

9. The function $f(x)$ is defined by $f(x) = x - \log_e(1 + x^2)$.

(a) Show that $f'(x)$ is never negative.

(b) Explain why the graph of $y = f(x)$ lies completely above the x-axis for $x > 0$.

(c) Hence prove that $e^x > 1 + x^2$, for all positive values of x.

10. Consider the function $y = e^x \left(1 - \dfrac{x}{10}\right)^{10}$.

(a) Find the two turning points of the graph of the function.

(b) Discuss the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.

(c) Sketch the graph of the function.

(d) From your graph, deduce that $e^x \leq \left(1 - \dfrac{x}{10}\right)^{-10}$, for $x < 10$.

(e) Hence show that $\left(\dfrac{11}{10}\right)^{10} \leq e \leq \left(\dfrac{10}{9}\right)^{10}$.

11. (a) Let a, b and c be the lengths of the sides of a triangle and let $\angle A$ be opposite side a. Use the cosine rule to help prove that $|b - c| \leq a \leq b + c$.
That is, prove that one side of a triangle is longer than the difference between the other two sides and shorter than the sum of the other two sides.

(b) Hence prove for any two complex numbers z and w that

$$\Big||z| - |w|\Big| \leq |z \pm w| \leq |z| + |w|\,.$$

(c) Under what circumstances is $\Big||z| - |w|\Big| = |z + w|$?

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12. In this question you may assume that simple exponential curves are concave up.

(a) Show by direct calculation that: (i) $6^6 < 3 \times 5^6$, (ii) $5 \times 6^6 < 2 \times 7^6$.

(b) The points $A\left(-\frac{1}{6}, 3^{-\frac{1}{6}}\right)$ and $B(0,1)$ lie on the exponential curve $y = 3^x$. The points B and $C\left(\frac{1}{6}, \left(\frac{5}{2}\right)^{\frac{1}{6}}\right)$ lie on the exponential curve $y = \left(\frac{5}{2}\right)^x$.

(i) Use part (a) to show that the gradient of chord AB is greater than 1 and the gradient of chord BC is less than 1.

(ii) Hence show that $\frac{5}{2} < e < 3$.

13. Let $|t| < 1$ and let N be a positive integer.

(a) Show that $1 + t^2 + t^4 + \ldots + t^{2N} < \dfrac{1}{1-t^2}$.

(b) Show that the difference between the two is $\dfrac{t^{2N+2}}{1-t^2}$.

(c) Integrate the result in part (a) between 0 and x, where $|x| < 1$. Hence show that:

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2N+1}}{2N+1} < \tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

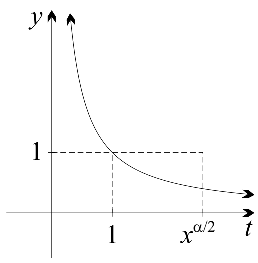
(d) Explain why $\displaystyle\int_0^x \frac{t^{2N+2}}{1-t^2}\,dt \le \int_0^x \frac{x^{2N+2}}{1-t^2}\,dt$.

(e) Use parts (b) to (d) to show that

$$\lim_{N\to\infty}\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2N+1}}{2N+1}\right) = \tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

(f) Hence find $\log 2$ correct to three decimal places.

14. The diagram shows the graph of $y = \dfrac{1}{t}$, for $t > 0$.



Let $x > 1$ and $\alpha > 0$.

(a) By considering upper and lower rectangles, show that

$$0 < \tfrac{1}{2}\alpha\log x < x^{\alpha/2}.$$

(b) Hence show that $\displaystyle\lim_{x\to\infty}\left(\frac{\log x}{x^\alpha}\right) = 0$, for all $\alpha > 0$.

15. (a) Let $n > 1$ and k be positive integers. Use lower rectangles to prove that

$$1 - \tfrac{1}{n} \le \int_{n^k}^{n^{k+1}} \frac{1}{x}\,dx.$$

(b) Hence prove that $\displaystyle\int_1^{n^k} \frac{1}{x}\,dx \to \infty$ as $k \to \infty$ regardless of the choice of n.

16. Consider the integral $\displaystyle\int_n^{n+x} \frac{1}{t}\,dt$.

(a) Use upper and lower rectangles to show that $\dfrac{x}{1+\frac{x}{n}} < n\log\left(1 + \frac{x}{n}\right) < x$.

(b) Hence show that $\displaystyle\lim_{n\to\infty}\left(1 + \tfrac{x}{n}\right)^n = e^x$ for any given value of x.

(c) Use trial and error to determine how big n needs to be so that $\left(1 + \frac{x}{n}\right)^n \doteqdot e^x$ correct to three decimal places when $x = 0{\cdot}1$.

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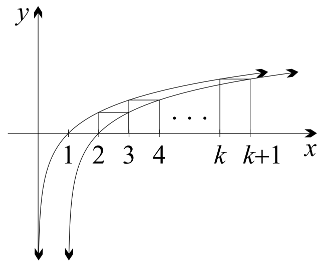
ENRICHMENT

17. The diagram shows the curves

$$y = \log x \qquad \text{and} \qquad y = \log(x-1)\,,$$



and $k-1$ rectangles constructed between $x = 2$ and $x = k+1$, where $k \ge 2$.

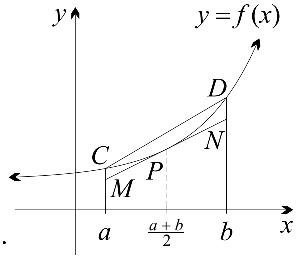
(a) Show that:

(i) $\displaystyle\int_2^{k+1} \log(x-1)\,dx = k\log k - k + 1$

(ii) $\displaystyle\int_2^{k+1} \log x\,dx = (k+1)\log(k+1) - \log 4 - k + 1$

(b) Deduce that $k^k < k!\,e^{k-1} < \frac{1}{4}(k+1)^{k+1}$, for all $k \ge 2$.

18. The diagram on the right shows the curve $y = f(x)$ in the interval $a \le x \le b$ where $f''(x) > 0$. The corresponding chord is CD and MN is tangent to $y = f(x)$ at P where $x = \frac{a+b}{2}$.



(a) Use areas to briefly explain why

$$(b-a)\,f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx < (b-a)\,\frac{f(a)+f(b)}{2}\,.$$

(b) Hence show that, for $n = 2, 3, 4, \ldots$,

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right).$$

(c) Deduce that

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

(d) Show that

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots.$$

(e) Hence show that $\displaystyle\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}$.

19. (a) Show that $\displaystyle\int_1^n \ln x\,dx = n\ln n - n + 1$.

(b) Use the trapezoidal rule on the intervals with endpoints $1, 2, 3, \ldots, n$ to show that

$$\int_1^n \ln x\,dx > \tfrac{1}{2}\ln n + \ln(n-1)!$$

(c) Hence show that $n! < n^{n+\frac{1}{2}}\,e^{1-n}$. NOTE: This is a preparatory lemma in the proof of Stirling's formula $n! \doteqdot \sqrt{2\pi}\,n^{n+\frac{1}{2}}\,e^{-n}$, which gives an approximation for $n!$ whose percentage error converges to 0 for large integers n.

20. (a) Prove that $\log_e x \le x - 1$ for $x > 0$.

(b) Suppose that $p_1,\, p_2,\, p_3,\, \ldots,\, p_n$ are positive real numbers whose sum is 1.

Prove that $\displaystyle\sum_{r=1}^{n} \log_e(np_r) \le 0$.

(c) Let $x_1,\, x_2,\, x_3,\, \ldots,\, x_n$ be positive real numbers.

Prove that $\dfrac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \ge (x_1x_2x_3\ldots x_n)^{\frac{1}{n}}$.

21. [A proof that e is irrational]

For positive integer values of n, let $S_n = 1 + \displaystyle\sum_{r=1}^{n} \frac{1}{r!}$.

(a) Prove by induction that $e - S_n = e\displaystyle\int_0^1 \frac{x^n}{n!}e^{-x}\,dx$ for all positive integer values of n.

(b) From (a) deduce that $0 < e - S_n < \dfrac{3}{(n+1)!}$ for all positive integer values of n.

(c) Use (b) to deduce that $(e - S_n)\,n!$ is NOT an integer for $n = 2,\, 3,\, 4,\, \ldots$

(d) Show that there cannot exist positive integers p and q such that $e = \dfrac{p}{q}$.

2G Chapter Review Exercise

Exercise 2G

1. Write down the converse of each statement, and state whether the converse is true or false.

(a) If a quadrilateral is cyclic, then its opposite angles are supplementary.

(b) If two numbers are both odd, then their sum is even.

(c) Every rhombus is a parallelogram.

2. Write down the negation of each statement.

(a) All mathematicians are intelligent.

(b) Suzie likes Physics and Chemistry.

(c) If I am on vacation, then I am not working.

3. Write down the contrapositive of each statement.

(a) If I am a bicycle, then I have two wheels.

(b) If a number is odd, then its last digit is not 6.

(c) A square has four equal sides.

4. Write down (in 'if... then' form) the two converse statements equivalent to each 'if and only if' statement.

(a) A number is even if and only if it is divisible by 2.

(b) A quadrilateral is a parallelogram if and only if its diagonals bisect each other.

(c) a is divisible by b if and only if $\exists\ c \in \mathbf{Z}$ such that $a = bc$.

5. Prove that:

(a) the sum of three consecutive integers is divisible by 3,

(b) the product of three consecutive even numbers is divisible by 8,

(c) the product of two consecutive even numbers is divisible by 8.

6. Prove that the remainder is 1 when an odd square number is divided by 4.

7. Prove that an integer is divisible by both 3 and 5 if it is divisible by 15.

8. If n is odd, use divisibility arguments to prove that $n^3 - n$ is divisible by 24.

9. If the integer n is *not* divisible by 3, prove that $n^2 + 2$ *is* divisible by 3.

10. (a) Show by expanding that if n is odd,

$$x^n + 1 = (x+1)(x^{n-1} - x^{n-2} + x^{n-3} - x^{n-4} + \cdots + x^2 - x + 1).$$

(b) Hence prove that if n is odd:

(i) $2^n + 1$ is divisible by 3,

(ii) $2^{mn} + 1$ is divisible by $2^m + 1$.

11. Prove by contradiction that:

(a) $\sqrt{7}$ is irrational,

(b) $\log_3 7$ is irrational.

12. By proving the contrapositive, prove that if a^2 is even then a is even.

13. (a) Prove that $x + y \ge 2\sqrt{xy}$ for $x, y > 0$.

(b) Hence prove that $(a+b)(1+ab) \ge 4ab$ for $a, b > 0$.

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14. Prove by induction that for all positive integer values of n:

(a) $\displaystyle\sum_{r=1}^{n}(r+1)\times 2^r = n\times 2^{n+1}$

(b) $\displaystyle\sum_{r=1}^{n} r^2(r+1) = \tfrac{1}{12}n(n+1)(n+2)(3n+1)$

15. Prove by induction that for all positive integer values of n:

(a) 6^n+4 is divisible by 5,

(b) n^3+2n is divisible by 3.

16. (a) Prove by induction that for all positive integer values of n:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\dots\left(1-\frac{1}{(n+1)^2}\right)=\frac{1}{2}\left(1+\frac{1}{n+1}\right)$$

(b) What is the value of $\displaystyle\lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\dots\left(1-\frac{1}{(n+1)^2}\right)$?

17. Prove by mathematical induction that:

(a) $n(n+2)$ is divisible by 4 for even values of n,

(b) 3^n+7^n is divisible by 10 for odd values of n,

(c) $4^n+5^n+6^n$ is divisible by 15 for odd values of n.

18. Use mathematical induction to prove that if $T_1=3$ and $T_n=T_{n-1}+4n$ for $n\ge 2$, then $T_n=2n^2+2n-1$ for $n\ge 1$.

19. For a certain sequence, $a_1=1$ and $a_{n+1}=\sqrt{2a_n+1}$ for $n\ge 1$.
Prove by induction that $a_n<3$ for $n\ge 1$.

20. Prove by induction that $n! > 3^n$ for $n\ge 7$.

21. Prove by induction that, for $n\ge 1$, the nth derivative of xe^{-x} is $(-1)^n(x-n)e^{-x}$.

22. Prove, for $n\ge 3$, that the exterior angle sum of a convex n-sided polygon is $360°$.

23. (a) Prove by induction that $2^n>n$, for all positive integers n.

(b) Hence show that $1<\sqrt[n]{n}<2$.

(c) Suppose that a and n are positive integers. It is known that if $\sqrt[n]{a}$ is rational, then it is an integer. What can we deduce about $\sqrt[n]{n}$, where n is a positive integer greater than 1?

24. (a) Use the fact that $\dfrac{x+y}{2}\ge\sqrt{xy}$, for $x,y>0$, to prove that $\dfrac{a+b+c+d}{4}\ge\sqrt[4]{abcd}$, for $a,b,c,d>0$.

(b) (i) Show that $\dfrac{a+b+c}{3}=\dfrac{1}{4}\left(a+b+c+\dfrac{a+b+c}{3}\right)$.

(ii) Hence prove that $\dfrac{a+b+c}{3}\ge\sqrt[3]{abc}$.

25. Suppose that abc is a 3-digit number, with $a-c>1$. If we subtract cba from abc and then add the result of this subtraction to its palindrome, prove that the answer is 1089.

Appendix: The Fundamental Theorem of Arithmetic

The fundamental theorem of arithmetic is not part of the Extension 2 course. Its proof is included here firstly because the theorem is so important in the study of number theory and secondly to justify its use in several proofs earlier in this chapter, most notably in proving that $\log_2 3$ is irrational. The proof of the theorem relies on several other results, which also need explanation and proof.

THEOREM: Suppose that two integers a and b have HCF $= d$.
Then $(a - bq)$ is divisible by d.

PROOF: Both a and b have factor d, so there exist integers m and n such that

$$a = md \qquad \text{and} \qquad b = nd\,.$$

Thus $a - bq = md - ndq$
$\qquad\qquad = d(m - nq)\,.$

Hence $(a - bq)$ is divisible by d.

Using the division algorithm, for integers a and b it is always possible to write

$$a = bq_1 + r_1 \qquad \text{where} \qquad 0 \le r_1 < b\,,$$

and where q_1 and r_1 are integers. It follows from the above proof that both b and r_1 are divisible by d, the HCF of a and b. Now repeat the division algorithm to get

$$b = r_1 q_2 + r_2 \qquad \text{where} \qquad 0 \le r_2 < r_1\,,$$

and where q_2 and r_2 are integers. Once again it follows that both r_1 and r_2 are divisible by d. Now continue the process to get a decreasing sequence of positive remainders $r_1, r_2, \ldots$ which must therefore terminate. Further, every term in this sequence is divisible by d. Consequently the last term in this sequence of remainders is d itself, the HCF of a and b. An example should help convince the reader of this.

Finding the HCF of 81 and 66:

$$81 = 66 \times 1 + 15 \qquad [1]$$
$$66 = 15 \times 4 + 6 \qquad [2]$$
$$15 = 6 \times 2 + 3 \qquad [3]$$
$$6 = 3 \times 2.$$

Thus the HCF of 81 and 66 is 3.

More significantly, when the division algorithm is written out like this, the result is a set of equations that can be used to write the HCF as a sum of multiples of the original numbers. It is simply a matter of working backwards through the equations. In this case:

$$\begin{aligned} 3 &= 15 - 6 \times 2 \quad \text{(from [3])} \\ &= 15 - (66 - 15 \times 4) \times 2 \quad \text{(from [2])} \\ &= 15 \times 9 - 66 \times 2 \\ &= (81 - 66) \times 9 - 66 \times 2 \quad \text{(from [1])} \\ &= 81 \times 9 - 66 \times 11. \end{aligned}$$

In other words, using these two algorithms, it is always possible to find the values of x and y such that $ax + by = d$. In this case, $x = 9$ and $y = -11$.

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It is appropriate to pause at this point to give a remarkable proof that $\sqrt{n}$ is either an integer or irrational whenever n is a positive integer.

THEOREM: If $n \in \mathbf{Z}$ then either $\sqrt{n} \in \mathbf{Z}$ or $\sqrt{n}$ is irrational.

PROOF: Suppose that $\sqrt{n}$ is rational. Then $\exists\, a, b \in \mathbf{Z}$ such that

$$\sqrt{n} = \frac{a}{b}$$

or $$b\sqrt{n} = a \qquad (**)$$

where $b \geq 1$ and the HCF of a and b is 1. Thus, from above, there exist integers x and y such that

$$ax + by = 1\,.$$

Multiply this equation by $\sqrt{n}$ to get

$$\sqrt{n} = (a\sqrt{n})x + (b\sqrt{n})y\,.$$

Now use the result of equation $(**)$ to get:

$$\sqrt{n} = (bn)x + (a)y\,.$$

The RHS of this last equation is a sum of products of integers, and must therefore be an integer. Hence if $\sqrt{n}$ is rational then it must be an integer. Otherwise $\sqrt{n}$ is irrational.

Here is another useful theorem that will be needed.

THEOREM: If the HCF of prime p and integer a is 1, and if integer ab is divisible by p, then b is divisible by p.

PROOF: Since the HCF of p and a is 1, $\exists\, x, y \in \mathbf{Z}$ such that

$$ax + py = 1$$

thus $$abx + pby = b\,.$$

But ab is divisible by p, so put $ab = mp$ to get

$$mpx + bpy = b$$

that is $$p(mx + by) = b\,,$$

by which b is divisible by p.

As a consequence of this proof, if an integer n is divisible by prime p then p divides at least one factor of n. The proof of this stronger result can be done by induction, and is left as an exercise. With these tools, it is now possible to prove the fundamental theorem of arithmetic for positive integers.

THEOREM: If a positive integer n can be written as the product of its primes in two ways, then one is a rearrangement of the other.

PROOF: Suppose, by way of contradiction there exists a positive integer n which can be written as the product of its prime factors in two different ways. Then let those products be

$$n = p_1 \times p_2 \times \ldots \times p_k = q_1 \times q_2 \times \ldots \times q_\ell\,,$$

where, for the sake of simplicity, it is assumed $k < \ell$. Then, by definition,

$$q_1 \times q_2 \times \ldots \times q_\ell \text{ is divisible by } p_1\,.$$

It follows from above that p_1 must divide one of these prime factors. Let this be q_j. But if one prime divides another then they must be equal. That is $p_1 = q_j$.

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Now re-order and re-label the primes so that $p_1 = q_1$. That is:

$$n = p_1 \times p_2 \times \ldots \times p_k = q_1 \times q_2 \times \ldots \times q_\ell \quad \text{with} \quad p_1 = q_1\,.$$

Cancel this prime p_1 so that

$$\frac{n}{p_1} = p_2 \times p_2 \times \ldots \times p_k = q_2 \times q_3 \times \ldots \times q_\ell\,.$$

Now repeat the above argument with $p_2, p_3, \ldots p_k$, (or write it out properly using induction) to get

$$\frac{n}{p_1 \times p_2 \times \ldots \times p_k} = 1 = q_{k+1} \times q_{k+2} \times \ldots \times q_\ell\,.$$

But a product of primes cannot equal 1. Hence $k = \ell$ and each prime p_i is equal to a corresponding prime q_i.

19 THE FUNDAMENTAL THEOREM OF ARITHMETIC: If a positive integer n can be written as two different products of primes

$$n = p_1 \times p_2 \times \ldots \times p_k = q_1 \times q_2 \times \ldots \times q_\ell\,,$$

then $k = \ell$ and the primes $p_1, p_2, \ldots, p_k$ are a re-arrangement of the primes $q_1, q_2, \ldots, q_\ell$.

Note that the above proof can be adapted to include negative integer values of n, and this is left as an exercise.

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3

Complex Numbers II: de Moivre and Euler

Chapter Overview: This chapter continues the study of complex numbers begun in Chapter 1. In that chapter it was shown that the Extension 1 work on polynomials could be broadened to include complex numbers. In this chapter, complex polynomial equations are investigated further by considering equations of the type $z^n - 1 = 0$, which have as their solutions the complex roots of unity.

The key to solving these equations is de Moivre's theorem, which is presented in Section A. One consequence of this theorem is that trigonometric identities can be quickly and easily developed, and some common identities are investigated in Section B. Section C deals with finding the complex roots of numbers, and deducing relationships between those roots. Complex polynomials are then used to develop Euler's famous result

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which is the focus of Section D. The chapter concludes with applications of this formula to trigonometry and roots of complex numbers.

3A Powers of Complex Numbers

Recall that when complex numbers are mulitplied the arguments are added, viz:

$$\arg(wz) = \arg(w) + \arg(z)\,.$$

Now put $w = z = \operatorname{cis}\theta$. That is, both are equal and have modulus 1. Then:

$$\begin{aligned} z^2 &= z \times z \\ &= \operatorname{cis}(\theta + \theta) \qquad \text{(adding arguments)} \\ &= \operatorname{cis} 2\theta\,. \end{aligned}$$

Next put $w = z^2$, so that

$$\begin{aligned} z^3 &= z^2 \times z \\ &= \operatorname{cis}(2\theta + \theta) \qquad \text{(again by adding arguments)} \\ &= \operatorname{cis} 3\theta\,. \end{aligned}$$

These initial calculations suggest the simple relationship

$$z^n = \operatorname{cis} n\theta\,,$$

whenever $|z| = 1$, at least for positive integers n. In fact the result is true for all integers, which is now proven.

de Moivre's Theorem: Let $z = \cos\theta + i\sin\theta$. It can be proven that

$$z^n = \cos n\theta + i\sin n\theta$$

for all integers n. The proof is in two parts, beginning with a proof by induction for $n \geq 0$. Conjugates are then used to extend the proof to negative integers.

PROOF: As always with proof by induction, first prove the result true for the starting value.

A. When $n = 0$

$$\begin{aligned} \text{LHS} &= z^0 \\ &= 1, \\ &\text{since } z \neq 0. \end{aligned} \qquad \begin{aligned} \text{RHS} &= \cos 0 + i\sin 0 \\ &= 1 + 0i \\ &= \text{LHS}. \end{aligned}$$

Hence the statement is true for $n = 0$.

B. Suppose that the result is true for some integer $k \geq 0$, that is

$$z^k = \cos k\theta + i\sin k\theta. \qquad (\dagger)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$z^{k+1} = \cos\big((k+1)\theta\big) + i\sin\big((k+1)\theta\big).$$

$$\begin{aligned} \text{LHS} &= z^k \times z \\ &= (\cos k\theta + i\sin k\theta) \times (\cos\theta + i\sin\theta) && \text{(by the hypothesis } (\dagger)) \\ &= \cos\big((k+1)\theta\big) + i\sin\big((k+1)\theta\big) && \text{(by the sum of arguments)} \\ &= \text{RHS}. \end{aligned}$$

Hence the result is true for $n = k + 1$.

C. It follows from parts A and B by mathematical induction that the statement is true for all integers $n \geq 0$.

D. Finally, extend the result to the negative integers.
To do this, consider the value of z^{-n} when n is a positive integer.

$$\begin{aligned} z^{-n} &= \left(z^{-1}\right)^n \\ &= (\,\overline{z}\,)^n && \text{(since } |z| = 1) \\ &= \big(\cos(-\theta) + i\sin(-\theta)\big)^n \\ &= \cos(-n\theta) + i\sin(-n\theta) && \text{(by part C, since } n \text{ is positive,)} \end{aligned}$$

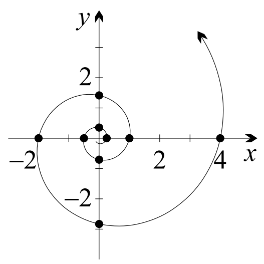
and the proof is complete.

1 DE MOIVRE'S THEOREM: Let $z = \cos\theta + i\sin\theta$ be a complex number with modulus 1. Then for all integers n,

$$z^n = \cos n\theta + i\sin n\theta.$$

One immediate consequence of the above theorem is that if $z = r\,\text{cis}\,\theta$ then $z^n = r^n\,\text{cis}\,n\theta$. Thus if $r > 1$ and $\theta > 0$ then as n increases so too does the modulus and argument of z^n. That is, the points representing z^n lie on an anticlockwise spiral.

By way of example, the table below shows the values of z^n in the case when $z = i\sqrt{2}$ for integer values of n between -4 and 4.



n	-4	-3	-2	-1	0	1	2	3	4
z^n	$\frac{1}{4}$	$i\frac{1}{2\sqrt{2}}$	$-\frac{1}{2}$	$-i\frac{1}{\sqrt{2}}$	1	$i\sqrt{2}$	-2	$-i2\sqrt{2}$	4

The corresponding points are shown in the graph above and are joined with a smooth curve. Notice that the spiral does not cut the axes at right angles.

A more practical application is to quickly simplify integer powers of complex numbers, as in the following example.

WORKED EXAMPLE 1: (a) Write $z = -\sqrt{3} + i$ in modulus-argument form.
(b) Hence express z^7 in factored real-imaginary form.

SOLUTION: (a) It should be clear that $z = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$.
(b) Using de Moivre's theorem,

$$\begin{aligned} z^7 &= 2^7(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6})^7 \\ &= 128(\cos\tfrac{35\pi}{6} + i\sin\tfrac{35\pi}{6}) \\ &= 128(\cos\tfrac{-\pi}{6} + i\sin\tfrac{-\pi}{6}) \\ &= 64(\sqrt{3} - i)\,. \end{aligned}$$

WORKED EXAMPLE 2: For which values of k is $(1+i)^k$ imaginary?

SOLUTION: Now $(1+i) = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$
so $(1+i)^k = \sqrt{2}^k(\cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4})$ (by de Moivre)
which is imaginary when $\frac{k\pi}{4}$ is an odd multiple of $\frac{\pi}{2}$.
Thus $\frac{k\pi}{4} = \frac{(2n+1)\pi}{2}$ where n is an integer,
that is $k = 4n + 2\,,$
hence $k = \ldots, -6, -2, 2, 6, 10, \ldots$

Exercise 3A

1. Write each expression in the form $\operatorname{cis} n\theta$:

(a) $(\cos\theta + i\sin\theta)^5$
(b) $(\cos\theta + i\sin\theta)^{-3}$
(c) $(\cos 2\theta + i\sin 2\theta)^4$
(d) $\cos\theta - i\sin\theta$
(e) $(\cos\theta - i\sin\theta)^{-7}$
(f) $(\cos 3\theta - i\sin 3\theta)^2$

2. Simplify as fully as possible:

(a) $\dfrac{(\cos\theta + i\sin\theta)^6(\cos\theta + i\sin\theta)^{-3}}{(\cos\theta - i\sin\theta)^4}$
(b) $\dfrac{(\cos 3\theta + i\sin 3\theta)^5(\cos 2\theta - i\sin 2\theta)^{-4}}{(\cos 4\theta - i\sin 4\theta)^{-7}}$

3. Write each expression in the form $a + ib$, where a and b are real:

(a) $(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^4$
(b) $(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^3$
(c) $(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^5$
(d) $(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})^{-2}$
(e) $(\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8})^{-6}$
(f) $(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12})^4$

4. (a) Write $1 + i$ in the form $r(\cos\theta + i\sin\theta)$.
(b) Hence, or otherwise, find $(1+i)^{17}$ in the form $a + ib$, where a and b are integers.

5. Let $z = 1 + i\sqrt{3}$.
(a) Express z in mod-arg form.
(b) Express z^{11} in the form $a + ib$, where a and b are real.

6. Let $z = -\sqrt{3} + i$.
(a) Find the values of $|z|$ and $\arg z$.
(b) Hence, or otherwise, show that $z^7 + 64z = 0$.

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7. (a) Express $\sqrt{3} - i$ in mod-arg form.
(b) Express $(\sqrt{3} - i)^7$ in mod-arg form.
(c) Hence express $(\sqrt{3} - i)^7$ in the form $x + iy$, where x and y are real.

8. (a) Express $-1 - i\sqrt{3}$ in mod-arg form.
(b) Express $(-1 - i\sqrt{3})^5$ in mod-arg form.
(c) Hence express $(-1 - i\sqrt{3})^5$ in the form $x + iy$, where x and y are real.

9. (a) Express $z = \sqrt{2} - i\sqrt{2}$ in mod-arg form.
(b) Hence write z^{22} in the form $a + ib$, where a and b are real.

DEVELOPMENT

10. Show that:
(a) $(1 + i)^{10}$ is purely imaginary
(b) $(1 - i\sqrt{3})^9$ is real
(c) $-1 + i$ is a fourth root of -4
(d) $-\sqrt{3} - i$ is a sixth root of -64

11. If k is a multiple of 4, prove that $(-1 + i)^k$ is real.

12. (a) Find the minimum value of the positive integer m for which $(\sqrt{3} + i)^m$ is:
(i) real, (ii) purely imaginary.
(b) Evaluate $(\sqrt{3} + i)^m$ for each of the above values of m.

13. (a) Prove that $(1 + i)^n + (1 - i)^n$ is real for all positive integer values of n.
(b) Determine the values of n for which $(1 + i)^n + (1 - i)^n = 0$.

14. Use de Moivre's theorem to prove that
$$(-\sqrt{3} + i)^n - (-\sqrt{3} - i)^n = 2^{n+1} \sin \tfrac{5\pi n}{6}\, i\,.$$

15. (a) Show that if n is divisible by 3 then $(1 + \sqrt{3}\,i)^{2n} + (1 - \sqrt{3}\,i)^{2n} = 2^{2n+1}$.
(b) Simplify the expression if n is not divisible by 3.

16. Show that $\left(\dfrac{1 + \cos 2\theta + i \sin 2\theta}{1 + \cos 2\theta - i \sin 2\theta}\right)^n = \operatorname{cis} 2n\theta$.

17. Prove that $(1 + \cos\alpha + i \sin\alpha)^k + (1 + \cos\alpha - i \sin\alpha)^k = 2^{k+1} \cos \tfrac{1}{2}k\alpha \cos^k \tfrac{1}{2}\alpha$.

ENRICHMENT

18. Let $z = \operatorname{cis} \frac{\pi}{n}$, where n is a positive integer. Show that:
(a) $1 + z + z^2 + \cdots + z^{2n-1} = 0$
(b) $1 + z + z^2 + \cdots + z^{n-1} = 1 + i \cot \frac{\pi}{2n}$

3B Trigonometric Identities

A useful application of De Moivre's theorem involves combining it with binomial expansions to obtain various trigonometric identities.

WORKED EXAMPLE 3:

(a) Express $\cos 3\theta$ in terms of powers of $\cos\theta$.
(b) Hence show that $x = \cos \frac{\pi}{9}$ is a solution of $8x^3 - 6x - 1 = 0$.
(c) Find the value of $\cos \frac{\pi}{9} \cos \frac{5\pi}{9} \cos \frac{7\pi}{9}$.

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SOLUTION:

(a) Let $z = \cos\theta + i\sin\theta$, then

$$z^3 = (\cos\theta + i\sin\theta)^3$$

so by de Moivre's theorem

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\,.$$

Take the real part to get

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)\\ &= 4\cos^3\theta - 3\cos\theta\,.\end{aligned}$$

(b) Let $x = \cos\theta$, so that

$$4x^3 - 3x = \cos 3\theta\,.$$

Thus $\quad 4x^3 - 3x = \frac{1}{2} \quad$ when $\theta = \frac{\pi}{9}$.

Hence $\quad 8x^3 - 6x - 1 = 0 \quad$ has solution $x = \cos\frac{\pi}{9}$.

(c) The roots of the cubic in (b) are $x = \cos\theta$ where $\cos 3\theta = \frac{1}{2}$. The solutions of $\cos 3\theta = \frac{1}{2}$ which give distinct values of $\cos\theta$ are $\theta = \frac{\pi}{9}$, $\frac{5\pi}{9}$ and $\frac{7\pi}{9}$. Hence, by the the product of the roots,

$$\cos\tfrac{\pi}{9}\cos\tfrac{5\pi}{9}\cos\tfrac{7\pi}{9} = \tfrac{1}{8}\,.$$

WORKED EXAMPLE 4:

(a) Let $z = \cos\theta + i\sin\theta$. Show that $z^n - z^{-n} = 2i\sin n\theta$.

(b) Expand $(z - z^{-1})^5$.

(c) Use parts (a) and (b) to show that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(d) Hence find $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^5\theta\,d\theta$.

SOLUTION:

(a)
$$\begin{aligned}z^n - z^{-n} &= z^n - \overline{z^n} && \text{(since } |z| = 1)\\ &= 2i\,\mathrm{Im}(z^n)\\ &= 2i\,\mathrm{Im}(\cos n\theta + i\sin n\theta) && \text{(by de Moivre)}\\ &= 2i\sin n\theta\,.\end{aligned}$$

(b) $(z - z^{-1})^5 = z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$.

(c) Rearranging part (b),

$$(z - z^{-1})^5 = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$$

so $\quad (2i\sin\theta)^5 = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \qquad$ (by part (a))

thus $\quad 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

(d) Dividing by 16 and integrating yields

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^5\theta\,d\theta &= \frac{1}{16}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 5\theta - 5\sin 3\theta + 10\sin\theta\,d\theta\\ &= \frac{1}{16}\left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\\ &= 0 - \frac{1}{16}\left(\frac{1}{5\sqrt{2}} - \frac{5}{3\sqrt{2}} - \frac{10}{\sqrt{2}}\right)\\ &= \frac{43\sqrt{2}}{120}\,.\end{aligned}$$

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Exercise **3B**

1. (a) Use the identity $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$ to show that:

(i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ (ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(b) Show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

2. Use similar methods to the previous question to show that:

(a) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(b) $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

(c) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

3. Let $z = \cos\theta + i\sin\theta$.

(a) Use de Moivre's theorem to show that $z^n + z^{-n} = 2\cos n\theta$.

(b) Show that $(z + z^{-1})^4 = (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$.

(c) Hence show that $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$.

4. Repeat the methods of the previous question to show that:

$$\sin^4\theta = \tfrac{1}{8}\cos 4\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}.$$

(Start by showing that $z^n - z^{-n} = 2i\sin n\theta$.)

DEVELOPMENT

5. (a) Use the methods of question 3 to show that

$$\cos^5\theta = \tfrac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta).$$

(b) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta$.

6. (a) Use de Moivre's theorem to prove the identity

$$\cos 6\alpha = 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1.$$

(b) Hence show that the polynomial equation $32x^6 - 48x^4 + 18x^2 - 1 = 0$ has roots $x = \cos\frac{n\pi}{12}$, for $n = 1, 3, 5, 7, 9, 11$.

(c) Use the product of these six roots to deduce that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4}$.

7. The identity $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ was derived in Question 2.

(a) Use this identity to show that the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ has distinct roots $x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$.

(b) Hence show that $\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$.

8. (a) Use de Moivre's theorem to show that

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

(b) Hence show that the equation $16x^5 - 20x^3 + 5x - 1 = 0$ has roots $x = 1$, $\sin\frac{\pi}{10}$, $\sin\frac{9\pi}{10}$, $\sin\frac{13\pi}{10}$, $\sin\frac{17\pi}{10}$.

(c) By equating coefficients, or otherwise, find the values of b and c for which $16x^4 + 16x^3 - 4x^2 - 4x + 1 = (4x^2 + bx + c)^2$, and hence explain why the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ has two double roots.

(d) Use part (b) to show that the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ has roots $x = \sin\frac{\pi}{10}$, $\sin\frac{9\pi}{10}$, $\sin\frac{13\pi}{10}$, $\sin\frac{17\pi}{10}$. Does this contradict part (c) which asserts that the equation has two double roots?

(e) Hence find exact values for $\sin\frac{\pi}{10}$ and $\sin\frac{3\pi}{10}$.

9. (a) Express $\cos 7\theta$ in the form $A\cos^7\theta + B\cos^5\theta + C\cos^3\theta + D\cos\theta$.

(b) Use the substitution $x = 4\cos^2\theta$ to show that the equation $x^3 - 7x^2 + 14x - 7 = 0$ has roots $x = 4\cos^2\frac{\pi}{14}, 4\cos^2\frac{3\pi}{14}, 4\cos^2\frac{5\pi}{14}$.

(c) Hence evaluate:

(i) $\cos^2\frac{\pi}{14} + \cos^2\frac{3\pi}{14} + \cos^2\frac{5\pi}{14}$ (ii) $\cos^4\frac{\pi}{14} + \cos^4\frac{3\pi}{14} + \cos^4\frac{5\pi}{14}$

10. (a) Show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

(b) Hence solve the equation $16\sin^5\theta = \sin 5\theta$ for $0 \le \theta < 2\pi$.

11. (a) Use de Moivre's theorem to show that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.

(b) Hence show that the equation $x^4 - 10x^2 + 5 = 0$ has roots $x = \pm\tan\frac{\pi}{5}, \pm\tan\frac{2\pi}{5}$.

(c) Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$ and that $\tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5} = 10$.

12. Let $z = \cos\theta + i\sin\theta$.

(a) Show that $2\cos n\theta = z^n + \dfrac{1}{z^n}$ and that $2i\sin n\theta = z^n - \dfrac{1}{z^n}$.

(b) Hence show that:

$$128\cos^3\theta\sin^4\theta = \left(z^7 + \frac{1}{z^7}\right) - \left(z^5 + \frac{1}{z^5}\right) - 3\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

(c) Deduce that $\cos^3\theta\sin^4\theta = \frac{1}{64}(\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta)$.

13. Consider the polynomial equation $5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$, which has four complex roots with modulus one.
Let $z = \operatorname{cis}\theta$.

(a) Show that $5\cos 2\theta - 11\cos\theta + 8 = 0$.

(b) Hence determine the four roots of the equation in the form $a + ib$, where a and b are real.

14. (a) Use de Moivre's theorem to express $\dfrac{\sin 8\theta}{\sin\theta\cos\theta}$ as a polynomial in s, where $s = \sin\theta$.

(b) Hence solve the equation $x^6 - 6x^4 + 10x^2 - 4 = 0$, leaving the roots in trigonometric form.

ENRICHMENT

15. Let n be a positive integer.

(a) Use de Moivre's theorem to show that:

$$\sin(2n+1)\theta = {}^{2n+1}C_1\cos^{2n}\theta\sin\theta - {}^{2n+1}C_3\cos^{2n-2}\theta\sin^3\theta + \cdots + (-1)^n\sin^{2n+1}\theta$$

(b) Hence show that the polynomial $P(x) = {}^{2n+1}C_1\,x^n - {}^{2n+1}C_3\,x^{n-1} + \cdots + (-1)^n$ has roots of the form $\cot^2\left(\dfrac{k\pi}{2n+1}\right)$ where $k = 1, 2, 3, \ldots, n$.

(c) Deduce that $\cot^2\left(\dfrac{\pi}{2n+1}\right) + \cot^2\left(\dfrac{2\pi}{2n+1}\right) + \cdots + \cot^2\left(\dfrac{n\pi}{2n+1}\right) = \dfrac{n(2n-1)}{3}$.

(d) Use the fact that $\cot\theta < \frac{1}{\theta}$ for $0 < \theta < \frac{\pi}{2}$ to show that:

$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right)\frac{(2n+1)^2}{2n(2n-1)} > \frac{\pi^2}{6}$$

3C Roots of Complex Numbers

Recall from a previous worked exercise that the points in the Argand diagram which represent z^n, where n is an integer, lie on a spiral whenever $|z| \neq 1$. When $|z| = 1$, it should be clear that the points lie on the unit circle. Further, if $z = \cos\theta + i\sin\theta$ then the angle at the origin subtended by successive points is

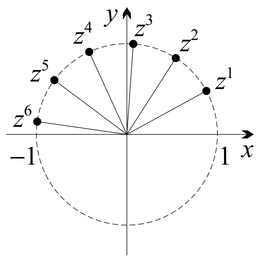
$$\begin{aligned}\arg(z^n) - \arg(z^{n-1}) &= \arg\left(\frac{z^n}{z^{n-1}}\right)\\ &= \arg(z)\\ &= \theta\,.\end{aligned}$$

That is, the angle is constant. Thus successive points are regularly spaced about the unit circle.

For example, the sketch on the right shows the points z^n for $n = 1, 2, 3, 4, 5, 6$, where $z = \cos\frac{1}{2} + i\sin\frac{1}{2}$. Note that

$$\arg(z) = \tfrac{1}{2} \doteqdot 28°39'\,,$$

which is the angle subtended at the origin by any pair of successive points. It should be clear that $2\pi \div \frac{1}{2} = 4\pi$ is irrational, and hence none of the points coincide, even for larger values of n. In that sense, this is not a very interesting example.



Roots of Unity: Significant configurations of points arise when equations of the form

$$z^n = 1$$

are solved. There are always n solutions and the points are equally spaced about the unit circle in the complex plane. These numbers are called the *roots of unity*, for obvious reasons.

2

> ROOTS OF UNITY: These are the real and complex solutions of the equation
>
> $$z^n = 1$$
>
> where n is an integer. When the solutions are plotted in the Argand diagram, the n points are equally spaced around the unit circle.

The situation is best demonstrated by example.

WORKED EXAMPLE 5:

(a) Solve $z^6 = 1$.

(b) Plot the solutions on the unit circle in the complex plane.

 (i) What is the angle subtended at the origin by successive roots?

 (ii) What regular polygon has these points as vertices?

(c) Let $\alpha = \text{cis}(-\frac{\pi}{3})$. Show that the list 1, α, α^2, α^3, α^4 and α^5 includes all six roots of $z^6 = 1$.

(d) Let $\beta = \text{cis}\,\frac{2\pi}{3}$. Which roots of $z^6 = 1$ can be written in the form β^m, where m is an integer? What polygon has these points as vertices?

SOLUTION:

(a) Let $z = \operatorname{cis}\theta$ and note that $1 = \operatorname{cis} 2k\pi$, where k is an integer. Thus

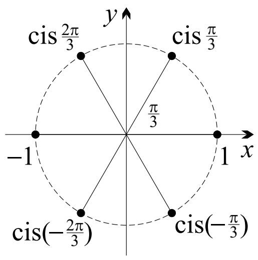
$$\begin{aligned} \operatorname{cis} 6\theta &= \operatorname{cis} 2k\pi && \text{(by de Moivre)} \\ \text{so} \quad 6\theta &= 2k\pi \\ \text{hence} \quad \theta &= \tfrac{k\pi}{3}. \end{aligned}$$

Apply the restriction $-\pi < \theta \le \pi$ to obtain all the distinct solutions. Thus

$$\begin{aligned} -\pi < \tfrac{k\pi}{3} &\le \pi \\ \text{so} \quad -3 < k &\le 3. \end{aligned}$$

Hence the six roots of $z^6 = 1$ are

$\operatorname{cis}(-\frac{2\pi}{3})$, $\operatorname{cis}(-\frac{\pi}{3})$, 1, $\operatorname{cis}\frac{\pi}{3}$, $\operatorname{cis}\frac{2\pi}{3}$ and -1.



(b) The graph on the right shows these six roots.

(i) Clearly the angle at the centre is $\frac{\pi}{3}$.

(ii) These are the vertices of a regular hexagon.

(c) Using de Moivre's theorem, the given list is:

$$1, \operatorname{cis}(-\tfrac{\pi}{3}), \operatorname{cis}(-\tfrac{2\pi}{3}), \operatorname{cis}(-\tfrac{3\pi}{3}) = -1,$$
$$\operatorname{cis}(-\tfrac{4\pi}{3}) = \operatorname{cis}\tfrac{2\pi}{3} \text{ and } \operatorname{cis}(-\tfrac{5\pi}{3}) = \operatorname{cis}\tfrac{\pi}{3}.$$

This is the same list as given in the answer to part (a), but simply in a different order.

(d) Now $\beta^m = \operatorname{cis}\frac{2m\pi}{3}$ by de Moivre's theorem. Hence $\arg(\beta^m)$ is a multiple of $\frac{2\pi}{3}$. Thus the only possible values that β^m may take are:

$$\operatorname{cis}(-\tfrac{2\pi}{3}), 1 \text{ and } \operatorname{cis}\tfrac{2\pi}{3}.$$

That is, only these three roots can be written as a power of β. The points in the Argand diagram form the vertices of an equilateral triangle.

There are several important features to observe in the worked exercise above. First, there are six solutions, as assured by the fundamental theorem of algebra. Secondly, the principal argument of 1 is deliberately not used in the working. The reasoning is that if the sixth power of a complex number is equal to one then 6θ, the argument of the sixth power, could exceed the limits of the principal argument before being reduced. This is borne out by the root $z = \operatorname{cis}(-\frac{2\pi}{3})$ for which $z^6 = \operatorname{cis}(-2\pi) = \operatorname{cis} 0$. Nevertheless, the values of the roots in the solution are written with their principal arguments.

3 FINDING THE ROOTS OF UNITY: To solve $z^n = 1$, begin by letting $z = \cos\theta + i\sin\theta$ and by noting that $1 = \cos 2k\pi + i\sin 2k\pi$. Thus, by de Moivre's theorem:

$$\cos n\theta + i\sin n\theta = \cos 2k\pi + i\sin 2k\pi$$

so that $n\theta = 2k\pi$. Then apply $-\pi < \theta \le \pi$ to find the restrictions on k.

Further, any power of a root will always coincide with one or other of the roots. In the case of α^m in the Worked Example, each of the other roots was visited one by one as m increased. In the case of β^m, only some of the other roots were obtained.

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Roots of Complex Numbers: The technique for finding roots of unity can be adapted to find the roots of any complex number. The trick is to allow both the modulus and the argument to be unknown. It is then a matter of equating moduli and arguments. Once again, the roots are equally spaced around a circle.

Worked Example 6: Find the cube roots of $8i$.

Solution: That is, solve the equation $z^3 = 8i$.

Begin by letting $z = r\,\mathrm{cis}\,\theta$ and putting $8i = 8\,\mathrm{cis}(\frac{\pi}{2} + 2k\pi)$, then:

$$r^3\,\mathrm{cis}\,3\theta = 8\,\mathrm{cis}(\tfrac{\pi}{2} + 2k\pi) \qquad \text{(by de Moivre.)}$$

Equate the moduli to get $r = 2$, and from the arguments

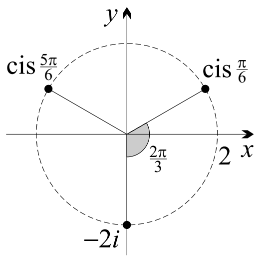
$$\theta = \tfrac{(4k+1)\pi}{6}\,.$$

For the principal argument in the solution

$$-\pi < \tfrac{(4k+1)\pi}{6} \le \pi$$

so $\quad -6 < 4k+1 \le 6$

thus $\quad k = -1, 0, 1$



and the three roots are: $2\,\mathrm{cis}(-\frac{\pi}{2})$, $2\,\mathrm{cis}\,\frac{\pi}{6}$ and $2\,\mathrm{cis}\,\frac{5\pi}{6}$.

That is $-2i$, $\sqrt{3} + i$ and $-\sqrt{3} + i$, which lie on the circle with radius 2.

The techniques can now be used to find the exact values of the trigonometric functions at certain rational multiples of π, as in the following worked exercise.

Worked Example 7: Consider the equation $z^5 + 1 = 0$.

(a) Find the roots of this equation and show them on the Argand diagram.

(b) Factorise $z^5 + 1$:

  (i) as a product of linear factors,

  (ii) as a product of linear and quadratic factors with real coefficients.

(c) Evauate $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$.

(d) Let α be a complex root of $z^5 + 1 = 0$, that is $\alpha \ne -1$.

  (i) Show that $1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 = 0$.

  (ii) Find a quadratic equation with roots $(\alpha^4 - \alpha)$ and $(\alpha^2 - \alpha^3)$.

(e) Put $\alpha = \mathrm{cis}\,\frac{\pi}{5}$ in part (d), and hence evaluate $\cos\frac{\pi}{5}$.

Solution:

(a) Let $z = \mathrm{cis}\,\theta$ and note that $-1 = \mathrm{cis}(2k+1)\pi$, where k is an integer. Thus

$$\mathrm{cis}\,5\theta = \mathrm{cis}(2k+1)\pi \qquad \text{(by de Moivre)}$$

so $\quad 5\theta = (2k+1)\pi$

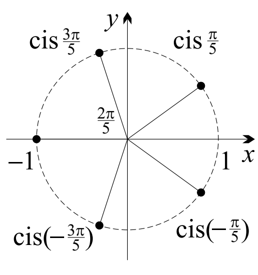
hence $\quad \theta = \frac{(2k+1)\pi}{5}$.

Apply the restriction $-\pi < \theta \le \pi$ to obtain all the distinct solutions. Thus

$$-\pi < \tfrac{(2k+1)\pi}{5} \le \pi$$

so $\quad -5 < (2k+1) \le 5$

or $\quad -3 < k \le 2$.



Hence the five roots are:

$$\mathrm{cis}(-\tfrac{3\pi}{5}),\ \mathrm{cis}(-\tfrac{\pi}{5}),\ \mathrm{cis}\,\tfrac{\pi}{5},\ \mathrm{cis}\,\tfrac{3\pi}{5} \text{ and } -1,$$

or in conjugate pairs,

$$\mathrm{cis}\,\tfrac{\pi}{5},\ \overline{\mathrm{cis}\,\tfrac{\pi}{5}},\ \mathrm{cis}\,\tfrac{3\pi}{5},\ \overline{\mathrm{cis}\,\tfrac{3\pi}{5}} \text{ and } -1\,.$$

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(b) Using the roots of the given equation,

$$\begin{aligned} z^5+1 &= (z+1)(z-\operatorname{cis}\tfrac{\pi}{5})(z-\overline{\operatorname{cis}\tfrac{\pi}{5}})(z-\operatorname{cis}\tfrac{3\pi}{5})(z-\overline{\operatorname{cis}\tfrac{3\pi}{5}}) \\ &= (z+1)(z^2-2z\cos\tfrac{\pi}{5}+1)(z^2-2z\cos\tfrac{3\pi}{5}+1)\,. \end{aligned}$$

(c) By the sum of the roots

$$\operatorname{cis}\tfrac{\pi}{5}+\overline{\operatorname{cis}\tfrac{\pi}{5}}+\operatorname{cis}\tfrac{3\pi}{5}+\overline{\operatorname{cis}\tfrac{3\pi}{5}}-1=0$$

hence $2\cos\frac{\pi}{5}+2\cos\frac{3\pi}{5}=1\,,$

that is $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}\,.$

(d) (i) Since α is a complex root,

$$\alpha^5+1=0$$

so $(\alpha+1)(1-\alpha+\alpha^2-\alpha^3+\alpha^4)=0$ (from GP theory)

thus $1-\alpha+\alpha^2-\alpha^3+\alpha^4=0$ (since $\alpha\neq -1$)

(ii) The sum of the roots is $-\alpha+\alpha^2-\alpha^3+\alpha^4=-1$ from part (i).
The product of the roots is

$$\begin{aligned} (\alpha^4-\alpha)(\alpha^2-\alpha^3) &= \alpha^6-\alpha^7-\alpha^3+\alpha^4 \\ &= -\alpha+\alpha^2-\alpha^3+\alpha^4 \qquad \text{(since } \alpha^5=-1\text{)} \\ &= -1\,. \end{aligned}$$

Hence the required quadratic is $z^2+z-1=0\,.$

(e) With $\alpha=\operatorname{cis}\frac{\pi}{5}$ the roots of the equation in part (d) are

$$\begin{aligned} \alpha^4-\alpha &= -\alpha^{-1}-\alpha \qquad \text{(since } \alpha^5=-1\text{)} \\ &= -(\overline{\alpha}+\alpha) \qquad \text{(since } |\alpha|=1\text{)} \\ &= -2\cos\tfrac{\pi}{5}\,, \end{aligned}$$

and

$$\begin{aligned} \alpha^2-\alpha^3 &= \alpha^2+\alpha^{-2} \qquad \text{(since } \alpha^5=-1\text{)} \\ &= \alpha^2+\overline{\alpha^2} \qquad \text{(since } |\alpha|=1\text{)} \\ &= 2\cos\tfrac{2\pi}{5}\,. \end{aligned}$$

Also, by direct calculation

$$z=\tfrac{-1-\sqrt{5}}{2} \text{ or } \tfrac{-1+\sqrt{5}}{2}\,,$$

hence, matching the positive and negative roots,

$$\cos\tfrac{\pi}{5}=\tfrac{1+\sqrt{5}}{4} \text{ and } \cos\tfrac{2\pi}{5}=\tfrac{-1+\sqrt{5}}{4}\,.$$

Exercise **3C**

1. (a) Find the three cube roots of unity, expressing the complex roots in both $r\operatorname{cis}\theta$ and $x+iy$ form. Use the restriction $-\pi<\theta\leq\pi$.

(b) Show that the points in the complex plane representing these three roots form an equilateral triangle.

(c) If ω is one of the complex roots, show that the other complex root is ω^2.

(d) Write down the values of:

(i) ω^3 (ii) $1+\omega+\omega^2$

(e) Show that:

(i) $(1+\omega^2)^3=-1$

(ii) $(1-\omega-\omega^2)(1-\omega+\omega^2)(1+\omega-\omega^2)=8$

(iii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9$

2. (a) Solve the equation $z^6 = 1$, expressing the complex roots in the form $a + ib$, where a and b are real.
(b) Plot these roots on an Argand diagram, and show that they form a regular hexagon.
(c) If α is the complex root with smallest positive principal argument, show that the other three complex roots are α^2, α^{-1} and α^{-2}.
(d) Show that $z^6 - 1 = (z^2 - 1)(z^4 + z^2 + 1)$.
(e) Hence write $z^4 + z^2 + 1$ as a product of quadratic factors with real coefficients.

3. (a) Find, in the form $a + ib$, the four fourth roots of -1.
(b) Hence write $z^4 + 1$ as a product of two quadratic factors with real coefficients.

4. (a) Find, in the form $a + ib$, the six roots of the equation $z^6 + 1 = 0$.
(b) Hence show that $z^6 + 1 = (z^2 + 1)(z^2 - \sqrt{3}\,z + 1)(z^2 + \sqrt{3}\,z + 1)$.
(c) Divide both sides of this identity by z^3, and then let $z = \operatorname{cis}\theta$ to show that:

$$\cos 3\theta = 4\cos\theta(\cos\theta - \cos\tfrac{\pi}{6})(\cos\theta - \cos\tfrac{5\pi}{6})$$

5. (a) Find, in mod-arg form, the five fifth roots of i.
(b) Find, in mod-arg form, the four fourth roots of $-i$.
(c) Find, in the form $a + ib$, the four fourth roots of $-8 - 8\sqrt{3}\,i$.
(d) Find, in mod-arg form, the five fifth roots of $16\sqrt{2} - 16\sqrt{2}\,i$.

DEVELOPMENT

6. (a) Find the five fifth roots of -1, writing the complex roots in mod-arg form.
(b) If α is the complex root with least positive principal argument, show that α^3, α^7 and α^9 are the other three complex roots.
(c) Show that $\alpha^7 = -\alpha^2$ and that $\alpha^9 = -\alpha^4$.
(d) Use the sum of the roots to show that $\alpha + \alpha^3 = 1 + \alpha^2 + \alpha^4$.

7. (a) Find the seven seventh roots of unity.
(b) By considering the sum of the roots, show that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$.
(c) Write $z^7 - 1$ as a product of one linear and three quadratic factors, all with real coefficients.
(d) If α is the complex seventh root of unity with the least positive principal argument, show that α^2, α^3, α^4, α^5 and α^6 are the other five complex roots.
(e) A certain cubic equation has roots $\alpha + \alpha^6$, $\alpha^2 + \alpha^5$ and $\alpha^3 + \alpha^4$. Use the relationships between the roots and coefficients to show that the equation is $x^3 + x^2 - 2x - 1 = 0$.

8. (a) (i) Find the five fifth roots of unity, writing the complex roots in mod-arg form.
(ii) Show that the points in the complex plane representing these roots form a regular pentagon.
(iii) By considering the sum of these five roots, show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.
(b) (i) Show that $z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$.
(ii) Hence show that $z^4 + z^3 + z^2 + z + 1 = (z^2 - 2\cos\frac{2\pi}{5}\,z + 1)(z^2 - 2\cos\frac{4\pi}{5}\,z + 1)$.
(iii) By equating the coefficients of z in this identity, show that $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$.
(c) (i) Use the substitution $x = u + \dfrac{1}{u}$ to show that the equation $x^2 + x - 1 = 0$ has roots $2\cos\frac{2\pi}{5}$ and $2\cos\frac{4\pi}{5}$.
(ii) Deduce that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$.

9. (a) Find the ninth roots of unity.

(b) Hence show that:

$$z^6 + z^3 + 1 = (z^2 - 2\cos\tfrac{2\pi}{9}\,z + 1)(z^2 - 2\cos\tfrac{4\pi}{9}\,z + 1)(z^2 - 2\cos\tfrac{8\pi}{9}\,z + 1)$$

(c) Deduce that:

$$2\cos 3\theta + 1 = 8(\cos\theta - \cos\tfrac{2\pi}{9})(\cos\theta - \cos\tfrac{4\pi}{9})(\cos\theta - \cos\tfrac{8\pi}{9})$$

10. Let $\omega = \operatorname{cis}\frac{2\pi}{9}$.

(a) Show that ω^k, where k is an integer, is a solution of the equation $z^9 = 1$.

(b) Show that $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1$.

(c) Hence show that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$.

(d) Deduce that $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$.

11. Let $\rho = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. The complex number $\alpha = \rho + \rho^2 + \rho^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where a and b are real.

(a) Prove that $1 + \rho + \rho^2 + \ldots + \rho^6 = 0$.

(b) The second root of the quadratic equation is β. Express β in terms of positive powers of ρ. Justify your answer.

(c) Find the values of the coefficients a and b.

(d) Deduce that $-\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2}$.

ENRICHMENT

12. (a) (i) Use de Moivre's theorem to express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(ii) Hence prove that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

(iii) Deduce that $\tan^{-1}\frac{24}{7} = 4\tan^{-1}\frac{1}{3}$.

(b) Hence find the four fourth roots of $7 + 24i$ in Cartesian form.

13. (a) Show that the equation $(z+1)^8 - z^8 = 0$ has roots $z = -\frac{1}{2}, -\frac{1}{2}\left(1 \pm i\cot\frac{k\pi}{8}\right)$, where $k = 1, 2, 3$.

(b) Hence show that:

$$(z+1)^8 - z^8 = \tfrac{1}{8}(2z+1)(2z^2+2z+1)(4z^2+4z+\operatorname{cosec}^2\tfrac{\pi}{8})(4z^2+4z+\operatorname{cosec}^2\tfrac{3\pi}{8})$$

(c) By making a suitable substitution into this identity, deduce that:

$$\cos^{16}\theta - \sin^{16}\theta = \tfrac{1}{16}\cos 2\theta(\cos^2 2\theta + 1)(\cos^2 2\theta + \cot^2\tfrac{\pi}{8})(\cos^2 2\theta + \cot^2\tfrac{3\pi}{8})$$

14. Suppose that $\omega^3 = 1$ and $\omega \neq 1$.
Let k be a positive integer.

(a) What are the two possible values of $1 + \omega^k + \omega^{2k}$?

(b) Use the binomial theorem to expand $(1+\omega)^n$ and $(1+\omega^2)^n$, where n is a positive integer.

(c) Let ℓ be the largest integer for which $3\ell \le n$.
Show that:

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{3\ell} = \tfrac{1}{3}\left(2^n + (1+\omega)^n + (1+\omega^2)^n\right)$$

(d) If n is a multiple of 6, show that:

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{n} = \tfrac{1}{3}\left(2^n + 2\right)$$

15. Consider the equation $(z+1)^{2n} + (z-1)^{2n} = 0$, where n is a positive integer.
(a) Show that every root of the equation is purely imaginary.
(b) Let the roots be represented by the points P_1, P_2, ...,P_{2n} in the Argand diagram, and let O be the origin.
Show that:

$$OP_1{}^2 + OP_2{}^2 + \cdots + OP_{2n}{}^2 = 2n(2n-1)$$

3D Exponential Form: Euler's Formula

It was established in Chapter 1 that integer powers of a complex variable z behave predictably, according to the normal rules of arithmetic and algebra. In particular, polynomials in z have been studied and polynomial equations have been solved, such as finding the nth roots of unity earlier in this chapter. It is appropriate to now turn attention to other functions and consider how they behave when the variable is a complex number. The calculus of functions of complex variables is not available in this course and so any investigation will naturally be severely retricted. Consequently, this section will concentrate on one significant result which can be derived through integer powers of z. That result is:

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

or more specifically, dividing through by e^x,

$$e^{iy} = \cos y + i \sin y\,.$$

This last identity is known as Euler's formula.

Extending the Exponential Function: In Year 9 the trigonometric functions were introduced using right angled triangles. In Years 10 and 11 those definitions were extended to include angles of any magnitude. This was done very carefully so that the new definitions still worked in right angled triangles.

Similarly, polynomial functions were introduced in Years 10 and 11. These were then extended to include complex polynomials in the first chapter on complex numbers. This was done carefully so that all the results were consistent with real polynomials.

This idea of extending functions will now be applied to the exponential function to allow complex numbers to be used. It will be done carefully so that the results for complex numbers are consistent with the real function e^x. The index rule for multiplication plays a key part in the process. Thus, for real numbers a and b:

$$e^a \times e^b = e^{a+b}\,.$$

The rule needs to be the same even when a and b are replaced with complex numbers. That is:

$$e^{z_1} \times e^{z_2} = e^{z_1+z_2}\,.$$

That means that when $z_1 = x$ is real and $z_2 = iy$ is imaginary, the result is

$$e^{x+iy} = e^x \times e^{iy}\,,$$

however, there is still the problem of understanding what e^{iy} means.

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Trying to evaluate $e^x \times e^{iy}$ poses all sorts of problems. If $y = 0$ and x is a rational number like $\frac{1}{2}$ then all is good, but what if x is irrational? Irrational powers have not been defined or studied at high school. Worse still, if $y \neq 0$ then the second factor has an imaginary index and, as yet, there is no way of knowing whether or not such quantities even exist.

Addressing the first objection; expressions with irrational powers, like $e^{\sqrt{2}}$, can be defined through sequences. In this case put

$$e^{\sqrt{2}} = \lim_{n\to\infty} e^{u_n} ,$$

where u_n is a sequence with limiting value $\sqrt{2}$.

The problem of imaginary indices can be overcome using a process called analytic continuation. This technique is a very powerful tool which is studied in detail at university level. In the context of this course, the process is greatly simplified. Essentially e^x is redefined using a polynomial and a limit. The polynomial is then evaluated replacing x with iy and the limit taken. Finally, the result is assumed to be the value of e^{iy} in order to be consistent with the definition of e^x.

Two methods are presented in this text for this task. One is given below and the other is given in a series of questions in the following exercise. Both methods are very difficult and it is not expected that they be memorised. Nevertheless they should be followed and understood.

Redefining the Exponential Function: Consider the following sequence of special polynomials, each with repeated zero at $x = -n$ and with degree n.

$$E_n(x) = \left(1 + \frac{x}{n}\right)^n \qquad \text{where } n \geq 1.$$

The first three such polynomials are:

$$\begin{aligned} E_1(x) &= 1 + x\,, \\ E_2(x) &= 1 + x + \frac{x^2}{4}\,, \\ E_3(x) &= 1 + x + \frac{x^2}{3} + \frac{x^3}{27}\,. \end{aligned}$$

It can be shown, for all real values of x, that $\lim_{n\to\infty} E_n(x) = e^x$. That is:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n .$$

A question in the exercise develops a proof of this. The value of e^{iy} is now defined to be the result when x is replaced by iy. That is:

$$\begin{aligned} e^{iy} &= \lim_{n\to\infty} E_n(iy) \\ &= \lim_{n\to\infty} \left(1 + \frac{iy}{n}\right)^n . \end{aligned}$$

The advantage of using this expression as the definition of e^{iy} is that $E_n(iy)$ is a complex polynomial and so it behaves predictably. The disadvantage of doing this is that the right hand side involves a limit. Fortunately, that limit can be greatly simplified.

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Euler's Formula: Let $\left(1+\dfrac{iy}{n}\right)=r(\cos\theta+i\sin\theta)$. Then:

$$E_n(iy)=\Big(r(\cos\theta+i\sin\theta)\Big)^n$$
$$=r^n(\cos n\theta+i\sin n\theta)\qquad\text{(by de Moivre's theorem,)}$$

hence $$e^{iy}=\lim_{n\to\infty}r^n(\cos n\theta+i\sin n\theta)\,.$$

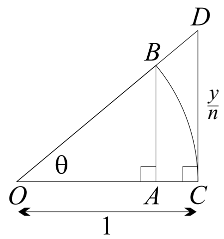
Now the modulus is given by $r^2=\big(1+(\frac{y}{n})^2\big)\,.$

Thus $$r^n=\big(1+(\tfrac{y}{n})^2\big)^{\frac{n}{2}}$$

and so $$\lim_{n\to\infty}r^n=\lim_{n\to\infty}\Big((1+\tfrac{y^2}{n^2})^{n^2}\Big)^{\frac{1}{2n}}$$
$$=\lim_{n\to\infty}\Big(e^{y^2}\Big)^{\frac{1}{2n}}\qquad\text{(by the definition of } e^x \text{ for real } x)$$
$$=1\,.$$

Notice that the limit was applied in two stages: first the bracketed term and then the index. Splitting a limit up like this can often lead to the wrong answer. Although it will not be proven here, splitting the limit gives the correct answer in this case because both limits exist and are finite.

Next consider $\lim\limits_{n\to\infty}(\cos n\theta+i\sin n\theta)$. The value of $n\theta$ can be determined in this limit from geometry.



In the diagram on the right, the point D represents the complex number $r(\cos\theta+i\sin\theta)$ and O is the origin. The real and imaginary axes have been omitted to simplify the diagram. Clearly $OC=1$ and $CD=\frac{y}{n}$. Sector OBC has been added and the point A is the foot of the perpendicular from B to OC.

Notice that $\triangle OAB$ and $\triangle OCD$ have equal angles and so are similar. From the hypotenuse of each, the similarity ratio is

$$OB:OD=1:\sqrt{1+\tfrac{y^2}{n^2}}\,,$$

and hence the ratio of areas is $1:(1+\frac{y^2}{n^2})$.

Now consider the areas of the two triangles and sector. It should be clear that

$$\big|\triangle OAB\big|\le\big|\text{sector } OBC\big|\le\big|\triangle OCD\big|\,.$$

Thus, using the usual formulae for $\big|\text{sector } OBC\big|$ and $\big|\triangle OCD\big|$, and applying the ratio of areas to determine $\big|\triangle OAB\big|$, this inequality gives

$$\frac{\frac{1}{2}\times1\times\frac{y}{n}}{1+\frac{y^2}{n^2}}\le\tfrac{1}{2}\theta\le\tfrac{1}{2}\times1\times\frac{y}{n}$$

and thus $$\frac{y}{1+\frac{y^2}{n^2}}\le n\theta\le y\,.$$

Hence in the limit as $n\to\infty$ this gives:

$$y\le\lim_{n\to\infty}n\theta\le y\,.$$

That is $$\lim_{n\to\infty}n\theta=y\,.$$

Finally, combining this limit with the one above for r^n yields the desired result:

$$\begin{aligned} e^{iy} &= \lim_{n\to\infty} r^n(\cos n\theta + i\sin n\theta) \\ &= 1 \times (\cos y + i\sin y)\,. \end{aligned}$$

That is $\quad e^{iy} = \cos y + i\sin y\,,$

exactly as stated at the start of this section. This is often referred to as Euler's formula, as it was first published by the Swiss mathematician Leonard Euler in 1748 in his *Introductio in analysin infinitorum*. It is a remarkable result. It shows that the trigonometric and exponential functions are all related by way of the imaginary number i. Because of this connection with trigonometry, the pronumeral y in Euler's formula is often replaced with an angle θ.

4 EULER'S FORMULA: For real numbers θ the value of $e^{i\theta}$ is defined to be

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This means there are now three ways to write a complex number z.

$$x + iy = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$$

The last expression is sometimes called the *complex exponential form*. Each of the last two expressions is called polar form, since each involves a radius r and an angle θ. As encountered in the previous chapter on complex numbers, the principal argument is normally used for θ.

A Check for Consistency: It is important that this new result is consistent with the definition of the exponential function for real variables. There are two essential characteristics: that $e^0 = 1$ and that $e^a \times e^b = e^{a+b}$. These characteristics are now checked. Firstly:

$$\begin{aligned} e^0 &= \cos 0 + i\sin 0 \\ &= 1 + i \times 0 \\ &= 1\,. \end{aligned}$$

Now let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then:

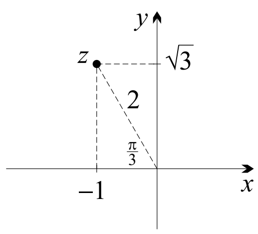
$$\begin{aligned} e^{z_1} \times e^{z_2} &= e^{x_1+iy_1} \times e^{x_2+iy_2} \\ &= e^{x_1}(\cos y_1 + i\sin y_1) \times e^{x_2}(\cos y_2 + i\sin y_2) \\ &= e^{x_1+x_2}\big(\cos(y_1+y_2) + i\sin(y_1+y_2)\big) \\ &= e^{x_1+x_2+i(y_1+y_2)} \\ &= e^{z_1+z_2} \qquad \text{(after re-ordering the index.)} \end{aligned}$$

Applications: Euler's formula has been derived and checked for consistency with the definition of the exponential function for real variables. It can now be confidently applied to solving various problems involving complex numbers.

WORKED EXAMPLE 8: Write $z = -1 + i\sqrt{3}$ in complex exponential form.

SOLUTION: The Argand diagram on the right shows the situation. Clearly $r = 2$ and $\theta = \frac{2\pi}{3}$. Hence $z = 2e^{\frac{2i\pi}{3}}$.

Notice that if $w = 2e^{\frac{14i\pi}{3}}$ then w and z represent the same point in the complex plane, because their arguments differ by a multiple of 2π. As usual, the principal value of the argument is given in the solution above.



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Worked Example 9: Show that $e^{i\pi} + 1 = 0$, known as Euler's identity.

Solution: Re-write the LHS in mod-arg form to get

$$\begin{aligned} e^{i\pi} + 1 &= \cos\pi + i\sin\pi + 1 \\ &= -1 + i \times 0 + 1 \\ &= 0 \end{aligned}$$

Worked Example 10: Use Euler's formula to write $\cos\theta$ in terms of e.

Solution: Euler's formula is

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

Replacing θ with $-\theta$, and using the symmetry of cosine and sine, yields

$$e^{-i\theta} = \cos\theta - i\sin\theta\,.$$

Adding these:

$$2\cos\theta = e^{i\theta} + e^{-i\theta}$$

or $$\cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\,.$$

Two important observations can be made at this point. The first is that the conjugate of $e^{i\theta}$ is its reciprocal $e^{-i\theta}$, as shown in the first few lines of working. This makes sense because it was previously established that

$$\frac{1}{\cos\theta + i\sin\theta} = \overline{(\cos\theta + i\sin\theta)}\,.$$

Secondly, the result for $\cos\theta$ now makes it algebraically clear why it is an even function. Replacing θ with $-\theta$:

$$\begin{aligned} \cos(-\theta) &= \tfrac{1}{2}\left(e^{-i\theta} + e^{i\theta}\right) \\ &= \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\ &= \cos\theta\,. \end{aligned}$$

This is hardly surprising. The exponential form of $\cos\theta$ was derived using the fact that $\cos\theta$ is even.

Euler and de Moivre: De Moivre's theorem can now be written using complex exponential form, though its proof is best done by relying on mod-arg form.

Worked Example 11: Prove the exponential form of de Moivre's theorem:

$$\left(e^{i\theta}\right)^n = e^{in\theta} \qquad \text{for integer values of } n\,.$$

Solution: First revert to mod-arg form:

$$\begin{aligned} \left(e^{i\theta}\right)^n &= (\cos\theta + i\sin\theta)^n && \text{(by Euler)} \\ &= \cos n\theta + i\sin n\theta && \text{(by de Moivre)} \\ &= e^{in\theta} && \text{(by Euler)} \end{aligned}$$

It is essential to note here that de Moivre's theorem was only proved for integer values of n. Hence the index rule for powers of complex exponentials is only valid when the power is an integer.

5 EULER AND DE MOIVRE: The complex exponential form of de Moivre's theorem is

$$\left(e^{i\theta}\right)^n = e^{in\theta} \qquad \text{provided } n \text{ is an integer.}$$

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Exercise **3D**

1. Simplify:

(a) $\left(e^{i\theta}\right)^3$

(b) $\left(e^{-i\theta}\right)^6$

(c) $\left(e^{2i\theta}\right)^4$

(d) $\left(e^{-5i\theta}\right)^{-2}$

2. Simplify as fully as possible:

(a) $e^{i\theta} \times e^{-2i\theta}$

(b) $\dfrac{e^{6i\theta}}{e^{3i\theta}}$

(c) $\left(e^{4i\theta}\right)^{-2} \times \left(e^{-2i\theta}\right)^{-5}$

(d) $\dfrac{\left(e^{2i\theta}\right)^3 \times \left(e^{-3i\theta}\right)^{-4}}{\left(e^{-i\theta}\right)^2}$

3. Express each of these complex numbers in exponential form.

(a) $2i$

(b) $1+i$

(c) -6

(d) $-1+\sqrt{3}\,i$

(e) $-3-3i$

(f) $2\sqrt{3}-2i$

4. Express each of these complex numbers in Cartesian form.

(a) $5e^{i\pi}$

(b) $e^{i\pi/3}$

(c) $4e^{-i\pi/2}$

(d) $2e^{5i\pi/6}$

(e) $2\sqrt{2}e^{-i\pi/4}$

(f) $4\sqrt{3}e^{-2i\pi/3}$

DEVELOPMENT

5. Let $z = 1+\sqrt{3}\,i$ and $w = 1-i$. Find, in exponential form:

(a) zw

(b) $\dfrac{w}{z}$

(c) z^3w

(d) $\dfrac{z^2}{w}$

6. By first converting to exponential form, find, in Cartesian form:

(a) $\left(\sqrt{3}+i\right)^6$

(b) $\left(-1+i\right)^5$

(c) $\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-8}$

(d) $\left(-3-3\sqrt{3}\,i\right)^4$

7. Suppose that $z = \dfrac{1+i}{\sqrt{2}}$ and $w = \dfrac{1-i}{\sqrt{2}}$.

Use the exponential forms of z and w to show that:

(a) $z^{10} - w^{10} = 2i$

(b) $1 + z + z^2 + z^3 + z^4 = \left(\sqrt{2}+1\right)i$

8. Use the exponential forms of $1+\sqrt{3}\,i$ and $1-i$ to show that:

(a) $\left(1+\sqrt{3}\,i\right)^5(1-i)^4 + \left(1-\sqrt{3}\,i\right)^5(1+i)^4 = -128$

(b) $\dfrac{\left(1+\sqrt{3}\,i\right)^5}{(1-i)^4} + \dfrac{\left(1-\sqrt{3}\,i\right)^5}{(1+i)^4} = -8$

9. If $z = e^{i\theta}$, prove that:

(a) $1 + z^4 = 2\cos 2\theta \operatorname{cis} 2\theta$

(b) $\dfrac{1+z^4}{1+z^{-4}} = \operatorname{cis} 4\theta$

10. If $z = re^{i\theta}$, prove that:

(a) $(1-i)z^2 = \sqrt{2}\,r^2e^{\frac{1}{4}i(8\theta-\pi)}$

(b) $\dfrac{1+\sqrt{3}\,i}{z} = \dfrac{2e^{\frac{1}{3}i(\pi-3\theta)}}{r}$

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11. Express the number in brackets in exponential form, and hence find the positive values of n for which:

(a) $(1+i)^n$ is real,
(b) $(1-i)^n$ is purely imaginary,
(c) $\left(\sqrt{3}-i\right)^n$ is real,
(d) $\left(1+\sqrt{3}\,i\right)^n$ is purely imaginary.

12. (a) Use the fact that $e^{i\theta}=\cos\theta+i\sin\theta$ to prove that:

(i) $e^{ni\theta}+e^{-ni\theta}=2\cos n\theta$
(ii) $e^{ni\theta}-e^{-ni\theta}=2i\sin n\theta$

(b) Hence find trigonometric expressions for:

(i) $e^{3i\theta}-e^{-3i\theta}$
(ii) $\left(e^{i\theta}+e^{-i\theta}\right)^2$
(iii) $\left(e^{i\theta}-e^{-i\theta}\right)^3$
(iv) $e^{2i\theta}+e^{i\theta}+2+e^{-i\theta}+e^{-2i\theta}$
(v) $e^{3i\theta}-e^{i\theta}+e^{-i\theta}-e^{-3i\theta}$

13. (a) Find an expression for $\cos\theta$ in terms of e.
(b) Hence show algebraically that $\cos\theta$ is an even function.
(c) Similarly, show that $\sin\theta$ is an odd function.
(d) Determine whether $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ are even, odd or neither.

14. Expand fully:

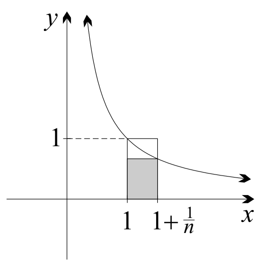
(a) $\left(z+2e^{i\pi/2}\right)\left(z-2e^{i\pi/2}\right)$
(b) $\left(z-e^{i\pi/3}\right)\left(z-e^{-i\pi/3}\right)$
(c) $(z+2)\left(z-2e^{i\pi/3}\right)\left(z-2e^{-i\pi/3}\right)$
(d) $\left(z-\sqrt{2}\,e^{i\pi/4}\right)\left(z-\sqrt{2}\,e^{-i\pi/4}\right)\left(z-\sqrt{2}\,e^{3i\pi/4}\right)\left(z-\sqrt{2}\,e^{-3i\pi/4}\right)$

15. Suppose that $re^{i\theta}=se^{i\phi}$, where both r and s are positive, and where both θ and ϕ are the principal values.

(a) Show algebraically that $r=s$.
[HINT: Take the modulus of both sides and use the fact that r and s are positive.]
(b) Show that $-2\pi<\phi-\theta<2\pi$, and hence show that $\theta=\phi$.
(c) Why is the result obvious from geometry?

ENRICHMENT

16. The diagram to the right shows $y=\dfrac{1}{x}$ for $x>0$. Consider the portion of this curve for $1\le x\le 1+\dfrac{1}{n}$, where n is a positive integer.



(a) By comparing areas, show that

$$\frac{1}{n+1}\le\log\left(1+\frac{1}{n}\right)\le\frac{1}{n}.$$

(b) Hence show that $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$.

(c) Use a suitable substitution in the result of part (b) to show that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x.$$

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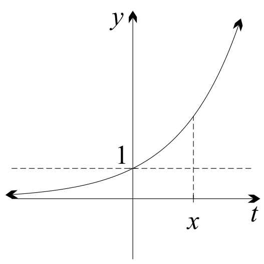
17. In this question we will derive an infinite series for e^x.



(a) The graph on the right clearly shows that $e^t \geq 1$ for $0 \leq t \leq x$. Integrate both sides of the inequality from 0 to x to show that $e^x \geq 1 + x$.

(b) From part (a), $e^t \geq 1 + t$ for $0 \leq t \leq x$. Integrate both sides of the inequation from 0 to x to show that $e^x \geq 1 + x + \frac{1}{2}x^2$.

(c) Continue to integrate in a similar fashion to show that

$$e^x \geq E(x) \text{ for } x \geq 0, \text{ where } E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(d) Now consider the function $h(x) = e^{-x} \times E(x)$. Show that $h'(x) = 0$ for all real x.

(e) Hence show that in fact $e^x = E(x)$ for all real x. This is called the power series of e^x.

18. In this question we will derive infinite series for $\cos x$ and $\sin x$. The previous question used integration. A different approach is used here.

(a) Suppose that $\cos x$ can be approximated with a polynomial. Since $\cos x$ is even, the polynomial should also be even, so let this be

$$c(x) = a_0 + a_2x^2 + a_4x^4 + ...$$

(i) Substitute $x = 0$ into $\cos x$ and $c(x)$ to find the value of a_0.

(ii) Given that $\dfrac{d^2}{dx^2}\cos x = -\cos x$, it makes sense to put $c''(x) = -c(x)$. Equate the coefficients of like powers of x to find a_2 and a_4.

(iii) Show that $c(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. This is called the power series of $\cos x$.

(b) Suppose that $\sin x$ can be approximated with a polynomial. Since $\sin x$ is odd, the polynomial should also be odd, so let this be

$$s(x) = a_1x + a_3x^3 + a_5x^5 + ...$$

(i) Substitute $x = 0$ into the derivatives of $\sin x$ and $s(x)$ to find the value of a_1.

(ii) Given that $\dfrac{d^2}{dx^2}\sin x = -\sin x$, it makes sense to put $s''(x) = -s(x)$. Equate the coefficients of like powers of x to find a_3 and a_5.

(iii) Show that $s(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. This is the power series of $\sin x$.

(c) Now consider the special function $h(x) = (c(x) - \cos x)^2 + (s(x) - \sin x)^2$.

(i) Show that $c'(x) = -s(x)$ and that $s'(x) = c(x)$.

(ii) Show that $h'(x) = 0$ and hence deduce that $h(x) = 0$ for all real x.

(iii) Explain why $c(x) = \cos x$ for all real x, and $s(x) = \sin x$ for all real x.

19. The function $E(x)$ in Question 16 involves powers of x. Hence it is expected to behave properly when x is replaced with the imaginary number $i\theta$.

(a) Let $e^{i\theta} = E(i\theta)$. Write out and simplify the power series for $e^{i\theta}$.

(b) Re-arrange this power series and hence show that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

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3E Applications of Exponential Form

Euler's formula is a powerful tool that can be used to write elegant solutions to problems involving exponential identities, trigonometric identities and roots of complex numbers.

Exponential Identities: A significant characteristic of the exponential form is that it is periodic. This is easy to prove using the corresponding trigonometric form.

Worked Example 12: Show that the exponential function $e^{i\theta}$ is periodic with period 2π. That is, prove the identity $e^{i(\theta+2k\pi)} = e^{i\theta}$ for all $\theta \in \mathbf{R}$.

Solution:
$$\begin{aligned} e^{i(\theta+2k\pi)} &= \cos(\theta+2k\pi) + i\sin(\theta+2k\pi) \\ &= \cos\theta + i\sin\theta \\ &= e^{i\theta} \end{aligned}$$

6 **The exponential function is periodic:** The exponential function $e^{i\theta}$ is periodic with period 2π. That is:

$$e^{i(\theta+2k\pi)} = e^{i\theta} \qquad \text{for all } \theta \in \mathbf{R}.$$

This result will be essential when using the exponential form to find the roots of complex numbers, later in this section.

Trigonometric Identities: The trigonometric identities developed in Section 3B can also be proven using exponential forms. In many cases, it is simply a matter of replacing z with $e^{i\theta}$. To demonstrate, compare the following with Question 3 of Exercise 3B.

Worked Example 13: Use the exponential form of $\cos\theta$ to show that

$$\cos^4\theta = \tfrac{1}{8}\Big(\cos 4\theta + 4\cos 2\theta + 3\Big).$$

Solution: First note that $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$. Thus:

$$\begin{aligned} \cos^4\theta &= \tfrac{1}{16}\left(e^{i\theta} + e^{-i\theta}\right)^4 \\ &= \tfrac{1}{16}\left(e^{i4\theta} + 4e^{i2\theta} + 6 + 4e^{-i2\theta} + e^{-i4\theta}\right) \\ &= \tfrac{1}{8}\Big(\tfrac{1}{2}(e^{i4\theta} + e^{-i4\theta}) + 4\times\tfrac{1}{2}(e^{i2\theta} + e^{-i2\theta}) + 3\Big) \\ &= \tfrac{1}{8}\Big(\cos 4\theta + 4\cos 2\theta + 3\Big). \end{aligned}$$

The solution above relied on the exponential form of $\cos\theta$, which was derived in Section 3D. The exponential form of $\sin\theta$ is equally important and its derivation is left as an exercise. The result for $\tan\theta$ is simply the ratio of these two.

7 **The exponential forms of sin and cos:** Euler's formula can be used to write these functions in exponential form. They are:

$$\cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \text{and} \qquad \sin\theta = \tfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

The above exponential forms of sine and cosine can also be used to confirm other trigonometric identities. Here is a harder example.

Worked Example 14: Use the exponential forms to verify that

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

Solution: To make the algebra easier, let $\lambda = \dfrac{\alpha+\beta}{2}$ and $\mu = \dfrac{\alpha-\beta}{2}$.
Next note that $(\lambda+\mu) = \alpha$ and $(\lambda-\mu) = \beta$. Then:

$$\begin{aligned}\text{RHS} &= -2 \times \tfrac{1}{2i}\left(e^{i\lambda} - e^{-i\lambda}\right) \times \tfrac{1}{2i}\left(e^{i\mu} - e^{-i\mu}\right)\\ &= \tfrac{1}{2}\left(e^{i(\lambda+\mu)} - e^{i(\lambda-\mu)} - e^{-i(\lambda-\mu)} + e^{-i(\lambda+\mu)}\right)\\ &= \tfrac{1}{2}\left(e^{i\alpha} - e^{i\beta} - e^{-i\beta} + e^{-i\alpha}\right)\\ &= \tfrac{1}{2}(e^{i\alpha} + e^{-i\alpha}) - \tfrac{1}{2}(e^{i\beta} + e^{-i\beta})\\ &= \text{LHS}.\end{aligned}$$

Roots of Complex Numbers: The exponential form can also be used to find roots of complex numbers. The periodic nature of $e^{i\theta}$ is a crucial part of the solution.

Worked Example 15: Find the cube roots of $2+2i$ in exponential form.

Solution: First note that $2+2i = 2^{\frac{3}{2}}e^{i\frac{\pi}{4}+2ik\pi}$. Let $z = r\,e^{i\theta}$ be a cube root,

then $\quad z^3 = 2+2i$

gives $\quad r^3e^{i3\theta} = 2^{\frac{3}{2}}e^{i\frac{\pi}{4}+2ik\pi}$ (by de Moivre)

Equating the moduli and arguments:

$$r^3 = 2^{\frac{3}{2}}$$

so $\quad r = \sqrt{2}$

and $\quad 3\theta = \frac{\pi}{4} + 2k\pi$

so $\quad \theta = \frac{\pi}{12} + \frac{2k\pi}{3}$

$$= \frac{\pi(1+8k)}{12}$$

hence, using the principal arguments ($k = -1, 0, 1$),

$$\theta = -\frac{7\pi}{12} \text{ or } \frac{\pi}{12} \text{ or } \frac{3\pi}{4}.$$

Finally, $\quad z = \sqrt{2}e^{-\frac{7i\pi}{12}}$ or $\sqrt{2}e^{\frac{i\pi}{12}}$ or $\sqrt{2}e^{\frac{3i\pi}{4}}$.

Exercise **3E**

1. Consider the equation $z^2 = 2i$, whose roots are the two square roots of $2i$.

(a) Write $2i$ in exponential form with principal argument.

(b) Hence write $2i$ in exponential form with a general argument.

(c) If $z = re^{i\theta}$, show that $r = \sqrt{2}$ and $\theta = \dfrac{(4k+1)\pi}{4}$.

(d) Hence write down the two roots of the equation in exponential form with principal arguments.

(e) Express the two roots in Cartesian form.

2. Consider the equation $z^4 = -1$, whose roots are the four fourth roots of -1.
(a) Write -1 in exponential form with principal argument.
(b) Hence write -1 in exponential form with a general argument.
(c) If $z = re^{i\theta}$, show that $r = 1$ and $\theta = \dfrac{(2k+1)\pi}{4}$.
(d) Hence write down the four roots of the equation in exponential form with principal arguments.
(e) Express the four roots in Cartesian form.

3. Consider the equation $z^3 = -i$, whose roots are the three cube roots of $-i$.
(a) Write $-i$ in exponential form with principal argument.
(b) Hence write $-i$ in exponential form with a general argument.
(c) If $z = re^{i\theta}$, show that $r = 1$ and $\theta = -\dfrac{(4k+1)\pi}{6}$.
(d) Hence write down the three roots of the equation in exponential form with principal arguments.
(e) Express the three roots in Cartesian form.

4. (a) Use the result $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $e^{ni\theta} + e^{-ni\theta} = 2\cos n\theta$.
(b) Show that $\left(e^{i\theta} + e^{-i\theta}\right)^3 = \left(e^{3i\theta} + e^{-3i\theta}\right) + 3\left(e^{i\theta} + e^{-i\theta}\right)$.
(c) Use parts (a) and (b) to derive the identity $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$.

5. Use similar methods to the previous question to prove the identity $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$.

DEVELOPMENT

6. (a) Show that $z^4 + 16 = \left(z - 2e^{i\pi/4}\right)\left(z - 2e^{-i\pi/4}\right)\left(z - 2e^{3i\pi/4}\right)\left(z - 2e^{-3i\pi/4}\right)$.
(b) Hence show that $z^4 + 16 = \left(z^2 - 2\sqrt{2}\,z + 4\right)\left(z^2 + 2\sqrt{2}\,z + 4\right)$.
(c) Confirm the factorisation in the previous part by writing $z^4 + 16$ as $\left(z^4 + 8z^2 + 16\right) - 8z^2$.

7. (a) Show that $z^5 + 1 = (z+1)\left(z - e^{i\pi/5}\right)\left(z - e^{-i\pi/5}\right)\left(z - e^{3i\pi/5}\right)\left(z - e^{-3i\pi/5}\right)$.
(b) Hence show that $z^5 + 1 = (z+1)\left(z^2 - \left(2\cos\frac{\pi}{5}\right)z + 1\right)\left(z^2 + \left(2\cos\frac{2\pi}{5}\right)z + 1\right)$.
(c) Deduce that $2\cos\frac{2\pi}{5} - 2\cos\frac{\pi}{5} + 1 = 0$, and hence find the exact values of $\cos\frac{\pi}{5}$ and $\cos\frac{2\pi}{5}$.

8. (a) Derive the exponential forms of $\cos\theta$ and $\sin\theta$ given in Box 7.
(b) Use these results to verify the following trigonometric identities.
(i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$
(ii) $\sin 2\theta = 2\cos\theta\sin\theta$
(iii) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
(iv) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

9. (a) Use the methods of question 4 to write $\cos^6\theta$ in terms of $\cos 6\theta, \cos 4\theta$ and $\cos 2\theta$.
(b) Hence show that $\displaystyle\int_0^{\frac{\pi}{4}} \cos^6\theta\, d\theta = \frac{15\pi + 44}{192}$.

10. (a) Find $\sin^3\theta$ and $\sin^5\theta$ in terms of sines of multiples of θ.
(b) Hence show that $\sin^3\theta\cos^2\theta = \frac{1}{16}(2\sin\theta + \sin 3\theta - \sin 5\theta)$.
(c) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3\theta\cos^2\theta\, d\theta$.

11. Consider the polynomial equation $5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$, which has four complex roots with modulus one, and let $z = e^{i\theta}$.
(a) Show that $e^{ni\theta} + e^{-ni\theta} = 2\cos n\theta$.
(b) Hence show that $5\cos 2\theta - 11\cos\theta + 8 = 0$.
(c) Find the four roots of the equation in Cartesian form.

12. Suppose that $1 - i = e^{a+ib}$, where $a, b \in \mathbf{R}$ and $-\frac{\pi}{2} < b < \frac{\pi}{2}$.
Find the exact values of a and b.

13. (a) Express $\cos(A+B) + \cos(A-B)$ in terms of $\cos A$ and $\cos B$.
(b) Hence prove that $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$.
(c) Similarly, prove that $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$.
(d) Given that α and β are real, deduce that $e^{i\alpha} + e^{i\beta} = 2\cos\frac{\alpha-\beta}{2}\, e^{\frac{1}{2}i(\alpha+\beta)}$.

14. (a) Use the exponential forms of $\cos\theta$ and $\sin\theta$ given in Box 7 to verify each identity.
(i) $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$ (ii) $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
(b) Write down the exponential form of $\tan\theta$.
(c) Hence verify that $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

15. (a) Write down the sum of the geometric series $z + z^2 + z^3 + \cdots + z^n$.
(b) Hence, by putting $z = e^{i\theta}$, show that:

$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta = \frac{\sin\frac{1}{2}n\theta\sin\frac{1}{2}(n+1)\theta}{\sin\frac{1}{2}\theta}$$

(c) Deduce that $\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} = \cot\frac{\pi}{2n}$.

ENRICHMENT

16. Use the exponential form of $\tan\theta$ found in Question 14 to verify that

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

17. Consider the equation $z^{2n+1} = 1$, where n is a positive integer.
(a) Find the roots of the equation, expressing them in exponential form.
(b) Hence show that

$$z^{2n} + z^{2n-1} + z^{2n-2} + \cdots + z^2 + z + 1 = \prod_{k=1}^{n}\left(z^2 - \left(2\cos\tfrac{2k\pi}{2n+1}\right)z + 1\right).$$

(c) Deduce that $2^n\sin\frac{\pi}{2n+1}\sin\frac{2\pi}{2n+1}\sin\frac{3\pi}{2n+1}\cdots\sin\frac{n\pi}{2n+1} = \sqrt{2n+1}$.

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3F Chapter Review Exercise

Exercise **3F**

1. Simplify:

(a) $(\cos\theta + i\sin\theta)^3(\cos 2\theta + i\sin 2\theta)^2$

(b) $\dfrac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^2}$

2. Evaluate $\dfrac{\left(e^{-i\frac{\pi}{7}}\right)^3}{\left(e^{i\frac{\pi}{7}}\right)^4}$.

3. (a) Write $1 - i$ in mod-arg form.

(b) Hence find $(1-i)^{13}$ in Cartesian form.

4. (a) Use de Moivre's theorem to evaluate $(\sqrt{3}+i)^{12} + (\sqrt{3}-i)^{12}$.

(b) If n is a positive integer:

(i) prove that $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n$ is real,

(ii) determine the values of n for which $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n$ is rational.

5. (a) Use de Moivre's theorem to find $\cos 6\theta$ and $\sin 6\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

(b) Hence show that $\tan 6\theta = \dfrac{2t\left(3 - 10t^2 + 3t^4\right)}{1 - 15t^2 + 15t^4 - t^6}$, where $t = \tan\theta$.

6. (a) Expand $\left(z + \dfrac{1}{z}\right)^4$ and $\left(z - \dfrac{1}{z}\right)^4$.

(b) By letting $z = \cos\theta + i\sin\theta$, prove that $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$.

7. Suppose that ω is a complex cube root of -1.

(a) Show that the other complex root is $-\omega^2$.

(b) Evaluate $(6\omega + 1)(6\omega^2 - 1)$.

8. Solve the equation $z^3 - 8i = 0$, writing the roots in the form $re^{i\theta}$.

9. Find, in mod-arg form:

(a) the three cube roots of $2 + 2i$,

(b) the six sixth roots of i.

10. Suppose that $z = 4\sqrt{3}e^{i\pi/3} - 4e^{5i\pi/6}$.

(a) Simplify z, writing your answer in exponential form.

(b) Show that $\dfrac{z}{8} + i\left(\dfrac{z}{8}\right)^2 + \left(\dfrac{z}{8}\right)^3 = 2i$.

(c) Find the three cube roots of z in exponential form.

11. (a) Show that $(z - z^{-1})^7 = (z^7 - z^{-7}) - 7(z^5 - z^{-5}) + 21(z^3 - z^{-3}) - 35(z - z^{-1})$.

(b) If $z = \cos\theta + i\sin\theta$, show that $z - z^{-1} = 2i\sin\theta$ and that $(z^n - z^{-n}) = 2i\sin n\theta$.

(c) Hence prove that $\sin^7\theta = \frac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)$.

(d) Find $\displaystyle\int \left(35\sin\theta - 64\sin^7\theta\right)\, d\theta$.

12. (a) Use de Moivre's theorem to prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(b) Hence solve the equation $16x^4 - 20x^2 + 5 = 0$, giving the roots in trigonometric form.

(c) Show that $\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \frac{\sqrt{5}}{4}$.

(d) If $u = 2x^2 - 1$, show that $4u^2 - 2u - 1 = 0$.

(e) Deduce the exact value of $\cos\frac{\pi}{5}$.

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13. (a) Derive the exponential forms of $\cos\theta$ and $\sin\theta$ given in Box 7.

(b) Use these results to verify each trigonometric identity.

(i) $2\cos^2\theta = 1 + \cos 2\theta$

(ii) $2\sin^2\theta = 1 - \cos 2\theta$

(iii) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

(iv) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

14. (a) Find the seven seventh roots of -1 in mod-arg form.

(b) Hence show that:

(i) $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$

(ii) $z^7 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$

(iii) $z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$
$= \left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$

(c) Divide both sides of the identity in (b)(iii) by z^3, and hence show that:

$$2\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1 = 8\left(\cos\theta - \cos\tfrac{\pi}{7}\right)\left(\cos\theta - \cos\tfrac{3\pi}{7}\right)\left(\cos\theta - \cos\tfrac{5\pi}{7}\right)$$

15. (a) Find the fifth roots of unity in exponential form.

(b) Let α be the complex fifth root of unity with the smallest positive argument, and suppose that $u = \alpha + \alpha^4$ and $v = \alpha^2 + \alpha^3$.

(i) Find the values of $u + v$ and $u - v$.

(ii) Deduce that $\cos\frac{2\pi}{5} = \frac{1}{4}\left(\sqrt{5} - 1\right)$.

16. Let $z = \cos\theta + i\sin\theta$ and suppose that n is a positive integer.

(a) Show that $z^n + z^{-n} = 2\cos n\theta$.

(b) Prove that $2\cos A\sin B = \sin(A+B) - \sin(A-B)$.

(c) Hence show that $\left(z^{2n} + z^{2n-2} + z^{2n-4} + \cdots + z^{-2n}\right)\sin\theta = \sin(2n+1)\theta$.

(d) Use the previous part and the result $\cos 3A = 4\cos^3 A - 3\cos A$ to prove the identity:

$$8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = \frac{\sin 7\theta}{\sin\theta}$$

17. Use the exponential forms of $\cos\theta$ and $\sin\theta$ given in Box 7 to verify that

$$\sin\alpha - \sin\beta = 2\cos\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2}$$

18. Suppose that n is an integer greater than 2 and ω is an nth root of unity, where $\omega \neq 1$.

(a) By expanding the left-hand side, show that

$$\left(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1}\right)(\omega - 1) = n.$$

(b) Using the identity $\dfrac{1}{z^2 - 1} = \dfrac{z^{-1}}{z - z^{-1}}$, or otherwise, prove that

$$\frac{1}{\cos 2\theta + i\sin 2\theta - 1} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta}.$$

(c) Hence, if $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, find the real part of $\dfrac{1}{\omega - 1}$.

(d) Deduce that $1 + 2\cos\frac{2\pi}{5} + 3\cos\frac{4\pi}{5} + 4\cos\frac{6\pi}{5} + 5\cos\frac{8\pi}{5} = -\frac{5}{2}$.

(e) By expressing the left-hand side of the result in part (iv) in terms of $\cos\frac{\pi}{5}$ and $\cos\frac{2\pi}{5}$, find the exact value of $\cos\frac{\pi}{5}$.

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4

Integration

Chapter Overview: The art of integration is a skill that all mathematicians must possess, as integrals arise in all areas of mathematics. For example, in the seemingly unrelated topic of prime numbers the integral $\displaystyle\int \frac{dx}{\log x}$ appears.

As integration is an art form, it requires plenty of practice to become proficient. Thus students are encouraged to attempt as many of the exercise questions as possible in the time they have available.

The work in this chapter builds on the content of the Mathematics Extension 1 course. A methodical approach is needed to study the material. In particular, it is important to be able to recognise the different forms of integrals, and to quickly determine which method is best used.

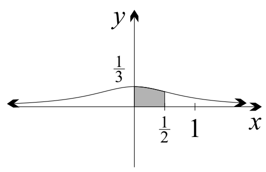
The first five sections are relatively straightforward, being based on algebraic manipulation. In Section 4F the new method of integration by parts is introduced, which is based on the product rule for differentiation. Section 4G covers various harder types of Trignometric integrals. Section 4H introduces the concept of integrals that can be referenced by an index, and the corresponding reduction formulae. The chapter concludes with a set of miscellaneous questions. These provide an opportunity to practise choosing the most efficient method to apply.

4A The Standard Integrals

Students will know that each examination is accompanied by a Reference Sheet, which includes various integrals. Most of these have already been encountered in the Mathematics Extension 1 course. The appendix to this chapter includes a table of similar integrals as well as some other common integrals that will be needed in this course. The ability to manipulate any integral formula in simple ways is expected of all students.

Worked Example 1: Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{3+4x^2}$.

Solution: Take out a factor of $\frac{1}{4}$ to get:



$$\int_0^{\frac{1}{2}} \frac{dx}{3+4x^2} = \frac{1}{4}\int_0^{\frac{1}{2}} \frac{dx}{(\frac{\sqrt{3}}{2})^2 + x^2}$$

$$= \frac{1}{4} \times \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right]_0^{\frac{1}{2}} \quad \text{(Reference Sheet)}$$

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$$= \frac{1}{2\sqrt{3}} \tan^{-1} \tfrac{1}{\sqrt{3}}$$
$$= \frac{\pi}{12\sqrt{3}}.$$

Exercise **4A**

1. Use the Integral Calculus section of the HSC Reference Sheet to determine each of these indefinite integrals.

(a) $\displaystyle\int e^{4x}\,dx$ (b) $\displaystyle\int \sin 5x\,dx$ (c) $\displaystyle\int \sec^2 \tfrac{1}{2}x\,dx$ (d) $\displaystyle\int \frac{1}{3x-4}\,dx$ (e) $\displaystyle\int \frac{2}{\sqrt{x}}\,dx$ (f) $\displaystyle\int 3^x\,dx$

2. Use the Integral Calculus section of the Reference Sheet to find:

(a) $\displaystyle\int \frac{1}{(2x-1)^2}\,dx$ (b) $\displaystyle\int \frac{1}{\sqrt{25-x^2}}dx$ (c) $\displaystyle\int x^2 e^{x^3}\,dx$ (d) $\displaystyle\int \frac{1}{9+x^2}\,dx$ (e) $\displaystyle\int \frac{4x+2}{x^2+x+1}\,dx$ (f) $\displaystyle\int 2x(x^2+1)^4 dx$

3. Use the Integral Calculus section of the HSC Reference Sheet to evaluate each of these definite integrals.

(a) $\displaystyle\int_0^4 e^{\frac{x}{2}}\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{6}} \sec^2 2x\,dx$ (c) $\displaystyle\int_{-4}^4 \frac{1}{16+x^2}\,dx$ (d) $\displaystyle\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx$ (e) $\displaystyle\int_{-2}^{-1} \frac{3}{2-3x}\,dx$ (f) $\displaystyle\int_0^{\frac{\pi}{4}} \cos x \sin^3 x\,dx$

DEVELOPMENT

4. Find:

(a) $\displaystyle\int -\frac{1}{x^2}\, e^{\frac{1}{x}}\,dx$ (b) $\displaystyle\int \frac{\cos 3x}{1+\sin 3x}\,dx$ (c) $\displaystyle\int x\sec^2 x^2\,dx$ (d) $\displaystyle\int 5^{2x}\,dx$ (e) $\displaystyle\int \frac{1+\sec^2 x}{x+\tan x}\,dx$ (f) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

5. Evaluate:

(a) $\displaystyle\int_0^4 (1-x)^3\,dx$ (b) $\displaystyle\int_0^1 \frac{x^2}{1+x^3}\,dx$ (c) $\displaystyle\int_0^1 \frac{dx}{1+3x^2}$ (d) $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx$ (e) $\displaystyle\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{4-9x^2}}$ (f) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}\,dx$

ENRICHMENT

6. Evaluate $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx.$

7. Use the derivative of $\dfrac{\ln x}{x}$ to find $\displaystyle\int \frac{\ln x}{x^2}\,dx.$

8. Use the derivative of $x\sin^{-1} x$ to show that $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1} x\,dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$

9. Use the derivative of $\tan^3 x$ to find $\displaystyle\int_0^{\frac{\pi}{4}} \tan^4 x\,dx.$

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4B Algebraic Manipulation

Many of the integrals encountered contain fractions which require some sort of rearrangement before proceeding. In the first worked example that follows the numerator is almost identical to the denominator.

Worked Example 2: Determine $\displaystyle\int \frac{x^2-1}{x^2+1}\,dx$.

Solution: Noting that $x^2 - 1 = (x^2+1) - 2$ the fraction may be separated.

$$\begin{aligned}\int \frac{x^2-1}{x^2+1}\,dx &= \int \frac{x^2+1}{x^2+1} - \frac{2}{x^2+1}\,dx\\ &= \int 1 - \frac{2}{x^2+1}\,dx\\ &= x - 2\tan^{-1} x + C\,.\end{aligned}$$

In harder problems long division may be required. In some cases the numerator is close to a multiple of the denominator, as in the next worked example.

Worked Example 3: Find $\displaystyle\int \frac{4x^3-2x^2+1}{2x-1}\,dx$.

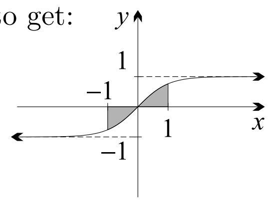
Solution:

$$\begin{aligned}\int \frac{4x^3-2x^2+1}{2x-1}\,dx &= \int \frac{2x^2(2x-1)+1}{2x-1}\,dx\\ &= \int 2x^2 + \frac{1}{2x-1}\,dx\\ &= \tfrac{2}{3}x^3 + \tfrac{1}{2}\log|2x-1| + C\,.\end{aligned}$$

Using a Common Factor: Some rational functions are easier to deal with after multiplication or division by a common factor. The result is a numerator which is the derivative of the denominator.

Worked Example 4: Evaluate $\displaystyle\int_{-1}^{1} \frac{e^{2x}-1}{e^{2x}+1}\,dx$.

Solution: Multiply numerator and denominator by e^{-x} to get:



$$\begin{aligned}\int_{-1}^{1} \frac{e^{2x}-1}{e^{2x}+1}\,dx &= \int_{-1}^{1} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx\\ &= \Big[\log(e^x+e^{-x})\Big]_{-1}^{1}\\ &= \log(e+e^{-1}) - \log(e^{-1}+e)\\ &= 0\,. \qquad \text{(Why is this obvious from the graph?)}\end{aligned}$$

Notice in the solution that the primitive $\log(e^x + e^{-x})$ is written without an absolute value. This is because $e^x + e^{-x}$ is always positive, and hence the absolute value function is redundant. In this text, the absolute value will normally be omitted whenever it is redundant.

Two New Integrals: The final two integrals in the appendix will be new to most readers. The result for the last integral is proven here using a very clever trick. Like the previous worked example, multiply through by a common factor.

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$$\begin{aligned}\int \frac{1}{\sqrt{x^2+a^2}}\,dx &= \int \frac{(x+\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}\,(x+\sqrt{x^2+a^2})}\,dx\\ &= \int \frac{(\frac{x}{\sqrt{x^2+a^2}}+1)}{(x+\sqrt{x^2+a^2})}\,dx\\ &= \int \frac{(1+\frac{x}{\sqrt{x^2+a^2}})}{(x+\sqrt{x^2+a^2})}\,dx\,.\end{aligned}$$

Looking carefully at the last line, notice that the numerator is the derivative of the denominator and hence

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log(x+\sqrt{x^2+a^2})+C\,.$$

The second last integral in the table may be done in a similar way, which is a question at the end of the exercise.

Exercise **4B**

1. Determine the following by rewriting the numerator in terms of the denominator.

(a) $\displaystyle\int \frac{x}{x-1}\,dx$ (b) $\displaystyle\int \frac{x-1}{x+1}\,dx$ (c) $\displaystyle\int \frac{x+1}{x-1}\,dx$

2. Evaluate the following.

(a) $\displaystyle\int_0^1 \frac{x-1}{x+1}\,dx$ (b) $\displaystyle\int_0^2 \frac{x}{2x+1}\,dx$ (c) $\displaystyle\int_0^1 \frac{3-x^2}{1+x^2}\,dx$

3. Evaluate the following. In each case, begin by rewriting the given fraction as two fractions by separating the terms in the numerator.

(a) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{1-x}{\sqrt{1-x^2}}\,dx$ (b) $\displaystyle\int_0^1 \frac{2x+1}{1+x^2}\,dx$ (c) $\displaystyle\int_0^1 \frac{1-x}{1+x^2}\,dx$ (d) $\displaystyle\int_0^2 \frac{1+x}{4+x^2}\,dx$

4. (a) Let $y=\log(x+\sqrt{x^2+a^2})$. Find and simplify $\dfrac{dy}{dx}$.

(b) Hence find a formula for $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx$.

(c) Use this formula to determine: (i) $\displaystyle\int \frac{1}{\sqrt{x^2+3}}\,dx$, (ii) $\displaystyle\int_{-4}^4 \frac{1}{\sqrt{x^2+9}}\,dx$

DEVELOPMENT

5. (a) Given that $x^3=x(x^2+1)-x$, determine $\displaystyle\int \frac{x^3}{x^2+1}\,dx$.

(b) Given that $x^3=(x^3+1)-1$ and that $x^3+1=(x+1)(x^2-x+1)$, determine $\displaystyle\int \frac{x^3}{x+1}\,dx$.

(c) Use similar approaches to those shown in parts (a) and (b) to determine the following.

(i) $\displaystyle\int \frac{x^3}{x-1}\,dx$ (iii) $\displaystyle\int \frac{1}{1+e^x}\,dx$ (v) $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$

(ii) $\displaystyle\int \frac{x^4}{x^2+1}\,dx$ (iv) $\displaystyle\int \frac{x}{\sqrt{2+x}}\,dx$ (vi) $\displaystyle\int \frac{x^3}{x^2+4}\,dx$

6. Evaluate these by first muliplying or dividing by an appropriate factor.

(a) $\displaystyle\int_1^2 \frac{e^{2x}+1}{e^{2x}-1}\,dx$ (b) $\displaystyle\int_0^1 \frac{e^x}{e^x+e^{-x}}\,dx$ (c) $\displaystyle\int_1^{\sqrt{3}} \frac{2+\frac{1}{x}}{x+\frac{1}{x}}\,dx$

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7. By using long division or otherwise, determine:

(a) $\displaystyle\int \frac{x^2+x+1}{x+1}\,dx$ (b) $\displaystyle\int \frac{x^3-2x^2+3}{x-2}\,dx$ (c) $\displaystyle\int \frac{(x+1)^2}{1+x^2}\,dx$

8. (a) Let $y = \log(x + \sqrt{x^2 - a^2})$, where $x > |a|$. Find and simplify $\dfrac{dy}{dx}$.

(b) Hence find a formula for $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx$, where $x > |a|$.

(c) Use this formula to determine: (i) $\displaystyle\int \frac{1}{\sqrt{x^2-5}}\,dx$ (ii) $\displaystyle\int_{\sqrt{5}}^{3} \frac{1}{\sqrt{x^2-4}}\,dx$

ENRICHMENT

9. Divide numerator and denominator by an appropriate factor to help determine

$$\int \frac{1}{x+\sqrt{x}}\,dx\,.$$

10. Use a similar approach to that shown in the text to prove that, for $|x| > |a|$,

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x + \sqrt{x^2-a^2}\right| + C\,.$$

4C Substitution

Many of the techniques used in integration are derived from differentiation. This is not so surprising since the two processes are essentially mutually inverse. One particularly useful technique is substitution which is the integration equivalent of the chain rule for differentiation, and is sometimes called the reverse chain rule.

The Integral Form of the Chain Rule: Suppose that F is a function of u, which is in turn a function of x. Further suppose that $F(u)$ is a primitive of $f(u)$. Differentiating F with respect to x and applying the chain rule gives:

$$\frac{d}{dx}F(u) = \frac{dF}{du} \times \frac{du}{dx}$$

so

$$\frac{d}{dx}F(u) = f(u) \times u'\,.$$

Integrating both sides of this result

$$\int \left(\frac{d}{dx}F(u)\right)\,dx = \int f(u) \times u'\,dx$$

or

$$F(u) + C = \int f(u) \times u'\,dx\,.$$

It is this last result which proves most useful for integration. Thus if an integrand can be expressed as a product, where one factor is a chain of functions $f(u)$ and the other factor is u' then the primitive can immediately be written down.

1 THE INTEGRAL FORM OF THE CHAIN RULE: If $F(u)$ is a primitive of $f(u)$ then

$$\int f(u) \times u'\,dx = F(u) + C\,.$$

Substitutions: In the simplest examples, the primitive can be determined mentally. For example, an easy integral involving the exponential function is

$$\int 2x\, e^{x^2}\, dx = e^{x^2} + C\,.$$

In harder examples a formal procedure should be followed.

WORKED EXAMPLE 5: Determine $\displaystyle\int \frac{x^2}{\sqrt{x^3+1}}\, dx$ by a suitable substitution.

SOLUTION: Let $I = \displaystyle\int \frac{x^2}{\sqrt{x^3+1}}\, dx$ and put $u = x^3 + 1$, then

$$\frac{du}{dx} = 3x^2$$

or $\quad \tfrac{1}{3}du = x^2\, dx \qquad$ (treating the derivative like a fraction.)

Thus $\quad I = \displaystyle\int \frac{1}{3\sqrt{u}}\, du$

$\quad = \tfrac{2}{3}\sqrt{u} + C\,.$

Hence $\quad I = \tfrac{2}{3}\sqrt{x^3+1} + C\,.$

Notice that the final step of the solution is a back substitution to get the integral I in terms of x. It is important to remember to do this.

Substitutions and Definite Integrals: It is equally important to follow this formal procedure when definite integrals are involved, paying particular attention to the limits of integration. However, the back substitution step is not needed.

WORKED EXAMPLE 6: Use a suitable substitution to find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)^3}\, dx\,.$

SOLUTION: Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)^3}\, dx$ and put $u = 1 + \cos x$ to get

$$\frac{du}{dx} = -\sin x$$

so $\quad -du = \sin x\, dx\,.$

When $\quad x = 0\,, \qquad u = 2\,,$

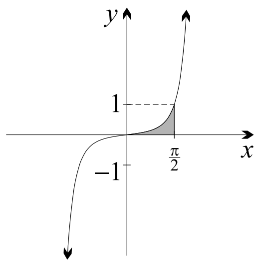
and when $\quad x = \tfrac{\pi}{2}\,, \qquad u = 1\,,$

thus $\quad I = \displaystyle\int_2^1 \frac{-1}{u^3}\, du$

$\quad = \left[\dfrac{1}{2u^2}\right]_2^1$

$\quad = \tfrac{1}{2} - \tfrac{1}{8}$

$\quad = \tfrac{3}{8}\,.$



The step where the limits are expressed in terms of the substitute variable is important. Had this step not been done then the wrong answer is obtained since

$$\int_0^{\frac{\pi}{2}} \frac{-1}{u^3}\, du = \left[\frac{1}{2u^2}\right]_0^{\frac{\pi}{2}}$$

which is undefined at the lower limit. Again notice that the derivative is treated like a fraction in the third line of the solution.

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Harder Examples: In simple examples like those above, candidates are expected to determine the appropriate substitution for themselves. In harder problems the substitution will be given. Implicit differentiation may also be required.

WORKED EXAMPLE 7: Use the substitution $u = \sqrt{x}$ to determine $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx\,.$

SOLUTION: Let $I = \displaystyle\int \frac{dx}{x+\sqrt{x}}$ and note that $u^2 = x$, so:

$$2u\,\frac{du}{dx} = 1$$

or $\quad 2u\,du = dx\,.$

Hence
$$\begin{aligned} I &= \int \frac{2u\,du}{u^2+u} \\ &= \int \frac{2\,du}{u+1} \\ &= 2\log(u+1) + C \\ &= 2\log(\sqrt{x}+1) + C\,. \end{aligned}$$

Take Care with Substitutions: There are many integrals which require a careful choice of substitution so as to avoid subsequent difficulties. For example, the correct choice of substitution in the previous worked example is $u = \sqrt{x}\,.$

On first inspection, it would seem to make no difference to make the alternate substitution $u^2 = x$, however observe what happens in the denominator.

$$x + \sqrt{x} = u^2 + \sqrt{u^2} = u^2 + |u|\,.$$

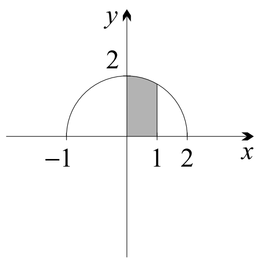
Thus in this case a new complication has been introduced, namely the absolute value function. In general, the best choice of substitution is of the form $u = f(x)$.

In some instances it is more natural to use a substitution of the form $x = g(u)$. Then the domain of u is restricted to avoid any later complication. This is often the case with trigonometric substitutions.

WORKED EXAMPLE 8: Evaluate $\displaystyle\int_0^1 \sqrt{4-x^2}\,dx$ using a suitable substitution.

SOLUTION: Let $I = \displaystyle\int_0^1 \sqrt{4-x^2}\ dx.$

The integrand is the upper semi-circle with radius 2,
so put $x = 2\sin\theta$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
This means that $\cos\theta \ge 0$.



Differentiating $\quad dx = 2\cos\theta\ d\theta\,.$

When $\quad x = 0\,,\quad \theta = 0\,,$

and when $\quad x = 1\,,\quad \theta = \sin^{-1}(\tfrac{1}{2}) = \tfrac{\pi}{6}\,.$

Thus
$$\begin{aligned} I &= \int_0^{\frac{\pi}{6}} 2\cos\theta\,\sqrt{4-4\sin^2\theta}\ d\theta \\ &= \int_0^{\frac{\pi}{6}} 4\cos\theta\,\sqrt{\cos^2\theta}\ d\theta \qquad \text{(by the Pythagorean identity)} \\ &= \int_0^{\frac{\pi}{6}} 4\cos^2\theta\ d\theta \qquad \text{(since } \cos\theta \ge 0\text{)} \end{aligned}$$

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$$= \int_0^{\frac{\pi}{6}} 2(1+\cos 2\theta)\,d\theta \qquad \text{(by the double-angle formula)}$$
$$= \Big[2\theta + \sin 2\theta\Big]_0^{\frac{\pi}{6}}$$
$$= \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}\,.$$

Notice that in the solution $\sqrt{\cos^2\theta} = |\cos\theta|$, but this is further simplified to $\cos\theta$ since it is known that $\cos\theta \ge 0$ in the specified domain. Further observe that the same outcome would have been achieved by substituting $\theta = \sin^{-1}\frac{x}{2}$, since the range of inverse sine is also $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. A trigonometric substitution should always be made in this manner, with the inverse function in mind.

Two Guidelines for Substitutions: The infinite variety of integrals that may be encountered makes it impractical to give a specific recipe for choosing the correct substitution. However the following two guidelines may help, and can be observed in practice in the previous worked examples.

- Try to replace the part of the integral which causes difficulty, such as the innermost function in a chain of functions. In particular, if the integral involves square-roots of sums or differences of squares then a trigonometric substitution is likely to work.
- It is better to use a substitution which is a function $u = f(x)$ rather than a relation $x = g(u)$. If a relation must be used then it is often best to restrict the domain in a similar manner to an inverse function.

Exercise **4C**

1. Find these integrals by the reverse chain rule, then do them again using a suitable substitution.

(a) $\displaystyle\int 2x(x^2+1)^4\,dx$ (c) $\displaystyle\int \frac{6x^2}{(1+x^3)^2}\,dx$ (e) $\displaystyle\int \frac{x}{\sqrt{x^2-2}}\,dx$

(b) $\displaystyle\int 3x^2(1+x^3)^6\,dx$ (d) $\displaystyle\int \frac{4x}{(3-x^2)^5}\,dx$ (f) $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

2. Use a suitable substitution where necessary to find:

(a) $\displaystyle\int \frac{\cos x}{\sin^3 x}\,dx$ (c) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$ (e) $\displaystyle\int \frac{x}{1+x^4}\,dx$

(b) $\displaystyle\int \frac{\sec^2 x}{(1+\tan x)^2}\,dx$ (d) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ (f) $\displaystyle\int \frac{x^2}{\sqrt{1-x^6}}\,dx$

3. Use a suitable substitution where necessary to evaluate:

(a) $\displaystyle\int_0^1 x^3(1+3x^4)^2\,dx$ (c) $\displaystyle\int_3^4 \frac{x+1}{\sqrt{x^2+2x+3}}\,dx$ (e) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\sec^2 x\,dx$

(b) $\displaystyle\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$ (d) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x\cos x\,dx$ (f) $\displaystyle\int_1^{e^2} \frac{\ln x}{x}\,dx$

DEVELOPMENT

4. (a) Use a suitable substitution to help evaluate $\displaystyle\int_0^1 x(x-1)^5\,dx$.

(b) How could this integral have been evaluated using just algebraic manipulation?

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5. Use the given substitution to find:

(a) $\displaystyle\int x\sqrt{x+1}\,dx$ [put $u=\sqrt{x+1}$]

(b) $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$ [put $u=1+\sqrt{x}$]

(c) $\displaystyle\int \frac{1}{1+x^{\frac{1}{4}}}\,dx$ [put $u=x^{\frac{1}{4}}$]

(d) $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$ [put $u=\sqrt{e^{2x}-1}$]

6. In each case, use the given substitution to evaluate the integral.

(a) $\displaystyle\int_0^1 \frac{2-x}{(2+x)^3}\,dx$ [put $u=2+x$]

(b) $\displaystyle\int_0^4 x\sqrt{4-x}\,dx$ [put $u=\sqrt{4-x}$]

(c) $\displaystyle\int_0^4 \frac{1}{5+\sqrt{x}}\,dx$ [put $u=\sqrt{x}$]

(d) $\displaystyle\int_4^{12} \frac{1}{(4+x)\sqrt{x}}\,dx$ [put $u=\sqrt{x}$]

7. In each case, use the given substitution to determine the primitive.

(a) $\displaystyle\int \frac{1}{(1+x)\sqrt{x}}\,dx$ [put $u=\sqrt{x}$]

(b) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$ [put $u=\sqrt{x+1}$]

8. In each case use the given trigonometric substitution to evaluate the integral. You may assume that $0\le\theta<\frac{\pi}{2}$.

(a) $\displaystyle\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$ [put $x=\tan\theta$]

(b) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ [put $x=2\sin\theta$]

(c) $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ [put $x=5\cos\theta$]

(d) $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$ [put $x=\tan\theta$]

9. (a) Use a suitable substitution to help evaluate $\displaystyle\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}}\,dx$.

(b) How could this integral have been evaluated using just algebraic manipulation?

10. (a) Use a suitable substitution to show that $\displaystyle\int_1^2 \sqrt{4-x^2}\,dx=\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$.

(b) Redo this problem by geometric means.

11. Let $\displaystyle I=\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx$.

(a) Use the substitution $u=\frac{\pi}{2}-x$ to show that $\displaystyle I=\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}\,dx$.

(b) By adding the two equal integrals, find the value of I.

12. (a) Show that $\displaystyle\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx=\frac{\pi}{2}$.

(b) Let $\displaystyle I=\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$.

(i) Use the substitution $u=\pi-x$ to show that $I=\frac{\pi^2}{2}-I$.

(ii) Hence evaluate I.

ENRICHMENT

13. (a) Use a trigonometric substitution to show that $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx=\frac{\pi}{12}-\frac{\sqrt{3}}{8}$.

(b) How could this integral have been evaluated using algebra then geometry?

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14. Consider the indefinite integral $I = \int \dfrac{dx}{x\sqrt{x^2-1}}$. Clearly the domain of the integrand is disjoint, being $x > 1$ or $x < -1$. Thus it seems appropriate to use a different substitution in each part of the domain.

(a) Find I for $x > 1$ by using the substitution $u = \sqrt{x^2-1}$.

(b) Find I for $x < -1$ by using the substitution $u = -\sqrt{x^2-1}$.

15. (a) Use a suitable substitution to determine $\displaystyle\int_{2+\epsilon}^{4} \frac{dx}{x^2\sqrt{x^2-4}}$, where $\epsilon > 0$.

(b) Take the limit of this result as $\epsilon \to 0^+$ and hence find $\displaystyle\int_{2}^{4} \frac{dx}{x^2\sqrt{x^2-4}}$.

4D Partial Fractions

In arithmetic, when given the sum of two fractions, the normal procedure is to combine them into a single fraction using the lowest common denominator. Thus

$$\frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

Unfortunately when the fractions are functions and integration is involved, this is exactly the wrong thing to do. Whilst it is true that

$$\frac{3}{x+2} + \frac{2}{x-1} = \frac{5x+1}{x^2+x-2},$$

when considering the corresponding integrals,

$$\int \frac{3}{x+2} + \frac{2}{x-1}\,dx = \int \frac{5x+1}{x^2+x-2}\,dx,$$

it should be clear that the left hand side is far simpler to determine than the right hand integral. So:

$$\begin{aligned}\int \frac{5x+1}{x^2+x-2}\,dx &= \int \frac{3}{x+2} + \frac{2}{x-1}\,dx \\ &= 3\log|x+2| + 2\log|x-1| + C.\end{aligned}$$

This example is typical of integrals of rational functions. It is easiest to first split the fraction into its simpler components. In mathematical terminology, the fraction is *decomposed into its partial fractions*.

A Theorem About Partial Fractions: Consider the rational function

$$\frac{P(x)}{A(x) \times B(x)},$$

where P, A and B are polynomials, with no common factors between any pair, and where $\deg P < \deg A + \deg B$. It is always possible to write

$$\frac{P(x)}{A(x) \times B(x)} = \frac{R_A(x)}{A(x)} + \frac{R_B(x)}{B(x)},$$

where the remainders R_A and R_B are polynomials with $\deg R_A < \deg A$ and $\deg R_B < \deg B$. However, the proof is beyond the scope of this course.

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Linear Factors: In the simplest examples, $A(x)$ and $B(x)$ are linear. Since the degrees of R_A and R_B are less, they must be constants, yet to be found.

WORKED EXAMPLE 9: (a) Decompose $\dfrac{x+1}{(x-1)(x+3)}$ into its partial fractions.

(b) Hence evaluate $\displaystyle\int_2^6 \frac{x+1}{(x-1)(x+3)}\,dx\,.$

SOLUTION: (a) Let $\dfrac{x+1}{(x-1)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+3}$, where A and B are unknown constants. Multiply this equation by $(x-1)(x+3)$ to get:

$$x+1 = A(x+3) + B(x-1)$$

or $$x+1 = (A+B)x + (3A-B)\,.$$

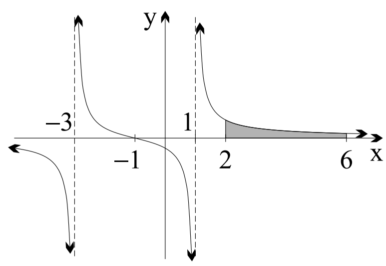
Equating coefficients of like powers of x yields the simultaneous equations

$$\begin{aligned} A+B &= 1 \\ 3A-B &= 1\,. \end{aligned}$$

These can be solved mentally to get $A = \frac{1}{2}$ and $B = \frac{1}{2}$. Thus

$$\frac{x+1}{(x-1)(x+3)} = \frac{(\frac{1}{2})}{x-1} + \frac{(\frac{1}{2})}{x+3}\,.$$

(b) Hence $$\begin{aligned} &\int_2^6 \frac{x+1}{(x-1)(x+3)}\,dx \\ &= \tfrac{1}{2}\int_2^6 \frac{1}{x-1} + \frac{1}{x+3}\,dx \\ &= \tfrac{1}{2}\Big[\log(x-1) + \log(x+3)\Big]_2^6 \\ &= \tfrac{1}{2}\Big((\log 5 + \log 9) - (\log 1 + \log 5)\Big) \\ &= \log 3\,. \end{aligned}$$



This method of equating coefficients of like powers of x is usually only convenient in straightforward examples like this one.

Finding the Constants by Substitution: A more general method of finding the unknown constants in partial fractions uses substitution. In many cases it is also a quicker method.

WORKED EXAMPLE 10: Decompose $\dfrac{3x-5}{(x-3)(x+1)}$ into partial fractions.

SOLUTION: Let $\dfrac{3x-5}{(x-3)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x+1}$, where A and B are unknown constants. Multiply this equation by $(x-3)(x+1)$ to get:

$$3x-5 = A(x+1) + B(x-3)\,.$$

When $x = 3$, $\quad 4 = 4A$

so $\quad A = 1\,.$

When $x = -1$, $\quad -8 = -4B$

so $\quad B = 2\,.$

Thus $$\frac{3x-5}{(x-3)(x+1)} = \frac{1}{x-3} + \frac{2}{x+1}\,.$$

The careful reader will have noticed a point of contention with the solution. The fraction is undefined when $x = 3$ and when $x = -1$, yet these values were used in the substitution steps. How can this be valid? The answer is that some of the detail of the solution has been omitted. Here is a more complete explanation.

Since $\quad \dfrac{3x-5}{(x-3)(x+1)} = \dfrac{A}{x-3} + \dfrac{B}{x+1}$ where $x \neq -1, 3$,

it follows that $\quad 3x - 5 = A(x+1) + B(x-3)$ where $x \neq -1, 3$.

Now this last equation is true whenever $x \neq -1, 3$. That is, it is a linear equation which is true for at least two other values of x. Hence it is an identity, and so it is true for all x, including $x = -1$ and $x = 3$. Thus these values can be substituted to determine A and B. It is not necessary to give this complete explanation as part of a solution, but students should be aware of it.

Numerators with Higher Degree: In slightly harder problems, the degree of the numerator is greater than or equal to the degree of the denominator. In such cases, the fraction should be expressed as a sum of a polynomial and the partial fractions. Long division may be used at this step, but it is often easier to use a polynomial with unknown coefficients, as in the following worked example.

WORKED EXAMPLE 11: Determine $\displaystyle\int \frac{x^3+x-3}{x^2-3x+2}\,dx$.

SOLUTION: First note that $\dfrac{x^3+x-3}{x^2-3x+2} = \dfrac{x^3+x-3}{(x-2)(x-1)}$,

so let $\quad \dfrac{x^3+x-3}{(x-2)(x-1)} = Ax + B + \dfrac{C}{x-2} + \dfrac{D}{x-1}$,

thus $\quad x^3 + x - 3 = (Ax+B)(x-2)(x-1) + C(x-1) + D(x-2)$.

Equating the coefficients of x^3, $A = 1$.

At $x = 1$ $\quad -1 = -D$ so $D = 1$.

At $x = 2$ $\quad 7 = C$.

At $x = 0$ $\quad -3 = 2B - 7 - 2$

so $\quad B = 3$.

Finally
$$\int \frac{x^3+x-3}{x^2-3x+2}\,dx = \int x + 3 + \frac{7}{x-2} + \frac{1}{x-1}\,dx$$
$$= \tfrac{1}{2}x^2 + 3x + 7\log|x-2| + \log|x-1| + C\,.$$

The Cover-up Rule: There is an even quicker method to determine the constants of the partial fractions, provided that the original denominator is a product of distinct linear factors, and provided that the degree of the numerator is less than the degree of the denominator. The trick is to multiply by just one linear factor at a time.

WORKED EXAMPLE 12: Express $\dfrac{7-5x}{(x+1)(x-2)(x-3)}$ in partial fractions form.

SOLUTION: Let
$$\frac{7-5x}{(x+1)(x-2)(x-3)} = \frac{C_1}{x+1} + \frac{C_2}{x-2} + \frac{C_3}{x-3}\,. \qquad (*)$$

$(*) \times (x+1)$ gives
$$\frac{7-5x}{(x-2)(x-3)} = C_1 + \frac{C_2(x+1)}{x-2} + \frac{C_3(x+1)}{x-3}$$

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so at $x = -1$ $\qquad C_1 = \dfrac{12}{(-3)(-4)} = 1\,.$

$(*) \times (x-2)$ gives $\qquad \dfrac{7-5x}{(x+1)(x-3)} = \dfrac{C_1(x-2)}{x+1} + C_2 + \dfrac{C_3(x-2)}{x-3}$

so at $x = 2$ $\qquad C_2 = \dfrac{-3}{3 \times (-1)} = 1\,.$

Finally $\qquad \dfrac{7-5x}{(x+1)(x-2)} = \dfrac{C_1(x-3)}{x+1} + \dfrac{C_2(x-3)}{x-2} + C_3$

so at $x = 3$ $\qquad C_3 = \dfrac{-8}{4 \times 1} = -2\,.$

Hence $\qquad \dfrac{7-5x}{(x+1)(x-2)(x-3)} = \dfrac{1}{x+1} + \dfrac{1}{x-2} - \dfrac{2}{x-3}\,.$

This method of finding the constants is sometimes called the *cover-up rule*. Look carefully at how the three constants are determined. For each constant, the matching linear factor is effectively omitted, or "covered up". Thus for C_1, $(x+1)$ is left out of the original fraction. For C_2, $(x-2)$ is excluded, and for C_3, $(x-3)$ is omitted from the original fraction. In each case, the resulting rational function is then evaluated at the corresponding value of x. With practice, most students should be able to determine the constants mentally using this method.

Proof of the Cover-up Rule — Extension: Here is a proof for the general case.

PROOF: Consider the rational function $\dfrac{P(x)}{Q(x)}$ where $\deg P < \deg Q$, and where $Q(x)$ is a product of distinct linear factors, that is

$$\begin{aligned} Q(x) &= C \times (x-a_1) \times (x-a_2) \times \ldots \times (x-a_n) \\ &= C \prod_{i=1}^{n} (x-a_i) \qquad \text{(note the use of product notation, } \textstyle\prod.) \end{aligned}$$

Let $\qquad \dfrac{P(x)}{Q(x)} = \dfrac{C_1}{x-a_1} + \dfrac{C_2}{x-a_2} + \ldots + \dfrac{C_k}{x-a_k} + \ldots + \dfrac{C_n}{x-a_n}$

Multiply this last equation by $(x-a_k)$ to get

$$\frac{P(x)(x-a_k)}{Q(x)} = \frac{C_1(x-a_k)}{x-a_1} + \frac{C_2(x-a_k)}{x-a_2} + \ldots + C_k + \ldots + \frac{C_n(x-a_k)}{x-a_n}\,.$$

Now take the limit as $x \to a_k$. All terms except C_k on the right hand side are zero and so:

$$\begin{aligned} C_k &= \lim_{x \to a_k} \frac{P(x)(x-a_k)}{Q(x)} \\ &= \lim_{x \to a_k} \frac{P(x)}{C \prod\limits_{\substack{i=1 \\ i \neq k}}^{n} (x-a_i)} \qquad \text{(that is, cancel the } k\text{th linear factor)} \end{aligned}$$

hence $\qquad C_k = \dfrac{P(a_k)}{C \prod\limits_{\substack{i=1 \\ i \neq k}}^{n} (a_k-a_i)}\,.$

The mathematical notation may seem difficult, but the result is exactly as before. To get the kth coefficient C_k, omit the kth linear factor from the denominator and evaluate the rest of the fraction at $x = a_k$.

Quadratic Factors: In certain instances, the denominator of the rational function being considered will have a quadratic factor with no real zero. For example, in

$$\frac{3x+10}{(x-2)(x^2+4)}$$

the quadratic factor (x^2+4) has no real zero. Thus the denominator of the rational function cannot be expressed as a product of real linear factors.

Nevertheless, the method for finding the partial fraction decomposition remains essentially the same. And since the only requirement is that the degree of the numerator is less than the degree of the denominator, it follows that for any quadratic factor the numerator can be a linear polynomial.

WORKED EXAMPLE 13: (a) Rewrite $\dfrac{3x+10}{(x-2)(x^2+4)}$ in its partial fractions.

(b) Hence determine $\displaystyle\int \frac{3x+10}{(x-2)(x^2+4)}\,dx$.

SOLUTION: (a) Let $\dfrac{3x+10}{(x-2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4}$, where A, B and C are unknown constants. Then

$$3x+10 = A(x^2+4) + (Bx+C)(x-2)$$

At $x=2$ $\quad 16 = 8A \quad$ so $\quad A = 2$.

Equating coefficients of x^2 yields

$$0 = 2 + B \quad \text{so} \quad B = -2\,.$$

At $x=0$ $\quad 10 = 8 - 2C$

so $\quad C = -1$.

Thus $\quad \dfrac{3x+10}{(x-2)(x^2+4)} = \dfrac{2}{x-2} - \dfrac{2x+1}{x^2+4}$.

(b) Hence $\displaystyle\int \frac{3x+10}{(x-2)(x^2+4)}\,dx = \int \frac{2}{x-2} - \frac{2x}{x^2+4} - \frac{1}{x^2+4}\,dx$

$$= 2\log|x-2| - \log|x^2+4| - \tfrac{1}{2}\tan^{-1}(\tfrac{x}{2}) + C\,.$$

2 PARTIAL FRACTIONS: Here is a summary of the techniques in this section.

- Make each numerator have degree one less than its denominator.
- Use the cover up rule when there are distinct linear factors.
- Substitution, simultaneous equations or equating coefficients may also be used to determine unknown constants.
- Use a polynomial with unknown coefficients or long division for numerators with higher degree.

Repeated Factors: Recall that a polynomial factor which has degree greater than one is called a repeated factor. For example in the denominator of the fraction

$$\frac{8-x}{(x-2)^2(x+1)},$$

the factor $(x-2)^2$ is a repeated factor since its index is two. When a partial fraction question involves repeated factors, normally the initial decomposition is given in the question and it is simply a matter of finding the values of the unknown constants.

WORKED EXAMPLE 14: (a) Find the real numbers A, B and C such that

$$\frac{8-x}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}\,.$$

(b) Hence evaluate $\displaystyle\int_0^1 \frac{8-x}{(x-2)^2(x+1)}\,dx\,.$

SOLUTION:

(a) Now $\quad 8-x = A(x-2)(x+1) + B(x+1) + C(x-2)^2\,.$

At $x=-1$ $\quad 9 = 9C \quad$ so $\quad C = 1\,.$

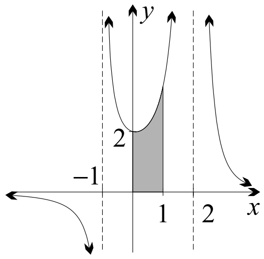
At $x=2$ $\quad 6 = 3B \quad$ so $\quad B = 2\,.$

At $x=3$ $\quad 5 = 4A + 8 + 1$

so $\quad A = -1\,.$

(b) Hence

$$\begin{aligned}
&\int_0^1 \frac{8-x}{(x-2)^2(x+1)}\,dx \\
&= \int_0^1 \frac{1}{x+1} + \frac{2}{(x-2)^2} - \frac{1}{x-2}\,dx \\
&= \left[\log(x+1) - \frac{2}{x-2} - \log|x-2|\right]_0^1 \\
&= (\log 2 + 2 - \log 1) - (\log 1 + 1 - \log 2) \\
&= 1 + 2\log 2\,.
\end{aligned}$$



Exercise **4D**

1. Decompose the following fractions into partial fractions.

(a) $\dfrac{2}{(x-1)(x+1)}$ (c) $\dfrac{4x}{x^2-9}$ (e) $\dfrac{x-1}{x^2+x-6}$

(b) $\dfrac{1}{(x-4)(x-1)}$ (d) $\dfrac{x}{x^2-3x+2}$ (f) $\dfrac{3x+1}{(x-1)(x^2+3)}$

2. Find:

(a) $\displaystyle\int \frac{2}{(x-4)(x-2)}\,dx$ (c) $\displaystyle\int \frac{3x-2}{(x-1)(x-2)}\,dx$ (e) $\displaystyle\int \frac{4x+5}{(2x+3)(x+1)}\,dx$

(b) $\displaystyle\int \frac{4}{x^2+4x+3}\,dx$ (d) $\displaystyle\int \frac{2x+10}{x^2+2x-3}\,dx$ (f) $\displaystyle\int \frac{10x}{2x^2-x-3}\,dx$

3. Evaluate:

(a) $\displaystyle\int_4^6 \frac{1}{x^2-4}\,dx$ (c) $\displaystyle\int_2^5 \frac{11}{2x^2+5x-12}\,dx$

(b) $\displaystyle\int_2^4 \frac{3}{x^2+x-2}\,dx$ (d) $\displaystyle\int_{-1}^0 \frac{1}{3x^2-4x+1}\,dx$

4. Determine:

(a) $\displaystyle\int \frac{x^2-2x+5}{(x-2)(x^2+1)}\,dx$ (b) $\displaystyle\int \frac{6-x}{(2x+1)(x^2+3)}\,dx$ (c) $\displaystyle\int \frac{x^2+x+3}{x^3+x}\,dx$

5. Find the value of:

(a) $\displaystyle\int_0^{\frac{1}{2}} \frac{1+2x-4x^2}{(x+1)(4x^2+1)}\,dx$ (b) $\displaystyle\int_{-1}^1 \frac{7-x}{(x+3)(x^2+1)}\,dx$ (c) $\displaystyle\int_1^{\sqrt{2}} \frac{x^2-4}{x^3+2x}\,dx$

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DEVELOPMENT

6. Find:

(a) $\displaystyle\int \frac{2x+3}{(x-1)(x-2)(2x-3)}\,dx$ (b) $\displaystyle\int \frac{4x+12}{x^3-6x^2+8x}\,dx$

7. Evaluate:

(a) $\displaystyle\int_2^7 \frac{3x+5}{(x-1)(x+2)(x+1)}\,dx$ (b) $\displaystyle\int_1^2 \frac{13x+6}{x^3-x^2-6x}\,dx$

8. (a) (i) Let $\dfrac{2x^2+1}{(x-1)(x+2)} = A + \dfrac{B}{x-1} + \dfrac{C}{x+2}$. Find the values of A, B and C.

(ii) Hence find $\displaystyle\int \frac{2x^2+1}{(x-1)(x+2)}\,dx$

(b) Use a similar technique to part (a) in order to find:

(i) $\displaystyle\int \frac{x^2-2x+3}{(x+1)(x-2)}\,dx$ (ii) $\displaystyle\int \frac{3x^2-66}{(x+4)(x-5)}\,dx$

9. (a) (i) Find the values of A, B, C and D such that

$$\frac{x^3-3x^2-4}{(x+1)(x-3)} = Ax + B + \frac{C}{x+1} + \frac{D}{x-3}\,.$$

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{x^3-3x^2-4}{(x+1)(x-3)}\,dx\,.$

(b) Use a similar method to evaluate $\displaystyle\int_2^4 \frac{x^3+4x^2+x-3}{(x+2)(x-1)}\,dx\,.$

10. (a) (i) Find the values of A, B and C such that

$$\frac{3x^2-10}{x^2-4x+4} = A + \frac{B}{x-2} + \frac{C}{(x-2)^2}\,.$$

(ii) Hence find $\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx.$

(b) (i) Find the integers A, B, C and D such that

$$\frac{3x+7}{(x-1)^2(x-2)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}\,.$$

(ii) Hence find $\displaystyle\int \frac{3x+7}{(x-1)^2(x-2)^2}\,dx\,.$

11. Show that:

(a) $\displaystyle\int_4^6 \frac{x^2-8}{x^3+4x}\,dx = \tfrac{3}{2}\ln 2 - 2\ln\tfrac{3}{2}.$ (b) $\displaystyle\int_0^2 \frac{1+4x}{(4-x)(x^2+1)}\,dx = \tfrac{1}{2}\ln 20.$

12. (a) Let $\dfrac{x^2-1}{x^4+x^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1}$. Find A, B, C and D.

(b) Hence show that $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{x^2-1}{x^4+x^2}\,dx = \tfrac{1}{3}(\pi - 2\sqrt{3}).$

13. Use appropriate methods to find:

(a) $\displaystyle\int \frac{x^2+1}{x^2-1}\,dx$ (c) $\displaystyle\int \frac{x^3+1}{x^3+x}\,dx$ (e) $\displaystyle\int \frac{x^3+5}{x^2+x}\,dx$

(b) $\displaystyle\int \frac{x^2+1}{x^2-x}\,dx$ (d) $\displaystyle\int \frac{x^2}{x^2-5x+6}\,dx$ (f) $\displaystyle\int \frac{x^4}{x^2-3x+2}\,dx$

ENRICHMENT

14. Use a similar approach to Question 10 for repeated factors to show that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{5x - x^2}{(x+1)^2(x-1)}\, dx = 4 - 3\ln 3\,.$$

15. (a) In the notation of the text, if $Q(x)$ is a product of distinct linear factors, one of which is $(x - a_k)$, then $C_k = \lim\limits_{x \to a_k} \dfrac{P(x)(x - a_k)}{Q(x)}$. Use this result to prove that

$$C_k = \frac{P(a_k)}{Q'(a_k)}\,.$$

[HINT: What is the value of $Q(a_k)$?]

(b) Use this formula to redo Questions 6(b) and 7(b).

4E Denominators with Quadratics

Many practical applications yield integrals with a quadratic in the denominator. In the simplest cases it is a matter of applying the following four results:

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \qquad \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \ln \left| x + \sqrt{x^2 - a^2} \right|$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a} \qquad \int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

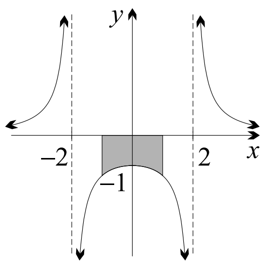
Another common integral is $\displaystyle\int \frac{dx}{x^2 - a^2}$. Although a formula exists for this, it is not part of the course. It is expected that candidates determine the primitive by use of partial fractions whenever this type of integral is encountered.

WORKED EXAMPLE 15: Evaluate $\displaystyle\int_{-1}^{1} \frac{4}{x^2 - 4}\, dx$.

SOLUTION: Now $\dfrac{4}{x^2 - 4} = \dfrac{4}{(x-2)(x+2)}$,

so let $\dfrac{4}{(x-2)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$.

Then by the cover-up rule $A = 1$ and $B = -1$.



$$\begin{aligned}
\text{Hence } \int_{-1}^{1} \frac{4}{x^2 - 4}\, dx &= \int_{-1}^{1} \frac{1}{x-2} - \frac{1}{x+2}\, dx \\
&= \Big[\log|x-2| - \log(x+2) \Big]_{-1}^{1} \\
&= (\log 1 - \log 3) - (\log 3 - \log 1) \\
&= -2\log 3\,.
\end{aligned}$$

Quadratics with Linear Terms: Frequently the quadratic will have a linear term, such as in $3 + 2x - x^2$. In these instances the method is to complete the square to obtain either the sum of two squares or the difference of two squares.

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WORKED EXAMPLE 16: Find $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$.

SOLUTION: Completing the square in the denominator:

$$\begin{aligned}\int \frac{1}{\sqrt{3+2x-x^2}}\,dx &= \int \frac{1}{\sqrt{4-(x-1)^2}}\,dx\\ &= \int \frac{1}{\sqrt{4-u^2}}\,du \qquad \text{where } u = x-1\\ &= \sin^{-1}\tfrac{u}{2} + C\\ &= \sin^{-1}\tfrac{x-1}{2} + C\,.\end{aligned}$$

Notice that the solution uses a substitution. This step may be omitted by using a result from the Mathematics Extension 1 course. Recall that if $F(x)$ is a primitive of $f(x)$ then

$$\int f(ax+b)\,dx = \tfrac{1}{a}F(ax+b) + C\,.$$

The result is a combination of shifting and stretching along the x-axis. A shift does not affect the area under a graph, but a stretch does, hence the factor $\frac{1}{a}$.

In Worked Example 16 above, $f(x) = \dfrac{1}{\sqrt{4-x^2}}$, the primitive is $F(x) = \sin^{-1}\frac{x}{2}$, with $a = 1$ and $b = 1$. Thus it is permissible to write

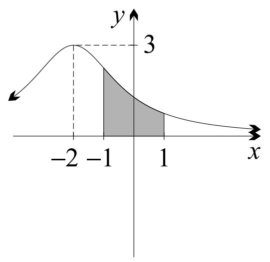
$$\int \frac{1}{\sqrt{4-(x-1)^2}}\,dx = \sin^{-1}\tfrac{x-1}{2} + C\,,$$

without showing any working. Here is a similar example.

WORKED EXAMPLE 17: Find the value of $\displaystyle\int_{-1}^{1} \frac{9}{7+4x+x^2}\,dx$.

SOLUTION: Completing the square in the denominator:

$$\begin{aligned}\int_{-1}^{1} \frac{9}{7+4x+x^2}\,dx &= \int_{-1}^{1} \frac{9}{3+(4+4x+x^2)}\,dx\\ &= \int_{-1}^{1} \frac{9}{3+(2+x)^2}\,dx\\ &= \tfrac{9}{\sqrt{3}}\Big[\tan^{-1}\tfrac{x+2}{\sqrt{3}}\Big]_{-1}^{1}\\ &= 3\sqrt{3}\left(\tfrac{\pi}{3}-\tfrac{\pi}{6}\right)\\ &= \tfrac{\pi\sqrt{3}}{2}\,.\end{aligned}$$



3	QUADRATICS WITH LINEAR TERMS: Complete the square, then use the result $$\int f(ax+b)\,dx = \tfrac{1}{a}F(ax+b) + C\,,$$ where $F(x)$ is the primitive of $f(x)$.

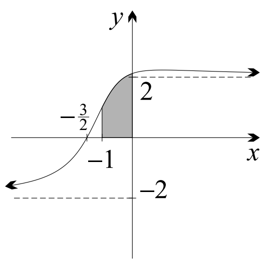
Linear Numerators: So far in all the worked examples the numerator has been a constant. When the numerator is linear it is best to carefully split it into two parts. The first term should be a multiple of the derivative of the quadratic in the denominator. The second term will then be a constant.

Worked Example 18: Determine $\displaystyle\int \frac{4x+3}{x^2+9}\,dx$.

Solution:
$$\begin{aligned}\int \frac{4x+3}{x^2+9}\,dx &= 2\int \frac{2x}{x^2+9}\,dx + \int \frac{3}{x^2+9}\,dx\\ &= 2\log(x^2+9) + \tan^{-1}\tfrac{x}{3} + C\,.\end{aligned}$$

In harder examples the quadratic will also contain a linear term. The following is such an example and requires the last integral formula in the appendix.

Worked Example 19: Evaluate $\displaystyle\int_{-1}^{0} \frac{2x+3}{\sqrt{x^2+2x+2}}\,dx$.



Solution:
$$\begin{aligned}&\int_{-1}^{0} \frac{2x+3}{\sqrt{x^2+2x+2}}\,dx\\ &= \int_{-1}^{0} \frac{2x+2}{\sqrt{x^2+2x+2}}\,dx + \int_{-1}^{0} \frac{1}{\sqrt{(x+1)^2+1}}\,dx\\ &= \Big[2\sqrt{x^2+2x+2}\Big]_{-1}^{0} + \Big[\log\big((x+1)+\sqrt{(x+1)^2+1}\,\big)\Big]_{-1}^{0}\\ &= 2\sqrt{2} - 2 + \log(1+\sqrt{2}) - \log 1\\ &= 2(\sqrt{2}-1) + \log(1+\sqrt{2})\,.\end{aligned}$$

4 | Linear numerators: When the numerator is linear it is best to split it into a multiple of the derivative of the quadratic in the denominator plus a constant.

Rationalising the Numerator: In much previous work it has been convenient to rationalise the denominator when a surd appears. In contrast, when calculus is involved it is often more convenient to rationalise the numerator instead.

Worked Example 20: Find $\displaystyle\int \sqrt{\frac{x+1}{x+7}}\,dx$.

Solution: Rationalising the numerator
$$\begin{aligned}\int \sqrt{\frac{x+1}{x+7}}\,dx &= \int \frac{x+1}{\sqrt{x^2+8x+7}}\,dx\\ &= \int \frac{x+4}{\sqrt{x^2+8x+7}}\,dx - \int \frac{3}{\sqrt{x^2+8x+7}}\,dx\\ &= \int \frac{x+4}{\sqrt{x^2+8x+7}}\,dx - \int \frac{3}{\sqrt{(x+4)^2-3^2}}\,dx\\ &= \sqrt{x^2+8x+7} - 3\log\Big|(x+4)+\sqrt{(x+4)^2-3^2}\,\Big| + C\,.\end{aligned}$$

Notice that in the first line of working, by rationalising, the numerator has become linear. This is typical of the questions done in this section.

5 | Rationalising the numerator: When calculus is involved it is often convenient to rationalise the numerator.

Exercise **4E**

NOTE: Two further standard integrals are required in this exercise:

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right) \qquad \text{and} \qquad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right|$$

1. Find these integrals. Parts (e) and (f) require the two new standard integrals above.

(a) $\displaystyle\int \frac{1}{9+x^2}\,dx$ (c) $\displaystyle\int \frac{1}{x^2-9}\,dx$ (e) $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx$

(b) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$ (d) $\displaystyle\int \frac{1}{9-x^2}\,dx$ (f) $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$

2. Determine the following. Parts (e) and (f) require the two new standard integrals above.

(a) $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$ (c) $\displaystyle\int \frac{1}{\sqrt{9+8x-x^2}}\,dx$ (e) $\displaystyle\int \frac{1}{\sqrt{x^2-6x+13}}\,dx$

(b) $\displaystyle\int \frac{1}{x^2-4x+20}\,dx$ (d) $\displaystyle\int \frac{1}{\sqrt{20-8x-x^2}}\,dx$ (f) $\displaystyle\int \frac{1}{\sqrt{4x^2+8x+6}}\,dx$

3. Evaluate the following. Parts (e) and (f) require the two new standard integrals above.

(a) $\displaystyle\int_1^3 \frac{1}{x^2-2x+5}\,dx$ (c) $\displaystyle\int_{-1}^0 \frac{1}{\sqrt{3-2x-x^2}}\,dx$ (e) $\displaystyle\int_{-1}^3 \frac{1}{\sqrt{x^2+2x+10}}\,dx$

(b) $\displaystyle\int_1^5 \frac{4}{x^2-6x+13}\,dx$ (d) $\displaystyle\int_0^1 \frac{3}{\sqrt{3+4x-4x^2}}\,dx$ (f) $\displaystyle\int_{\frac{1}{2}}^1 \frac{2}{\sqrt{x^2-x+1}}\,dx$

DEVELOPMENT

4. Find:

(a) $\displaystyle\int \frac{2x+1}{x^2+2x+2}\,dx$ (c) $\displaystyle\int \frac{x}{\sqrt{6x-x^2}}\,dx$ (e) $\displaystyle\int \frac{x}{\sqrt{x^2+2x+10}}\,dx$

(b) $\displaystyle\int \frac{x}{x^2+2x+10}\,dx$ (d) $\displaystyle\int \frac{x+3}{\sqrt{4-2x-x^2}}\,dx$ (f) $\displaystyle\int \frac{x+3}{\sqrt{x^2-2x-4}}\,dx$

5. Find the value of:

(a) $\displaystyle\int_0^2 \frac{x+1}{x^2+4}\,dx$ (c) $\displaystyle\int_1^2 \frac{2x-3}{x^2-2x+2}\,dx$ (e) $\displaystyle\int_{-1}^3 \frac{1-2x}{\sqrt{x^2+2x+3}}\,dx$

(b) $\displaystyle\int_1^2 \frac{x+1}{x^2-4x+5}\,dx$ (d) $\displaystyle\int_{-1}^0 \frac{x}{\sqrt{3-2x-x^2}}\,dx$ (f) $\displaystyle\int_0^1 \frac{x+3}{\sqrt{x^2+4x+1}}\,dx$

6. Determine each primitive.

(a) $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx$ (b) $\displaystyle\int \sqrt{\frac{3-x}{2+x}}\,dx$ (c) $\displaystyle\int \sqrt{\frac{x-1}{x+1}}\,dx$

7. Evaluate:

(a) $\displaystyle\int_{-1}^0 \sqrt{\frac{1-x}{x+3}}\,dx$ (b) $\displaystyle\int_{-1}^0 \sqrt{\frac{x+2}{1-x}}\,dx$ (c) $\displaystyle\int_0^1 \sqrt{\frac{x+1}{x+3}}\,dx$

ENRICHMENT

8. (a) Why is it not valid to evaluate $\displaystyle\int_0^2 \sqrt{\frac{x}{4-x}}\,dx$ using the techniques of this section?

(b) Nevertheless, show that its value is $\displaystyle\lim_{\epsilon\to 0^+}\int_\epsilon^2 \sqrt{\frac{x}{4-x}}\,dx = \pi - 2\,.$

9. (a) Show that $x^3+3x^2+5x+1=(x+1)(x^2+2x+2)+(x-1)$.
(b) Hence or otherwise show that

$$\int_{-1}^{0}\frac{x^3+3x^2+5x+1}{\sqrt{x^2+2x+2}}\,dx=\tfrac{1}{3}(5\sqrt{2}-4)-2\ln(1+\sqrt{2})\,.$$

4F Integration by Parts

Whilst there are well known and relatively simple formulae for the derivatives of products and quotients of functions, there are no such general formulae for the integrals of products and quotients. Nevertheless, as was found in the previous two sections, certain quotients can be integrated relatively easily. In this section, a method of integration is developed that can be applied to certain types of products. It begins with the product rule for differentiation.

Now $$\frac{d}{dx}(uv)=u'v+u\,v'\,.$$

Swapping sides and integrating yields

$$\int u'v\,dx+\int u\,v'\,dx=uv\,,$$

hence $$\int u\,v'\,dx=uv-\int u'v\,dx\,.$$

This last equation provides a way to rearrange an integral of one product into an integral of a different product. The formula is applied with the aim that the new integral is in some way simpler. The process is called *integration by parts.*

WORKED EXAMPLE 21: Use integration by parts to find $\int xe^x\,dx$.

SOLUTION:

Let $I=\int xe^x\,dx$

$=\int u\,v'\,dx$,

where $u=x$ and $v'=e^x$

so $u'=1$ and $v=e^x$.

Hence $I=uv-\int u'v\,dx$

$=xe^x-\int e^x\,dx$

$=xe^x-e^x+C$

or $I=e^x(x-1)+C$.

Notice the lack of any constant of integration until the process is finished.

6

INTEGRATION BY PARTS: The integral of the product $u\,v'$ can be rearranged using integration by parts, viz:

$$\int u\,v'\,dx=uv-\int u'v\,dx\,.$$

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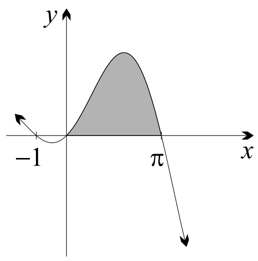
Reducing Polynomials: When one of the factors of the integrand is a polynomial, it is common to let u be that polynomial. In that way the new integral, which depends on u', will contain a polynomial of lesser degree. That is, the aim is to reduce the degree of the polynomial.

WORKED EXAMPLE 22: Evaluate $\displaystyle\int_0^{\pi} (x+1)\sin x\, dx$.



SOLUTION:

Let $\displaystyle I = \int_0^{\pi} (x+1)\sin x\, dx$

$\displaystyle\quad = \int_0^{\pi} u\,v'\, dx\,,$

where $u = (x+1)$ and $v' = \sin x$

so $u' = 1$ and $v = -\cos x$.

Thus $\displaystyle I = \Big[uv\Big]_0^{\pi} - \int_0^{\pi} u'v\, dx$

$\displaystyle\quad = \Big[-(x+1)\cos x\Big]_0^{\pi} + \int_0^{\pi} \cos x\, dx$

$\displaystyle\quad = (\pi+1) + 1 + \Big[\sin x\Big]_0^{\pi},$

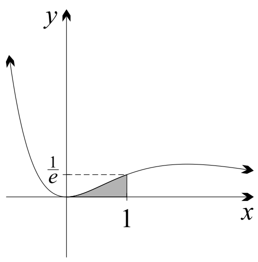
hence $I = \pi + 2$.

7	REDUCING POLYNOMIALS: Integration by parts may be used to reduce the degree of a polynomial. Let u be that polynomial and v' the other factor.

Repeated Applications: It may be necessary to apply integration by parts more than once in order to complete the process of integration. In simpler examples it may be possible to do some of the steps mentally.

WORKED EXAMPLE 23: Evaluate $\displaystyle\int_0^{1} x^2e^{-x}\, dx$.

SOLUTION:



Let $\displaystyle I = \int_0^1 x^2e^{-x}\, dx$

and put $u = x^2$ and $v' = e^{-x}$

so $u' = 2x$ and $v = -e^{-x}$.

Then $\displaystyle I = \Big[-x^2e^{-x}\Big]_0^1 + \int_0^1 2xe^{-x}\, dx$ (by parts.)

The second term is another integral of a product.

So put $u = 2x$ and $v' = e^{-x}$

with $u' = 2$ and $v = -e^{-x}$.

Thus $\displaystyle I = -e^{-1} + \left(\Big[-2xe^{-x}\Big]_0^1 + \int_0^1 2e^{-x}\, dx\right)$ (by parts again)

$\displaystyle\quad = -e^{-1} - 2e^{-1} - \Big[2e^{-x}\Big]_0^1$

$\quad = 2 - 5e^{-1}$.

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Exceptions with Polynomials: Although it is common to reduce the degree of a polynomial using integration by parts, there are many exceptions. In this course these exceptions typically involve the logarithm function.

Worked Example 24: Determine $\displaystyle\int x \log x \, dx$.

Solution:

Let $\displaystyle I = \int x \log x \, dx$

and put $u = \log x$ and $v' = x$

so $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$.

$$\begin{aligned}\text{Thus } I &= \tfrac{1}{2}x^2 \log x - \int \tfrac{1}{2}x^2 \times \tfrac{1}{x}\, dx \qquad \text{(by parts)}\\ &= \tfrac{1}{2}x^2 \log x - \int \tfrac{1}{2}x \, dx\\ &= \tfrac{1}{2}x^2 \log x - \tfrac{1}{4}x^2 + C\end{aligned}$$

or $I = \frac{1}{4}x^2(2\log x - 1) + C$.

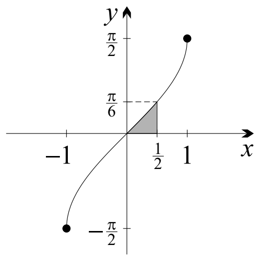
Integrands where $v' = 1$: The prime number 5 has only two distinct factors, namely 1×5. A function may be treated like a prime in a similar way:

$$\sin^{-1} x = 1 \times \sin^{-1} x\,.$$

This somewhat artificial form of factoring is applied to facilitate integration by parts. It is then usual to put u equal to the function and $v' = 1$.

Worked Example 25: Find the value of $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1} x \, dx$.

Solution:



$$\begin{aligned}\text{Let } I &= \int_0^{\frac{1}{2}} \sin^{-1} x \, dx\\ &= \int_0^{\frac{1}{2}} 1 \times \sin^{-1} x \, dx\,.\end{aligned}$$

Put $u = \sin^{-1} x$ and $v' = 1$

so $u' = \dfrac{1}{\sqrt{1-x^2}}$ and $v = x$.

$$\begin{aligned}\text{Thus } I &= \Big[x \sin^{-1} x\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\, dx \qquad \text{(by parts)}\\ &= \Big[x\sin^{-1} x + \sqrt{1-x^2}\,\Big]_0^{\frac{1}{2}}\\ &= \left(\tfrac{1}{2} \times \tfrac{\pi}{6} + \sqrt{\tfrac{3}{4}}\right) - (0+1)\\ &= \tfrac{\pi}{12} + \tfrac{\sqrt{3}}{2} - 1\,.\end{aligned}$$

The careful reader will have seen that there is a much simpler way to do this integral. The key is in the diagram. The area shaded is the difference between the areas of the rectangle, width $\frac{1}{2}$ and height $\frac{\pi}{6}$, and the unshaded portion. That unshaded portion involves a very simple integral along the y-axis.

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This example highlights the importance of a diagram to see the most efficient solution. In this case, it would be better to integrate along the y-axis than to use integration by parts. It is left as an exercise to show the result is the same.

A Recurrence of the Integral: Integration by parts may lead to a recurrence of the original integral. It is then simply a matter of collecting like terms.

WORKED EXAMPLE 26: Find a primitive of $e^x \sin x$.

SOLUTION:

Let $\quad I = \int e^x \sin x \, dx$

and put $\quad u = \sin x \quad$ and $\quad v' = e^x$

so $\quad u' = \cos x \quad$ and $\quad v = e^x$.

Then $\quad I = e^x \sin x - \int e^x \cos x \, dx \qquad$ (by parts)

Now put $\quad u = \cos x \quad$ and $\quad v' = e^x$

so $\quad u' = -\sin x \quad$ and $\quad v = e^x$.

Thus $\quad I = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) \qquad$ (by parts again)

$\quad = e^x(\sin x - \cos x) - I$

or $\quad 2I = e^x(\sin x - \cos x)$

hence $\quad I = \frac{1}{2}e^x(\sin x - \cos x) + C \quad$ is the general primitive.

In this example it was important to apply the method consistently. Notice that u was always the trigonometric function and v' was always the exponential function. As an exercise to highlight the significance of these choices, repeat the worked exercise but put $u = e^x$ and $v' = \cos x$ at the second integration by parts.

As a final note, there is no constant of integration in the second last line of the solution to Worked Example 26, yet the last line includes a constant. There is nothing to be alarmed about here. The process has led to a specific primitive $I = \frac{1}{2}e^x(\sin x - \cos x)$ and, as with any indefinite integral, the general primitive is then obtained by simply adding a constant at the last step.

Exercise **4F**

1. Find:

(a) $\int xe^x \, dx$ (c) $\int (x+1)e^{3x} \, dx$ (e) $\int (x-1)\sin 2x \, dx$

(b) $\int xe^{-x} \, dx$ (d) $\int x \cos x \, dx$ (f) $\int (2x-3)\sec^2 x \, dx$

2. Evaluate:

(a) $\int_0^{\pi} x \sin x \, dx$ (c) $\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$ (e) $\int_0^1 (1-x)e^{-x} \, dx$

(b) $\int_0^{\frac{\pi}{2}} x \cos x \, dx$ (d) $\int_0^1 xe^{2x} \, dx$ (f) $\int_{-2}^0 (x+2)e^x \, dx$

3. Use integration by parts with $v' = 1$ to find:

(a) $\int \ln x \, dx$ (b) $\int \ln(x^2) \, dx$ (c) $\int \cos^{-1} x \, dx$

4. Find the value of:
(a) $\displaystyle\int_0^1 \tan^{-1} x\,dx$ (b) $\displaystyle\int_1^e \ln x\,dx$ (c) $\displaystyle\int_1^e \ln\sqrt{x}\,dx$

5. In each case use integration by parts to increase the power of x.
(a) $\displaystyle\int x\ln x\,dx$ (b) $\displaystyle\int x^2\ln x\,dx$ (c) $\displaystyle\int \frac{\ln x}{x^2}\,dx$

DEVELOPMENT

6. Use repeated applications of integration by parts in order to find:
(a) $\displaystyle\int x^2e^x\,dx$ (b) $\displaystyle\int x^2\cos x\,dx$ (c) $\displaystyle\int (\ln x)^2\,dx$

7. These integrals are more naturally done by substitution. However, they can also be done by parts. Use integration by parts and then redo each integral using a suitable substitution, in order to compare the efficiency and ease of each method.
(a) $\displaystyle\int_0^1 x(x-1)^5\,dx$ (Q4) (b) $\displaystyle\int_0^1 x\sqrt{x+1}\,dx$ (c) $\displaystyle\int_0^4 x\sqrt{4-x}\,dx$ (Q6b)

8. Determine: (a) $\displaystyle\int e^x\cos x\,dx$ (b) $\displaystyle\int e^{-x}\sin x\,dx$

9. Evaluate: (a) $\displaystyle\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} e^x\sin 2x\,dx$

10. Use integration by parts to evaluate:
(a) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \sin^{-1} x\,dx$ (b) $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \cos^{-1} x\,dx$ (c) $\displaystyle\int_0^1 4x\tan^{-1} x\,dx$

11. Show that:
(a) $\displaystyle\int_0^{\pi} x^2\cos 2x\,dx = \tfrac{\pi}{2}$
(b) $\displaystyle\int_0^{\pi} x^2\sin\tfrac{1}{2}x\,dx = 8\pi - 16$
(c) $\displaystyle\int_1^e \sin(\ln x)\,dx = \tfrac{1}{2}e(\sin 1 - \cos 1) + \tfrac{1}{2}$
(d) $\displaystyle\int_1^e \cos(\ln x)\,dx = \tfrac{1}{2}e(\sin 1 + \cos 1) - \tfrac{1}{2}$

12. (a) Determine $\displaystyle\int x\ln x\,dx$. (b) Hence find $\displaystyle\int x(\ln x)^2\,dx$.

13. Use trigonometric identities and then integration by parts to show that:
(a) $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\cos x\,dx = \tfrac{\pi}{8}$
(b) $\displaystyle\int_0^{\frac{\pi}{2}} x\sin^2 x\,dx = \tfrac{1}{16}(\pi^2+4)$
(c) $\displaystyle\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx = \tfrac{\pi}{4} - \tfrac{\pi^2}{32} - \tfrac{1}{2}\ln 2$
(d) $\displaystyle\int_0^{\pi} x^2(\cos^2 x - \sin^2 x)\,dx = \tfrac{\pi}{2}$

ENRICHMENT

14. Determine formulae for the following:
(a) $\displaystyle\int \sqrt{a^2-x^2}\,dx$ (b) $\displaystyle\int \ln(x+\sqrt{x^2+a^2}\,)\,dx$ (c) $\displaystyle\int \ln(x+\sqrt{x^2-a^2}\,)\,dx$

15. Determine:
(a) $\displaystyle\int x\sin x\cos 3x\,dx$ (b) $\displaystyle\int x\cos 2x\cos x\,dx$ (c) $\displaystyle\int e^x\sin 2x\cos x\,dx$

16. Evaluate: (a) $\displaystyle\int_0^{\frac{1}{2}} x\sin^{-1} x\,dx$ (b) $\displaystyle\int_0^1 x^2\tan^{-1} x\,dx$

17. Let s be a positive constant. Show that $\displaystyle\lim_{N\to\infty}\int_0^N te^{-st}\,dt = \frac{1}{s^2}$.

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4G Trigonometric Integrals

Trigonometric integrals arise frequently in practical applications. This section contains those integrals more commonly encountered, and is grouped by type.

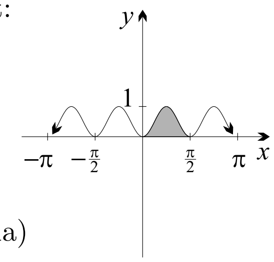
Powers of Cosine and Sine: There are two methods for the integral

$$\int \cos^m x \sin^n x \, dx$$

depending on whether the constants m and n are odd or even. If both are even then it is best to use the double angle identities.

WORKED EXAMPLE 27: Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} 4\cos^2 x \sin^2 x \, dx$

SOLUTION: Apply the double angle formula for sine to get:



$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} 4\cos^2 x \sin^2 x \, dx \\
&= \int_0^{\frac{\pi}{2}} \sin^2 2x \, dx \\
&= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} 1 - \cos 4x \, dx \qquad \text{(cosine double angle formula)} \\
&= \tfrac{1}{2}\Big[x - \tfrac{1}{4}\sin 4x\Big]_0^{\frac{\pi}{2}} \\
&= \tfrac{\pi}{4}\,.
\end{aligned}$$

In the second method one or both of m and n is odd. Work with cosine if m is odd, otherwise work with sine. The odd index of the chosen trigonometric function can be reduced to 1 via the Pythagorean identity, $\cos^2 x + \sin^2 x = 1$. It is then a matter of making a substitution for the other trigonometric function. The result is a polynomial integral.

WORKED EXAMPLE 28: Determine $\displaystyle\int \cos^3 x \sin^2 x \, dx\,.$

SOLUTION:

$$\begin{aligned}
\text{Let} \qquad I &= \int \cos^3 x \sin^2 x \, dx \\
&= \int \cos x (1 - \sin^2 x)\sin^2 x \, dx \qquad \text{(by Pythagoras.)} \\
\text{Put} \qquad u &= \sin x\,, \\
\text{so that} \qquad du &= \cos x \, dx\,, \\
\text{then} \qquad I &= \int (1 - u^2)u^2 \, du \\
&= \int u^2 - u^4 \, du \\
&= \tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 + C \\
&= \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C\,.
\end{aligned}$$

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8

> POWERS OF COSINE AND SINE: Given an integral of the form $\int \cos^m x \sin^n x \, dx$:
> - if m and n are both even then use the double angle identities,
> - if either m or n is odd then use the Pythagorean identity and a substitution.

Powers of Secant and Tangent: There are three general methods for the integral

$$\int \sec^m x \tan^n x \, dx\,,$$

again depending on whether the constants m and n are odd or even. There are also two special cases which should be dealt with first.

When $m = 0$ and $n = 1$ the situation is trivial, viz:

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x}\, dx \\ &= -\log|\cos x| + C\,.\end{aligned}$$

A very clever trick is required for the other special case when $m = 1$ and $n = 0$.

$$\begin{aligned}\int \sec x \, dx &= \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\, dx \\ &= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\, dx \\ &= \log|\sec x + \tan x| + C\,.\end{aligned}$$

Notice that in both special cases the result is a logarithmic function since the numerator of the integrand is the derivative of the denominator.

9

> THE INTEGRALS OF THE TANGENT AND SECANT FUNCTIONS:
>
> $$\int \tan x \, dx = -\log|\cos x| + C$$
>
> $$\int \sec x \, dx = \log|\sec x + \tan x| + C$$

Now for the general cases. If m and n are both even then separate out a factor of $\sec^2 x$ and substitute $u = \tan x$ to get a polynomial integral. The Pythagorean identity $1 + \tan^2 x = \sec^2 x$ may be required, particularly when $m = 0$.

WORKED EXAMPLE 29: Find $\int \tan^4 x \, dx$.

SOLUTION:

$$\begin{aligned}\int \tan^4 x \, dx &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx && \text{(by Pythagoras)} \\ &= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x - 1 \, dx && \text{(by Pythagoras again)} \\ &= \int u^2 \, du - \int \sec^2 x \, dx + \int 1 \, dx && \text{where } u = \tan x \\ &= \tfrac{1}{3}u^3 - \tan x + x + C \\ &= \tfrac{1}{3}\tan^3 x - \tan x + x + C\,.\end{aligned}$$

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WORKED EXAMPLE 30: Show that $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan^2 x\, dx = \tfrac{8}{15}$.

SOLUTION: Let $I = \displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x \tan^2 x\, dx$

so $\quad I = \displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\,(\tan^2 x + 1)\tan^2 x\, dx \qquad$ (by Pythagoras.)

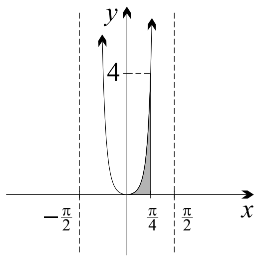
Put $\quad u = \tan x$,

then $\quad I = \displaystyle\int_0^{1} (u^2+1)u^2\, du$

$\quad = \displaystyle\int_0^{1} u^4 + u^2\, du$

$\quad = \Big[\tfrac{1}{5}u^5 + \tfrac{1}{3}u^3\Big]_0^1$

$\quad = \tfrac{8}{15}$.



If n is odd then factor out the term $\sec x \tan x$ and substitute $u = \sec x$ to obtain a polynomial integral. The Pythagorean identity may be required.

WORKED EXAMPLE 31: Determine the value of $\displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x \tan x\, dx$.

SOLUTION: Let $I = \displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x \tan x\, dx$,

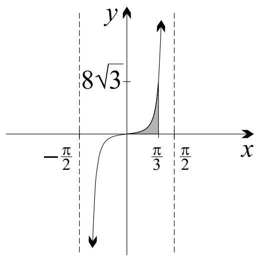
so $\quad I = \displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x \times \sec x \tan x\, dx$.

Put $\;u = \sec x$,

then $I = \displaystyle\int_1^{2} u^2\, du$

$\quad = \Big[\tfrac{1}{3}u^3\Big]_1^2$

$\quad = \tfrac{7}{3}$.



Whenever m is odd and n is even it is best to integrate by parts. Once again the Pythagorean identity may be required.

WORKED EXAMPLE 32: Find $\displaystyle\int \sec^3 x\, dx$.

SOLUTION:

Let $\quad I = \displaystyle\int \sec^3 x\, dx$

$\quad = \displaystyle\int \sec^2 x \times \sec x\, dx$.

Put $\quad u = \sec x \quad$ and $\quad v' = \sec^2 x$

so $\quad u' = \sec x \tan x \quad$ and $\quad v = \tan x$.

Then $\quad I = \sec x \tan x - \displaystyle\int \sec x \tan^2 x\, dx \qquad$ (by parts)

$\quad = \sec x \tan x - \displaystyle\int \sec x(\sec^2 x - 1)\, dx \qquad$ (by Pythagoras.)

Thus $\quad I = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \,,$

so $\quad I = \sec x \tan x - I + \log|\sec x + \tan x| \qquad$ (from the special case)

or $\quad 2I = \sec x \tan x + \log|\sec x + \tan x| \,,$

hence $\quad I = \frac{1}{2}\big(\sec x \tan x + \log|\sec x + \tan x|\big) + C \,.$

10 POWERS OF SECANT AND TANGENT: Given an integral of the form $\int \sec^m x \tan^n x \, dx$:

- if m and n are both even then factor out $\sec^2 x$ and substitute $u = \tan x$
- if n is odd then factor out the term $\sec x \tan x$ and substitute $u = \sec x$
- if m is odd and n is even then use integration by parts

Products to Sums: There are three standard formulae for converting products of trigonometric functions to sums. These will be familiar to some readers and are easily proved by expanding each right hand side.

11 PRODUCTS TO SUMS:

$$\sin A \cos B = \tfrac{1}{2}\big(\sin(A - B) + \sin(A + B)\big)$$
$$\cos A \cos B = \tfrac{1}{2}\big(\cos(A - B) + \cos(A + B)\big)$$
$$\sin A \sin B = \tfrac{1}{2}\big(\cos(A - B) - \cos(A + B)\big)$$

These formulae can be applied to simplify an integral, as in the following example.

WORKED EXAMPLE 33: Find $\int \cos 3x \cos 2x \, dx$.

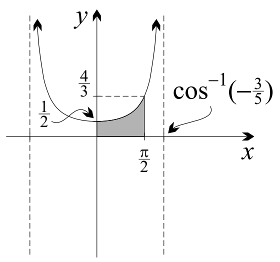
SOLUTION: $\int \cos 3x \cos 2x \, dx = \frac{1}{2}\int \cos x + \cos 5x \, dx \qquad$ (products to sums)

$\qquad = \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C \,.$

The t-substitution: The t-substitution, namely $t = \tan\frac{x}{2}$, should be well known to all readers, being part of the Mathematics Extension 1 course. The aim here is to transform a trigonometric integrand into a rational function.

WORKED EXAMPLE 34: Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{4}{3 + 5\cos x} \, dx = \log 3 \,.$

SOLUTION: Let $\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{4}{3 + 5\cos x} \, dx \,.$



Put $\qquad t = \tan\frac{x}{2}$

then $\qquad \dfrac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\qquad = \frac{1}{2}(1 + t^2) \,.$

So $\qquad dx = \dfrac{2\,dt}{1 + t^2} \,,$

with $\qquad \cos x = \dfrac{1 - t^2}{1 + t^2} \,.$

Thus
$$\begin{aligned} I &= \int_0^1 \frac{4}{3+5\,\frac{1-t^2}{1+t^2}} \times \frac{2\,dt}{1+t^2} \\ &= \int_0^1 \frac{8}{8-2t^2}\,dt \\ &= \int_0^1 \frac{4}{4-t^2}\,dt\,. \end{aligned}$$

Let $\dfrac{4}{(2+t)(2-t)} = \dfrac{A}{2+t} + \dfrac{B}{2-t}$ (partial fractions)

then $A = 1$ and $B = 1$ (by the cover-up method.)

Hence
$$\begin{aligned} I &= \int_0^1 \frac{1}{2+t} + \frac{1}{2-t}\,dt \\ &= \Big[\log(2+t) - \log(2-t)\Big]_0^1 \\ &= \log 3\,. \end{aligned}$$

12 THE t-SUBSTITUTION: Use this to transform a trigonometric integrand into a rational function. Let $t = \tan\frac{x}{2}$, then:

$$\tan x = \frac{2t}{1-t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \sin x = \frac{2t}{1+t^2} \quad \text{with} \quad dx = \frac{2\,dt}{1+t^2}$$

Exercise 4G

1. Find:

(a) $\int \cos x\,dx$ (b) $\int \sin x\,dx$ (c) $\int \tan x\,dx$ (d) $\int \cot x\,dx$

2. Find each of the following integrals by substituting either $u = \sin x$ or $u = \cos x$. You may also need to apply the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

(a) $\int \cos x \sin^2 x\,dx$ (c) $\int \sin^3 x\,dx$ (e) $\int \cos^5 x\,dx$

(b) $\int \cos^2 x \sin x\,dx$ (d) $\int \cos^3 x\,dx$ (f) $\int \sin^3 x \cos^3 x\,dx$

3. Use the double angle formulae to evaluate:

(a) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x\,dx$ (c) $\int_0^{\pi} \sin^2 x \cos^2 x\,dx$

4. Use the substitution $u = \tan x$ to find the following. You may also need to apply the Pythagorean identity $1 + \tan^2 x = \sec^2 x$.

(a) $\int \sec^2 x\,dx$ (b) $\int \tan^2 x\,dx$ (c) $\int \sec^4 x\,dx$ (d) $\int \tan^4 x\,dx$

DEVELOPMENT

5. Evaluate:

(a) $\int_0^{\frac{\pi}{2}} \cos^3 x \sin x\,dx$ (c) $\int_0^{\frac{\pi}{3}} \sin^3 x \cos x\,dx$ (e) $\int_0^{\pi} \sin^3 x \cos^2 x\,dx$

(b) $\int_0^{\frac{\pi}{6}} \cos^3 x\,dx$ (d) $\int_0^{\frac{\pi}{3}} \sin^5 x\,dx$ (f) $\int_0^{\frac{\pi}{4}} \sin^2 x \cos^3 x\,dx$

6. Determine:
(a) $\displaystyle\int \cos^4 x\,dx$ (b) $\displaystyle\int \sin^4 x\,dx$ (c) $\displaystyle\int \sin^4 x\cos^4 x\,dx$

7. Show that:
(a) $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\tan^2 x\,dx = \sqrt{3}$ (c) $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x\tan x\,dx = \tfrac{3}{4}$
(b) $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x\tan^3 x\,dx = 2\tfrac{2}{9}$ (d) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^5 x\,dx = \tfrac{1}{4}(2\ln 2 - 1)$

8. Use the t-substitution to evaluate:
(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx$ (c) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5+3\sin x}\,dx$

9. In each case use a suitable trigonometric substitution to evaluate the integral.
(a) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\displaystyle\int_0^1 x^3\sqrt{1+x^2}\,dx$ (c) $\displaystyle\int_0^1 x^2\sqrt{1-x^2}\,dx$

10. Let $I = \displaystyle\int \sin x\cos x\,dx$.
(a) Find I using a suitable substitution. (b) Find I by the double angle formulae.
(c) Show that the answers to parts (a) and (b) are equivalent.

11. Evaluate:
(a) $\displaystyle\int_0^{\frac{\pi}{4}} \left(\tan^3 x + \tan x\right)\,dx$ (b) $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\cos x - \cos^3 x\right)\,dx$

12. Evaluate:
(a) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x\sec^2 x\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x\sec^4 x\,dx$

13. Find these integrals by first converting the products to sums.
(a) $\displaystyle\int \sin 3x\cos x\,dx$ (b) $\displaystyle\int \cos 3x\sin x\,dx$ (c) $\displaystyle\int \cos 6x\cos 2x\,dx$

14. Evaluate these definite integrals by first converting the products to sums.
(a) $\displaystyle\int_0^{\frac{\pi}{4}} \sin 3x\sin x\,dx$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} \cos 4x\cos 2x\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{3}} \sin 4x\cos 2x\,dx$

15. Use the substitution $t = \tan\frac{x}{2}$ to determine:
(a) $\displaystyle\int \frac{1}{1+\cos x}\,dx$ (b) $\displaystyle\int \frac{1}{1+\sin x-\cos x}\,dx$ (c) $\displaystyle\int \frac{1}{3\sin x+4\cos x}\,dx$

16. (a) Use the t-substitution to show that $\displaystyle\int \sec x\,dx = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$
(b) Show that the primitive in (a) is equivalent to $\ln|\sec x + \tan x| + C$.

ENRICHMENT

17. In the chapter on complex numbers it was shown that $(\operatorname{cis}\theta)^3 = \operatorname{cis}3\theta$. Use this result to help determine $\displaystyle\int \cos^3\theta\,d\theta$.

18. Use integration by parts and the fact that $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$ to show that

$$\int_0^{\frac{\pi}{4}} \sec^3 x\,dx = \tfrac{1}{\sqrt{2}} + \tfrac{1}{2}\ln\left(1+\sqrt{2}\right).$$

4H Reduction Formulae

Readers will be familiar with sequences and series, such as the odd numbers,

$$1,\ 3,\ 5,\ 7,\ \ldots \qquad \text{or} \qquad u_n = 2n-1\,,$$

or the powers of 2,

$$1,\ 2,\ 4,\ 8,\ \ldots \qquad \text{or} \qquad u_n = 2^{n-1}\,.$$

In this section, sequences of integrals are considered, such as the sequence

$$\int_0^{\frac{\pi}{2}} \sin x\,dx\,,\ \int_0^{\frac{\pi}{2}} \sin^2 x\,dx\,,\ \int_0^{\frac{\pi}{2}} \sin^3 x\,dx\,,\ \ldots \quad \text{or} \quad I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx\,.$$

Of particular interest are the equations which relate the terms of the sequence.

Continuing with this example, it can be shown that

$$I_n = \tfrac{n-1}{n} \times I_{n-2} \qquad \text{for} \qquad n \ge 2\,.$$

Such equations are called *reduction formulae*, because they enable the index to be reduced, in this case from n to $n-2$. In practical terms, this means that if one of the integrals in the sequence is known then other terms can be simply calculated from it without the need for further integration. Returning to the example above, since

$$I_1 = \int_0^{\frac{\pi}{2}} \sin x\,dx = 1\,,$$

it follows that $\quad I_3 = \tfrac{2}{3}I_1 = \tfrac{2}{3}\,,$

and $\quad I_5 = \tfrac{4}{5}I_3 = \tfrac{8}{15}\,.$

This is obviously a significant saving of effort since it was not necessary to find the primitives of $\sin^3 x$ and $\sin^5 x$ in order to evaluate I_3 and I_5. It should be clear from this that reduction formulae are of particular importance.

Note that the convention is to evaluate the sequence index *before* the integral is evaluated. Thus, once again using the same example,

$$\begin{aligned} I_0 &= \int_0^{\frac{\pi}{2}} 1\,dx \\ &= \tfrac{\pi}{2}\,. \end{aligned}$$

Identities: In a few cases the reduction formula can be generated by use of an identity, as in the following example.

WORKED EXAMPLE 35: Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\,dx\,.$

(a) Show that $I_n = \dfrac{1}{n-1} - I_{n-2}$ for $n \ge 2\,.$

(b) Evaluate I_1 and hence find I_5.

SOLUTION:

(a)

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,(\sec^2 x - 1)\,dx \qquad \text{(by Pythagoras, for } n \ge 2) \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,\sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx \end{aligned}$$

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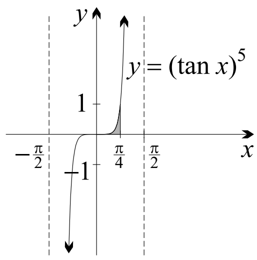
$$\text{thus} \quad I_n = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$$
$$= \frac{1}{n-1} - I_{n-2} \qquad \text{for } n \geq 2\,.$$

(b)
$$I_1 = \int_0^{\frac{\pi}{4}} \tan x\, dx$$
$$= \Big[-\log(\cos x)\Big]_0^{\frac{\pi}{4}}$$
$$= \tfrac{1}{2}\log 2\,.$$



Thus $I_3 = \frac{1}{2} - I_1$
$= \frac{1}{2} - \frac{1}{2}\log 2\,,$

and $I_5 = \frac{1}{4} - I_3$
$= \frac{1}{2}\log 2 - \frac{1}{4}\,.$

By Parts: Many examples of reduction formulae use integration by parts.

Worked Example 36: Let $I_n = \displaystyle\int_1^e (\log x)^n\, dx$

(a) Show that $I_n = e - nI_{n-1}$ for $n \geq 1$.

(b) Hence show that $I_3 = 6 - 2e$.

Solution:

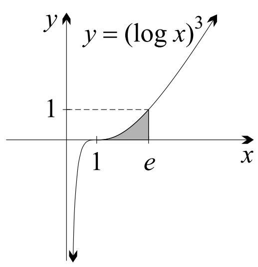
(a)
$$I_n = \int_1^e 1 \times (\log x)^n\, dx$$
$$= \Big[x(\log x)^n\Big]_1^e - \int_1^e x \times \tfrac{n}{x}(\log x)^{n-1}\, dx \qquad \text{(by parts)}$$
$$= (e - 0) - n\int_1^e (\log x)^{n-1}\, dx$$
$$= e - nI_{n-1}\,.$$

(b)
$$I_0 = \int_1^e 1\, dx$$
$$= e - 1\,.$$

Thus $I_1 = e - I_0$
$= 1\,,$

$I_2 = e - 2I_1$
$= e - 2\,,$

and $I_3 = e - 3I_2$
$= 6 - 2e\,.$



By Parts with an Identity: Some examples use integration by parts and an identity.

Worked Example 37: Let $I_n = \displaystyle\int_0^1 x^2(1-x^2)^n\, dx\,.$

(a) Use the identity $x^2 \equiv 1 - (1-x^2)$ to show that $I_n = \frac{2n}{2n+3} I_{n-1}$ for $n \geq 1$.

(b) Evaluate I_0 and hence find I_3.

SOLUTION:

(a) Apply integration by parts first to get:

$$\begin{aligned} I_n &= \int_0^1 x^2(1-x^2)^n\,dx \\ &= \Big[\tfrac{1}{3}x^3(1-x^2)^n\Big]_0^1 - \int_0^1 \tfrac{1}{3}x^3 \times (-2nx)(1-x^2)^{n-1}\,dx \\ &= 0 + \tfrac{2n}{3}\int_0^1 x^2 \times x^2(1-x^2)^{n-1}\,dx \\ &= \tfrac{2n}{3}\int_0^1 x^2(1-x^2)^{n-1} - x^2(1-x^2)^n\,dx \qquad \text{(by the identity)} \end{aligned}$$

so $I_n = \tfrac{2n}{3}I_{n-1} - \tfrac{2n}{3}I_n$.

thus $\tfrac{2n+3}{3}I_n = \tfrac{2n}{3}I_{n-1}$

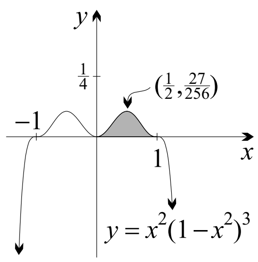
or $I_n = \tfrac{2n}{2n+3}I_{n-1}$ for $n \ge 1$.

(b)
$$\begin{aligned} I_0 &= \int_0^1 x^2\,dx \\ &= \tfrac{1}{3}. \end{aligned}$$

Thus $I_1 = \tfrac{2}{5}I_0 = \tfrac{2}{15}$,

$I_2 = \tfrac{4}{7}I_1 = \tfrac{8}{105}$,

and $I_3 = \tfrac{6}{9}I_2 = \tfrac{16}{315}$.



Exercise 4H

1. (a) Given that $I_n = \int \tan^n x\,dx$, prove that $I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$ for $n \ge 2$.
   (b) Hence show that $I_6 = \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C$

2. (a) If $I_n = \int x^n e^x\,dx$, show that $I_n = x^n e^x - nI_{n-1}$ for $n \ge 1$.
   (b) Hence show that $\int x^3 e^x\,dx = (x^3 - 3x^2 + 6x - 6)e^x + C$.

3. (a) If $I_n = \int_1^e x(\ln x)^n\,dx$, show that $I_n = \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}$ for $n \ge 1$.
   (b) Find I_0 and hence show that $I_4 = \frac{1}{4}(e^2 - 3)$.

4. Let $u_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$.
   (a) Use integration by parts with $v' = \cos x$ to show that
   $$u_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^2 x \cos^{n-2} x\,dx \qquad \text{for } n \ge 2.$$
   (b) Hence show that $u_n = (n-1)(u_{n-2} - u_n)$.
   (c) Deduce that $u_n = \frac{n-1}{n}u_{n-2}$ for $n \ge 2$, and hence evaluate u_5.

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DEVELOPMENT

5. Let $T_n = \displaystyle\int_0^{\frac{\pi}{4}} \sec^n x\, dx\,.$

(a) Show that $T_n = \dfrac{(\sqrt{2})^{n-2}}{n-1} + \dfrac{n-2}{n-1}T_{n-2}$ for $n \ge 2$.
You will need to use integration by parts and a trigonometric identity.

(b) Deduce that $T_6 = \frac{28}{15}$.

6. Let $C_n = \displaystyle\int_0^{\frac{\pi}{2}} x^n \cos x\, dx\,,$ where $n \ge 0\,.$

(a) Prove that $C_n = (\frac{\pi}{2})^n - n(n-1)C_{n-2}$, for $n \ge 2\,.$ (b) Hence evaluate C_6.

7. Suppose that $I_n = \displaystyle\int \frac{x^n}{1+x^2}\, dx.$

(a) Use algebraic manipulation to show that $I_n = \dfrac{x^{n-1}}{n-1} - I_{n-2}.$

(b) Hence find $\displaystyle\int \frac{x^5}{1+x^2}\, dx.$

8. (a) If $I_n = \displaystyle\int_0^1 (1-x^2)^n\, dx$, show that $I_n = \dfrac{2n}{2n+1}\, I_{n-1}$ for $n \ge 1$.

(b) Evaluate I_0 and hence find I_4.

9. (a) If $u_n = \displaystyle\int_0^1 x(1-x^3)^n\, dx\,,$ show that $u_n = \dfrac{3n}{3n+2}\, u_{n-1}$ for $n \ge 1$.

(b) Show that $u_0 = \frac{1}{2}$ and hence evaluate u_4.

10. Suppose that $J_n = \displaystyle\int \frac{x^n}{\sqrt{1-x^2}}\, dx\,.$

(a) Show that $J_n = \frac{1}{n}\Big((n-1)J_{n-2} - x^{n-1}\sqrt{1-x^2}\Big)$ for $n \ge 2$.

[HINT: Do this by parts with $u = x^{n-1}$ and $v' = \dfrac{x}{\sqrt{1-x^2}}$.]

(b) Hence determine $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\, dx\,.$

11. Let $u_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x\, dx.$

(a) Show that $u_n = \left(\frac{n-1}{n+2}\right) u_{n-2}$, for $n \ge 2\,.$
[HINT: Do this by parts with $u = \sin^{n-1} x$ and $v' = \sin x \cos^2 x$.]

(b) Hence show that $u_4 = \frac{\pi}{32}\,.$

12. Consider the integral $I_n = \displaystyle\int_0^1 \frac{x^n}{\sqrt{1+x}}\, dx\,.$

(a) Show that $I_0 = 2\sqrt{2} - 2\,.$

(b) Show that $I_{n-1} + I_n = \displaystyle\int_0^1 x^{n-1}\sqrt{1+x}\, dx$ for $n \ge 1$.

(c) Use integration by parts to show that $I_n = \dfrac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$ for $n \ge 1$.

(d) Hence evaluate $I_2\,.$

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13. (a) Show that $(1+t^2)^{n-1} + t^2(1+t^2)^{n-1} = (1+t^2)^n$.

(b) Put $P_n = \displaystyle\int_0^x \left(1+t^2\right)^n dt$. Use integration by parts and part (a) to show that

$$P_n = \tfrac{1}{2n+1}\Big(\left(1+x^2\right)^n x + 2nP_{n-1}\Big) \qquad \text{for } n \geq 1.$$

(c) Hence determine P_4:

(i) by the reduction formula, (ii) by using the binomial theorem.

(d) Hence write $1 + \frac{4}{3}x^2 + \frac{6}{5}x^4 + \frac{4}{7}x^6 + \frac{1}{9}x^8$ in powers of $(1+x^2)$.

14. Let $T_n = \displaystyle\int_0^1 x^n\sqrt{1-x}\,dx$.

(a) Deduce the reduction formula $T_n = \dfrac{2n}{2n+3}\,T_{n-1}$ for $n \geq 1$.

(b) Show that $T_3 = \frac{32}{315}$.

(c) Use the reduction formula to help prove by induction that $T_n = \dfrac{n!(n+1)!}{(2n+3)!}\,4^{n+1}$.

ENRICHMENT

15. Let $I_n = \displaystyle\int_0^1 (1-x^2)^n\,dx$ and $J_n = \displaystyle\int_0^1 x^2(1-x^2)^n\,dx$.

(a) Apply integration by parts to I_n to show that $I_n = 2n\,J_{n-1}$ for $n \geq 1$.

(b) Hence show that $I_n = \dfrac{2n}{2n+1}\,I_{n-1}$ for $n \geq 1$.

(c) Show that $J_n = I_n - I_{n+1}$, and hence deduce that $J_n = \dfrac{1}{2n+3}\,I_n$.

(d) Hence write down a reduction formula for J_n in terms of J_{n-1}.

16. For $n = 0, 1, 2, \ldots$ let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n\theta\,d\theta$.

(a) Show that $I_1 = \frac{1}{2}\ln 2$.

(b) Show that, for $n \geq 2$, $I_n + I_{n-2} = \dfrac{1}{n-1}$.

(c) For $n \geq 2$, explain why $I_n < I_{n-2}$, and deduce that

$$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}.$$

(d) Use the reduction formula in part (b) to find I_5, and hence deduce that $\frac{2}{3} < \ln 2 < \frac{3}{4}$.

17. Suppose that $I_n = \displaystyle\int_0^{\frac{\pi}{3}} \cos nx \sec x\,dx$.

(a) Prove that $I_n = \dfrac{2}{n-1}\sin\dfrac{(n-1)\pi}{3} - I_{n-2}$ for $n \geq 2$.

[HINT: Write nx as $(n-2)x + 2x$.]

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \cos 5x \sec x\,dx$.

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4I Miscellaneous Integrals

As was stated in the chapter overview, integration is an art form and requires much practice. In particular, it is important to be able to recognise the different forms of integrals, and to quickly determine which method is best used. To that end, this section has been included. The exercise contains a mixture of all integral types. Some questions can be done by more than one method. It is up to the reader to determine which method is the most efficient.

Exercise 4I

1. Evaluate:

(a) $\displaystyle\int_{-1}^{1} \frac{x^2}{(5+x^3)^2}\,dx$

(b) $\displaystyle\int_{0}^{\pi} x\sin x\,dx$

(c) $\displaystyle\int_{2}^{3} \frac{2x+2}{(x+3)(x-1)}\,dx$

(d) $\displaystyle\int_{0}^{2} \frac{x-1}{x+1}\,dx$

(e) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{3\cos x}{\sin^4 x}\,dx$

(f) $\displaystyle\int_{0}^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\,dx$

2. Find:

(a) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$

(b) $\displaystyle\int \frac{1+x}{1+x^2}\,dx$

(c) $\displaystyle\int \sin x\cos^4 x\,dx$

(d) $\displaystyle\int \frac{1}{2x^2+3x+1}\,dx$

(e) $\displaystyle\int x^3\ln x\,dx$

(f) $\displaystyle\int \sin^3 2x\,dx$

(g) $\displaystyle\int \frac{1}{x^2+6x+25}\,dx$

(h) $\displaystyle\int 3x\cos 3x\,dx$

(i) $\displaystyle\int \frac{x}{\sqrt{4+x}}\,dx$

3. Show that:

(a) $\displaystyle\int_{0}^{1} x^2e^{-x}\,dx = 2-\tfrac{5}{e}$

(b) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin^3 x\cos^5 x\,dx = \tfrac{1}{24}$

(c) $\displaystyle\int_{0}^{1} \frac{x}{(x+1)(x^2+1)}\,dx = \tfrac{1}{8}(\pi-2\ln 2)$

(d) $\displaystyle\int_{0}^{\frac{1}{2}} (1-x^2)^{-\frac{3}{2}}\,dx = \tfrac{1}{\sqrt{3}}$

(e) $\displaystyle\int_{0}^{1} \frac{1-x^2}{1+x^2}\,dx = \tfrac{\pi}{2}-1$

(f) $\displaystyle\int_{2}^{4} \frac{x}{\sqrt{6x-8-x^2}}\,dx = 3\pi$

(g) $\displaystyle\int_{0}^{1} \frac{\sqrt{x}}{1+x}\,dx = \tfrac{1}{2}(4-\pi)$

(h) $\displaystyle\int_{0}^{\sqrt{3}} \tan^{-1} x\,dx = \tfrac{\pi}{\sqrt{3}}-\ln 2$

(i) $\displaystyle\int_{0}^{\frac{\pi}{4}} \sin 2x\cos 3x\,dx = \tfrac{1}{10}(3\sqrt{2}-4)$

(j) $\displaystyle\int_{0}^{\pi} e^{-x}\cos x\,dx = \tfrac{1}{2}(1+e^{-\pi})$

DEVELOPMENT

4. (a) Find the rational numbers A, B and C such that

$$\frac{x-1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}\,.$$

(b) Hence show that $\displaystyle\int_{0}^{1} \frac{x^3+x}{x^3+1}\,dx = 1-\tfrac{2}{3}\ln 2.$

5. Use integration by parts to show that $\displaystyle\int x^3e^{-x^2}\,dx = -\tfrac{1}{2}e^{-x^2}(1+x^2)+C\,.$

6. (a) Evaluate $\displaystyle\int_{0}^{\frac{\pi}{3}} \sec^4 x\,dx\,.$

(b) Hence evaluate $\displaystyle\int_{0}^{\frac{\pi}{3}} \sec^6 x\,dx\,.$

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7. In each case let $t = \tan\frac{x}{2}$ in order to show that:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\,dx = \tfrac{1}{4}\ln 3$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\cos x - 2\sin x + 3}\,dx = \tfrac{\pi}{4}$

8. (a) Find the values of A, B, C and D such that

$$\frac{4t}{(1+t)^2(1+t^2)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{Ct+D}{1+t^2}\,.$$

(b) Hence use the t-substitution to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,dx\,.$

9. Use the substitution $u = \sqrt[6]{x}$ to show that $\displaystyle\int_1^{64} \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = 11 - 6\ln\tfrac{3}{2}\,.$

10. Find $\displaystyle\int \sqrt{a^2-x^2}\,dx$ using:

(a) the substitution $\theta = \sin^{-1}\frac{x}{a}$, (b) integration by parts.

11. (a) Show that $\displaystyle\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}\,dx = \tfrac{1}{2}(\pi + \ln\tfrac{27}{16})\,.$

(b) Hence find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x+2\sin x}\,dx$ using the substitution $t = \tan\frac{x}{2}$.

12. (a) Find integers P and Q such that

$$8\sin x + \cos x - 2 = P(3\sin x + 2\cos x - 1) + Q(3\cos x - 2\sin x)\,.$$

(b) Hence find $\displaystyle\int \frac{8\sin x+\cos x-2}{3\sin x+2\cos x-1}\,dx\,.$

13. (a) If $T_n = \displaystyle\int_0^{\pi} \sin^n x\,dx$, show that $T_n = \frac{n-1}{n}T_{n-2}$ for $n \ge 2$.

(b) Hence show that $T_5T_6 = \frac{\pi}{3}$.

14. (a) Let $I_n = \displaystyle\int_1^{e} (\ln x)^n\,dx$ and show that $I_n = e - nI_{n-1}$ for $n \ge 1$.

(b) Hence evaluate I_3.

ENRICHMENT

15. Let $I_n = \displaystyle\int_0^1 \frac{x^{n-1}}{(x+1)^n}\,dx$, for $n = 1, 2, 3, \ldots$

(a) Show that $I_1 = \ln 2\,.$

(b) Use integration by parts to show that $I_{n+1} = I_n - \dfrac{1}{n\,2^n}\,.$

(c) The maximum value of $\dfrac{x}{x+1}$, for $0 \le x \le 1$, is $\frac{1}{2}$.
Use this fact to show that $I_{n+1} < \frac{1}{2}I_n\,.$

(d) Deduce that $I_n < \dfrac{1}{n\,2^{n-1}}\,.$

(e) Use the reduction formula in part (b) and the inequality in part (d) to show that

$$\tfrac{2}{3} < \ln 2 < \tfrac{17}{24}\,.$$

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16. Given that $\displaystyle\int_0^{\pi} \frac{1}{5+3\cos x}\,dx = \tfrac{\pi}{4}$, show that $\displaystyle\int_0^{\pi} \frac{\cos x + 2\sin x}{5+3\cos x}\,dx = \tfrac{1}{12}(16\ln 2 - \pi)$.

17. (a) Use the substitution $u = t - t^{-1}$ to show that $\displaystyle\int \frac{1+t^2}{1+t^4}\,dt = \tfrac{1}{\sqrt{2}}\tan^{-1}\tfrac{\sqrt{2}(t^2-1)}{2t} + C$.

(b) Alternatively, use the result $(1+t^4) = (1+t^2)^2 - (\sqrt{2}\,t)^2$ and partial fractions to show that $\displaystyle\int \frac{1+t^2}{1+t^4}\,dt = \tfrac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\,t+1) + \tfrac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\,t-1) + C$.

(c) The formulae in parts (a) and (b) agree when applied to the integral $\displaystyle\int_p^q \frac{1+t^2}{1+t^4}\,dt$ provided $pq > 0$. When $pq \le 0$ the formula in (a) is incorrect. Why might that be?

18. Consider the two new functions $\cosh x = \tfrac{1}{2}(e^x + e^{-x})$ and $\sinh x = \tfrac{1}{2}(e^x - e^{-x})$.
Show that $\displaystyle\int_0^{\ln 2} \frac{1}{5\cosh x - 3\sinh x}\,dx = \tfrac{1}{2}\tan^{-1}\tfrac{1}{3}$.

19. Suppose that $I = \displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$.

(a) Use the substitution $x = \tan\theta$ to show that $I = \displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$.

(b) Next, use the substitution $u = \tfrac{\pi}{4} - \theta$ to show that $I = \displaystyle\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$.

(c) Finally, deduce that $I = \tfrac{1}{8}\pi\ln 2$.

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4J Chapter Review Exercise

Exercise 4J

1. Find:

(a) $\displaystyle\int xe^{x^2}\,dx$

(b) $\displaystyle\int \frac{3x}{x^2+1}\,dx$

(c) $\displaystyle\int x(1+x^2)^5\,dx$

(d) $\displaystyle\int \cos^3 x \sin x\,dx$

(e) $\displaystyle\int \frac{4x}{x^2-2x-3}\,dx$

(f) $\displaystyle\int xe^{-2x}\,dx$

2. Find:

(a) $\displaystyle\int \tan^2 x\,dx$

(b) $\displaystyle\int \frac{x}{\sqrt{3+x}}\,dx$

(c) $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$

(d) $\displaystyle\int x\cos\tfrac{1}{3}x\,dx$

(e) $\displaystyle\int \frac{x+2}{x+1}\,dx$

(f) $\displaystyle\int \frac{3x^2+2}{x^3+x}\,dx$

3. Use the given substitution to find:

(a) $\displaystyle\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx$ [put $x = 2\sin\theta$]

(b) $\displaystyle\int \frac{e^x}{e^{2x}-1}\,dx$ [put $u = e^x$]

(c) $\displaystyle\int \frac{1}{2+\sqrt{x}}\,dx$ [put $u = \sqrt{x}$]

(d) $\displaystyle\int \frac{1}{5+4\cos x}\,dx$ [put $t = \tan\frac{1}{2}x$]

4. Evaluate:

(a) $\displaystyle\int_{-1}^{2} x^2\sqrt{x^3+1}\,dx$

(b) $\displaystyle\int_{4}^{5} \frac{2x}{x^2-4x+3}\,dx$

(c) $\displaystyle\int_{0}^{\frac{\pi}{3}} \sin^3 x\,dx$

(d) $\displaystyle\int_{0}^{1} \frac{8x}{3+4x}\,dx$

(e) $\displaystyle\int_{0}^{1} x^2\sqrt{1-x}\,dx$

(f) $\displaystyle\int_{0}^{\frac{\pi}{4}} \sin 5x\cos 3x\,dx$

5. Evaluate:

(a) $\displaystyle\int_{0}^{\frac{\pi}{3}} x\sin 3x\,dx$

(b) $\displaystyle\int_{0}^{2} \frac{3-7x}{\sqrt{4x-x^2}}\,dx$

(c) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin^2 x\cos^3 x\,dx$

(d) $\displaystyle\int_{0}^{3} \frac{x^2+x+18}{x^3+9x^2+9x+81}\,dx$

(e) $\displaystyle\int_{2}^{4} \sqrt{16-x^2}\,dx$

(f) $\displaystyle\int_{0}^{\frac{1}{2}} e^{2x}\sin \pi x\,dx$

6. Use the given substitution to find:

(a) $\displaystyle\int_{8}^{15} \frac{1}{(x-3)\sqrt{x+1}}\,dx$ [put $u = \sqrt{x+1}$]

(b) $\displaystyle\int_{0}^{\frac{\pi}{3}} \frac{1}{9-8\sin^2 x}\,dx$ [put $t = \tan x$]

(c) $\displaystyle\int_{0}^{2} \sqrt{x(4-x)}\,dx$ [put $x = 4\sin^2\theta$]

(d) $\displaystyle\int_{0}^{\frac{\pi}{3}} \frac{1}{\cos x}\,dx$ [put $t = \tan\frac{1}{2}x$]

7. (a) Given that $I_n = \displaystyle\int_0^1 x^n e^x\,dx$, prove that $I_n = e - nI_{n-1}$.

(b) Hence show that $I_5 = 120 - 44e$.

8. (a) Derive a reduction formula for I_n in terms of I_{n-1} given $I_n = \displaystyle\int x^3(\ln x)^n\,dx$.

(b) Hence show that $\displaystyle\int x^3(\ln x)^3\,dx = \frac{1}{128}x^4\Big(32(\ln x)^3 - 24(\ln x)^2 + 12\ln x - 3\Big) + C.$

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9. (a) If $I_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx$, prove that $I_{2n} = \dfrac{2n-1}{2n} I_{2n-2}$.

(b) Hence find $\int_0^{\frac{\pi}{2}} \sin^6 x \, dx$.

10. (a) If $I_n = \int_0^1 (1+x^2)^n \, dx$, prove that $(2n+1)I_n = 2^n + 2nI_{n-1}$.

(b) If $J_n = \int_0^{\frac{\pi}{4}} \sec^{2n} \theta \, d\theta$, show that $J_n = I_{n-1}$ and hence find a reduction formula for J_n.

(c) Evaluate $\int_0^{\frac{\pi}{4}} \sec^6 \theta \, d\theta$.

11. (a) Given that $I_n = \int \dfrac{\sin 2nx}{\sin x} \, dx$, prove that $I_n = \dfrac{2}{2n-1} \sin(2n-1)x + I_{n-1}$.

(b) Hence find $\int_0^{\frac{\pi}{2}} \dfrac{\sin 6x}{\sin x} \, dx$.

12. (a) Use the substitution $u = a - x$ to prove that $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$.

(b) Hence show that $\int_0^{\pi} \dfrac{x \sin x}{3 + \sin^2 x} \, dx = \dfrac{\pi \ln 3}{4}$.

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Appendix: A Short Table of Integrals

Here is a short table of integrals. Each integral in the first group is a simplified form of one found in the examination Reference Sheet. Students may find this table helpful whilst studying this chapter, but should aim to become proficient at using the examination Reference Sheet as soon as possible.

The group of three extra integrals at the bottom are only occasionally needed in Mathematics Extension 2, and do not appear in the examination Reference Sheet. Students are expected to be able to apply unfamiliar integral formulae like these when they are given.

COMMON INTEGRALS

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b}\,dx = \frac{1}{a}\,e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

OTHER INTEGRALS

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax + C\,, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right) + C\,, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right) + C$$

5

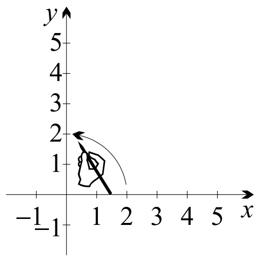
Vectors

Chapter Overview: The work on vectors, begun in the Extension 1 course, is now broadened to include vectors in three dimensions. Once again, the focus is on geometric problems involving points, lines and angles. Three dimensions often makes these problems more complicated, but in many cases the theory developed in Extension 1 applies in a similar way.

A good knowledge of the coordinate system is essential to better understand how vectors work in three dimensions. The first section in this chapter introduces that coordinate system and presents some of its basic features. Section 5B formally extends vectors to three dimensions, and adds the new basis vector $\underset{\sim}{k}$ to the list of standard basis vectors, $\underset{\sim}{i}$ and $\underset{\sim}{j}$. The dot product is reviewed in Section 5C. The definition is extended to three dimensions, and several applications are considered in Section 5D. Then follows a section on vector proofs in geometry. Section 5F develops the vector equation of a line. This is necessarily a long section, as it compares and contrasts the vector equations with previous work in coordinate geometry. The chapter concludes with a number of other applications of vectors, including circles, spheres, planes and some simple curves.

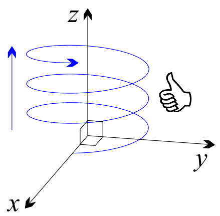
5A Coordinates in Three Dimensions

Right-handed Coordinates: The familiar two-dimensional coordinate system has the x-axis drawn horizontally with positive to the right, and has the y-axis drawn vertically with positive up the page. Consequently, a rotation about the origin from the positive x-axis to the positive y-axis is anticlockwise. The configuration is called a *right-handed* system. This is because when a person's right hand is placed with the edge of the palm resting on the page, such as when holding a pencil, the fingers curl in the same anticlockwise direction. The situation is shown in the diagram on the right.

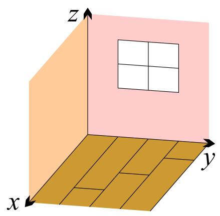


The Coordinate Axes in Three Dimensions: A third axis is required for drawing graphs in three dimensions, which is labelled the z-axis. The direction of the z-axis needs to be determined. Following the conventions established for two dimensions, the three axes should be at right angles to each other, and oriented in a right-handed way.

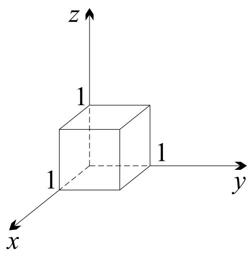
Start with an xy-coordinate plane. Then imagine that a right-handed screw is placed upright at the origin. The direction the screw advances when turned anticlockwise, from x-axis to y-axis, is taken to be the positive direction of the z-axis. This is also the direction of the thumb on the right hand, with fingers curled anticlockwise. The configuration is shown in the diagram on the right.

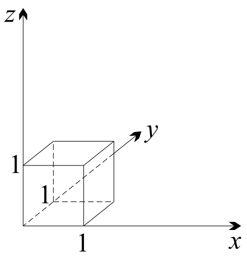


Another way to view this is to sit on the floor in the corner of a room where two walls and the floor meet, all at right angles to each other. To the right, along the line where one wall meets the floor, is the positive x-axis. To the left, along the line where the other wall meets the floor, is the positive y-axis. The positive z-axis is up the vertical line where the two walls meet.



How to draw the axes: Drawing a good representation of a three dimensional object on a page requires perspective, which may be difficult for those who are not artistically talented. Mathematicians use two common approaches to draw simple objects like cubes and rectangular prisms. Here they are showing the unit cube (with edge length 1) with one corner at the origin.





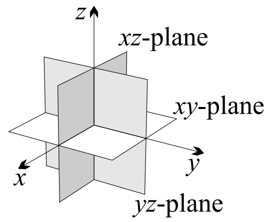
In the first instance, the y-axis is horizontal and the z-axis is vertical as though they lie in the plane of the page. The x-axis is drawn at about 40° to the horizontal, below and to the left of the origin on the page. This is a simplistic attempt to show perspective, as though the x-axis is coming up out of the page. It represents what is seen when standing in a room looking at the corner where two walls and the floor all meet. It has the advantage of being easy to draw, but is not a correct representation of perspective, and so some figures will look distorted when drawn this way.

In the second case, the x-axis is horizontal and the z-axis is vertical as though they lie in the plane of the page. The y-axis is drawn at about 40° to the horizontal, above and to the right of the origin on the page. This is another simplistic attempt to show perspective, as though the y-axis is heading into the page. It represents what is seen when looking into a glass display cabinet near the bottom left corner. Again, it has the advantage of being easy to draw, but is not a correct representation of perspective, and so some figures will look distorted.

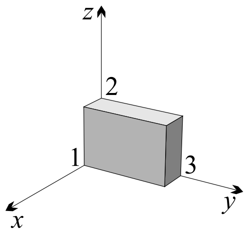
It is worthwhile practising these two approaches so that realistic looking pictures and graphs can be drawn quickly. Of course, those who are artistically talented may wish to draw the axes and other objects in genuine perspective. In many instances, the diagrams in this text have been drawn in true perspective.

The Coordinate Planes: Each pairing of the variables x, y and z corresponds to a coordinate plane. Thus there are three coordinate planes in three dimensions: the xy-plane, the xz-plane and the yz-plane. Each pair of planes meets at right angles along a line which is the common axis. Thus the xy-plane and yz-plane meet at the y-axis. All three planes meet at a single point, the origin.

In one dimension, the origin divides the number line into two opposite rays. In two dimensions, the two axes divide the plane into four regions called quadrants. In three dimensions, the three coordinate planes divide space into eight regions called octants. This is demonstrated in the diagram of the coordinate planes on the right.



Planes Parallel with the Coordinate Planes: Recall that in two dimensions the equation $x = 1$ represented a line parallel with the y-axis. In three dimensions it is a plane parallel with the yz-plane. Likewise the equation $y = 3$ is a plane parallel with the xz-axis. The equation $z = 2$ is a plane parallel with the xy-plane. Notice that in each case, the two letters not used in the equation indicate the parallel plane. Further observe that the three planes, together with the coordinate planes, form the six faces of a rectangular prism, as shown in the diagram.



The Coordinates of Points: Notice that the three planes $x = 1$, $y = 3$ and $z = 2$ in the last example intersect at a point. The coordinates of this point are written $(1, 3, 2)$. That is, the coordinates of a point are written as an ordered triple, in alphabetic order (x, y, z). Also notice that this point is in the octant where all three coordinates are positive. This is called the first octant. Though rarely needed in this course, octants are numbered from 1 to 4 with $z > 0$, corresponding to the four quadrants in two dimensions, and from 5 to 8 with $z < 0$.

1 THE COORDINATES OF POINTS: The coordinates of a point in three dimensions are written as an ordered triple, in alphabetic order, (x, y, z). The point with coordinates (a, b, c) is where the three planes $x = a$, $y = b$ and $z = c$ intersect.

Pythagoras: The formula for the distance between two points can be extended to three dimensions. Consider the distance OP, from the origin to the arbitrary point $P = (x, y, z)$ in the diagram below. Let Q be the foot of the perpendicular to the xy-plane, and let R be the foot of the perpendicular from Q to the x-axis. Using Pythagoras' theorem in $\triangle OPQ$,

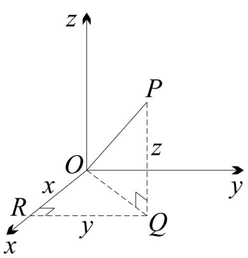
$$OP^2 = OQ^2 + z^2\,,$$

and in $\triangle OQR$,

$$OQ^2 = x^2 + y^2\,.$$

Combining these two gives the result

$$OP^2 = x^2 + y^2 + z^2\,.$$



Now suppose that the distance is measured to $S(x_1, y_1, z_1)$ instead. Then, by the shifting results established in Extension 1,

$$SP^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 .$$

Notice that the change of equation for shifting in three dimensions works in the same way as in two dimensions. That is, in order to shift up by z_1, replace z with $(z - z_1)$. Finally, for the specific point $T = (x_0, y_0, z_0)$, the distance formula in three dimensions is

$$ST^2 = (x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2 .$$

2 THE DISTANCE FORMULA: The distance from $S(x_1, y_1, z_1)$ to $T = (x_0, y_0, z_0)$ is

$$ST^2 = (x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2 .$$

WORKED EXAMPLE 1: Find the distance from $A(1, -2, 1)$ to $B(-1, 3, 5)$.

SOLUTION: Applying the formula above,

$$\begin{aligned} AB^2 &= (-1 - 1)^2 + (3 + 2)^2 + (5 - 1)^2 \\ &= 4 + 25 + 16 \\ &= 45 \end{aligned}$$

Hence $AB = 3\sqrt{5}$

Midpoint Formula: The midpoint formula in two dimensions can be stated in words as the averages of the coordinates. In three dimensions, the statement is the same. Thus if $M(x, y, z)$ is the midpoint of the interval joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2} \quad \text{and} \quad z = \frac{z_1 + z_2}{2}.$$

The usual proof of the formula in two dimensions uses congruent triangles. That method of proof works equally well in three dimensions, and is left as an exercise. Later in this chapter it will be possible to prove the result in three dimensions using vectors.

WORKED EXAMPLE 2: Find the midpoint of AB in the first worked example.

SOLUTION: Taking the averages of the coordinates, the midpoint is $(0, \frac{1}{2}, 3)$.

3 THE MIDPOINT FORMULA: Let $M(x, y, z)$ be the midpoint of the interval joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2} \quad \text{and} \quad z = \frac{z_1 + z_2}{2}.$$

Graphing Objects and Relations: Graphing relations in three dimensions can be notoriously difficult. Fortunately, often all that is needed in this course is a simple rectangular prism, so plot the vertices and then join them.

WORKED EXAMPLE 3: Draw the rectangular prism with faces parallel with the coordinate planes, for which the end points of one space diagonal are the origin and the point $(-2, 2, 4)$. Add this space diagonal to the diagram and find the centre of the prism.

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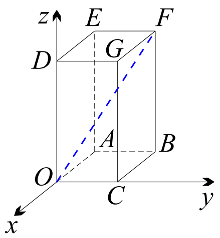
Note: a space diagonal joins opposite vertices and passes through the solid. In this case, because of symmetry, it passes through the centre of the prism.



SOLUTION: The eight vertices are: $O(0,0,0)$, $A(-2,0,0)$, $C(0,2,0)$, $B(-2,2,0)$, $D(0,0,4)$, $E(-2,0,4)$, $G(0,2,4)$ and $F(-2,2,4)$.

The space diagonal OF is shown dashed and blue in the diagram. The centre of the prism is the midpoint of OF, $(-1,1,2)$.

Exercise **5A**

1. Name the three coordinate planes and write down their corresponding equations.

2. In which octant does each of these points lie?

(a) $(-2,3,1)$ (c) $(2,-3,1)$ (e) $(2,-3,-1)$

(b) $(2,3,-1)$ (d) $(-2,3,-1)$ (f) $(-2,-3,1)$

3. Suppose that P is the point $(3,2,5)$. Write down the coordinates of the image of P under each of these transformations. (Assume that the orientation of the axes is the same as the diagram on the top of page 2.)

(a) P is translated 6 units down,

(b) P is translated 8 units backwards,

(c) P is translated 10 units to the right,

(d) P is translated 5 units forwards and 7 units up,

(e) P is translated 3 units to the left and 4 units down,

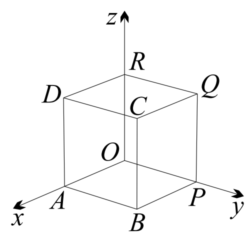
(f) P is reflected in the xy-plane,

(g) P is reflected in the yz-plane,

(h) P is reflected in the xz-plane,

(i) P is rotated about the x-axis through $180°$.

4. The diagram shows a cube of side 2 units.



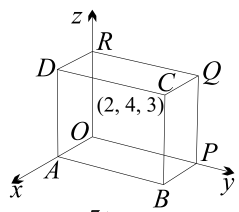
(a) Write down the coordinates of its vertices.

(b) Find the length of a face diagonal.

(c) Hence find the length of a space diagonal of the cube.

(d) Write down the equations of the planes corresponding to the faces of the prism.

5. The diagram shows a rectangular prism with vertex C which has coordinates $(2,4,3)$.



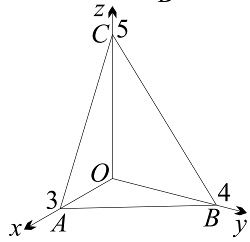
(a) Write down the coordinates of A, B, D, P, Q and R.

(b) Find the length of the face diagonal OB.

(c) Hence find the length of the space diagonal BR.

(d) Write down the equations of the planes corresponding to the faces of the prism.

6. The diagram shows a triangular pyramid $OABC$.



(a) Find the area of the base OAB.

(b) Hence find the volume of the pyramid.

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DEVELOPMENT

7. A triangle has vertices $O(0,0,0), A(2,6,3)$ and $B(-3,5,-8)$.

(a) Find the lengths of the three sides.

(b) Hence show that the triangle is right-angled.

8. A triangle has vertices $A(3,-1,-3), B(1,-5,7)$ and $C(-1,3,3)$. If M and N are the midpoints of AB and AC respectively, show that MN is half the length of BC.

9. An interval has endpoints $P(-6,-8,14)$ and $Q(-10,20,22)$.

(a) Find the midpoint M of the interval PQ.

(b) Hence find the points X and Y that divide the interval PQ in the ratios $1:3$ and $3:1$ respectively.

10. Show that the points $P(1,0,0), Q(-3,-1,1)$ and $R(-2,3,4)$ lie on a sphere with centre $C(-1,1,2)$.

11. If the distance from the point $(x, x+5, x-2)$ to the point $(1,0,-1)$ is $2\sqrt{6}$ units, find the value of x.

12. A triangle has vertices $A(4,2,6), B(-2,0,2)$ and $C(10,-2,4)$.

(a) Show that the triangle is isosceles.

(b) Show that the exact area of the triangle is $6\sqrt{19}\,\mathrm{u}^2$.

13. The equation $3x+4y+6z=12$ represents a plane in three dimensions.

(a) What is the equation of the intersection of this plane with the xy-plane?

(b) Write down the equations of the intersection of the given plane with each of the other coordinate planes.

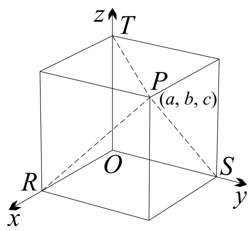
(c) What type of geometric object is each of these intersections?

(d) What geometric fact is confirmed in this question about two non-parallel planes?

ENRICHMENT

14. The diagram shows a rectangular prism with vertex P which has coordinates (a,b,c).

Use the converse of Pythagoras' theorem to prove that angles PRO, PSO and PTO are right-angles.



15. Consider the surface in three dimensional space with equation $x^2+y^2=4$.

(a) What is the significance of the fact that there is no term in z?

(b) What is the intersection of the surface with the horizontal plane $z=k$?

(c) Describe the surface.

(d) Sketch the surface.

16. Consider the paraboloid with equation $z=x^2+y^2$.

(a) Explain why $z \geq 0$.

(b) What is the intersection of the paraboloid with the horizontal plane $z=k$?

(c) What is the intersection of the paraboloid with the xz-plane?

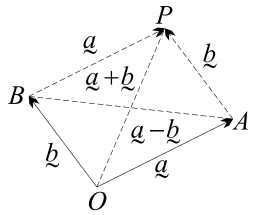
(d) Sketch the paraboloid.

5B Vectors in Three Dimensions

Vectors were developed carefully in the Extension 1 course and so many of the results are unchanged in three dimensions. Nevertheless, here is a quick review of that work.

A Review of Vectors: A vector has two characteristics, a length and a direction. Thus there is a strong connection between a vector and a directed line segment. There are three ways to denote a vector, $\overrightarrow{OA}$, $\underset{\sim}{a}$, or $\mathbf{a}$. The first two notations will be used in this text. The length of vector $\underset{\sim}{a}$ is indicated by $|\underset{\sim}{a}|$. The zero vector $\underset{\sim}{0}$ has zero length, that is $|\underset{\sim}{0}| = 0$, and has no specified direction.

Vectors may be treated as pronumerals when combined with real numbers in addition and subtraction, such as $3\underset{\sim}{a} + 2\underset{\sim}{b}$. Addition and subtraction may be represented as the diagonals of a parallelogram. Thus in $OAPB$, where $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$, the diagonals are $\overrightarrow{OP} = \underset{\sim}{a} + \underset{\sim}{b}$ and $\overrightarrow{BA} = \underset{\sim}{a} - \underset{\sim}{b}$.



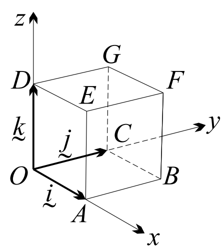
The product of a number and a vector, such as $3\underset{\sim}{a}$, is called scalar multiplication. The number itself is called a scalar. The result is a new vector. If $\underset{\sim}{b} = \lambda\underset{\sim}{a}$ and if $\lambda > 0$ then $\underset{\sim}{b}$ has the same direction as $\underset{\sim}{a}$ but $|\underset{\sim}{b}| = \lambda|\underset{\sim}{a}|$. If $\lambda < 0$ then $\underset{\sim}{b}$ has the opposite direction. When $\lambda = 0$, $\underset{\sim}{b} = \underset{\sim}{0}$ and $|\underset{\sim}{b}| = 0$.

All these results are also true in three dimensions, however there is an issue with drawing vectors in three dimensions. In two dimensions, the direction of a vector is simply shown by a directed line interval. When a three dimensional problem is drawn on a page, there is potential for ambiguity. The most obvious examples of this are optical illusions. Consequently, many problems will rely more heavily on component form, column vectors and ordered triples, instead of diagrams.

Worked Example 4: The diagram shows the unit cube $OABCDEFG$. Let $\overrightarrow{OA} = \underset{\sim}{i}$, $\overrightarrow{OC} = \underset{\sim}{j}$ and $\overrightarrow{OD} = \underset{\sim}{k}$. By writing each expression in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$, show that

$$\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BF} = \overrightarrow{OC} + \overrightarrow{CG} + \overrightarrow{GF}$$

and describe the resultant vector.



Solution:

$$\begin{aligned} \text{LHS} &= \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} \\ \text{RHS} &= \underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{i} \\ &= \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k} \qquad \text{(addition of vectors is commutative)} \\ &= \text{LHS}\,. \end{aligned}$$

The resultant vector is $\overrightarrow{OF}$ which is the space diagonal of the unit cube.

Component Form in 3D: The vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ in the last worked example are the familiar basis vectors used in two dimensions. In three dimensions, a third vector must be included in the basis, which is the unit vector $\underset{\sim}{k}$, aligned with the z-axis. These three vectors, $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$, are the standard right-handed orthonormal basis vectors in three dimensions.

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Corresponding to each point $A(x, y, z)$ in three dimensions is the position vector $\underset{\sim}{a} = \overrightarrow{OA}$ which can be written in component form as

$$\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}\,.$$

Like the two dimensional situation, vectors in three dimensions can be combined by taking the same combination of their components. Thus for the two vectors $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$, and for the two scalars λ and μ,

$$\lambda\underset{\sim}{u} + \mu\underset{\sim}{v} = (\lambda x_1 + \mu x_2)\underset{\sim}{i} + (\lambda y_1 + \mu y_2)\underset{\sim}{j} + (\lambda z_1 + \mu z_2)\underset{\sim}{k}\,.$$

WORKED EXAMPLE 5: Find $2\underset{\sim}{u} - \underset{\sim}{v}$ when $\underset{\sim}{u} = \underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$.

SOLUTION:
$$\begin{aligned} 2\underset{\sim}{u} - \underset{\sim}{v} &= (2\times 1 - 2)\underset{\sim}{i} + (2\times 4 + 1)\underset{\sim}{j} + (2\times(-3) - 1)\underset{\sim}{k} \\ &= 9\underset{\sim}{j} - 7\underset{\sim}{k}\,. \end{aligned}$$

Of course, such simple calculations will be done mentally in future.

4 BASIS VECTORS IN 3D: These are the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$, aligned with each of the coordinate axes.

Column Vectors in 3D: In many instances, manipulating vectors in component form is unwieldy, and it is better to use column vectors, as in two dimensions. Of course, in three dimensions the column vectors have one more component. Thus, the vector $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is written either using brackets

$$\underset{\sim}{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{or using parentheses} \qquad \underset{\sim}{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Like the situation in two dimensions, there is a strong link between the vector $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ and the point $A(x, y, z)$, and $\underset{\sim}{a}$ may be called the position vector of A. In order to avoid confusion between a column vector and the coordinates of a point, square-brackets are mostly used for column vectors in this text.

It immediately follows that in this notation the basis vectors are

$$\underset{\sim}{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \underset{\sim}{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \underset{\sim}{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and the zero vector is $\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Vector combinations can now be written more compactly, with all components grouped in the one place rather than spread across the page. Thus for the two vectors $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$, and the scalars λ and μ,

$$\begin{aligned} \lambda\underset{\sim}{u} + \mu\underset{\sim}{v} &= \lambda\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \mu\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda x_1 + \mu x_2 \\ \lambda y_1 + \mu y_2 \\ \lambda z_1 + \mu z_2 \end{bmatrix}. \end{aligned}$$

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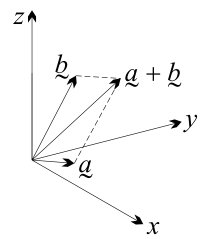
5

COLUMN VECTORS IN 3D:
- The vector $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is written as
$$\underset{\sim}{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{or} \qquad \underset{\sim}{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$
- The basis vectors are
$$\underset{\sim}{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \underset{\sim}{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \underset{\sim}{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$
- Vectors are combined in the natural way. Thus,
$$\lambda \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \mu \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 + \mu x_2 \\ \lambda y_1 + \mu y_2 \\ \lambda z_1 + \mu z_2 \end{bmatrix}.$$

WORKED EXAMPLE 6: Given $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$, determine $\underset{\sim}{a} + \underset{\sim}{b}$. Draw the situation, showing the vectors as position vectors, and indicating the parallelogram for addition of vectors.

SOLUTION:
$$\underset{\sim}{a} + \underset{\sim}{b} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$



The diagram on the right shows the parallelogram in true perspective. In this case, the parallelogram is easily seen because $\underset{\sim}{a}$ and $\underset{\sim}{b}$ lie in a vertical plane. In many problems the parallelogram will be harder to observe.

The Magnitude of a Vector: Suppose that the point $A(x, y, z)$ corresponds to the position vector $\underset{\sim}{a} = \overrightarrow{OA}$. That is $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$. Then it is clear that $|\underset{\sim}{a}| = |OA|$, the distance from the origin to the point A. The distance formula in three dimensions was developed in Section 5A, and applying it here gives
$$|\underset{\sim}{a}|^2 = x^2 + y^2 + z^2.$$
Notice that if $z = 0$ then this reduces to the familiar two dimensional formula.

WORKED EXAMPLE 7: Find the unit vector in the same direction as
$$\underset{\sim}{a} = -2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}.$$

SOLUTION: Clearly $|\underset{\sim}{a}|^2 = 14$, so the unit vector with the same direction is
$$\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{14}} \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}.$$

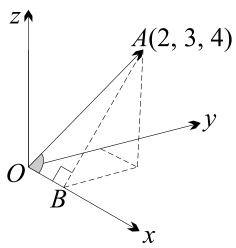
6 THE MAGNITUDE OF A VECTOR: The magnitude of $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is given by

$$|\underset{\sim}{a}|^2 = x^2 + y^2 + z^2\,.$$

In two dimensions, the direction of a non-zero vector was often associated with the angle it made with an axis. This is harder to visualise in three dimensions, but the angle a vector makes with each of the three axes can still be calculated.

WORKED EXAMPLE 8: Calculate the angle that $\underset{\sim}{a} = \overrightarrow{OA}$ makes with the x-axis, where $A = (2, 3, 4)$. Give your answer correct to the nearest degree.

SOLUTION: The situation is sketched on the right. The angle required is $\angle AOB$, which lies in the sloping plane of $\triangle AOB$.



$$|\underset{\sim}{a}|^2 = 29$$

so $\quad OA = \sqrt{29}$

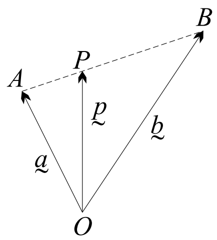
and $\quad OB = 2 \quad$ (the x-coordinate of A.)

Thus $\quad \angle AOB = \cos^{-1} \frac{2}{\sqrt{29}}$

$\qquad \doteqdot 68^\circ \quad$ (to the nearest degree.)

Ratio Division: Vectors can be used to derive a formula for the coordinates of a point that divides a given interval into a specified ratio. Suppose that the position vectors of the points A and B are $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Further suppose that AB is divided into the ratio $k : \ell$ by the point P with position vector $\underset{\sim}{p}$. The diagram below on the right shows the situation.

From the given ratio it follows that $AP = \frac{k}{k+\ell}AB$ and so



$$\begin{aligned}\overrightarrow{AP} &= \tfrac{k}{k+\ell}\overrightarrow{AB}\\ &= \tfrac{k}{k+\ell}(\underset{\sim}{b} - \underset{\sim}{a})\,.\end{aligned}$$

Hence, by vector addition,

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP}\\ &= \underset{\sim}{a} + \tfrac{k}{k+\ell}(\underset{\sim}{b} - \underset{\sim}{a})\\ &= (1 - \tfrac{k}{k+\ell})\underset{\sim}{a} + \tfrac{k}{k+\ell}\underset{\sim}{b}\\ &= \tfrac{\ell}{k+\ell}\underset{\sim}{a} + \tfrac{k}{k+\ell}\underset{\sim}{b}\,.\end{aligned}$$

That is $\quad \underset{\sim}{p} = \frac{1}{k+\ell}(\ell\underset{\sim}{a} + k\underset{\sim}{b})\,.$

Once the components of $\underset{\sim}{p}$ have been calculated by this formula, the coordinates of P can immediately be written down.

WORKED EXAMPLE 9:

(a) Find the coordinates of P that divides interval AB in the ratio $1 : 2$, where $A = (1, -3, 2)$ and $B = (-5, 6, -1)$.

(b) Check the result by calculating $\left|\overrightarrow{AB}\right|$ and $\left|\overrightarrow{AP}\right|$.

SOLUTION:

(a) In this case $k = 1$ and $\ell = 2$, so from the above result

$$\begin{aligned} \underset{\sim}{p} &= \frac{1}{3}\left(2 \times \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ 6 \\ -1 \end{bmatrix}\right) \\ &= \frac{1}{3}\begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Hence $P = (-1, 0, 1)$.

(b) First, using subtraction of vectors

$$\begin{aligned} \overrightarrow{AB} &= \underset{\sim}{b} - \underset{\sim}{a} \\ &= \begin{bmatrix} -6 \\ 9 \\ -3 \end{bmatrix} \end{aligned}$$

thus $AB^2 = 36 + 81 + 9$
$= 126$

and $AB = 3\sqrt{14}$.

Likewise $\overrightarrow{AP} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$

so $AP = \sqrt{14}$
$= \frac{1}{3}AB$ (as expected.)

7 | RATIO DIVISION: Suppose that interval AB is divided into the ratio $k : \ell$ by point P. Further suppose that the corresponding position vectors are $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{p}$. The coordinates of P can be found by the vector equation

$$\underset{\sim}{p} = \tfrac{1}{k+\ell}(\ell \underset{\sim}{a} + k\underset{\sim}{b}) \,.$$

When P lies between A and B, as in the above derivation of the formula, it is said that P divides AB internally. When P lies outside the interval AB it is said that P divides AB externally. The only change to the formula in this case is that one of k or ℓ is made negative. The proof of this result is left as an exercise.

Exercise **5B**

1. For the given point P in each part, express the vector $\overrightarrow{OP}$, where O is the origin, first (i) as a column vector, then (ii) as a component vector.

(a) $P(2, -3, 5)$ (b) $P(-4, 0, 13)$ (c) $P(a, -2a, -3a)$

2. Find the length of $\underset{\sim}{a}$ and a unit vector in the direction of $\underset{\sim}{a}$ if:

(a) $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{k}$ (b) $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

3. Find $|\underset{\sim}{v}|$ and find $\hat{\underset{\sim}{v}}$ in column vector form given: (a) $\underset{\sim}{v} = -1\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$ (b) $\underset{\sim}{v} = 5\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$

4. If $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$ and $\underset{\sim}{q} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$, find:

(a) $2\underset{\sim}{p} + \underset{\sim}{q}$ (b) $|2\underset{\sim}{p} + \underset{\sim}{q}|$ (c) $\underset{\sim}{p} - 5\underset{\sim}{q}$ (d) $|\underset{\sim}{p} - 5\underset{\sim}{q}|$

5. The points P and Q have position vectors $2\underset{\sim}{i} + 7\underset{\sim}{j} - \underset{\sim}{k}$ and $5\underset{\sim}{i} - 5\underset{\sim}{j} + 3\underset{\sim}{k}$ respectively. Find:

(a) $\overrightarrow{PQ}$ (b) $\overrightarrow{QP}$ (c) the distance PQ

6. The position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are $\begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$ respectively. Find:

(a) $\overrightarrow{BA}$ (b) $\overrightarrow{AB}$ (c) $|\overrightarrow{AB}|$

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7. Given that $\underset{\sim}{a} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -2 \\ -4 \\ 4 \end{bmatrix}$, find λ_1 and λ_2 such that $\lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{b} = \begin{bmatrix} 14 \\ 26 \\ -18 \end{bmatrix}$.

8. Given that $\underset{\sim}{a} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ and $\underset{\sim}{c} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$, find λ_1, λ_2 and λ_3 such that $\lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{b} + \lambda_3\underset{\sim}{c} = \begin{bmatrix} -7 \\ -14 \\ 7 \end{bmatrix}$.

9. Points A, B, C and D have position vectors $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 8 \\ -7 \end{bmatrix}$ respectively.

(a) Show that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

(b) Determine whether $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are parallel.

10. Use vectors to show that the points $A(-2, -1, 0)$, $B(0, 5, -2)$ and $C(4, 17, -6)$ are collinear.

11. Given the points $A(5, 4, 7)$, $B(7, -1, -4)$, $C(-1, -3, -5)$ and $D(-3, 2, 6)$, use vectors to show that $ABCD$ is a parallelogram.

12. The points A, B and C have position vectors $3\underset{\sim}{i} - 8\underset{\sim}{j} - 2\underset{\sim}{k}$, $2\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}$ and $-2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ respectively. Find the position vector of the point D so that $ABCD$ is a parallelogram.

13. Suppose that A is the point $(2, 1, 3)$. Follow the method of Worked Example 8 to determine, to the nearest degree, the respective angles that $\overrightarrow{OA}$ makes with the x, y and z axes.

14. The points A and B have position vectors $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 5 \\ -8 \end{bmatrix}$ respectively. Find the position vector of the point that:

(a) divides the line segment AB internally in the ratio $1 : 2$,

(b) divides the line segment AB externally in the ratio $1 : 2$.

15. The points A and B have position vectors $-4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ and $6\underset{\sim}{i} - 8\underset{\sim}{j} + 10\underset{\sim}{k}$ respectively. Find the position vector of the point that:

(a) divides the line segment AB internally in the ratio $2 : 3$,

(b) divides the line segment AB externally in the ratio $2 : 3$.

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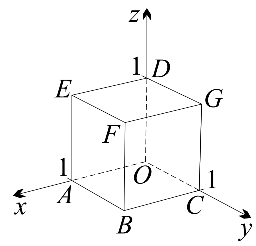
16. The diagram shows a cube of side length one unit.



(a) Write $\overrightarrow{AG}$ in component form.

(b) Find $|\overrightarrow{AG}|$.

(c) Use vectors to find $|\overrightarrow{OH}|$, where H is the centre of the square face $BCGF$.

17. Suppose that $\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Carefully write proofs of these distributive laws.

(a) $(\lambda_1 + \lambda_2)\underset{\sim}{a} = \lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{a}$, where $\lambda_1, \lambda_2 \in \mathbf{R}$. (b) $\lambda(\underset{\sim}{a} + \underset{\sim}{b}) = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}$, where $\lambda \in \mathbf{R}$.

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18. Three vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ in 3-dimensions are said to be *linearly independent* if the only solution to the equation $\lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{b} + \lambda_3\underset{\sim}{c} = \underset{\sim}{0}$ is the trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Determine whether or not each set of vectors is linearly independent.

(a) $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \underset{\sim}{c} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ (b) $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \underset{\sim}{c} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

19. The points $A(-2, 2, 5)$, $B(5, 7, 3)$ and $C(-3, 4, 1)$ are three vertices of a parallelogram. Find the three possibilities for the point D, the fourth vertex of the parallelogram.

20. Suppose that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero and non-parallel. Show that if $\lambda\underset{\sim}{a} + \mu\underset{\sim}{b} = \ell\underset{\sim}{a} + m\underset{\sim}{b}$ then $\lambda = \ell$ and $\mu = m$.

5C The Dot Product

Recall from Mathematics Extension 1 that the dot product is also called the scalar product. The reason for this is that the result is a scalar. However, in this text the term dot product is used, so as to avoid potential confusion with scalar multiplication of a vector.

The Dot Product in Geometric Form: Geometrically, the dot product in two dimensions is identical to three dimensions. Both situations involve the angle between two position vectors. There are three points associated with the position vectors and in three dimensions there is only one plane that passes through those three points. Thus, despite the extra freedom of configuration available in three dimensions, the action takes place in a plane. In other words, the dot product is always a two dimensional operation regardless of the situation.

Since there is no difference, any result established geometrically in two dimensions can equally be applied in three dimensions. In particular, for non-zero position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, the dot product is defined to be:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$$

where θ is the non-reflex angle between them. But if either vector is zero, then the dot product is zero. Thus,

$$\underset{\sim}{a} \cdot \underset{\sim}{0} = 0\,.$$

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8 THE DOT PRODUCT IN GEOMETRIC FORM: The geometric definition of the dot product for three dimensions is the same as for two dimensions.

- $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$ whenever $\underset{\sim}{a} \neq \underset{\sim}{0}$ and $\underset{\sim}{b} \neq \underset{\sim}{0}$
- $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ whenever $\underset{\sim}{a} = \underset{\sim}{0}$ or $\underset{\sim}{b} = \underset{\sim}{0}$

Other useful results that follow from the two-dimensional case include:

Magnitude: $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$, thus $|\underset{\sim}{a}| = \sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}}$

Commutative Law: $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$

Associative Law: $\lambda(\underset{\sim}{a} \cdot \underset{\sim}{b}) = (\lambda\underset{\sim}{a}) \cdot \underset{\sim}{b}$

Distributive Law: $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

The Cosine Rule: $|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b}$

All these results apply to any vectors. Of particular importance, however, is the following result for non-zero perpendicular vectors.

For all $\underset{\sim}{a} \neq \underset{\sim}{0}$ and $\underset{\sim}{b} \neq \underset{\sim}{0}$, $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ if and only if $\underset{\sim}{a} \perp \underset{\sim}{b}$.

9 PERPENDICULAR VECTORS: For all $\underset{\sim}{a} \neq \underset{\sim}{0}$ and $\underset{\sim}{b} \neq \underset{\sim}{0}$, $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ if and only if $\underset{\sim}{a} \perp \underset{\sim}{b}$.

From the geometric definition of the dot product in Box 7, and from the range of $\cos\theta$, it follows that

$$-|\underset{\sim}{a}|\,|\underset{\sim}{b}| \leq \underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a}|\,|\underset{\sim}{b}|\,,$$

which is sometimes called the *Cauchy-Schwarz inequality*. Thus the dot product and magnitude of a vector are consistent with the absolute value function for real numbers, for which

$$-|x| \leq x \leq |x|\,.$$

The vector inequality can also be derived using the dot product and the fact that the magnitude of a vector must be positive.

Consider the quadratic function $Q(t) = |\underset{\sim}{a} - t\underset{\sim}{b}|^2$. Clearly $Q(t) \geq 0$ for all $t \in \mathbf{R}$.

Now
$$\begin{aligned} Q(t) &= (\underset{\sim}{a} - t\underset{\sim}{b}) \cdot (\underset{\sim}{a} - t\underset{\sim}{b}) \\ &= |\underset{\sim}{a}|^2 - 2t\underset{\sim}{a} \cdot \underset{\sim}{b} + t^2|\underset{\sim}{b}|^2 \end{aligned}$$

The coefficient of t^2 is positive so $Q(t)$ has a minimum value at

$$t = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2}$$

for which
$$Q_{\min} = |\underset{\sim}{a}|^2 - \frac{(\underset{\sim}{a} \cdot \underset{\sim}{b})^2}{|\underset{\sim}{b}|^2}\,.$$

Hence
$$0 \leq |\underset{\sim}{a}|^2 - \frac{(\underset{\sim}{a} \cdot \underset{\sim}{b})^2}{|\underset{\sim}{b}|^2}\,.$$

Rearranging, $(\underset{\sim}{a} \cdot \underset{\sim}{b})^2 \leq |\underset{\sim}{a}|^2|\underset{\sim}{b}|^2$

and hence $-|\underset{\sim}{a}|\,|\underset{\sim}{b}| \leq \underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a}|\,|\underset{\sim}{b}|\,.$

The Triangle Inequality In Vector Form: The triangle inequality has been derived for real numbers and for complex numbers. In vector form, this inequality is

$$\Big|\,|\underset{\sim}{a}| - |\underset{\sim}{b}|\,\Big| \leq |\underset{\sim}{a} + \underset{\sim}{b}| \leq |\underset{\sim}{a}| + |\underset{\sim}{b}|\,.$$

Here is a proof of the right hand inequality. Begin by squaring the middle term.

$$\begin{aligned}|\underset{\sim}{a}+\underset{\sim}{b}|^2 &= (\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b})\\ &= |\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2\,.\end{aligned}$$

And so, from the Cauchy-Schwarz inequality developed above,

$$\begin{aligned}|\underset{\sim}{a}+\underset{\sim}{b}|^2 &\le |\underset{\sim}{a}|^2 + 2|\underset{\sim}{a}|\,|\underset{\sim}{b}| + |\underset{\sim}{b}|^2\\ &= (|\underset{\sim}{a}| + |\underset{\sim}{b}|)^2\,.\end{aligned}$$

Hence $\quad |\underset{\sim}{a}+\underset{\sim}{b}| \le |\underset{\sim}{a}| + |\underset{\sim}{b}|\,.$

The proof of the left hand inequality is a question in the exercise.

Worked Example 10: Verify the inequality $|\underset{\sim}{a}+\underset{\sim}{b}| \le |\underset{\sim}{a}| + |\underset{\sim}{b}|$ for the vectors

$$\underset{\sim}{a} = \begin{bmatrix}1\\-2\\3\end{bmatrix} \quad\text{and}\quad \underset{\sim}{b} = \begin{bmatrix}2\\3\\1\end{bmatrix}.$$

Solution: It should be clear in this case that $|\underset{\sim}{a}| = |\underset{\sim}{b}|$, and

$$\begin{aligned}|\underset{\sim}{a}|^2 &= \underset{\sim}{a}\cdot\underset{\sim}{a}\\ &= 1+4+9\\ &= 14,\end{aligned}$$

so $\quad |\underset{\sim}{a}| + |\underset{\sim}{b}| = 2\sqrt{14}\,.$

Also $\quad \underset{\sim}{a}+\underset{\sim}{b} = \begin{bmatrix}3\\1\\4\end{bmatrix}$

thus $\quad |\underset{\sim}{a}+\underset{\sim}{b}|^2 = 26\,.$

Now $\quad |\underset{\sim}{a}+\underset{\sim}{b}|^2 = 26 \quad$ and $\quad \Big(|\underset{\sim}{a}| + |\underset{\sim}{b}|\Big)^2 = 56\,,$

so $\quad |\underset{\sim}{a}+\underset{\sim}{b}|^2 \le \Big(|\underset{\sim}{a}| + |\underset{\sim}{b}|\Big)^2,$

hence $\quad |\underset{\sim}{a}+\underset{\sim}{b}| \le |\underset{\sim}{a}| + |\underset{\sim}{b}|\,.$

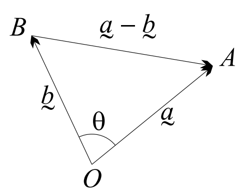
10 THE TRIANGLE INEQUALITY IN VECTOR FORM: For all $\underset{\sim}{a}$ and $\underset{\sim}{b}$,

$$\Big||\underset{\sim}{a}| - |\underset{\sim}{b}|\Big| \le |\underset{\sim}{a}+\underset{\sim}{b}| \le |\underset{\sim}{a}| + |\underset{\sim}{b}|\,.$$

The Dot Product in Component Form: Suppose $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are the non-zero vectors

$$\underset{\sim}{a} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} \quad\text{and}\quad \underset{\sim}{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}.$$

Notice the use of subscripts in the components, with 1 for the x-component, 2 for y, and 3 for z. This notation may be new to some readers, but it is commonly used by mathematicians when dealing with vectors.



As in two dimensions, the component form of the dot product is obtained by applying the cosine rule to the associated triangle, as shown in the diagram on the right.

Thus $\quad \left|\overrightarrow{BA}\right|^2 = \left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{OB}\right|^2 - 2\left|\overrightarrow{OA}\right|\,\left|\overrightarrow{OB}\right|\cos\theta$

so $\quad |\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos\theta$ [1]

but $\quad |\underset{\sim}{a} - \underset{\sim}{b}|^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2$

$$= \left(a_1{}^2 + a_2{}^2 + a_3{}^2\right) + \left(b_1{}^2 + b_2{}^2 + b_3{}^2\right) - 2\left(a_1b_1 + a_2b_2 + a_3b_3\right)$$

thus $\quad |\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\left(a_1b_1 + a_2b_2 + a_3b_3\right)$. [2]

Equating the right hand sides of [1] and [2], it is clear that

$$|\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3\,,$$

that is, $\quad \underset{\sim}{a}\cdot\underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3\,.$

Clearly the last line is valid even if $\underset{\sim}{a} = \underset{\sim}{0}$ or $\underset{\sim}{b} = \underset{\sim}{0}$. Consequently this algebraic expression is sometimes used as the definition of the dot product, instead of the geometric definition. Also notice that if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ lie in the xy-plane then

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = a_1b_1 + a_2b_2 \qquad \text{(because } a_3 = b_3 = 0 \text{ in the } xy\text{-plane.)}$$

In other words, the component formula in three dimensions is consistent with the two dimensional formula.

WORKED EXAMPLE 11: Verify the inequality $-|\underset{\sim}{a}|\,|\underset{\sim}{b}| \le \underset{\sim}{a}\cdot\underset{\sim}{b} \le |\underset{\sim}{a}|\,|\underset{\sim}{b}|$ for

$$\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k} \quad \text{and} \quad \underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}\,.$$

SOLUTION: From Worked Example 10, $|\underset{\sim}{a}| = |\underset{\sim}{b}| = \sqrt{14}$, so

$$|\underset{\sim}{a}|\,|\underset{\sim}{b}| = 14.$$

Also $\quad \underset{\sim}{a}\cdot\underset{\sim}{b} = 2 - 6 + 3$

$\quad = -1.$

Thus $\quad -14 \le -1 \le 14$

that is, $-|\underset{\sim}{a}|\,|\underset{\sim}{b}| \le \underset{\sim}{a}\cdot\underset{\sim}{b} \le |\underset{\sim}{a}|\,|\underset{\sim}{b}|$, as expected.

11 THE DOT PRODUCT IN COMPONENT FORM:

Let $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$ then

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3\,,$$

which is valid even when $\underset{\sim}{a} = \underset{\sim}{0}$ or $\underset{\sim}{b} = \underset{\sim}{0}$.

Although it will not be needed very often in this course, this last formula may also be written using summation notation.

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = \sum_{i=1}^{3} a_ib_i\,.$$

WORKED EXAMPLE 12: Consider the three vectors

$$\underset{\sim}{u} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix},\ \underset{\sim}{v} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix},\ \underset{\sim}{w} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Use appropriate dot products to show that the three vectors are orthonormal. That is, they are mutually perpendicular, and each is a unit vector.

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Solution: First calculate the magnitudes of the vectors.

$$|\underset{\sim}{u}|^2 = \underset{\sim}{u}\cdot\underset{\sim}{u} = \tfrac{1}{2}+\tfrac{1}{2}+0 = 1 \qquad |\underset{\sim}{v}|^2 = \underset{\sim}{v}\cdot\underset{\sim}{v} = \tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{2} = 1 \qquad |\underset{\sim}{w}|^2 = \underset{\sim}{w}\cdot\underset{\sim}{w} = \tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{2} = 1$$

Thus all three are unit vectors. Now check they are perpendicular.

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = -\tfrac{1}{2\sqrt{2}}+\tfrac{1}{2\sqrt{2}}+0 = 0 \qquad \underset{\sim}{u}\cdot\underset{\sim}{w} = \tfrac{1}{2\sqrt{2}}-\tfrac{1}{2\sqrt{2}}+0 = 0 \qquad \underset{\sim}{v}\cdot\underset{\sim}{w} = -\tfrac{1}{4}-\tfrac{1}{4}+\tfrac{1}{2} = 0$$

Hence the three vectors are unit vectors and mutually perpendicular.

Worked Example 13: Find any values of λ for which $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular,

$$\text{where} \quad \underset{\sim}{a} = \begin{bmatrix}\lambda\\1\\2\end{bmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{bmatrix}\lambda-1\\2\\-4\end{bmatrix}$$

Solution: The vectors will be perpendicular if $\underset{\sim}{a}\cdot\underset{\sim}{b}=0$.

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = \lambda^2-\lambda+2-8$$

thus $\lambda^2-\lambda-6=0\,,$

so $\lambda = -2 \text{ or } 3\,.$

When $\lambda=-2$ the vectors are $\underset{\sim}{a} = \begin{bmatrix}-2\\1\\2\end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix}-3\\2\\-4\end{bmatrix}$,

and when $\lambda=3$ the vectors are $\underset{\sim}{a} = \begin{bmatrix}3\\1\\2\end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix}2\\2\\-4\end{bmatrix}$.

Exercise 5C

1. Find the value of $\underset{\sim}{a}\cdot\underset{\sim}{b}$ if:
(a) $|\underset{\sim}{a}|=4$, $|\underset{\sim}{b}|=6$ and the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 45°,
(b) $|\underset{\sim}{a}|=5$, $|\underset{\sim}{b}|=8$ and the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 120°.

2. Find $\underset{\sim}{a}\cdot\underset{\sim}{b}$ given:
(a) $\underset{\sim}{a} = 3\underset{\sim}{i}-\underset{\sim}{j}+5\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i}+6\underset{\sim}{j}+\underset{\sim}{k}$
(b) $\underset{\sim}{a} = \begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$
(c) $\underset{\sim}{a} = a_1\underset{\sim}{i}+a_2\underset{\sim}{j}+a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i}+b_2\underset{\sim}{j}+b_3\underset{\sim}{k}$

3. Given $\underset{\sim}{a} = a_1\underset{\sim}{i}+a_2\underset{\sim}{j}+a_3\underset{\sim}{k}$, prove that $\underset{\sim}{a}\cdot\underset{\sim}{a} = |\underset{\sim}{a}|^2$.

4. Suppose that $\underset{\sim}{a} = 2\underset{\sim}{i}-7\underset{\sim}{j}+3\underset{\sim}{k}$ and $\underset{\sim}{b} = -4\underset{\sim}{i}+\underset{\sim}{j}+5\underset{\sim}{k}$.
(a) Find $\underset{\sim}{a}\cdot\underset{\sim}{b}$.
(b) What can we conclude about $\underset{\sim}{a}$ and $\underset{\sim}{b}$?

5. Given $\underset{\sim}{a} = \begin{bmatrix}13\\23\\7\end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix}2\\1\\-7\end{bmatrix}$ and $\underset{\sim}{c} = \begin{bmatrix}3\\-2\\1\end{bmatrix}$, show that $\underset{\sim}{a}$ is perpendicular to both $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

6. Given $\underset{\sim}{a} = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 8 \\ 4 \\ -10 \end{bmatrix}$, find:

(a) $\underset{\sim}{a} \cdot \underset{\sim}{a}$ (b) $2\underset{\sim}{b} \cdot \underset{\sim}{b}$ (c) $\underset{\sim}{a} \cdot \underset{\sim}{b}$ (d) $\underset{\sim}{a} \cdot \left(\underset{\sim}{a} + \underset{\sim}{b}\right)$

DEVELOPMENT

7. Confirm that the Cauchy-Schwarz inequality is satisfied in each case below.

(a) $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$ (b) $\underset{\sim}{a} = -\underset{\sim}{i} + 3\underset{\sim}{j}, \underset{\sim}{b} = -6\underset{\sim}{j} + 2\underset{\sim}{k}$

8. Confirm that the triangle inequality holds for each part in the previous question.

9. The points A, B, C and D have respective position vectors $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \\ -5 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$.

Show that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular.

10. Find any values of λ for which $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

(a) $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -2 \\ -5 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 3 \\ \lambda \\ -2 \end{bmatrix}$ (b) $\underset{\sim}{a} = \begin{bmatrix} -4 \\ \lambda + 3 \\ 2 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} \lambda \\ 5 \\ -\lambda^2 \end{bmatrix}$

11. Find a vector that is perpendicular to both $\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$.

12. Given $\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\lambda \in \mathbf{R}$, prove that $\underset{\sim}{a} \cdot (\lambda b) = \lambda \left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right)$.

13. Let $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$, $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$ and $\underset{\sim}{c} = c_1\underset{\sim}{i} + c_2\underset{\sim}{j} + c_3\underset{\sim}{k}$.
Prove the distributive law: $\underset{\sim}{a} \cdot \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$.

14. A set of three vectors in three dimensions is called *orthonormal* if each is a unit vector and the three vectors are mutually orthogonal (that is, perpendicular). In each part show that the three vectors are orthonormal.

(a) $\underset{\sim}{u} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \underset{\sim}{v} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \underset{\sim}{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(b) $\underset{\sim}{u} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \underset{\sim}{v} = \dfrac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ \sqrt{6} \\ -1 \end{bmatrix}, \underset{\sim}{w} = \dfrac{1}{2\sqrt{2}}\begin{bmatrix} -\sqrt{3} \\ \sqrt{2} \\ \sqrt{3} \end{bmatrix}$

15. Given three non-zero vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, prove that:

(a) if $\underset{\sim}{a} \cdot \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{b} \cdot \left(\underset{\sim}{a} - \underset{\sim}{c}\right)$ then $\underset{\sim}{c} \cdot \left(\underset{\sim}{a} + \underset{\sim}{b}\right) = 0$,

(b) if $\left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right)\underset{\sim}{c} = \left(\underset{\sim}{b} \cdot \underset{\sim}{c}\right)\underset{\sim}{a}$ then $\underset{\sim}{a}$ and $\underset{\sim}{c}$ are parallel or $\underset{\sim}{b}$ is perpendicular to both.

16. Given two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, prove that:

(a) if $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$ are perpendicular then $|\underset{\sim}{a}| = |\underset{\sim}{b}|$,

(b) if $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a} - \underset{\sim}{b}|$ then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

17. The points A, B and C have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively relative to the origin O. If $\overrightarrow{AB} \perp \overrightarrow{OC}$ and $\overrightarrow{BC} \perp \overrightarrow{OA}$, prove that $\overrightarrow{AC} \perp \overrightarrow{OB}$.

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18. Given that $|\underset{\sim}{a}| = 2$, $|\underset{\sim}{b}| = 3$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 5$, find $|\underset{\sim}{a} + \underset{\sim}{b}|$.

19. Given that $|\underset{\sim}{u}| = 2\sqrt{2}$, $|\underset{\sim}{v}| = 2\sqrt{3}$ and $\underset{\sim}{u} \cdot \underset{\sim}{v} = -4$, find $|\underset{\sim}{u} - \underset{\sim}{v}|$.

20. In the notes, the right hand inequality of the triangle inequality is proven. Adopt a similar approach to prove the left hand inequality, that is, $\big||\underset{\sim}{a}| - |\underset{\sim}{b}|\big| \le |\underset{\sim}{a} + \underset{\sim}{b}|$.

ENRICHMENT

21. Two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are such that $\underset{\sim}{a} + \underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$ and $|\underset{\sim}{b}| = |\underset{\sim}{a}|\sqrt{2}$. Show that $2\underset{\sim}{a} + \underset{\sim}{b}$ is perpendicular to $\underset{\sim}{b}$.

22. A vector in 3-dimensional space makes angles of α, β and γ with the x, y and z axes respectively. Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

5D Applications of the dot product

The most obvious use of the dot product is to find the angle between two position vectors. Rearranging the geometric formula for the dot product gives

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}\,.$$

The right hand side is evaluated using the algebraic definition in Box 11. Note that there will only ever be one solution to the trigonometric equation. This is because the geometry of the situation requires that the angle must be non-reflex. That is, $0° \le \theta \le 180°$, and so

$$\theta = \cos^{-1}\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}\right).$$

Worked Example 14: Find the angle at the origin subtended by AB for the points $A = (1, 1, 2)$ and $B = (-2, 3, -1)$. Round the answer to the nearest degree.

Solution: Let $\angle AOB = \theta$, and let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$, then

$$\underset{\sim}{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix},$$

$$\begin{aligned}
&\text{so} & |\underset{\sim}{a}|^2 &= 1 + 1 + 4 \\
&\text{or} & |\underset{\sim}{a}| &= \sqrt{6}\,, \\
&\text{and} & |\underset{\sim}{b}|^2 &= 4 + 9 + 1 \\
&\text{so} & |\underset{\sim}{b}| &= \sqrt{14} \\
&\text{with} & \underset{\sim}{a} \cdot \underset{\sim}{b} &= -2 + 3 - 2 \\
& & &= -1\,. \\
&\text{Thus} & \cos\theta &= \frac{-1}{2\sqrt{21}} \\
&\text{and hence} & \theta &\doteqdot 96° \quad \text{(to the nearest degree.)}
\end{aligned}$$

Direction Cosines: In two dimensions, it is easy to demonstrate the direction of a vector by drawing a graph. If the angles with the axes are needed then they can be measured from that graph. The problem with this is that any measurement is just an approximation, and in three dimensions the angles cannot be measured from a graph because perspective distorts the angles.

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One way to deal with the angles between a vector and the coordinate axes is to use the dot product. Suppose that the angle θ between $\underset{\sim}{a}$ and the unit vector $\widehat{\underset{\sim}{u}}$ is required, where $\widehat{\underset{\sim}{u}}$ is aligned with one of the coordinate axes. Then

$$\underset{\sim}{a} \cdot \widehat{\underset{\sim}{u}} = |\underset{\sim}{a}| \times 1 \times \cos\theta$$

so
$$\cos\theta = \tfrac{1}{|\underset{\sim}{a}|}\underset{\sim}{a} \cdot \widehat{\underset{\sim}{u}}\,.$$

But $\tfrac{1}{|\underset{\sim}{a}|}\underset{\sim}{a}$ is simply the unit vector $\widehat{\underset{\sim}{a}}$ so

$$\cos\theta = \widehat{\underset{\sim}{a}} \cdot \widehat{\underset{\sim}{u}}\,.$$

Thus replacing the unit vector $\widehat{\underset{\sim}{u}}$ with each basis vector gives the cosine of the angle between $\widehat{\underset{\sim}{a}}$ and the corresponding coordinate axis. Hence they are called *direction cosines*. But

$$\widehat{\underset{\sim}{a}} \cdot \underset{\sim}{i} = \widehat{a}_1\,, \qquad \widehat{\underset{\sim}{a}} \cdot \underset{\sim}{j} = \widehat{a}_2\,, \qquad \widehat{\underset{\sim}{a}} \cdot \underset{\sim}{k} = \widehat{a}_3\,.$$

In other words, the direction cosines of a vector are simply the components of the corresponding unit vector. It is therefore not surprising that the direction of a vector in three dimensions is completely determined if the three direction cosines are known.

WORKED EXAMPLE 15: Evaluate the direction cosines for $\underset{\sim}{a} = \overrightarrow{OA}$, where A is the point $(2, 3, 4)$. Hence give the angle to each axis correct to the nearest degree.

SOLUTION: First calculate $|\underset{\sim}{a}|$.

$$\begin{aligned}|\underset{\sim}{a}|^2 &= 4 + 9 + 16\\ &= 29\,.\end{aligned}$$

Hence the direction cosines of $\underset{\sim}{a}$ are:

$$\widehat{a}_1 = \tfrac{2}{\sqrt{29}}\,, \quad \widehat{a}_2 = \tfrac{3}{\sqrt{29}}\,, \quad \text{and} \quad \widehat{a}_3 = \tfrac{4}{\sqrt{29}}\,.$$

The corresponding angles with the axes are:

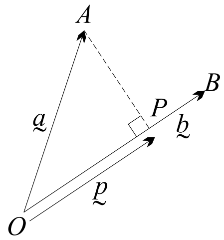
$$\angle AOX \doteqdot 68^\circ\,, \quad \angle AOY \doteqdot 56^\circ \quad \text{and} \quad \angle AOZ \doteqdot 42^\circ\,.$$

12 DIRECTION COSINES: The direction cosines of a vector $\underset{\sim}{a}$ are the components of the unit vector $\widehat{\underset{\sim}{a}}$, viz:

$$\widehat{\underset{\sim}{a}} \cdot \underset{\sim}{i} = \widehat{a}_1\,, \qquad \widehat{\underset{\sim}{a}} \cdot \underset{\sim}{j} = \widehat{a}_2\,, \qquad \widehat{\underset{\sim}{a}} \cdot \underset{\sim}{k} = \widehat{a}_3\,.$$

Projections in 3D: Like the dot product, a projection involves two non-zero vectors and hence the action takes place in a plane. Thus the results for projections established in two dimensions apply equally in three dimensions. Nevertheless, here is a quick derivation of the formula for the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$ in three dimensions. Once again, there is only one plane which passes through all three points. Thus it is possible to find a point P in OB such that $\triangle AOP$ lies in this plane and $OB \perp PA$.



Now
$$\overrightarrow{OP} = \lambda\underset{\sim}{b} \qquad (OP||OB)$$

so
$$\begin{aligned}\overrightarrow{PA} &= \overrightarrow{OA} - \overrightarrow{OP}\\ &= \underset{\sim}{a} - \lambda\underset{\sim}{b}\,.\end{aligned}$$

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$$
\begin{array}{lll}
\text{Also} & \overrightarrow{OB} \cdot \overrightarrow{PA} = 0 & (OB \perp PA.) \\
\text{Hence} & \underset{\sim}{b} \cdot (\underset{\sim}{a} - \lambda \underset{\sim}{b}) = 0 & \\
\text{or} & \underset{\sim}{b} \cdot \underset{\sim}{a} - \lambda \underset{\sim}{b} \cdot \underset{\sim}{b} = 0 & \\
\text{thus} & \lambda = \left(\dfrac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \right). &
\end{array}
$$

The vector $\overrightarrow{OP} = \lambda \underset{\sim}{b}$ is called the *projection* of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$, and is written

$$\operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \left(\frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \right) \underset{\sim}{b}\,.$$

WORKED EXAMPLE 16: Find the projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$ for $A = (4, 2, -3)$ and $B = (-1, 1, 1)$.

SOLUTION: Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{b} = \overrightarrow{OB}$. Clearly $\underset{\sim}{b} \cdot \underset{\sim}{b} = 3$.

$$
\begin{aligned}
\underset{\sim}{a} \cdot \underset{\sim}{b} &= -4 + 2 - 3 \\
&= -5\,,
\end{aligned}
$$

so $$\operatorname{proj}_{\underset{\sim}{a}} \underset{\sim}{b} = -\frac{5}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

13 **PROJECTIONS IN 3D:** The formula is the same as for two dimensions, viz:

$$\operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b}\,.$$

The above formula is often the most convenient way to calculate the projected vector. However, it can also be determined from the unit vector $\widehat{\underset{\sim}{b}}$ by observing that $\underset{\sim}{b} \cdot \underset{\sim}{b} = |\underset{\sim}{b}|^2$. Thus:

$$
\begin{array}{ll}
& \operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{|\underset{\sim}{b}|} \dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|} \\
\text{so} & \operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \left(\widehat{\underset{\sim}{b}} \cdot \underset{\sim}{a} \right) \widehat{\underset{\sim}{b}} \\
\text{or} & \operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{a} = (|\underset{\sim}{a}| \cos\theta) \widehat{\underset{\sim}{b}}\,.
\end{array}
$$

From this alternative formula, it is clear that the length of the projection is

$$|\operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{a}| = |\underset{\sim}{a}| \cos\theta\,,$$

which could have been deduced just as easily from trigonometry in $\triangle OAP$.

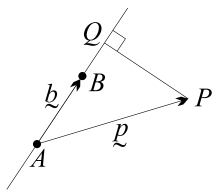
A special case of projection onto a unit vector occurs when $\widehat{\underset{\sim}{b}}$ is one of the basis vectors. For example, when $\widehat{\underset{\sim}{b}} = \underset{\sim}{k}$,

$$
\begin{aligned}
\operatorname{proj}_{\underset{\sim}{k}} \underset{\sim}{a} &= |\underset{\sim}{a}| \cos\theta \underset{\sim}{k} \\
&= |\underset{\sim}{a}| \frac{a_3}{|\underset{\sim}{a}|} \underset{\sim}{k} \qquad \text{(from the direction cosine)} \\
&= a_3 \underset{\sim}{k}\,.
\end{aligned}
$$

That is, the projection of a vector onto a basis vector gives the corresponding component vector. Of course, this is intuitively obvious, but at least the example demonstrates that the projection does what is expected.

14 | PROJECTIONS AND COMPONENTS: The projection of a vector onto a basis vector gives the corresponding component vector.

Perpendicular Distance: A projection can be used to find the perpendicular distance between a point and a line. The diagram to the right shows line AB with point P not on the line. Let Q be the point on the line that is closest to P, then $PQ \perp AB$. More significantly $\overrightarrow{AQ}$ is the projection of $\overrightarrow{AP}$ onto $\overrightarrow{AB}$. The distance that is required is then $|\overrightarrow{PQ}| = |\overrightarrow{AQ} - \overrightarrow{AP}|$.



For simplicity, put $\overrightarrow{AB} = \underset{\sim}{b}$ and $\overrightarrow{AP} = \underset{\sim}{p}$. Then

$$\overrightarrow{AQ} = \operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{p}$$

thus $$\overrightarrow{PQ} = \operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{p} - \underset{\sim}{p}$$

and hence $|\overrightarrow{PQ}| = \left|\operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{p} - \underset{\sim}{p}\right|$.

WORKED EXAMPLE 17: Find the perpendicular distance from $P = (2, 1, 0)$ to the line through $A = (-1, 0, 2)$ and $B = (1, 1, 3)$.

SOLUTION: Let $\underset{\sim}{p} = \overrightarrow{AP} = 3\underset{\sim}{i} + 1\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \overrightarrow{AB} = 2\underset{\sim}{i} + 1\underset{\sim}{j} + 1\underset{\sim}{k}$. Then

$$|\underset{\sim}{b}|^2 = 6$$
$$\underset{\sim}{b} \cdot \underset{\sim}{p} = 5$$

so $$\operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{p} = \frac{5}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Thus if Q is the point on AB nearest to P then

$$\begin{aligned} \overrightarrow{PQ} &= \operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{p} - \underset{\sim}{p} \\ &= \frac{5}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} -8 \\ -1 \\ 17 \end{bmatrix} \end{aligned}$$

hence $$|\overrightarrow{PQ}| = \tfrac{\sqrt{354}}{6} \qquad \text{(about } 3{\cdot}1358.)$$

15 | PERPENDICULAR DISTANCE: To find the perpendicular distance from the point P to the line AB, first let $\underset{\sim}{b} = \overrightarrow{AB}$ and $\underset{\sim}{p} = \overrightarrow{AP}$. Then

$$\text{distance} = \left|\operatorname{proj}_{\underset{\sim}{b}} \underset{\sim}{p} - \underset{\sim}{p}\right|.$$

Although the above derivation of the distance formula assumed that P was not on the line, the formula also works when P is on the line. The proof of this result is left as an exercise.

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Exercise 5D

1. If $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, find:

(a) $\underset{\sim}{a} \cdot \underset{\sim}{b}$ (b) $|\underset{\sim}{a}|$ and $|\underset{\sim}{b}|$ (c) the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

2. Find $\cos\theta$, where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(a) $\underset{\sim}{a} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ (b) $\underset{\sim}{a} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

3. If θ is the angle between $\underset{\sim}{a} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}$, show that $\cos\theta = \dfrac{3}{2\sqrt{143}}$.

4. Find, correct to the nearest degree, the angle between $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ given:

(a) $\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\underset{\sim}{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (b) $\underset{\sim}{v}_1 = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$, $\underset{\sim}{v}_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$

5. Suppose that $\underset{\sim}{a} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$. Use the standard result $\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a} = \left(\dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b}$ to confirm that the respective projections of $\underset{\sim}{a}$ onto each of the standard basis vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are the components of $\underset{\sim}{a}$, that is, $3\underset{\sim}{i}$, $-2\underset{\sim}{j}$ and $5\underset{\sim}{k}$.

DEVELOPMENT

6. Find the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ given:

(a) $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$ (b) $\underset{\sim}{a} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

7. Find the length of the projection $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ given:

(a) $\underset{\sim}{a} = 2i + 3\underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}$ (b) $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$

8. $\triangle ABC$ has vertices $A(2, 7, -12)$, $B(-1, 5, -5)$ and $C(4, 1, -4)$.

(a) Write $\overrightarrow{BA}$ and $\overrightarrow{BC}$ in component form.

(b) Hence show that $\angle ABC = 90^\circ$.

(c) Find the side lengths of the triangle and show that they satisfy Pythagoras' theorem.

9. $\triangle ABC$ has vertices $A(3, -3, 1)$, $B(-2, 1, 2)$ and $C(4, 0, -1)$.

(a) Write $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as column vectors.

(b) Hence find $\angle BAC$ correct to the nearest degree.

10. $\triangle PQR$ has vertices $P(-4, -1, 6)$, $Q(-5, 3, 4)$ and $R(-3, 4, -7)$. Use the scalar product to find $\angle PQR$ to the nearest minute.

11. $\triangle ABC$ has vertices $A(1, 0, -1)$, $B(1, 1, 1)$ and $C(0, 1, -1)$.

(a) Show that $\cos\angle ACB = \frac{1}{\sqrt{10}}$.

(b) Use the formula area $= \frac{1}{2}ab\sin C$ to find the area of $\triangle ABC$.

12. Let P, A and B be the points $(-4, 3, -1)$, $(3, 2, 1)$ and $(0, -4, 1)$ respectively.

(a) Find $\overrightarrow{AP}$ and $\overrightarrow{AB}$.

(b) Find $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{p}$, where $\overrightarrow{AP} = \underset{\sim}{p}$ and $\overrightarrow{AB} = \underset{\sim}{b}$.

(c) Find the perpendicular distance d from P to the line AB using $d = \left|\text{proj}_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$.

13. Use the approach of the previous question to find the perpendicular distance from the point P to the line through A and B.

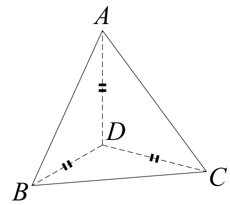
(a) $P = (3, -2, 1), A = (1, -11, -4), B = (9, 3, 8)$

(b) $P = (0, 0, 3), A = (1, 2, 1), B = (4, 0, 0)$

14. A cube with side length a has vertices at $O(0, 0, 0)$, $A(a, 0, 0)$, $B(a, a, 0)$, $C(0, a, 0)$, $D(0, 0, a)$, $E(a, 0, a)$, $F(a, a, a)$ and $G(0, a, a)$. Use vector methods to show that the acute angle between the diagonals AG and CE is $\arccos \frac{1}{3}$.

15. A rectangular prism is 1 unit by 2 units by 3 units. By giving its vertices appropriate coordinates, find the three possible values of $\cos\theta$, where θ is the acute angle between a pair of diagonals of the prism.

16. The diagram shows a triangular pyramid. Its base is a right-angled isosceles triangle, and its perpendicular height AD is equal to the length of the equal sides of the base. By giving the vertices of the pyramid appropriate coordinates, use vector methods to show that the acute angle between the front face ABC and the base BCD is $\cos^{-1} \frac{1}{\sqrt{3}}$.



17. The position vectors of the vertices of a tetrahedron (that is, a triangular pyramid) $ABCD$ are $\overrightarrow{OA} = -5\underset{\sim}{i} + 22\underset{\sim}{j} + 5\underset{\sim}{k}$, $\overrightarrow{OB} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, $\overrightarrow{OC} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\overrightarrow{OD} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ respectively.

(a) Find $\angle CBD$.

(b) Show that AB is perpendicular to both BC and BD.

(c) Calculate the volume of the tetrahedron.

18. The points A and B have position vectors $\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\overrightarrow{OB} = 6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$. The point P lies on AB and $AP : PB = \lambda : 1 - \lambda$, where $\lambda \in \mathbf{R}$.

(a) Show that $\overrightarrow{OP} = (2 + 4\lambda)\underset{\sim}{i} + (1 - 4\lambda)\underset{\sim}{j} + (4\lambda - 2)\underset{\sim}{k}$.

(b) Find the value of λ for which $\overrightarrow{OP}$ and $\overrightarrow{AB}$ are perpendicular.

(c) Find the value of λ for which $\angle AOP = \angle BOP$.

19. Find the possible values of λ if the angle between $\underset{\sim}{a} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -2 \\ -4 \\ \lambda \end{bmatrix}$ is $\cos^{-1} \frac{4}{21}$.

20. The points A, B and P have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{p} = \lambda\underset{\sim}{a} + (1 - \lambda)\underset{\sim}{b}$, where $\lambda \in \mathbf{R}$.

(a) Prove that A, B and P are collinear.

(b) Given that $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{b} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$ and O is the origin, find the two values of λ for which $\angle AOP = 60°$.

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21. The position vectors of the points A, B and C are $\overrightarrow{OA} = 9\underset{\sim}{i} + 7\underset{\sim}{j} - \underset{\sim}{k}$, $\overrightarrow{OB} = 3\underset{\sim}{i} - 11\underset{\sim}{j} + 5\underset{\sim}{k}$ and $\overrightarrow{OC} = 5\underset{\sim}{i} - 5\underset{\sim}{j} - \underset{\sim}{k}$.

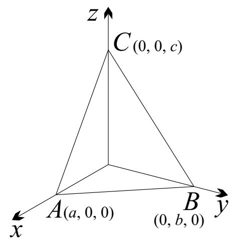
(a) Find the area of $\triangle ABC$.

(b) Suppose that D lies on AB so that $AD : AB = 1 : 3$ and E is the midpoint of CD. Find the position vectors of D and E.

(c) Show that $\overrightarrow{OE}$ is perpendicular to the plane ABC by showing that it is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(d) Hence find the volume of the tetrahedron $OABC$.

22. The diagram on the right shows a triangular pyramid $OABC$. Let $|\triangle XYZ|$ denote the area of $\triangle XYZ$.
Prove that $|\triangle AOB|^2 + |\triangle BOC|^2 + |\triangle COA|^2 = |\triangle ABC|^2$.



5E Vector Proofs in Geometry

Whilst vectors provide a powerful tool for tackling problems, in many instances in geometry the Euclidean proofs studied in Years 7 to 10 are by far the simplest and best. However, there are a few cases where vector proofs are both efficient and elegant. This short section considers some of those proofs. As a general rule, the problems involve lengths or right angles.

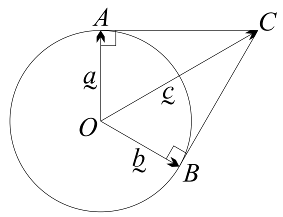
WORKED EXAMPLE 18: Point C is outside a circle with centre O. The points of contact of the two tangents from C to the circle are A and B. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. Prove the following.

(a) Tangents CA and CB subtend equal angles at the centre O.

(b) $CA = CB$.

SOLUTION: Let the radius of the circle be r.

(a) The angle between a radius and tangent at the point of contact is 90°.



Hence $(\underset{\sim}{c} - \underset{\sim}{a}) \cdot \underset{\sim}{a} = 0$

thus
$$\begin{aligned} \underset{\sim}{c} \cdot \underset{\sim}{a} &= \underset{\sim}{a} \cdot \underset{\sim}{a} \\ &= r^2 . \end{aligned}$$

Likewise $\underset{\sim}{c} \cdot \underset{\sim}{b} = r^2$.

Thus $\underset{\sim}{c} \cdot \underset{\sim}{a} = \underset{\sim}{c} \cdot \underset{\sim}{b}$

or $|\underset{\sim}{c}| r \cos\angle AOC = |\underset{\sim}{c}| r \cos\angle BOC$

so $\angle AOC = \angle BOC$.

(b) Next consider the square of the length of AC.

$$\begin{aligned} |\underset{\sim}{c} - \underset{\sim}{a}|^2 &= (\underset{\sim}{c} - \underset{\sim}{a}) \cdot (\underset{\sim}{c} - \underset{\sim}{a}) \\ &= |\underset{\sim}{c}|^2 - 2\underset{\sim}{c} \cdot \underset{\sim}{a} + |\underset{\sim}{a}|^2 \\ &= |\underset{\sim}{c}|^2 - 2\underset{\sim}{c} \cdot \underset{\sim}{b} + |\underset{\sim}{b}|^2 \qquad \text{(by part (a))} \\ &= |\underset{\sim}{c} - \underset{\sim}{b}|^2 . \end{aligned}$$

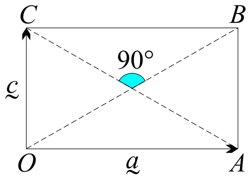
Hence $|\underset{\sim}{c} - \underset{\sim}{a}| = |\underset{\sim}{c} - \underset{\sim}{b}|$,

that is, $AC = BC$.

Exercise 5E

1. Use vectors to prove that a quadrilateral is a parallelogram if its diagonals bisect each other.

2. The diagram on the right shows rectangle $OABC$ with diagonals that are perpendicular. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

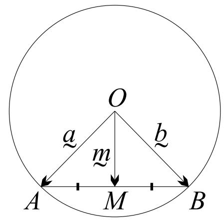


(a) What is the value of $\underset{\sim}{a} \cdot \underset{\sim}{c}$?

(b) Write $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.

(c) Hence prove that $OABC$ is a square.

3. In the diagram on the right, O is the centre of the circle and OM bisects the chord AB. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OM} = \underset{\sim}{m}$.



(a) Show that $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$.

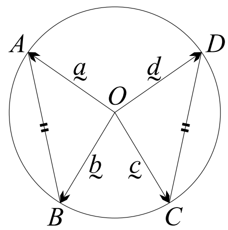
(b) Show that $\big(\underset{\sim}{m} - \underset{\sim}{a}\big) \cdot \big(\underset{\sim}{m} - \underset{\sim}{a}\big) = \big(\underset{\sim}{m} - \underset{\sim}{b}\big) \cdot \big(\underset{\sim}{m} - \underset{\sim}{b}\big)$.

(c) Hence prove that $OM \perp AB$.

4. Using a similar approach, prove the converse of the theorem in the previous question.

DEVELOPMENT

5. In the diagram on the right, AB and CD are equal chords of a circle with centre O. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$ and $\overrightarrow{OD} = \underset{\sim}{d}$.

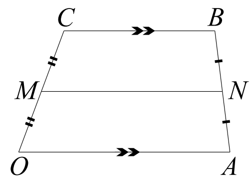


(a) Explain why $|\underset{\sim}{b} - \underset{\sim}{a}|^2 = |\underset{\sim}{d} - \underset{\sim}{c}|^2$.

(b) Use part (a) to show that $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{c} \cdot \underset{\sim}{d}$.

(c) Hence show that $\angle AOB = \angle COD$.

6. The diagram on the right shows trapezium $OABC$ with $OA||CB$. The midpoints of the non-parallel sides OC and AB are M and N respectively. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.



(a) Show that $\overrightarrow{MN} = \frac{1}{2}\big(\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}\big)$.

(b) Explain why $\underset{\sim}{b} - \underset{\sim}{c} = k\,\underset{\sim}{a}$, where k is a constant.

(c) Hence show that MN is parallel to OA and CB.

7. A quadrilateral is a four sided figure in a plane. In this question, suppose that the definition of a quadrilateral is extended to include the case in three dimensions where the four vertices do not all lie in a plane. Let $OABC$ be such a quadrilateral with vertices at $O(0, 0, 0)$, $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$.

Given that M, N, P and R are the respective midpoints of OA, AB, BC and CO, use vectors to prove that $MNPR$ is a parallelogram.

8. The cube $OABCDEFG$ has vertices $O(0, 0, 0)$, $A(a, 0, 0)$, $B(a, a, 0)$, $C(0, a, 0)$, $D(0, a, a)$, $E(0, 0, a)$, $F(a, 0, a)$ and $G(a, a, a)$, where $a > 0$.

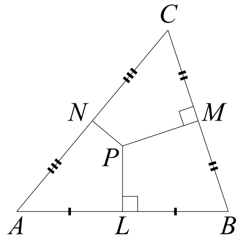
(a) Use a scalar product to find the angle between the two face diagonals OF and OD.

(b) What type of special triangle is $\triangle OFD$?

(c) What special name is given to the triangular pyramid $OBDF$?

(d) Find $\angle FXD$, where X is the centre of the cube. (This angle is the bonding angle of carbon tetrachloride.)

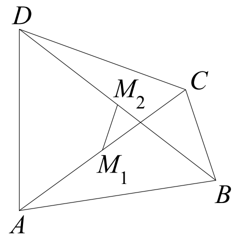
9. In the diagram on the right, PL and PM are the perpendicular bisectors of sides AB and BC of $\triangle ABC$, and N is the midpoint of AC. Let $\overrightarrow{AB} = \underset{\sim}{u}$, $\overrightarrow{BC} = \underset{\sim}{v}$ and $\overrightarrow{PL} = \underset{\sim}{w}$.



(a) Find $\overrightarrow{PN}$ in terms of $\underset{\sim}{v}$ and $\underset{\sim}{w}$.

(b) Hence prove that the three perpendicular bisectors of the sides of a triangle are concurrent by proving that $\overrightarrow{PN} \cdot \overrightarrow{AC} = 0$.

10. [A Theorem of Euler] The diagram shows convex quadrilateral $ABCD$ in two dimensions. The midpoint of diagonal AC is M_1 and the midpoint of diagonal BD is M_2. It is known that



$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4(M_1M_2)^2\,.$$

That is, the sum of squares of the sides is equal to the sum of squares of the diagonals plus 4 times the square of the distance between their midpoints.

Use the result $|\underset{\sim}{v}|^2 = \underset{\sim}{v} \cdot \underset{\sim}{v}$ to write a vector proof of this theorem. Begin by letting $\overrightarrow{AB} = \underset{\sim}{b}$, $\overrightarrow{AC} = \underset{\sim}{c}$ and $\overrightarrow{AD} = \underset{\sim}{d}$.

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11. Suppose that $OABC$ is a trapezium with parallel sides OA and CB such that $OA = 3\,CB$. If E and F are the respective midpoints of the diagonals OB and AC, use vectors to prove that $EFBC$ is a parallelogram.

12. As in Question 7, suppose that the definition of a quadrilateral is extended to include the case in three dimensions where the four vertices do not all lie in a plane. Use a vector approach to prove that the sum of the squares of the sides of such a quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the distance between the midpoints of the diagonals. That is, repeat Question 10 in three dimensions.

[HINT: Take the origin as one of the vertices and then specify general coordinates for the other three vertices.]

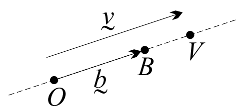
5F The Vector Equation of a Line

One of the advantages of vectors is that the derivation of certain results and the equations of certain objects are the same both in two dimensions and in three dimensions. For example, the formula for the projection of one vector onto another is always

$$\text{proj}_{\underset{\sim}{b}}\,\underset{\sim}{a} = \left(\frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right) \underset{\sim}{b}\,.$$

The focus of this section is on the vector equations of lines, as they too have the same form in both two dimensions and three dimensions.

Lines Through the Origin: Let O be the origin and let B be another point with position vector $\underset{\sim}{b}$. Consider the line OB. Let R be a variable point in OB with position vector $\underset{\sim}{r}$. It follows that $OV||OB$ and hence



$$\underset{\sim}{r} = \lambda\underset{\sim}{b}.$$

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Thus the position vector of every point in OB is obtained as λ varies, as shown in the diagram above, and this is the vector equation of the line OB.

The proper name for the equation is a *parametric vector equation*, since it involves a *parameter* λ. Notice that the argument used to derive this equation is valid regardless of whether the situation is in two dimensions or in three dimensions.

WORKED EXAMPLE 19: Find the vector equation of the line through the origin and the point $B(2,3)$.

SOLUTION: The point B has position vector $\underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$. Let $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$. Then the equation of OB is

$$\underset{\sim}{r} = \lambda \underset{\sim}{b},$$

that is $\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$

16 LINES THROUGH THE ORIGIN: The line through the origin and another point B with position vector $\underset{\sim}{b}$ has vector equation

$$\underset{\sim}{r} = \lambda \underset{\sim}{b}.$$

The Parametric Equations: The components of the vector equation in the last worked example are

$$x = 2\lambda$$
$$y = 3\lambda\,.$$

These are called the *parametric equations* of the straight line. They form a pair of simultaneous equations with parameter λ. The parameter can be eliminated to give the familiar Cartesian equation

$$y = \tfrac{3}{2}x\,.$$

In the worked example there is no restriction on the domain of the parameter, and so the whole line is obtained. However, setting $0 \le \lambda \le 4$ would yield

$$y = \tfrac{3}{2}x \quad \text{with} \quad 0 \le x \le 8 \quad (\text{since } x = 2\lambda.)$$

That is, the result is the portion of the line between the origin and $(8, 12)$.

There are many curves which can be defined through parametric equations. Some of these will be encountered in the next section. In simple cases like the one above, the parameter is eliminated and any restrictions noted in order to find the Cartesian equation.

WORKED EXAMPLE 20: Determine the Cartesian equation of the line

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mu \begin{bmatrix} 2 \\ -4 \end{bmatrix} \quad \text{with} \quad -1 \le \mu \le 1.$$

SOLUTION: The parametric equations are

$$x = 2\mu \quad \text{with} \quad -2 \le x \le 2$$

and $\quad y = -4\mu = -2(2\mu)\,.$

Hence the result is the line segment

$$y = -2x\,, \quad -2 \le x \le 2.$$

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Note that in this example the parameter was labelled μ. Any symbol can be used. However, the Greek letter λ is often used for problems involving lines because it is equivalent to the letter ℓ.

The Direction Vector and the Gradient: In two dimensions, the direction vector $\underset{\sim}{b}$ of the vector equation $\underset{\sim}{r} = \lambda\underset{\sim}{b}$ and the gradient m of the Cartesian equation $y = mx$ are closely related. The vector $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j}$ specifies how x and y change as λ varies. In particular, the coordinates of R for $\lambda = 0$ and 1 are:

$$\lambda = 0, R = (0,0) \qquad \text{and} \qquad \lambda = 1, R = (b_1, b_2).$$

From this it is clear that b_2 is the rise and b_1 is the run. Hence the gradient is

$$m = \frac{b_2}{b_1} \qquad \text{provided} \qquad b_1 \neq 0\,.$$

The restriction $b_1 \neq 0$ means that the line cannot be vertical.

Thus it is sometimes convenient to put $b_1 = 1$ so that $b_2 = m$ and the gradient appears explicitly in the vector equation. This gives

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ m \end{bmatrix}.$$

WORKED EXAMPLE 21: Write the equation of the line $y = -2x$ in vector form.

SOLUTION: $\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

17 THE DIRECTION VECTOR AND THE GRADIENT: The two equations of a non-vertical line through the origin in two dimensions are $\underset{\sim}{r} = \lambda\underset{\sim}{b}$ and $y = mx$. From this

$$m = \frac{b_2}{b_1} \quad (b_1 \neq 0), \quad \text{and putting } b_1 = 1 \text{ gives} \quad \underset{\sim}{b} = \begin{bmatrix} 1 \\ m \end{bmatrix}.$$

The Direction Vector and General Form: Whilst the connection between the direction vector and the gradient is important, it fails for vertical lines in two dimensions. The situation is made much clearer when the vector equation is compared with the general form of the equation of a straight line.

As before, let $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j}$. Then $\underset{\sim}{r} = \lambda\underset{\sim}{b}$ has parametric equations

$$x = b_1\lambda$$
$$y = b_2\lambda$$

from which it follows that

$$b_1 y = b_2 x\,.$$

So in general form the equation is

$$b_2 x - b_1 y = 0\,.$$

Now compare this with the usual notation for general form,

$$Ax + By = 0\,.$$

The simplest solution is that $b_1 = -B$ and $b_2 = A$. Since the right hand side of each equation is zero, another solution is $b_1 = B$ and $b_2 = -A$.

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Either solution will work because $-B\underset{\sim}{i}+A\underset{\sim}{j}$ and $B\underset{\sim}{i}-A\underset{\sim}{j}$ are opposite vectors, and opposite vectors are parallel. Thus the two solutions give parallel lines through the origin, that is, the same line. Hence it is a matter of convenience which direction vector is used when solving a problem. Using these two identities, it is easy to switch between vector form and Cartesian form.

WORKED EXAMPLE 22: Write down the vector form of the line $3y = 4x$.

SOLUTION: The general form of the equation is $4x - 3y = 0$.
In this case put $b_1 = 3$ and $b_2 = 4$, so the equation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

18 **THE DIRECTION VECTOR AND GENERAL FORM:** If the vector equation $\underset{\sim}{r} = \lambda\underset{\sim}{b}$ represents the same line as $Ax + By = 0$ then either

$$\underset{\sim}{b} = \begin{bmatrix} -B \\ A \end{bmatrix} \qquad \text{or} \qquad \underset{\sim}{b} = \begin{bmatrix} B \\ -A \end{bmatrix}.$$

The advantage of using Box 18 to convert between vector and Cartesian form is that it works for all lines, including vertical lines. When deciding which direction vector formula to use, choose the one which gives fewer negatives in the answer.

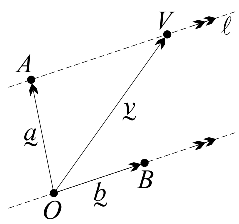
WORKED EXAMPLE 23: Find the vector equations of the x-axis and y-axis.

SOLUTION: The equation of the x-axis is $y = 0$. Choosing the second solution in Box 18 gives $\underset{\sim}{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underset{\sim}{i}$. Thus the vector equation of the x-axis is $\underset{\sim}{r} = \lambda\underset{\sim}{i}$.

The equation of the y-axis is $x = 0$. Choosing the first solution in Box 18 gives $\underset{\sim}{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underset{\sim}{j}$. Thus the vector equation of the y-axis is $\underset{\sim}{r} = \mu\underset{\sim}{j}$.

In fact both answers were obvious from the definitions of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

The Line Through a Given Point: Let O be the origin and let A and B be two other points with position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Consider the line ℓ through A parallel with OB. Let R be a variable point in ℓ with position vector $\underset{\sim}{r}$. It follows that $AR \| OB$ and hence



$$\underset{\sim}{r} - \underset{\sim}{a} = \lambda\underset{\sim}{b}.$$

Thus the position vector of every point in ℓ is obtained as λ varies, as shown in the diagram above, and this is the parametric vector equation of the line.

It is important to observe that the equation $\underset{\sim}{r} - \underset{\sim}{a} = \lambda\underset{\sim}{b}$ is consistent with the way a function changes when its graph is shifted. The line OB with equation $\underset{\sim}{r} = \lambda\underset{\sim}{b}$ has been shifted so that it passes through the point A with position vector $\underset{\sim}{a}$. Consequently the variable vector $\underset{\sim}{r}$ has been replaced with $\underset{\sim}{r} - \underset{\sim}{a}$. Despite this significant result, it is more common to write the equation as

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}.$$

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Worked Example 24: Find the vector equation of the line through A parallel with OB, where $A = (-2, -1, 3)$ and $B = (1, 0, 1)$. Then determine whether or not $C = (0, -1, 4)$ is on this line.

Solution: The vector equation is simply

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The point C will be on the line if there is a solution to

$$\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

so $$\lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

or $$\lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly this last equation has no solution and hence C does not lie on the line.

19 The line through a given point: The vector equation of the line with direction $\underset{\sim}{b}$ which passes through a point with position vector $\underset{\sim}{a}$ is

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}.$$

Alternatively, shift $\underset{\sim}{r} = \lambda \underset{\sim}{b}$ by $\underset{\sim}{a}$ to get $\underset{\sim}{r} - \underset{\sim}{a} = \lambda \underset{\sim}{b}$.

Worked Example 25: Show that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ m \end{bmatrix}$ is equivalent to the gradient–intercept formula in two dimensional coordinate geometry.

Solution: The parametric equations are

$$x = \lambda$$
$$y = \lambda m + c$$

thus by substitution it is clear that

$$y = mx + c\,,$$

which is the gradient–intercept formula in coordinate geometry.

The Line Through Two Given Points: In the two point method of coordinate geometry, the first step is to use the given points to determine the gradient of the line. Likewise, for the vector equation, the first step is to find the direction vector of the line. Let the points A and B have position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Then the direction is given by

$$\overrightarrow{AB} = (\underset{\sim}{b} - \underset{\sim}{a})\,.$$

It is now a trivial matter to write down the vector equation of AB, which is

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})\,.$$

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WORKED EXAMPLE 26: Consider the points $A(-1,-2,3)$ and $B(-2,1,0)$.

(a) Evaluate $\overrightarrow{AB}$.

(b) Hence determine the vector equation of AB.

SOLUTION: Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

(a)
$$\begin{aligned}\overrightarrow{AB} &= \underset{\sim}{b} - \underset{\sim}{a}\\ &= \begin{bmatrix}-2\\1\\0\end{bmatrix} - \begin{bmatrix}-1\\-2\\3\end{bmatrix}\\ &= \begin{bmatrix}-1\\3\\-3\end{bmatrix}.\end{aligned}$$

(b) Thus the line is
$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-1\\-2\\3\end{bmatrix} + \lambda\begin{bmatrix}-1\\3\\-3\end{bmatrix}.$$

20 THE LINE THROUGH TWO GIVEN POINTS: Suppose that A has position vector $\underset{\sim}{a}$ and B has position vector $\underset{\sim}{b}$. The vector equation of the line AB is

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a}).$$

Line Segments: A line segment is simply the result of restricting the parameter in the vector equation of a line. The most significant example is the case of the line through two points, viz

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a}).$$

Re-arrange this equation to get

$$\underset{\sim}{r} = (1-\lambda)\underset{\sim}{a} + \lambda\underset{\sim}{b}$$

or
$$\underset{\sim}{r} = \tfrac{1}{(1-\lambda)+\lambda}\Big((1-\lambda)\underset{\sim}{a} + \lambda\underset{\sim}{b}\Big)$$

which is a special case of the ratio division formula in given Box 6. That is, the point R divides the interval AB in the ratio $\lambda : (1-\lambda)$. Thus as λ varies between 0 and 1, the point R occupies every location in the interval AB. Hence the equation of the line segment is

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a}) \quad \text{for} \quad 0 \le \lambda \le 1.$$

Clearly $\lambda = 0$ gives $\underset{\sim}{r} = \underset{\sim}{a}$, the position vector of A, and $\lambda = 1$ gives $\underset{\sim}{r} = \underset{\sim}{b}$, the position vector of B. Thus as λ increases, the point R moves from A to B.

WORKED EXAMPLE 27: Find the vector equation of the line segment AB where $A = (-1,2)$ and $B = (3,3)$.

SOLUTION: In this case $\overrightarrow{AB} = \begin{bmatrix}4\\1\end{bmatrix}$, so the equation is

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix} + \lambda\begin{bmatrix}4\\1\end{bmatrix} \qquad \text{for } 0 \le \lambda \le 1.$$

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Parallel and Perpendicular Lines: Consider the lines with vector equations

$$\underset{\sim}{r}_1 = \underset{\sim}{a}_1 + \lambda_1 \underset{\sim}{b}_1 \quad \text{and} \quad \underset{\sim}{r}_2 = \underset{\sim}{a}_2 + \lambda_2 \underset{\sim}{b}_2 \,.$$

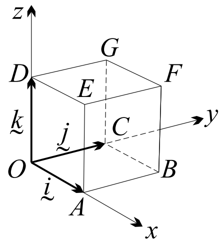
If the lines are parallel then their direction vectors must also be parallel. That is

$$\underset{\sim}{b}_1 = \mu \underset{\sim}{b}_2 \,.$$

If the lines intersect and they are perpendicular then their direction vectors must also be perpendicular. That is

$$\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_2 = 0 \,.$$

It may seem strange to include the qualification that the lines intersect when discussing perpendicular lines. It is a necessary qualification because two direction vectors can be perpendicular in three dimensions but the lines do not meet. A simple example occurs in the unit cube shown on the right. Clearly the lines OA and EF do not intersect. Yet their direction vectors are $\underset{\sim}{i}$ and $\underset{\sim}{j}$, and $\underset{\sim}{i} \cdot \underset{\sim}{j} = 0$. That is, the direction vectors are perpendicular.



In some instances it is useful to say that the lines OA and EF are perpendicular, but this is very unusual. If the need ever arises to do this, it must be made clear in any working that the lines do not intersect.

The proper name for lines in three dimensions that are not parallel and do not meet is *skew lines*. However, the direction vectors do not need to be perpendicular in skew lines. For example, AB and EG form another pair of skew lines in the unit cube shown above.

WORKED EXAMPLE 28:

(a) Find the point where the following lines intersect.

$$\underset{\sim}{r}_1 = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\underset{\sim}{r}_2 = \begin{bmatrix} -5 \\ 2 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

(b) Hence show that the two lines form a right angle at the point of intersection.

SOLUTION:

(a) At the point of intersection $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$. Equating the first two components:

$$-1 + \lambda = -5 + 3\mu$$

and $\quad -\lambda = 2 - \mu \,.$

Adding these gives

$$-1 = -3 + 2\mu \,,$$

so $\mu = 1$ and hence $\lambda = -1$. Thus from $\underset{\sim}{r}_2$ the point of intersection is

$$\begin{bmatrix} -5 \\ 2 \\ 5 \end{bmatrix} + 1 \times \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} .$$

As a check, substituting $\lambda = -1$ into $\underset{\sim}{r}_1$ gives the same point.

(b) Taking the dot product of the direction vectors:

$$\begin{aligned}\begin{bmatrix}1\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}3\\-1\\-2\end{bmatrix} &= 3+1-4\\ &= 0\end{aligned}$$

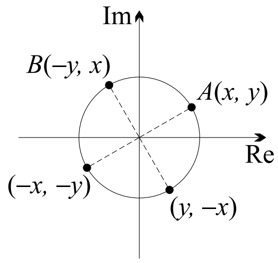
and hence the lines are perpendicular.

Note that the check in the final line of part (a) above is essential, as will be demonstrated in a later example with skew lines.

21 PARALLEL AND PERPENDICULAR LINES: Let $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ be the directions of two lines.

- If the lines are parallel then $\underset{\sim}{b}_1 = \mu\underset{\sim}{b}_2$.
- If the lines intersect and are perpendicular then $\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_2 = 0$.
- Lines in three dimensions that are not parallel and do not intersect are called skew lines.

Perpendicular Vectors in Two Dimensions: For a given vector in two dimensions it is easy to find a perpendicular vector. Suppose $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$ is the position vector of the point A in the Argand diagram. Thus A represents the complex number $\alpha = x + iy$. The result of rotating A by $90°$ about the origin is $\beta = i\alpha$, and $\beta = -y + ix$. Let this represent B with position vector $\underset{\sim}{b} = -y\underset{\sim}{i} + x\underset{\sim}{j}$. It follows that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, as can be checked by the dot product.



Continuing this process, the complex numbers α, $i\alpha$, $i^2\alpha$ and $i^3\alpha$ correspond to the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $-\underset{\sim}{a}$ and $-\underset{\sim}{b}$. That is, the vectors

$$\begin{bmatrix}x\\y\end{bmatrix},\quad \begin{bmatrix}-y\\x\end{bmatrix},\quad \begin{bmatrix}-x\\-y\end{bmatrix}\quad \text{and}\quad \begin{bmatrix}y\\-x\end{bmatrix}$$

form a sequence where each subsequent vector is the result of a rotation by $90°$. Further, every vector in this sequence has the same magnitude. This feature proves useful in certain problems.

The next worked example also requires a direction vector for a line in general form. In this case, the line does not pass through the origin. Fortunately, the result in Box 18 also applies to lines written in the form $Ax + By + C = 0$. That is, the direction vector is either $-B\underset{\sim}{i} + A\underset{\sim}{j}$ or $B\underset{\sim}{i} - A\underset{\sim}{j}$.

WORKED EXAMPLE 29: What is the vector equation of the line perpendicular to $2x - 3y + 4 = 0$ which passes through the point $(-5, 6)$?

SOLUTION: From Box 18, a direction vector of the given line is $\begin{bmatrix}3\\2\end{bmatrix}$.

Thus a vector perpendicular to the given line is $\begin{bmatrix}2\\-3\end{bmatrix}$.

Hence the required line is

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-5\\6\end{bmatrix} + \lambda\begin{bmatrix}2\\-3\end{bmatrix}.$$

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22	PERPENDICULAR VECTORS IN TWO DIMENSIONS: In the sequence of vectors $$\begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} -y \\ x \end{bmatrix}, \quad \begin{bmatrix} -x \\ -y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y \\ -x \end{bmatrix},$$ each subsequent vector is the result of a rotation by 90°. Further, every vector in this sequence has the same magnitude.

Skew Lines and Inconsistent Equations: Skew lines do not intersect. Trying to find a point of intersection by equating components yields a set of inconsistent simultaneous equations.

WORKED EXAMPLE 30: Determine whether or not the following lines intersect.

$$\underset{\sim}{r}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\underset{\sim}{r}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

SOLUTION: Equating the first two components:

$$2 + \lambda = 2 + \mu$$

and $$-1 + 2\lambda = 2 - \mu\,.$$

Adding these gives

$$1 + 3\lambda = 4$$

by which $\lambda = 1$, and so $\mu = 1$. Substituting these into the z-components gives

$$2 - 1 \neq -1 - 2.$$

Since the equations are inconsistent, the lines do not intersect.

23	SKEW LINES AND INCONSISTENT EQUATIONS: Equating the components of skew lines yields a set of inconsistent simultaneous equations.

Exercise **5F**

NOTE: Throughout this exercise λ and μ are real parameters.

1. A line ℓ passes through the point $(-1, 3)$ and has direction vector $2\underset{\sim}{i} - \underset{\sim}{j}$.

(a) Sketch the line.

(b) Write down a vector equation for the line.

(c) What is the gradient of the line?

(d) What is the Cartesian equation of the line?

2. A line has Cartesian equation $y = \frac{2}{3}x - 4$.

(a) Find the position vector of the point on the line where $x = 3$.

(b) Use the gradient of the line to find a direction vector for the line.

(c) Hence write down a vector equation of the line.

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3. (a) Consider the line with equation $x - 3y + 12 = 0$.
  (i) Use Box 18 to write down a direction vector for this line.
  (ii) Write down the position vector of an intercept.
  (iii) Hence write down the vector equation of the line.
(b) Follow the same method to find the vector equation of these lines.
  (i) $x + 3y = 6$ (ii) $y = 3$ (iii) $x = -5$

4. By eliminating λ from a pair of parametric equations, find the Cartesian equation of each line.
(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ (b) $\underset{\sim}{r} = 5\underset{\sim}{i} + 2\underset{\sim}{j} + \lambda(-2\underset{\sim}{i} + 3\underset{\sim}{j})$

5. Determine whether or not each point lies on the line $\underset{\sim}{r} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.
(a) $(2, -8)$ (b) $(-13, 17)$ (c) $(8, -20)$

6. Write down a vector equation for the line that:
(a) passes through the point $(7, 0, -5)$ and is parallel to the vector $-4\underset{\sim}{i} - 6\underset{\sim}{j} + 9\underset{\sim}{k}$,
(b) passes through the point $(3, 4, 5)$ and has direction vector $\begin{bmatrix} -6 \\ -7 \\ -8 \end{bmatrix}$.

7. Write down a vector equation for the line that:
(a) passes through the point $(3, -2, -4)$ and is parallel to the line
$$\underset{\sim}{r} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k} + \lambda(5\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}),$$
(b) passes through the point $(-1, -1, 2)$ and is parallel to the line
$$\underset{\sim}{r} = \tfrac{1}{3}\underset{\sim}{i} - \tfrac{1}{3}\underset{\sim}{j} - \underset{\sim}{k} + \lambda(\tfrac{1}{6}\underset{\sim}{i} + \tfrac{1}{3}\underset{\sim}{j} + \tfrac{1}{2}\underset{\sim}{k}).$$

8. Determine whether or not each point lies on the line $\underset{\sim}{r} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 3 \\ -6 \end{bmatrix}$.
(a) $(8, -13, 11)$ (b) $(-4, 5, -25)$

DEVELOPMENT

9. (a) Consider the line with Cartesian equation $x + 2y - 4 = 0$.
  (i) Write down a direction vector for this line.
  (ii) Thus write down a direction vector which is perpendicular to this.
  (iii) Hence write down the vector equation of the perpendicular line through $(2, -3)$.
(b) Likewise find the vector equation of the line perpendicular to $x - y + 3 = 0$ which passes through $(1, -2)$

10. Given $\underset{\sim}{r} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\underset{\sim}{a} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 1 \\ m \end{bmatrix}$, show that the vector equation $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$ is equivalent to the point–gradient formula $y - y_1 = m(x - x_1)$.

11. Find a vector equation for the line AB given:
(a) $A(4, 3)$ and $B(6, 0)$ (b) $A(-7, 5)$ and $B(-13, -8)$

12. Find a vector equation for the line PQ given:

(a) $P(-1,3,1)$ and $Q(2,4,5)$ (b) $P(7,-11,14)$ and $Q(17,9,-16)$

13. Find the parametric vector equation of the interval AB where:

(a) $A=(1,-2)$, $B=(5,4)$ (b) $A=(-1,1,-2)$, $B=(2,3,-1)$

14. Show that the lines $\underset{\sim}{r} = \begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ -15 \\ -24 \end{bmatrix}$ and $\underset{\sim}{r} = 2\underset{\sim}{i} - 5\underset{\sim}{j} - 4\underset{\sim}{k} + \lambda(-4\underset{\sim}{i} + 10\underset{\sim}{j} + 16\underset{\sim}{k})$ are parallel.

15. Find the point of intersection of each pair of lines.

(a) $\underset{\sim}{r} = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$

(b) $\underset{\sim}{r} = \begin{bmatrix} 7 \\ -3 \\ 8 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix} -2 \\ 1 \\ 10 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix}$

16. Show that the lines $\underset{\sim}{r}_1 = \underset{\sim}{i} - \underset{\sim}{k} + \lambda(2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})$ and $\underset{\sim}{r}_2 = \underset{\sim}{i} + \underset{\sim}{j} + \mu(-4\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k})$ are skew. (You must show that they are not parallel and do not intersect.)

17. In each case either show that the given lines are skew or find their point of intersection.

(a) $\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\underset{\sim}{v}_2 = \begin{bmatrix} -2 \\ -2 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) $\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\underset{\sim}{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

18. The line ℓ_1 passes through the points $(2,0,1)$ and $(-1,3,4)$, while the line ℓ_2 passes through $(-1,3,0)$ and $(4,-2,5)$.

(a) Find the point of intersection of ℓ_1 and ℓ_2.

(b) Find, to the nearest tenth of a degree, the acute angle between ℓ_1 and ℓ_2.

19. Find the value of a for which the lines $\underset{\sim}{r} = 2\underset{\sim}{i} + 9\underset{\sim}{j} + 13\underset{\sim}{k} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$ and $\underset{\sim}{r} = a\underset{\sim}{i} + 7\underset{\sim}{j} - 2\underset{\sim}{k} + \mu(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})$ intersect.

20. Consider the line with vector equation $\underset{\sim}{r} = \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and let P be the point $(-2,3)$.

(a) Write down the coordinates of any two points A and B on the line.

(b) Find $\overrightarrow{AP}$ and $\overrightarrow{AB}$.

(c) Find $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{p}$, where $\overrightarrow{AP} = \underset{\sim}{p}$ and $\overrightarrow{AB} = \underset{\sim}{b}$.

(d) Find the perpendicular distance d from P to the given line using $d = \left|\text{proj}_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$.

21. Repeat the previous question for the point $P(1,-1,1)$ and the line $\underset{\sim}{r} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

22. Consider the lines $\underset{\sim}{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\underset{\sim}{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

(a) Explain why the lines are parallel.
(b) Write down the position vector of a point on $\underset{\sim}{v}_2$.
(c) Use the method in Question 17 to find the distance between these parallel lines.

23. Suppose that $\underset{\sim}{a} = -\underset{\sim}{i} - 2\underset{\sim}{j}$, $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, and let A, B, C and D be the points in the Cartesian plane with respective position vectors $\underset{\sim}{a}$, $\underset{\sim}{a}+\underset{\sim}{b}$, $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}$ and $\underset{\sim}{a}+\underset{\sim}{c}$.

(a) Show that $ABCD$ is a rhombus.
(b) Find vector equations for the bisectors of angles DAB and ABC.

24. Suppose that the points A, B, C and D have respective position vectors $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ relative to an origin O, and M is the midpoint of AC.

(a) Find the position vector of M.
(b) Find a vector equation for the line BD.
(c) Show that M lies on the line BD.
(d) Find the ratio $BM : MD$.

25. The points A and B have position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Describe the part of the line AB that is represented by the vector equation $\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$ if:

(a) $0 \le \lambda \le 1$ (b) $\lambda \ge 1$ (c) $\lambda \le 0$

ENRICHMENT

26. A line ℓ has vector equation $\underset{\sim}{r} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, P is a variable point on ℓ and A is the point $(1, 1, 1)$.

(a) Find $|\overrightarrow{AP}|$ in terms of λ.
(b) Hence find the minimum distance from A to ℓ.
(c) Show that the minimum distance found in (ii) is the perpendicular distance.
(d) Repeat (ii), this time using the dot product of $\overrightarrow{AP}$ and the direction vector of ℓ.
(e) Repeat (ii) yet again, this time by choosing a point B on ℓ and finding the length of the projection of BA onto ℓ.

5G Vector Equations of Circles, Spheres and Planes

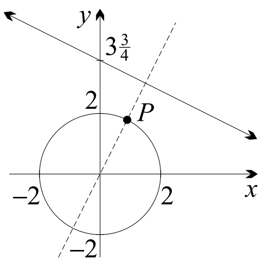
Circles in Two Dimensions: The equation of a circle in two dimensions is most easily found by using the geometric definition. Thus, the variable point V with position vector $\underset{\sim}{v}$ will lie on the circle with radius r and centre the origin if

$$|\underset{\sim}{v}| = r\,.$$

Notice that this equation is consistent with the equation of a circle in the Argand diagram, viz $|z| = r$. This is hardly surprising, since vectors have already been used in conjunction with complex numbers in an earlier chapter.

WORKED EXAMPLE 31: Determine the point on the circle with centre the origin and radius 2 which is closest to the line $2x + 4y - 15 = 0$.

SOLUTION: From the diagram on the right, the point P lies in the first quadrant, and lies on the line through the origin perpendicular to the given line.



By Box 18, a direction vector of the given line is $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

Hence by Box 22 a perpendicular vector is $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

The corresponding unit vector is $\dfrac{1}{\sqrt{20}} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Double this to get $\overrightarrow{OP} = \dfrac{2}{\sqrt{20}} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ with radius 2.

Hence the required point is $P = \left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right)$.

Circles with Other Centres: Now suppose that the circle $|\underset{\sim}{v}| = r$ is shifted so that the centre is at C with position vector $\underset{\sim}{c}$. Then, from the shifting results,

$$|\underset{\sim}{v} - \underset{\sim}{c}| = r\,.$$

Once again, the equation is consistent with $|z - \alpha| = r$, which is a circle with centre α in the Argand diagram.

WORKED EXAMPLE 32: The line $\underset{\sim}{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ intersects the circle with centre $\underset{\sim}{c} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and radius 3 at P and Q. The midpoint of chord PQ is M. Find the coordinates of M.

SOLUTION: The equation of the circle is $|\underset{\sim}{v} - \underset{\sim}{c}| = 3$. Solving simultaneously with the equation of the line gives

$$\left| \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right|^2 = 3^2$$

so $\qquad (1 + \lambda)^2 + (3 + 2\lambda)^2 = 9\,.$

Expanding brackets and collecting terms:

$$5\lambda^2 + 14\lambda + 1 = 0\,.$$

Then P and Q correspond to the roots of this equation.

The midpoint corresponds to the average of the roots, which is

$$\begin{aligned} \lambda &= \tfrac{1}{2} \times \tfrac{-14}{5} \\ &= -\tfrac{7}{5}\,. \end{aligned}$$

Hence the midpoint has position vector

$$\underset{\sim}{m} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \tfrac{7}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and so $\quad M = \left(\frac{3}{5}, -\frac{9}{5} \right)$

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> CIRCLES IN TWO DIMENSIONS: Let $\underset{\sim}{v}$ be the position vector of a variable point on the circle with centre $\underset{\sim}{c}$ and radius r. Then
>
> $$\left|\underset{\sim}{v} - \underset{\sim}{c}\right| = r\,.$$

Spheres: The form of the vector equation of a sphere is identical to that of a circle in two dimensions. Thus the sphere with centre the origin and radius r is

$$\left|\underset{\sim}{v}\right| = r\,.$$

Likewise, the sphere with centre $\underset{\sim}{c}$ is

$$\left|\underset{\sim}{v} - \underset{\sim}{c}\right| = r\,,$$

where, of course, each vector has three components since the situation is now three dimensional.

WORKED EXAMPLE 33: Find the cartesian equation of the sphere with centre $\underset{\sim}{c} = -\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ which passes through $\underset{\sim}{a} = 2\underset{\sim}{i} + 1\underset{\sim}{j} + 5\underset{\sim}{k}$.

SOLUTION: The radius is given by

$$\begin{aligned} r^2 &= \left|\underset{\sim}{a} - \underset{\sim}{c}\right|^2 \\ &= 3^2 + 2^2 + 6^2 \\ &= 49. \end{aligned}$$

The equation of the sphere is $\left|\underset{\sim}{v} - \underset{\sim}{c}\right|^2 = r^2$, so in Cartesian form this gives

$$(x+1)^2 + (y+1)^2 + (z+1)^2 = 49$$

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> SPHERES: In three dimensions, let $\underset{\sim}{v}$ be the position vector of a variable point on the sphere with centre $\underset{\sim}{c}$ and radius r. Then
>
> $$\left|\underset{\sim}{v} - \underset{\sim}{c}\right| = r\,.$$

Circles and Parameters: Consider the vector equation

$$\underset{\sim}{v} = \underset{\sim}{c} + \underset{\sim}{a}\cos\theta + \underset{\sim}{b}\sin\theta$$

where $|\underset{\sim}{a}| = |\underset{\sim}{b}| = r$ and $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$. This is the parametric vector equation of a circle with centre $\underset{\sim}{c}$ and radius r. The form of the equation is the same in both two and three dimensions. In the simplest case, $\underset{\sim}{a} = \underset{\sim}{i}$ and $\underset{\sim}{b} = \underset{\sim}{j}$ in two dimensions.

WORKED EXAMPLE 34: Find the Cartesian equation of the curve with vector equation $\underset{\sim}{v} = (\underset{\sim}{i} + 2\underset{\sim}{j}) + \underset{\sim}{i}\cos\theta + \underset{\sim}{j}\sin\theta$.

SOLUTION: First rewrite the equation as

$$\left|\underset{\sim}{v} - \underset{\sim}{i} - 2\underset{\sim}{j}\right| = \left|\underset{\sim}{i}\cos\theta + \underset{\sim}{j}\sin\theta\right|$$

so $\quad (x-1)^2 + (y-2)^2 = \cos^2\theta + \sin^2\theta$

that is $\quad (x-1)^2 + (y-2)^2 = 1\,.$

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26 | CIRCLES AND PARAMETERS: Let $|\underset{\sim}{a}| = |\underset{\sim}{b}| = r$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$. Then

$$\underset{\sim}{v} = \underset{\sim}{c} + \underset{\sim}{a}\cos\theta + \underset{\sim}{b}\sin\theta$$

is the parametric vector equation of a circle with centre $\underset{\sim}{c}$ and radius r.

The proof that the equation represents a circle is easy in two dimensions.

$$\begin{aligned}
& \underset{\sim}{v} - \underset{\sim}{c} = \underset{\sim}{a}\cos\theta + \underset{\sim}{b}\sin\theta \\
\text{thus}\quad & \left|\underset{\sim}{v} - \underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\cos\theta + \underset{\sim}{b}\sin\theta\right|^2 \\
& \qquad = (\underset{\sim}{a}\cos\theta + \underset{\sim}{b}\sin\theta)\cdot(\underset{\sim}{a}\cos\theta + \underset{\sim}{b}\sin\theta) \\
& \qquad = |\underset{\sim}{a}|^2\cos^2\theta + 2\underset{\sim}{a}\cdot\underset{\sim}{b}\cos\theta\sin\theta + |\underset{\sim}{b}|^2\sin^2\theta \\
& \qquad = |\underset{\sim}{a}|^2\cos^2\theta + |\underset{\sim}{b}|^2\sin^2\theta \qquad (\text{since } \underset{\sim}{a}\cdot\underset{\sim}{b} = 0) \\
& \qquad = r^2(\cos^2\theta + \sin^2\theta) \qquad (\text{since } |\underset{\sim}{a}| = |\underset{\sim}{b}| = r) \\
\text{hence}\quad & \left|\underset{\sim}{v} - \underset{\sim}{c}\right|^2 = r^2\,. \\
\text{That is}\quad & \left|\underset{\sim}{v} - \underset{\sim}{c}\right| = r\,,
\end{aligned}$$

which is the equation of a circle in two dimensions.

Putting $\theta = 0$ in this equation gives $\underset{\sim}{v} = \underset{\sim}{c} + \underset{\sim}{a}$, and when $\theta = \frac{\pi}{2}$ the equation becomes $\underset{\sim}{v} = \underset{\sim}{c} + \underset{\sim}{b}$. Thus as θ increase, the variable point with position vector $\underset{\sim}{v}$ rotates around the circle with the same orientation as the angle from $\underset{\sim}{a}$ to $\underset{\sim}{b}$. In Worked Example 34, $\underset{\sim}{i}$ to $\underset{\sim}{j}$ is anticlockwise, so the circle is traversed in an anticlockwise direction, starting at $(2, 2)$ when $\theta = 0$.

WORKED EXAMPLE 35: Consider the vector equation $\underset{\sim}{v} = \underset{\sim}{c} + \underset{\sim}{a}\cos\theta + \underset{\sim}{b}\sin\theta$ where $\underset{\sim}{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\underset{\sim}{c} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

(a) Show that this is a circle by finding its Cartesian equation.

(b) Where on the circle is $\theta = 0$ and in which direction is the circle traversed as θ increases?

SOLUTION:

(a) Once again, use the vector magnitudes to solve this:

$$\begin{aligned}
\left|\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right|^2 &= \left|\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos\theta + \begin{bmatrix} -1 \\ -1 \end{bmatrix}\sin\theta\right|^2 \\
\text{so}\qquad x^2 + (y-2)^2 &= (\cos\theta - \sin\theta)^2 + (-\cos\theta - \sin\theta)^2 \\
&= \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta \\
&\quad + \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta \\
&= 2(\cos^2\theta + \sin^2\theta)
\end{aligned}$$

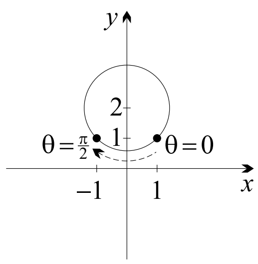
that is $x^2 + (y-2)^2 = 2\,.$

This is the equation of a circle with centre $(0, 2)$ and radius $\sqrt{2}$.

(b) When $\theta = 0$, $\underset{\sim}{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

when $\theta = \frac{\pi}{2}$, $\underset{\sim}{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

so the circle is being traversed clockwise, as seen in the diagram on the right.



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Other Parametric Curves: The parametric vector equation of a circle is one of a myriad of useful curves defined using vectors. Vector equations are often easier to apply in three dimensions because the corresponding Cartesian equations are too complicated or cannot easily be found. The spiral used in the diagram at the beginning of the chapter to demonstrate right-handed coordinate systems is such an example. Its vector equation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2\cos\theta \\ 2\sin\theta \\ \frac{1}{2\pi}\theta \end{bmatrix} \qquad \text{for} \qquad 0 \le \theta \le 6\pi\,.$$

In contrast, curves in two dimensions can often be written in Cartesian form by solving the simultaneous equations formed from the components of the vector equation. When those equations involve trigonometric functions, the solution is often found using various trigonometric identities, like the circle above.

An amazing application of parameters is a *Bezier curve*. This is truly modern mathematics, being developed independently by mathematicians at two French car manufacturers in 1959 and 1960. Originally developed for technical drawings of automotive parts, they are now extensively used in many other applications, particularly those involving computer graphics. The curves are remarkably simple to construct, being based on the ratio division of intervals.

The following worked example uses a simple quadratic Bezier curve. Given three points in vector form, the Bezier curve joins the first and last with a smooth curve by cleverly using the middle point. In this case, the curve is a parabola.

WORKED EXAMPLE 36: Let $\underset{\sim}{a}_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\underset{\sim}{a}_1 = \begin{bmatrix} (\frac{1}{2}) \\ 3 \end{bmatrix}$ and $\underset{\sim}{a}_2 = \begin{bmatrix} 2 \\ (\frac{3}{2}) \end{bmatrix}$.

(a) Find vector equations of the point B_0 which divides A_0A_1 in the ratio $t : 1-t$, and the point B_1 which divides A_1A_2 in the same ratio.

(b) Hence find the vector equation of the point P which divides B_0B_1 in the ratio $t : 1-t$, for $0 \le t \le 1$.

(c) Show that $P = A_0$ when $t = 0$, and that $P = A_1$ when $t = 1$.

(d) Find the Cartesian equation of P.

SOLUTION:

(a) Using the formula in Box 6:

$$\underset{\sim}{b}_0 = (1-t)\underset{\sim}{a}_0 + t\underset{\sim}{a}_1$$
$$\underset{\sim}{b}_1 = (1-t)\underset{\sim}{a}_1 + t\underset{\sim}{a}_2$$

(b) Likewise:

$$\begin{aligned} \underset{\sim}{p} &= (1-t)\underset{\sim}{b}_0 + t\underset{\sim}{b}_1 \\ &= (1-t)^2\underset{\sim}{a}_0 + (1-t)t\underset{\sim}{a}_1 \\ &\qquad\qquad + t(1-t)\underset{\sim}{a}_1 + t^2\underset{\sim}{a}_2 \end{aligned}$$

that is $\underset{\sim}{p} = (1-t)^2\underset{\sim}{a}_0 + 2(1-t)t\underset{\sim}{a}_1 + t^2\underset{\sim}{a}_2 \quad$ for $0 \le t \le 1$.

This is the quadratic Bezier Curve from A_0 to A_2 using A_1.

(c) At $t = 0$, the second and third terms are clearly zero, leaving $\underset{\sim}{p} = \underset{\sim}{a}_0$.
At $t = 1$, the first and second terms are clearly zero, leaving $\underset{\sim}{p} = \underset{\sim}{a}_1$.

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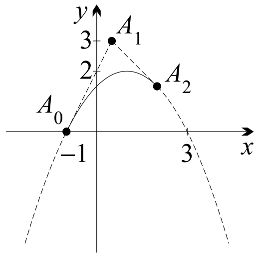
(d) First write the vector equation in component form.

$$\begin{bmatrix} x \\ y \end{bmatrix} = (1-t)^2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 2(1-t)t \begin{bmatrix} (\frac{1}{2}) \\ 3 \end{bmatrix} + t^2 \begin{bmatrix} 2 \\ (\frac{3}{2}) \end{bmatrix}$$

Now writing out the components as simultaneous equations:

$$x = -(1-t)^2 + (1-t)t + 2t^2$$
$$y = 6(1-t)t + \tfrac{3}{2}t^2\,.$$



Expanding and simplifying these equations gives:

$$\begin{aligned} x &= 3t - 1 \\ 2y &= 12t - 9t^2 \\ &= 3t(4-3t) \end{aligned}$$

so $\quad 2y = (x+1)(4-(x+1)) \quad$ (from A_0 to A_2)

that is $\quad y = \frac{1}{2}(x+1)(3-x)$ for $-1 \le x \le 2\,.$

The curve is shown as the solid curve on the right between A_0 and A_2. The dashed part of the curve shows the rest of the parabola. Notice that the intervals A_0A_1 and A_1A_2 are tangent to the parabola. This is always the case with a quadratic Bezier curve.

Extension — Planes: The parametric vector equation of a circle in Box 26 has two direction vectors and one parameter. Although the circle lies in a plane, only the points on the circle can be obtained using that parameter. In order to reach all points in a plane, two parameters are needed. A familiar example in three dimensions is the xy-plane. Every point in the plane can be reached using

$$\underset{\sim}{v} = x\underset{\sim}{i} + y\underset{\sim}{j}\,.$$

In this case, the parameters are the coordinates x and y, and the direction vectors are the standard basis vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

In general, let O be the origin and let A and B be two other points with position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, and where $\underset{\sim}{a} \ne \lambda\underset{\sim}{b}$. That is, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are not parallel. There is only one plane which passes through these three points, and the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ lie in that plane. Although it will not be proven in this course, the position vector of every point in this plane through the origin can be written as

$$\underset{\sim}{v} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{b}\,.$$

One way to think of this is that λ and μ give the coordinates of each point for the non-standard basis vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Worked Example 37: The point $(3, 2, 1)$ lies in the plane through the origin with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Find the value of λ and μ.

Solution: The component parametric equations are:

$$\begin{aligned} 3 &= \lambda + \mu \\ 2 &= \lambda + 2\mu \\ 1 &= \lambda + 3\mu \end{aligned}$$

It should be clear that $\lambda = 4$ and $\mu = -1$.

Now shift the plane $\underset{\sim}{v} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{b}$ so that it passes through the point P with position vector $\underset{\sim}{p}$, and so that it remains parallel with the direction vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Then applying shifting results gives the new equation

$$\underset{\sim}{v} - \underset{\sim}{p} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{b}\,.$$

This is more commonly written as

$$\underset{\sim}{v} = \underset{\sim}{p} + \lambda\underset{\sim}{a} + \mu\underset{\sim}{b}\,.$$

WORKED EXAMPLE 38: A plane passes through the points $A(1, 2, 1)$, $B(-1, 0, 1)$ and $C(0, -1, 2)$. Find its equation:

(a) in vector form, (b) in Cartesian form.

SOLUTION:

(a) Two direction vectors for the plane are

$$\overrightarrow{BA} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \overrightarrow{BC} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

thus, using point A, the equation of the plane is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(b) The component equations are

$$\begin{aligned} x &= 1 + 2\lambda + \mu & [1] \\ y &= 2 + 2\lambda - \mu & [2] \\ z &= 1 \qquad\quad + \mu & [3] \end{aligned}$$

Now take [1] − [3] and [2] + [3] to get

$$\begin{aligned} x - z &= 2\lambda \\ y + z &= 3 + 2\lambda \end{aligned}$$

hence $y + z = 3 + x - z$

or $x - y - 2z = -3\,.$

27 PLANES: Let $\underset{\sim}{a} \neq \lambda\underset{\sim}{b}$. That is, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are not parallel. The parametric vector equation of a plane through the point P with position vector $\underset{\sim}{p}$ and parallel with the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is

$$\underset{\sim}{v} = \underset{\sim}{p} + \lambda\underset{\sim}{a} + \mu\underset{\sim}{b}\,.$$

Exercise **5G**

NOTE: Throughout this exercise λ and μ are real parameters.

1. A circle has centre $(6, -9)$ and radius $2\sqrt{7}$. Write down:

(a) a Cartesian equation for the circle,

(b) a vector equation for the circle,

(c) a pair of parametric equations for the circle.

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2. A sphere has centre $(-2, 7, -4)$ and radius 9. Write down:
(a) a Cartesian equation for the sphere,
(b) a vector equation for the sphere.

3. Write down a Cartesian equation for:
(a) the circle $\left| \underset{\sim}{r} - \begin{bmatrix} -5 \\ -10 \end{bmatrix} \right| = 3\sqrt{5}$,
(b) the sphere $\left| \underset{\sim}{r} - \begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix} \right| = 11$.

4. Write down a vector equation and a Cartesian equation for the circle with parametric equations $x = 5 + 2\sqrt{2}\cos\theta$ and $y = -3 + 2\sqrt{2}\sin\theta$ for $0 \le \theta < 2\pi$.

5. Find a vector equation for:
(a) the circle $x^2 + y^2 - 6x + 8y = 0$,
(b) the sphere $x^2 + y^2 + z^2 + x - 2y - 5z = 0$.

6. Show that the point $P(8, -5, 2)$ lies on the surface of the sphere $\left| \underset{\sim}{r} - \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix} \right| = 7$.

7. Determine whether $A(-4, -5, 6)$ lies inside or outside the sphere $\left| \underset{\sim}{r} - \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix} \right| = 3\sqrt{15}$.

8. A circle has vector equation $\big(\underset{\sim}{r} - (2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})\big) \cdot \big(\underset{\sim}{r} - (2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})\big) = 20$. Write down the centre and radius of the circle.

9. A circle has vector equation $\underset{\sim}{r}(t) = (2\cos t + 1)\underset{\sim}{i} + (2\sin t - 1)\underset{\sim}{j}$.
(a) Write down a pair of parametric equations for the circle.
(b) Eliminate the parameter to find the Cartesian equation of the circle.

DEVELOPMENT

10. A function is defined by the vector equation $\underset{\sim}{r}(t) = (t - 2)\underset{\sim}{i} + (t^2 - 2)\underset{\sim}{j}$, for $t \ge 0$.
(a) Find the Cartesian equation of the function.
(b) Hence state the domain of the function.
(c) Sketch the graph of the function.

11. Suppose that the point A has position vector $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j}$.
(a) Write down a vector equation for the circle passing through A with centre at the origin.
(b) Explain why the tangent to the circle at A has vector equation $\big(\underset{\sim}{r} - (3\underset{\sim}{i} - \underset{\sim}{j})\big) \cdot (3\underset{\sim}{i} - \underset{\sim}{j}) = 0$.
(c) Hence find the Cartesian equation of the tangent.

12. A line has equation $\underset{\sim}{r} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and a circle has equation $\left| \underset{\sim}{r} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right| = \sqrt{13}$.
(a) Write down a pair of parametric equations representing the line.
(b) Hence find the points of intersection of the line and the circle.

13. Two spheres have equations $\left| \underset{\sim}{r} - \begin{bmatrix} 5 \\ -6 \\ 3 \end{bmatrix} \right| = 7$ and $\left| \underset{\sim}{r} - \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix} \right| = 5$. Show that the spheres touch each other at a single point.

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14. Two spheres have equations $|\underset{\sim}{r}| = 3$ and $\left| \underset{\sim}{r} - \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \right| = 4$.

(a) Explain why the circle of intersection of the two spheres is parallel to the xy-plane.

(b) Determine the centre and radius of the circle of intersection.

15. A line ℓ and a sphere $\mathcal{S}$ have equations $\underset{\sim}{r} = \begin{bmatrix} -3 \\ 16 \\ -9 \end{bmatrix} + \lambda \begin{bmatrix} 7 \\ -12 \\ 3 \end{bmatrix}$ and $(x-3)^2 + (y+4)^2 + (z+2)^2 = 81$ respectively. Find the points where ℓ intersects $\mathcal{S}$.

16. A line has vector equation $\underset{\sim}{r} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

(a) Write down a set of three parametric equations representing the line.

(b) Hence determine the point of intersection of the line and the plane $2x + 4y - z = 55$.

17. Find the Cartesian equation corresponding to each of the following vector equations.

(a) $\underset{\sim}{r}(t) = \frac{1}{2}\left(e^t + e^{-t}\right)\underset{\sim}{i} + \frac{1}{2}\left(e^t - e^{-t}\right)\underset{\sim}{j}$ (b) $\underset{\sim}{r}(t) = (2\sin t)\underset{\sim}{i} + (2\sin t \tan t)\underset{\sim}{j}$

ENRICHMENT

18. The vector equation of a plane has the form $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b} + \mu\underset{\sim}{c}$, where $\underset{\sim}{a}$ is the position vector of a point in the plane, and $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are non-parallel direction vectors (that is, non-parallel vectors that are both parallel to the plane).
Suppose that a plane $\mathcal{P}$ passes through the points $A(1,-1,0)$, $B(2,3,1)$ and $C(3,4,-2)$.

(a) Find a vector equation for $\mathcal{P}$.

(b) Find the Cartesian equation of $\mathcal{P}$.

19. Show that the vector $a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ is perpendicular to the plane $ax + by + cz = d$ by showing that it is perpendicular to two non-parallel direction vectors of the plane.

20. Sketch the curve defined parametrically by the vector equation:

(a) $\underset{\sim}{r}(t) = (t - 2\sin t)\underset{\sim}{i} + t^2\underset{\sim}{j}$ (b) $\underset{\sim}{r}(t) = (3\cos t)\underset{\sim}{i} + (3\sin t)\underset{\sim}{j} + t\underset{\sim}{k}$

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5H Chapter Review Exercise

Exercise 5H

NOTE: Throughout this exercise λ and μ are real parameters.

1. Find the length of $\underset{\sim}{a}$ and a unit vector in the direction of $\underset{\sim}{a}$ given $\underset{\sim}{a} = 6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$

2. The points A and B have position vectors $3\underset{\sim}{i} - \underset{\sim}{j} - 6\underset{\sim}{k}$ and $-2\underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k}$ respectively. Find:
   (a) $\overrightarrow{AB}$ (b) $\overrightarrow{BA}$ (c) the distance AB

3. Show that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel for the points $A(6, 12, 7)$, $B(10, 2, -15)$, $C(-4, 1, 5)$ and $D(-2, -4, -6)$.

4. Use vectors to show that the points $A(2, 3, -1)$, $B(5, -1, 1)$ and $C(-4, 11, -5)$ are collinear.

5. Given $\underset{\sim}{a} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$, find:
   (a) $\underset{\sim}{a} \cdot \underset{\sim}{a}$ (b) $\underset{\sim}{b} \cdot \underset{\sim}{b}$ (c) $\underset{\sim}{a} \cdot \underset{\sim}{b}$ (d) $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

6. The points A, B, C and D have position vectors $\begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ respectively. Show that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular.

7. Find the value of λ for which $\underset{\sim}{a} = (\lambda + 4)\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + (\lambda - 4)\underset{\sim}{j} + \underset{\sim}{k}$ are perpendicular.

8. Find the exact value of $\cos\theta$, where θ is the acute angle between the vectors $\underset{\sim}{a} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

9. Find the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$ given $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$.

10. Let P, A and B be the points $(2, 3, 1)$, $(1, 0, -2)$ and $(0, -1, 1)$ respectively.
    (a) Find $\overrightarrow{AP}$ and $\overrightarrow{AB}$.
    (b) Find $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{p}$, where $\overrightarrow{AP} = \underset{\sim}{p}$ and $\overrightarrow{AB} = \underset{\sim}{b}$.
    (c) Find the perpendicular distance d from P to the line AB using $d = \left|\text{proj}_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$.

11. $\triangle XYZ$ has vertices $X(-5, 7, 3)$, $Y(5, -2, 6)$ and $Z(3, -5, -4)$. Use the scalar product to find $\angle XYZ$ correct to the nearest degree.

12. Use vectors to prove each of these theorems.
    (a) The midpoints of the sides of a rhombus are the vertices of a rectangle.
    (b) The midpoints of the sides of a rectangle are the vertices of a rhombus.

13. Write the linear equation $y = 2x + 3$ in parametric vector form by letting $x = \lambda$.

14. Find the Cartesian equation of the line $\underset{\sim}{r} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + \lambda(3\underset{\sim}{i} + \underset{\sim}{j})$.

15. A line ℓ has vector equation $\underset{\sim}{r} = -6\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k} + \lambda(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$. Find the points of intersection of ℓ with the yz, xz and xy planes.

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16. Determine whether or not each point lies on the line $\underset{\sim}{r} = \begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} -5 \\ 2 \\ 7 \end{bmatrix}$.

(a) $(-4, 0, 13)$ (b) $(16, -8, -17)$

17. Find a parametric vector equation for the line PQ given $P(1, 1, -1)$ and $Q(2, -1, 2)$.

18. Find the point of intersection of each pair of lines.

(a) $\underset{\sim}{r} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ and $\underset{\sim}{r} = \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$

(b) $\underset{\sim}{r} = -7\underset{\sim}{i} - 1\underset{\sim}{j} + 7\underset{\sim}{k} + \lambda(2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k})$ and $\underset{\sim}{r} = 9\underset{\sim}{i} - 4\underset{\sim}{j} - 16\underset{\sim}{k} + \lambda(4\underset{\sim}{i} - 3\underset{\sim}{j} - 5\underset{\sim}{k})$

19. A sphere has centre $(3, -4, 2)$ and radius $\sqrt{7}$. Write down:

(a) a Cartesian equation for the sphere,

(b) a vector equation for the sphere.

20. Show that the point $P(5, -1, 4)$ lies outside the sphere $\left| \underset{\sim}{r} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right| = 7$.

21. Find a vector equation for the sphere $x^2 + y^2 + z^2 - 4x - 10y + 12z + 41 = 0$.

22. Find the points of intersection of the line $\underset{\sim}{r} = 4\underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k} + \lambda(2\underset{\sim}{i} + 13\underset{\sim}{j} - 11\underset{\sim}{k})$ and the sphere $(x + 2)^2 + (y - 3)^2 + (z + 4)^2 = 125$.

23. Find the Cartesian equation corresponding to each of the following vector equations.

(a) $\underset{\sim}{r}(t) = (2t)\,\underset{\sim}{i} + \left(\dfrac{2}{1 + t^2}\right)\underset{\sim}{j}$

(b) $\underset{\sim}{r}(t) = \left(\dfrac{2t}{1 + t^2}\right)\underset{\sim}{i} + \left(\dfrac{1 - t^2}{1 + t^2}\right)\underset{\sim}{j}$

(c) $\underset{\sim}{r}(t) = (\sin t)\,\underset{\sim}{i} + (\sin 2t)\,\underset{\sim}{j}$

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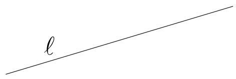
Appendix: Some Geometry in 3D

The three fundamental objects of geometry are points, lines and planes. Because they are so fundamental, it is difficult to give precise definitions of them, just as it is difficult to define precisely what a number is. The following descriptions of points, lines and planes don't really say what they are, but will enable discussions about them with some agreement about what is meant.

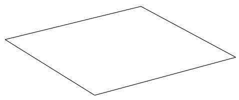
Point: A *point* marks a position. As it is a location, it has no size. In diagrams, points are marked with a dot or cross-hair, like those on the right. Italic capital letters are used to label points.

Line: A *line* has no breadth, but extends infinitely in opposite directions. In diagrams, lines are drawn with a ruler, like the one on the right. Italic lower case letters are used to label lines.



The word *line* always means *straight line*, and does not include curves. A line can also be thought of as the path of a point moving in a fixed direction, like the path of a thin beam of light. It is assumed that a straight line is the shortest path between two points.

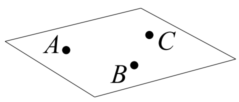
Plane: A *plane* has no thickness, but extends infinitely in all directions. In diagrams, a plane is drawn as though it is a sheet of paper shown in perspective, like on the right. In this text, a region within a plane will usually be labelled by its vertices.



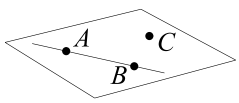
In the same way that a line is the path traced by a point moving in a fixed direction, a plane is the path traced by the line moving in a second fixed direction. A practical example is the edge of a ruler, representing the line, as it is dragged across a desk, representing the plane that is traced out.

The word *plane* always means *flat plane*, and does not include curved surfaces like cylinders, cones or spheres. A line joining any two distinct points in a plane lies entirely within it.

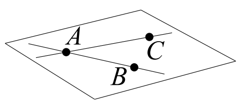
Defining a Plane: It will be assumed that there is only one plane which passes through three distinct fixed points. The three points are said to *define the plane*. It is why a tripod never wobbles, regardless of how uneven the ground it is placed on.



Now consider the line AB and the point C. Since the same three points have been used, it is clear that there is only one plane which passes through a line and a point which is not on the line. Thus a plane can also be defined in this way.



Further, consider the two lines AB and AC. Since the same three points have been used, it is clear that there is only one plane which passes through two lines that intersect in a single point. Thus a plane can also be defined in this way.



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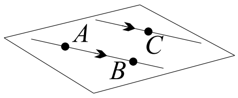
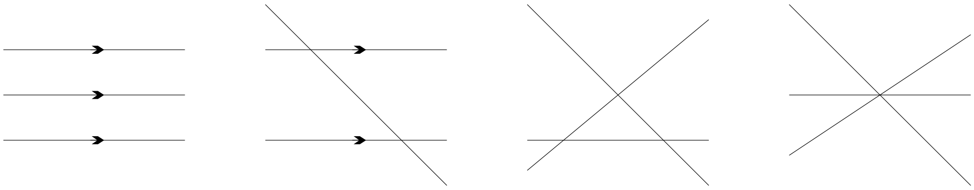
Finally consider the line AB and the line parallel with AB through C. Once again, the same three points have been used so it is clear that there is only one plane which passes through two parallel lines. Thus a plane can also be defined in this way.

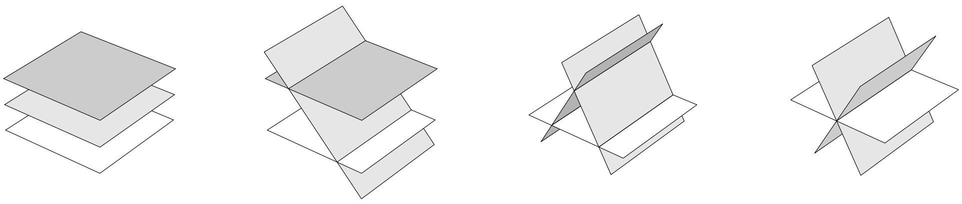


Arrangements of Lines in a Plane: There are precisely two arrangements of lines in planes. Either they are parallel or they intersect. This is demonstrated in the diagrams above for the last two ways to define a plane. There are four arrangements when there are three lines, and these are shown below.

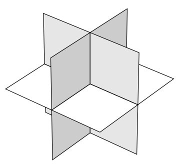


Arrangements of Planes in Space: There is a similar arrangement of planes in space corresponding to each of the arrangements of lines in a plane. Two planes in space can either be parallel or intersect in a line, as shown below.

And here are the arrangements of three planes in space corresponding to the arrangements shown above of three lines in a plane.



In addition, three planes in space can also intersect at a point, as shown on the right. This diagram shows the special case when each plane is at right angles to the other two. Three planes in space intersecting at a point do not have to be at right angles to each other. A simple example is a tetrahedron, where pairs of triangular faces are at 71° to each other, yet each group of three meets at a common vertex.



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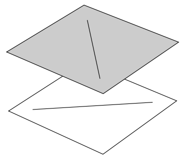
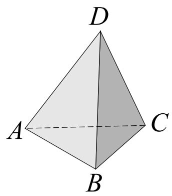
Arrangements of Two Lines in Space: As noted above, if two distinct lines are parallel then they define a plane, and two lines which intersect in a point also define a plane. There is a third possible configuration of two lines in space where neither are the lines parallel nor do they intersect.

Consider two parallel planes. For any line drawn in one plane, it is always possible to draw a line in the other which is not parallel with the given line. An example of the situation is shown on the right. Pairs of lines like these which are not parallel and do not intersect are called *skew lines*.



Once again, a tetrahedron provides a practical example of the situation. The opposite edges AB and CD are clearly not parallel and clearly do not intersect, and hence they are skew lines. As an exercise, try to identify any pair of parallel lines, each pair of intersecting lines, and each pair of skew lines. Then extend the exercise to each of the Platonic solids.



There is much more that could be said about geometry in three dimensions. What has been included here are the essential concepts and configurations that are likely needed in the study of points, lines and planes through vectors.

6

Mechanics

Chapter Overview: Some practical applications of calculus in mechanics are considered in this chapter. Simple mathematical models enable solutions to be found in terms of the familiar functions of this course.

Problems later in the chapter require some knowledge of Newton's laws of motion, and that acceleration be written as a function of x. Preparation for that is done in Section 6A. Next, simple harmonic motion is presented, firstly in terms of time in Section 6B, and then in terms of displacement in Section 6C.

The work in Sections 6D and 6E takes the first step in making the theory of motion more realistic by introducing a resistive force. Horizontal motion with friction, and vertical motion in a resisting medium with constant gravity are considered. In the following section, projectile motion is first reviewed and then extended to include a resistive force proportional to the velocity, based on the work done in 6E. The chapter concludes with a collection of various problems which either extend the theory learnt to new situations or are harder examples of applications of that theory. One specific example investigated is the simple harmonic motion approximation for a pendulum.

6A Forces and Acceleration

In the problems encountered in this chapter, the equations of motion are either specified explicitly or given indirectly, such as by a balance of forces. Whilst this is not a course in physics, a basic understanding of the laws of motion is required.

Some Assumptions: A significant simplification is the assumption that an object may be represented by a point mass, often called a particle. If the scale of the motion is large compared with the object, such as in the case of a ball bearing thrown 5 m, then this assumption is reasonable. A second significant assumption is that air is an ideal fluid and is not particulate in nature. At low to medium speeds this is a satisfactory assumption.

A third assumption is that the forces due to the orbit of the earth around the sun and due to the rotation of the earth on its axis are negligible in the problems being considered. By way of example, in the problem of projectile motion without air resistance, the acceleration due to gravity at the surface of the earth is about $9{\cdot}8\,\text{m/s}^2$. The acceleration due to the orbit of the earth is about $6 \times 10^{-4}\,\text{m/s}^2$, and at the equator the acceleration due to the rotation of the earth is about $0{\cdot}034\,\text{m/s}^2$. Thus this third assumption seems reasonable.

Newton's Laws of Motion: The equations of motion encountered in this course are all derived from Sir Isaac Newton's laws of motion, contained in his book *Principia*, published in 1686. It is written in Latin, the scientific language of the day. An early translation of the laws is as follows:

Law I Every body continues in its state of rest or of uniform motion in a straight line except in so far as it be compelled by impressed force to change that state.

Law II The rate of change of momentum is proportional to the impressed force and takes place in the direction in which the force acts.

Law III To every action there is an equal and opposite reaction.

In the second law, momentum is defined to be mv, the product of mass with velocity. The standard units used are kilograms, metres and seconds. The unit of force is called the *newton*, with $1\,\text{N} = 1\,\text{kg}\,\text{m}\,\text{s}^{-2}$. If these SI units are used then, in the second law, the constant of proportionality is 1. Thus

$$F = \frac{d}{dt}(mv)\,,$$

and if the mass is constant then

$$F = ma \quad \text{where} \quad a = \dot{v}\,.$$

If other units are used then these equations must be appropriately modified.

It may seem strange to state that the mass is constant, but in many cases the mass is certainly not constant, such as a rocket as its fuel is burnt. In this course, however, the mass is always assumed to be constant.

1 NEWTON'S SECOND LAW OF MOTION: When a force F newtons is applied to a constant mass m kg, which is free to move, the resulting acceleration is $a\,\text{m}\,\text{s}^{-2}$, where

$$F = ma$$

In many situations it is simply a matter of integrating the force equation in order to find the other details of the motion.

WORKED EXAMPLE 1: A body of mass 4 kg is acted upon by a variable force $F = 48(5 - t)$ newtons for 5 seconds. If the body starts from rest at $x = 0$ then what is its final velocity and how far has it travelled?

SOLUTION: From Newton's second law, after dividing through by the mass,

$$\frac{dv}{dt} = 60 - 12t\,.$$

Integrating, $v = 60t - 6t^2 + C\,.$

At $t = 0$ the velocity is $v = 0$, so $C = 0$ and

$$v = 60t - 6t^2\,.$$

Hence at $t = 5$, $v = 150$ m/s.

Integrating again gives the displacement:

$$x = 30t^2 - 2t^3 + D\,.$$

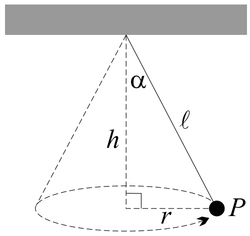
At $t = 0$ the body is at $x = 0$, so $D = 0$ and

$$x = 30t^2 - 2t^3\,.$$

Hence at $t = 5$, $x = 500$ m.

Resolution of Forces: When two or more forces act on a body, the problem can often be solved by resolving the forces. Typically the forces are resolved horizontally and vertically to determine the equations of motion. Usually a force diagram, called a *free-body diagram*, is helpful at this step. The sum of the components can then be used in Newton's second law.

WORKED EXAMPLE 2: In a conical pendulum, particle P of mass m hangs from a ceiling on a wire of length ℓ. The wire makes a constant angle α with the vertical and P moves around a circle with constant angular velocity ω. Let r be the radius of the circular motion and let h be the height of the cone traced out by the wire.

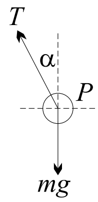


(a) Draw a free-body diagram of the situation.

(b) The horizontal force required to keep the particle in circular motion is $mr\omega^2$, directed towards the axis of the cone. By resolving the forces horizontally and vertically, obtain an expression for ω in terms of ℓ, α and g.

(c) The period of the motion is $\frac{2\pi}{\omega}$. Show that when $\ell = \frac{g}{\pi^2}$ (about $0{\cdot}994$ m) the period is always less than 2 seconds.

SOLUTION: Let T be the tension in the wire.



(a) The diagram is shown on the right.

(b) The vertical component of tension is $T\cos\alpha$. The only other vertical force is due to gravity and is $-mg$, the minus sign indicating a downwards force. As there is no vertical acceleration, the sum of the vertical forces is zero. Thus

$$T\cos\alpha - mg = 0$$

or $$T\cos\alpha = mg \qquad (1)$$

The horizontal component of tension keeps the particle in circular motion. So

$$T\sin\alpha = mr\omega^2 \qquad (2)$$

Now in the cone $\sin\alpha = \frac{r}{\ell}$ so from equation (2)

$$T\tfrac{r}{\ell} = mr\omega^2$$

or $$T = m\ell\omega^2$$

Substitute this result into (1) to get

$$m\ell\omega^2\cos\alpha = mg$$

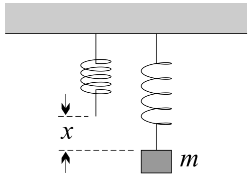
so $$\omega^2 = \frac{g}{\ell\cos\alpha}$$

or $$\omega = \sqrt{\frac{g}{\ell\cos\alpha}}\,.$$

(c) From the given formula for the period,

$$\text{period} = 2\pi\sqrt{\frac{\ell\cos\alpha}{g}}$$
$$= 2\sqrt{\cos\alpha} \qquad (\text{when } \ell = \tfrac{g}{\pi^2}\,.)$$

Hence the the period is always less than 2 seconds, with period $\to 2^-$ seconds as $\alpha \to 0^+$.



Other Forms of Acceleration: There are some problems where the equation of motion is more naturally expressed using displacement. For example, it is easy to measure the displacement x of one end of a spring when a mass m is hung from it. It is found that x is proportional to the force so the resulting equation is

$$\dot{v} = -kx \qquad \text{(for some constant } k.)$$

In order to do anything meaningful with this equation it is necessary to rewrite the acceleration $\dot{v}$ as a derivative in x instead of t. This is done as follows.

$$\begin{aligned} \frac{dv}{dt} &= \frac{dv}{dx} \times \frac{dx}{dt} && \text{(by the chain rule)} \\ &= \frac{dv}{dx} \times v && \text{(by the definition of velocity)} \\ \text{thus} \quad \frac{dv}{dt} &= v\frac{dv}{dx}\,. \end{aligned}$$

Putting this into the equation of motion for the spring gives

$$v\frac{dv}{dx} = -kx$$

which is a variable separable differential equation.

Worked Example 3: A spring is hung from a ceiling and extended a distance a metres then released from rest. It is found that the equation of motion for the spring is $\ddot{x} = -x$, where x metres is the displacement of the end of the spring at time t seconds. Show that $-a \le x \le a$.

Solution: Replace the acceleration $\ddot{x}$ with $v\dfrac{dv}{dx}$ and double to get

$$2v\frac{dv}{dx} = -2x$$

which is variable separable. Next integrate both sides to get

$$v^2 = C - x^2$$

for some constant C. But at $t = 0$, $x = a$ and $v = 0$ so

$$0 = C - a^2$$

thus $C = a^2$, and rearranging the equation gives

$$x^2 + v^2 = a^2\,.$$

Graphing v against x, this is the equation of a circle, and hence $-a \le x \le a$. Note that this does not prove that the end of the spring ever reaches $x = -a$. That will be done later in this chapter.

Notice that the equation was doubled in the first line of the solution above. This made the integration easier. There is another approach which is often used.

$$\begin{aligned} \frac{d^2x}{dt^2} &= v\frac{dv}{dx} \\ &= \tfrac{1}{2}\left(v\,\frac{dv}{dx} + \frac{dv}{dx}\,v\right) \\ &= \tfrac{1}{2}\frac{d}{dx}(v \times v) && \text{(by the product rule)} \\ \text{that is} \quad \frac{d^2x}{dt^2} &= \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)\,. \end{aligned}$$

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WORKED EXAMPLE 4: A particle starts at the origin with velocity $v = 1\,\text{m/s}$. It is found that its displacement x metres at time t seconds satisfies $\ddot{x} = e^{-2x}$.

(a) Find an expression for v^2 in terms of x.

(b) Explain why the velocity must always be positive.

(c) Hence show that $v \to \sqrt{2}$ as $t \to \infty$.

SOLUTION:

(a) From above

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = e^{-2x}$$

so $$\frac{d}{dx}\left(v^2\right) = 2e^{-2x}$$

thus $$v^2 = C - e^{-2x}$$

for some constant C. Now apply the conditions $v(0) = 1$ and $x(0) = 0$.

$$1 = C - 1$$

so $$C = 2$$

and hence $$v^2 = 2 - e^{-2x}\,.$$

(b) Now at $t = 0$, $x = 0$ and $v = 1$, so the particle begins moving to the right. That is, x becomes positive and v is positive. In order for v to change sign, it must first be zero, which cannot happen for $x \geq 0$. Hence v remains positive.

(c) Since $v^2 = 2 - e^{-2x}$, $v \geq 1$ for all time and hence as $t \to \infty$ so too $x \to \infty$.

Thus $$\lim_{t\to\infty} v^2 = \lim_{x\to\infty} v^2 = \lim_{x\to\infty} 2 - e^{-2x} = 2\,,$$

and since $v > 0$, take the positive square root to get $\lim_{t\to\infty} v = \sqrt{2}$.

2 **OTHER FORMS OF ACCELERATION:** The four common forms of acceleration are

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)\,.$$

In each problem, choose the form most suitable for integration.

Integrating Twice: When acceleration is written as $\frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$, integration will often yield v^2 as a function of x, such as $v^2 = a^2 - x^2$ in Worked Example 3. Rewriting this equation:

$$\left(\frac{dx}{dt}\right)^2 = a^2 - x^2,$$

which is a non-linear first order differential equation. Such equations are usually too difficult to solve in this course, and no further progress is possible.

If, however, the sign of v can be established then the appropriate square root can be taken and a second integration performed. The sign of v may be determined either mathematically (as in Worked Example 4) or, more typically, from the physics of the situation.

Worked Example 5: Experiments suggest that acceleration due to gravity is inversely proportional to the square of the distance to the centre of the planet. Thus, if x is altitude and R is the radius of the earth then

$$\frac{d^2x}{dt^2} = \frac{-k}{(x+R)^2} \qquad \text{where } k \text{ is a positive constant.}$$

The negative sign indicates the acceleration is downwards.

An object is dropped from $x = R$. Let g be the acceleration due to gravity at the surface of the earth.

(a) Show that the object hits the ground with speed $\sqrt{Rg}$.

(b) Find an expression for the time taken in terms of x, R and g.

Solution: Given that $\ddot{x} = -g$ at $x = 0$, it follows that $-g = -kR^{-2}$. Hence $k = R^2 g$.

(a) Rewrite the given differential equation as

$$\frac{d}{dx}\left(v^2\right) = -2k(x+R)^{-2}$$

so $$v^2 = 2k(x+R)^{-1} + C$$

Using the initial condition $v = 0$ at $x = R$,

$$C = -kR^{-1}.$$

Hence
$$\begin{aligned} v^2 &= \frac{2k}{x+R} - \frac{k}{R} \\ &= \frac{k}{R} \times \frac{2R-(x+R)}{x+R} \\ &= Rg\left(\frac{R-x}{R+x}\right) \qquad \text{(from the value of } k \text{ above.)} \end{aligned}$$

Clearly at $x = 0$, $v^2 = Rg$ and hence the speed is $\sqrt{Rg}$.

(b) From the physical situation, the velocity is always negative. Thus

$$v = -\sqrt{Rg} \times \sqrt{\frac{R-x}{R+x}}$$

so taking reciprocals and rearranging yields

$$\sqrt{Rg} \times \frac{dt}{dx} = -\sqrt{\frac{R+x}{R-x}}.$$

Integrating,
$$\begin{aligned} \sqrt{Rg} \times t &= -\int \frac{R+x}{\sqrt{R^2-x^2}}\,dx \\ &= -\int \left(\frac{R}{\sqrt{R^2-x^2}} + \frac{x}{\sqrt{R^2-x^2}}\right) dx \end{aligned}$$

so $$\sqrt{Rg}\ t = -R\sin^{-1}\left(\tfrac{x}{R}\right) + \sqrt{R^2-x^2} + D\,.$$

But $x = R$ at $t = 0$ so

$$0 = -R\sin^{-1} 1 + D$$

and $$D = \tfrac{\pi R}{2}\,.$$

Hence $$t = \sqrt{\frac{R}{g}}\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{x}{r}\right) + \sqrt{1 - \frac{x^2}{R^2}}\right).$$

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Exercise 6A

1. In each part, the velocity v is given as a function of x. It is known that $x = 1$ when $t = 0$. Express: (i) t in terms of x, (ii) x in terms of t.

(a) $v = 6$ (c) $v = -6x^3$ (e) $v = 1 + x^2$
(b) $v = -6x^{-2}$ (d) $v = e^{-2x}$ (f) $v = \cos^2 x$

2. In each motion of the previous question, find $\ddot{x}$ using the result $\ddot{x} = \dfrac{d}{dx}(\tfrac{1}{2}v^2)$.

3. In each part, the acceleration $\ddot{x}$ is given as a function of x. By replacing $\ddot{x}$ with $\dfrac{d}{dx}(\tfrac{1}{2}v^2)$, express v^2 in terms of x given that $v = 0$ when $x = 0$.

(a) $\ddot{x} = 6x^2$ (b) $\ddot{x} = \dfrac{1}{e^x}$ (c) $\ddot{x} = \dfrac{1}{2x+1}$ (d) $\ddot{x} = \dfrac{1}{4+x^2}$

4. In each part, the acceleration $\ddot{x}$ is given as a function of v. By replacing $\ddot{x}$ with $\dfrac{dv}{dt}$, express t in terms of v.

(a) $\ddot{x} = \dfrac{2}{v^2}$ and when $t = 0$, $v = 0$
(b) $\ddot{x} = v^2$ and when $t = 0$, $v = \tfrac{1}{2}$
(c) $\ddot{x} = 2 + v$ and when $t = 0$, $v = 1$

5. In each part, the acceleration $\ddot{x}$ is given as a function of v. By replacing $\ddot{x}$ with $v\,\dfrac{dv}{dx}$, express x in terms of v.

(a) $\ddot{x} = \dfrac{v^2}{4}$ and when $x = 0$, $v = 1$
(b) $\ddot{x} = \dfrac{3}{v}$ and when $x = 0$, $v = 6$
(c) $\ddot{x} = 2 + v$ and when $x = 0$, $v = 0$

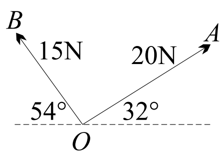
6. A particle of mass m moves in a straight line subject to a force F. At time t, the displacement of the particle is x and the velocity is v. The particle was initially at rest at the origin.

(a) If $F = 6t - 4$ and $m = 2$, find x when $t = 4$.
(b) If $F = 2x + 1$ and $m = 1{\cdot}5$, find the positive value of v when $x = 3$.
(c) If $F = \dfrac{1}{v+2}$ and $m = 0{\cdot}25$, find t when $v = 4$.
(d) If $F = \dfrac{1}{v+2}$ and $m = 0{\cdot}5$, find x when $v = 3$.

DEVELOPMENT

7. Three forces act on an object of mass 2 kg. These forces are represented by the vectors $12\underset{\sim}{i}+23\underset{\sim}{j}$, $9\underset{\sim}{i}-7\underset{\sim}{j}$ and $-5\underset{\sim}{i}+14\underset{\sim}{j}$. Calculate the magnitude and direction of the acceleration of the object.

8. The diagram on the right shows two forces of magnitude 20 N and 15 N represented by the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



(a) Express $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as component vectors.
(b) Calculate the magnitude of the resultant of the two forces, correct to the nearest newton.
(c) Determine the direction of the resultant, correct to the nearest degree.

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9. Tom can paddle his canoe with a force of 20 N. He starts paddling from a point on the south bank of a river and steers the canoe at 90° to the bank. He experiences a force of 6 N acting due east due to the current, and he also has to contend with a force of 4 N due to a breeze blowing from the north-east.

(a) Express the resultant force on Tom and his canoe as a component vector.

(b) Hence find the magnitude (in newtons to one decimal place) and direction (in degrees to one decimal place) of the resultant force.

10. [A formula from physics] A particle moves with constant acceleration a, so that its equation of motion is $\ddot{x} = a$. Its initial velocity is u. After t seconds, its velocity is v and its displacement is s. Use $\dfrac{d}{dx}(\frac{1}{2}v^2)$ for acceleration to show that $v^2 = u^2 + 2as$.

11. A ball is thrown vertically upwards at 20 m/s. Taking $g = 10\text{m/s}^2$, upwards as positive, and the ground as the origin of displacement, the equation of motion is then $\ddot{x} = -10$.

(a) Show that $v^2 = 400 - 20x$, and find the greatest height.

(b) Explain why $v = \sqrt{400 - 20x}$ while the ball is rising.

(c) Integrate to find the displacement–time function, and find how long it takes the ball to reach its greatest height.

12. Assume that a bullet, fired at 1 km/s, moves through the air with deceleration proportional to the square of the velocity, so that $\ddot{x} = -kv^2$ for some positive constant k.

(a) If the velocity after 100 metres is 10 m/s, use $\ddot{x} = v\,\dfrac{dv}{dx}$ to find x as a function of v, then find how far the bullet has travelled when its velocity is 1 m/s.

(b) If the velocity after 1 second is 10 m/s, use $\ddot{x} = \dfrac{dv}{dt}$ to find at what time the bullet has velocity 1 m/s.

13. A particle has acceleration $\ddot{x} = e^{-x}$, and initially $v = 2$ and $x = 0$. Find v^2 as a function of x, and explain why v is always positive and at least 2. Then briefly explain what happens as time goes on.

14. A particle has velocity $v = 6 - 2x$, and initially the particle was at the origin.

(a) Find the acceleration at the origin.

(b) Show that $t = -\frac{1}{2}\ln\left|1 - \frac{1}{3}x\right|$, and hence find x as a function of t.

(c) Describe the behaviour of the particle as $t \to \infty$.

15. An object is initially at rest at the origin. It moves in a straight line away from the origin with acceleration $2(1 + v)\,\text{m/s}^2$. Find, correct to 3 significant figures:

(a) how long it takes for the velocity to reach 20 m/s,

(b) the distance travelled when the velocity reaches 20 m/s.

16. A particle P of mass m starts from the origin O with velocity u and moves in a straight line. When $OP = x$, where $x \ge 0$, the velocity v of P is given by $v = u + \dfrac{x}{k}$, where k is a positive constant.

(a) Prove that the force acting on P is at all times proportional to v, and state the constant of proportionality.

(b) Given that the velocity is $3u$ at the point A, find:

(i) the distance OA in terms of k and u.

(ii) the time, in terms of k, taken by P to move from O to A.

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17. A particle of mass 0·5 kg is acted upon by a force $F = \left(x - \frac{1}{2}\right)$ newtons. Initially the particle is at rest 5 metres on the positive side of the origin.

(a) Find v^2 in terms of x, and hence explain why the particle can never be at the origin.

(b) Find where the speed of the particle is $2\sqrt{5}$ m/s, justifying your answer, and describe the subsequent motion.

18. A particle of mass 2 kg is subject to a force of $6x^2$ newtons. Initially the particle is at $x = 1$ with velocity $-\sqrt{2}$ m/s.

(a) Find v^2 as a function of x.

(b) Then find the displacement–time function, and briefly describe the motion.

19. The acceleration of a particle moving in a straight line is given by $\ddot{x} = 3(1 - x^2)$, where x metres is the displacement of the particle. Initially the particle was at the origin with velocity 4 m/s.

(a) Find v^2 as a function of x.

(b) Does the particle ever change direction? Justify your answer with clear reasoning.

ENRICHMENT

20. Newton's law of gravitation says that an object falling towards a planet has acceleration $\ddot{x} = -kx^{-2}$, for some positive constant k, where x is the distance from the centre of the planet. Show that if the body starts from rest at a distance D from the centre, then its speed at a distance x from the centre is $\sqrt{\dfrac{2k(D-x)}{Dx}}$.

21. A projectile is fired vertically upwards with speed V from the surface of the Earth.

(a) Assuming the same equation of motion as in the previous question, and ignoring air resistance, show that $k = gR^2$, where R is the radius of the Earth.

(b) Find v^2 in terms of x and hence find the maximum height of the projectile above the centre of the earth.

(c) [The escape velocity from the Earth] Given that $R = 6400$ km and $g = 9{\cdot}8$ m/s^2, find the least value of V so that the projectile will never return.

22. The velocity of a particle moving on the positive x-axis is given by $v = \left(8 - 3e^{-2t}\right)$ m/s.

(a) Show that $\displaystyle\int \frac{v}{8-v}\,dv = \int 2\,dx$.

(b) Find, correct to 3 significant figures, the distance travelled by the particle as its speed increases from 0 m/s to 7 m/s.

6B Simple Harmonic Motion and Time

Many things naturally exhibit repeated patterns or oscillations. The human body provides many examples such as the heart beating, the lungs breathing or the vocal chords vibrating. Elsewhere in nature there are the waves on a beach, the tides of the ocean or the phases of the moon. Man-made phenomena include the regular motion of a piston in a steam engine or combustion engine, or the regular ebb and flow of traffic caused by the phases of traffic lights. The mathematics behind each of these examples is extremely complicated, but each can be based on the oscillations observed in the graphs of the sine and cosine functions. This is what will be studied in the next two sections.

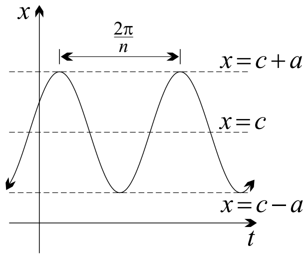
Simple Harmonic Motion: The shape of the graph of $x = \sin t$ is called *sinusoidal.* A translation, dilation or any combination of these two does not change the basic shape. Thus the graphs of functions like $x = 3\cos(2t + \frac{\pi}{2})$ are also sinusoidal. Whenever the graph of displacement x versus time t for a particular motion is sinusoidal, that motion is called *simple harmonic motion*, or *SHM* for short.

Accommodating the combinations of translations and dilations, both vertically and horizontally, the equation of simple harmonic motion is commonly written in one of two ways:

$$x = a\cos(nt + \alpha) + c,$$
$$x = a\sin(nt + \beta) + c.$$

Sometimes a third equation is used and that will be discussed later.

The physical meanings of the values a, n and c are the same in both these equations. The constant c is called the *centre* of motion, as the wave oscillates symmetrically either side of the horizontal line $x = c$, as seen in the graph on the right. The value a is the *amplitude* of the motion because the wave moves at most a away from the centre. Thus x always lies in the range $c - a \le x \le c + a$. The constant n is used to calculate the *period* $= \frac{2\pi}{n}$, which is the time between two peaks or two troughs.



At any time t the quantities $(nt + \alpha)$ and $(nt + \beta)$ are each called the phase of the motion, as their values determine the location, rather like the way that time dictates the phases of the moon. The constant angles α and β are each called the *initial phase* of the motion, as they determine the initial displacement. When $t = 0$ the two forms of the equation give

$$a\cos\alpha + c = a\sin\beta + c$$

and so $\cos\alpha = \sin\beta.$

Hence the angles are complementary and $\beta = \frac{\pi}{2} - \alpha$, though it is not necessary to memorise this formula.

Worked Example 6: A particle is moving in simple harmonic motion according to the equation $x = 2\sin(\frac{\pi}{2}t + \frac{\pi}{6}) + 3$.

(a) Write down its centre, amplitude and extremes of displacement.

(b) Determine the period, initial phase and location at $t = 0$.

(c) Determine when the particle is next at the same location.

Solution:

(a) The centre is $x = 3$, the amplitude is 2 so the extremes are $x = 1$ and $x = 5$.

(b) The period is $2\pi \div \frac{\pi}{2} = 4$, with initial phase $\frac{\pi}{6}$. At $t = 0$, $x = 2\sin\frac{\pi}{6} + 3 = 4$.

(c) Solving $x = 4$ gives

$$\begin{aligned} & & 2\sin(\tfrac{\pi}{2}t + \tfrac{\pi}{6}) + 3 &= 4 \\ &\text{thus} & \sin(\tfrac{\pi}{2}t + \tfrac{\pi}{6}) &= \tfrac{1}{2} \\ &\text{so} & \tfrac{\pi}{2}t + \tfrac{\pi}{6} &= \tfrac{\pi}{6}, \tfrac{5\pi}{6}, \tfrac{13\pi}{6}, \ldots \\ &\text{or} & \tfrac{\pi}{2}t &= 0, \tfrac{2\pi}{3}, 2\pi, \ldots \\ &\text{that is} & t &= 0, \tfrac{4}{3}, 4, \ldots \end{aligned}$$

Hence the particle is next at $x = 4$ when $t = \frac{4}{3}$.

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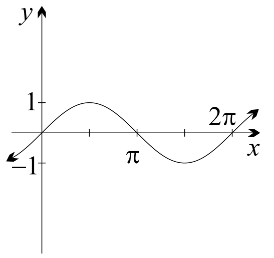
The function in this last example is the one graphed above in the discussion. By chance, both the values of a and c are integers whilst both n and β are irrational. In general, there are no restrictions except that each is a real number.

3

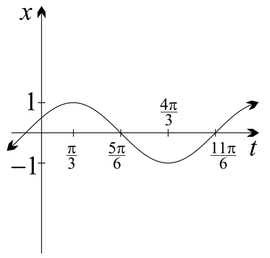
> SIMPLE HARMONIC MOTION: A particle with displacement x at time t is in simple harmonic motion if either
>
> $$x = a\cos(nt + \alpha) + c \qquad \text{or} \qquad x = a\sin(nt + \beta) + c\,.$$
>
> The centre of motion is c, the amplitude is a and the period is $\frac{2\pi}{n}$. The quantities $(nt+\alpha)$ and $(nt+\beta)$ are each called the phase of the motion. Each angle α or β is called the initial phase.

SHM and Transformations: As stated earlier, motion which is simple harmonic may be the result of shifts or dilations. The transformations applied to $x = \cos t$ in order to obtain $x = a\cos(nt+\alpha)+c$ are investigated in Extension 1. Here the horizontal transformations will be done first and the vertical second. Of course, that order can be reversed.

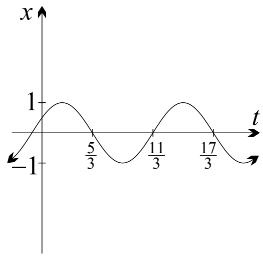
First shift $x = \cos t$ left by α to get $x = \cos(t+\alpha)$. Both of these functions have period 2π. Now stretch horizontally by factor $\frac{1}{n}$ to get $x = \cos(nt+\alpha)$ which has period $\frac{2\pi}{n}$. Next stretch the wave vertically by factor a, giving $x = a\cos(nt+\alpha)$. Finally shift the graph up by c so that the centre, also called the mean position, is at $x = c$. A similar sequence of transformations is applied to $x = \sin t$. The five graphs that follow show such a sequence from $x = \sin t$ to $x = 2\sin(\frac{\pi}{2}t + \frac{\pi}{6}) + 3$.



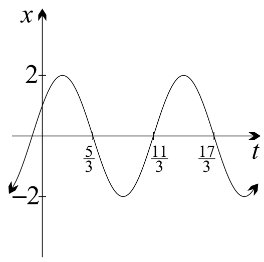
$x = \sin t$



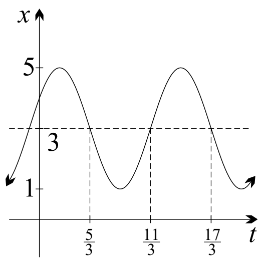
shift left by $\frac{\pi}{6}$
$x = \sin(t + \frac{\pi}{6})$



stretch t by $\frac{2}{\pi}$
$x = \sin(\frac{\pi}{2}t + \frac{\pi}{6})$



stretch x by 2
$x = 2\sin(\frac{\pi}{2}t + \frac{\pi}{6})$



shift up by 3
$x = 2\sin(\frac{\pi}{2}t + \frac{\pi}{6}) + 3$

In a few cases it may be convenient to apply the horizontal transformations in the opposite order by writing $x = a\cos\big(n(t + \frac{\alpha}{n})\big) + c$. In this case a horizontal stretch by factor $\frac{1}{n}$ is followed by a shift left by $\frac{\alpha}{n}$. The value $\frac{\alpha}{n}$ is sometimes called the phase shift.

Choosing the Origin: In many practical problems the location of the origin is not specified or may be changed without significantly altering the problem. In such problems, the origin should be chosen so that the equation of simple harmonic motion is one of the following four cases.

$x = a\cos nt$ — the motion starts at the top and moves downwards,
$x = -a\cos nt$ — the motion starts at the bottom and moves upwards,
$x = a\sin nt$ — the motion starts at the centre and moves upwards,
$x = -a\sin nt$ — the motion starts at the centre and moves downwards.

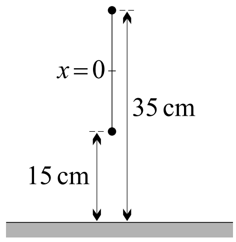
Notice that in each case $c = 0$. In the first case $\alpha = 0$, and in the second case $\alpha = \pi$ which accounts for the negative sign. Likewise, in the third case $\beta = 0$ and in the fourth case $\beta = \pi$.

Worked Example 7: A weight is hung from a stand on a table by a spring and set in vertical motion. The weight oscillates between 15cm and 35 cm above the table, and it takes 2 seconds to complete one cycle. It is found that the motion of the weight may be modelled by simple harmonic motion.

(a) What is its height above the table $\frac{3}{4}$s after it passes through the lowest point?
(b) For how long in each cycle is the weight at or below 18cm? Approximate the answer correct to two decimal places.

Solution: Let the centre of motion be $x = 0$, 25cm above the table. The problem involves time after the bottom of the wave, so let $t = 0$ there. The amplitude of the motion is $a = 10$ and the period is $2 = \frac{2\pi}{n}$ so $n = \pi$. Hence put

$$x = -10\cos\pi t$$



(a) At $t = \frac{3}{4}$ the displacement is

$$\begin{aligned} x &= -10\cos\tfrac{3\pi}{4} \\ &= 5\sqrt{2} \end{aligned}$$

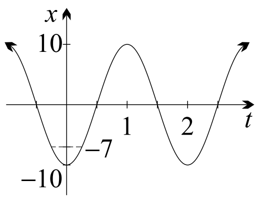
hence the height is $25 + 5\sqrt{2} \doteqdot 32{\cdot}07$ cm.

(b) The question is equivalent to solving

$$-10\cos\pi t \le -7$$

or $\cos\pi t \ge 0.7$

For equality, the first positive and negative solutions are $t \doteqdot \pm 0{\cdot}253$, thus the weight is at or below 18cm for about $0{\cdot}51$ seconds each cycle.



4 **Choosing the origin:** Whenever possible, choose the origin of displacement and time so that the equation of motion is one of the following cases:
$x = a\cos nt$ — the motion starts at the top and moves downwards,
$x = -a\cos nt$ — the motion starts at the bottom and moves upwards,
$x = a\sin nt$ — the motion starts at the centre and moves upwards,
$x = -a\sin nt$ — the motion starts at the centre and moves downwards.

Speed and Acceleration: In simple harmonic motion the displacement x is given as a function of time t. Hence it is easy to differentiate to find the velocity and acceleration. For simplicity here, assume that the centre of motion is the origin.

Let	$x = a\cos(nt+\alpha)$	Let	$x = a\sin(nt+\beta)$
then	$\dot{x} = -na\sin(nt+\alpha)\,,$	then	$\dot{x} = na\cos(nt+\beta)\,,$
and	$\ddot{x} = -n^2a\cos(nt+\alpha)\,.$	and	$\ddot{x} = -n^2a\sin(nt+\beta)\,.$
Hence	$\ddot{x} = -n^2x\,.$	Hence	$\ddot{x} = -n^2x\,.$

Several important observations can be made from this algebra. The maximum speed is $|\dot{x}| = na$, which occurs when $|\sin(nt+\alpha)| = 1$, or $(nt+\alpha) = \frac{\pi}{2} + k\pi$. At these values $\cos(nt+\alpha) = 0$ and so $x = 0$. That is, the maximum speed of an object in SHM occurs as it passes through the centre of motion. Further, the minimum speed is zero, which occurs when $(nt+\alpha) = k\pi$. At these values $|x| = a$. That is, the minimum speed corresponds with the turning points at the extremes of the motion.

The maximum acceleration is $|\ddot{x}| = n^2a$, which occurs when $|\cos(nt+\alpha)| = 1$, or $(nt+\alpha) = k\pi$. At these values $|x| = a$. That is, the maximum acceleration of an object in SHM corresponds with the extremes of the motion, as it is changing direction. The minimum acceleration is zero when $(nt+\alpha) = \frac{\pi}{2} + k\pi$. At these values $|x| = 0$. That is, the minimum acceleration occurs as the object passes through the centre of motion.

WORKED EXAMPLE 8: An object is in simple harmonic motion with a period of $\frac{\pi}{3}$ seconds. At a certain moment its position is 3 cm above the centre of motion, and it is moving towards the centre of motion with speed 24 cm/s.

(a) Determine its equation of motion and hence find its maximum speed.

(b) Find when it next has the same displacement, correct to two decimal places.

SOLUTION: Let the centre of motion be $x = 0$. Let the moment it is observed correspond with $t = 0$. From the period, $n = 2\pi \div \frac{\pi}{3}$. That is $n = 6$. Since it is initially moving downwards, the velocity is initially negative.

Let $\quad x = a\cos(6t+\alpha)$

then $\quad \dot{x} = -6a\sin(6t+\alpha)\,.$

At $t = 0$, from the given information,

$$a\cos\alpha = 3 \qquad (1)$$

and $\quad -6a\sin\alpha = -24$

or $$a\sin\alpha = 4\,. \qquad (2)$$

Squaring and adding, it is clear that

$$a^2 = 25$$

thus $\quad a = 5 \qquad (a > 0.)$

From equations (1) and (2) it follows that α is acute, so

$$\alpha = \cos^{-1}\tfrac{3}{5} \doteqdot 0{\cdot}9273$$

(a) The maximum speed is $|\dot{x}| = 6a = 30$ cm/s.

(b) The object will return to its original position when

$$5\cos(6t+\alpha) = 5\cos\alpha$$

so $\quad \cos(6t+\alpha) = \cos\alpha$

The first time this happens is, by the symmetries of cosine, when

$$6t+\alpha = 2\pi - \alpha$$

thus $\quad t = \tfrac{1}{3}(\pi - \alpha) \doteqdot 0{\cdot}74$ s.

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> SPEED AND ACCELERATION:
> - The maximum speed is at the centre of motion where the acceleration is zero.
> - The maximum acceleration occurs at the extremes of the motion where the speed is zero. That is, at the stationary points.

The Differential Equation for Simple Harmonic Motion: Above it was shown:

for $x = a\cos(nt + \alpha)$ and for $x = a\sin(nt + \beta)$

$\ddot{x} = -n^2 a\cos(nt + \alpha)$, $\ddot{x} = -n^2 a\sin(nt + \beta)$.

Look carefully at the expressions for displacement and acceleration. It should be clear that in both cases the acceleration is related to the displacement by the autonomous second order linear differential equation

$$\ddot{x} = -n^2 x\,.$$

Applying Newton's formula for force gives $F = -mn^2x$. This means that the force required to keep an object in SHM is proportional to its displacement. The negative sign indicates that the force is directed towards the centre of motion. The equation may be used to test whether or not a given motion is simple harmonic with centre the origin. If the centre is $x = c$ instead then this equation becomes

$$\ddot{x} = -n^2(x - c)\,,$$

and the proof of this is left as an exercise.

6

> THE DIFFERENTIAL EQUATION FOR SIMPLE HARMONIC MOTION: If an object is moving in simple harmonic motion with centre $x = c$ and period $\frac{2\pi}{n}$ then
>
> $$\ddot{x} = -n^2(x - c)\,.$$
>
> Consequently the force associated with this motion is directed towards the centre. This equation may be used as a test for simple harmonic motion.

Another Form of Simple Harmonic Motion: In some problems where the initial displacement and velocity are known, it may be best to use the following form:

$$x = A\cos nt + B\sin nt\,.$$

It is easy to show by differentiation that this is an equation for SHM.

$$\dot{x} = -nA\sin nt + nB\cos nt$$

and $$\ddot{x} = -n^2A\cos nt - n^2B\sin nt$$

$$= -n^2(A\cos nt + B\sin nt)$$

thus $$\ddot{x} = -n^2x\,.$$

By the test in Box 6, this differential equation confirms that the motion is simple harmonic with centre $x = 0$ and period $\frac{2\pi}{n}$.

WORKED EXAMPLE 9: Once again, an object is in simple harmonic motion with a period of $\frac{\pi}{3}$ seconds. At a certain moment its position is 3 cm above the centre of motion, and it is moving towards the centre of motion with speed 24 cm/s. Let the equation of motion be $x = A\cos nt + B\sin nt$.

(a) Determine the values of n, A and B.

(b) Use the t-formulae to find when the object is next at its original position.

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SOLUTION:

(a) As in the previous worked example, $n = 6$, and

$$x = A\cos 6t + B\sin 6t$$

so $\dot{x} = -6A\sin 6t + 6B\cos 6t\,.$

At $t = 0$ the displacement is $x = 3$ so

$$A = 3$$

and the velocity is $\dot{x} = -24$ so

$$6B = -24$$

thus $B = -4\,.$

Hence $x = 3\cos 6t - 4\sin 6t\,.$

(b) The object will return at the first positive solution of

$$3\cos 6t - 4\sin 6t = 3\,.$$

Put $\tau = \tan 3t$, then this becomes

$$3 \times \frac{1-\tau^2}{1+\tau^2} - 4 \times \frac{2\tau}{1+\tau^2} = 3$$

thus $3 - 3\tau^2 - 8\tau = 3 + 3\tau^2$

or $6\tau^2 + 8\tau = 0\,.$

Hence for the first positive solution

$$\tau = -\tfrac{4}{3}\,,$$

viz $t = \tfrac{1}{3}(\pi - \tan^{-1}\tfrac{4}{3}) \doteqdot 0{\cdot}74\,\text{s}$ (as before.)

7 ANOTHER FORM OF SIMPLE HARMONIC MOTION: In some problems for which the initial displacement and velocity are known, it may be best to use

$$x = A\cos nt + B\sin nt\,.$$

This corresponds to simple harmonic motion with centre $x = 0$ and period $\frac{2\pi}{n}$.

Exercise **6B**

1. A particle is moving in simple harmonic motion about the origin. Its displacement x cm after t seconds is given by $x = 12\cos\frac{\pi}{2}t$.

(a) What are the amplitude and period of the motion?

(b) Differentiate to find v and $\ddot{x}$ as functions of t, and then show that $\ddot{x} = -\frac{\pi^2}{4}x$.

(c) What are the initial displacement and velocity of the particle?

(d) When is the particle first at the origin?

(e) How long is it between visits to the origin?

2. A particle moves so that its displacement x metres after t seconds is given by $x = 2\sin 4\pi t$.

(a) Write down the amplitude and period of the motion.

(b) Sketch the displacement function for $0 \le t \le 1$.

(c) Find the velocity and acceleration as functions of time.

(d) Find the acceleration as a function of displacement and hence show that the particle is moving in simple harmonic motion.

(e) Find the first two times at which the particle is at rest, and find the acceleration at each of these times.

(f) What is the greatest speed of the particle?

3. A particle is moving in simple harmonic motion with displacement $x = \frac{4}{\pi} \sin \pi t$, in units of metres and seconds.
(a) Show that the particle is initially at the origin.
(b) Differentiate to find v and $\ddot{x}$ as functions of time, and show that $\ddot{x} = -\pi^2 x$.
(c) What are the amplitude and period of the motion?
(d) What are the maximum distance of the particle from the origin and the maximum speed?
(e) Sketch one period of the graphs of x, v and $\ddot{x}$ against time.
(f) Find the next two times the particle is at the origin, and the velocities then.
(g) Find the first two times the particle is stationary, and the accelerations then.

4. In each of the following parts find a and n, and hence write the displacement function in the form $x = a \sin nt$.
(a) A particle moving in SHM with centre the origin and period π seconds starts from the origin with velocity $4\,\text{m/s}$.
(b) A particle moving in SHM with centre the origin and amplitude 6 metres starts from the origin with velocity $4\,\text{m/s}$.

5. (a) A particle's displacement is given by $x = b \sin nt + c \cos nt$, where $n > 0$. Find v and $\ddot{x}$ as functions of t. Then show that $\ddot{x} = -n^2 x$, and hence that the motion is simple harmonic.
(b) By substituting into the functions for x and v:
(i) find b and c if initially the particle is at rest at $x = 3$,
(ii) find b, c and n, and the first time the particle reaches the origin, if the particle is initially at rest at $x = 5$, and the period is 1 second.

6. A particle moving in a straight line started from the origin with velocity $4\pi\,\text{cm/s}$. Its displacement after t seconds is given by $x = a \sin \pi t$.
(a) Prove that the motion is SHM.
(b) Find the value of a.
(c) Find the first two times that the speed of the particle is $2\pi\,\text{cm/s}$.

7. A particle's displacement is $x = 12 - 2\cos 3t$, in units of centimetres and seconds.
(a) Differentiate to find v and $\ddot{x}$ as functions of t, show that the particle is initially stationary at $x = 10$, and sketch the displacement–time graph.
(b) What are the amplitude, period and centre of the motion?
(c) In what interval is the particle moving, and how long does it take to go from one end to the other?
(d) Find the first two times after time zero when the particle is closest to the origin, and the speed and acceleration then.
(e) Find the first two times when the particle is at the centre, and the speed and acceleration then.

8. A particle is moving in SHM according to the equation $x = 6 \sin(2t + \frac{\pi}{2})$.
(a) What are the amplitude, period and initial phase?
(b) Find $\dot{x}$ and $\ddot{x}$, and show that $\ddot{x} = -n^2 x$, for some $n > 0$.
(c) Find the first two times when the particle is at the origin, and the velocity then.
(d) Find the first two times when the velocity is maximum, and the position then.
(e) Find the first two times the particle returns to its initial position, and its velocity and acceleration then.

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DEVELOPMENT

9. A particle is oscillating in simple harmonic motion about the origin with period 24 seconds and amplitude 120 metres. Initially it is at the origin with positive velocity.

(a) Write down functions for x and v, and state the maximum speed.

(b) What is the first time when it is 30 metres: (Answer correct to four significant figures.) (i) to the right of the origin, (ii) to the left of the origin?

(c) Find the first two times its speed is half its maximum speed.

10. A particle moves in simple harmonic motion about the origin with period $\frac{\pi}{2}$ seconds. Initially the particle is at rest 4 cm to the right of O.

(a) Write down displacement–time and velocity–time functions.

(b) Find how long the particle takes to move from its initial position to: (i) a point 2 cm to the right of O, (ii) a point 2 cm on the left of O.

(c) Find the first two times when the speed is half the maximum speed.

11. The equation of motion of a particle is $x = \sin^2 t$. Use a double-angle identity to put the equation in the form $x = x_0 - a\cos nt$, and state the centre, amplitude, range and period of the motion.

12. A particle moves according to $x = 3 - 2\cos^2 2t$, in units of centimetres and seconds.

(a) Use a double-angle identity to put the equation in the form $x = x_0 - a\cos nt$.

(b) Find the centre of motion, the amplitude, the range of the motion and the period.

(c) What is the maximum speed of the particle, and when does it first occur?

13. The displacement x cm of a particle after t seconds is given by $x = 2 + 3\cos t + 3\sqrt{3}\sin t$.

(a) Find $\ddot{x}$ and hence prove that the motion is simple harmonic.

(b) Where is the centre of motion?

(c) What is the period?

(d) Express $3\cos t + 3\sqrt{3}\sin t$ in the form $A\cos(x - \theta)$, where $A > 0$ and $0 < \theta < \frac{\pi}{2}$.

(e) Hence state the amplitude and the initial phase.

(f) Within what interval does the particle oscillate?

14. A particle's displacement is given by $x = b\sin nt + c\cos nt$, where $n > 0$. Find v as a function of t. Then find n, c and b, and the first two times the particle is at the origin, if:

(a) the period is 4π, the initial displacement is 6 and the initial velocity is 3,

(b) the period is 6 and when $t = 0$, $x = -2$ and $\dot{x} = 3$. (In this part write the times correct to 3 decimal places.)

15. Given that $x = a\sin(nt + \alpha)$ (in units of metres and seconds), write v as a function of time. Find a, n and α if $a > 0$, $n > 0$, $0 \le \alpha < 2\pi$ and:

(a) the period is 6 seconds, and initially $x = 0$ and $v = 5$,

(b) the period is 3π seconds, and initially $x = -5$ and $v = 0$,

(c) the period is 2π seconds and initially $x = 1$ and $v = -1$.

16. Given that $x = a\cos(2t + \alpha)$, find a and α if $a > 0$, $-\pi < \alpha \le \pi$ and:

(a) initially $x = 0$ and $v = 6$,

(b) initially $x = 1$ and $v = -2\sqrt{3}$.

17. A particle is moving in simple harmonic motion according to $x = a\cos(\frac{\pi}{8}t + \alpha)$, where $a > 0$ and $0 \le \alpha < 2\pi$. When $t = 2$ it passes through the origin, and when $t = 4$ its velocity is 4 cm/s in the negative direction. Find the amplitude a and the initial phase α.

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18. A particle is moving in simple harmonic motion with period 8π seconds according to $x = a\sin(nt + \alpha)$, where x is the displacement in metres, and $a > 0$ and $0 \le \alpha < 2\pi$. When $t = 1$, $x = 3$ and $v = -1$. Find a and α correct to four significant figures.

19. A particle moving in simple harmonic motion has period $\frac{\pi}{2}$ seconds. Initially the particle is at $x = 3$ with velocity $v = 16\,\text{m/s}$.

(a) Find x as a function of t in the form $x = b\sin nt + c\cos nt$.

(b) Find x as a function of t in the form $x = a\cos(nt - \varepsilon)$, where $a > 0$ and $0 \le \varepsilon < 2\pi$.

(c) Find the amplitude and the maximum speed of the particle.

(d) Find the first time the particle is at the origin, using each of the above displacement functions in turn. Prove that the two answers obtained are the same.

20. The temperature at each instant of a day can be modelled by a simple harmonic function oscillating between 9° at 4:00 am and 19° at 4:00 pm. Find, correct to the nearest minute, the times between 4:00 am and 4:00 pm when the temperature is:

(a) 14° (b) 11° (c) 17°

21. The rise and fall in sea level due to tides can be modelled by simple harmonic motion. On a certain day, a channel is 10 metres deep at 9:00 am when it is low tide, and 16 metres deep at 4:00 pm when it is high tide. If a ship needs 12 metres of water to sail down a channel safely, at what times (correct to the nearest minute) between 9:00 am and 9:00 pm can the ship pass through?

22. Show that for any particle moving in simple harmonic motion, the ratio of the average speed over one oscillation to the maximum speed is $2 : \pi$.

ENRICHMENT

23. The motion of a particle in a straight line is governed by the displacement function

$$x = 4\sin(3t + \tfrac{\pi}{6}) + 2\sin 3t.$$

(a) Prove that the motion is simple harmonic.

(b) Find the amplitude.

24. A particle moving in SHM about the origin starts at $x = 1$. At the end of each of the first two seconds the particle is at $x = 5$. Prove that the period of the motion is $\dfrac{2\pi}{\cos^{-1}\frac{3}{5}}$.

(Let the displacement function be $x = a\cos(nt + \alpha)$.)

6C Simple Harmonic Motion and Displacement

The focus of the previous section was on the displacement-time function for simple harmonic motion. In this section it is the differential equation

$$\ddot{x} = -n^2(x - c)$$

that takes centre stage. Firstly it will be used as a test for simple harmonic motion. Then the identity $\dfrac{d^2x}{dt^2} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$ will be used with integration to find the velocity as a function of displacement.

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Simple Harmonic Motion and the Differential Equation: In the last section it was shown that if a particle is in simple harmonic motion with centre $x = 0$ and period $\frac{2\pi}{n}$ then

$$\ddot{x} = -n^2 x\,.$$

There is a clever way to prove the corresponding result when the centre is not zero. Let $y = x + c$, then the particle is in simple harmonic motion with centre $y = c$. Now

$$x = y - c$$

so $\quad \dot{x} = \dot{y}$

and $\quad \ddot{x} = \ddot{y}\,.$

Hence by substitution, the above differential equation becomes

$$\ddot{y} = -n^2(y - c)\,.$$

Finally, since the choice of pronumeral was arbitrary, if a particle is in simple harmonic motion with centre $x = c$ and period $\frac{2\pi}{n}$ then

$$\ddot{x} = -n^2(x - c)\,.$$

This equation may be taken as a test for simple harmonic motion. That is, if the displacement of a particle satisfies this equation then the particle is in simple harmonic motion. The proof that the motion is simple harmonic is given later.

WORKED EXAMPLE 10: The displacement x cm of a particle at time t seconds satisfies $\ddot{x} = -4(x - 3)$. The particle is initially at $x = 3$ with velocity 6 cm/s. Determine the displacement-time function and find when the particle is first at the origin.

SOLUTION: Since $\ddot{x} = -2^2(x - 3)$, the particle is in simple harmonic motion with centre $x = 3$ and period π. The particle starts at its centre with positive velocity, so put

$$x = 3 + a\sin 2t\,.$$

Thus $\quad \dot{x} = 2a\cos 2t$

so at $t = 0$ $\quad 6 = 2a$

and hence $\quad x = 3 + 3\sin 2t\,.$

The particle will be at the origin when

$$\sin 2t = -1$$

or $\quad 2t = \ldots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

so the first positive solution is

$$t = \frac{3\pi}{4}\,.$$

8 SIMPLE HARMONIC MOTION AND THE DIFFERENTIAL EQUATION: If the displacement of a particle satisfies the equation

$$\ddot{x} = -n^2(x - c)$$

then it is in simple harmonic motion with centre $x = c$ and period $\frac{2\pi}{n}$.

The proof that the motion must be simple harmonic is relatively straightforward when the centre is $x = 0$. The substitution $y = x + c$ can then be used to extend the proof to motion with other centres.

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Proof: Suppose that $\ddot{x} = -n^2x$. Regardless of the values of $x(0)$ and $\dot{x}(0)$, it is always possible to find a and α such that $x(0) = a\sin\alpha$ and $\dot{x}(0) = na\cos\alpha$. (The proof of this is left as an exercise.) Now consider the function

$$u = x - a\sin(nt+\alpha)$$

Firstly, $$u(0) = x(0) - a\sin\alpha = 0\,.$$

Next, $$\dot{u} = \dot{x} - na\cos(nt+\alpha)$$

so $$\dot{u}(0) = \dot{x}(0) - na\cos\alpha = 0\,.$$

Further, $$\ddot{u} = \ddot{x} - n^2a\sin(nt+\alpha) = -n^2x - n^2a\sin(nt+\alpha)$$

so $$\ddot{u} = -n^2u\,.$$

Thus $$\frac{d}{du}\left(\tfrac{1}{2}\dot{u}^2\right) = -n^2u$$

or $$\frac{d}{du}\left(\dot{u}^2\right) = -2n^2u$$

hence $$\dot{u}^2 = C - n^2u^2 \quad \text{for some constant } C.$$

Now apply the initial conditions $u(0) = \dot{u}(0) = 0$ to get $C = 0$.

Thus $$\dot{u}^2 = -n^2u^2\,.$$

But u is a real function and squares of reals are either positive or zero. Thus the only solution to this equation is that $u = 0$ for all t. Hence

$$x = a\sin(nt+\alpha)$$

and the motion is simple harmonic.

It is now possible to revisit Worked Example 3, concerning the motion of a spring. In that problem,

$$\ddot{x} = -x$$

and thus the motion is simple harmonic with centre $x = 0$. The initial conditions are $x(0) = -a$ and $\dot{x}(0) = 0$, so it follows that the displacement-time function is

$$x = -a\cos t\,.$$

It is now clear that x takes all values in the range $-a \le x \le a$ as t varies.

Velocity as a Function of Displacement: Starting with the differential equation for acceleration

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -n^2(x-c)$$

or $$\frac{d}{dx}\left(v^2\right) = -2n^2(x-c)$$

thus $$v^2 = D - n^2(x-c)^2\,.$$

Now at each extreme of motion, the velocity is zero, thus

$$0 = D - n^2a^2.$$

That is $$D = n^2a^2$$

and $$v^2 = n^2a^2 - n^2(x-c)^2$$

or $$v^2 = -n^2\left((x-c)^2 - a^2\right).$$

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Notice that the term inside the outer brackets is a monic quadratic with its square completed. This formula can be used to determine the centre, period and amplitude of the motion, however it must be derived each time, and not quoted. In practice it is better to first check that the motion is simple harmonic by differentiation, thus finding the centre and n. Then set $v = 0$ to find the range and amplitude.

WORKED EXAMPLE 11: The motion of a particle satisfies $v^2 = -2x^2 + 8x + 10$. Show that the motion is simple harmonic then determine the period, centre of motion, range and amplitude.

SOLUTION: Firstly, differentiate to find the acceleration.

$$\begin{aligned}\ddot{x} &= \frac{d}{dx}\left(-x^2 + 4x - 5\right)\\ &= -2x + 4\\ &= -2(x-2)\,.\end{aligned}$$

Hence the motion is simple harmonic with $n = \sqrt{2}$. The period is $\pi\sqrt{2}$ and centre is $x = 2$. Now put $v = 0$ to find the range.

$$x^2 - 4x - 5 = 0$$

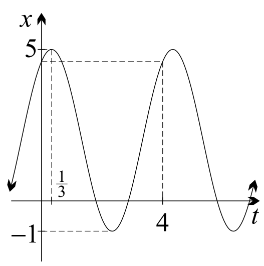
or $$(x+1)(x-5) = 0\,,$$

so the extremes of the motion are $x = -1$ and $x = 5$. Taking the average confirms the centre is $x = 2$. Taking the difference and halving gives amplitude $a = 3$.

9	VELOCITY AS A FUNCTION OF DISPLACEMENT: First check that the motion is simple harmonic by differentiation, and so find the centre and n. Then set $v = 0$ to find the range and amplitude.

Finding the Equation of Motion from the Graph: It is easy to determine the equation of simple harmonic motion from its graph. It is simply a matter of determining the shifts and stretches applied to sine or cosine. A quick example is included here as a reminder of the process.

WORKED EXAMPLE 12: The motion of a body is plotted in the graph on the right. Assuming the motion to be simple harmonic, find the displacement-time function.



SOLUTION: The period is $4 = \frac{2\pi}{n}$ hence $n = \frac{\pi}{2}$.
The extremes of motion are $x = -1$ and $x = 5$.
The centre is the mean $c = \frac{5+-1}{2} = 2$.
The amplitude is half the range $= \frac{5+1}{2} = 3$.
Hence $x = 2 + 3\sin(\frac{\pi}{2}t + \alpha)$ with α acute so that the initial velocity is positive.

Now $\dot{x} = \frac{3\pi}{2}\cos(\frac{\pi}{2}t + \alpha)$

and the velocity is zero at the extreme when $t = \frac{1}{3}$. Thus

$$0 = \cos(\tfrac{\pi}{6} + \alpha)$$

whereby $\frac{\pi}{6} + \alpha = \frac{\pi}{2}$

so $\alpha = \frac{\pi}{3}\,.$

Hence $x = 2 + 3\sin\left(\frac{\pi}{6}(3t+2)\right)\,.$

Exercise **6C**

1. The displacement x metres of a particle after t seconds is given by $x = 3\cos 2t$.
(a) Find expressions for v and $\ddot{x}$ as functions of t, and for $\ddot{x}$ and v^2 in terms of x.
(b) Find the velocity and acceleration of the particle at $x = 2$.

2. The motion of a particle is governed by the equation $\ddot{x} = -9x$ (in units of metres and seconds). The particle is stationary when $x = 5$.
(a) Integrate to find an equation for v^2.
(b) Find the velocity and acceleration when $x = 3$.
(c) What is the speed at the origin?

3. A particle is oscillating according to the equation $\ddot{x} = -16x$ (in units of centimetres and seconds), and its speed at the origin is 24 cm/s.
(a) Integrate to find an equation for v^2.
(b) What are the amplitude and the period?
(c) Find the speed and acceleration when $x = 2$.

4. A particle moves in SHM according to the equation $\ddot{x} = -4x$ (in units of metres and seconds). The amplitude is 6 metres.
(a) Find the velocity–displacement equation, the period and the maximum speed.
(b) Find the simplest form of the displacement–time equation if initially the particle is:
(i) stationary at $x = 6$,
(ii) stationary at $x = -6$,
(iii) at the origin with positive velocity,
(iv) at the origin with negative velocity.

5. (a) The motion of a ball on the end of a spring is modelled by the equation $\ddot{x} = -256x$ (in units of centimetres and seconds). The ball is pulled down 2 cm from the origin and released. Find the speed at the centre of motion.
(b) The motion of another ball on the end of a spring is modelled by $\ddot{x} + \frac{1}{4}x = 0$ (in units of centimetres and seconds), and its speed at the equilibrium position is 4 cm/s. How far was it pulled down from the origin before it was released?

6. [In these questions use the formula $v^2 = n^2(a^2 - x^2)$.]
(a) A particle moving in simple harmonic motion has period $\frac{\pi}{2}$ minutes, and it starts from the mean position with velocity 4 m/min. Find the amplitude.
(b) The motion of a buoy floating on top of the waves can be modelled as simple harmonic motion with period 3 seconds. If the waves rise and fall 2 metres about their mean position, find the buoy's greatest speed.

DEVELOPMENT

7. A particle oscillates in SHM between two points A and B that are 20 cm apart. The period is 8 seconds. Let O be the midpoint of AB.
(a) Find the maximum speed and maximum acceleration.
(b) Find the velocity and acceleration when the particle is 6 cm from O.

8. The amplitude of a particle moving in simple harmonic motion is 5 metres, and its acceleration when it is 2 metres from its mean position is 4 m/s^2. Find the speed of the particle when it is at its mean position, and also when it is 4 metres from its mean position.

9. A particle is moving in SHM with period π seconds and maximum speed 8 m/s. Find the amplitude, and find the speed when the particle is 3 metres from its mean position.

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10. A particle moves in a straight line so that its acceleration is proportional to its displacement x from the origin O. When 4 cm on the positive side of O, its velocity is 20 cm/s and its acceleration is $-6\frac{2}{3}$ cm/s^2. Find the amplitude of the motion.

11. A particle moves in simple harmonic motion with centre O, and passes through O with speed $10\sqrt{3}$ cm/s. Determine the speed of the particle when it is halfway between its mean position and an endpoint.

12. A particle moving in simple harmonic motion about the origin starts at the origin with velocity V. Prove that the particle first comes to rest after travelling a distance of V/n.

13. [The general case] Suppose that a particle is moving in simple harmonic motion with amplitude a and equation of motion $\ddot{x} = -n^2 x$, where $n > 0$.

(a) Use integration to prove that $v^2 = n^2(a^2 - x^2)$.

(b) Find expressions for: (i) the speed at the origin, (ii) the speed and acceleration halfway between the origin and the maximum displacement.

14. (a) A particle moves in a straight line according to the equation $v^2 = -9x^2 + 18x + 27$. Prove that the motion is simple harmonic, and find the centre of motion, the period and the amplitude.

(b) Repeat part (a) for:

(i) $v^2 = 80 + 64x - 16x^2$

(ii) $v^2 = -9x^2 + 108x - 180$

(iii) $v^2 = -2x^2 - 8x - 6$

(iv) $v^2 = 8 - 10x - 3x^2$

15. (a) Show that the motion defined by $x = \sin^2 5t$ (in units of metres and minutes) satisfies $\ddot{x} = -n^2(x - c)$, for some c and some $n > 0$, by:

(i) first writing the displacement function as $x = \frac{1}{2} - \frac{1}{2}\cos 10t$,

(ii) differentiating x directly without any use of double-angle identities.

(b) Find the centre, range and period of the motion, and the next time it visits the origin.

16. A particle moves in simple harmonic motion according to the equation $\ddot{x} = -9(x - 7)$, in units of centimetres and seconds. Its amplitude is 7 cm.

(a) Find the centre of motion, and hence explain why the velocity at the origin is zero.

(b) Integrate to find v^2 as a function of x, complete the square in this expression, and hence find the maximum speed.

(c) Explain how, although the particle is stationary at the origin, it nevertheless moves away from the origin.

17. A particle is moving according to the equation $x = 4\cos 3t - 6\sin 3t$.

(a) Prove that the acceleration is proportional to the displacement but oppositely directed, and hence that the motion is simple harmonic.

(b) Find the period, amplitude and maximum speed of the particle, and find the magnitude of the acceleration when the particle is halfway between its mean position and one of its extreme positions.

18. The motion of a particle is governed by the equation by $x = 3 + \sin 4t + \sqrt{3}\cos 4t$.

(a) Prove that $\ddot{x} = -16(x - 3)$, and write down the centre and period of the motion.

(b) Express the motion in the form $x = x_0 + a\sin(4t + \alpha)$, where $a > 0$ and $0 \le \alpha < 2\pi$.

(c) Find the first three times that the particle is at the centre, and its speed there.

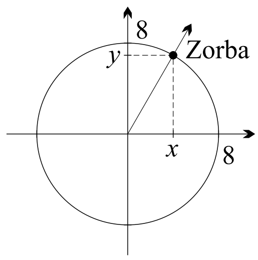
19. A particle moves according to the equation $x = 10 + 8\sin 2t + 6\cos 2t$.

(a) Prove that the motion is simple harmonic, and find the centre of motion, the period and the amplitude.

(b) Find, correct to four significant figures, when the particle first reaches the origin.

ENRICHMENT

20. [Simple harmonic motion is the projection of circular motion onto a diameter.] A Ferris wheel of radius 8 metres mounted in the north–south plane is turning anticlockwise at 1 revolution per minute. At time zero, Zorba is level with the centre of the wheel and north of it.



(a) Let x and y be Zorba's horizontal distance north of the centre and height above the centre respectively.
Show that $x = 8\cos 2\pi t$ and $y = 8\sin 2\pi t$.

(b) Find expressions for $\dot{x}$, $\dot{y}$, $\ddot{x}$ and $\ddot{y}$, and show that $\ddot{x} = -4\pi^2 x$ and $\ddot{y} = -4\pi^2 y$.

(c) Find how far (in radians) the wheel has turned during the first revolution when:

(i) $x : y = \sqrt{3} : 1$ (ii) $\dot{x} : \dot{y} = -\sqrt{3} : 1$ (iii) $\dot{x} = \dot{y}$

21. A particle moves in simple harmonic motion according to $\ddot{x} = -n^2 x$.

(a) Prove that $v^2 = n^2(a^2 - x^2)$, where a is the amplitude of the motion.

(b) The particle has speeds v_1 and v_2 when the displacements are x_1 and x_2 respectively. Show that the period T is given by

$$T = 2\pi\sqrt{\frac{{x_1}^2 - {x_2}^2}{{v_2}^2 - {v_1}^2}}\,,$$

and find a similar expression for the amplitude.

(c) The particle has speeds of 8 cm/s and 6 cm/s when it is 3 cm and 4 cm respectively from O. Find the amplitude, the period and the maximum speed of the particle.

22. A particle moving in simple harmonic motion has amplitude a and maximum speed V. Find its velocity when $x = \frac{1}{2}a$, and its displacement when $v = \frac{1}{2}V$. Prove also the more general results

$$|v| = V\sqrt{1 - x^2/a^2} \qquad \text{and} \qquad |x| = a\sqrt{1 - v^2/V^2}\,.$$

23. Two balls on elastic strings are moving vertically in simple harmonic motion with the same period 2π and with centres level with each other. The second ball was set in motion α seconds later, where $0 \le \alpha < 2\pi$, with twice the amplitude, so their equations are

$$x_1 = \sin t \qquad \text{and} \qquad x_2 = 2\sin(t - \alpha).$$

Let $x = \sin t - 2\sin(t - \alpha)$ be the height of the first ball above the second.

(a) Show that $\ddot{x} = -x$, and hence that x is also simple harmonic with period 2π.

(b) Show that the greatest vertical difference A between the balls is $A = \sqrt{5 - 4\cos\alpha}$. What are the maximum and minimum values of A, and what form does x then have?

(c) Show that the balls are level when $\tan t = \dfrac{4T}{1 - 3T^2}$, where $T = \tan\frac{1}{2}\alpha$. How many times are they level in the time interval $0 \le t < 2\pi$?

(d) If the distance between the balls is known to be greatest when $t = 0$, what values could α have, and what forms does x have?

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6D Horizontal Resisted Motion

If an object moves horizontally then gravity may be effectively ignored. Typically the object is not entirely free to move as there is usually a resistive force, due to friction for example, which acts in the opposite direction to the velocity. In the problems of this section, the equations of motion will either be specified in the question or will need to be determined by balancing forces.

Two Common Integrals: A large number of problems encountered in this section and the next result in integrals of the form

$$\int \frac{v'}{a+bv}\,dx \qquad \text{or} \qquad \int \frac{v\,v'}{a+bv^2}\,dx\,.$$

Notice that in both cases the numerator is a multiple of the derivative of the denominator. Hence the results are logarithmic functions. Thus

$$\begin{aligned}\int \frac{v'}{a+bv}\,dx &= \frac{1}{b}\times\int\frac{bv'}{a+bv}\,dx\\ &= \tfrac{1}{b}\log(a+bv)+C\,.\end{aligned}$$

for some constant C. And in the second instance

$$\begin{aligned}\int \frac{v\,v'}{a+bv^2}\,dx &= \frac{1}{2b}\times\int\frac{2bv\,v'}{a+bv^2}\,dx\\ &= \tfrac{1}{2b}\log(a+bv^2)+D\,.\end{aligned}$$

Notice that in both cases no absolute values are used. In the practical problems that require these integrals, the quantities will generally be positive, and so no absolute value is needed. Every question should be routinely checked, however, and the absolute values re-inserted if they are required. These formulae will be used in the examples without further explanation.

WORKED EXAMPLE 13: A rowing eight crosses the finish line with speed 5·5m/s, and stops rowing. In a greatly simplified mathematical model, the boat is slowed by two drag forces. The skin drag is due to the surface area in contact with the water and is equal to $\frac{1}{10}mv^2$. The form drag is due to the shape of the boat pushing the water aside and is equal to $\frac{1}{100}mv$. Thus the force equation is

$$m\ddot{x} = -\frac{m(v+10v^2)}{100}$$

where x is the distance past the finish line, and v is the speed of the boat.

(a) Find v as a function of x.

(b) How far, correct to the nearest metre, does the boat eventually travel as it comes to a stop?

SOLUTION:

(a) Divide through by the mass m and use $\ddot{x} = v'v$ to get

$$v'v = -\frac{v+10v^2}{100}\,,$$

then re-arrange and integrate with respect to x:

$$\int\frac{10v'}{1+10v}\,dx = \int -\frac{1}{10}\,dx\,.$$

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So $\qquad \log(1+10v) = -\dfrac{x}{10} + C \qquad$ (for some constant C.)

At $x=0$ the boat speed is $v = 5{\cdot}5$ so

$$C = \log(56)$$

hence $\log\left(\dfrac{1+10v}{56}\right) = -\dfrac{x}{10}$.

Thus $\qquad \dfrac{1+10v}{56} = e^{-\frac{x}{10}}$

or $\qquad v = \frac{1}{10}\left(56e^{-\frac{x}{10}} - 1\right)$.

(b) Let $v \to 0^+$ in the third last line above to get

$$-\frac{x}{10} = \log\left(\frac{1}{56}\right)$$

thus $\qquad x = 10\log(56)$

$\qquad\qquad \doteqdot 40\,\text{m}$

In fact, according to this mathematical model, the boat never stops moving and only approaches this distance in the limit as $t \to \infty$. As an exercise, find v then x as functions of t and hence explain why this is the case.

Generalised Solutions: It is often efficient to solve a certain problem once, using suitable pronumerals in place of the various constants. Once this generalised solution is found, it is simply a matter of substituting the values of the constants to get the final solution. In the next worked example it is a particularly useful technique as a ratio of speeds is specified.

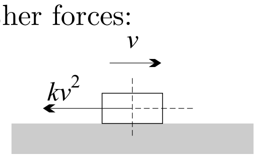
WORKED EXAMPLE 14: An object of mass m moves horizontally, starting at the origin with velocity $V_0 > 0$. It experiences a resistance due to friction which is proportional to the square of its speed and in the opposite direction.

(a) Find an expression for the velocity v in terms of the displacement x.

(b) For a certain 5 kg object the resistance is equal to $\frac{1}{10}v^2$. Find how far the object has travelled when it has slowed to $\frac{1}{4}$ of its initial speed.

SOLUTION:

(a) Let k be the constant of proportionality for the frictional force. Since it is opposite in direction to the velocity, and there are no other forces:



$$m\ddot{x} = -kv^2$$

or $\qquad v\,\dfrac{dv}{dx} = -\dfrac{k}{m}\,v^2$

so $\qquad \dfrac{v'}{v} = -\dfrac{k}{m}$.

Integrating with respect to x:

$$\int \frac{v'}{v}\,dx = -\int \frac{k}{m}\,dx\,.$$

Thus $\qquad \log v = -\frac{k}{m}x + C \qquad$ (for some constant C)

or $\qquad v = e^{-\frac{k}{m}x + C}$

so $\qquad v = Ae^{-kx/m} \qquad$ (where $A = e^C$.)

At $x = 0 \qquad V_0 = A$

thus $\qquad v = V_0 e^{-kx/m}$.

(b) Here $k = \frac{1}{10}$, $m = 5$ and $v = \frac{1}{4}V_0$. Thus

$$\tfrac{1}{4} = e^{-x/50}$$

or $\quad e^{x/50} = 4\,.$

Thus $\quad x = 50\log(4)$

$$\doteqdot 69\,\text{m}.$$

Other Methods: Numerous methods can be used to solve the equations encountered. The above worked example is now extended to demonstrate two of those methods appropriate to the course. In the first case, the derivative is treated like a fraction, and in the second case, definite integrals are used. Teachers and students are encouraged to investigate other techniques, such as separation of variables, using an integrating factor, and treating (v^2) like a variable.

WORKED EXAMPLE 15: Starting with $v = V_0 e^{-kx/m}$, find the displacement as a function of time.

SOLUTION: First replace v with the derivative $\dfrac{dx}{dt}$.

$$\frac{dx}{dt} = V_0 e^{-kx/m}.$$

Now treat the derivative like a fraction to get

$$\frac{dt}{dx} = \frac{e^{kx/m}}{V_0}\,.$$

Thus $\quad t = \dfrac{me^{kx/m}}{kV_0} + C\,.$

Recall that at $t = 0$, $x = 0$, so

$$0 = \frac{m}{kV_0} + C$$

or $\quad C = -\dfrac{m}{kV_0}$

hence $\quad t = \dfrac{m}{kV_0}\left(e^{kx/m} - 1\right)\,.$

Finally, re-arrange this equation to get,

$$e^{kx/m} = \frac{kV_0 t + m}{m}$$

or $\quad x = \dfrac{m}{k}\log\left(\dfrac{kV_0 t + m}{m}\right)\,.$

WORKED EXAMPLE 16: Starting with $m\dot{v} = -kv^2$, find the time taken for the object to reduce in speed from V_0 to $\frac{1}{2}V_0$.

SOLUTION: Once again, treat the derivative like a fraction and re-arrange.

$$\frac{dt}{dv} = -\frac{m}{kv^2}\,.$$

Thus the time can be written as a primitive function of v as follows:

$$T(v) = \int\left(-\frac{m}{kv^2}\right)dv\,.$$

In this question, the time required is the value of $t = T(\frac{1}{2}V_0) - T(V_0)\,.$

But, by the Fundamental Theorem of Calculus, this is just a definite integral.

$$\begin{aligned}\text{Thus}\qquad t &= \int_{V_0}^{\frac{1}{2}V_0} \left(-\frac{m}{kv^2}\right) dv \\ &= \left[\frac{m}{kv}\right]_{V_0}^{\frac{1}{2}V_0} \\ &= \frac{2m}{kV_0} - \frac{m}{kV_0} \\ &= \frac{m}{kV_0}\,.\end{aligned}$$

Notice that no constant of integration needed to be found because a definite integral was used. Now that the method of definite integrals has been established, the technique can be used in future problems without the need for the detailed explanation given above.

Exercise **6D**

1. A certain drag-racing car of mass M kg is capable of a top speed of 288 km/h. After it reaches this top speed, two different retarding forces combine to bring it to rest. First there is a constant braking force of magnitude $\frac{2}{3}M$ newtons. Secondly there is a resistive force of magnitude $\frac{1}{180}Mv^2$ newtons, where v m/s is the speed of the car, acting against a parachute released from the rear-end of the vehicle. Let x metres be the distance of the car from the point at which the two retarding forces are activated.

(a) Show that $x = 90 \ln\left(\dfrac{120 + 80^2}{120 + v^2}\right)$.

(b) Hence calculate, to the nearest metre, the distance that the drag-racing car travels as it is brought from its top speed to rest.

2. A monorail of mass 10 000 kg is pulling out of a station S. Its motor provides a propelling force of magnitude 10 000 Newtons, and as it moves it experiences a resistive force of magnitude $100v^2$ Newtons, where v metres per second is its velocity.

(a) Show that the maximum speed the monorail can attain is 36 km/h.

(b) Show that $x = 50 \ln\left(\dfrac{100}{100 - v^2}\right)$, where x metres is the distance the monorail has travelled from S.

(c) What percentage (to the nearest per cent) of its maximum speed has the monorail reached when it has travelled 50 metres?

3. A particle of mass m kg experiences a resistance of kv^2 newtons when moving along the x-axis, where k is a positive constant and v is the speed of the particle in metres per second. The maximum speed attainable by the particle is u metres per second under a variable propelling force of $\dfrac{P}{v}$ newtons, where P is a positive constant.

(a) Show that $k = \dfrac{P}{u^3}$.

(b) Show that $\dfrac{dv}{dx} = \dfrac{P}{m}\left(\dfrac{1}{v^2} - \dfrac{v}{u^3}\right)$.

(c) Prove that the distance travelled as the speed changes from $\frac{1}{3}u$ m/s to $\frac{2}{3}u$ m/s is $\dfrac{mu^3 \ln \frac{26}{19}}{3P}$ metres.

(d) When the brakes are applied, the propelling force is no longer in operation. If the maximum force exerted by the brakes is B Newtons, prove that the minimum distance travelled in coming to rest from a speed of u m/s is $\dfrac{mu^3}{2P} \ln\left(1 + \dfrac{P}{Bu}\right)$ metres.

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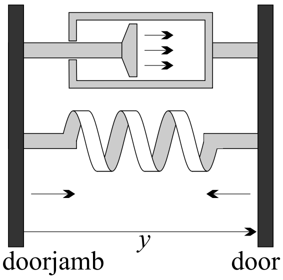
DEVELOPMENT

4. A simple model of a door closing mechanism is a spring and dashpot. The spring pulls the door closed and the dashpot, a gas or oil filled piston, resists the motion, which ensures that the door does not close too fast and slam shut. A schematic diagram is shown on the right.



For a particular door with mass m, the force exerted by the spring is $-2my$ and the resistance from the dashpot is $-3m\dot{y}$, where y is the displacement of the door from the doorjamb.

(a) What is the significance of the minus sign in each force?

(b) Write down the equation of motion for the door.

(c) Show that, if $y = f(t)$ and $y = g(t)$ are both solutions to the equation of motion, and if A and B are constants, then $y = A\,f(t) + B\,g(t)$ is also a solution.

(d) It is known that the function $y = e^{kt}$ is a solution of the differential equation.

Show that the only possible values of the constant k are $k = -1$ and $k = -2$.

(e) From parts (c) and (d), a solution of the differential equation is

$$y = Ae^{-2t} + Be^{-t}.$$

When $t = 0$, it is known that $y = 0$ and $\dfrac{dy}{dt} = 1$. Find the values of A and B.

5. The engines on a submarine of mass m deliver a maximum driving force of F newtons. The water resists the motion with a force proportional to the square of the speed v.

(a) Explain why $\dfrac{dv}{dt} = \dfrac{1}{m}\left(F - kv^2\right)$ where k is a positive constant.

(b) The submarine increases its speed from v_1 to v_2. Show that the distance travelled during this period is

$$\frac{m}{2k} \times \log_e\left(\frac{F - kv_1{}^2}{F - kv_2{}^2}\right).$$

6. As a particle of unit mass moves in a straight line, the only force acting on it is a resistance, which is in the opposite direction to its velocity, v. The size of this force is $v + v^3$. Initially the particle is at the origin and has velocity Q, where $Q > 0$.

(a) Use partial fractions to show that the time t is given by $t = \frac{1}{2}\log_e\left(\dfrac{Q^2(1+v^2)}{v^2(1+Q^2)}\right)$.

(b) Hence find v^2 as a function of t.

(c) Determine the limit of v as $t \to \infty$, and hence explain why v is always positive.

(d) Show that the velocity is related to the displacement x by the formula

$$x = \tan^{-1} Q - \tan^{-1} v\,,$$

and hence find $\lim\limits_{t\to\infty} x$.

(e) Does it follow that $x = \tan^{-1}\left(\dfrac{Q - v}{1 + Qv}\right)$? Justify your answer.

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7. (a) Find the values of A and B such that $\dfrac{1}{(2-v)(3+v)} = \dfrac{A}{(2-v)} + \dfrac{B}{(3+v)}$.

(b) A body with mass $m = 4{\cdot}5 \times 10^6$ kg is acted upon by a force of $10^4(6-v)$ N where v is its speed in metres per second. It also experiences a resistance proportional to the square of its speed. That is:

$$\frac{dv}{dt} = \frac{10^4}{m}\left(6 - v - kv^2\right).$$

(i) Its maximum speed is 2 m/s. Find k.

(ii) Show that the body attains a speed of $1{\cdot}5$ m/s, starting from rest, in a little over 2 minutes and 41 seconds.

8. When a jet aircraft touches down two different retarding forces combine to slow it down. If the aircraft has mass M kg and speed v m/s then there is a constant frictional force of $2M$ newtons due to the brakes and a force of $\frac{1}{10\,000}Mv^2$ newtons due to the reverse thrust of the engines. The reverse thrust does not take effect until 3 seconds after touchdown.

Let x be the distance in metres of the jet from its point of touchdown and let t be the time in seconds after touchdown.

(a) The jet's landing speed is 72 m/s. Show that $v = 66$ and $x = 207$ at the instant the reverse thrust of the engines takes effect.

(b) Show that when $t > 3$, $x = 207 + 5000 \ln\left(\dfrac{20000 + 66^2}{20000 + v^2}\right)$.

(c) Reverse thrust is shut down when the aircraft reaches a speed of 36 m/s. How far from the point of touchdown, correct to the nearest metre, does this happen?

(d) The brakes alone are then used to reach the taxi speed of 7 m/s. How far from the point of touchdown, correct to the nearest metre, does the plane reach its taxi speed?

9. A box of mass m is pushed across a floor with a constant force mP. As the box moves it is also retarded by a force due to friction of mkv, where v is its velocity and k is a positive constant. The box is initially travelling with velocity V_i.

(a) Write down a force balance equation and hence show that the speed which results in zero nett force is $V_0 = \frac{P}{k}$.

(b) Integrate once to show that $v = V_0\left(1 - \left(1 - \frac{V_i}{V_0}\right)e^{-kt}\right)$. Hence find $\lim\limits_{t\to\infty} v$.

(c) Draw a graph of v versus t in the case when: (i) $V_i > V_0$ (ii) $V_i < V_0$

(d) Find the time taken for the box to accelerate from $v = \frac{1}{3}V_0$ to $v = \frac{2}{3}V_0$.

10. A particle is moving along the x-axis. Its acceleration is given by

$$\frac{d^2x}{dt^2} = \frac{5-2x}{x^3}$$

and it starts from rest at $x = 1$.

(a) Explain why the particle starts moving in the positive x direction.

(b) Let v be the velocity of the particle. Show that $v = \dfrac{\sqrt{x^2+4x-5}}{x}$ for $x \ge 1$.

(c) Describe the behaviour of the velocity of the particle after the particle passes $x = \frac{5}{2}$.

11. An object is moving across a flat surface for which the friction is proportional to $v^{3/2}$ and in the opposite direction. That is, the acceleration is given by the equation

$$v\frac{dv}{dx} = -kv^{3/2}\,,$$

where v is the velocity when the object is at x, and k is a positive constant. Initially the particle is at the origin with speed $V_0 > 0$.

(a) (i) Integrate with respect to x and hence show that $\sqrt{\dfrac{v}{V_0}} = 1 - \dfrac{kx}{2\sqrt{V_0}}$.

(ii) What values can x take, and where is the particle when it stops moving?

(b) (i) Find x as a function of the time t.

(ii) Review your answer to part (a)(ii) in light of this result.

ENRICHMENT

12. When a certain object is pushed across a polished floor, the resistance due to friction is proportional to $v^{3/2}$, where v is its velocity. The resulting equation for acceleration is

$$\frac{dv}{dt} = 1 - v^{3/2}\,.$$

Initially the object is at rest.

(a) Use the substitution $u = \sqrt{v}$ to show that $\dfrac{dt}{du} = \dfrac{2u}{1-u^3}$.

(b) Use partial fractions to show that

$$\frac{dt}{du} = \frac{2}{3}\left(\frac{1}{1-u} + \frac{1}{2}\times\frac{1+2u}{1+u+u^2} - \frac{3}{2}\times\frac{1}{(u+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right).$$

(c) Hence show that $t = \dfrac{2}{3}\left(\dfrac{1}{2}\times\log\left(\dfrac{1+\sqrt{v}+v}{(1-\sqrt{v})^2}\right) - \sqrt{3}\tan^{-1}\left(\dfrac{\sqrt{3v}}{\sqrt{v}+2}\right)\right)$.

6E Vertical Resisted Motion

In this section it is assumed that when a particle of mass m travels vertically through a fluid such as air, water or oil, only two forces act on the particle, namely a force due to gravity and a resistance R to the motion. Typically it is found that the resistance is proportional to a power of the speed $|v|$. Thus

$$|R| = mk|v|^n \qquad \text{for some constant } k\,.$$

Although n can take various values, only the cases $n = 1$ and $n = 2$ will be studied in any detail in this course.

It is also observed that the resistance is opposite in direction to the velocity v. Thus a more convenient form of the resistance is:

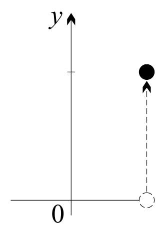
$$R = -mk|v|^{n-1}v\,.$$

Normally this is further simplified by selecting a coordinate system in which the velocity is positive, so that $|v| = v$ and

$$R = -mkv^n\,.$$

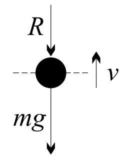
For simplicity, the case of a projectile fired vertically is considered here. The trajectory is naturally divided into two parts, the journey up and the journey down. These parts will be considered separately.

The Upward Journey: Suppose that a particle of mass m is fired upwards with initial velocity V_0 through a resisting medium under the influence of gravity g. Let y be the height of the particle above the point of projection at time t. That is, upwards is the positive direction as shown in the diagram on the right. Also note that $y = 0$ when $t = 0$.



10	COORDINATES FOR THE UPWARDS JOURNEY: In most circumstances it is best to put the origin at the initial position with upwards as the positive direction.

The only forces acting on the particle are the resistive force and the weight. Thus the net force F is



$$
\begin{aligned}
F &= -mg - mkv^n \\
\text{or} \quad m\ddot{y} &= -mg - mkv^n \\
\text{so} \quad \ddot{y} &= -(g + kv^n)\,.
\end{aligned}
$$

Thus the differential equation and set of initial conditions for the motion are:

$$\ddot{y} = -(g + kv^n)\,,$$

whilst at $t = 0$,

$$
\begin{aligned}
\dot{y} &= V_0 \\
\text{and} \quad y &= 0\,.
\end{aligned}
$$

In any given problem, at each step, the differential equation is integrated and then the initial conditions are used to determine any unknown constant of integration.

11	THE UPWARD JOURNEY: For a given value of n, integrate the differential equation $$\ddot{y} = -(g + kv^n)\,.$$ Determine any constant of integration by applying the initial conditions. $$\text{At } t = 0\,, \quad \dot{y} = V_0, \quad y = 0$$

WORKED EXAMPLE 17: A particle is projected upwards in a medium for which the resistance to the motion is one fifth of the mass times the velocity and opposite in direction. The initial velocity is $30\,\text{m/s}$. Let y be its height in metres above the point of projection after t seconds. Use $g \doteqdot 10\,\text{m/s}^2$, so that

$$\ddot{y} = -(10 + \tfrac{1}{5}v)\,.$$

Find y as a function of the velocity v and hence find its maximum height.

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SOLUTION: Noting that $\ddot{y} = v\dfrac{dv}{dy}$, it follows that

$$v\frac{dv}{dy} = -\frac{50+v}{5}$$

so $$\frac{v}{50+v}\,\frac{dv}{dy} = -\frac{1}{5}$$

or $$\left(1 - \frac{50}{50+v}\right)\frac{dv}{dy} = -\frac{1}{5}\,.$$

Integrate with respect to y to get:

$$\int \left(v' - \frac{50v'}{50+v}\right) dy = -\int \frac{1}{5}\,dy\,.$$

Thus $\quad v - 50\log(50+v) = -\tfrac{1}{5}y + C\,.$

At $t = 0$, the particle is at the origin and $v = 30$, so

$$C = 30 - 50\log 80$$

Hence $\quad v - 50\log(50+v) = -\tfrac{1}{5}y + 30 - 50\log 80\,.$

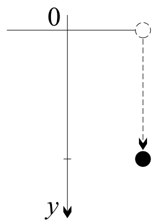
Rearrange this equation to get, after a few lines of algebra,

$$y = 5\left(30 - v + 50\log\left(\frac{50+v}{80}\right)\right)\,.$$

At the maximum height $v = 0$, thus

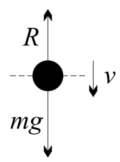
$$y_{\max} = 5\left(30 + 50\log\tfrac{5}{8}\right) \qquad \text{(about 32·5 metres.)}$$

The Downward Journey: Suppose that a particle of mass m is allowed to fall from rest through a resisting medium under the influence of gravity g. Let y be the distance *below* its initial position at time t. That is, downwards is the positive direction as shown in the diagram on the right. Also note that $y = 0$ and $\dot{y} = 0$ when $t = 0$.



12 COORDINATES FOR THE DOWNWARDS JOURNEY: In most circumstances it is best to put the origin at the initial position with downwards as the positive direction.

The only forces acting on the particle are the resistive force and the weight. Thus the net force F is



$$F = mg - mkv^n$$

or $\quad m\ddot{y} = mg - mkv^n$

so $\quad \ddot{y} = g - kv^n\,.$

Thus the differential equation of motion is:

$$\ddot{y} = g - kv^n$$

whilst at $t = 0$, the initial conditions are

$$\dot{y} = 0$$

and $\quad y = 0\,.$

In any given problem, at each step, the differential equation is integrated and then the intial conditions are used to determine any unknown constant of integration.

13 THE DOWNWARDS JOURNEY: For a given value of n, integrate the differential equation

$$\ddot{y} = g - kv^n \, .$$

Determine any constant of integration by applying the initial conditions. In the case where the object is dropped from rest these are:

$$\text{at } t = 0\,, \qquad \begin{aligned} \dot{y} &= 0 \\ y &= 0 \end{aligned}$$

WORKED EXAMPLE 18: A particle falls from rest through a medium for which the resistance is proportional to the velocity and opposite in direction. Let y be the distance in metres below its initial position after t seconds. Assume that

$$\ddot{y} = g - kv \, .$$

(a) Find v as a function of t.

(b) Hence show that the terminal velocity is $\lim\limits_{t\to\infty} v = \frac{g}{k}$.

(c) Find y as a function of t.

(d) Show that if h is large enough then the approximate time to reach the ground at $y = h$ is given by $t \doteqdot \frac{kh}{g} + \frac{1}{k}$.

SOLUTION:

(a) Since $\ddot{y} = \dot{v}$ it follows that

$$\dot{v} = g - kv$$

so $$\frac{k\dot{v}}{g - kv} = k \, .$$

Integrate with respect to time to get:

$$\int \frac{-k\dot{v}}{g - kv}\, dt = \int -k \, dt$$

thus $\log(g - kv) = -kt + C_1$.

At $t = 0$, the velocity is zero, thus

$$C_1 = \log g \, .$$

Hence $\log(g - kv) = -kt + \log g$.

Rearrange this equation to get

$$\log\left(\frac{g - kv}{g}\right) = -kt$$

so $$\frac{g - kv}{g} = e^{-kt}$$

or $$v = \tfrac{g}{k}\left(1 - e^{-kt}\right) \, .$$

(b) $$\lim_{t\to\infty} v = \lim_{t\to\infty} \tfrac{g}{k}\left(1 - e^{-kt}\right) = \tfrac{g}{k}(1 - 0) = \tfrac{g}{k} \, .$$

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(c) From part (a), $\dot{y} = \frac{g}{k}\left(1 - e^{-kt}\right)$. Integrate with respect to time to get:

$$y = \tfrac{g}{k}\left(t + \tfrac{1}{k}e^{-kt}\right) + C_2\,.$$

At $t = 0$ $\quad 0 = \frac{g}{k}(0 + \frac{1}{k}) + C_2$

so $\quad C_2 = -\frac{g}{k^2}\,.$

Hence $\quad y = \frac{g}{k}\left(t + \frac{1}{k}\left(e^{-kt} - 1\right)\right)\,.$

(d) If h is large enough then t is also large, so that $e^{-kt} \doteqdot 0$. Thus

$$h \doteqdot \tfrac{g}{k}\left(t - \tfrac{1}{k}\right)$$

and $\quad t \doteqdot \frac{kh}{g} + \frac{1}{k}\,.$

Terminal Velocity: The terminal velocity of an object is most often associated with falling bodies, but is sometimes referred to in other problems. As demonstrated in the above worked example, one method of finding the terminal velocity is to find v as a function of t and then determine the limit

$$V_T = \lim_{t\to\infty} v(t)\,.$$

A simpler approach is to recognise that the terminal velocity corresponds to zero acceleration. Thus in the above worked example

$$0 = g - kV_T$$

so $\quad V_T = \dfrac{g}{k}$

14 | TERMINAL VELOCITY: This is easily found by putting $\ddot{y} = 0$.

An Object Thrown Down: In this case, the equation of motion is the same as for the downward journey above. Let the initial speed be U. There are three cases of note: $0 \le U < V_T$, $U = V_T$ and $U > V_T$. It can be shown that when $U = V_T$ the velocity never changes, so that the displacement function is simply

$$y = V_T\, t\,.$$

Students should ensure they are able to prove this result for themselves. The other two cases are dealt with in the exercise and the following example.

WORKED EXAMPLE 19: An object is thrown downward with initial speed twice its terminal velocity. Its motion is affected by gravity and air resistance, which is proportional to the square of its speed. Put the origin at the point of projection with downward as positive. The velocity v satisfies the equation

$$\dot{v} = g - kv^2$$

with the initial condition that $v(0) = 2V_T$, where $V_T = \sqrt{\frac{g}{k}}$.

(a) Let $w = \dfrac{v}{V_T}$. Show that w satisfies the equation

$$\dot{w} = -(w^2 - 1)\sqrt{gk}$$

and state the initial condition for w.

(b) Use partial fractions to help find t as a function of w.

(c) Hence find the time taken for the object to reduce in speed from $2V_T$ to $\frac{11}{10}V_T$.

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Solution:

(a) It should be clear that $v = V_T\, w$ and $\dot{v} = V_T\, \dot{w}$, thus

$$\begin{aligned} V_T\, \dot{w} &= g - kV_T{}^2\, w^2 \\ &= g - gw^2 \end{aligned}$$

so
$$\begin{aligned} \dot{w} &= (1 - w^2)\frac{g}{V_T} \\ &= -(w^2 - 1)\sqrt{gk}\,. \end{aligned}$$

The initial condition is $v(0) = 2V_T$, so from the substitution it follows that

$$V_T\, w(0) = 2V_T\,,$$

hence $\quad w(0) = 2\,.$

(b) Treating the derivative like a fraction

$$\frac{dt}{dw} = \frac{-1}{(w^2 - 1)\sqrt{gk}}$$

or
$$\sqrt{gk}\,\frac{dt}{dw} = \frac{-1}{(w + 1)(w - 1)}$$

Let
$$\frac{-1}{(w + 1)(w - 1)} = \frac{A}{w + 1} + \frac{B}{w - 1}$$

then by the cover-up rule $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

And so
$$2\sqrt{gk}\,\frac{dt}{dw} = \frac{1}{w + 1} - \frac{1}{w - 1}\,.$$

Now integrate to get

$$2t\sqrt{gk} = \log(w + 1) - \log(w - 1) + C\,,$$

and from the initial condition

$$0 = \log 3 + C$$

hence $2t\sqrt{gk} = \log(w + 1) - \log(w - 1) - \log 3\,,$

or
$$t = \frac{1}{2\sqrt{gk}} \log\left(\frac{w + 1}{3(w - 1)}\right)\,.$$

(c) When $v = \frac{11}{10}V_T$ it follows that $w = \frac{11}{10}$ so

$$\begin{aligned} t &= \frac{1}{2\sqrt{gk}} \log\left(\frac{11 + 10}{3(11 - 10)}\right) \\ &= \frac{\log 7}{2\sqrt{gk}}\,. \end{aligned}$$

Notice that the equation in part (b) can be solved for w to get

$$w = \frac{3e^{2t\sqrt{gk}} + 1}{3e^{2t\sqrt{gk}} - 1}$$

so that $\displaystyle v = V_T \times \frac{3e^{2t\sqrt{gk}} + 1}{3e^{2t\sqrt{gk}} - 1}\,.$

It should be clear that the numerator of the fraction is always greater than the denominator. Hence the velocity is always greater than the terminal velocity. More specifically, the velocity is always decreasing, quickly at first and then at a decreasing rate as it approaches the limiting value V_T from above. That is, the graph of $v(t)$ is concave up with a horizontal asymptote.

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Exercise **6E**

1. An object of mass 5 kg is projected vertically upwards with velocity 40 m/s and experiences a resistive force in Newtons of magnitude $0{\cdot}2v^2$, where v is the velocity of the object at time t seconds. Assume that $g = 10\,\text{m/s}^2$.

(a) Show that $\dfrac{dv}{dt} = \dfrac{-250 - v^2}{25}$.

(b) Find, correct to the nearest tenth of a second, the time to reach its maximum height.

(c) Use $\dfrac{dv}{dt} = v\dfrac{dv}{dy}$ to help find the maximum height, correct to the nearest metre.

2. An object of mass 0·5 kg is projected upwards with velocity 40 m/s and experiences a resistive force in Newtons of magnitude $0{\cdot}2v$, where v is the velocity of the object at time t seconds. Assume that $g = 10\,\text{m/s}^2$.

(a) Show that $\ddot{x} = \dfrac{-50 - 2v}{5}$.

(b) Show that the object takes $\frac{5}{2}\ln\frac{13}{5}$ seconds to reach its maximum height.

(c) Show that the maximum height reached is $(100 + \frac{125}{2}\ln\frac{5}{13})$ metres.

3. An object of mass 100 kg is found to experience a resistive force, in newtons, of one-tenth the square of its velocity, in metres per second, when it moves through the air. Suppose that the object falls from rest under gravity, and take $g = 9{\cdot}8\,\text{m/s}^2$.

(a) Show that its terminal velocity is about 99 m/s.

(b) If the object reaches 80% of its terminal velocity before striking the ground, show that the point from which it was dropped was about 511 metres above the ground.

DEVELOPMENT

4. (a) An object of mass 1 kg is projected vertically upwards from the ground at 20 m/s. The body is under the effect of both gravity and air resistance which, at any time, has a magnitude of $\frac{1}{40}v^2$, where v is the velocity at time t. Put $g = 10\,\text{m/s}^2$, and take upwards as the positive direction.

(i) Show that the greatest height reached by the object is $20\ln 2$ metres.

(ii) Show that the time taken to reach this greatest height is $\frac{\pi}{2}$ seconds.

(b) Having reached its greatest height the particle falls back to its starting point. The particle is still under the effect of both gravity and air resistance. Take downwards as the positive direction.

(i) Write down the equation of motion of the object as it falls.

(ii) Find the speed of the object when it returns to its starting point.

5. An object is projected downwards with initial velocity V_0. The air resistance at speed v has magnitude mkv, where k is a positive constant. Take downwards as the positive direction.

(a) Show that $t = \frac{1}{k}\log_e\left(\frac{g-kV_0}{g-kv}\right)$.

(b) Hence show that $v = \frac{g}{k}(1 - e^{-kt}) + V_0\,e^{-kt}$, and that the terminal velocity is $\frac{g}{k}$.

(c) Integrate again to show that $x = \frac{g}{k}t + \frac{kV_0-g}{k^2}\left(1 - e^{-kt}\right)$.

(d) Suppose that the terminal velocity of this object is 20 m/s, and that $g = 10\,\text{m/s}^2$. The object is thrown vertically downwards from a lookout at the top of a cliff at precisely the terminal velocity. At the same instant, a similar object is dropped from the same height. Show that the distance between the two falling objects after t seconds is $40(1 - e^{-\frac{1}{2}t})$ metres, and hence state the limiting distance between them.

6. A particle of mass 10 kg is found to experience a resistive force, in newtons, of one-tenth of the square of its velocity in metres per second when it moves through the air. The particle is projected vertically upwards from a point O with a velocity of u metres per second. The point A, vertically above O, is the highest point reached by the particle before it starts to fall to the ground again. Assume that $g = 10\,\text{m/s}^2$.

(a) Show that the particle takes $\sqrt{10}\tan^{-1}\frac{u}{10\sqrt{10}}$ seconds to reach A from O.

(b) Show that the height OA is $50\log_e(\frac{1000+u^2}{1000})$ metres.

(c) Let w be the velocity of the particle when it returns to O. Show that $w^2 = \dfrac{1000u^2}{1000+u^2}$.

7. (a) A particle of mass m falls from rest, from a point O, in a medium whose resistance is mkv, where k is a positive constant and v is the velocity at time t.

(i) Prove that the terminal velocity V is $V = \dfrac{g}{k}$.

(ii) Prove that the speed at time t is given by $V\left(1 - e^{-kt}\right)$.

(b) An identical particle is projected upwards from O with initial velocity U in the same medium. Suppose that both particles begin their motion simultaneously.

(i) Prove that the second particle reaches its maximum height at $t = \frac{1}{k}\ln\frac{g+kU}{g}$.

(ii) Prove that the speed of the first particle when the second particle reaches its maximum height is $\dfrac{UV}{U+V}$.

8. A particle P_1 of mass m kg is dropped from point A and falls towards point B, which is directly underneath A. At the instant when P_1 is dropped, a second particle P_2, also of mass m kg, is projected upwards from B towards A with an initial velocity equal in magnitude to twice the terminal velocity of P_1. Each particle experiences a resistance of magnitude mkv as it moves, where $v\,\text{ms}^{-1}$ is the velocity and k is a constant.

(a) Show that the terminal velocity of P_1 is $\dfrac{g}{k}$, where g is acceleration due to gravity.

(b) Show that the time taken for particle P_2 to reach velocity v is $t = \dfrac{1}{k}\ln\left(\dfrac{3g}{g+kv}\right)$.

(c) Suppose that the particles collide at the instant when P_1 has reached 30% of its terminal velocity. Show that the velocity of P_2 when they collide is $\dfrac{11g}{10k}\,\text{ms}^{-1}$.

9. (a) Consider the function $f(x) = x - \dfrac{g^2}{x} - 2g\ln\left(\dfrac{x}{g}\right)$, for $x \ge g$.

(i) Evaluate $f(g)$.

(ii) Show that $f'(x) = \left(1 - \dfrac{g}{x}\right)^2$.

(iii) Explain why $f(x) > 0$ for $x > g$.

(b) A body is moving vertically through a resisting medium, with resistance proportional to its speed. The body is initially fired upwards from the origin with speed V_0. Let y metres be the height of the object above the origin at time t seconds, and let g be the constant acceleration due to gravity. Thus

$$\frac{d^2y}{dt^2} = -(g+kv)\,, \quad \text{where} \quad k > 0\,.$$

You may assume that this equation is valid for all $t \ge 0$.

(i) Find v as a function of t, and hence show that

$$k^2 y = (g + kV_0)(1 - e^{-kt}) - gkt\,.$$

(ii) Find T, the time taken to reach the maximum height.

(iii) Show that when $t = 2T$,

$$k^2 y = (g + kV_0) - \frac{g^2}{g + kV_0} - 2g \ln\left(\frac{g + kV_0}{g}\right).$$

(iv) Use this result and part (a) to show that the downwards journey takes longer.

10. An object of mass 1 kg is dropped from a lookout on top of a high cliff. Let the acceleration due to gravity be $10\,\text{m/s}^2$.

(a) At first, air resistance causes a deceleration of magnitude $\frac{1}{10}v$, where v m/s is the speed of the object t seconds after it is dropped.

(i) Taking downwards as positive, explain why its equation of motion is

$$\ddot{x} = 10 - \tfrac{1}{10}v\,,$$

where x is the distance that the object has fallen in the first t seconds.

(ii) Show that $\dfrac{dv}{dx} = \dfrac{100 - v}{10v}$, and hence show that the speed V of the object when it is 40 metres below the lookout satisfies the equation

$$V + 100 \log_e\left(1 - \tfrac{V}{100}\right) + 4 = 0.$$

(b) After the object has fallen 40 metres and reached this speed V, a very small parachute opens, and air resistance now causes a deceleration to its motion of magnitude $\frac{1}{10}v^2$.

(i) Taking downwards as positive, write an expression for the new acceleration $\ddot{x}$ of the object, where x now is the distance that the object has fallen in the first t seconds after the parachute opens.

(ii) Show that $v^2 = 100 - (100 - V^2)e^{-\frac{1}{5}x}$, and hence find the terminal velocity of the object.

(iii) Show that t seconds after the parachute opens,

$$t = \tfrac{1}{2} \log_e \frac{(v + 10)(V - 10)}{(v - 10)(V + 10)}\,.$$

(iv) The solution to the equation in part (ii) of part (a) is $V \doteqdot 25{\cdot}7\,\text{m/s}$. How long after the parachute opens does the object reach 105% of its terminal velocity?

11. A projectile is fired with velocity $V = 30\,\text{m/s}$ at an angle of 45° to the horizontal. Air resistance is proportional to the velocity. Thus the equations of motion are

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = -g - k\frac{dy}{dt}\,,$$

where it is known that $k = \frac{1}{3}$. Take $g = 10\,\text{m/s}^2$.

(a) Show that, at time t, the horizontal displacement is $x = 45(1 - e^{-t/3})\sqrt{2}$.

(b) Find a similar expression for the vertical displacement y as a function of t.

(c) Hence show that $y = (1 + \sqrt{2})x - 90 \log\left(\dfrac{45\sqrt{2}}{45\sqrt{2} - x}\right)$.

(d) Accurately plot the function in (c) and use it to estimate the range.

(e) By way of comparison, show the trajectory for no air resistance on the same graph.

ENRICHMENT

12. The case of vertical resisted motion where $n = 1$ can be solved without the need to divide the journey into two parts, and was assumed in Question 9(b). Use the equation

$$\ddot{y} = -(g + kv)$$

with initial conditions $v(0) = V_0$ and $y(0) = 0$ to show that the correct terminal velocity $V_T = -\frac{g}{k}$ is obtained. Also show that the height at any time after the particle reaches its maximum height is consistent with the result given in the text for the downward journey.

13. The equations of motion for a projectile with air resistance proportional to the square of the speed are

$$\dot{u} = -k(u^2 + v^2)^{\frac{1}{2}}\, u \quad \text{and} \quad \dot{v} = -g - k(u^2 + v^2)^{\frac{1}{2}}\, v\,,$$

where $u = \dot{x}$ is the horizontal component of velocity and $v = \dot{y}$ is the vertical component of velocity. There is no known solution to this pair of equations. Nevertheless the trajectory can be approximated and plotted by following the steps below.

In this case, the initial speed of the projectile is 30 m/s, the angle of projection is 45°, and $k = \frac{1}{90}$. Take $g = 10\,\text{m/s}^2$. Give your answers correct to 2 decimal places.

(a) (i) Recall that, from first principles, $\dfrac{du}{dt} = \lim\limits_{\delta t \to 0} \dfrac{u(t + \delta t) - u(t)}{\delta t}$. Hence, for δt small enough, $\dfrac{du}{dt} \doteqdot \dfrac{u(t + \delta t) - u(t)}{\delta t}$. Show that

$$u(t + \delta t) \doteqdot u(t)\left(1 - k\,\delta t\left(u(t)^2 + v(t)^2\right)^{\frac{1}{2}}\right).$$

(ii) Find similar expressions for $x(t + \delta t)$, $y(t + \delta t)$ and $v(t + \delta t)$.

(b) It should be clear that $u(0) = v(0) = 15\sqrt{2}$, and that $x(0) = y(0) = 0$.

(i) Use part (a)(i) with $t = 0$ and $\delta t = 0{\cdot}1$ to find the approximate value of $u(0{\cdot}1)$.

(ii) Similarly find the approximate values of $v(0{\cdot}1)$, $x(0{\cdot}1)$ and $y(0{\cdot}1)$.

(c) What are the approximate values of $u(0{\cdot}2)$, $v(0{\cdot}2)$, $x(0{\cdot}2)$ and $y(0{\cdot}2)$?

(d) Use a spreadsheet or appropriate mathematical software to continue to find x, y, u and v in time steps of $0{\cdot}1$. Hence plot the trajectory of the projectile. What is the approximate range, correct to the nearest metre?

6F Projectile Motion

In this section, harder questions on projectile motion are considered. There are also some questions involving projectile motion with air resistance, however, these are very limited as solutions can only be found in one case, when the resistance is proportional to the velocity. There is no new theory, but the following two examples indicate the difficulty of the questions that may be asked.

WORKED EXAMPLE 20: A fielder in a women's cricket team finds she can catch a ball with greatest ease when the height of the ball is between y_1 and y_2, where $y_2 < y_1$. A ball is hit which has range R and reaches height $h > y_1$. The fielder will catch the ball on its downwards trajectory at height y, $y_2 \le y \le y_1$. If the fielder positions herself to catch the ball at distance x from the point of projection, what is the length of the interval that x can lie within? Give the answer in terms of y_1, y_2, R and h. Assume air resistance is negligible.

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SOLUTION: This is a trick question, which is about roots of quadratic equations, not projectile motion. Since the range is R the equation of the trajectory may be written as

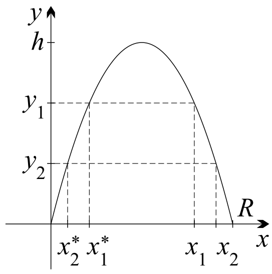
$$y = ax(R - x)$$

The vertex of the parabola is $(\frac{1}{2}R, h)$ so

$$h = \tfrac{1}{4}aR^2$$

thus $\quad a = \dfrac{4h}{R^2}\,.$

From the diagram, the closest the fielder can be is at x_1, where x_1 and x_1^* are roots of the quadratic equation



$$ax(R - x) = y_1$$

viz $\quad ax^2 - aRx + y_1 = 0\,.$

Now $\quad (x_1 - x_1^*)^2 = (x_1 + x_1^*)^2 - 4x_1x_1^*$

and using the sums and products of quadratic roots

$$\begin{aligned}(x_1 - x_1^*)^2 &= R^2 - 4\frac{y_1}{a}\\ &= R^2 - \frac{y_1R^2}{h}\,.\end{aligned}$$

Hence, taking the positive square root

$$(x_1 - x_1^*) = R\sqrt{1 - \frac{y_1}{h}}.$$

Likewise $\quad (x_2 - x_2^*) = R\sqrt{1 - \dfrac{y_2}{h}}.$

From the symmetry of the parabola, the required distance is

$$\begin{aligned}(x_2 - x_1) &= \tfrac{1}{2}\Big((x_2 - x_2^*) - (x_1 - x_1^*)\Big)\\ &= \frac{R}{2}\left(\sqrt{1 - \frac{y_2}{h}} - \sqrt{1 - \frac{y_1}{h}}\right).\end{aligned}$$

WORKED EXAMPLE 21: A projectile is fired with velocity vector $\underset{\sim}{V} = 15\underset{\sim}{i} + 30\underset{\sim}{j}$. Air resistance is proportional to the velocity. Thus the equations of motion are

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = -g - k\frac{dy}{dt}\,,$$

where $k = \frac{1}{5}$ and $g \doteqdot 10$.

Find $x(t)$ and $y(t)$. Hence determine the Cartesian equation of motion.

SOLUTION: This time the question is a genuine projectile motion problem. The differential equation for the horizontal component is:

$$\frac{d(\dot{x})}{dt} = -\tfrac{1}{5}\dot{x}$$

which is the equation for exponential decay, thus

$$\dot{x} = Ae^{-t/5}.$$

At $t = 0 \quad 15 = A$

hence $\quad \dfrac{dx}{dt} = 15e^{-t/5}.$

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Integrating a second time

$$x = B - 75e^{-t/5}$$

and at $t = 0$ the projectile is at the origin so

$$B = 75$$

thus $\quad x = 75(1 - e^{-t/5}).$

Now solve the differential equation for the vertical component of motion.

$$\frac{d(\dot{y})}{dt} = -\tfrac{1}{5}(50 + \dot{y})$$

which is shifted exponential decay, thus,

$$\dot{y} = Ce^{-t/5} - 50$$

At $t = 0$, $30 = C - 50$, so $C = 80$ and hence

$$\frac{dy}{dt} = 80e^{-t/5} - 50$$

Integrating a second time

$$y = -400e^{-t/5} - 50t + D$$

and at $t = 0$ the projectile is at the origin so

$$D = 400$$

and $$y = 400(1 - e^{-t/5}) - 50t\,. \qquad (1)$$

Now from above

$$x = 75(1 - e^{-t/5}) \qquad (2)$$

so $$75e^{-t/5} = 75 - x$$

or $$e^{t/5} = \frac{75}{75 - x}$$

and $$t = 5\log\left(\frac{75}{75 - x}\right). \qquad (3)$$

Thus combining (1), (2) and (3) gives: $\quad y = \dfrac{400x}{75} - 250\log\left(\dfrac{75}{75 - x}\right).$

Exercise **6F**

1. A projectile is fired with velocity $V = 40$ m/s on a horizontal plane at an angle of elevation $\alpha = 60°$. Take $g = 10\,\text{m/s}^2$, and let the origin be the point of projection.

(a) Show that $\dot{x} = 20$ and $\dot{y} = -10t + 20\sqrt{3}$, and find x and y.

(b) Find the flight time, and the horizontal range of the projectile.

(c) Find the maximum height reached, and the time taken to reach it.

(d) An observer claims that the projectile would have had a greater horizontal range if its angle of projection had been halved. Investigate this claim by reworking the question with $\alpha = 30°$.

2. A projectile is fired at a speed of 39 m/s and at an angle of elevation of $\tan^{-1}\frac{12}{5}$. It just clears a tower at a horizontal distance of 30 m from the point of projection.

(a) Obtain, by integration, expressions for $\dot{x}, \dot{y}, x$ and y, taking $g = 10\,\text{m/s}^2$.

(b) Find the height of the tower.

(c) Find, in m/s correct to one decimal place, the speed of the projectile as it clears the tower.

(d) Does the projectile reach its greatest height before or after it clears the tower?

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3. A pebble is thrown from the top of a vertical cliff with velocity $20\,\mathrm{m/s}$ at an angle of elevation of $30°$. The cliff is 75 metres high and overlooks a river.

(a) Derive expressions for the horizontal and vertical components of the displacement of the pebble from the top of the cliff after t seconds. (Take $g = 10\,\mathrm{m/s^2}$.)

(b) Find the time it takes for the pebble to hit the water and the distance from the base of the cliff to the point of impact.

(c) Find the greatest height that the pebble reaches above the river.

(d) Find the values of $\dot{x}$ and $\dot{y}$ at the instant when the pebble hits the water. Hence find the speed (to the nearest m/s) and the acute angle below the horizontal (to the nearest degree) at which the pebble hits the water.

(e) The path of the pebble is a parabolic arc. By eliminating t from the equations for x and y, find its equation in Cartesian form.

4. A plane is flying horizontally at $363{\cdot}6\,\mathrm{km/h}$ and its altitude is 600 metres. It is to drop a food parcel onto a large cross marked on the ground in a remote area.

(a) Convert the speed of the plane into metres per second.

(b) Derive expressions for the horizontal and vertical components of the food parcel's displacement from the point where it was dropped. (Take $g = 10\,\mathrm{m/s^2}$.)

(c) Show that the food parcel will be in the air for $2\sqrt{30}$ seconds.

(d) Find the speed and angle at which the food parcel will hit the ground.

(e) At what horizontal distance from the cross, correct to the nearest metre, should the plane drop the food parcel?

5. Ming hit a golf ball from level ground with initial speed $50\,\mathrm{m/s}$ at an angle of $45°$above the horizontal. The ball hit the clubhouse 75 metres away. Take $g = 10\,\mathrm{m/s^2}$.

(a) Show that the ball hit the clubhouse after $\frac{3}{2}\sqrt{2}$seconds at a point $52{\cdot}5$ metres above the ground.

(b) Show that the velocity of the ball when it struck the clubhouse was $5\sqrt{58}\,\mathrm{m/s}$ at an angle of $\tan^{-1}\frac{2}{5}$ above the horizontal.

6. Jeffrey hit a golf ball that was lying on level ground. Two seconds into its flight, the ball just cleared a 28-metre-tall tree which was $24\sqrt{5}$ metres from where the ball was hit. Let $V\,\mathrm{m/s}$ be the initial speed of the ball, and let θ be the angle above the horizontal at which the ball was hit. Take $g = 10\,\mathrm{m/s^2}$.

(a) Show that the horizontal and vertical components of the displacement of the ball from its initial position are $x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$.

(b) Show that $V\cos\theta = 12\sqrt{5}$ and $V\sin\theta = 24$.

(c) By squaring and adding, find V. Then find θ, correct to the nearest minute.

(d) Find, correct to the nearest metre, how far Jeffrey hit the ball.

DEVELOPMENT

7. A ball is thrown from level ground at an initial speed of $V\,\mathrm{m/s}$ and at an angle of projection of α above the horizontal. Assume that, t seconds after release, the horizontal and vertical displacements are given by $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$.

(a) Show that the trajectory has Cartesian equation $y = \dfrac{x}{\cos^2\alpha}\left(\sin\alpha\cos\alpha - \dfrac{gx}{2V^2}\right)$.

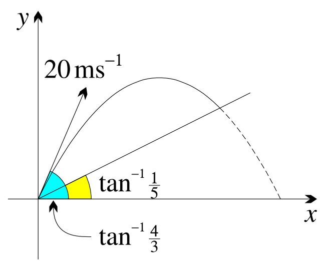
(b) Hence show that the horizontal range is $\dfrac{V^2\sin 2\alpha}{g}$.

(c) When $V = 30\,\text{m/s}$, the ball lands 45 metres away. Take $g = 10\,\text{m/s}^2$.

(i) Find the two possible values of α.

(ii) A 2-metre-high fence is placed 40 metres from the thrower. Examine each trajectory to see whether the ball will still travel 45 metres.

8. A ball is thrown with initial velocity $20\,\text{m/s}$ at an angle of elevation of $\tan^{-1}\frac{4}{3}$.



(a) Show that the parabolic path of the ball has parametric equations $x = 12t$ and $y = 16t - 5t^2$.

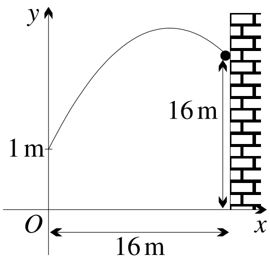
(b) Hence find the horizontal range of the ball, and its greatest height.

(c) Suppose that, as shown opposite, the ball is thrown up a road inclined at $\tan^{-1}\frac{1}{5}$ to the horizontal. Show that:

(i) the ball is about 9 metres above the road when it reaches its greatest height,

(ii) the time of flight is $2{\cdot}72$ seconds, and find, correct to the nearest tenth of a metre, the distance the ball has been thrown up the road.

9. Sofia threw a ball with velocity $20\,\text{m/s}$ from a point exactly one metre above the level ground she was standing on. The ball travelled towards a wall of a tall building 16 metres away. The plane in which the ball travelled was perpendicular to the wall. The ball struck the wall 16 metres above the ground. Take $g = 10\,\text{m/s}^2$.



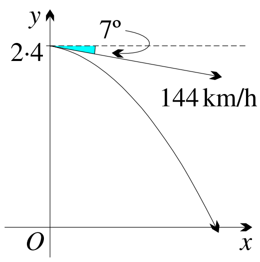
(a) Let the origin be the point on the ground directly below the point from which the ball was released. Show that, t seconds after the ball was thrown, $x = 20t\cos\theta$ and $y = -5t^2 + 20t\sin\theta + 1$, where θ is the angle above the horizontal at which the ball was originally thrown.

(b) The ball hit the wall after T seconds. Show that $4 = 5T\cos\theta$ and $3 = 4T\sin\theta - T^2$.

(c) Hence show that $16\tan^2\theta - 80\tan\theta + 91 = 0$.

(d) Hence find the two possible values of θ, correct to the nearest minute.

10. Glenn the fast bowler runs in to bowl and releases the ball $2{\cdot}4$ metres above the ground with speed $144\,\text{km/h}$ at an angle of 7° below the horizontal. Take the origin to be the point where the ball is released, and take $g = 10\,\text{m/s}^2$.



(a) Show that the coordinates of the ball t seconds after its release are given by

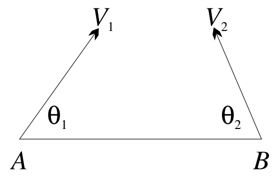
$$x = 40t\cos 7^\circ, \quad y = 2{\cdot}4 - 40t\sin 7^\circ - 5t^2.$$

(b) How long will it be (to the nearest $0{\cdot}01$ seconds) before the ball hits the pitch?

(c) Calculate the angle (to the nearest degree) at which the ball will hit the pitch.

(d) The batsman is standing 19 metres from the point of release. If the ball lands more than 5 metres in front of him, it will be classified as a ‘short-pitched’ delivery. Is this particular delivery short-pitched?

11. Two particles P_1 and P_2 are projected simultaneously from the points A and B, where AB is horizontal. The motion takes place in the vertical plane through A and B. The initial velocity of P_1 is V_1 at an angle θ_1 to the horizontal, and the initial velocity of P_2 is V_2 at an angle θ_2 to the horizontal. You may assume that the equations of motion of a particle projected with velocity V at an angle θ to the horizontal are $x = Vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$.



(a) Show that the condition for the particles to collide is $V_1\sin\theta_1 = V_2\sin\theta_2$.

(b) Suppose that $AB = 200$ metres, $V_1 = 30\,\text{m/s}$, $\theta_1 = \sin^{-1}\frac{4}{5}$, $\theta_2 = \sin^{-1}\frac{3}{5}$, $g = 10\,\text{m/s}^2$ and that the particles collide.

(i) Show that $V_2 = 40\,\text{m/s}$, and that the particles collide after 4 seconds.

(ii) Find the height of the point of collision above AB.

(iii) Find, correct to the nearest degree, the obtuse angle between the directions of motion of the particles at the instant they collide.

12. A particle is projected from the origin with initial speed $30\,\text{m/s}$ at an angle of $60°$ above the horizontal. It is subject to gravity as well as air resistance proportional to $\frac{1}{6}$ of its velocity in both the horizontal and vertical directions, so that $\ddot{x} = -\frac{1}{6}\dot{x}$ and $\ddot{y} = -10 - \frac{1}{6}\dot{y}$.

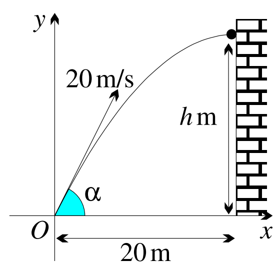
(a) Show that $\dot{x} = 15e^{-\frac{1}{6}t}$ and $\dot{y} = 15\left((4+\sqrt{3})e^{-\frac{1}{6}t} - 4\right)$.

(b) Show that $x = 90\left(1 - e^{-\frac{1}{6}t}\right)$.

(c) Show that the particle reaches its greatest height when $t = 6\ln\left(\dfrac{4+\sqrt{3}}{4}\right)$.

(d) Find, correct to the nearest metre, the horizontal distance travelled when the particle reaches its greatest height.

13. A cricketer hits the ball from ground level with a velocity of magnitude $20\,\text{m/s}$ at an angle of elevation α. It flies towards a high wall 20 metres away on level ground. Take the origin at the point where the ball was hit, and take $g = 10\,\text{m/s}^2$.



(a) Show that the ball hits the wall when $h = 20\tan\alpha - 5\sec^2\alpha$.

(b) Show that $\dfrac{d}{d\alpha}(\sec\alpha) = \sec\alpha\tan\alpha$.

(c) Show that the maximum value of h occurs when $\tan\alpha = 2$.

(d) Find the maximum height.

(e) Find the speed and angle (to the nearest minute) at which the ball hits the wall.

14. A stone is propelled at an angle of θ above the horizontal from the top of a vertical cliff 40 metres above a lake. The speed of propulsion is $20\,\text{m/s}$. Take $g = 10\,\text{m/s}^2$.

(a) Show that $x(t)$ and $y(t)$, the horizontal and vertical components of the stone's displacement from the top of the cliff, are given by

$$x(t) = 20t\cos\theta, \quad y(t) = -5t^2 + 20t\sin\theta.$$

(b) If the stone hits the lake at time T seconds, show that

$$\big(x(T)\big)^2 = 400T^2 - (5T^2 - 40)^2.$$

(c) Hence find, by differentiation, the value of T that maximises $\big(x(T)\big)^2$, and then find the value of θ that maximises the distance between the foot of the cliff and the point where the stone hits the lake.

15. A particle P_1 is projected from the origin with speed V at an angle of elevation θ.

(a) Assuming the usual equations of motion, show that the particle reaches a maximum height of $\dfrac{V^2 \sin^2 \theta}{2g}$.

(b) A second particle P_2 is projected from the origin with velocity $\frac{3}{2}V$ at an angle $\frac{1}{2}\theta$ to the horizontal. The two particles reach the same maximum height.

(i) Show that $\theta = \cos^{-1} \frac{1}{8}$. (ii) Do the two particles take the same time to reach this maximum height? Justify your answer.

16. (a) Prove that the horizontal range of a projectile is $\dfrac{V^2 \sin 2\alpha}{g}$, where V is the initial speed, α is the angle of projection and $g\,\text{m/s}^2$ is the acceleration due to gravity.

(b) A garden sprinkler sprays water symmetrically about its vertical axis at a constant speed of V m/s. The initial direction of the spray varies continuously between $15°$ and $45°$ above the horizontal.

(i) Explain why, from a fixed point O on level ground, the sprinkler will wet an annulus with centre O, inner radius $\dfrac{V^2}{2g}$ metres and outer radius $\dfrac{V^2}{g}$ metres.

(ii) Deduce that by appropriately locating the sprinkler relative to a rectangular garden 6 m by 3 m, the entire garden can be watered provided that $\dfrac{V^2}{2g} \ge 1 + \sqrt{7}$.

17. A particle is projected from the origin with speed V at an angle of α above the horizontal. It is subject to both gravity and an air resistance proportional to its velocity, so that its horizontal and vertical components of acceleration while it is rising are given by $\ddot{x} = -k\dot{x}$ and $\ddot{y} = -g - k\dot{y}$.

(a) Show that $\dot{x} = V \cos\alpha\, e^{-kt}$ and $\dot{y} = \left(\dfrac{g}{k} + V \sin\alpha\right) e^{-kt} - \dfrac{g}{k}$.

(b) Hence show that $x = \dfrac{V\cos\alpha}{k}\left(1 - e^{-kt}\right)$ and $y = \left(\dfrac{g}{k^2} + \dfrac{V\sin\alpha}{k}\right)\left(1 - e^{-kt}\right) - \dfrac{g}{k}t$.

(c) When the particle reaches its greatest height, show that it has travelled a horizontal distance of $\dfrac{V^2 \sin 2\alpha}{2\left(g + Vk\sin\alpha\right)}$.

ENRICHMENT

18. A projectile is fired from the origin with velocity V and angle of elevation α, where α is acute. Assume the usual equations of motion.

(a) Let $k = \dfrac{V^2}{2g}$. Show that the Cartesian equation of the parabolic path of the projectile can be written as

$$x^2 \tan^2 \alpha - 4kx \tan\alpha + (4ky + x^2) = 0.$$

(b) Show that the projectile can pass through the point (X, Y) in the first quadrant by firing at two different initial angles α_1 and α_2 only if $X^2 < 4k^2 - 4kY$.

(c) Suppose that $\tan\alpha_1$ and $\tan\alpha_2$ are the two real roots of the quadratic equation in $\tan\alpha$ in part (a). Show that $\tan\alpha_1 \tan\alpha_2 > 1$, and hence explain why it is impossible for α_1 and α_2 both to be less than $45°$.

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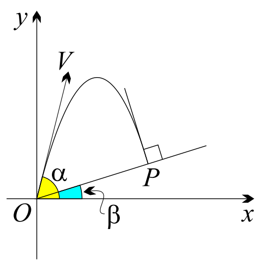
19. The diagram shows the parabolic path of a particle that is projected from the origin O with speed V at an angle of α above the horizontal. It lands at P, which lies on a plane inclined at an angle of β to the horizontal. When the particle hits the plane, its direction of motion is perpendicular to the plane.



Let $OP = d$, and assume that the horizontal and vertical displacements of the particle from O are given by $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$.

(a) Write down the coordinates of P in terms of d and β.

(b) By substituting the coordinates of P into the displacement equations, show that

$$d = \frac{2V^2\cos^2\alpha}{g\cos^2\beta}\left(\tan\alpha\cos\beta - \sin\beta\right).$$

(c) By resolving the horizontal and vertical components of the velocity at P, show that

$$\cot\beta = \frac{gd\cos\beta}{V^2\cos^2\alpha} - \tan\alpha.$$

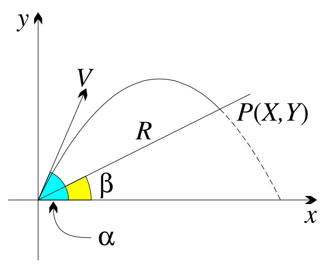
(d) Deduce that $\tan\alpha = \cot\beta + 2\tan\beta$.

20. (a) Consider the function $y = 2\sin(x-\theta)\cos x$.

(i) Show that $\dfrac{dy}{dx} = 2\cos(2x-\theta)$. (ii) Hence, or otherwise, show that

$$2\sin(x-\theta)\cos x = \sin(2x-\theta) - \sin\theta.$$

(b) A projectile is fired from the origin with velocity V at an angle of α to the horizontal up a plane inclined at β to the horizontal. Assume that the horizontal and vertical components of the projectile's displacement are given by $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$.



(i) If the projectile strikes the plane at (X, Y), show that

$$X = \frac{2V^2\cos^2\alpha(\tan\alpha - \tan\beta)}{g}.$$

(ii) Hence show that the range R of the projectile up the plane is given by

$$R = \frac{2V^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}.$$

(iii) Use part (a)(ii) to show that the maximum possible value of R is $\dfrac{V^2}{g(1+\sin\beta)}$.

(iv) If the angle of inclination of the plane is 14°, at what angle to the horizontal should the projectile be fired in order to attain the maximum possible range?

21. A tall building stands on level ground. The nozzle of a water sprinkler is positioned at a point P on the ground at a distance d from a wall of the building. Water sprays from the nozzle with speed V and the nozzle can be pointed in any direction from P.

(a) If $V > \sqrt{gd}$, prove that the water can reach the wall above ground level.

(b) Suppose that $V = 2\sqrt{gd}$. Show that the portion of the wall that can be sprayed with water is a parabolic segment of height $\frac{15}{8}d$ and area $\frac{5\sqrt{15}}{2}d^2$.

6G Miscellaneous Problems

There is no new mathematical theory presented in this section. Instead it is a collection of harder problems deemed to be within the scope of the syllabus. The list of problems presented here is not intended to be exhaustive. Teachers and students are encouraged to find other practical applications of the mathematics they have encountered, and to solve those problems. When attempting questions, students are encouraged to draw a diagram, where appropriate, as often a problem is more simply solved from the diagram.

Worked Example 22: A car of mass m is travelling at a constant speed V_0 m/s. The brakes are applied, resulting in a constant deceleration until the car comes to a stop. Given that the car stops in ℓ m, find the force exerted by the brakes.

Solution: Let v m/s be the speed of the car x m beyond where the brakes are first applied. Since deceleration is involved, let the acceleration of the car be $-a$, with $a > 0$, so that

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -a\,.$$

Integrating, $\quad \tfrac{1}{2}v^2 = -ax + C\,.$

At $x = \ell$, $v = 0$ so $C = a\ell$. Thus the velocity is given by

$$\tfrac{1}{2}v^2 = -ax + a\ell\,.$$

At $x = 0$, $\quad \tfrac{1}{2}{V_0}^2 = a\ell$

thus $\quad a = \dfrac{{V_0}^2}{2\ell}\,.$

Hence the force required to stop the car is:

$$\begin{aligned} F &= m(-a) \\ &= -\frac{m{V_0}^2}{2\ell}\,. \end{aligned}$$

Notice that for any given force, the distance is proportional to the square of the speed. Thus a car going just 10% above the speed limit will require a distance 21% greater to stop. This is obviously a significant issue for road safety.

Variable Gravity: One particular type of problem that can be solved in this course is motion with variable gravity, as shown in previously Worked Example 5 and here in the following example.

Worked Example 23: The acceleration experienced by an object due to gravity is inversely proportional to the square of the radius r metres between the centre of the earth and the centre of the object, and is directed towards the centre of the earth. Thus, ignoring all other forces, the equation of motion is:

$$\frac{d^2r}{dt^2} = -\frac{k}{r^2}\,.$$

Assume that the earth is a sphere of radius R and that the acceleration due to gravity at the surface of the earth is g. The object is projected vertically with initial velocity V_0.

(a) The smallest initial velocity for which the object never returns to the earth is called the escape velocity. Show that the escape velocity is $V_0 = \sqrt{2gR}$.

(b) Given that $V_0 = \sqrt{2gR}$, find the time taken to reach an altitude of R.

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SOLUTION:

(a) Begin by finding the value of k. At the surface of the earth $r = R$, so

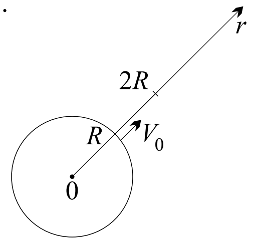
$$-g = -\frac{k}{R^2}$$

hence $\quad k = gR^2\,.$

Next, re-write the acceleration in terms of velocity v and radius r.

$$\frac{d}{dr}\left(v^2\right) = -\frac{2gR^2}{r^2}$$

so $\quad v^2 = \dfrac{2gR^2}{r} + C_1 \quad$ (for some constant C_1) .



When $t = 0$, $r = R$ and $v = V_0$ so

$$V_0{}^2 = 2gR + C_1$$

thus $\quad C_1 = V_0{}^2 - 2gR$

and $\quad v^2 = \dfrac{2gR^2}{r} + V_0{}^2 - 2gR\,.$

Since the object never returns, r must always increase, otherwise $\ddot{r} < 0$. So take the limit as $r \to \infty$ to get

$$\lim_{r\to\infty} v^2 = V_0{}^2 - 2gR$$

which must be positive or zero. (Squares cannot be negative.) Thus

$$V_0{}^2 - 2gR \ge 0$$

hence $\quad V_0 \ge \sqrt{2gR}\,.$

That is, the escape velocity is $V_0 = \sqrt{2gR}$.

For the earth, these quantities are approximately:
$g = 10\,\text{m/s}^2$, $R = 6400\,\text{km}$ and $V_0 = 8\sqrt{2} \doteqdot 11{\cdot}3\,\text{km/s}$.

(b) In the case of the escape velocity, $v^2 = \frac{2gR^2}{r}$, which is continuous for $r > 0$, and is never zero. Hence v can never change sign. Since it is initially positive, it must remain positive. Thus take the positive square root to get

$$\frac{dr}{dt} = \sqrt{\frac{2gR^2}{r}}$$

or $\quad r^{\frac{1}{2}} \times \dfrac{dr}{dt} = R\sqrt{2g}\,.$

Integrating, $\quad \frac{2}{3}r^{\frac{3}{2}} = Rt\sqrt{2g} + C_2 \qquad$ (for some constant C_2 .)

When $t = 0$, $r = R$ so

$$C_2 = \tfrac{2}{3}R^{\frac{3}{2}}\,.$$

Thus $\quad Rt\sqrt{2g} = \frac{2}{3}\left(r^{\frac{3}{2}} - R^{\frac{3}{2}}\right)$

and $\quad t = \frac{\sqrt{2}}{3R\sqrt{g}}\left(r^{\frac{3}{2}} - R^{\frac{3}{2}}\right)\,.$

Hence at $r = 2R$ the time is

$$\begin{aligned} t &= \tfrac{\sqrt{2}}{3R\sqrt{g}}(2R\sqrt{2R} - R\sqrt{R}\,) \\ &= \tfrac{\sqrt{R}}{3\sqrt{g}}(4 - \sqrt{2}\,) \qquad \text{(about 690 s for earth.)} \end{aligned}$$

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Projectile Motion: There is an enormous number of different questions that can be asked on this topic. Some of those included in the subsequent exercise may also be appropriate for Mathematics Extension 1 candidates. Equally, many worthwhile questions have been omitted simply for the sake of space.

WORKED EXAMPLE 24: An object is projected from the origin O at an angle α with initial speed V. At a particular point $P(x, y)$ on the trajectory, the gradient of OP is $\tan\beta$ and the gradient of the tangent to the trajectory at P is $\dfrac{dy}{dx} = -\tan\beta$. You may assume that

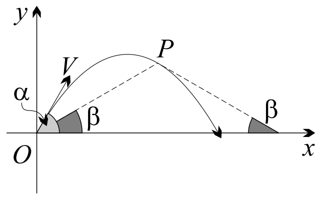
$$\begin{aligned} & y = Vt\sin\alpha - \tfrac{1}{2}gt^2 \\ \text{and} \quad & x = Vt\cos\alpha\,. \end{aligned}$$

(a) Show that the time taken to reach P is $\dfrac{4V\sin\alpha}{3g}$.

(b) Hence show that $\tan\alpha = 3\tan\beta$.

SOLUTION:

(a) At P $\quad \tan\beta = \dfrac{y}{x} \quad$ and $\quad \dfrac{dy}{dx} = -\tan\beta$



$$\begin{aligned} \text{so} \quad & \frac{y}{x} = -\frac{dy}{dx}\,. \\ \text{Also} \quad & \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \\ \text{thus} \quad & \frac{y}{x} = -\frac{\dot{y}}{\dot{x}} \\ \text{so} \quad & \frac{Vt\sin\alpha - \frac{1}{2}gt^2}{Vt\cos\alpha} = -\frac{V\sin\alpha - gt}{V\cos\alpha} \\ \text{or} \quad & V\sin\alpha - \tfrac{1}{2}gt = gt - V\sin\alpha \\ \text{thus} \quad & 2V\sin\alpha = \tfrac{3}{2}gt \\ \text{and} \quad & t = \frac{4V\sin\alpha}{3g}\,. \end{aligned}$$

(b) Using the value of t found in part (a),

$$\begin{aligned} \tan\beta &= -\frac{dy}{dx} \\ &= \frac{g\left(\frac{4V\sin\alpha}{3g}\right) - V\sin\alpha}{V\cos\alpha} \\ &= \frac{4V\sin\alpha - 3V\sin\alpha}{3V\cos\alpha} \\ \text{thus } \tan\beta &= \frac{\sin\alpha}{3\cos\alpha} \\ \text{or} \quad \tan\alpha &= 3\tan\beta\,. \end{aligned}$$

Simple Harmonic Motion: A harder application of simple harmonic motion is in the small angle approximation of the motion of a pendulum. It is also an example of how a force may be resolved into its components in different directions, rather like the resolution of the initial velocity in projectile motion.

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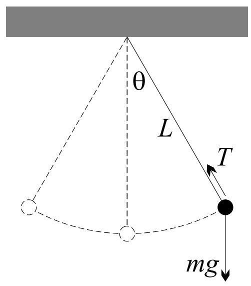
WORKED EXAMPLE 25: A mass m is suspended from a fixed point on a light inextensible wire of length L. The ball is set in motion, swinging back and forth along an arc in a vertical plane. At time t the angle that the wire makes with the vertical is θ. The only forces acting on the ball are the tension T acting along the wire normal to the arc, and gravity acting vertically.



(a) Resolve the force due to gravity normally to find T.

(b) Next, resolve the force tangentially to show that $\dfrac{d^2\theta}{dt^2} = -\dfrac{g\sin\theta}{L}$.

(c) Under what circumstances can the equation in part (b) be approximated with

$$\frac{d^2\theta}{dt^2} = -\frac{g\theta}{L}\,?$$

(d) At $t = 0$, the ball is released from rest at an angle $\theta = \theta_0$, where θ_0 is positive. Use the result of part (c) to show that $\theta = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\,t\right)$.

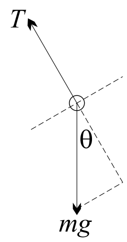
(e) What is the speed of the ball at the bottom of the swing?

SOLUTION:

(a) Since the length of the wire does not change it follows that the sum of the normal forces is zero. Thus

$$T - mg\cos\theta = 0$$

or $$T = mg\cos\theta\,.$$



(b) The only force acting tangentially is the component of weight, which is $mg\sin\theta$. From observation, the force acts in the opposite direction to the displacement, θL. Hence

$$m\frac{d^2}{dt^2}\Big(\theta L\Big) = -mg\sin\theta$$

so $$\frac{d^2\theta}{dt^2} = -\frac{g\sin\theta}{L}\,.$$

(c) When θ is small, it follows that $\sin\theta \doteqdot \theta$ so

$$\frac{d^2\theta}{dt^2} = -\frac{g\theta}{L}\,.$$

(d) The previous result is the equation of simple harmonic motion with $n^2 = \frac{g}{L}$. Hence the general solution is

$$\theta = a\cos\left(\sqrt{\tfrac{g}{L}}\,t + \phi\right) \qquad \text{where} \qquad -\tfrac{\pi}{2} \le \phi \le \tfrac{\pi}{2}$$

so $$\dot\theta = -a\sqrt{\tfrac{g}{L}}\sin\left(\sqrt{\tfrac{g}{L}}\,t + \phi\right)\,.$$

At $t = 0$, the velocity is zero. That is $\dot\theta L = 0$, so

$$0 = -a\sqrt{\tfrac{g}{L}}\sin(\phi)$$

hence $\phi = 0$ (for ϕ in the specified domain)

and $\theta = a\cos\left(\sqrt{\tfrac{g}{L}}\,t\right)$.

At $t = 0$, the initial angle is $\theta_0 = a\cos 0$

hence $a = \theta_0$

and $\theta = \theta_0\cos\left(\sqrt{\tfrac{g}{L}}\,t\right)$.

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Here is a simpler argument. The motion is SHM and starts from rest hence a cosine solution is appropriate, since the cosine function has a stationary point at $t = 0$. It follows that the initial displacement is the amplitude.

(e) At the bottom of the swing $\theta = 0$, so

$$\cos\left(\sqrt{\tfrac{g}{L}}\,t\right) = 0$$

that is $$\sqrt{\tfrac{g}{L}}\,t = \tfrac{\pi}{2} + k\pi\,.$$

The speed is $v = \left|\dot{\theta}L\right|$, so

$$\begin{aligned} v &= \left|-\theta_0 L\sqrt{\tfrac{g}{L}}\sin\left(\sqrt{\tfrac{g}{L}}\,t\right)\right| \\ &= \left|-\theta_0 L\sqrt{\tfrac{g}{L}}\sin\left(\tfrac{\pi}{2} + k\pi\right)\right| \qquad \text{(at the bottom of the swing)} \\ &= \theta_0\sqrt{gL}\,. \end{aligned}$$

Exercise **6G**

1. An object falls from the top of a building. An office worker sees the object fall past a window 2 m high in $\frac{1}{5}$ s. Assume that air resistance is negligible and use $g = 10\,\text{m/s}^2$. Let y be the height of the object above the bottom of the window at time t.

(a) Show that $y = H - 5t^2$, where H is the distance from the bottom of the window to the top of the building.

(b) Use part (a) and the information given in the question to form a pair of equations. Solve these simultaneously to find the value of H.

2. A projectile is fired vertically under the influence of gravity alone. Let the acceleration due to gravity be $g \doteqdot 10\,\text{m/s}^2$. Thus the height y_1 at time t satisfies $\ddot{y}_1 = -10$. The projectile is fired with initial speed of 16 m/s. One second later another projectile is fired from the same point.

(a) Find the time for the first projectile to reach its maximum height. Hence find this maximum height.

(b) Let the initial speed of the second projectile be v. Show that its height is given by $y_2 = -5(t-1)^2 + v(t-1)$.

(c) Find v given that the two collide when the first has reached its maximum height.

3. Ignoring air resistance, a hot-air balloon has two forces acting on it: a constant buoyancy force B and the force due to gravity. Suppose that when the mass of the balloon is M it descends with a downward acceleration $d > 0$. Ballast of mass m is thrown out and the result is that the balloon ascends with upward acceleration $a > 0$.

(a) Write down two force balance equations for the information given.

(b) Hence show that $m = M\dfrac{a+d}{a+g}$.

4. The acceleration due to gravity at distance r from the centre of the moon is given by $\dfrac{d^2r}{dt^2} = -\dfrac{gR^2}{r^2}$, where $R \doteqdot 1\,737$ km is its radius and $g \doteqdot 1{\cdot}625\,\text{m/s}^2$ is the acceleration due to gravity at the surface of the moon. A projectile is fired vertically from the surface with initial velocity V_0.

(a) Derive an expression for v^2, the square of the velocity of the projectile, in terms of r.

(b) The maximum altitude of the projectile is R. Determine V_0.

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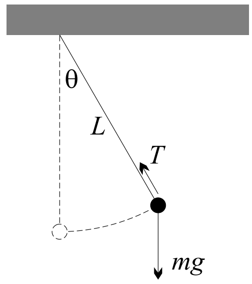

5. The bob of a pendulum is released from rest. The initial angle between the wire and the vertical is θ_0 and the tension along the wire is T_0. At time t after it is released, the angle between the wire and the vertical is θ and the tension is T. You may assume that the only forces acting on the bob are due to the tension acting along the wire and gravity acting vertically. Let the mass of the bob be m and the length of the wire be L.

(a) Resolve the forces on the bob normally and tangentially to obtain the equations of motion.

(b) Let $\omega = \dfrac{d\theta}{dt}$. Show that $\dfrac{d^2\theta}{dt^2} = \dfrac{d}{d\theta}\left(\tfrac{1}{2}\omega^2\right)$.

(c) Hence show that $\omega^2 = \frac{2}{mL}(T - T_0)$.

6. The deck of a ship is 1 m below a wharf at low tide and is 0·4 m above the wharf at high tide. On a certain day the ship has a precious cargo that can only be safely offloaded when the deck is above the level of the wharf. On that day, low tide is at 1:10 am and again at 1:35 pm. Given that the tide can be modelled with simple harmonic motion, between what times in the morning can the cargo be safely offloaded?

DEVELOPMENT

7. (a) Show that $\tan(\frac{\pi}{4} + \frac{\alpha}{2}) = \dfrac{\sin\alpha + 1}{\cos\alpha}$.

(b) The Cartesian equation of a projectile fired at angle β and velocity V is

$$y = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta\,.$$

Write this equation as a quadratic in $\tan\beta$.

(c) In a game of netball the Goal Attack is about to shoot for goal. The line joining the net and the point of projection of the ball makes an angle α with the horizontal. Let h be the height of the net above the point of projection. Let V be the minimum speed required to score a goal. That is, there is only one angle of projection β at which the shot may be taken with speed V.

(i) Let the coordinates of the net be (x, y). Show that $V^4 - 2gyV^2 - g^2x^2 = 0$.

(ii) At the net, $y = h$ and $x = h\cot\alpha$. Show that $V^2 = gh(1 + \operatorname{cosec}\alpha)$.

(iii) Write down an expression for $\tan\beta$ and hence use part (a) to show that $\beta = \frac{\pi}{4} + \frac{\alpha}{2}$.

8. The Cartesian equation of a projectile fired at angle θ and velocity V may be written as

$$\frac{gx^2}{2V^2}\tan^2\theta - x\tan\theta + \left(y + \frac{gx^2}{2V^2}\right) = 0\,. \qquad (*)$$

(a) Let this quadratic equation for $\tan\theta$ have two solutions, $\tan\theta_1$ and $\tan\theta_2$.
Show that $\tan(\theta_1 + \theta_2) = -\dfrac{x}{y}$.

(b) Suppose that for a certain point $P(a, b)$ there is only one solution for θ.
Prove that this angle is $\theta = \frac{\pi}{4} + \frac{\alpha}{2}$, where $\tan\alpha = \dfrac{b}{a}$.

(c) Each point that can be hit by the projectile lies on or below a parabola. By considering the discriminant of equation $(*)$, find and describe that parabola.

9. A number of ball bearings are fired at the same moment, from the same point and with the same speed V but at different angles of projection in the same vertical plane. Ignoring air resistance, the time equations for the position of each ball bearing are

$$x = Vt\cos\alpha_j \qquad \text{and} \qquad y = Vt\sin\alpha_j - \tfrac{1}{2}gt^2$$

where α_j is the angle of projection of the j^{th} ball bearing. Show that at any time, all the ball bearings lie on a circle, and find the centre and radius of this circle.
[HINT: Eliminate α_j from the above equations.]

10. A thrill-seeker goes bungy-jumping from a point O on a bridge above a river. Let x be the distance below O and let v be the velocity of the thrill-seeker towards the river.

(a) The person free falls for a distance ℓ. Thus $\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = g$ for $0 \le x \le \ell$.
Show that $v^2 = 2g\ell$ when $x = \ell$.

(b) For $x > \ell$, the bungy cord begins to slow the person down according to the equation

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = g - gk(x - \ell)$$

where k is a positive constant.

(i) Show that $v^2 = 2gx - gk(x - \ell)^2$ for $x > \ell$.

(ii) Show that the furthest the thrill seeker reaches is

$$x_{\text{max}} = \tfrac{1}{k}\left(1 + k\ell + \sqrt{1 + 2k\ell}\right).$$

(iii) Find x_{max} in terms of ℓ when $k\ell = 4$.

11. An object is fired vertically upwards with initial speed V_0 m/s from ground level. Ignoring air resistance, the acceleration due to gravity varies with height x above ground level according to the equation

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \frac{-10R^2}{(R + x)^2},$$

where R is the average radius of the earth, and v is the velocity of the object.

(a) What is the significance of the negative sign?

(b) Find an expression for h, the maximum height reached, in terms of R and V_0.

(c) Given that $R = 6{\cdot}4 \times 10^6$ m and $V_0 = 500$ m/s, evaluate h correct to the nearest metre.

12. The temperature B of a beaker containing a hot chemical, and the temperature W of a cooler water bath in which the beaker is placed, both satisfy Newton's law of cooling. Thus the temperatures are related by the equations

$$\frac{dB}{dt} = -k(B - W) \qquad \text{and} \qquad \frac{dW}{dt} = \tfrac{3}{4}k(B - W),$$

where k is a positive constant.

(a) By differentiating $\tfrac{3}{4}B + W$, show that $\tfrac{3}{4}B + W = C$, where C is a constant.

(b) Initially the beaker is 120°C and the water bath is 22°C. Find C and hence show that

$$\frac{dB}{dt} = -k(\tfrac{7}{4}B - 112).$$

(c) Solve the above equation to find B as a function of time.

(d) After ten minutes the temperature of the beaker is 92°C. Find the temperature of the beaker after a further ten minutes.

(e) What is the eventual temperature of the water bath?

13. A model for the population, P, of elephants in Serengeti National Park is

$$P = \frac{21\,000}{7 + 3e^{-t/3}}$$

where t is the time in years after the end of 2008.

(a) Show that P satisfies the differential equation

$$\frac{dP}{dt} = \frac{P}{3}\left(1 - \frac{P}{3000}\right).$$

(b) What was the population at the end of 2008?

(c) What does the model predict that the eventual population will be?

(d) What was the annual percentage rate of growth at the end of 2008?

14. (a) Show that the focal length of the parabola $y = bx - ax^2$ is $\frac{1}{4a}$.

(b) The usual Cartesian equation of the trajectory of a projectile is

$$y = x\tan\alpha - x^2 \times \frac{g\sec^2\alpha}{2V^2}.$$

(i) Find the coordinates of the vertex of this trajectory.

(ii) Find the focal length of the trajectory.

(iii) Hence find the equation of its directrix, and the coordinates of the focus.

15. As water empties from a water cooler through a small hole at the bottom it is found that the depth of the water satisfies the equation

$$\frac{dy}{dt} = -k\sqrt{y},$$

where k is a positive constant and y is the depth of water. Initially the depth of the water is y_0 and it takes T seconds to fully drain.

(a) Show that $\sqrt{\dfrac{y}{y_0}} = 1 - \dfrac{t}{T}$ for $0 \le t \le T$.

(b) Suppose that it takes 15 seconds for half the water to drain out. How long does it take to empty the full cooler?

16. A bead of mass m slides along a wire in the shape of the curve $y = \frac{3}{2}x^{2/3}$, where $0 \le x \le 1$. At time t, let the bead be at $P(x, y)$, where x and y are functions of t. The coordinates of P satisfy the equation

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 + mgy = E,$$

where E and g are constants.

(a) When $t = 0$ the bead is released from rest at the point $(1, \frac{3}{2})$. Find E.

(b) Show that $\dot{x} = x^{1/3}\dot{y}$, and hence show that $\dot{y}^2 = \dfrac{3g(3-2y)}{3+2y}$.

(c) Find $\dot{y}$ and $\dot{x}$ when the bead is at the origin.

(d) It is known that

$$\int_0^\alpha \sqrt{\frac{1+u}{1-u}}\,du = \sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2} \qquad \text{for} \qquad 0 \le \alpha \le 1.$$

Use this result to find the time it takes for the bead to travel from $(1, \frac{3}{2})$ to the origin.

(e) As an exercise, use the methods of integration in Chapter 4 to prove the above result. What is peculiar about the case $\alpha = 1$?

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ENRICHMENT

17. In this question you may assume that the trajectory of a projectile is a parabola, and that its directrix is independent of α, the angle of projection. Suppose it is found that there is only one angle at which a projectile may be fired from point A to pass through point P.

(a) Use geometry to show that the focus of the trajectory is on the line AP.

(b) Now prove the result algebraically. You may need the results of Question 14.

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6H Chapter Review Exercise

Exercise **6H**

1. In each part v is given as a function of x, and it is known that $x = 1$ when $t = 0$. Express: (i) t in terms of x, (ii) x in terms of t.

(a) $v = 2x - 1$ (b) $v = -6x^2$

2. In each motion of the previous question, find $\ddot{x}$ in terms of x.

3. In each part the acceleration $\ddot{x}$ is given as a function of x. Express v^2 in terms of x given that $v = 0$ when $x = 0$.

(a) $\ddot{x} = 6$ (b) $\ddot{x} = \sin 6x$

4. Given that $\ddot{x} = 4v$, and $v = 2$ when $t = 0$, find t as a function of v.

5. Given that $\ddot{x} = \dfrac{1}{3v}$, and $v = 1$ when $x = 0$, find x as a function of v.

6. A particle of mass m moves in a straight line subject to a force F. At time t, the displacement of the particle is x and the velocity is v. The particle was initially at rest at the origin.

(a) If $F = 18t$ and $m = 3$, find x when $t = 2$.

(b) If $F = 4x - 3$ and $m = 2$, find the positive value of v when $x = 4$.

(c) If $F = \dfrac{1}{1 + v}$ and $m = 0{\cdot}5$, find t when $v = 6$.

(d) If $F = 2 + v^2$ and $m = 2$, find x when $v = 2$.

7. Two forces F_1 and F_2 are acting on a particle P of mass 2 kg. F_1 has magnitude 40 N and acts in the north-easterly direction, while F_2 has components 40 N due north and 20 N due east. Calculate the magnitude and direction (both to 3 significant figures) of the resultant acceleration of P.

8. A plane lands on a runway at 100 m/s. It then brakes with a constant deceleration until it stops 2 km down the runway.

(a) Explain why the equation of motion is $\ddot{x} = -k$, for some positive constant k. By integrating with respect to x, find k, and find v^2 as a function of x.

(b) Find: (i) the velocity after 1 km, (ii) where it is when the velocity is 50 m/s.

(c) Explain why, during the braking, $v = \sqrt{10\,000 - 5x}$ rather than $v = -\sqrt{10\,000 - 5x}$.

(d) Integrate to find the displacement–time function, and find how long it takes to stop.

9. When fired under water, a torpedo moves with deceleration proportional to its velocity, so that $\ddot{x} = -kv$ for some positive constant k. Its initial speed is $\frac{1}{2}$ km/s, and its speed after it has gone 50 metres is 250 m/s.

(a) Use $\ddot{x} = v\dfrac{dv}{dx}$ to find v as a function of x, then find x as a function of t.

(b) Show that it takes $\frac{1}{5}\ln 2$ seconds to go the first 50 metres, and describe the subsequent motion of the torpedo.

10. A particle moves with acceleration $\ddot{x} = -\frac{1}{2}e^{-x}$ m/s^2.

(a) Initially it is at the origin with velocity 1 m/s. Find an expression for v^2.

(b) Explain why v is always positive, and hence find the displacement as a function of time.

(c) Describe what happens to the particle as $t \to \infty$.

11. The acceleration of a particle moving in a straight line is given by $\ddot{x} = a + bv^2$, where a and b are positive constants. Initially the particle was at rest at the origin. Show that:

(a) $v^2 = \dfrac{a}{b}\left(e^{2bx} - 1\right)$ (b) $v = \dfrac{\sqrt{a}}{\sqrt{b}}\tan\left(\sqrt{ab}\,t\right)$

12. A particle is oscillating in simple harmonic motion with acceleration $\ddot{x} = -16x$. At time $t = 0$, $x = 1$ and $\dot{x} = 4$.

(a) Show by integrating that $v^2 = 16(2 - x^2)$.

(b) What is the maximum displacement of the particle?

(c) Find x as a function of t in the form $x = a\cos(nt + \alpha)$.

13. The displacement x at time t of a particle moving on the x-axis is given by

$$x = 5 + \sqrt{3}\sin 3t - \cos 3t.$$

(a) Prove that the motion of the particle is simple harmonic.

(b) Express $\sqrt{3}\sin 3t - \cos 3t$ in the form $R\sin(3t - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(c) What are the amplitude and the centre of the motion?

(d) When does the particle first reach its maximum speed?

14. A particle is moving in simple harmonic motion. Its maximum speed and acceleration are $2\,\text{m/s}$ and $6\,\text{m/s}^2$ respectively. Find the amplitude and period of the motion.

15. A piston in a car engine is moving up and down in simple harmonic motion with an amplitude of 5 cm. The mass of the piston is 0·5 kg and it is making 60 oscillations per second. Find:

(a) the maximum speed of the piston, in m/s correct to one decimal place,

(b) the maximum force acting on the piston, correct to the nearest newton.

16. A particle is oscillating on the x-axis about the origin. It passes through the point A, where $x = 0{\cdot}01$ metres, with speed $0{\cdot}09\,\text{m/s}$ and it passes through the point B, where $x = -0{\cdot}02$ metres, with speed $0{\cdot}06\,\text{m/s}$.

(a) Show that the amplitude of the motion is $\frac{\sqrt{10}}{125}$ metres.

(b) Show that the motion has period $\frac{2\pi}{\sqrt{15}}$ seconds.

(c) Find, correct to the nearest hundredth of a second, the time that the particle takes to move directly from A to B.

17. An aircraft of mass M kg moves along a horizontal runway, starting from rest. The aircraft's engines exert a constant thrust of T newtons and, when the speed of the aircraft is v m/s, it experiences a resistance of magnitude kv^2, where $k > 0$.

When the speed of the aircraft is V m/s, show that it has travelled

$$\frac{M}{2k}\ln\left(\frac{T}{T - kV^2}\right) \text{ metres.}$$

18. An object of mass 2 kg is projected vertically upwards with initial speed 20 m/s. It experiences air resistance of magnitude $\frac{1}{10}v^2$ newtons, where v m/s is the speed of the object after t seconds. Take $g = 10\,\text{m/s}^2$.

(a) Show that the object reaches a maximum height of $10\ln 3$ metres.

(b) Find, correct to 3 significant figures, the speed of the object when it reaches half its maximum height.

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19. A child drops an air-filled balloon of mass 30 grams from a bridge 6 metres above a river. The balloon experiences air resistance of magnitude $0{\cdot}6v$ newtons, where v m/s is the speed of the balloon after t seconds. Take $g = 10\,\text{m/s}^2$.

(a) Write down an equation of motion for the balloon's descent.

(b) Show that the terminal speed of the balloon is $0{\cdot}5\,\text{m/s}$.

(c) Show that $v = \frac{1}{2}\left(1 - e^{-20t}\right)$.

(d) How long does it take, to 2 significant figures, for the balloon to reach half its terminal speed?

20. A particle of mass m falls from rest in a medium in which the resistance to its motion is mkv, where k is a constant and v is the speed of the particle. As the particle falls, v approaches a limiting value of V.

(a) Show that $\dfrac{dv}{dt} = \dfrac{g}{V}(V - v)$.

(b) Find the time it takes for the particle to reach a speed of $\frac{1}{2}V$.

(c) Show that the distance fallen when a speed of $\frac{1}{2}V$ is reached is $\dfrac{V^2}{g}\left(\ln 2 - \dfrac{1}{2}\right)$.

21. A particle P_1 is projected vertically upwards from a point A with initial speed U. At the same instant, a second particle P_2, also of mass m, is dropped from a point B directly above A. The distance H between A and B is equal to the maximum height that P_1 would reach were it not to collide with P_2. As the particles move, they each experience air resistance of magnitude mkv^2, where $k > 0$ and v is the speed. At the instant the particles collide, P_2 has reached exactly 50% of its terminal speed. Let y_1 be the distance of P_1 above A, and y_2 the distance of P_2 below B.

(a) Show that $V = \sqrt{\dfrac{g}{k}}$.

(b) Show that $y_1 = \dfrac{1}{2k}\ln\left(\dfrac{g + kU^2}{g + k{v_1}^2}\right)$, where v_1 is the speed of P_1.

(c) Hence show that $H = \dfrac{1}{2k}\ln\left(1 + \dfrac{U^2}{V^2}\right)$.

(d) Given that $y_2 = \dfrac{1}{2k}\ln\left|\dfrac{g}{g - k{v_2}^2}\right|$, show that, at the instant the particles collide, P_2 has fallen a distance of $\dfrac{1}{2k}\ln\dfrac{4}{3}$.

(e) Deduce that the speed of P_1 at the instant the particles collide is $\dfrac{V}{\sqrt{3}}$.

22. A projectile has initial velocity vector $48\underset{\sim}{i} + 36\underset{\sim}{j}$.

(a) Obtain, by integration, expressions for $\dot{x}, \dot{y}, x$ and y, taking $g = 10\,\text{m/s}^2$.

(b) Find the maximum height reached by the projectile.

(c) Find the horizontal range of the projectile.

(d) Find, as a component vector, the velocity of the particle after 1.6 seconds.

23. A particle is projected from the origin and follows a parabolic path with parametric equations $x = 12t$ and $y = 9t - 5t^2$ (where x and y are in metres).

(a) Show that the Cartesian equation of the path is $y = \frac{3}{4}x - \frac{5}{144}x^2$.

(b) Find the horizontal range R and the greatest height H.

(c) Find the gradient at $x = 0$, and hence find the angle of projection.

(d) Find $\dot{x}$ and $\dot{y}$ when $t = 0$. Hence, by resolving these components, find the initial velocity, and confirm the angle of projection.

(e) Find when the particle is 4 metres high, and the horizontal displacement then.

24. Eve tossed an apple to Adam who was sitting near him. Adam caught it at the same height that Eve released it from. Suppose that the initial velocity of the apple was $V = 5\,\text{m/s}$, at an angle of $\tan^{-1} 2$ above the horizontal.

(a) Find the initial values of $\dot{x}$ and $\dot{y}$.

(b) Find $\dot{x}$, x, $\dot{y}$ and y by integrating $\ddot{x} = 0$ and $\ddot{y} = -10$, taking the origin to be the point of release of the apple.

(c) Find the greatest height above the point of release reached by the apple.

(d) Show that the apple was in the air for $\frac{2}{5}\sqrt{5}$ seconds, and hence find the horizontal distance travelled by the apple.

(e) Find $\dot{x}$ and $\dot{y}$ at the time Adam caught the apple. Then show that the final speed was the same as the initial speed, and the final direction was the opposite of the initial direction.

(f) The path of the apple is a parabolic arc. By eliminating t from the equations for x and y, find its equation in Cartesian form.

25. A bullet is fired horizontally at $200\,\text{m/s}$ from a window 45 metres above the level ground below. It doesn't hit anything and falls harmlessly to the ground.

(a) Write down the initial values of $\dot{x}$ and $\dot{y}$.

(b) Taking $g = 10\,\text{m/s}^2$ and the origin at the window, find $\dot{x}$, x, $\dot{y}$ and y. Hence find the Cartesian equation of the parabolic path.

(c) Find the horizontal distance that the bullet travels.

(d) Find, correct to the nearest minute, the angle below the horizontal at which the bullet hits the ground.

26. A ball is thrown up a hill which is inclined at 30° to the horizontal. The initial velocity of the ball is $30\,\text{m/s}$ at an angle of 45° above the horizontal. Suppose that the ball is thrown from the origin O at the foot of the hill, and it lands at a point P on the side of the hill.

(a) Starting with $\ddot{x} = 0$ and $\ddot{y} = -10$, show that the parabolic path of the ball has parametric equations $x = 15\sqrt{2}\,t$ and $y = 15\sqrt{2}\,t - 5t^2$.

(b) Find the Cartesian equation of the parabolic path.

(c) Hence show that $OP = 60\left(\sqrt{3} - 1\right)$ metres.

27. A cannon can fire a shell with an initial speed V and a variable angle of elevation α. Assume that t seconds after being fired, the horizontal and vertical displacements x and y of the shell are given by $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$.

(a) Show that the Cartesian equation of the shell's path may be written as

$$gx^2\tan^2\alpha - 2xV^2\tan\alpha + (2yV^2 + gx^2) = 0.$$

(b) Suppose that $V = 200\,\text{m/s}$, $g = 10\,\text{m/s}^2$ and the shell hits a target positioned 3 km horizontally and 0·5 km vertically from the cannon. Show that $\tan\alpha = \dfrac{4 \pm \sqrt{3}}{3}$, and hence find the two possible values of α, correct to the nearest minute.

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28. A small paintball is fired from the origin with velocity 14 m/s towards an 8 m high wall. The origin is at ground level and is 10 m from the base of the wall. Suppose that the paintball was fired at an angle of θ above the horizontal.

(a) Starting with $\ddot{x} = 0$ and $\ddot{y} = -9{\cdot}8$, show that $x = 14t\cos\theta$ and $y = 14t\sin\theta - 4{\cdot}9t^2$.

(b) Show that the trajectory of the paintball has Cartesian equation

$$y = mx - \left(\frac{m^2+1}{40}\right)x^2, \qquad \text{where } m = \tan\theta.$$

(c) If the paintball hits the wall at height h metres, show that $m = 2 \pm \sqrt{3 - 0{\cdot}4h}$ and hence determine the maximum possible value of h.

(d) Suppose that there is a large hole in the wall, the bottom of which is 3·9 metres above the ground and the top of which is 5·9 metres above the ground. Determine the values of m for which the paintball will pass through the hole.

29. A particle of mass m is thrown from the top, O, of a tall building with initial velocity u at an angle of α above the horizontal. The particle experiences the effect of gravity as well as a resistance proportional to its velocity in both the horizontal and vertical directions. The equations of motion are given by $\ddot{x} = -k\dot{x}$ and $\ddot{y} = -k\dot{y} - g$, where k is a constant and g is the acceleration due to gravity.

(a) Show that $\dot{x} = ue^{-kt}\cos\alpha$.

(b) Show, by differentiating, that $\dot{y} = \dfrac{1}{k}\big((ku\sin\alpha + g)\,e^{-kt} - g\big)$ satisfies the vertical equation of motion.

(c) Find the value of t at which the particle reaches its maximum height.

(d) What is the limiting value of the horizontal displacement?

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Answers to Exercises

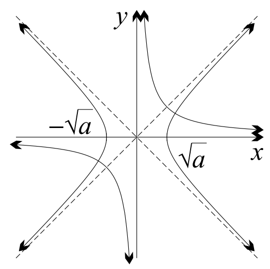
Chapter One

Exercise **1A** (Page 8)

1(a) -1 **(b)** 1 **(c)** $-i$ **(d)** i
(e) i **(f)** -1 **(g)** 1 **(h)** 0
2(a) $-2i$ **(b)** $3-i$ **(c)** $1+i$ **(d)** $5+3i$ **(e)** $-3-2i$
3(a) $12-2i$ **(b)** $-6+2i$ **(c)** $1+5i$ **(d)** $7-11i$
4(a) $-5+4i$ **(b)** $5+5i$ **(c)** $14+5i$ **(d)** $-26+82i$
(e) $24+10i$ **(f)** $-5-12i$ **(g)** $2+11i$ **(h)** -4
(i) $28-96i$
5(a) 5 **(b)** 17 **(c)** 29 **(d)** 65
6(a) $-i$ **(b)** $1-2i$ **(c)** $3+2i$ **(d)** $1-2i$ **(e)** $-1+3i$
(f) $-\frac{1}{5}+\frac{3}{5}i$
7(a) $-2-i$ **(b)** $4-3i$ **(c)** $3+7i$ **(d)** 3 **(e)** $-3+4i$
8(a) $6+2i$ **(b)** 18 **(c)** $19-22i$ **(d)** $8-i$ **(e)** $1+2i$
9(a) $22+19i$ **(b)** $6+15i$ **(c)** $4-2i$ **(d)** $2-3i$
(e) 6
10(a) $x=3$ and $y=-2$ **(b)** $x=2$ and $y=-1$
(c) $x=6$ and $y=2$ **(d)** $x=\frac{14}{5}$ and $y=\frac{3}{5}$
(e) $x=\frac{35}{2}$ and $y=-\frac{39}{2}$
11(a) $\frac{9}{10}-\frac{13}{10}i$ **(b)** 1 **(c)** $-\frac{8}{29}$ **(d)** $-4-\frac{5}{2}i$
16(a) $\frac{x-iy}{x^2+y^2}$ **(b)** $\frac{x^2-y^2-2ixy}{(x^2+y^2)^2}$ **(c)** $\frac{x^2+y^2-1+2iy}{(x+1)^2+y^2}$

Exercise **1B** (Page 16)

1(a) $z=\pm 3i$ **(b)** $z=2\pm 4i$ **(c)** $z=-1\pm 2i$
(d) $z=3\pm i$ **(e)** $z=\frac{1}{2}\pm\frac{1}{4}i$ **(f)** $z=-\frac{3}{2}\pm 2i$
2(a) $(z-6i)(z+6i)$ **(b)** $(z-2\sqrt{2}\,i)(z+2\sqrt{2}\,i)$
(c) $(z-1-3i)(z-1+3i)$ **(d)** $(z+2-i)(z+2+i)$
(e) $(z-3+\sqrt{5}\,i)(z-3-\sqrt{5}\,i)$
(f) $(z+\frac{1}{2}-\frac{\sqrt{3}}{2}\,i)(z+\frac{1}{2}+\frac{\sqrt{3}}{2}\,i)$
3(a) $z^2+2=0$ **(b)** $z^2-2z+2=0$
(c) $z^2+2z+5=0$ **(d)** $z^2-4z+7=0$
4(a) $\pm(1+i)$ **(b)** $\pm(2+i)$ **(c)** $\pm(-1+3i)$
(d) $\pm(6+i)$ **(e)** $\pm(2+3i)$ **(f)** $\pm(5-i)$ **(g)** $\pm(1-4i)$
(h) $\pm(5-4i)$
5(a) $\pm(1-2i)$ **(b)** $z=2-i$ or $1+i$
6(a) $\pm(1+3i)$ **(b)** $z=4+i$ or $3-2i$
7(a) $z=1-i$ or i **(b)** $z=-3+2i$ or $-2i$ **(c)** $z=4+i$ or $2-i$ **(d)** $z=-2+i$ or $\frac{1}{2}(3-i)$ **(e)** $z=-5+i$ or $3-2i$ **(f)** $z=3+i$ or $-1-3i$
8(a) $w=-1$ **(b)** $a=-6$ and $b=13$
(c) $k=8-i$ and the other root is $2+3i$.
9 $z=\pm(2+i)$
10(a) $\cos\theta+i\sin\theta$ or $\cos\theta-i\sin\theta$
11(a) $z=-1$ or $\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ **(b)** $z=i$ or $\pm\frac{\sqrt{3}}{2}-\frac{1}{2}i$
12(a) $z=\omega$ satisfies the equation. **(c)** They are complex conjugates.
13(a) $\overline{\alpha}$
14(a)(i)



15(a) $\pm\frac{1}{\sqrt{2}}(1-i)$ **(b)** $\pm\sqrt{2}(1+2i)$ **(c)** $\pm(\sqrt{3}+i)$
(d) $\pm\sqrt{2}(3-2i)$
(e) $\pm\left(\sqrt{\sqrt{5}+1}-i\sqrt{\sqrt{5}-1}\right)$
16(a) $-2-i\pm\left(\sqrt{\sqrt{2}+1}+i\sqrt{\sqrt{2}-1}\right)$
(b) $1+i\pm\left(\sqrt{\sqrt{5}-1}-i\sqrt{\sqrt{5}+1}\right)$
(c) $-1+i\sqrt{3}\pm\left(\sqrt{2}-i\sqrt{6}\right)$
(d) $\frac{1}{2}\left(-1+i\pm\left(\sqrt{\sqrt{13}+2}-i\sqrt{\sqrt{13}-2}\right)\right)$
19 The term $b/|b|$ is the sign of b.
It is 1 when $b>0$, and -1 when $b<0$.

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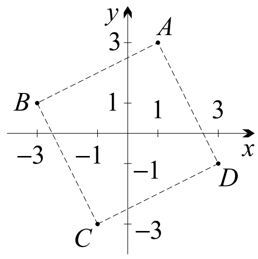
Exercise **1C** (Page 21)

1(a) $(2, 0)$ **(b)** $(0, 1)$ **(c)** $(-3, 5)$ **(d)** $(2, -2)$
(e) $(-5, -5)$ **(f)** $(-1, 2)$

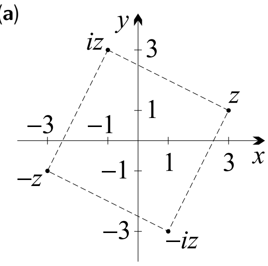
2(a) $-3 + 0i = -3$ **(b)** $0 + 3i = 3i$ **(c)** $7 - 5i$
(d) $a + bi$

3(a)

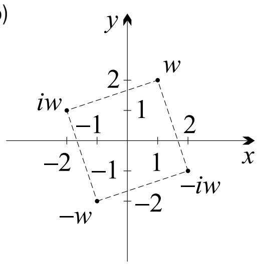


(b) A square. **(c)** An anticlockwise rotation of $90°$ about the origin.

4(a)

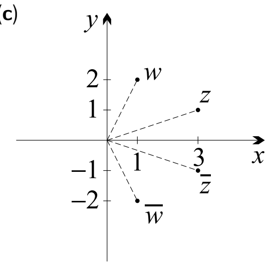


(b)

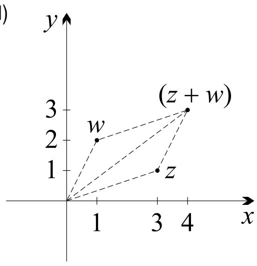


In (a) and (b) the points form a square.

(c)



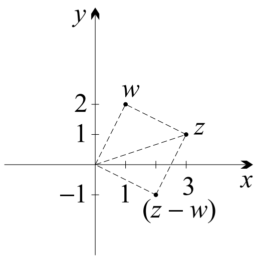
(d)



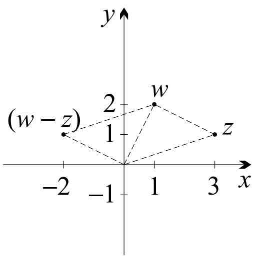
Conjugate pairs are reflections in the real axis.

With O the points form a parallelogram.

(e)

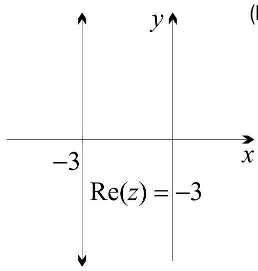


(f)

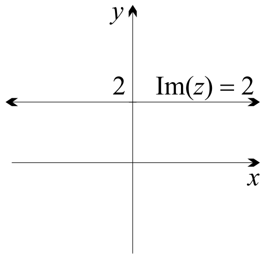


Again, in (e) and (f) the points are the vertices of a parallelogram.

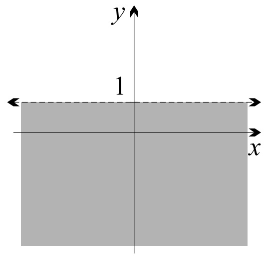
5(a)



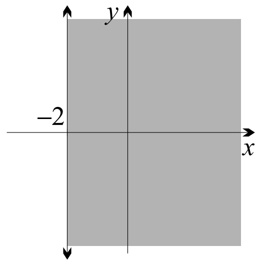
(b)



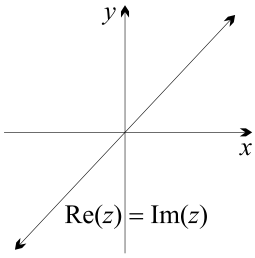
(c)



(d)



(e)



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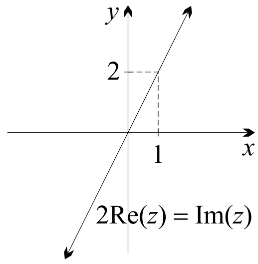

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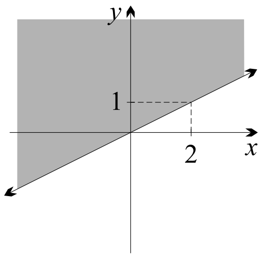

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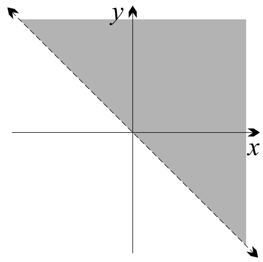

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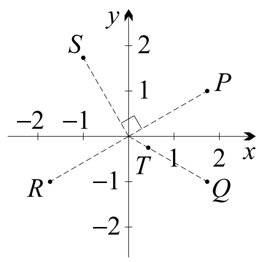

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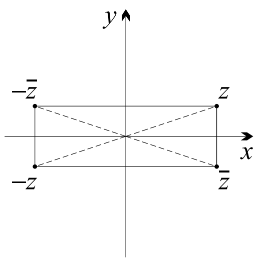


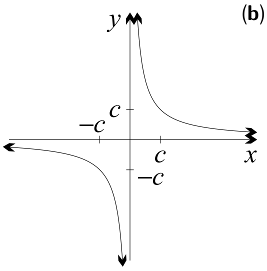
10(c) right-isosceles

11 It is the circle centre $(0, -1)$ with radius 1, omitting the origin.

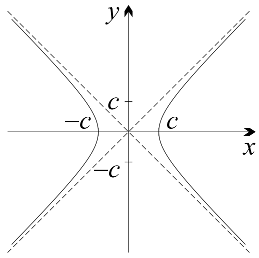
12 It is the circle centre $(3, 0)$ with radius 3, omitting the origin.

14 It is a parabola with focus the origin and directrix $x = 1$.

15(a)



(b)



Exercise 1D (Page 28)

1(a) 3 **(b)** 5 **(c)** $\sqrt{2}$ **(d)** 2 **(e)** 5 **(f)** 17

2(a) π **(b)** $\frac{\pi}{2}$ **(c)** $-\frac{\pi}{4}$ **(d)** $\frac{\pi}{3}$ **(e)** $\frac{3\pi}{4}$ **(f)** $-\frac{5\pi}{6}$

3(a) $2\operatorname{cis}\frac{\pi}{2}$ **(b)** $4\operatorname{cis}\pi$ **(c)** $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ **(d)** $2\operatorname{cis}\left(-\frac{\pi}{6}\right)$
(e) $2\operatorname{cis}\frac{2\pi}{3}$ **(f)** $\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

4(a) $5\operatorname{cis}(0{\cdot}93)$ **(b)** $13\operatorname{cis}(-0{\cdot}39)$
(c) $\sqrt{5}\operatorname{cis}(2{\cdot}68)$ **(d)** $\sqrt{10}\operatorname{cis}(-1{\cdot}89)$

5(a) 3 **(b)** $-5i$ **(c)** $2\sqrt{2}+2\sqrt{2}\,i$ **(d)** $3\sqrt{3}-3i$
(e) $-\sqrt{2}+\sqrt{2}\,i$ **(f)** $-1-\sqrt{3}\,i$

6(a) $\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **(b)** $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ **(c)** $\sqrt{2}\operatorname{cis}\frac{3\pi}{4}$
(d) $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ **(e)** $2\operatorname{cis}\left(-\frac{\pi}{2}\right)$ **(f)** $\frac{1}{\sqrt{2}}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

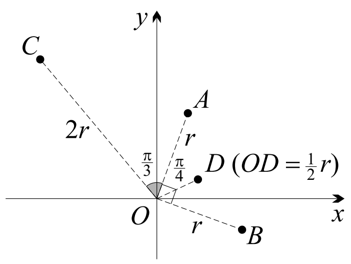
7(a) $10\operatorname{cis}\frac{\pi}{3}$ **(b)** $9\operatorname{cis}3\theta$ **(c)** $2\operatorname{cis}\frac{\pi}{3}$ **(d)** $\frac{3}{2}\operatorname{cis}\alpha$
(e) $16\operatorname{cis}\frac{2\pi}{5}$ **(f)** $8\operatorname{cis}\frac{6\pi}{7}$

8(a) $2\sqrt{2}$ **(b)** $3\sqrt{2}$ **(c)** 6 **(d)** $4\sqrt{3}$ **(e)** 5 **(f)** $\sqrt{5}$

9(a) $\frac{\pi}{4}$ **(b)** $-\frac{3\pi}{4}$ **(c)** $-\frac{\pi}{3}$ **(d)** $\frac{\pi}{6}$ **(e)** $0{\cdot}93\ (53^\circ)$
(f) $-2{\cdot}03\ (-117^\circ)$

10(a) i **(b)** -1 **(c)** $\frac{1}{2}(1+i\sqrt{3})$ **(d)** $\dfrac{1}{\sqrt{2}}(-1+i)$
(e) $\frac{1}{2}(-\sqrt{3}+i)$ **(f)** $-i$ **(g)** $\frac{1}{\sqrt{2}}(1-i)$
(h) $-\frac{1}{2}(1+i\sqrt{3})$

11



13(a) $z_1 = 2\operatorname{cis}\frac{\pi}{6}$ and $z_2 = 4\operatorname{cis}\frac{\pi}{4}$ **(b)** $z_1z_2 = 8\operatorname{cis}\frac{5\pi}{12}$ and $\frac{z_2}{z_1} = 2\operatorname{cis}\frac{\pi}{12}$

14 $z_1 = 2\operatorname{cis}\frac{5\pi}{6}$, $z_2 = \sqrt{2}\operatorname{cis}(-\frac{3\pi}{4})$,
$z_1z_2 = 2\sqrt{2}\operatorname{cis}\frac{\pi}{12}$ and $\frac{z_2}{z_1} = \frac{\sqrt{2}}{2}\operatorname{cis}\frac{5\pi}{12}$

15(a) $\frac{1}{2}\Big((\sqrt{3}+1)+i(\sqrt{3}-1)\Big)$ **(b)** $\sqrt{2}\operatorname{cis}\frac{\pi}{12}$
(c) $\frac{1}{2\sqrt{2}}(\sqrt{3}+1)$

16(a) $\sqrt{2}$ **(b)** $\frac{\pi}{4}$ **(c)** $1+i$

24 $z+w = 2\cos\left(\frac{\theta-\phi}{2}\right)\operatorname{cis}\left(\frac{\theta+\phi}{2}\right)$

25(a) When $\operatorname{Im}(z)=0$.

Exercise 1E (Page 34)

1(a) $7+4i$ **(b)** $-3+2i$ **(c)** $3-2i$

2(a) $-3+4i$ **(b)** $1+7i$ **(c)** $-4-3i$ **(d)** $-7+i$

3 $-3+6i$

4(a) B represents $1+3i$, C represents $-1+2i$
(b) $-\sqrt{2}+2\sqrt{2}\,i$

5(a) $4+3i$ **(b)** $-3+4i$ **(c)** $2+7i$

6(a) $-5+12i$ **(b)** $-3-4i$

8 E represents w_2-w_1, F represents $i(w_2-w_1)$, C represents $w_2+i(w_2-w_1)$ and D represents $w_1+i(w_2-w_1)$.

9(a) Vectors BA and BC represent z_1-z_2 and z_3-z_2 respectively, and BA is the anticlockwise rotation of BC through 90° about B. So $z_1-z_2 = i(z_3-z_2)$. Squaring both sides gives the result.
(b) $z_1-z_2+z_3$

10(a) $2\omega i$ **(b)** $\frac{1}{2}\omega(1+2i)$

11 -2 and $1-\sqrt{3}\,i$

12(a) $w=-4+3i$ or $4-3i$ **(b)** $w=-1+7i$ or $7+i$ **(c)** $w=\frac{1}{2}(7+i)$ or $\frac{1}{2}(-1+7i)$

13 $-2+2i$, $12i$, 4

18(a) $z_1 = 2\operatorname{cis}\frac{\pi}{2}$, $z_2 = 2\operatorname{cis}\frac{\pi}{3}$ **(c)(i)** $\frac{5\pi}{12}$ **(ii)** $\frac{11\pi}{12}$

19(c) The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

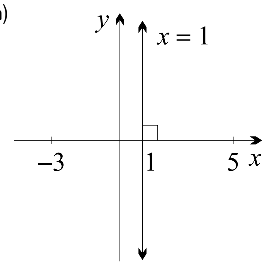
20(c) parallelogram **(d)** $\arg\frac{w}{z}=\frac{\pi}{2}$, so $\frac{w}{z}$ is purely imaginary.

23 Use the converse of the opposite angles of a cyclic quadrilateral.

24 Take the argument of the fraction in the hint. The result is $\arg\left(\dfrac{z_3}{z_2}\right)-\arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)$. These are the angles at 0 and z_1 which, by the angles in the same segment theorem, are equal. Finally, use the result of Question 24.

Exercise 1F (Page 42)

1(a)



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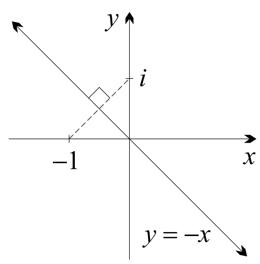
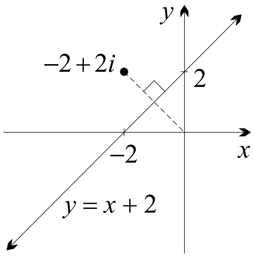


(c)



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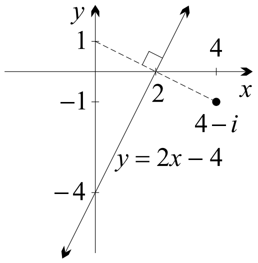
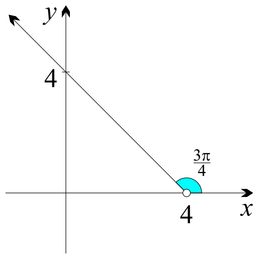


2(a)



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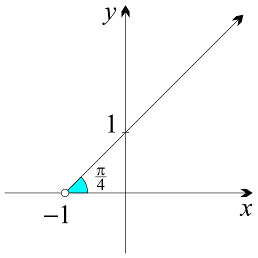

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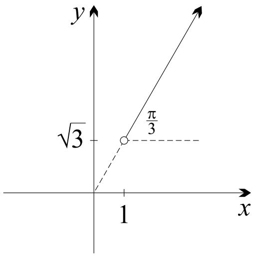

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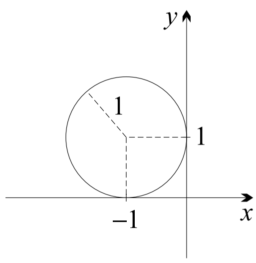

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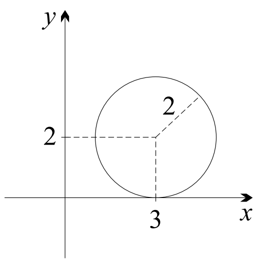

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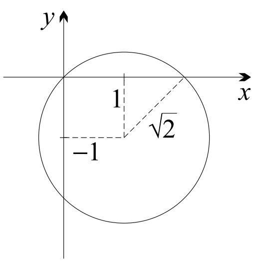
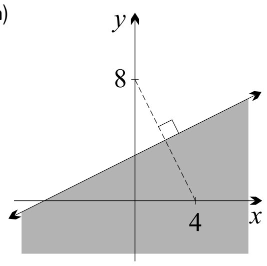


4(a)



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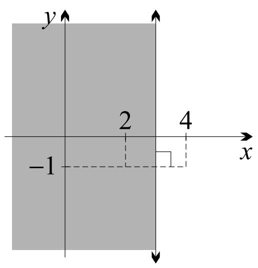

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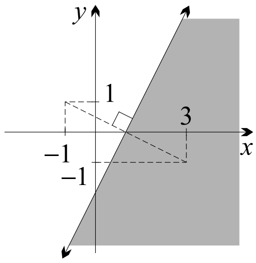

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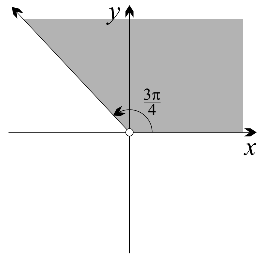

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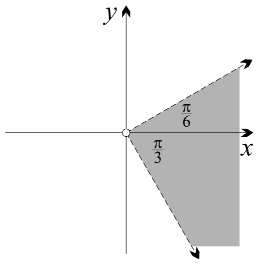

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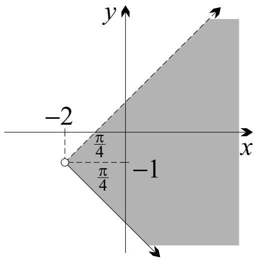

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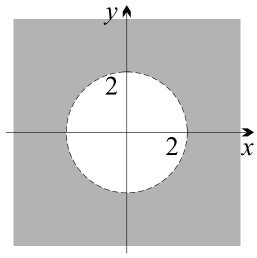

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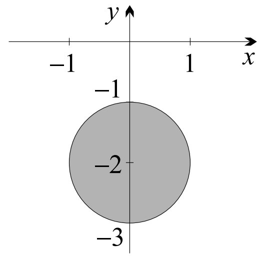
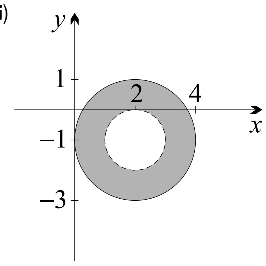
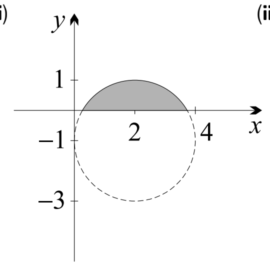


(i)



5(a)(i)



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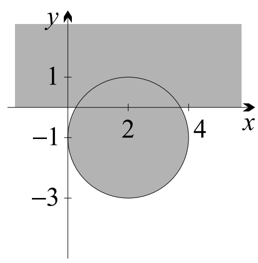

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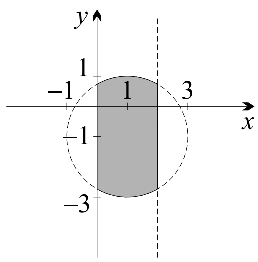

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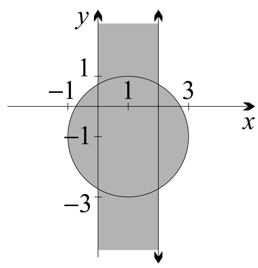

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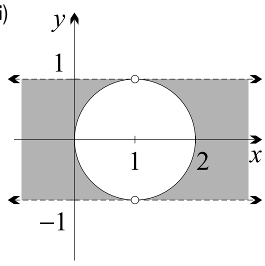

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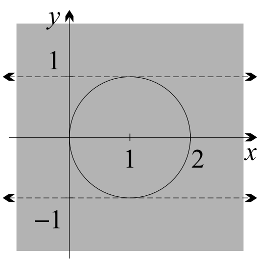
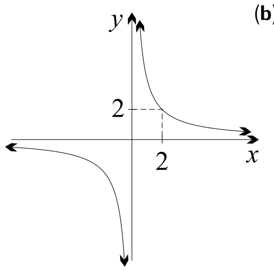


6(a)



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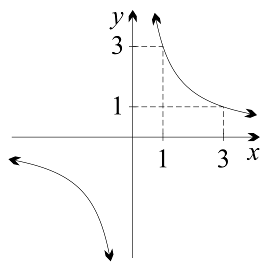

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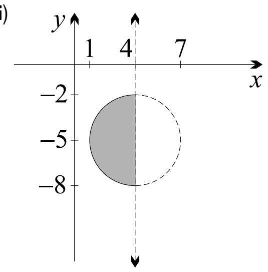

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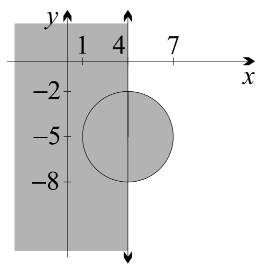
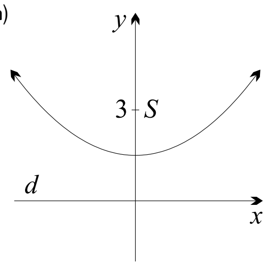


7(a)



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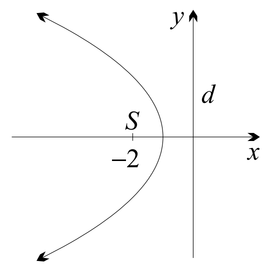

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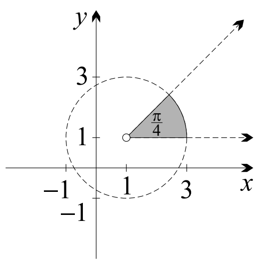

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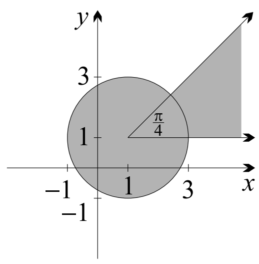

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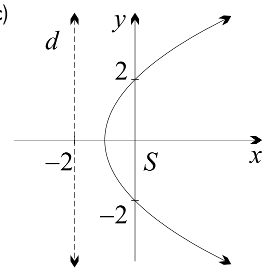

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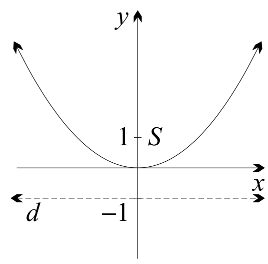

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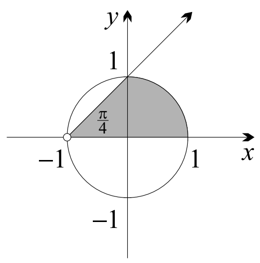

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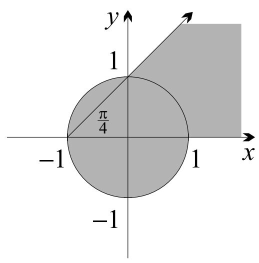
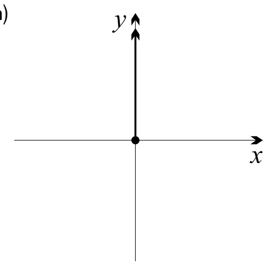


8(a)



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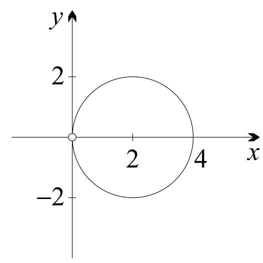

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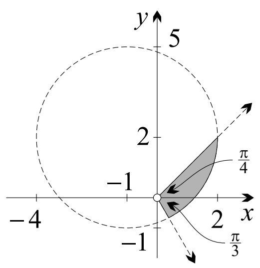

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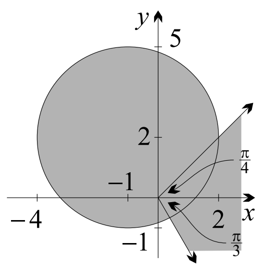

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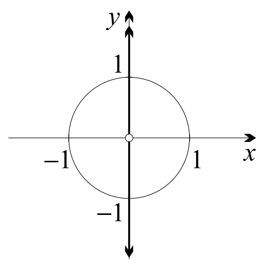

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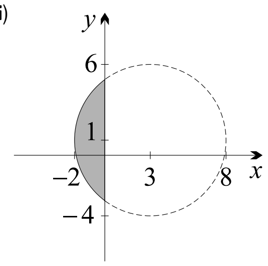

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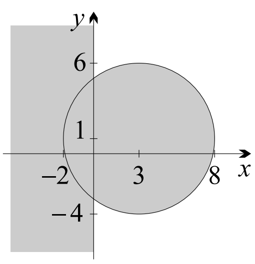
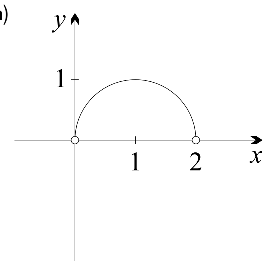


9(a)



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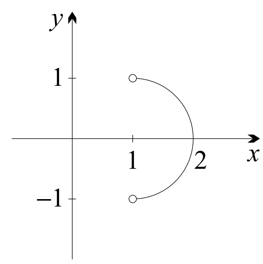
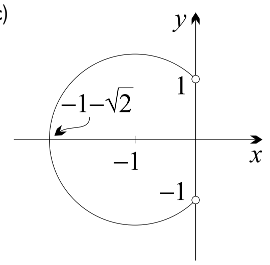


(c)


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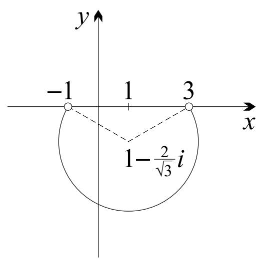

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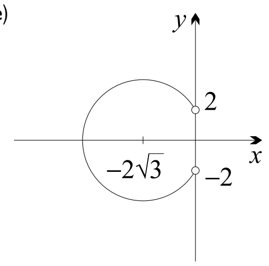

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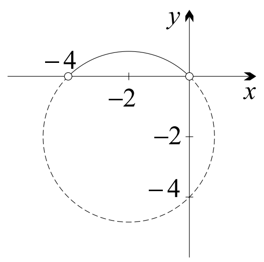
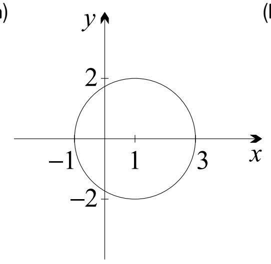
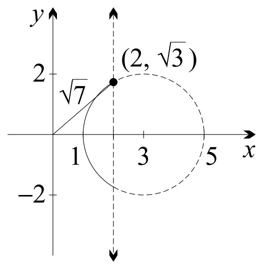


10(b) $\sqrt{3}\,\text{cis}\,\frac{\pi}{3} = \frac{\sqrt{3}}{2}(1 + i\sqrt{3}\,)$

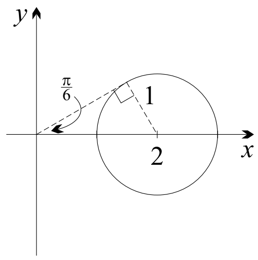
11(a) $\arg(z + 3) = \frac{\pi}{3}$ **(b)** $|z| = \frac{3\sqrt{3}}{2}$, $\arg z = \frac{5\pi}{6}$

(c) $-\frac{9}{4} + \frac{3\sqrt{3}}{4}i$

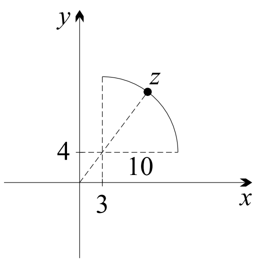
12(a)


(b)


3 and 1

13(a)(i)


(b) This is simply part (a) shifted left by 2.

14(a)


(b) 15 **(c)** $9 + 12i$

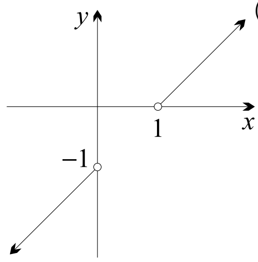
15(b)(i) $|z + 2| = 2$, centre -2, radius 2

(ii) $|z - (1 + i)| = 1$, centre $1 + i$, radius 1

(iii) $|z - 1| = 1$, centre 1, radius 1

16(a) The line through 1 and i, omitting i.

(b) The circle with diameter joining 1 and i, omitting these two points.

17(a)


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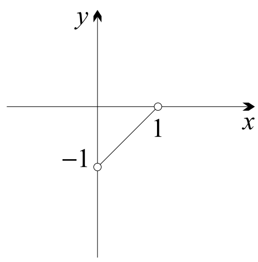
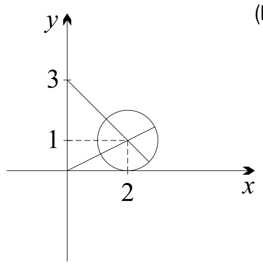


18(a)


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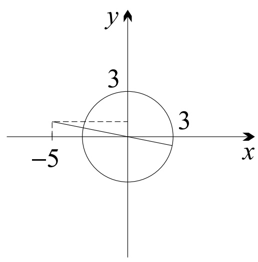

(i) $\sqrt{5}+1$ and $\sqrt{5}-1$ $\sqrt{26}+3$ and $\sqrt{26}-3$

(ii) $2\sqrt{2}+1$ and $2\sqrt{2}-1$

(c)(i) $\Big||z_0| - r\Big| \le |z| \le |z_0| + r$

(ii) $\Big||z_0 - z_1| - r\Big| \le |z - z_1| \le |z_0 - z_1| + r$

19(a) straight line external to z_1 and z_2 **(b)** major arc **(c)** semi-circle **(d)** minor arc **(e)** interval between z_1 and z_2

20 It is the circle with the interval joining z_1 and z_2 as diameter.

21(b) The graph is the perpendicular bisector of the line joining z_1 and z_2.

Exercise **1G** (Page 51)

1(a) $(x - 2)(x + 1 - \sqrt{3}\,)(x + 1 + \sqrt{3}\,)$

(b) $(x - 1)(x + 2 - \sqrt{2}\,)(x + 2 + \sqrt{2}\,)$

(c) $(x - 1)(x - 1 - \sqrt{5}\,)(x - 1 + \sqrt{5}\,)$

2(a) The coefficients of $P(x)$ are real, so complex zeroes occur in conjugate pairs. **(b)** 6

3(a) $1 + 2i$; the coefficients of $P(x)$ are real, so complex zeroes occur in conjugate pairs.

(c) $P(x) = (x + 2)(x^2 - 2x + 5)$

4(a) $3i$; the coefficients of $P(z)$ are real, so complex zeroes occur in conjugate pairs. **(b)** $z^2 + 9$

(c) $P(z) = (2z + 3)(z^2 + 9)$

5(b) 0; the coefficients of $P(z)$ are real, so complex zeroes occur in conjugate pairs.

(c)(i) $P(z) = (2z - 1)(z - 3 - i)(z - 3 + i)$

(ii) $P(z) = (2z - 1)(z^2 - 6z + 10)$

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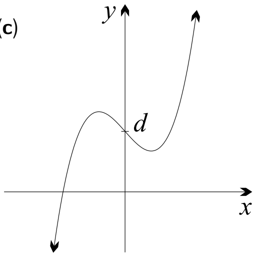
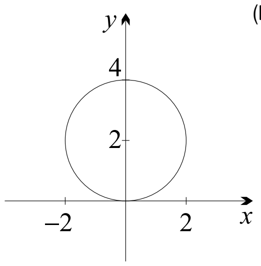
6(a) The coefficients of $Q(x)$ are real, so complex zeroes occur in conjugate pairs. **(b)** $3+\sqrt{5}$, $3-\sqrt{5}$
(c)(i) $(x-2i)(x+2i)(x-3-\sqrt{5}\,)(x-3+\sqrt{5}\,)$
(ii) $(x^2+4)(x-3-\sqrt{5}\,)(x-3+\sqrt{5}\,)$
(iii) $(x^2+4)(x^2-6x+4)$
7(a) $x=1\pm 3i$, 3 or -2 **(b)** $x=1\pm i$ or $2\pm i$
8(b) $x=3,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i$
9(a) $a=3$ **(b)** $b=1$
(c) $(z^2-6z+10)(z^2-6z+13)$
10(b) $k=3$
11(b) $m=7$, $n=-4$
12(a) $-7-4i$ **(b)(i)** $-7+4i$ **(ii)** $2x-7$
13(b) $P(z)=\frac{1}{2}(z^4-2)(2z^4-1)$ so one root is $z=\sqrt[4]{2}$.
(c) $\sqrt[4]{2},\ \frac{1}{\sqrt[4]{2}},\ -\sqrt[4]{2},\ -\frac{1}{\sqrt[4]{2}}$, and $i\sqrt[4]{2},\ \frac{1}{\sqrt[4]{2}}\,i,\ -i\sqrt[4]{2},\ -\frac{1}{\sqrt[4]{2}}\,i$
14(a) $P(x)$ has minimum value B, when $x=0$. Since $B>0$, it follows that $P(x)>0$ for all real values of x. **(b)** $-ic$, $-id$; the coefficients of $P(x)$ are real, so complex zeroes occur in conjugate pairs.
15(a) They form a conjugate pair, since $P(x)$ has real coefficients. **(c)**



16(a) The minimum stationary point is at $x=1$. $f(1)=k-2>0$. Hence the graph of $f(x)$ has only one x-intercept which lies to the left of the maximum stationary point at $x=-1$.
(b) $f(x)$ has real coefficients **(d)** -14, $7\pm 12i$
17 HINT: consider $P(x)-P'(x)$
18(b) $-1+2i$ is a double zero of $P(z)$ **(c)** The coefficients of $P(z)$ are real and $-1+2i$ counts as two of the zeroes of $P(z)$, so its conjugate $-1-2i$ must also count as two zeroes.
(d) $P(z)=(z+1-2i)^2(z+1+2i)^2=(z^2+2z+5)^2$
22(b) $(z-\alpha)^2(z-\overline{\alpha})^2$ is a factor. **(c)** HINT: Begin by writing: $P(z)=\big(z-2\,\mathrm{Re}(\alpha)+|\alpha|^2\big)^2\times Q(z)$

Review Exercise **1H** (Page 53)

1(a) $1-5i$ **(b)** $18-26i$ **(c)** $5+2i$
2(a) $(z+10i)(z-10i)$ **(b)** $(z+5-3i)(z+5+3i)$
3(a) $z=4+3i$ or $4-3i$ **(b)** $z=-\frac{1}{2}+\frac{3}{4}i$ or $-\frac{1}{2}-\frac{3}{4}i$
4(a) $\pm(3-2i)$ **(b)** $\pm(3+\sqrt{2}\,i)$
5(a) $z=2+i$ or $3-i$ **(b)** $z=2+3i$ or $4-2i$
6 $\overline{3i}=-3i$ is also a zero, so $(z-3i)(z+3i)=z^2+9$ is a factor.
7(a) The coefficients of $P(z)$ are real. **(b)** 4
(c) $P(z)=(z-4)(z^2-4z+29)$
8(a) $\sqrt{2}\,\mathrm{cis}(-\frac{\pi}{4})$ **(b)** $6\,\mathrm{cis}\,\frac{5\pi}{6}$
9(a) $4i$ **(b)** $-\sqrt{3}-\sqrt{3}\,i$
10(a) $6\,\mathrm{cis}\,\frac{5\pi}{6}$ **(b)** $2\,\mathrm{cis}\,5\theta$ **(c)** $9\,\mathrm{cis}\,6\alpha$
11(a)



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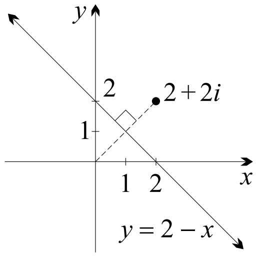

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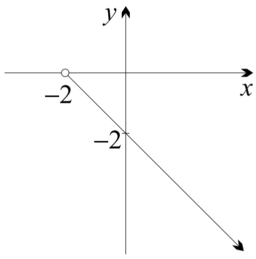

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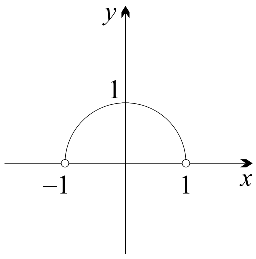

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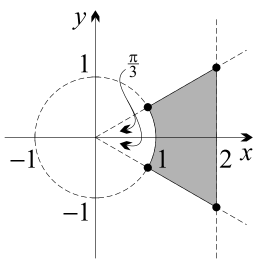
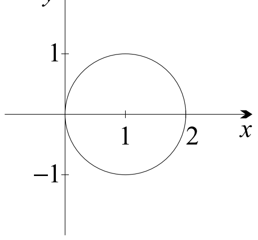
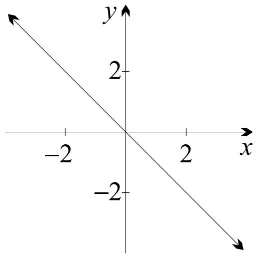


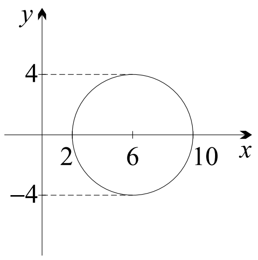
13(a) $\frac{1}{2}(\sqrt{3}-1)+\frac{1}{2}(\sqrt{3}+1)i$ **(b)** $z=2\,\mathrm{cis}\,\frac{2\pi}{3}$ and $w=\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}$ **(c)** $\sqrt{2}\,\mathrm{cis}\,\frac{5\pi}{12}$
14(a)



(b)



(c)



15(a) $3+5i$ **(b)** $-5+3i$ **(c)** $-1+i$
16(a) $1-5i,\ 7+3i$ **(b)** $3+6i,\ -3-2i$
18(a) $\frac{7\pi}{24}$ **(b)** $\frac{19\pi}{24}$
21 Use similar triangles.
22(b) $\sqrt{5}+1$

Chapter Two

Exercise **2A** (Page 64)

1(a) equality **(b)** implication **(c)** equivalence **(d)** for all **(e)** there exists
2(a) If a triangle has two equal angles, then it has two equal sides. True. **(b)** If the square of a number is odd, then the number is odd. True. **(c)** If I have four legs, then I am a horse. False. **(d)** If a number is even, then it ends with the digit 6. False. **(e)** Every rhombus is a square. False. **(f)** If $n \geq 0$, then $\sqrt{n} \in \mathbf{R}$. True.
3(a) True **(b)** False **(c)** False **(d)** False **(e)** True **(f)** True
4(a) Not all cars are red. (Alternatively, some cars are not red.) **(b)** $a \leq b$ or $a \not> b$ **(c)** Hillary does not like steak or she does not like pizza. **(d)** Bill and Dave are both wrong. **(e)** I live in Tasmania and I don't live in Australia. **(f)** Nikhil doesn't study and he passes.
(g) $x < -3$ or $x > 8$ **(h)** $-5 \leq x < 0$
5(a) If my plants do not grow, then I haven't watered them. **(b)** If you live in Melbourne, then you live in Australia. **(c)** If a triangle does not have three equal angles, then it does not have three equal sides. **(d)** If I like motorists, then I do not like cycling. **(e)** If a number is even, then the previous number is odd. **(f)** If $\frac{1}{a} \geq \frac{1}{b}$, then $a \leq b$ or a and b are not both positive.
6(a) If a number is divisible by both 3 and 5, then it is divisible by 15. Conversely, if a number is divisible by 15, then it is divisible by both 3 and 5. **(b)** If a triangle has two equal sides, then it has two equal angles. Conversely, if a triangle has two equal angles, then it has two equal sides. **(c)** If the only divisors of the integer n, where $n > 1$, are 1 and n, then n is prime. Conversely, if n is prime, then its only divisors are 1 and n. **(d)** If a quadrilateral has a pair of opposite sides that are equal and parallel, then it is a parallelogram. Conversely, if a quadrilateral is a parallelogram, then it has a pair of opposite sides that are equal and parallel.
7(a) true **(b)** false, $3 \times (-1) < -1$ **(c)** true **(d)** false, $\frac{1}{2} > \left(\frac{1}{2}\right)^2$ **(e)** false, $|-(-1)| \neq -1$ **(f)** true

8(a) false, $2 > -3$ but $2^2 < (-3)^2$ **(b)** false, $(-3)^2 > 2^2$ but $-3 < 2$ **(c)** true **(d)** true **(e)** false, $|2 + (-1)| < |2| + |-1|$ **(f)** true

9(a) $\Rightarrow$ **(b)** $\Leftrightarrow$ **(c)** $\Leftrightarrow$ **(d)** $\Rightarrow$ **(e)** $\Leftrightarrow$ **(f)** $\Rightarrow$

10(a) false **(b)** false **(c)** true **(d)** true

11(a) If Jack does Extension 2 Mathematics then he is crazy. **(b)** Jack does Extension 2 Mathematics and he is not crazy. **(c)** If Jack does not do Extension 2 Mathematics then he is crazy. **(d)** If Jack is not crazy then he will do Extension 2 Mathematics. **(e)** If Jack does not do Extension 2 Mathematics then he is not crazy. **(f)** If Jack does Extension 2 Mathematics then he is not crazy.

12(a) For each integer there always exists a larger integer. **(b)** The sum of any positive real number and its reciprocal is greater than or equal to two.

13(a) true **(b)** false **(c)** true **(d)** true

14(a) They are both false.

(b) If $1 < 0$ then 1 is a negative number. True

15(a) yes — consider the contrapositive of (1)

(b) unknown — studying hard is a sufficient condition for passing, but it is not a necessary condition

16 If either Anna or Bryan passed, then Chris passed. So since the statement is negated, Chris failed.

17 Pender is the driver.

Exercise 2B (Page 68)

1(a) [Hint: An even number has the form $2n$, where $n \in \mathbf{Z}$.] **(b)** [Hint: An odd number has the form $2n + 1$, where $n \in \mathbf{Z}$.]

4(a) [Hint: If b is divisible by a, then $b = ka$ for some $k \in \mathbf{Z}$.]

6(a) [Hint: Let the consecutive integers be $n - 1$, n, $n + 1$ and $n + 2$.]

8 [Hint: Find a pair of simultaneous equations.]

10 [Hint: Factorise the expression, then explain why it is divisible by both 2 and 3.]

13 [Hint: Let the consecutive integers range from $n - 3$ to $n + 3$.]

14(b) No. There will always be a remainder of $\frac{n}{2}$.

15 [Hint: A 4-digit number with digits a, b, c, d has value $1000a + 100b + 10c + d$.]

16(c) $\Big[10x + y = 13m\Big] \Leftrightarrow \Big[x + 4y = 13(4m - 3x)\Big]$

18(a) A factor of n is of the form $p^c q^d$, where $c \in \{0, 1, 2, \ldots, a\}$ and $d \in \{0, 1, 2, \ldots, b\}$. So there are $a + 1$ possible values for c and $b + 1$ possible values for d. So by the multiplication principle, there are $(a + 1)(b + 1)$ possible factors of n. **(b)** 40

19 [Hint: Consider the expression $(a - c)(b - d)$.]

Exercise 2C (Page 72)

3(a) If a is even then a^2 is even.

10(a) An odd number lies between two consecutive multiples of 4. It is one more than the smaller multiple of 4 or one less than the larger multiple of 4.

16(a) If p is not prime then $\exists\ a, b \in \mathbf{Z}^+$ such that $p|ab \Rightarrow p\nmid a$ and $p \nmid b$.

Exercise 2D (Page 78)

6(b) Use part (a) three times.

8(a) Use Question 6(b) with $p = a^2$ and so on.

(b) In part (a) replace a^2 with ab and so on.

(c) Use parts (a) and (b).

9(c) Use part (b).

(d) Use part (c) with $a^3 = x$ and so on.

11(a) $\frac{a^4}{b^4} - \frac{4a^2}{b^2} + 6 - \frac{4b^2}{a^2} + \frac{b^4}{a^4}$

12(a) Use the given AM/GM inequality twice on the RHS.

13(d)(i) The triangle inequality: the length of any side is less than (or equal to if the points are collinear) the sum of the other two sides.

(ii) Use part (i) three times then add.

14(b) Expand the LHS and use part (a).

(c)(i) Begin with LHS – RHS.

16(a) Begin with LHS – RHS.

18(c) When $z = kw$, with $k > 0$, or when either $z = 0$ or $w = 0$.

Exercise 2E (Page 83)

8(a) $x > 1 + \sqrt{2}$ or $x < 1 - \sqrt{2}$

25(a) Ben may pair $(n - 1)$ other players. In each case there are $(n - 2)$ remaining players. In each case, the number of derangements for those players is D_{n-2}. Hence multiply to get $(n - 1)D_{n-2}$.

(d) $D_1 = 0$, $D_2 = 1$

Exercise 2F (Page 88)

1(a) $|\triangle OAB| = \frac{1}{4}\text{ cm}^2$, 3 cm^2

(b)(ii) $|\triangle OGH| = (2-\sqrt{3})\text{ cm}^2$,
dodecagon area $= 12(2-\sqrt{3})\text{ cm}^2$

2(a) $\frac{\pi}{36}(4+\sqrt{3})$

3(a) $\frac{3}{4}$ and $\frac{2}{3}$ square units.

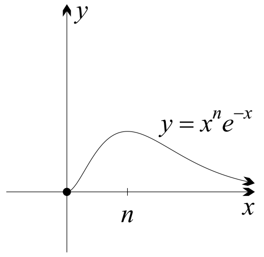
4(a) $(1-e^{-1})$ sq. units **(b)** $\frac{1}{2}(1+e^{-1})$ sq. units

(c) $e^{-\frac{1}{2}}$ sq. units

6(b) It diverges to infinity.

7(c) $\left(\frac{a+2b}{3}, \frac{\ln a + 2\ln b}{3}\right)$

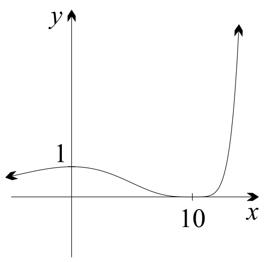
8(b)



10(a) $(0,1)$ is a maximum turning point, $(10,0)$ is a minimum turning point.

(b) $y \to \infty$ as $x \to \infty$, and $y \to 0$ as $x \to -\infty$.

(c)



11(c) When $z = kw$, with $k < 0$, or when either $z = 0$ or $w = 0$.

12(a)(i) $6^6 = 46\,656$, $3 \times 5^6 = 46\,875$

(ii) $5 \times 6^6 = 233\,280$, $2 \times 7^6 = 235\,298$

13(f) $0{\cdot}693$

16(c) $n = 9$

18(b) In part (a), put $f(x) = x^{-2}$, $a = (n-1)$ and $b = n$.

20(a) Put $f(x) = x - 1 - \log x$.
Show that $y = f(x)$ is concave up for all $x > 0$.
Show that $f(x)$ has a global minimum at $x = 1$.

(c) Put $p_r = \frac{x_r}{x_1+x_2+\ldots+x_n}$ so that $\sum_{r=1}^{n} p_r = 1$.
Also $np_r = \frac{x_r}{\mu}$, where $\mu = \frac{x_1+x_2+\ldots+x_n}{n}$.

Review Exercise 2G (Page 93)

1(a) If the opposite angles of a quadrilateral are supplementary, then it is cyclic. True. **(b)** If two numbers have an even sum, then they are both odd. False. **(c)** Every parallelogram is a rhombus. False.

2(a) Not all mathematicians are intelligent.

(b) Suzie does not like Physics or she does not like Chemistry. **(c)** I am on vacation and I am working.

3(a) If I don't have two wheels, then I am not a bicycle. **(b)** If the last digit of a number is 6, then the number is even. **(c)** If a quadrilateral does not have four equal sides, then it is not a square.

4(a) If a number is divisible by 2, then it is even. Conversely, if a number is even, then it is divisible by 2. **(b)** If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. Conversely, the diagonals of a parallelogram bisect each other. **(c)** If $\exists\, c \in \mathbf{Z}$ such that $a = bc$, then a is divisible by b. Conversely, if a is divisible by b, then $\exists\, c \in \mathbf{Z}$ such that $a = bc$.

16(b) $\frac{1}{2}$

23(c) From part (b), $\sqrt[n]{n}$ is not an integer, so it is not rational.

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Chapter Three

Exercise 3A (Page 100)

1(a) $\operatorname{cis} 5\theta$ **(b)** $\operatorname{cis}(-3\theta)$ **(c)** $\operatorname{cis} 8\theta$ **(d)** $\operatorname{cis}(-\theta)$
(e) $\operatorname{cis} 7\theta$ **(f)** $\operatorname{cis}(-6\theta)$
2(a) $\operatorname{cis} 7\theta$ **(b)** $\operatorname{cis}(-5\theta)$
3(a) -1 **(b)** $-i$ **(c)** $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ **(d)** $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$
(e) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ **(f)** $\frac{1}{2} + \frac{\sqrt{3}}{2}i$
4(a) $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ **(b)** $256 + 256i$
5(a) $2\operatorname{cis}\frac{\pi}{3}$ **(b)** $1024 - 1024\sqrt{3}\,i$
6(a) $2,\ \frac{5\pi}{6}$
7(a) $2\operatorname{cis}\left(-\frac{\pi}{6}\right)$ **(b)** $128\operatorname{cis}\frac{5\pi}{6}$ **(c)** $-64\sqrt{3} + 64i$
8(a) $2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$ **(b)** $32\operatorname{cis}\frac{2\pi}{3}$ **(c)** $-16 + 16\sqrt{3}\,i$
9(a) $2\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **(b)** $2^{22}i$
12(a)(i) 6 **(ii)** 3 **(b)** $-64,\ 8i$
13(b) $n = 2,\ 6,\ 10,\ \dots$
15(b) -2^{2n}

Exercise 3B (Page 102)

5(b) $\frac{8}{15}$
8(c) $b = 2,\ c = -1$
(d) No, since $\sin\frac{\pi}{10} = \sin\frac{9\pi}{10}$ and $\sin\frac{13\pi}{10} = \sin\frac{17\pi}{10}$
(e) $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$, $\sin\frac{3\pi}{10} = \frac{\sqrt{5}+1}{4}$
9(a) $64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$
(c)(i) $\frac{7}{4}$ **(ii)** $\frac{21}{16}$
10(b) $\theta = 0,\ \frac{\pi}{6},\ \frac{5\pi}{6},\ \pi,\ \frac{7\pi}{6},\ \frac{11\pi}{6}$
13(b) $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ or $\frac{3}{5} \pm \frac{4}{5}i$
14(a) $8(1 - 10s^2 + 24s^4 - 16s^6)$
(b) $x = 2\sin\frac{n\pi}{8}$ for $n = \pm1, \pm2, \pm3$

Exercise 3C (Page 108)

1(a) $\operatorname{cis} 0 = 1$, $\operatorname{cis}\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$,
$\operatorname{cis}\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ **(d)(i)** 1 **(ii)** 0
2(a) $z = \pm1,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \frac{1}{2} - \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$
$-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ **(e)** $(z^2 - z + 1)(z^2 + z + 1)$
3(a) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,\ -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$
(b) $(z^2 - \sqrt{2}\,z + 1)(z^2 + \sqrt{2}\,z + 1)$
4(a) $i,\ -i,\ \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ \frac{\sqrt{3}}{2} - \frac{1}{2}i,\ -\frac{\sqrt{3}}{2} + \frac{1}{2}i,\ -\frac{\sqrt{3}}{2} - \frac{1}{2}i$
5(a) $\operatorname{cis}\left(-\frac{7\pi}{10}\right),\ \operatorname{cis}\left(-\frac{3\pi}{10}\right),\ \operatorname{cis}\frac{\pi}{10},\ \operatorname{cis}\frac{\pi}{2} = i,\ \operatorname{cis}\frac{9\pi}{10}$
(b) $\operatorname{cis}\left(-\frac{5\pi}{8}\right),\ \operatorname{cis}\left(-\frac{\pi}{8}\right),\ \operatorname{cis}\frac{3\pi}{8},\ \operatorname{cis}\frac{7\pi}{8}$
(c) $1 + \sqrt{3}\,i,\ -1 - \sqrt{3}\,i,\ \sqrt{3} - i,\ -\sqrt{3} + i$
(d) $2\operatorname{cis}\left(-\frac{17\pi}{20}\right),\ 2\operatorname{cis}\left(-\frac{9\pi}{20}\right),\ 2\operatorname{cis}\left(-\frac{\pi}{20}\right),\ 2\operatorname{cis}\frac{7\pi}{20},$
$2\operatorname{cis}\frac{3\pi}{4}$
6(a) $-1,\ \operatorname{cis}\frac{\pi}{5},\ \operatorname{cis}\left(-\frac{\pi}{5}\right),\ \operatorname{cis}\frac{3\pi}{5},\ \operatorname{cis}\left(-\frac{3\pi}{5}\right)$
7(a) $1,\ \operatorname{cis}\left(\pm\frac{2\pi}{7}\right),\ \operatorname{cis}\left(\pm\frac{4\pi}{7}\right),\ \operatorname{cis}\left(\pm\frac{6\pi}{7}\right)$
(c) $(z - 1) \times (z^2 - 2\cos\frac{2\pi}{7}\,z + 1)\times$
$(z^2 - 2\cos\frac{4\pi}{7}\,z + 1) \times (z^2 - 2\cos\frac{6\pi}{7}\,z + 1)$
8(a)(i) $1,\ \operatorname{cis}\frac{2\pi}{5},\ \operatorname{cis}\left(-\frac{2\pi}{5}\right),\ \operatorname{cis}\frac{4\pi}{5},\ \operatorname{cis}\left(-\frac{4\pi}{5}\right)$
9(a) $\operatorname{cis}\frac{2k\pi}{9}$ for $k = -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4$
11(b) $\beta = \rho^3 + \rho^5 + \rho^6$ **(c)** $a = 1, b = 2$
12(a)(i) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$,
$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$
(b) $\pm\frac{1}{\sqrt{2}}(3 + i),\ \pm\frac{1}{\sqrt{2}}(1 - 3i)$
14(a) 3, when k is a multiple of 3, 0 otherwise.
(b) $(1 + \omega)^n = \sum_{r=0}^{n}\binom{n}{r}\omega^r$ and
$(1 + \omega^2)^n = \sum_{r=0}^{n}\binom{n}{r}\omega^{2r}$
15(a) The roots are $-i\cot\frac{(2k-1)\pi}{4n}$
for $k = 1,\ 2,\ 3,\ \dots,\ 2n$.

Exercise 3D (Page 115)

1(a) $e^{3i\theta}$ **(b)** $e^{-6i\theta}$ **(c)** $e^{8i\theta}$ **(d)** $e^{10i\theta}$
2(a) $e^{-i\theta}$ **(b)** $e^{3i\theta}$ **(c)** $e^{2i\theta}$ **(d)** $e^{20i\theta}$
3(a) $2e^{i\pi/2}$ **(b)** $\sqrt{2}\,e^{i\pi/4}$ **(c)** $6e^{i\pi}$ **(d)** $2e^{2i\pi/3}$
(e) $3\sqrt{2}\,e^{-3i\pi/4}$ **(f)** $4e^{-i\pi/6}$
4(a) -5 **(b)** $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ **(c)** $-4i$ **(d)** $-\sqrt{3}+i$ **(e)** $2-2i$
(f) $-2\sqrt{3} - 6i$
5(a) $2\sqrt{2}\,e^{i\pi/12}$ **(b)** $\frac{1}{\sqrt{2}}e^{-7i\pi/12}$ **(c)** $8\sqrt{2}\,e^{3i\pi/4}$
(d) $2\sqrt{2}\,e^{11i\pi/12}$
6(a) -64 **(b)** $4 - 4i$ **(c)** $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$
(d) $-648 - 648\sqrt{3}\,i$
11(a) n is divisible by 4 **(b)** $n = 2, 6, 10, \dots$
(c) n is divisible by 6 **(d)** $n = \frac{3}{2}, \frac{9}{2}, \frac{15}{2}, \dots$
12(b)(i) $2i\sin 3\theta$ **(ii)** $4\cos^2\theta$ **(iii)** $-8i\sin^3\theta$
(iv) $2\cos\theta\,(2\cos\theta + 1)$ **(v)** $2\,(\sin 3\theta - \sin\theta)\,i$
13(a) $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ **(d)** $\tan\theta$, $\cot\theta$ and $\operatorname{cosec}\theta$ are odd, $\sec\theta$ is even
14(a) $z^2 + 4$ **(b)** $z^2 - z + 1$ **(c)** $z^3 + 8$ **(d)** $z^4 + 4$
15(b) $e^{i(\phi-\theta)} = 1$ with $-2\pi < \phi - \theta < 2\pi$.
This has only one solution, which is $\phi - \theta = 0$.
(c) If two complex numbers are equal, then they represent the same point in the Argand diagram. Hence the moduli are equal and the principal arguments are equal.

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Exercise **3E** (Page 120)

1(a) $2i = 2e^{i\pi/2}$ **(b)** $2i = 2e^{i\left(\frac{\pi}{2}+2k\pi\right)}$, $k \in \mathbf{Z}$
(d) $z = \sqrt{2}e^{-3i\pi/4}, \sqrt{2}e^{i\pi/4}$
(e) $z = -1-i, 1+i$
2(a) $-1 = e^{i\pi}$ **(b)** $-1 = e^{i(\pi+2k\pi)}$, $k \in \mathbf{Z}$
(d) $z = e^{-3i\pi/4}, e^{-i\pi/4}, e^{i\pi/4}, e^{3i\pi/4}$
(e) $z = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$
3(a) $i = e^{-i\pi/2}$ **(b)** $-i = e^{-i(\pi/2+2k\pi)}$, $k \in \mathbf{Z}$
(d) $z = e^{i\pi/2}, e^{-i\pi/6}, e^{-5i\pi/6}$
(e) $z = i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i$
7(c) $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}, \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$
9(a) $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$
10(a) $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$,
$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$ **(c)** $\frac{47}{480}$
11(c) $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ or $\frac{3}{5} \pm \frac{4}{5}i$
12 $a = \frac{1}{2}\ln 2, b = -\frac{\pi}{4}$
13(a) $2\cos A\cos B$
14(b) $\tan\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$
15(a) $\frac{z^{n+1}-z}{z-1}$
17(a) $z = e^{\frac{2k\pi}{2n+1}\,i}$ for $k = 0, 1, 2, \ldots, 2n$

Review Exercise **3F** (Page 123)

1(a) $\operatorname{cis} 7\theta$ **(b)** $\operatorname{cis} 6\theta$
2 -1
3(a) $\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **(b)** $-64+64i$
4(a) 2^{13} **(b)(ii)** n is even or a multiple of 3
5(a) $\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$,
$6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$
6(a) $z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$,
$z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$
7(b) -43
8 $z = 2e^{-i\pi/2}, 2e^{i\pi/6}, 2e^{5i\pi/6}$
9(a) $\sqrt{2}\operatorname{cis}\frac{k\pi}{12}$ for $k = -7, 1, 9$
(b) $\operatorname{cis}\frac{k\pi}{12}$ for $k = -11, -7, -3, 1, 5, 9$
10(a) $8e^{i\pi/6}$ **(c)** $2e^{-11i\pi/18}, 2e^{i\pi/18}, 2e^{13i\pi/18}$
11(d) $-7\cos 3\theta + \frac{7}{5}\cos 5\theta - \frac{1}{7}\cos 7\theta + C$
12(b) $x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$ **(e)** $\frac{\sqrt{5}+1}{4}$
14(a) $\operatorname{cis}\frac{k\pi}{7}$ for $k = -5, -3, -1, 1, 3, 5, 7$
15(a) $e^{ki\pi/5}$ for $k = -4, -2, 0, 2, 4$ **(b)(i)** -1 and $\sqrt{5}$
18(c) $-\frac{1}{2}$ **(e)** $\frac{1+\sqrt{5}}{4}$

Chapter Four

Exercise **4A** (Page 126)

1(a) $\frac{1}{4}e^{4x} + C$ **(b)** $-\frac{1}{5}\cos 5x + C$ **(c)** $2\tan\frac{1}{2}x + C$
(d) $\frac{1}{3}\ln|3x-4| + C$ **(e)** $4\sqrt{x} + C$ **(f)** $\frac{3^x}{\ln 3} + C$
2(a) $-\frac{1}{2(2x-1)} + C$ **(b)** $\sin^{-1}\frac{x}{5} + C$
(c) $\frac{1}{3}e^{x^3} + C$ **(d)** $\frac{1}{3}\tan^{-1}\frac{x}{3} + C$
(e) $2\ln(x^2+x+1) + C$ **(f)** $\frac{1}{5}(x^2+1)^5 + C$
3(a) $2(e^2-1)$ **(b)** $\frac{\sqrt{3}}{2}$ **(c)** $\frac{\pi}{8}$ **(d)** $\frac{\pi}{4}$ **(e)** $\ln\frac{8}{5}$ **(f)** $\frac{1}{16}$
4(a) $e^{\frac{1}{x}} + C$ **(b)** $\frac{1}{3}\ln(1+\sin 3x) + C$
(c) $\frac{1}{2}\tan x^2 + C$ **(d)** $\frac{5^{2x}}{2\ln 5} + C$
(e) $\ln|x+\tan x| + C$ **(f)** $\sin^{-1}e^x + C$
5(a) -20 **(b)** $\frac{1}{3}\ln 2$ **(c)** $\frac{\pi}{3\sqrt{3}}$ **(d)** $\frac{1}{2}\ln\frac{e^2+1}{2}$ **(e)** $\frac{\pi}{18}$
(f) $\ln 2$
6 $\ln 2$
7 $-\frac{1+\ln x}{x} + C$
9 $\frac{\pi}{4} - \frac{2}{3}$

Exercise **4B** (Page 128)

1(a) $x + \ln|x-1| + C$ **(b)** $x - 2\ln|x+1| + C$
(c) $x + 2\ln|x-1| + C$
2(a) $1 - \ln 4$ **(b)** $1 - \frac{1}{4}\ln 5$ **(c)** $\pi - 1$
3(a) $\frac{\pi}{3} - \frac{1}{2}$ **(b)** $\frac{\pi}{4} + \ln 2$ **(c)** $\frac{1}{4}(\pi - \ln 4)$
(d) $\frac{\pi}{8} + \frac{1}{2}\ln 2$
4(a) $\dfrac{1}{\sqrt{x^2+a^2}}$ **(b)** $\log(x + \sqrt{x^2+a^2}) + C$
(c)(i) $\log\left(x + \sqrt{x^2+3}\right) + C$ **(ii)** $2\log 3$
5(a) $\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C$
(b) $\frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C$
(c)(i) $\frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + C$
(ii) $\frac{x^3}{3} - x + \tan^{-1}x + C$ **(iii)** $x - \ln(1+e^x) + C$
(iv) $\frac{2}{3}(x-4)\sqrt{2+x} + C$ **(v)** $-\frac{2}{3}(2+x)\sqrt{1-x} + C$
(vi) $\frac{1}{2}x^2 - 2\ln(x^2+4) + C$
6(a) $\ln(e+e^{-1})$ **(b)** $\frac{1}{2}\ln\frac{e^2+1}{2}$ **(c)** $\frac{\pi}{12} + \ln 2$
7(a) $\frac{1}{2}x^2 + \ln|x+1| + C$ **(b)** $\frac{1}{3}x^3 + 3\ln|x-2| + C$
(c) $x + \ln(1+x^2) + C$
8(a) $\dfrac{1}{\sqrt{x^2-a^2}}$ **(b)** $\log(x + \sqrt{x^2-a^2}) + C$
(c)(i) $\log\left(x + \sqrt{x^2-5}\right) + C$
(ii) $\log\left(\frac{3+\sqrt{5}}{1+\sqrt{5}}\right) = \log\left(\frac{1+\sqrt{5}}{2}\right)$
9 $2\ln(1+\sqrt{x}) + C$

Exercise **4C** (Page 132)

1(a) $\frac{1}{5}(x^2+1)^5+C$ **(b)** $\frac{1}{7}(1+x^3)^7+C$
(c) $-\frac{2}{1+x^3}+C$ **(d)** $\frac{1}{2(3-x^2)^4}+C$
(e) $\sqrt{x^2-2}+C$ **(f)** $\frac{1}{2}\sqrt{1+x^4}+C$
2(a) $\frac{-1}{2\sin^2 x}+C$ **(b)** $\frac{-1}{1+\tan x}+C$ **(c)** $\frac{1}{3}(\ln x)^3+C$
(d) $2\sin\sqrt{x}+C$ **(e)** $\frac{1}{2}\tan^{-1}x^2+C$
(f) $\frac{1}{3}\sin^{-1}x^3+C$
3(a) $\frac{7}{4}$ **(b)** $2-\sqrt{3}$ **(c)** $3(\sqrt{3}-\sqrt{2})$
(d) $\frac{1}{5}$ **(e)** $\frac{1}{3}$ **(f)** 2
4(a) $-\frac{1}{42}$ **(b)** Begin by writing $x=(x-1)+1$.
5(a) $\frac{2}{15}(3x-2)(1+x)\sqrt{1+x}+C$
(b) $2\left(1+\sqrt{x}-\ln(1+\sqrt{x})\right)+C$
(c) $4\left(x^{\frac{1}{4}}-\frac{1}{2}\sqrt{x}+\frac{1}{3}x^{\frac{3}{4}}-\ln(1+x^{\frac{1}{4}})\right)+C$
(d) $\tan^{-1}\sqrt{e^{2x}-1}+C$
6(a) $\frac{1}{9}$ **(b)** $\frac{128}{15}$ **(c)** $4+10\ln\frac{5}{7}$ **(d)** $\frac{\pi}{12}$
7(a) $2\tan^{-1}\left(\sqrt{x}\right)+C$ **(b)** $\frac{2}{3}(x-2)\sqrt{x+1}+C$
8(a) $\frac{x}{\sqrt{1+x^2}}+C$ **(b)** $2\sin^{-1}\frac{x}{2}-\frac{1}{2}x\sqrt{4-x^2}+C$
(c) $-\frac{\sqrt{25-x^2}}{25x}+C$ **(d)** $-\frac{1}{x}\sqrt{1+x^2}+C$
9(a) $\frac{2}{3}$ **(b)** Begin by writing $x^3=x(x^2+1)-x$.
10(b) The region is half a segment.
11(b) $\frac{\pi}{4}$
12(b)(ii) $\frac{\pi^2}{4}$
13(b) Begin by writing $x^2=1-(1-x^2)$.
14(a) $\tan^{-1}\sqrt{x^2-1}+C_1$ **(b)** $\tan^{-1}\sqrt{x^2-1}+C_2$
15(a) $\frac{\sqrt{3}}{8}-\frac{\sqrt{\epsilon(4+\epsilon)}}{4(2+\epsilon)}$ **(b)** $\frac{\sqrt{3}}{8}$

Exercise **4D** (Page 139)

1(a) $\frac{1}{x-1}-\frac{1}{x+1}$ **(b)** $\frac{1}{3(x-4)}-\frac{1}{3(x-1)}$ **(c)** $\frac{2}{x-3}+\frac{2}{x+3}$
(d) $\frac{2}{x-2}-\frac{1}{x-1}$ **(e)** $\frac{1}{5(x-2)}+\frac{4}{5(x+3)}$ **(f)** $\frac{1}{x-1}+\frac{2-x}{x^2+3}$
2(a) $\ln|x-4|-\ln|x-2|+C$
(b) $2\ln|x+1|-2\ln|x+3|+C$
(c) $4\ln|x-2|-\ln|x-1|+C$
(d) $3\ln|x-1|-\ln|x+3|+C$
(e) $\ln|x+1|+\ln|2x+3|+C$
(f) $2\ln|x+1|+3\ln|2x-3|+C$
3(a) $\frac{1}{4}\ln\frac{3}{2}$ **(b)** $\ln 2$ **(c)** $\ln\frac{14}{3}$ **(d)** $\frac{1}{2}\ln 2$
4(a) $\ln|x-2|-2\tan^{-1}x+C$
(b) $\ln|2x+1|-\frac{1}{2}\ln(x^2+3)+C$
(c) $\tan^{-1}x+3\ln|x|-\ln(x^2+1)+C$
5(a) $\frac{\pi}{4}-\ln\frac{3}{2}$ **(b)** $\pi+\ln 2$ **(c)** $\ln 4-\frac{3}{2}\ln 3$
6(a) $5\ln|x-1|+7\ln|x-2|-12\ln|2x-3|+C$
(b) $\frac{3}{2}\ln|x|-5\ln|x-2|+\frac{7}{2}\ln|x-4|+C$
7(a) $\frac{5}{3}\ln 3-\ln 2$ **(b)** $2\ln 3-8\ln 2$
8(a)(i) $A=2,\ B=1,\ C=-3$
(ii) $2x+\ln|x-1|-3\ln|x+2|+C$
(b)(i) $x+\ln|x-2|-2\ln|x+1|+C$
(ii) $3x+2\ln|x+4|+\ln|x-5|+C$
9(a)(i) $A=1,\ B=-1,\ C=2,\ D=-1$
(ii) $\ln 3+\ln 2-\frac{1}{2}$ **(b)** $12+\ln 2$
10(a)(i) $A=3,\ B=12,\ C=2$
(ii) $3x+12\ln|x-2|-\frac{2}{x-2}+C$
(b)(i) $A=23,\ B=10,\ C=-23,\ D=13$
(ii) $23\ln\left|\frac{x-1}{x-2}\right|-\frac{10}{x-1}-\frac{13}{x-2}+C$
12(a) $A=0,\ B=-1,\ C=0,\ D=2$
13(a) $x+\ln|x-1|-\ln|x+1|+C$
(b) $x+2\ln|x-1|-\ln|x|+C$
(c) $x-\tan^{-1}x+\ln|x|-\frac{1}{2}\ln(x^2+1)+C$
(d) $x+9\ln|x-3|-4\ln|x-2|+C$
(e) $\frac{1}{2}x^2-x+5\ln|x|-4\ln|x+1|+C$
(f) $\frac{1}{3}x^3+\frac{3}{2}x^2+7x+16\ln|x-2|-\ln|x-1|+C$

Exercise **4E** (Page 143)

1(a) $\frac{1}{3}\tan^{-1}\frac{x}{3}+C$ **(b)** $\sin^{-1}\frac{x}{3}+C$
(c) $\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|+C$ **(d)** $\frac{1}{6}\ln\left|\frac{3+x}{3-x}\right|+C$
(e) $\ln\left|x+\sqrt{9+x^2}\right|+C$ **(f)** $\ln\left|x+\sqrt{x^2-9}\right|+C$
2(a) $\tan^{-1}(x+2)+C$ **(b)** $\frac{1}{4}\tan^{-1}\left(\frac{x-2}{4}\right)+C$
(c) $\sin^{-1}\frac{x-4}{5}+C$ **(d)** $\sin^{-1}\frac{x+4}{6}+C$
(e) $\ln\left|x-3+\sqrt{x^2-6x+13}\right|+C$
(f) $\frac{1}{2}\ln\left|x+1+\sqrt{x^2+2x+\frac{3}{2}}\right|+C$
3(a) $\frac{\pi}{8}$ **(b)** π **(c)** $\frac{\pi}{6}$ **(d)** $\frac{\pi}{2}$ **(e)** $\ln 3$ **(f)** $\ln 3$
4(a) $\ln(x^2+2x+2)-\tan^{-1}(x+1)+C$
(b) $\frac{1}{2}\ln(x^2+2x+10)-\frac{1}{3}\tan^{-1}\frac{x+1}{3}+C$
(c) $-\sqrt{6x-x^2}+3\sin^{-1}\frac{x-3}{3}+C$
(d) $-\sqrt{4-2x-x^2}+2\sin^{-1}\frac{x+1}{\sqrt{5}}+C$
(e) $\sqrt{x^2+2x+10}-\ln\left|x+1+\sqrt{x^2+2x+10}\right|+C$
(f) $\sqrt{x^2-2x-4}+4\ln\left|x-1+\sqrt{x^2-2x-4}\right|+C$
5(a) $\frac{1}{2}\ln 2+\frac{\pi}{8}$ **(b)** $\frac{1}{4}(3\pi-\ln 4)$ **(c)** $\ln 2-\frac{\pi}{4}$ **(d)** $2-\sqrt{3}-\frac{\pi}{6}$ **(e)** $3\ln(3+2\sqrt{2})-4\sqrt{2}$
(f) $\ln\left(1+\sqrt{\frac{2}{3}}\right)+\sqrt{6}-1$
6(a) $\sin^{-1}x-\sqrt{1-x^2}+C$
(b) $\sqrt{6+x-x^2}+\frac{5}{2}\sin^{-1}\frac{2x-1}{5}+C$
(c) $\sqrt{x^2-1}-\ln\left|x+\sqrt{x^2-1}\right|+C$
7(a) $\frac{\pi}{3}+\sqrt{3}-2$ **(b)** $3\sin^{-1}\frac{1}{3}$
(c) $2\sqrt{2}-\sqrt{3}+\ln\left(\frac{2+\sqrt{3}}{3+2\sqrt{2}}\right)$
8(a) $\frac{x}{\sqrt{4x-x^2}}$ is undefined at $x=0$.

Exercise **4F** (Page 148)

1(a) $e^x(x-1)+C$ **(b)** $-e^{-x}(x+1)+C$
(c) $\frac{1}{9}e^{3x}(3x+2)+C$ **(d)** $x\sin x+\cos x+C$
(e) $-\frac{1}{2}(x-1)\cos 2x+\frac{1}{4}\sin 2x+C$
(f) $(2x-3)\tan x+2\ln(\cos x)+C$
2(a) π **(b)** $\frac{\pi}{2}-1$ **(c)** $\frac{\pi}{4}-\frac{1}{2}\ln 2$ **(d)** $\frac{1}{4}(e^2+1)$
(e) e^{-1} **(f)** $1+e^{-2}$
3(a) $x(\ln x-1)+C$ **(b)** $2x(\ln x-1)+C$
(c) $x\cos^{-1}x-\sqrt{1-x^2}+C$
4(a) $\frac{\pi}{4}-\frac{1}{2}\ln 2$ **(b)** 1 **(c)** $\frac{1}{2}$
5(a) $\frac{1}{4}x^2(2\ln x-1)+C$ **(b)** $\frac{1}{9}x^3(3\ln x-1)+C$
(c) $-\frac{1}{x}(\ln x+1)+C$
6(a) $(2-2x+x^2)e^x+C$
(b) $x^2\sin x+2x\cos x-2\sin x+C$
(c) $x(\ln x)^2-2x\ln x+2x+C$
7(a) $-\frac{1}{42}$ **(b)** $\frac{4}{15}(1+\sqrt{2})$ **(c)** $\frac{128}{15}$
8(a) $\frac{1}{2}e^x(\cos x+\sin x)+C$
(b) $-\frac{1}{2}e^{-x}(\cos x+\sin x)+C$
9(a) $\frac{1}{5}(e^{\pi}-2)$ **(b)** $\frac{1}{5}(e^{\frac{\pi}{4}}+2)$
10(a) $\frac{1}{2\sqrt{3}}(\pi-\sqrt{3})$ **(b)** $\frac{\sqrt{3}\pi}{2}$ **(c)** $\pi-2$
12(a) $\frac{1}{4}x^2(2\ln x-1)+C$
(b) $\frac{1}{4}x^2\left(2(\ln x)^2-2\ln x+1\right)+C$
14(a) $\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\sin^{-1}(\frac{x}{a})\right)+C$
(b) $x\ln\left|x+\sqrt{x^2+a^2}\right|-\sqrt{x^2+a^2}+C$
(c) $x\ln\left|x+\sqrt{x^2-a^2}\right|-\sqrt{x^2-a^2}+C$
15(a) $\frac{1}{32}(\sin 4x-4x\cos 4x+8x\cos 2x-4\sin 2x)+C$
(b) $\frac{1}{18}(3x\sin 3x+\cos 3x+9x\sin x+9\cos x)+C$
(c) $\frac{1}{20}e^x(\sin 3x-3\cos 3x+5\sin x-5\cos x)+C$
16(a) $\frac{1}{48}(3\sqrt{3}-\pi)$ **(b)** $\frac{1}{12}(\pi+2\ln 2-2)$

Exercise **4G** (Page 154)

1(a) $\sin x+C$ **(b)** $-\cos x+C$ **(c)** $-\ln|\cos x|+C$
(d) $\ln|\sin x|+C$
2(a) $\frac{1}{3}\sin^3 x+C$ **(b)** $-\frac{1}{3}\cos^3 x+C$
(c) $\frac{1}{3}\cos^3 x-\cos x+C$ **(d)** $\sin x-\frac{1}{3}\sin^3 x+C$
(e) $\frac{1}{5}\sin^5 x-\frac{2}{3}\sin^3 x+\sin x+C$
(f) $\frac{1}{4}\sin^4 x-\frac{1}{6}\sin^6 x+C$
3(a) $\frac{\pi}{4}$ **(b)** $\frac{\pi}{12}$ **(c)** $\frac{\pi}{8}$
4(a) $\tan x+C$ **(b)** $\tan x-x+C$
(c) $\frac{1}{3}\tan^3 x+\tan x+C$ **(d)** $\frac{1}{3}\tan^3 x-\tan x+x+C$
5(a) $\frac{1}{4}$ **(b)** $\frac{11}{24}$ **(c)** $\frac{9}{64}$ **(d)** $\frac{53}{480}$ **(e)** $\frac{4}{15}$ **(f)** $\frac{7}{60\sqrt{2}}$
6(a) $\frac{1}{32}(\sin 4x+8\sin 2x+12x)+C$
(b) $\frac{1}{32}(\sin 4x-8\sin 2x+12x)+C$
(c) $\frac{1}{1024}(24x-8\sin 4x+\sin 8x)+C$
8(a) 1 **(b)** $\frac{1}{3}\ln 2$ **(c)** $\frac{1}{2}(\tan^{-1}2+\tan^{-1}\frac{1}{2})=\frac{\pi}{4}$
9(a) $\frac{\pi}{4}$ **(b)** $\frac{2}{15}(1+\sqrt{2})$ **(c)** $\frac{\pi}{16}$
10(a) $\frac{1}{2}\sin^2 x+C_1$ **(b)** $-\frac{1}{4}\cos 2x+C_2$
11(a) $\frac{1}{2}$ **(b)** $\frac{\sqrt{3}}{4}$
12(a) $\frac{1}{2}$ **(b)** $\frac{4}{3}$
13(a) $-\frac{1}{8}\cos 4x-\frac{1}{4}\cos 2x+C$
(b) $-\frac{1}{8}\cos 4x+\frac{1}{4}\cos 2x+C$
(c) $\frac{1}{16}\sin 8x+\frac{1}{8}\sin 4x+C$
14(a) $\frac{1}{4}$ **(b)** $\frac{1}{6}$ **(c)** $\frac{3}{8}$
15(a) $\tan\frac{x}{2}+C$ **(b)** $\ln\left|\frac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right|+C$
(c) $\frac{1}{5}\ln\left|\frac{1+2\tan\frac{x}{2}}{2-\tan\frac{x}{2}}\right|+C$
17 $\frac{1}{3}\sin 3\theta+\sin^3\theta+C$

Exercise **4H** (Page 158)

3(b) $\frac{1}{2}(e^2-1)$
4(c) $\frac{8}{15}$
6(b) $(\frac{\pi}{2})^6-30(\frac{\pi}{2})^4+360(\frac{\pi}{2})^2-720$
7(b) $\frac{x^4}{4}-\frac{x^2}{2}+\frac{1}{2}\ln(1+x^2)+C$
8(b) $I_0=1,\ I_4=\frac{128}{315}$
9(b) $u_4=\frac{243}{1540}$
10(b) $J_2=-\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x+C$
12(d) $\frac{1}{15}(14\sqrt{2}-16)$
13(d) $\frac{1}{9}\Big((1+x^2)^4+\frac{8}{7}(1+x^2)^3+\frac{48}{35}(1+x^2)^2$
$+\frac{192}{105}(1+x^2)+\frac{384}{105}\Big)$
15(d) $J_n=\dfrac{2n}{2n+3}\,J_{n-1}$
16(d) $I_5=\frac{1}{4}(2\ln 2-1)$
17(b) $\frac{\pi}{3}-\frac{3\sqrt{3}}{4}$

Exercise **4I** (Page 161)

1(a) $\frac{1}{36}$ **(b)** π **(c)** $\ln\frac{12}{5}$ **(d)** $2-2\ln 3$ **(e)** $2\sqrt{2}-1$
(f) $\frac{\pi}{18}$
2(a) $\sqrt{1+x^2}+C$ **(b)** $\tan^{-1}x+\frac{1}{2}\ln(1+x^2)+C$
(c) $-\frac{1}{5}\cos^5 x+C$ **(d)** $\ln\left|\frac{2x+1}{x+1}\right|+C$
(e) $\frac{1}{4}x^4\ln x-\frac{1}{16}x^4+C$ **(f)** $\frac{1}{6}\cos^3 2x-\frac{1}{2}\cos 2x+C$
(g) $\frac{1}{4}\tan^{-1}\frac{x+3}{4}+C$ **(h)** $x\sin 3x+\frac{1}{3}\cos 3x+C$
(i) $\frac{2}{3}(x-8)\sqrt{4+x}+C$
4(a) $A=-\frac{2}{3},\ B=\frac{2}{3},\ C=-\frac{1}{3}$
6(a) $2\sqrt{3}$ **(b)** $\frac{24\sqrt{3}}{5}$
8(a) $A=0,\ B=-2,\ C=0,\ D=2$ **(b)** $\frac{\pi}{2}-1$
10 $\frac{1}{2}a^2\sin^{-1}\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$
11(b) $\frac{1}{10}(\pi+\ln\frac{27}{16})$
12(a) $P=2,\ Q=-1$
(b) $2x-\ln|3\sin x+2\cos x-1|+C$
14(b) $6-2e$
17(c) If $pq\le 0$, then $0\in[p,q]$ and u is undefined at $t=0$.

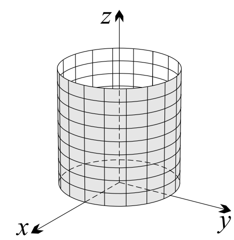
Review Exercise **4J** (Page 164)

1(a) $\frac{1}{2}e^{x^2}+C$ **(b)** $\frac{3}{2}\ln(x^2+1)+C$ **(c)** $\frac{1}{12}(1+x^2)^6+C$ **(d)** $-\frac{1}{4}\cos^4 x+C$ **(e)** $3\ln|x-3|+\ln|x+1|+C$ **(f)** $-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}+C$
2(a) $\tan x - x + C$ **(b)** $\frac{2}{3}(3+x)^{\frac{3}{2}}-6(3+x)^{\frac{1}{2}}+C$
(c) $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)$
(d) $3x\sin\frac{1}{3}x+9\cos\frac{1}{3}x+C$ **(e)** $x+\ln|x+1|+C$
(f) $2\ln|x|+\frac{1}{2}\ln(x^2+1)+C$
3(a) $\frac{x}{4\sqrt{4-x^2}}+C$ **(b)** $\frac{1}{2}\ln\left(\frac{e^x-1}{e^x+1}\right)+C$
(c) $2\sqrt{x}-4\ln(2+\sqrt{x})+C$
(d) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{1}{2}x\right)+C$
4(a) 6 **(b)** $\ln 6$ **(c)** $\frac{5}{24}$ **(d)** $2-\frac{3}{2}\ln\frac{7}{3}$ **(e)** $\frac{16}{105}$ **(f)** $\frac{1}{4}$
5(a) $\frac{\pi}{9}$ **(b)** $14-\frac{11\pi}{2}$ **(c)** $\frac{2}{15}$ **(d)** $\frac{\pi}{12}+\ln\frac{4}{3}$
(e) $\frac{8\pi}{3}-2\sqrt{3}$ **(f)** $\frac{2e+\pi}{4+\pi^2}$
6(a) $\frac{1}{2}\ln\frac{5}{3}$ **(b)** $\frac{\pi}{18}$ **(c)** π **(d)** $\ln(2+\sqrt{3})$
8(a) $I_n=\frac{1}{4}x^4(\ln x)^n-\frac{1}{4}nI_{n-1}$
9(b) $\frac{5\pi}{32}$
10(b) $(2n-1)J_n=2^{n-1}+2(n-1)J_{n-1}$ **(c)** $\frac{28}{15}$
11(b) $\frac{26}{15}$

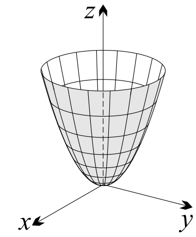
Chapter Five

Exercise **5A** (Page 171)

1 The xy-plane with equation $z=0$, the xz-plane with equation $y=0$ and the yz-plane with equation $x=0$.
2(a) 2nd **(b)** 5th **(c)** 4th **(d)** 6th **(e)** 8th **(f)** 3rd
3(a) $(3,2,-1)$ **(b)** $(-5,2,5)$ **(c)** $(3,12,5)$
(d) $(8,2,12)$ **(e)** $(3,-1,1)$ **(f)** $(3,2,-5)$
(g) $(-3,2,5)$ **(h)** $(3,-2,5)$ **(i)** $(3,-2,-5)$
4(a) $A(2,0,0), B(2,2,0), C(2,2,2), D(2,0,2), O(0,0,0), P(0,2,0), Q(0,2,2), R(0,0,2)$
(b) $2\sqrt{2}$ **(c)** $2\sqrt{3}$ **(d)** $x=0, y=0, z=0, x=2, y=2, z=2$
5(a) $A(2,0,0), B(2,4,0), D(2,0,3), P(0,4,0), Q(0,4,3), R(0,0,3)$
(b) $2\sqrt{5}$ **(c)** $\sqrt{29}$ **(d)** $x=0, y=0, z=0, x=2, y=4, z=3$
6(a) $6\,\mathrm{u}^2$ **(b)** $10\,\mathrm{u}^3$
7(a) $OA=7, OB=7\sqrt{2}, AB=7\sqrt{3}$
(b) $OA^2+OB^2=AB^2$, so $\angle AOB=90°$.
8 $BC=2\sqrt{21}, MN=\sqrt{21}=\frac{1}{2}BC$
9(a) $M=(-8,6,18)$
(b) $X=(-7,-1,16)$ and $Y=(-9,13,20)$
11 $x=-1$
12(a) $AB=AC=2\sqrt{14}$
13(a) $3x+4y=12$ **(b)** $3x+6z=12$ with the xz-plane, $4y+6z=12$ with the yz-plane **(c)** a line **(d)** They intersect in a line.
15(a) z can take any real value.
(b) circle $x^2+y^2=4$
(c) It is the curved surface of a cylinder with radius 2. The diagram shows the portion from $z=0$ to $z=4$.
(d)



16(a) A sum of squares can never be negative.
(b) circle $x^2+y^2=k$.
(c) parabola $z=x^2$.
(d)



Exercise 5B (Page 177)

1(a) $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ **(b)** $\begin{bmatrix} -4 \\ 0 \\ 13 \end{bmatrix}, -4\underset{\sim}{i} + 13\underset{\sim}{k}$

(c) $\begin{bmatrix} a \\ -2a \\ -3a \end{bmatrix}, a\underset{\sim}{i} - 2a\underset{\sim}{j} - 3a\underset{\sim}{k}$

2(a) $|\underset{\sim}{a}| = 5$, $\widehat{\underset{\sim}{a}} = \frac{4}{5}\underset{\sim}{i} - \frac{3}{5}\underset{\sim}{k}$

(b) $|\underset{\sim}{a}| = 3$, $\widehat{\underset{\sim}{a}} = \frac{1}{3}\underset{\sim}{i} + \frac{2}{3}\underset{\sim}{j} - \frac{2}{3}\underset{\sim}{k}$

3(a) $|\underset{\sim}{v}| = 3\sqrt{2}$, $\widehat{\underset{\sim}{v}} = \dfrac{1}{6}\begin{bmatrix} -\sqrt{2} \\ -4\sqrt{2} \\ \sqrt{2} \end{bmatrix}$

(b) $|\underset{\sim}{v}| = 5\sqrt{2}$, $\widehat{\underset{\sim}{v}} = \dfrac{1}{10}\begin{bmatrix} 5\sqrt{2} \\ 3\sqrt{2} \\ -4\sqrt{2} \end{bmatrix}$

4(a) $\begin{bmatrix} 5 \\ -10 \\ 23 \end{bmatrix}$ **(b)** $\sqrt{654}$ **(c)** $\begin{bmatrix} 19 \\ 28 \\ -38 \end{bmatrix}$ **(d)** $\sqrt{2589}$

5(a) $3\underset{\sim}{i} - 12\underset{\sim}{j} + 4\underset{\sim}{k}$ **(b)** $-3\underset{\sim}{i} + 12\underset{\sim}{j} - 4\underset{\sim}{k}$ **(c)** 13

6(a) $\begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}$ **(b)** $\begin{bmatrix} -8 \\ -3 \\ 2 \end{bmatrix}$ **(c)** $\sqrt{77}$

7 $\lambda_1 = 2, \lambda_2 = -4$

8 $\lambda_1 = -1, \lambda_2 = 3, \lambda_3 = -2$

9(a) $\overrightarrow{CD} = -3\overrightarrow{AB}$ **(b)** They are not parallel.

10 $\overrightarrow{BC} = 2\overrightarrow{AB}$, so AB and BC are parallel.

12 $-\underset{\sim}{i} - 14\underset{\sim}{j} - 6\underset{\sim}{k}$

13 $58°, 74°, 37°$

14(a) $\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 \\ -7 \\ 4 \end{bmatrix}$

15(a) $-5\underset{\sim}{j} + 7\underset{\sim}{k}$ **(b)** $-24\underset{\sim}{i} + 7\underset{\sim}{j} - 5\underset{\sim}{k}$

16(a) $\overrightarrow{AG} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ **(b)** $\sqrt{3}$ **(c)** $\frac{\sqrt{6}}{2}$

18(a) No, since $-\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$ **(b)** Yes

19 $(-10, -1, 3), (4, 9, -1), (6, 5, 7)$

Exercise 5C (Page 183)

1(a) $12\sqrt{2}$ **(b)** -20

2(a) 5 **(b)** $x_1x_2 + y_1y_2 + z_1z_2$ **(c)** $a_1b_1 + a_2b_2 + a_3b_3$

4(a) 0 **(b)** They are perpendicular.

6(a) 126 **(b)** 360 **(c)** -48 **(d)** 78

7(a) $-21 \le 8 \le 21$ **(b)** $-20 \le -18 \le 20$

8(a) $4 \le \sqrt{74} \le 10$ **(b)** $\sqrt{10} \le \sqrt{14} \le 3\sqrt{10}$

10(a) $\lambda = 8$ **(b)** $\lambda = -\frac{5}{2}$ or 3

11 One such vector is $\underset{\sim}{i} + 7\underset{\sim}{j} + 3\underset{\sim}{k}$.

18 $\sqrt{23}$

19 $2\sqrt{7}$

Exercise 5D (Page 188)

1(a) 3 **(b)** Both $\sqrt{6}$. **(c)** $\frac{\pi}{3}$

2(a) $\frac{2}{3}$ **(b)** $\frac{\sqrt{2}}{3}$

4(a) $44°$ **(b)** $87°$

6(a) $\frac{2}{9}\underset{\sim}{i} - \frac{2}{9}\underset{\sim}{j} - \frac{1}{9}\underset{\sim}{k}$ **(b)** $\begin{bmatrix} \frac{8}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$

7(a) $\frac{8}{3\sqrt{5}}$ **(b)** $\frac{5}{3}$

8(a) $\overrightarrow{BA} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 7\underset{\sim}{k}$, $\overrightarrow{BC} = 5\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$

(c) $AB = \sqrt{62}$, $BC = \sqrt{42}$, $AC = \sqrt{104} = 2\sqrt{26}$

9(a) $\overrightarrow{AB} = \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}, \overrightarrow{AC} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ **(b)** $78°$

10 $117°49'$

11(b) $\frac{3}{2}\,\text{u}^2$

12(a) $\overrightarrow{AP} = \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix}, \overrightarrow{AB} = \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$

(c) 7 units

13(a) 3 units **(b)** $\frac{5\sqrt{70}}{14}$ units

15 $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

17(a) $90°$ **(c)** $\frac{220}{3}\,\text{u}^2$

18(a) $\overrightarrow{OP} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$ **(b)** $\lambda = \frac{1}{12}$ **(c)** $\lambda = \frac{3}{10}$

19 $\lambda = 4$ or $-\frac{44}{65}$

20(b) $\lambda = \frac{2}{3}$ or $\frac{4}{3}$

21(a) $12\sqrt{10}\,\text{u}^2$ **(b)** $\overrightarrow{OD} = 7\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, $\overrightarrow{OE} = 6\underset{\sim}{i} - 2\underset{\sim}{j}$

(d) $80\,\text{u}^3$

Exercise 5E (Page 191)

2(a) 0 **(b)** $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$, $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

3(a) equal radii

5(a) equal chords

6(b) $\overrightarrow{CB} \parallel \overrightarrow{OA}$

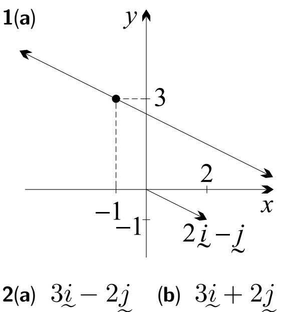
8(a) $60°$ **(b)** equilateral **(c)** regular tetrahedron (all 4 faces are equilateral triangles)

(d) $\cos^{-1}\left(-\frac{1}{3}\right) \doteqdot 109°28'$

9(a) $\frac{1}{2}\underset{\sim}{v} + \underset{\sim}{w}$

Exercise 5F (Page 201)

1(a)



(b) Using basis vectors: $\underset{\sim}{r} = (-\underset{\sim}{i} + 3\underset{\sim}{j}) + \lambda(2\underset{\sim}{i} - \underset{\sim}{j})$, $\lambda \in \mathbf{R}$

(c) $-\frac{1}{2}$

(d) $y = -\frac{1}{2}x + \frac{5}{2}$

2(a) $3\underset{\sim}{i} - 2\underset{\sim}{j}$ **(b)** $3\underset{\sim}{i} + 2\underset{\sim}{j}$

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(c) $\underset{\sim}{r} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \lambda(3\underset{\sim}{i} + 2\underset{\sim}{j}), \lambda \in \mathbf{R}$

3(a)(i) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(ii) The x- and y-intercepts are $\begin{bmatrix} -12 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

(iii) Using the y-intercept, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(b)(i) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

(ii) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(iii) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

4(a) $y = 4x + 17$ **(b)** $3x + 2y = 19$

5(a) yes **(b)** yes **(c)** no

6(a) $\underset{\sim}{r} = 7\underset{\sim}{i} - 5\underset{\sim}{k} + \lambda(-4\underset{\sim}{i} - 6\underset{\sim}{j} + 9\underset{\sim}{k})$

(b) $\underset{\sim}{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} -6 \\ -7 \\ -8 \end{bmatrix}$

7(a) $\underset{\sim}{r} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k} + \lambda(5\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k})$

(b) $\underset{\sim}{r} = -\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$

8(a) yes **(b)** yes

9(a)(i) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **(iii)** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

11(a) $\underset{\sim}{r} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + \lambda(2\underset{\sim}{i} - 3\underset{\sim}{j})$

(b) $\underset{\sim}{r} = -7\underset{\sim}{i} + 5\underset{\sim}{j} + \lambda(6\underset{\sim}{i} + 13\underset{\sim}{j})$

Note that there are many possible answers.

12(a) $\underset{\sim}{r} = -\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k} + \lambda(3\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k})$

(b) $\underset{\sim}{r} = 7\underset{\sim}{i} - 11\underset{\sim}{j} + 14\underset{\sim}{k} + \lambda(10\underset{\sim}{i} + 20\underset{\sim}{j} - 30\underset{\sim}{k})$

13(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad 0 \le \lambda \le 1$

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad 0 \le \lambda \le 1$

15(a) $(1, 2, 0)$ **(b)** $(3, -2, 6)$

17(a) $-\underset{\sim}{i} + \underset{\sim}{k}$ **(b)** skew

18(a) $(1, 1, 2)$ **(b)** 70.5°

19 $a = -3$

20(d) $\frac{11\sqrt{5}}{5}$ units

21 $\frac{\sqrt{145}}{5}$ units

22(a) $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ have the same direction vector. **(b)** $\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ **(c)** $\sqrt{5}$

23(b) For $\angle BAD$: $\underset{\sim}{r} = (-\underset{\sim}{i} - 2\underset{\sim}{j}) + \lambda(\underset{\sim}{i} + \underset{\sim}{j})$, for $\angle ABC$: $\underset{\sim}{r} = 2\underset{\sim}{i} + \mu(-\underset{\sim}{i} + \underset{\sim}{j})$

24(a) $\overrightarrow{OM} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$ **(b)** $\underset{\sim}{r} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$

(d) $3 : 1$

25(a) The interval or line segment AB. **(b)** The ray with endpoint B in the direction of $\underset{\sim}{b} - \underset{\sim}{a}$. **(c)** The ray with endpoint A in the direction of $\underset{\sim}{a} - \underset{\sim}{b}$.

26(a) $\sqrt{2\lambda^2 - 8\lambda + 10}$ **(b)** $\sqrt{2}$

Exercise **5G** (Page 210)

1(a) $(x - 6)^2 + (y + 9)^2 = 28$

(b) $\left| \underset{\sim}{r} - \begin{bmatrix} 6 \\ -9 \end{bmatrix} \right| = 2\sqrt{7}$

(c) $x = 6 + 2\sqrt{7}\cos\theta,\ y = -9 + 2\sqrt{7}\sin\theta$

2(a) $(x + 2)^2 + (y - 7)^2 + (z + 4)^2 = 81$

(b) $\left| \underset{\sim}{r} - \begin{bmatrix} -2 \\ 7 \\ -4 \end{bmatrix} \right| = 9$

3(a) $(x + 5)^2 + (y + 10)^2 = 45$

(b) $(x - 3)^2 + (y + 1)^2 + (z - 8)^2 = 121$

4 $\left| \underset{\sim}{r} - \begin{bmatrix} 5 \\ -3 \end{bmatrix} \right| = 2\sqrt{2},\ (x - 5)^2 + (y + 3)^2 = 8$

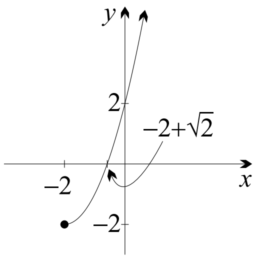
5(a) $\left| \underset{\sim}{r} - \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right| = 5$ **(b)** $\left| \underset{\sim}{r} - \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix} \right| = \frac{\sqrt{30}}{2}$

7 inside

8 centre $(2, 1, -1)$, radius $2\sqrt{5}$

9(a) $x = 2\cos t + 1, y = 2\sin t - 1$ **(b)** $(x - 1)^2 + (y + 1)^2 = 4$

10(a) $y = (x + 2)^2 - 2$ **(c)**



(b) $[-2, \infty)$

11(a) $|\underset{\sim}{r}| = \sqrt{10}$ **(b)** The radius and tangent are perpendicular. **(c)** $y = 3x - 10$

12(a) $x = 3\lambda + 1,\ y = 2\lambda - 1$

(b) $(-2, -3)$ and $(4, 1)$

14(a) Both spheres have centres on the z-axis.

(b) centre $(0, 0, \frac{9}{5})$, radius $\frac{12}{5}$

15 $(4, 4, -6)$ and $(11, -8, -3)$

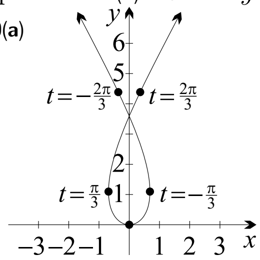
16(a) $x = 3\lambda - 2,\ y = 4\lambda + 3,\ z = 5\lambda + 4$

(b) $(7, 15, 19)$

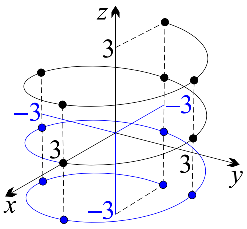
17(a) $x^2 - y^2 = 1$ **(b)** $y = \pm\frac{x^2}{\sqrt{4 - x^2}}$

18(a) $\underset{\sim}{r} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$ is a possible equation. **(b)** $13x - 4y + 3z = 17$

20(a)



(b)



$t < 0$: blue, $t \geq 0$: black

Review Exercise **5H** (Page 213)

1 $|\underset{\sim}{a}| = 7$, $\hat{\underset{\sim}{a}} = \frac{6}{7}\underset{\sim}{i} - \frac{3}{7}\underset{\sim}{j} + \frac{2}{7}\underset{\sim}{k}$

2(a) $-5\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$ **(b)** $5\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}$ **(c)** $3\sqrt{10}$

5(a) 50 **(b)** 44 **(c)** 8 **(d)** 110

7 $\lambda = -1$

8 $\cos\theta = \frac{6}{\sqrt{102}}$

9 $\frac{44}{29}\underset{\sim}{i} - \frac{33}{29}\underset{\sim}{j} - \frac{22}{29}\underset{\sim}{k}$

10(a) $\overrightarrow{AP} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$, $\overrightarrow{AB} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$ **(b)** $\frac{5}{11}\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

(c) $\frac{2\sqrt{506}}{11}$ units

11 81°

13 $\underset{\sim}{r} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

14 $x - 3y = 14$

15 $(0, 7, -3), (-14, 0, 11), (-3, \frac{11}{2}, 0)$

16(a) no **(b)** yes

17 $\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} + \lambda(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k})$

18(a) $(2, 3, -5)$ **(b)** $(-3, 5, -1)$

19(a) $(x-3)^2 + (y+4)^2 + (z-2)^2 = 7$

(b) $\left| \underset{\sim}{r} - \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix} \right| = \sqrt{7}$

21 $\left| \underset{\sim}{r} - \begin{bmatrix} 2 \\ 5 \\ -6 \end{bmatrix} \right| = 2\sqrt{6}$

22 $(4, -5, 1)$ and $(6, 8, -10)$

23(a) $y = \frac{8}{4+x^2}$ **(b)** $x^2 + y^2 = 1$, where $y \neq -1$.

(c) $y = \pm 2x\sqrt{1 - x^2}$

Chapter Six

Exercise **6A** (Page 223)

1(a) $t = \frac{1}{6}(x-1)$, $x = 6t + 1$

(b) $t = \frac{1}{18}(1 - x^3)$, $x = (1 - 18t)^{\frac{1}{3}}$

(c) $t = \frac{1}{12}(x^{-2} - 1)$, $x = (12t + 1)^{-\frac{1}{2}}$

(d) $t = \frac{1}{2}(e^{2x} - e^2)$, $x = \frac{1}{2}\ln(2t + e^2)$

(e) $t = \tan^{-1} x - \frac{\pi}{4}$, $x = \tan(t + \frac{\pi}{4})$

(f) $t = \tan x - \tan 1$, $x = \tan^{-1}(t + \tan 1)$

2(a) $\ddot{x} = 0$ **(b)** $\ddot{x} = -72x^{-5}$ **(c)** $\ddot{x} = 108x^5$

(d) $\ddot{x} = -2e^{-4x}$ **(e)** $\ddot{x} = 2x(1 + x^2)$

(f) $\ddot{x} = -2\cos^3 x \sin x$

3(a) $v^2 = 4x^3$ **(b)** $v^2 = 2(1 - e^{-x})$ **(c)** $v^2 = \ln|2x + 1|$ **(d)** $v^2 = \tan^{-1} \frac{1}{2}x$

4(a) $t = \frac{v^3}{6}$ **(b)** $t = 2 - \frac{1}{v}$

(c) $t = \ln\left|\frac{2+v}{3}\right|$

5(a) $x = 4\ln|v|$ **(b)** $x = \frac{1}{9}v^3 - 24$

(c) $x = v + 2\ln\left|\frac{2}{2+v}\right|$

6(a) 16 **(b)** 4 **(c)** 4 **(d)** 9

7 $17\,\text{m/s}^2$ at an angle of $\tan^{-1}\frac{15}{8}$ above the horizontal

8(a) $\overrightarrow{OA} = (20\cos 32^\circ)\underset{\sim}{i} + (20\sin 32^\circ)\underset{\sim}{j}$,
$\overrightarrow{OB} = (-15\cos 54^\circ)\underset{\sim}{i} + (15\sin 54^\circ)\underset{\sim}{j}$

(b) 24 N **(c)** 70° above the positive horizontal direction or 020° T

9(a) $(6 - 2\sqrt{2})\underset{\sim}{i} + (20 - 2\sqrt{2})\underset{\sim}{j}$ **(b)** 17·5 N, 79·5° above the horizontal

11(a) 20 metres **(b)** Upwards is positive, so while the stone is rising its velocity is positive.

(c) $t = 2 - \frac{\sqrt{400 - 20x}}{10}$, $x = 20t - 5t^2$, 2 seconds

12(a) $x = 150 - \dfrac{50\ln|v|}{\ln 10}$, $x = 150$ metres

(b) $t = \dfrac{1000}{99}\left(\dfrac{1}{v} - \dfrac{1}{1000}\right)$, $t = 10\frac{1}{11}$ s

13 $v^2 = 6 - 2e^{-x}$. The acceleration is always positive, and the velocity is initially 2. Hence the velocity is always increasing with minimum 2. The particle continues to accelerate, but with limiting velocity $\sqrt{6}$.

14(a) $\ddot{x} = -12$ **(b)** $x = 3(1 - e^{-2t})$ **(c)** As $t \to \infty$, the particle moves to the limiting position $x = 3$.

15(a) 1·52 s **(b)** 8·48 m

16(a) $\frac{m}{k}$ **(b)(i)** $2ku$ **(ii)** $k\ln 3$

17(a) $v^2 = 2(x-5)(x+4)$. v^2 cannot be negative.

(b) $x = 6$ m ($x = -5$ is impossible, because the particle can never pass through the origin). The particle moves forwards with increasing velocity.

18(a) $v^2 = 2x^3$ **(b)** $x = \dfrac{2}{(t+\sqrt{2}\,)^2}$. The particle starts at $x = 1$ and moves backwards towards the origin, its speed having limit zero, and position having limit the origin.

19(a) $v^2 = 6x - 2x^3 + 16$ **(b)** Yes. (Sketch the graph of v^2 against x.)

21(b) $v^2 = V^2 + 2gR^2\,(1/x - 1/R)$, $H = 2gR^2/(2gR - V^2)$ **(c)** $11{\cdot}2\,\text{km/s}$

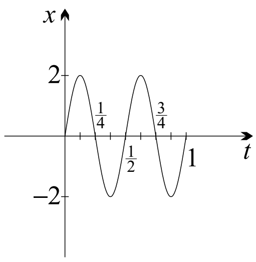
22(b) $12\ln 2 - 3{\cdot}5 \doteqdot 4{\cdot}82\,\text{m}$

Exercise **6B** (Page 232)

1(a) $12\,\text{cm}$, 4 seconds

(b) $v = -6\pi\sin\frac{\pi}{2}t$, $\ddot{x} = -3\pi^2\cos\frac{\pi}{2}t$ **(c)** $12\,\text{cm}$, $0\,\text{cm/s}$ **(d)** After 1 second **(e)** 2 seconds

2(a) $2\,\text{m}$, $\frac{1}{2}\,\text{s}$ **(b)**



(c) $v = 8\pi\cos 4\pi t$, $\ddot{x} = -32\pi^2\sin 4\pi t$

(d) $\ddot{x} = -16\pi^2 x$

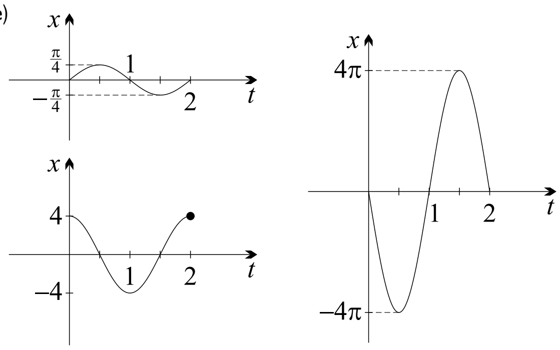
(e) $\ddot{x} = -32\pi^2$ at $t = \frac{1}{8}$ and $\ddot{x} = 32\pi^2$ at $t = \frac{3}{8}$

(f) $8\pi\,\text{m/s}$

3(b) $v = 4\cos\pi t$, $\ddot{x} = -4\pi\sin\pi t$

(c) $a = \frac{4}{\pi}$, $T = 2$ seconds **(d)** $\frac{4}{\pi}\,\text{m}$, $4\,\text{m/s}$

(e)



(f) $t = 1$ (when $v = -4\,\text{m/s}$) and $t = 2$ (when $v = 4\,\text{m/s}$) **(g)** $t = \frac{1}{2}$ (when $\ddot{x} = -4\pi\,\text{m/s}^2$) and $t = \frac{3}{2}$ (when $\ddot{x} = 4\pi\,\text{m/s}^2$)

4(a) $x = 2\sin 2t$

(b) $x = 6\sin\frac{2}{3}t$

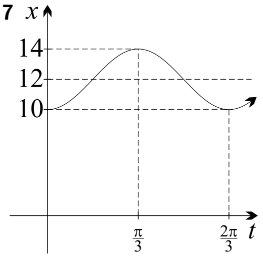
5(a) $v = bn\cos nt - cn\sin nt$, $\ddot{x} = -bn^2\sin nt - cn^2\cos nt = -n^2 x$

(b)(i) $c = 3$ and $b = 0$, so $x = 3\cos nt$.

(ii) $x = 5\cos 2\pi t$, $\frac{1}{4}\,\text{s}$

6(b) $a = 4$ **(c)** $\frac{1}{3}\,\text{s}$, $\frac{2}{3}\,\text{s}$

7



(a) $v = 6\sin 3t$, $\ddot{x} = 18\cos 3t$

(b) $a = 2$, $T = \frac{2\pi}{3}$ seconds, centre $x = 12$

(c) $10 \le x \le 14$, $\frac{\pi}{3}$ seconds

(d) $t = \frac{2\pi}{3}$ and $t = \frac{4\pi}{3}$. At both times, $|v| = 0$ and $\ddot{x} = 18\,\text{cm/s}^2$. **(e)** $t = \frac{\pi}{6}$ and $t = \frac{\pi}{2}$. At both times, $|v| = 6\,\text{cm/s}$ and $\ddot{x} = 0\,\text{cm/s}^2$.

8(a) amplitude: 6, period: π, initial phase: $\frac{\pi}{2}$

(b) $\dot{x} = 12\cos(2t + \frac{\pi}{2})$, $\ddot{x} = -24\sin(2t + \frac{\pi}{2})$, $\ddot{x} = -4x$, so $n = 2$.

(c) $t = \frac{\pi}{4}$ when $v = -12$, $t = \frac{3\pi}{4}$ when $v = 12$

(d) $t = \frac{3\pi}{4}$ and $t = \frac{7\pi}{4}$, when $x = 0$

(e) $t = \pi$ and $t = 2\pi$, when $v = 0$ and $\ddot{x} = -24$

9(a) $x = 120\sin\frac{\pi}{12}t$, $v = 10\pi\cos\frac{\pi}{12}t$, $10\pi\,\text{m/s}$

(b)(i) $\frac{12}{\pi}\sin^{-1}\frac{1}{4} \doteqdot 0{\cdot}9652$ seconds

(ii) $12 + \frac{12}{\pi}\sin^{-1}\frac{1}{4} \doteqdot 12{\cdot}97$ seconds **(c)** 4 seconds and 8 seconds

10(a) $x = 4\cos 4t$, $v = -16\sin 4t$ **(b)(i)** $\frac{\pi}{12}\,\text{s}$ **(ii)** $\frac{\pi}{6}\,\text{s}$

(c) $\frac{\pi}{24}$ seconds and $\frac{5\pi}{24}$ seconds

11 $x = \frac{1}{2} - \frac{1}{2}\cos 2t$, $x_0 = \frac{1}{2}$, $\frac{1}{2}$, $0 \le x \le 1$, π

12(a) $x = 2 - \cos 4t$ **(b)** $x_0 = 2$, $1\,\text{cm}$, $1 \le x \le 3$, $\frac{\pi}{2}\,\text{s}$ **(c)** $4\,\text{cm/s}$ when $t = \frac{\pi}{8}$

13(b) $x = 2$ **(c)** $2\pi\,\text{s}$ **(d)** $6\cos(t - \frac{\pi}{3})$ **(e)** 6, $-\frac{\pi}{3}$

(f) $-4 \le x \le 8$

14 $v = bn\cos nt - cn\sin nt$

(a) $n = \frac{1}{2}, c = 6, b = 6$, $\frac{3\pi}{2}\,\text{s}$ and $\frac{7\pi}{2}\,\text{s}$

(b) $n = \frac{\pi}{3}, c = -2, b = \frac{9}{\pi}$, about $0{\cdot}582\,\text{s}$ and $3{\cdot}582\,\text{s}$

15 $v = an\cos(nt + \alpha)$ **(a)** $n = \frac{\pi}{3}$, $\alpha = 0$, $a = \frac{15}{\pi}$

(b) $n = \frac{2}{3}$, $\alpha = \frac{3\pi}{2}$, $a = 5$ **(c)** $n = 1$, $\alpha = \frac{3\pi}{4}$, $a = \sqrt{2}$

16(a) $a = 3, \alpha = -\frac{\pi}{2}$

(b) $a = 2, \alpha = \frac{\pi}{3}$

17 $a = \frac{32\sqrt{2}}{\pi}$, $\alpha = \frac{\pi}{4}$

18 $a = 5$, $\alpha \doteqdot 2{\cdot}248$

19(a) $x = 4\sin 4t + 3\cos 4t$ **(b)** $x = 5\cos(4t - \varepsilon)$, where $\varepsilon = \tan^{-1}\frac{4}{3}$ **(c)** $5\,\text{m}$, $20\,\text{m/s}$

(d) $t = \frac{\pi}{4} - \frac{1}{4}\tan^{-1}\frac{3}{4}$, $t = \frac{\pi}{8} + \frac{1}{4}\tan^{-1}\frac{4}{3}$

20(a) 10:00 am **(b)** 7:33 am **(c)** 12:27 pm

21 11:45 am to 8:15 pm

23(b) $2\sqrt{5 + 2\sqrt{3}}$

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Exercise **6C** (Page 238)

1(a) $v = -6\sin 2t$, $\ddot{x} = -12\cos 2t$, $\ddot{x} = -4x$, $v^2 = 4(9 - x^2)$ **(b)** $\pm 2\sqrt{5}\,\text{m/s}$, $-8\,\text{m/s}^2$

2(a) $v^2 = 9(25 - x^2)$

(b) $v = \pm 12\,\text{m/s}$, $\ddot{x} = -27\,\text{m/s}^2$ **(c)** $15\,\text{m/s}$

3(a) $v^2 = 16(36 - x^2)$ **(b)** $6\,\text{cm}$, $\frac{\pi}{2}$ seconds

(c) $|v| = 16\sqrt{2}\,\text{cm/s}$, $\ddot{x} = -32\,\text{cm/s}^2$

4(a) $v^2 = 4(36 - x^2)$, π seconds, $12\,\text{m/s}$

(b)(i) $x = 6\cos 2t$ **(ii)** $x = -6\cos 2t$

(iii) $x = 6\sin 2t$ **(iv)** $x = -6\sin 2t$

5(a) $32\,\text{cm/s}$ **(b)** $8\,\text{cm}$

6(a) $a = 1$ metre **(b)** $\frac{4\pi}{3}\,\text{m/s}$

7(a) $\frac{5\pi}{2}\,\text{cm/s}$, $\frac{5\pi^2}{8}\,\text{cm/s}^2$

(b) $\pm 2\pi\,\text{cm/s}$, $\pm\frac{3\pi^2}{8}\,\text{cm/s}^2$

8 $5\sqrt{2}\,\text{m/s}$, $3\sqrt{2}\,\text{m/s}$

9 4, $2\sqrt{7}\,\text{m/s}$

10 $v^2 = -\frac{5}{3}(x^2 - 16^2)$, so the amplitude $= 16$.

11 $15\,\text{cm/s}$

13(b)(i) When $x = 0$, $|v| = an$.

(ii) When $x = \frac{1}{2}a$, $|v| = \frac{1}{2}\sqrt{3}\,an$ and $\ddot{x} = -\frac{1}{2}an^2$.

14(a) $\ddot{x} = -9(x - 1)$, centre: $x = 1$, period: $\frac{2\pi}{3}$, amplitude: 2 **(b)(i)** $\ddot{x} = -16(x - 2)$, centre: $x = 2$, period: $\frac{\pi}{2}$, amplitude: 3 **(ii)** $\ddot{x} = -9(x - 6)$, centre: $x = 6$, period: $\frac{2\pi}{3}$, amplitude: 4

(iii) $\ddot{x} = -2(x + 2)$, centre: $x = -2$, period: $\pi\sqrt{2}$, amplitude: 1 **(iv)** $\ddot{x} = -3(x + \frac{5}{3})$, centre: $x = -\frac{5}{3}$, period: $2\pi/\sqrt{3}$, amplitude: $2\frac{1}{3}$

15(a)(i) $\ddot{x} = 50\cos 10t = -100(x - \frac{1}{2})$

(ii) $\ddot{x} = -50(2\sin^2 5t - 1) = -100(x - \frac{1}{2})$

(b) centre: $x = \frac{1}{2}$, range: $0 \le x \le 1$, period: $\frac{\pi}{5}$ minutes, $t = \frac{\pi}{5}$

16(a) centre: $x = 7$. Since the amplitude is 7, the extremes of motion are $x = 0$ and $x = 14$, and the particle is stationary there.

(b) $v^2 = 9\Big(49 - (x - 7)^2\Big)$, $21\,\text{cm/s}$

(c) Although the particle is stationary for an instant, its acceleration at that time is positive (it is actually $63\,\text{m/s}^2$), and so the velocity immediately changes and the particle moves away.

17(a) $\ddot{x} = -9x$ **(b)** period: $\frac{2\pi}{3}$, amplitude: $2\sqrt{13}$, maximum speed $6\sqrt{13}$, $|\ddot{x}| = 9\sqrt{13}$

18(a) $x = 3$, $\frac{\pi}{2}$ **(b)** $x = 3 + 2\sin(4t + \frac{\pi}{3})$

(c) $t = \frac{\pi}{6}, \frac{5\pi}{12}, \frac{2\pi}{3}$, $|v| = 8$

19(a) $\ddot{x} = -4(x - 10)$, centre: $x = 10$, period: π, amplitude: 10

(b) $\frac{3\pi}{4} - \frac{1}{2}\tan^{-1}\frac{3}{4}$ $(= \pi - \tan^{-1} 2 \doteqdot 2{\cdot}034)$

20(b) $\dot{x} = -16\pi\sin 2\pi t$, $\dot{y} = 16\pi\cos 2\pi t$, $\ddot{x} = -32\pi^2\cos 2\pi t$, $\ddot{y} = -32\pi^2\sin 2\pi t$

(c)(i) $\frac{\pi}{6}$ or $\frac{7\pi}{6}$ **(ii)** $\frac{\pi}{3}$ or $\frac{4\pi}{3}$ **(iii)** $\frac{3\pi}{4}$ or $\frac{7\pi}{4}$

21(b) $a = \sqrt{\dfrac{{v_2}^2{x_1}^2 - {v_1}^2{x_2}^2}{{v_1}^2 - {v_2}^2}}$

(c) $5\,\text{cm}$, π seconds, $10\,\text{cm/s}$

22 $v = \frac{1}{2}V\sqrt{3}$ or $v = -\frac{1}{2}V\sqrt{3}$, $x = \frac{1}{2}a\sqrt{3}$ or $x = -\frac{1}{2}a\sqrt{3}$

23(b) When $\alpha = \pi$, $A = 3$ and $x = 3\sin t$. When $\alpha = 0$, $A = 1$ and $x = -\sin t$.

(c) twice **(d)** When $\alpha = \frac{\pi}{3}$, $x = \sqrt{3}\cos t$. When $\alpha = \frac{5\pi}{3}$, $x = -\sqrt{3}\cos t$.

Exercise **6D** (Page 245)

1(b) 360 metres

2(c) 80%

4(a) The force of the spring is directed towards the origin.The resistive force is in the opposite direction to the velocity. **(b)** $\ddot{y} + 3\dot{y} + 2y = 0$

(e) $A = -1$, $B = 1$

6(b) $v^2 = \frac{Q^2}{(1+Q^2)e^{2t} - Q^2}$ **(c)** $\lim\limits_{t\to\infty} v = 0$. Since $v \ne 0$, it can never change sign. **(d)** $\tan^{-1} Q$

(e) Yes. Since Q and v have the same sign, $\tan^{-1} Q - \tan^{-1} v = \tan^{-1}\frac{Q - v}{1 + Qv}$. In contrast, evaluate each side when $Q = \sqrt{3}$ and $v = -\sqrt{3}$.

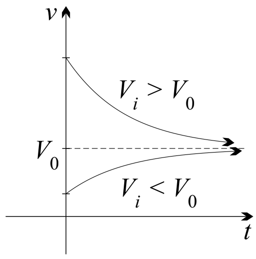
7(a) $A = B = \frac{1}{5}$ **(b)(i)** 1 **(ii)** $t = \frac{m}{5\times 10^4}\log\left(\frac{6+2v}{6-3v}\right)$

8(c) 878 metres **(d)** 1190 metres

9(a) $F = mP - mkv$

(b) $v = \frac{P}{k}$

(d) $\frac{1}{k}\log 2$



10(a) At $x = 1$ the acceleration is positive.

(c) The velocity approaches 1 from above.

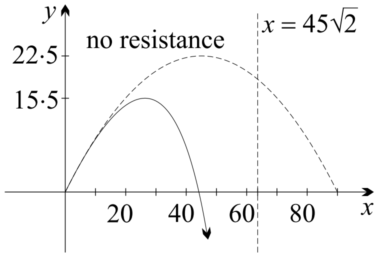
11(a)(ii) $0 \le x \le \frac{2\sqrt{V_0}}{k}$, and $v = 0$ when $x = \frac{2\sqrt{V_0}}{k}$.

(b)(i) $x = \frac{2\sqrt{V_0}}{k}\left(1 - \frac{2}{2 + kt\sqrt{V_0}}\right)$

(ii) $\lim\limits_{t\to\infty} x = \frac{2\sqrt{V_0}}{k}$, and $\lim\limits_{t\to\infty} v = 0$.

Exercise 6E (Page 254)

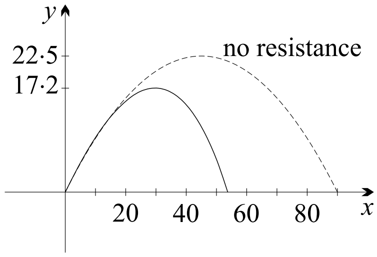
1(b) 1·9 seconds **(c)** 25 metres
4(b)(i) $v\dfrac{dv}{dy} = \dfrac{400 - v^2}{40}$ **(ii)** $10\sqrt{2}\,\text{m/s}$
5(d) 40 metres
9(a)(i) 0 **(iii)** $f'(x) > 0$ for $x > g$. That is, f is increasing for $x > g$. **(b)(ii)** $T = \frac{1}{k}\ln\left(\dfrac{g + kV_0}{g}\right)$
10(b)(i) $\ddot{x} = 10 - \frac{1}{10}v^2$ **(ii)** $10\,\text{m/s}$ **(iv)** $t \doteqdot 1{\cdot}446$
11(b) $y = 45(2 + \sqrt{2})(1 - e^{-t/3}) - 30t$ **(d)** $R \doteqdot 44$



Here is the trajectory for $0 \le t \le 4$. The dotted line is the trajectory for no air resistance.
13(a)(ii) $x(t + \delta t) \doteqdot x(t) + \delta t\, u(t)$

$y(t + \delta t) \doteqdot y(t) + \delta t\, v(t)$

$v(t + \delta t) \doteqdot v(t)\left(1 - k\,\delta t\left(u(t)^2 + v(t)^2\right)^{\frac{1}{2}}\right) - g\delta t$
(b)(i) $u(0{\cdot}1) \doteqdot 20{\cdot}51$
(ii) $v(0{\cdot}1) \doteqdot 19{\cdot}51$, $x(0{\cdot}1) \doteqdot 2{\cdot}12$, $y(0{\cdot}1) \doteqdot 2{\cdot}12$
(c) $u(0{\cdot}2) \doteqdot 19{\cdot}86$, $v(0{\cdot}2) \doteqdot 17{\cdot}89$, $x(0{\cdot}2) \doteqdot 4{\cdot}17$, $y(0{\cdot}2) \doteqdot 4{\cdot}07$ **(d)** $R \doteqdot 54$



Exercise 6F (Page 259)

1(a) $x = 20t$, $y = -5t^2 + 20\sqrt{3}t$ **(b)** $4\sqrt{3}$ seconds, $80\sqrt{3}$ metres **(c)** $2\sqrt{3}$ seconds, 60 metres
(d) It is false. The horizontal range would not have changed, although the flight time would have been 4 seconds and the maximum height would have been 20 metres.
2(a) $\dot{x} = 15, \dot{y} = -10t + 36, x = 15t, y = -5t^2 + 36t$
(b) 52 m **(c)** 21.9 m/s **(d)** after
3(a) $x = 10\sqrt{3}t$, $y = -5t^2 + 10t$
(b) 5 s, $50\sqrt{3}$ metres **(c)** 80 metres
(d) 44 m/s, 67° **(e)** $y = -\frac{1}{60}x^2 + \frac{1}{\sqrt{3}}x$
4(a) 101 m/s **(b)** $x = 101t$, $y = -5t^2$ **(d)** 149 m/s, $\tan^{-1}\frac{20\sqrt{30}}{101} \doteqdot 47°19'$ below the horizontal
(e) 1·106 km
6(c) $V = 36$, $\theta \doteqdot 41°49'$ **(d)** 129 metres
7(c)(i) $\alpha = 15°$ or $75°$
(ii) It will if $\alpha = 75°$, but not if $\alpha = 15°$.
8(b) range: 38·4 metres, height: 12·8 metres
(c)(ii) 33·3 metres
9(d) $60°15'$ or $72°54'$
10(b) 0·36 s **(c)** 12° **(d)** No, it lands 4·72 metres in front of him.
11(b)(ii) 16 metres **(iii)** 112°
12(d) 27 m
13(d) 15 metres **(e)** 10 m/s, $63°26'$
14(c) $T = 4$, $\theta = 30°$
15(b)(ii) Yes. The vertical components of their initial velocities are equal, and they are both subject to the same force (gravity) acting in the vertical direction.
18(c) For $0° < \alpha < 45°$, $0 < \tan\alpha < 1$. Hence if α_1 and α_2 are both less than 45°, then the two roots of the quadratic both lie between 0 and 1. But the product of these roots is greater than 1, so α_1 and α_2 cannot both be less than 45°.
19(a) $(d\cos\beta, d\sin\beta)$
20(b)(iv) 52°

Exercise 6G (Page 269)

1(b) 6·05 m
2(a) $1\frac{3}{5}$ s, $12\frac{4}{5}$ m **(c)** $24\frac{1}{3}$ m/s
3(a) $B - Mg = -Md$, $B - (M - m)g = (M - m)a$
4(a) $v^2 = {V_0}^2 + \dfrac{2gR^2}{r} - 2gR$
(b) $V_0 = \sqrt{gR} \doteqdot 1680\,\text{m/s}$
5(a) $T = mg\cos\theta$, $\dfrac{d^2\theta}{dt^2} = -\dfrac{g\sin\theta}{L}$
6 5:09 am to 9:36 am
7(b) $\dfrac{gx^2}{2V^2}\tan^2\beta - x\tan\beta + \left(y + \dfrac{gx^2}{2V^2}\right) = 0$
(c)(iii) $\tan\beta = \dfrac{V^2}{gh\cot\alpha}$
8(c) $x^2 = -\frac{2V^2}{g}(y - \frac{V^2}{2g})$ with focal length $\frac{V^2}{2g}$ and vertex $(0, \frac{V^2}{2g})$
9 centre $= (0, -\frac{1}{2}gt^2)$, radius $= Vt$
10(b)(iii) 2ℓ
11(a) The acceleration is downwards.
(b) $h = \frac{R{V_0}^2}{20R - {V_0}^2}$ **(c)** 12 524 m
12(b) $C = 112$ **(c)** $B = 4(16 + 14e^{-7kt/4})$

(d) $78°\text{C}$ **(e)** $64°\text{C}$
13(b) 2100 **(c)** 3000 **(d)** 10%
14(b)(i) $\left(\frac{V^2\sin 2\alpha}{2g}, \frac{V^2\sin^2\alpha}{2g}\right)$ **(ii)** $\frac{V^2\cos^2\alpha}{2g}$
(iii) $y = \frac{V^2}{2g}$, $S = \left(\frac{V^2\sin 2\alpha}{2g}, -\frac{V^2\cos 2\alpha}{2g}\right)$
Notice that the directrix is independent of α.
15(b) $15(2+\sqrt{2}\,)$ s, which is about 51·2s.
16(a) $\frac{3}{2}mg$ **(c)** $\dot{y}^2 = 3g$, and since y is decreasing to that point on the curve, $\dot{y} = -\sqrt{3g}\,.$ $\dot{x} = 0$.
(d) $\frac{\sqrt{3}}{2\sqrt{g}}\left(\sin^{-1}\frac{1}{4} + 1 - \frac{\sqrt{15}}{4}\right)$
(e) The integral is improper.

Review Exercise **6H** (Page 274)

1(a) $t = \frac{1}{2}\ln|2x-1|$, $x = \frac{1}{2}(e^{2t}+1)$
(b) $t = \frac{1}{6}(x^{-1}-1)$, $x = (1+6t)^{-1}$
2(a) $\ddot{x} = 2(2x-1)$ **(b)** $\ddot{x} = 72x^3$
3(a) $v^2 = 12x$
(b) $v^2 = \frac{1}{3}(1-\cos 6x)$
4 $t = \frac{1}{4}\ln\frac{v}{2}$
5 $x = v^3 - 1$
6(a) 8 **(b)** $2\sqrt{5}$ **(c)** 12 **(d)** $\ln 3$
7 $41{\cdot}8\,\text{m/s}^2$, $035{\cdot}3°\text{T}$
8(a) $\ddot{x} = -\frac{5}{2}\,\text{m/s}$, $v^2 = 10\,000 - 5x$
(b)(i) $v = 50\sqrt{2}\,\text{m/s}$ **(ii)** $x = 1500$ metres
(c) The plane is still moving forwards while it is braking. **(d)** $x = 100t - \frac{5}{4}t^2$, 40 seconds
9(a) $v = 500 - 5x$, $x = 100\left(1 - e^{-5t}\right)$
(b) The torpedo moves to a limiting position of $x = 100$ as the velocity decreases to zero.
10(a) $v^2 = e^{-x}$ **(b)** v is initially positive, and is never zero. $x = 2\ln\frac{1}{2}(t+2)$ **(c)** As $t \to \infty$, $x \to \infty$ (slowly) and $v \to 0$.
12(b) $\sqrt{2}$ **(c)** $x = \sqrt{2}\cos\left(4t + \frac{\pi}{4}\right)$
13(b) $2\sin\left(3t - \frac{\pi}{6}\right)$ **(c)** the amplitude is 2, the centre is $x = 5$ **(d)** $t = \frac{\pi}{18}$
14 $\frac{2}{3}$ m, $\frac{2\pi}{3}$ s
15(a) $18{\cdot}8\,\text{m/s}$ **(b)** $3553\,\text{N}$
16(c) $0{\cdot}34\,\text{s}$
18(b) $12\ {\cdot}1\,\text{m/s}$
19(a) $\ddot{x} = 10 - 20v$ **(d)** $0{\cdot}035\,\text{s}$
20(b) $\frac{V}{g}\ln 2$
22(a) $\dot{x} = 48, \dot{y} = -10t + 36, x = 48t, y = -5t^2 + 36t$ **(b)** $64{\cdot}8\,\text{m}$ **(c)** $345{\cdot}6\,\text{m}$ **(d)** $48\underset{\sim}{i} + 20\underset{\sim}{j}$
23(b) $R = 21{\cdot}6$ metres, $H = 4{\cdot}05$ metres
(c) $\tan^{-1}\frac{3}{4}$ **(d)** $15\,\text{m/s}$
(e) $t = 0{\cdot}8$, when $x = 9{\cdot}6$, and $t = 1$, when $x = 12$
24(a) Initially, $\dot{x} = \sqrt{5}$ and $\dot{y} = 2\sqrt{5}\,.$ **(b)** $\dot{x} = \sqrt{5}\,, x = t\sqrt{5}\,, \dot{y} = -10t + 2\sqrt{5}\,, y = -5t^2 + 2t\sqrt{5}$
(c) 1 metre **(d)** 2 metres **(e)** $\dot{x} = \sqrt{5}\,, \dot{y} = -2\sqrt{5}\,,$ $v = 5\,\text{m/s}$, $\theta = -\tan^{-1}2$ **(f)** $y = 2x - x^2$
25(a) $\dot{x} = 200$, $\dot{y} = 0$ **(b)** $\dot{x} = 200$, $x = 200t$, $\dot{y} = -10t$, $y = -5t^2$, $y = -\frac{1}{8000}x^2$ **(c)** 600 metres
(d) $8°32'$
26(b) $y = x - \frac{1}{90}x^2$
27(b) $62°22'$ or $37°5'$
28(c) $7{\cdot}5\,\text{m}$ **(d)** $0{\cdot}8 \le m \le 1{\cdot}2$ or $2{\cdot}8 \le m \le 3{\cdot}2$
29(c) $\frac{1}{k}\ln\left(1 + \frac{ku}{g}\sin\alpha\right)$ **(d)** $\frac{u}{k}\cos\alpha$

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