INTRODUCTION TO DIFFERENTIAL TOPOLOGY

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Preface

These are notes for the lecture course *"Differential Geometry II"* held by the second author at ETH Zürich in the spring semester of 2018. A prerequisite is the foundational chapter about smooth manifolds in [35] as well as some basic results about geodesics and the exponential map. For the benefit of the reader we summarize some of the relevant background material in the first chapter and in the appendix. The lecture course covered the content of Chapters 1 to 7 (except §1.6 and §6.5).

The first half of the book deals with degree theory and the Poincaré–Hopf theorem, the Pontryagin construction, intersection theory, and Lefschetz numbers. In this part we follow closely the beautiful exposition of Milnor in [26]. For the additional material on intersection theory and Lefschetz numbers a useful reference is the book by Guillemin and Pollack [12].

The second half of the book is devoted to differential forms and de Rham cohomology. It begins with an elementary introduction into the subject and continues with some deeper results such as Poincaré duality, the Čech–de Rham complex, and the Thom Isomorphism Theorem. Many of our proofs in this part are taken from the classical textbook of Bott and Tu [3] which is also a highly recommended reference for a deeper study of the subject (including sheaf theory, homotopy theory, and characteristic classes).

Here is a description of the content of the book, chapter by chapter. Chapter 1 is devoted to degree theory modulo two and covers the content of the first four chapters of Milnor's book [26]. It begins with a recap of the basic definitions of smooth manifolds and smooth maps in the intrinsic setting, followed by the introduction of tangent spaces, derivatives, regular values, and a proof of the Fundamental Theorem of Algebra. The chapter continues with the Theorem of Sard, an introduction to the concepts of a submanifold and a manifold with boundary, a proof of the Brouwer Fixed Point Theorem, and a proof of Sard's Theorem for smooth maps. It then introduces the degree modulo two and, as applications, includes the Borsuk–Ulam Theorem and Brouwer's Invariance of Domain Theorem.

Chapter 2 introduces the Brouwer degree and proves the Poincaré–Hopf Theorem, following [26, §5 & §6]. It begins with an introduction to oriented manifolds and then defines the degree of a smooth map between compact oriented manifolds of the same dimension. As an application it introduces the index of an isolated zero of a vector field and proves the Poincaré–Hopf Theorem, which asserts that the sum of the indices of the zeros of a vector field with only isolated zeros is independent of the choice of the vector field and hence is a topological invariant of the underlying smooth manifold. The proof that this invariant is the Euler characteristic (the alternating sum of the Betti numbers) is deferred to Chapter 6.

Chapter 3 studies maps from a compact manifold without boundary to a sphere of smaller or equal dimension, following [26, §7]. In this situation the degree is replaced by the Pontryagin construction of a framed submanifold. The submanifold is the pre-image of a regular value and the framing is a trivialization of the normal bundle, determined by a positive basis of the tangent space of the regular value. The main theorem asserts that two such maps are homotopic if and only if the corresponding Pontryagin manifolds are framed cobordant. In the case of equal dimension this can be used to prove the Hopf Degree Theorem, which asserts that two maps from a compact connected oriented smooth manifold without boundary to a sphere of the same dimension are smoothly homotopic if and only if they have the same degree. As an application we include a proof of a theorem of Hopf which assert that manifolds with Euler characteristic zero admit vector fields without zeros.

Chapter 4 deals with the subject of intersection theory, which can be thought of as an extension of degree theory to a setting where the target manifold has a larger dimension than the source manifold and the regular value is replaced by a submanifold of the target which is transverse to the map. If the dimensions match, this gives rise to an intersection number of a map and a submanifold of the target, generalizing the degree. The chapter begins with an introduction to Thom–Smale transversality and then defines intersection numbers, both modulo two and as integers in the oriented case, and shows that these numbers are invariant under homotopy. The next topic is the self-intersection number of a submanifold of middle dimension. This number can be expressed as the sum of the indices of the zeros of a section of the normal bundle with only isolated zeros, in analogy to the Poincaré–Hopf Theorem. The final section of this chapter introduces the Lefschetz number of a smooth map from a manifold to itself and establishes its homotopy invariance. For compact oriented manifolds without boundary the Lefschetz number of a map can be defined as the intersection number of the graph with the diagonal. However, orientability is not required and the boundary can be nonempty. Then the Lefschetz number of a map is defined as the sum of the fixed point indices, provided that the fixed points are all isolated and there are no fixed points on the boundary. The proof that the Lefschetz number is the alternating sum of the traces on cohomology is again deferred to Chapter 6. This concludes the first half of the book.

Chapter 5 begins with the exterior algebra of a vector space, followed by an introduction to differential forms on smooth manifolds and a proof of Stokes' Theorem. An important result with many applications is Cartan's formula for the Lie derivative of a differential form in the direction of a vector field. It can be used to prove that a differential form of top degree on a compact connected oriented smooth manifold without boundary is exact if and only if its integral vanishes. Combining degree theory with integration we prove that the integral of the pullback is the product of the integral with the degree. The Gauß–Bonnet formula can then be derived as a corollary of the Poincaré–Hopf Theorem. The chapter closes with an introduction to Moser isotopy for volume forms.

The de Rham cohomology groups are already introduced in Chapter 5 and are examined in earnest in Chapter 6. The chapter begins with the Poincaré Lemma, followed by the Mayer–Vietoris sequence, a proof that the de Rham cohomology groups are finite-dimensional for smooth manifolds that admit finite good covers, a proof of the Künneth formula for the de Rham cohomology of the product of two manifolds, and the Mayer–Vietoris sequence for the compactly supported de Rham cohomology. The chapter then moves on to prove Poincaré duality. This is used to establish the homological formulas for the Lefschetz number and the Euler characteristic. The chapter closes with a section on the Čech–de Rham complex, which was not part of the lecture course but is useful for proving de Rham's theorem.

Chapter 7 introduces vector bundles over smooth manifolds, establishes the Thom Isomorphism Theorem, and defines the Euler class of an oriented vector bundle. Applications include the product structure on the de Rham cohomology of complex projective space, and a result about the relation between the Thom class and intersection theory, which is used in the proof of the homological formula for the Lefschetz number.

Chapter 8 was not part of the lecture course. It introduces connections on vector bundles and their curvature, explains Chern–Weil theory, and defines the Chern classes as the main example. The chapter also discusses some applications of the Chern classes to questions in geometry and topology, and then gives an outlook to some deeper results in low-dimensional topology which go beyond the scope of the present book.

The appendix explains some useful background material for differential topology. The first section shows that second countable locally compact Hausdorff spaces are paracompact, the second section establishes partitions of unity, and the third section proves that every second countable Hausdorff manifold with boundary admits an embedding into Euclidean space. This allows us to use the extrinsic setting whenever that is convenient. The fourth section classifies smooth 1-manifolds, following the exposition of Milnor [26]. This classification is an essential tool in differential topology. The last two sections summarize some results about Riemannian metrics and geodesics (proved in [35]) that are useful for the construction of tubular neighborhoods and of finite good covers.

In closing, we note that the sections of the book (except for the exercise sections as well as §1.6, §6.5, and Chapter 8) are essentially in one-to-one correspondence with the lectures of the course. Thus Chapter 1 was covered in five lectures, Chapter 2 in three lectures, Chapter 3 in two and a half lectures, Chapter 4 in three and a half lectures, Chapters 5 and 6 in four lectures each, and Chapter 7 in three lectures. The last lecture discussed some of the material in §8.5 and §8.6 as an outlook.

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30 July 2025 *Joel W. Robbin and Dietmar A. Salamon*

Contents

Chapter 1

Degree Theory Modulo Two

In this and the following two chapters we follow closely the beautiful book *"Topology from the Differentiable Viewpoint"* by Milnor [26]. Milnor's masterpiece of mathematical exposition cannot be improved. The only excuse we can offer for including the material in this book is for completeness of the exposition. There are, nevertheless, two minor points in which the first three chapters of this book differ from [26]. The first is that our exposition uses both the extrinsic and the intrinsic notion of a smooth manifold, switching between them whenever it is convenient for the proofs or the examples. The basic definitions are included in §1.1 and the proofs of some foundational theorems such as the existence of partitions of unity and of embeddings into Euclidean spaces are relegated to the appendix. For a more extensive discussion of these concepts the reader is referred to the two introductory chapters of [35] which are understood as prerequisites for the present book. A second minor point of departure from Milnor's text is the inclusion of the Borsuk–Ulam Theorem in §1.5 and of the Brower Invariance of Domain Theorem in §1.6 at the end of the present chapter. In particular, the proof of the Borsuk–Ulam Theorem already uses some of the arguments that reappear in the proof of the Poincaré–Hopf Theorem in Chapter 2.

The other four sections of the present chapter follow closely the first four chapters of Milnor's book [26]. After the introductory section (§1.1), which includes a proof of the Fundamental Theorem of Algebra, we discuss the Theorem of Sard, introduce manifolds with boundary, and prove the Brouwer Fixed Point Theorem in §1.2, include a proof of Sard's Theorem in §1.3, and introduce the degree modulo two of a smooth map in §1.4. Throughout we assume that the reader is familiar with first year analysis and the basic notions of point set topology.

1.1 Smooth Manifolds and Smooth Maps

The subject of oifferential topology is concerned with smooth manifolds and smooth maps between them. Before giving the formal definitions, let us recall some basic terms. The m-dimensional Euclidean space $\mathbb{R}^m$ consists of all m-tuples $x = (x_1, \dots, x_m)$ of real numbers and is endowed with the topology induced by the Euclidean norm $|x| := \sqrt{x_1^2 + \cdots + x_m^2}$.

Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. A map $f : U \to V$ is called **smooth** iff all its partial derivatives $\partial^k f/\partial x_{i_1} \cdots \partial x_{i_k}$ exist and are continuous. For a smooth map $f = (f_1, \dots, f_n) : U \to V$ and a point $x \in U$ the **derivative of f at x** is the linear map $df(x) : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$df(x)\xi := \left.\frac{d}{dt}\right|_{t=0} f(x + t\xi) = \lim_{t\to 0} \frac{f(x + t\xi) - f(x)}{t}, \qquad \xi \in \mathbb{R}^m.$$

This linear map is represented by the **Jacobian matrix** of f at x which will also be denoted by

$$df(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_m}(x) \end{pmatrix} \in \mathbb{R}^{n\times m}.$$

Thus we use the same notation for the Jacobian matrix and for the linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ determined by this matrix. The derivative satisfies the **chain rule**: If $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^p$ are open sets and $f : U \to V$ and $g : V \to W$ are smooth maps, then $g \circ f : U \to W$ is smooth and

$$d(g \circ f)(x) = dg(f(x)) \circ df(x) : \mathbb{R}^m \to \mathbb{R}^p$$

for every $x \in U$. Moreover the identity map $\mathrm{id}_U : U \to U$ is always smooth and its derivative at every point is the identity map of $\mathbb{R}^m$. This implies that, if $f : U \to V$ is a **diffeomorphism** (i.e. f is bijective and f and f^{-1} are both smooth), then its derivative at every point is an invertible linear map and so $m = n$. The following theorem is a partial converse.

Theorem 1.1.1 (**Inverse Function Theorem**). *Let $\Omega \subset \mathbb{R}^m$ be an open set and let $f : \Omega \to \mathbb{R}^m$ be a smooth map. If the derivative $df(x) : \mathbb{R}^m \to \mathbb{R}^m$ at $x \in \Omega$ is invertible, then there exists an open neighborhood $U \subset \Omega$ of x such that $f(U)$ is open and f maps U diffeomorphically onto $f(U)$.*

Proof. See for example [37, Appendix C]. □

Note that the map $f : \Omega \to \mathbb{R}^m$ in Theorem 1.1.1 may not be injective on all of Ω, even if its derivative is everywhere invertible. An example is the exponential map from the complex plane to itself.

1.1.1 Smooth Manifolds

Here is the basic definition of the class of spaces studied in this book.

Definition 1.1.2 (Smooth m-Manifold). *Let $m \in \mathbb{N}_0$. A* **smooth m-manifold** *is a topological space M, equipped with an open cover $\{U_\alpha\}_{\alpha\in A}$ and a collection of homeomorphisms $\phi_\alpha : U_\alpha \to \Omega_\alpha$ onto open sets $\Omega_\alpha \subset \mathbb{R}^m$, such that for each pair $\alpha, \beta \in A$ the transition map*

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.1.1}$$

is smooth (see Figure 1.1). The homeomorphisms ϕ_α are called **coordinate charts***, their inverses $\psi_\alpha := \phi_\alpha^{-1} : \Omega_\alpha \to U_\alpha$ are called* **parametrizations (of U_α)***, and the collection $\mathscr{A} := \{U_\alpha, \phi_\alpha\}_{\alpha\in A}$ is called an* **atlas***.*

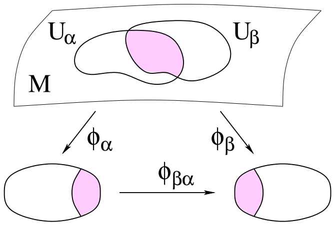

Figure 1.1: Coordinate charts and transition maps.

Let $(M, \mathscr{A} = \{U_\alpha, \phi_\alpha\}_{\alpha\in A})$ be a smooth m-manifold. Then a subset $U \subset M$ is open if and only if $\phi_\alpha(U \cap U_\alpha)$ is an open subset of $\mathbb{R}^m$ for every $\alpha \in A$. Thus the topology on M is uniquely determined by the atlas. A homeomorphism $\phi : U \to \Omega$ from an open set $U \subset M$ to an open set $\Omega \subset \mathbb{R}^m$ is called **compatible with the atlas** $\mathscr{A}$ iff the transition map $\phi_\alpha \circ \phi^{-1} : \phi(U \cap U_\alpha) \to \phi_\alpha(U \cap U_\alpha)$ is a diffeomorphism for each α. The atlas $\mathscr{A}$ is called **maximal** iff it contains every coordinate chart that is compatible with all its members. Thus every atlas $\mathscr{A}$ is contained in a unique maximal atlas $\overline{\mathscr{A}}$, consisting of all coordinate charts $\phi : U \to \Omega$ that are compatible with $\mathscr{A}$. Such a maximal atlas is also called a **smooth structure** on the topological space M. We do not distinguish the manifolds $(M, \mathscr{A})$ and $(M, \mathscr{A}')$ if the corresponding maximal atlasses agree, i.e. if the charts of $\mathscr{A}'$ are all compatible with $\mathscr{A}$ (and vice versa) or, equivalently, if the union $\mathscr{A} \cup \mathscr{A}'$ is again a smooth atlas. If this holds, we say that $\mathscr{A}$ and $\mathscr{A}'$ induce the same smooth structure on M. Since the manifolds in this book are all smooth, we will often use the phrase "*let M be an m-manifold*" instead of "*let M be a smooth m-manifold*".

Example 1.1.3. The Euclidean space $M = \mathbb{R}^n$ is a smooth manifold, where the standard smooth structure is determined by an atlas consisting of a single coordinate chart with $U_\alpha = \Omega_\alpha = \mathbb{R}^m$ and $\phi_\alpha : U_\alpha \to \Omega_\alpha$ the identity map.

Example 1.1.4. The m-sphere

$$S^m := \left\{ x \in \mathbb{R}^{m+1} \,|\, x_1^2 + \cdots + x_{m+1}^2 = 1 \right\}$$

is a smooth manifold with the coordinate charts $\phi_\pm : U_\pm \to \mathbb{R}^m$ given by the stereographic projections from the north and south pole (see Exercise 1.7.1):

$$U_\pm := S^n \setminus \{(0, \ldots, 0, \mp 1)\}, \qquad \phi_\pm(x) := \left(\frac{x_1}{1 \pm x_{m+1}}, \ldots, \frac{x_n}{1 \pm x_{m+1}} \right).$$

Example 1.1.5. The **real m-torus** is the space $\mathbb{T}^m := \mathbb{R}^m / \mathbb{Z}^m$ equipped with the quotient topology. Thus $x, y \in \mathbb{R}^m$ are equivalent iff $x - y \in \mathbb{Z}^m$. Denote by $\pi : \mathbb{R}^m \to \mathbb{T}^m$ the canonical projection which assigns to each vector $x \in \mathbb{R}^m$ its equivalence class $\pi(x) := [x] := x + \mathbb{Z}^m$. Then a set $U \subset \mathbb{T}^m$ is open if and only if $\pi^{-1}(U)$ is an open subset of $\mathbb{R}^m$. An atlas is given by the open sets $U_\alpha := \pi(\Omega_\alpha)$, where $\Omega_\alpha := \{x \in \mathbb{R}^m \mid |x - \alpha| < 1/2\}$, and the coordinate charts $\phi_\alpha([x]) := x$ for $\alpha \in \mathbb{R}^m$ and $x \in \Omega_\alpha$

Example 1.1.6. The **complex projective space** $\mathbb{C}\mathrm{P}^n$ is the set

$$\mathbb{C}\mathrm{P}^n = \left\{ \ell \subset \mathbb{C}^{n+1} \,|\, \ell \text{ is a 1-dimensional complex subspace} \right\}$$

of complex lines in $\mathbb{C}^{n+1}$. It can be identified with the quotient

$$\mathbb{C}\mathrm{P}^n = \left(\mathbb{C}^{n+1} \setminus \{0\} \right) / \mathbb{C}^*$$

of the space of nonzero vectors in $\mathbb{C}^{n+1}$ modulo the action of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of nonzero complex numbers. The equivalence class of a nonzero vector $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ will be denoted by

$$[z] = [z_0 : z_1 : \cdots : z_n] := \{\lambda z \,|\, \lambda \in \mathbb{C}^*\}$$

and the associated line is $\ell = \mathbb{C}z$. An atlas on $\mathbb{C}\mathrm{P}^n$ is given by the open cover $U_i := \{[z_0 : \cdots : z_n] \,|\, z_i \neq 0\}$ for $i = 0, 1, \ldots, n$ and the coordinate charts $\phi_i : U_i \to \mathbb{C}^n$ are

$$\phi_i([z_0 : \cdots : z_n]) := \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i} \right). \tag{1.1.2}$$

Exercise: Prove that each ϕ_i is a homeomorphism and the transition maps are holomorphic. Prove that the manifold topology is the quotient topology, i.e. if $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathrm{P}^n$ denotes the obvious projection, then a subset $U \subset \mathbb{C}\mathrm{P}^n$ is open if and only if $\pi^{-1}(U)$ is an open subset of $\mathbb{C}^{n+1} \setminus \{0\}$.

Example 1.1.7. The **real projective space** $\mathbb{R}\mathrm{P}^n$ is the set

$$\mathbb{R}\mathrm{P}^n = \left\{\ell \subset \mathbb{R}^{n+1} \,|\, \ell \text{ is a 1-dimensional linear subspace}\right\}$$

of real lines in $\mathbb{R}^{n+1}$. It can again be identified with the quotient

$$\mathbb{R}\mathrm{P}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right)/\mathbb{R}^*$$

of the space of nonzero vectors in $\mathbb{R}^{n+1}$ modulo the action of the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ of nonzero real numbers, and the equivalence class of a nonzero vector $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ will be denoted by

$$[x] = [x_0 : x_1 : \cdots : x_n] := \{\lambda x \,|\, \lambda \in \mathbb{R}^*\}.$$

An atlas on $\mathbb{R}\mathrm{P}^n$ is given by the open cover $U_i := \{[x_0 : \cdots : x_n] \,|\, x_i \neq 0\}$ and the coordinate charts $\phi_i : U_i \to \mathbb{R}^n$ are again given by (1.1.2), with z_j replaced by x_j. The arguments in Example 1.1.6 show that these coordinate charts form an atlas and the manifold topology is the quotient topology. The transition maps are real analytic diffeomorphisms.

Example 1.1.8. Consider the **complex Grassmannian**

$$\mathrm{G}_k(\mathbb{C}^n) := \{V \subset \mathbb{C}^n \,|\, v \text{ is a } k\text{-dimensional complex linear subspace}\}.$$

This set can again be described as a quotient space

$$\mathrm{G}_k(\mathbb{C}^n) \cong \mathcal{F}_k(\mathbb{C}^n)/\mathrm{U}(k).$$

Here $\mathcal{F}_k(\mathbb{C}^n) := \left\{D \in \mathbb{C}^{n\times k} \,|\, D^*D = \mathbb{1}\right\}$ denotes the set of unitary k-frames in $\mathbb{C}^n$ and the group $\mathrm{U}(k)$ acts on $\mathcal{F}_k(\mathbb{C}^n)$ contravariantly by $D \mapsto Dg$ for $g \in \mathrm{U}(k)$. The projection

$$\pi : \mathcal{F}_k(\mathbb{C}^n) \to \mathrm{G}_k(\mathbb{C}^n)$$

sends a matrix $D \in \mathcal{F}_k(\mathbb{C}^n)$ to its image

$$V := \pi(D) := \mathrm{im}\ D.$$

A subset $U \subset \mathrm{G}_k(\mathbb{C}^n)$ is open if and only if $\pi^{-1}(U)$ is an open subset of $\mathcal{F}_k(\mathbb{C}^n)$. Every k-dimensional subspace $V \subset \mathbb{C}^n$ determines an open set $U_V \subset \mathrm{G}_k(\mathbb{C}^n)$ consisting of all k-dimensional subspaces of $\mathbb{C}^n$ that can be represented as graphs of linear maps from V to $V^\perp$. This set of graphs can be identified with the space $\mathrm{Hom}^{\mathbb{C}}(V, V^\perp)$ of complex linear maps from V to $V^\perp$ and hence with $\mathbb{C}^{(n-k)\times k}$. This leads to an atlas on $\mathrm{G}_k(\mathbb{C}^n)$ with holomorphic transition maps and shows that $\mathrm{G}_k(\mathbb{C}^n)$ is a manifold of complex dimension $k(n-k)$. **Exercise:** Verify the details of this construction. Find explicit formulas for the coordinate charts and their transition maps. Carry this over to the real setting. Show that $\mathbb{C}\mathrm{P}^n$ and $\mathbb{R}\mathrm{P}^n$ are special cases.

Example 1.1.9 (The real line with two zeros). A topological space M is called **Hausdorff** if any two points in M can be separated by disjoint open neighborhoods. The following example shows that a manifold need not be a Hausdorff space. Consider the quotient space $M := \mathbb{R} \times \{0,1\}/\equiv$ where $[x,0] \equiv [x,1]$ for $x \neq 0$. An atlas on M consists of two coordinate charts $\phi_0 : U_0 \to \mathbb{R}$ and $\phi_1 : U_1 \to \mathbb{R}$ where $U_i := \{[x,i] \,|\, x \in \mathbb{R}\}$ and $\phi_i([x,i]) := x$ for $i = 0,1$. Thus M is a 1-manifold. The topology on M is not Hausdorff, because the points $[0,0]$ and $[0,1]$ cannot be separated by disjoint open neighborhoods.

Example 1.1.10 (A 2-manifold without a countable atlas). Consider the vector space $X = \mathbb{R} \times \mathbb{R}^2$ with the equivalence relation

$$[t_1, x_1, y_2] \equiv [t_2, x_2, y_2] \iff \begin{array}{l} \text{either } y_1 = y_2 \neq 0,\ t_1 + x_1 y_1 = t_2 + x_2 y_2 \\ \text{or } y_1 = y_2 = 0,\ t_1 = t_2,\ x_1 = x_2. \end{array}$$

For $y \neq 0$ we have $[0, x, y] \equiv [t, x - t/y, y]$, however, each point $(x, 0)$ on the x-axis gets replaced by the uncountable set $\mathbb{R} \times \{(x,0)\}$. Our manifold is the quotient space

$$M := X/\equiv$$

with the topology induced by the atlas defined below. (This is not the quotient topology.) The coordinate charts are parametrized by the reals. For $t \in \mathbb{R}$ the coordinate chart $\phi_t : U_t \to \mathbb{R}^2$ is given by

$$U_t := \{[t, x, y] \,|\, x, y \in \mathbb{R}\}, \qquad \phi_t([t, x, y]) := (x, y).$$

A subset $U \subset M$ is open, by definition, iff $\phi_t(U \cap U_t)$ is an open subset of $\mathbb{R}^2$ for every $t \in \mathbb{R}$. With this topology each ϕ_t is a homeomorphism from U_t onto $\mathbb{R}^2$ and M admits a countable dense subset

$$S := \{[0, x, y] \,|\, x, y \in \mathbb{Q}\}.$$

However, there is no atlas on M consisting of countably many charts. (Each coordinate chart can contain at most countably many of the points $[t, 0, 0]$.) The function $f : M \to \mathbb{R}$ given by $f([t, x, y]) := t + xy$ is smooth and each point $[t, 0, 0]$ is a critical point of f with value t. Thus f has no regular value. **Exercise:** M is a path-connected Hausdorff space.

Throughout this book we will tacitly assume that manifolds are Hausdorff and second countable. This excludes pathological examples such as Example 1.1.9 and Example 1.1.10. Theorem A.3.1 shows that smooth manifolds whose topology is Hausdorff and second countable are precisely those that can be embedded in Euclidean space.

1.1.2 Smooth Maps

Smooth maps between smooth manifolds are defined as foillows.

Definition 1.1.11 (**Smooth Map**). *A map $f : M \to N$ between smooth manifolds $(M, \{(U_\alpha, \phi_\alpha)\}_{\alpha\in A})$ and $(N, \{(V_\beta, \psi_\beta)\}_{\beta\in B})$ is called* **smooth** *iff it is continuous and the map*

$$f_{\beta\alpha} := \psi_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(V_\beta) \tag{1.1.3}$$

is smooth for every $\alpha \in A$ and every $\beta \in B$. It is called a **diffeomorphism** *iff it is bijective and f and f^{-1} are smooth. Two manifolds M and N are called* **diffeomorphic** *iff there exists a diffeomorphism $f : M \to N$.*

It follows directly from the definition that compositions of smooth maps are smooth, and that the identity map on each manifold is smooth.

Example 1.1.12. The map $\mathbb{T}^1 \to S^1 : [t] \mapsto (\cos(2\pi t), \sin(2\pi t))$ is a diffeomorphism.

Example 1.1.13. The map $f : S^2 \to \mathbb{C}\mathrm{P}^1$ defined by

$$f(x) := \begin{cases} [1 + x_3 : x_1 + \mathbf{i}x_2], & \text{if } x \neq (0,0,-1), \\ [0 : 1], & \text{if } x = (0,0,-1), \end{cases}$$

for $x = (x_1, x_2, x_3) \in S^2$ is a diffeomorphism whose inverse is given by

$$f^{-1}([z_0 : z_1]) = \left(\frac{2\mathrm{Re}(\bar{z}_0 z_1)}{|z_0|^2 + |z_1|^2}, \frac{2\mathrm{Im}(\bar{z}_0 z_1)}{|z_0|^2 + |z_1|^2}, \frac{|z_0|^2 - |z_1|^2}{|z_0|^2 + |z_1|^2} \right)$$

for $[z_0 : z_1] \in \mathbb{C}\mathrm{P}^1$. Note that

$$f(x) = [1 : h(x)] \qquad \text{for } x \in S^2 \setminus \{(0,0,-1)\},$$

where

$$h : S^2 \setminus \{(0,0,-1)\} \to \mathbb{C}$$

is the stereographic projection from the south pole, given by

$$h(x) := \frac{x_1 + \mathbf{i}x_2}{1 + x_3}.$$

(See Exercise 1.7.1.)

Example 1.1.14. Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d$$

be a polynomial with complex coefficients. Then the map $f : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ defined by

$$f([z_0 : z_1]) := \left[z_0^d : a_0 z_0^d + a_1 z_0^{d-1} z_1 + \cdots + a_{d-1} z_0 z_1^{d-1} + a_d z_1^d\right]$$

for $[z_0 : z_1] \in \mathbb{C}\mathrm{P}^1$ is smooth.

Example 1.1.15. Let $A \in \mathbb{Z}^{n \times m}$ and le $b \in \mathbb{R}^n$. Then the map $x \mapsto Ax + b$ descends to a smooth map $f : \mathbb{T}^m \to \mathbb{T}^n$.

Smooth manifolds and smooth maps between them form a category whose isomorphisms are diffeomorphisms. The subject of differential topology can roughly be described as the study of those properties of smooth manifolds that are invariant under diffeomorphisms. A longstanding open problem in the field is of whether every smooth four-manifold that is homeomorphic to the four-sphere is actually diffeomorphic to the four-sphere. This is known as the **four-dimensional smooth Poincaré conjecture**.

1.1.3 Tangent Spaces and Derivatives

To carry over the notion of a derivative to smooth maps between manifolds we need the concept of a tangent space.

Definition 1.1.16 (Tangent Space). *Let $(M, \{U_\alpha, \phi_\alpha\}_{\alpha \in A})$ be a smooth m-manifold, let $(N, \{V_\beta, \psi_\beta\}_{\beta \in B})$ be a smooth n-manifold, and let $p \in M$.*

(i) *The* **tangent space of** M **at** p *is the quotient space*

$$T_pM := \bigcup_{p \in U_\alpha} \{\alpha\} \times \mathbb{R}^m / \overset{p}{\sim}, \tag{1.1.4}$$

where the union is over all $\alpha \in A$ with $p \in U_\alpha$ and

$$(\alpha, \xi) \overset{p}{\sim} (\beta, \eta) \qquad \Longleftrightarrow \qquad d\left(\phi_\beta \circ \phi_\alpha^{-1}\right)(x)\xi = \eta, \quad x := \phi_\alpha(p).$$

The equivalence class of a pair $(\alpha, \xi) \in A \times \mathbb{R}^m$ with $p \in U_\alpha$ is denoted by $[\alpha, \xi]_p$. The quotient space T_pM is a real vector space of dimension m.

(ii) *Let $f : M \to N$ be a smooth map. The* **derivative of** f **at** p *is the linear map $df(p) : T_pM \to T_{f(p)}N$ defined by*

$$df(p)[\alpha, \xi]_p := [\beta, df_{\beta\alpha}(x)\xi]_{f(p)}, \qquad x := \phi_\alpha(p), \tag{1.1.5}$$

for $\alpha \in A$ with $p \in U_\alpha$ and $\beta \in B$ with $f(p) \in V_\beta$, where the map $f_{\beta\alpha}$ is given by equation (1.1.3) in Definition 1.1.11.

Remark 1.1.17. **(i)** Think of $N = \mathbb{R}^n$ as a manifold with a single coordinate chart $\psi_\beta = \mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$. For every $q \in N = \mathbb{R}^n$ the tangent space T_qN is then canonically isomorphic to $\mathbb{R}^n$ via (1.1.4). Thus the derivative of a smooth map $f : M \to \mathbb{R}^n$ at $p \in M$ is a linear map $df(p) : T_pM \to \mathbb{R}^n$, and the formula (1.1.5) reads

$$df(p)[\alpha, \xi]_p = d(f \circ \phi_\alpha^{-1})(x)\xi$$

for $p \in U_\alpha$, $x := \phi_\alpha(p)$, and $\xi \in \mathbb{R}^m$.

(ii) The formula in part (i) applies to maps defined on open subsets of M. In particular, for $f = \phi_\alpha : U_\alpha \to \mathbb{R}^m$ we have $d\phi_\alpha(p)[\alpha, \xi]_p = \xi$. Thus the derivative $d\phi_\alpha(p) : T_pM \to \mathbb{R}^m$ is the canonical vector space isomorphism determined by α. When the coordinate chart $\phi_\alpha : U_\alpha \to \Omega_\alpha$ is understood from the context, it is customary to use the notation

$$\frac{\partial}{\partial x_i}(p) := [\alpha, e_i]_p \in T_pM \tag{1.1.6}$$

for $p \in U_\alpha$ and $i = 1, \dots, m$, where $e_1, \dots, e_m$ is the standard basis of $\mathbb{R}^m$.

(iii) For a smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ define the **derivative**

$$\dot\gamma(0) \in T_pM$$

as the equivalence class

$$\dot\gamma(0) := \left[\alpha, \tfrac{d}{dt}\big|_{t=0} \phi_\alpha(\gamma(t))\right]_p.$$

In the notation of Definition 1.1.16 the vector $\dot\gamma(0) \in T_{\gamma(0)}M$ is the image of the vector $1 \in T_0\mathbb{R} = \mathbb{R}$ under the linear map $d\gamma(0) : T_0\mathbb{R} \to T_{\gamma(0)}M$.

(iv) For every $p \in M$ and every tangent vector $v \in T_pM$ there exists a smooth curve $\gamma : \mathbb{R} \to M$ such that

$$\gamma(0) = p, \qquad \dot\gamma(0) = v.$$

To see this, choose a coordinate chart $\phi_\alpha : U_\alpha \to \Omega_\alpha$ such that $p \in U_\alpha$, define $x := \phi_\alpha(p)$ and $\xi := d\phi_\alpha(p)v$, choose a constant $\varepsilon > 0$ such that

$$x + t\xi \in \Omega_\alpha \qquad \text{for } -\varepsilon < t < \varepsilon,$$

and define

$$\gamma(t) := \phi_\alpha^{-1}\left(x + \frac{\varepsilon t}{\sqrt{\varepsilon^2 + t^2}}\xi\right)$$

for $t \in \mathbb{R}$.

The Inverse Function Theorem

A fundamental property of the derivative is the **chain rule**. It asserts that, if $f : M \to N$ and $g : N \to P$ are smooth maps between smooth manifolds, then the derivative of the composition $g \circ f : M \to P$ at $p \in M$ is the composition of the derivatives, i.e.

$$d(g \circ f)(p) = dg(q) \circ df(p), \qquad q := f(p) \in N.$$

In other words, to every commutative triangle

$$\begin{tikzcd} & N \arrow[dr, "g"] & \\ M \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & P \end{tikzcd} .$$

of smooth maps between smooth manifolds M, N, P and every $p \in M$ there corresponds a commutative triangle of linear maps

$$\begin{tikzcd} & T_qN \arrow[dr, "dg(q)"] & \\ T_pM \arrow[ur, "df(p)"] \arrow[rr, "d(g\circ f)(p)"] & & T_rP \end{tikzcd} ,$$

where $q := f(p) \in N$ and $r := g(q) \in P$. A second fundamental observation is that the derivative of the identity map $f = \mathrm{id}_M : M \to M$ at each point $p \in M$ is the identity map of the tangent space, i.e.

$$d\mathrm{id}_M(p) = \mathrm{id}_{T_pM}$$

for all $p \in M$.

Lemma 1.1.18. *Let $f : M \to N$ be a diffeomorphism between smooth manifolds and let $p \in M$. Then the derivative*

$$df(p) : T_pM \to T_{f(p)}N$$

is a vector space isomorphism. Thus M and N have the same dimension.

Proof. Denote the inverse map by $g := f^{-1} : N \to M$ and let $q := f(p) \in N$. Then $g \circ f = \mathrm{id}_M$ and so

$$dg(q) \circ df(p) = d(g \circ f)(p) = \mathrm{id}_{T_pM}$$

by the chain rule. Likewise $df(p) \circ dg(q) = d(f \circ g)(q) = \mathrm{id}_{T_qN}$ and so $df(p)$ is a vector space isomorphism with the inverse $dg(q) : T_qN \to T_pM$. This proves Lemma 1.1.18. □

A partial converse of Lemma 1.1.18 is the inverse function theorem.

Theorem 1.1.19 (Inverse Function Theorem). *Let M and N be smooth m-manifolds and let $f : M \to N$ be a smooth map. Let $p_0 \in M$ and suppose that the derivative $df(p_0) : T_{p_0}M \to T_{f(p_0)}N$ is a vector space isomorphism. Then there exists an open neighborhood $U \subset M$ of p_0 such that $V := f(U)$ is an open subset of N and the restriction $f|_U : U \to V$ is a diffeomorphism.*

Proof. For smooth maps between open subsets of Euclidean space this is the assertion of Theorem 1.1.1. The general case follows by applying the special case to the map $f_{\beta\alpha}$ in Definition 1.1.11. $\square$

1.1.4 Regular Values

Let $f : M \to N$ be a smooth map between smooth manifolds.

Definition 1.1.20 (Regular value). *An element $p \in M$ is a called a* **regular point of** *f iff $df(p) : T_pM \to T_qN$ is surjective and is called a* **critical point of** *f iff $df(p)$ is not surjective. An element $q \in N$ is called a* **regular value of** *f iff the set $f^{-1}(q)$ contains only regular points and is called a* **critical value of** *f iff it is not a regular value. Denote the set of critical points of f by*

$$\mathcal{C}_f := \left\{ p \in M \,|\, df(p) : T_pM \to T_{f(p)}N \text{ is not surjective} \right\}.$$

Then $f(\mathcal{C}_f) \subset N$ is the set of critical values of f and its complement

$$\mathcal{R}_f := N \setminus f(\mathcal{C}_f)$$

is the set of regular values of f.

Denote by $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ the set of nonnegative integers. If A is a finite set, denote by $\#A \in \mathbb{N}_0$ the number of elements of A.

Lemma 1.1.21. *Let $f : M \to N$ be a smooth map between smooth manifolds and assume that M is compact. Then the following holds.*

(i) *The set $\mathcal{R}_f \subset N$ of regular values of f is open.*

(ii) *If $\dim(M) = \dim(N)$ and $q \in \mathcal{R}_f$, then $f^{-1}(q)$ is a finite set.*

(iii) *If $\dim(M) = \dim(N)$, then the function*

$$\mathcal{R}_f \to \mathbb{N}_0 : q \mapsto \#f^{-1}(q) \tag{1.1.7}$$

is locally constant.

Proof. The set $\mathcal{C}_f$ of critical points of f is always a closed subset of M. If M is compact, it follows that $\mathcal{C}_f$ is a compact subset of M, hence its image $f(\mathcal{C}_f)$ is a compact and therefore closed subset of N, and hence the set $\mathcal{R}_f = N \setminus f(\mathcal{C}_f)$ of regular values of f is open. This proves (i).

If $\dim(M) = \dim(N)$ and q is a regular value of f, then $f^{-1}(q)$ is a discrete subset of M. Namely, if $p \in f^{-1}(q)$, then $df(p) : T_pM \to T_qN$ is bijective, hence by the Inverse Function Theorem 1.1.19 there exists a neighborhood $U \subset M$ of p such that $f|_U$ is injective, and so

$$U \cap f^{-1}(q) = \{p\}.$$

Since M is compact, every discrete subset of M is finite. This proves (ii).

To prove part (iii), fix a regular value q of f. If $\#f^{-1}(q) = 0$, then q belongs to the open set $N \setminus f(M)$ on which the function (1.1.7) vanishes. Now assume

$$k := \#f^{-1}(q) > 0$$

and write

$$f^{-1}(q) = \{p_1, \dots, p_k\}.$$

The Inverse Function Theorem 1.1.19 asserts that for each i there exist open neighborhoods $U_i \subset M$ of p_i and $V_i \subset N$ of q such that $f|_{U_i}$ is a diffeomorphism from U_i to V_i. Shrinking the U_i, if necessary, we may assume that

$$U_i \cap U_j = \emptyset \qquad \text{for } i \neq j.$$

Then the set

$$V := V_1 \cap \dots \cap V_k \setminus f\bigl(M \setminus (U_1 \cup \dots \cup U_k)\bigr)$$

is open, satisfies $q \in V \subset \mathcal{R}_f$, and $\#f^{-1}(q') = k$ for all $q' \in V$. This proves Lemma 1.1.21. □

The hypothesis that M is compact cannot be removed in Lemma 1.1.21. For example the function $f(x) := e^x \sin(x)$ (from $M = \mathbb{R}$ to $N = \mathbb{R}$) has zero as a regular value in the closure of the set of singular values, and $f^{-1}(0)$ is an infinite set. The inclusion of the open unit disk into the complex plane is an example where every complex number is a regular value but the function (1.1.7) is not locally constant.

1.1.5 The Fundamental Theorem of Algebra

As an application of these notions we prove the Fundamental Theorem of Algebra, which asserts that every nonconstant polynomial has a zero.

Let $p : \mathbb{C} \to \mathbb{C}$ be a nonconstant polynomial. Thus there exist a positive integer d and complex numbers $a_0, a_1, \dots, a_d \in \mathbb{C}$ such that $a_d \neq 0$ and

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d$$

for all $z \in \mathbb{C}$. Define the map $f : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ by

$$f([1 : z]) := [1 : p(z)]$$

for $z \in \mathbb{C}$ and by

$$f([0 : 1]) := [0 : 1].$$

By Example 1.1.14 this map is smooth. Moreover, it has only a finite number of critical points; namely, the map f fails to be a local diffeomorphism near $[1 : z] \in \mathbb{C}\mathrm{P}^1$ if and only if z is a zero of the derivative polynomial

$$p'(z) = \sum_{k=1}^{d} k a_k z^{k-1},$$

and p' has only finitely many zeros because it is not identically zero. Thus the set $\mathcal{R}_f = \mathbb{C}\mathrm{P}^1 \setminus f(\mathcal{C}_f)$ of regular values of f is the complement of a finite subset of $\mathbb{C}\mathrm{P}^1 \cong S^2$ (see Example 1.1.13), and so is connected. This implies that the locally constant function

$$\mathcal{R}_f \to \mathbb{N}_0 : q \mapsto \#f^{-1}(q)$$

in part (iii) of Lemma 1.1.21 must actually be constant. Since $\#f^{-1}(q)$ cannot be zero everywhere, we conclude that it is zero nowhere, hence

$$\mathcal{R}_f \subset f(\mathbb{C}\mathrm{P}^1),$$

and hence f is surjective. In particular,

$$\#f^{-1}([1 : 0]) > 0.$$

Thus there exists a complex number $z \in \mathbb{C}$ such that $p(z) = 0$ and this proves the fundamental theorem of algebra.

1.2 The Theorem of Sard and Brown

In §1.1.5 we have seen that the set of singular values of a polynomial map from $\mathbb{C}\mathrm{P}^1$ to itself is finite. In general, the set of singular values of a smooth map may be infinite, however, it has Lebesgue measure zero in each coordinate chart. This is the content of Sard's Theorem [38], proved in 1942 after earlier work by A.P. Morse [30].

Theorem 1.2.1 (Sard). *Let $U \subset \mathbb{R}^m$ be an open set, let $f : U \to \mathbb{R}^n$ be a smooth map, and denote the set of critical points of f by*

$$\mathcal{C} := \{x \in U \,|\, \text{the derivative } df(x) : \mathbb{R}^m \to \mathbb{R}^n \text{ is not surjective}\}.$$

Then the set $f(\mathcal{C}) \subset \mathbb{R}^n$ of critical values of f has Lebesgue measure zero.

The proof of Theorem 1.2.1 is deferred to §1.3.

Since a set of Lebesgue measure zero connot contain any nonempty open set, it follows from Theorem 1.2.1 that the set $\mathbb{R}^n \setminus f(\mathcal{C})$ of regular values of f is dense in $\mathbb{R}^n$. This was proved by A.P. Brown [6, Thm 3-III] in 1935 and rediscovered by Dubovitskii [9] in 1953 and by Thom [40] in 1954.

Theorem 1.2.1 is not sharp. It actually suffices to assume that f is a C^ℓ-map, where $\ell \geq 1 + \max\{0, m - n\}$. The proof of this stronger version can be found in [1]. For the applications in this book it suffices to assume that f is smooth as in Theorem 1.2.1. The proof in §1.3 is taken from Milnor [26] and requires the existence of many derivatives.

Corollary 1.2.2 (Sard–Brown). *Let M be a smooth m-manifold (whose topology is second countable and Hausdorff), let N be a smooth n-manifold, let $f : M \to N$ be a smooth map, and let $\mathcal{C}_f \subset M$ be the set of critical points of f (where the derivative $df(p) : T_pM \to T_{f(p)}N$ is not surjective). Then the set $f(\mathcal{C}_f)$ of critical values of f has Lebesgue measure zero in each coordinate chart and the set $\mathcal{R}_f := N \setminus f(\mathcal{C}_f)$ of regular values of f is dense in N.*

Proof. Since M is paracompact by Theorem A.1.4, it admits a countable atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$. Let $\psi : V \to \Omega \subset \mathbb{R}^n$ be a coordinate chart on N and, for each $\alpha \in A$, define the map $f_\alpha := \psi \circ f \circ \phi_\alpha^{-1} : \Omega_\alpha := \phi_\alpha(U_\alpha \cap f^{-1}(V)) \to \Omega$ and denote by $\mathcal{C}_\alpha \subset \Omega_\alpha$ the set of critical points of f_α. By Theorem 1.2.1 the set $f_\alpha(\mathcal{C}_\alpha) \subset \mathbb{R}^n$ has Lebesgue measure zero for every $\alpha \in A$. Since A is countable, the set $\psi(f(\mathcal{C}_f) \cap V) = \bigcup_{\alpha \in A} f_\alpha(\mathcal{C}_\alpha) \subset \Omega$ has Lebesgue measure zero. Hence the set $\psi(\mathcal{R}_f \cap V) = \Omega \setminus \psi(f(\mathcal{C}_f) \cap V)$ is dense in Ω. Since this holds for each coordinate chart on N, it follows that $\mathcal{R}_f$ is dense in N. This proves Corollary 1.2.2. $\square$

1.2.1 Submanifolds

To exploit the existence of regular values in Corollary 1.2.2 we will need the observation that the preimage of every regular value is a submanifold. As a preparatory step it is convenient at this point to slightly digress and carefully introduce the concept of a submanifold and its tangent spaces. Following Milnor [26], we begin by extending the notion of a smooth map to maps between subsets of manifolds that are not necessarily open.

Definition 1.2.3 (Smooth Map). *Let $X \subset M$ and $Y \subset N$ be arbitrary subsets of smooth manifolds. A map $f : X \to Y$ is called* **smooth** *iff for every $p_0 \in X$ there exists an open neighborhood $U \subset M$ of p_0 and a smooth map $F : U \to N$ that agrees with f on $U \cap X$. It is called a* **diffeomorphism** *iff it is bijective and f and f^{-1} are smooth. The sets X and Y are called* **diffeomorphic** *iff there exists a diffeomorphism $f : X \to Y$.*

Definition 1.2.4 (Submanifold). *Let M be a smooth m-manifold and let d be an integer such that $0 \le d \le m$. A subset $P \subset M$ is called a* **d-dimensional submanifold of** *M iff every point in P has an open neighborhood $U \subset M$ such that $U \cap M$ is diffeomorphic to an open set $\Omega \subset \mathbb{R}^d$. A diffeomorphism $\psi : \Omega \to U \cap P$ is called a* **parametrization of** *$U \cap P$.*

We will next introduce tangent spaces of a submanifold (Lemma 1.2.5) and prove that preimages of regular values are submanifolds (Lemma 1.2.7). In the remainder of the present subsection we will continue the digression by examining embeddings (Lemma 1.2.10) and constructing coordinate charts which map a submanifold to a subspace (Lemma 1.2.11).

Returning to the main theme of [26] we will then introduce manifolds with boundary (§1.2.2) and prove the Brouwer Fixed Point Theorem (§1.2.3).

Lemma 1.2.5. *Let $P \subset M$ be a d-dimensional submanifold of a smooth m-manifold M and let $\psi : \Omega \to U \cap P$ be parametrization of the intersection of P with an open set $U \subset M$, defined on an open set $\Omega \subset \mathbb{R}^d$. Let $x \in \Omega$ and $p := \psi(x)$. Then the set*

$$\begin{aligned} T_pP &:= \left\{ \dot\gamma(0) \;\middle|\; \begin{array}{l} \gamma : I \to M \textit{ is a smooth curve} \\ \textit{on an interval } I \subset \mathbb{R} \textit{ such that} \\ 0 \in I,\ \#I > 1,\ \gamma(I) \subset P,\ \gamma(0) = p \end{array} \right\} \\ &= \operatorname{im} d\psi(x) \end{aligned} \tag{1.2.1}$$

is a d-dimensional subspace of T_pM, called the **tangent space of P at p.**

Proof. We prove that the derivative $d\psi(x) : \mathbb{R}^m \to T_pM$ is injective. By assumption the map $\psi^{-1} : U \cap P \to \Omega$ is smooth. Hence, by Definition 1.2.3, there exists a smooth map $\phi : U' \to \mathbb{R}^d$ on an open neighborhood $U' \subset M$ of p that agrees with ψ^{-1} on $U' \cap U \cap P$. Hence $\Omega' := \psi^{-1}(U') \subset \Omega$ is an open set in $\mathbb{R}^d$ and $\phi \circ \psi|_{\Omega'} = \mathrm{id}_{\Omega'}$. Since $x \in \Omega'$ and $\psi(x) = p$, it follows that $d\phi(p) \circ d\psi(x) = \mathrm{id}_{\mathbb{R}^d}$ and so $d\psi(x)$ is injective.

We prove that $\operatorname{im} d\psi(x) \subset T_pP$. Let $\xi \in \mathbb{R}^d$ and choose $\varepsilon > 0$ such that

$$x + t\xi \in \Omega \qquad \text{for } -\varepsilon < t < \varepsilon.$$

Define the curve $\gamma : I := (-\varepsilon, \varepsilon) \to M$ by $\gamma(t) := \psi(x + t\xi)$ for $t \in I$. Then $\gamma(I) \subset P$ and $\gamma(0) = \psi(x) = p$, and hence $d\psi(x)\xi = \dot{\gamma}(0) \in T_pM$. Thus we have proved that $\operatorname{im} d\psi(x) \subset T_pM$.

We prove that $T_pM \subset \operatorname{im} d\psi(x)$. Let $v \in T_pM$ and choose a smooth curve $\gamma : I \to M$ on an interval $I \subset \mathbb{R}$ such that

$$0 \in I, \qquad \#I > 1, \qquad \gamma(I) \subset P, \qquad \gamma(0) = p, \qquad \dot{\gamma}(0) = v.$$

Shrinking I, if necessary, we may assume that $\gamma(I) \subset U$. Since $\psi : \Omega \to U \cap P$ is a diffeomorphism, it follows that the curve $c := \psi^{-1} \circ \gamma : I \to \Omega$ is smooth and hence $v = \dot{\gamma}(0) = d\psi(c(0))\dot{c}(0) = d\psi(x)\dot{c}(0)$. Thus $T_pM \subset \operatorname{im}\, d\psi(x)$.

Since $d\psi(x)$ is injective, it follows that $T_pM = \operatorname{im} d\psi(x)$ is a d-dimensional linear subspace of T_pM. This proves Lemma 1.2.5. □

Remark 1.2.6. **(i)** Let $P \subset M$ be a d-dimensional submanifold of a smooth m-manifold M. Then P is a smooth d-manifold in its own right. The topology on P is the relative topology as a subset of M and the coordinate charts are the diffeomorphisms $\phi = \psi^{-1} : U \cap P \to \Omega \subset \mathbb{R}^d$ in Definition 1.2.4. With this structure the inclusion $\iota : P \to M$ is a smooth map (it is actually an embedding as in Definition 1.2.8) and the tangent space $T_pP \subset T_pM$ in (1.2.1) is the image of the derivative of this inclusion at p.

(ii) The definition of the tangent space T_pP in (1.2.1) was chosen in a slightly more cumbersome form than necessary. Namely, the interval I can be replaced by $\mathbb{R}$, if one replaces the curve γ by $\gamma(t\varepsilon/\sqrt{\varepsilon^2 + t^2})$ for $\varepsilon > 0$ sufficiently small. The reason for choosing the formulation in (1.2.1) is that it carries over verbatim to submanifolds with boundary.

(iii) The notion of a submanifold of M in Definition 1.2.4 coincides with Milnor's notion of a submanifold of a Euclidean space $M = \mathbb{R}^m$, and the proof of Lemma 1.2.5 mimics the discussion in [26, page 5].

(iv) In the formula (1.2.1) for T_pP the first term on the right is evidently independent of the parametrization and the second term shows that T_pP is a linear subspace of T_pM depending smoothly on p.

Lemma 1.2.7. *Let $f : M \to N$ be a smooth map from an m-manifold M to an n-manifold N, where $n \le m$, and let $q \in N$ be a regular value of f. Then $P := f^{-1}(q)$ is a smooth $(m-n)$-dimensional submanifold of M and*

$$T_pP = \ker\, df(p) \qquad \text{for all } p \in P. \tag{1.2.2}$$

Proof. This is [26, §2, Lemmas 1&2]. Let $d := m - n$ and let $p_0 \in P$. Then the derivative $df(p_0)$ is surjective and so its kernel has dimension d. Thus there exists a linear map $\Lambda : T_{p_0}M \to \mathbb{R}^d$ whose restriction to $\ker df(p_0)$ is bijective. By Exercise 1.7.3 choose a smooth map $h : M \to \mathbb{R}^d$ such that

$$h(p_0) = 0, \qquad dh(p_0) = \Lambda.$$

Choose a coordinate chart $\psi : V \to \mathbb{R}^n$ on an open neighborhood $V \subset N$ of q such that $\psi(q) = 0$, choose an open neighborhood $U \subset M$ of p_0 such that $f(U) \subset V$, and define the map $\phi : U \to \mathbb{R}^d \times \mathbb{R}^n = \mathbb{R}^m$ by

$$\phi(p) := \big(h(p), \psi(f(p))\big) \qquad \text{for } p \in U.$$

Then $d\phi(p_0)$ is bijective, so the Inverse Function Theorem 1.1.19 applies. Thus, shrinking U if necessary, we may assume that $\Omega := \phi(U) \subset \mathbb{R}^m$ is open and $\phi : U \to \Omega$ is a diffeomorphism. Moreover, for $p \in U$ we have

$$\begin{aligned}\phi(p) \in \mathbb{R}^d \times 0 &\iff \psi(f(p)) = 0\\ &\iff f(p) = q\\ &\iff p \in P.\end{aligned}$$

Thus

$$\Omega_0 := \{x \in \mathbb{R}^d \,|\, (x, 0) \in \Omega\} \subset \mathbb{R}^d$$

is open set and the map $h : U \cap P \to \Omega_0$ is a diffeomorphism with the inverse

$$\Omega_0 \to U \cap P : x \mapsto \phi^{-1}(x, 0).$$

This shows that P is a d-dimensional submanifold of M.

Let $v \in T_pP$ and choose a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(\mathbb{R}) \subset P$ and $\gamma(0) = p$, $\dot\gamma(0) = v$. Then $f(\gamma(t)) = 0$ for all t and hence, dIfferentiating this equation at $t = 0$, we obtain $df(p)v = 0$. Thus $T_pM \subset \ker df(p)$. Since both spaces have dimension d, this proves (1.2.2) and Lemma 1.2.7. □

Before continuing with the program of [26] in §1.2.2, we introduce here the notion of an embedding, prove that the image of an embedding is a submanifold, and relate the above definition of a submanifold to one that is more standard in the intrinsic setting, by constructing coordinate charts which map the submanifold to a linear subspace. These results are not essential to the story and can be skipped at first reading.

Definition 1.2.8 (Embedding). *Let M be an m-manifold and let Q be a d-manifold, where $0 \le d \le m$. A smooth map $f : Q \to M$ is called*

- *an* **immersion** *iff $df(q) : T_qQ \to T_{f(q)}M$ is injective for all $q \in Q$,*
- **proper** *iff for every compact set $K \subset f(Q)$ the set $f^{-1}(K)$ is compact,*
- *an* **embedding** *iff it is a proper injective immersion.*

Remark 1.2.9. In our definition of proper maps it is important that the compact set K is required to be contained in the image of f. The literature also contains a stronger definition of proper which requires that $f^{-1}(K)$ is a compact subset of Q for every compact subset $K \subset M$, whether or not K is contained in the image of f. This holds if and only if the map f is proper in the sense of Definition 1.2.8 and has a closed image. (Exercise!)

Lemma 1.2.10. *Let Q be a nonempty d-manifold and let $f : Q \to M$ be an embedding. Then $P := f(Q)$ is a submanifold of M with the tangent spaces*

$$T_{f(q)}P = \operatorname{im} df(q) \qquad \text{for } q \in Q. \tag{1.2.3}$$

Proof. Let $q_0 \in Q$. Since f is an immersion, it follows from the Inverse Function Theorem 1.1.19 that there exists an open neighborhood $V \subset Q$ of q_0 such that f restricts to a diffeomorphism from V to $f(V)$ (in the sense of Definition 1.2.3). Since Q is a d-manifold we may choose V to be diffeomorphic to an open set $\Omega \subset \mathbb{R}^d$. Since f is injective and proper, it is a homeomorphism from Q to $P = f(Q)$ with respect to the relative topology on P. Hence there exists an open set $U \subset M$ such that $U \cap P = f(V)$. Thus $U \cap P$ is diffeomorphic to V and hence to Ω. This shows that P is a submanifold of M. That its tangent spaces are given by (1.2.3) follows directly from (1.2.1). This proves Lemma 1.2.10 □

Lemma 1.2.11. *Let M be a smooth m-manifold, let d be an integer such that $0 \le d \le m$, and let $P \subset M$. Then the following are equivalent.*

(i) *P is a smooth d-dimensional submanifold of M.*

(ii) *For every $p \in P$ there exists a diffeomorphism $\phi : U \to \Omega$ from an open neighborhood $U \subset M$ of p to an open set $\Omega \subset \mathbb{R}^m$ (see Figure 1.2) such that*

$$\phi(U \cap P) = \Omega \cap (\mathbb{R}^d \times \{0\}). \tag{1.2.4}$$

Proof. We prove that (ii) implies (i). Let $\iota : \mathbb{R}^d \to \mathbb{R}^m$ be the canonical inclusion given by $\iota(x) := (x, 0)$, and let $\pi : \mathbb{R}^m \to \mathbb{R}^d$ be the canonical projection given by $\pi(x) := (x_1, \dots, x_d)$. Let $\phi : U \to \Omega$ be as in (ii). Then $\Omega_0 := \iota^{-1}(\Omega) \subset \mathbb{R}^d$ is open and the map $\phi^{-1} \circ \iota : \Omega_0 \to U \cap P$ is a diffeomorphism with the inverse $\pi \circ \phi : U \cap P \to \Omega_0$. Thus (ii) implies (i).

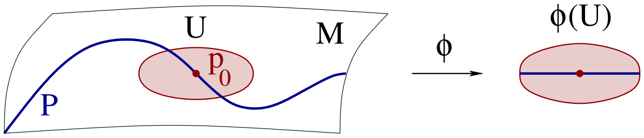

Figure 1.2: A coordinate chart adapted to a submanifold.

We prove that (i) implies (ii). Since (ii) is a local statement, it suffices to consider the case $M = \mathbb{R}^m$. Let $P \subset \mathbb{R}^m$ be a d-dimensional submanifold and let $p_0 \in P$. Then there exist a constant $r > 0$, a smooth map

$$\psi_0 : B_r^d := \left\{ x \in \mathbb{R}^d \,|\, |x| < r \right\} \to \mathbb{R}^m,$$

and an open set $U_r \subset M$ such that $\psi_0(0) = p_0$ and

$$\psi_0 : B_r^d \to \psi_0(B_r^d) = U_r \cap P$$

is a diffeomorphism. By Lemma 1.2.5 the derivative $d\psi_0(0) : \mathbb{R}^d \to \mathbb{R}^m$ is injective. Hence there exist vectors $e_{d+1}, \dots, e_m \in \mathbb{R}^m$ such that

$$\operatorname{im} d\psi_0(0) \oplus \operatorname{span}\{e_{d+1}, \dots, e_m\} = \mathbb{R}^m.$$

Define the map $\psi : B_r^m \to \mathbb{R}^m$ by

$$\psi(x) := \psi_0(x_1, \dots, x_d) + x_{d+1}e_{d+1} + \cdots + x_m e_m \qquad \text{for } x \in B_r^d.$$

Then $d\psi(0)$ is invertible. Hence, by the Inverse Function Theorem 1.1.1, we may assume, after shrinking r and U_r if necessary, that $\psi(B_r^m)$ is open and ψ maps B_r^m diffeomorphically onto $\psi(B_r^m)$. Now choose $0 < \rho < r$ such that $\psi(B_\rho^m) \subset U_r$. Then

$$\psi(B_\rho^m \cap (\mathbb{R}^d \times \{0\})) = \psi(B_\rho^m) \cap P. \tag{1.2.5}$$

Namely, if $x \in B_\rho^m$ satisfies $\psi(x) \in P$, then $\psi(x) \in U_r \cap P = \psi_0(B_r^d)$, hence there exists a vector $\xi \in B_r^d$ such that $\psi(x) = \psi_0(\xi) = \psi(\xi, 0)$, and since ψ is injective on B_r^m it follows that $x = (\xi, 0)$, hence $\xi \in B_\rho^d$, and hence $x \in B_\rho^m \cap (\mathbb{R}^d \times \{0\})$. This proves (1.2.5).

It follows from (1.2.5) that the map $\phi := \psi^{-1} : U \to \Omega$ satisfies (1.2.4) with $\Omega := B_\rho^m$ and $U := \psi(B_\rho^m)$. This proves Lemma 1.2.11. □

Remark 1.2.12. Let $P \subset M$ and $\phi : U \to \Omega$ be as in (1.2.4). Then

$$T_pP = d\phi(p)^{-1}(\mathbb{R}^d \times \{0\}) \qquad \text{for all } p \in U \cap P. \tag{1.2.6}$$

Define $f := (\phi_{d+1}, \dots, \phi_m) : U \to \mathbb{R}^{m-d}$. Then zero is a regular value of f and $f^{-1}(0) = U \cap P$. Hence it follows from equation (1.2.2) in Lemma 1.2.7 that $T_pM = \ker df(p) = d\phi(p)^{-1}(\mathbb{R}^d \times \{0\})$ for all $p \in U \cap P$.

1.2.2 Manifolds with Boundary

This section introduces the concept of a manifold with boundary. Fix a positive integer m and introduce the notations

$$\begin{aligned} \mathbb{H}^m &:= \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m \,\middle|\, x_m \geq 0 \right\}, \\ \partial\mathbb{H}^m &:= \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m \,\middle|\, x_m = 0 \right\} \end{aligned} \tag{1.2.7}$$

for the m-dimensional upper half space and its boundary.

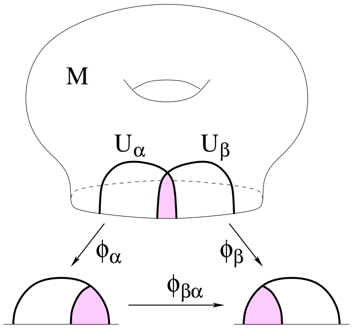

Figure 1.3: A manifold with boundary.

Definition 1.2.13. *A* **smooth m-manifold with boundary** *consists of a (second countable Haudorff) topological space M, an open cover $\{U_\alpha\}_{\alpha\in A}$ of M, and a collection of homeomorphisms*

$$\phi_\alpha : U_\alpha \to \Omega_\alpha$$

onto open subsets $\Omega_\alpha \subset \mathbb{H}^m$, one for every $\alpha \in A$, such that, for every pair $\alpha, \beta \in A$, the **transition map**

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism (see Figure 1.3). The homeomorphisms $\phi_\alpha : U_\alpha \to \Omega_\alpha$ are called **coordinate charts***, the collection $\{U_\alpha, \phi_\alpha\}_{\alpha\in A}$ is called an* **atlas of** *M. An element $p \in M$ is called a* **boundary point** *iff it satisfies*

$$p \in U_\alpha \qquad \Longrightarrow \qquad \phi_\alpha(p) \in \partial\mathbb{H}^m \tag{1.2.8}$$

for all $\alpha \in A$. The set

$$\partial M := \left\{ p \in M \,\middle|\, \phi_\alpha(p) \in \partial\mathbb{H}^m \text{ for every } \alpha \in A \text{ with } p \in U_\alpha \right\} \tag{1.2.9}$$

of all boundary points is called the **boundary of** *M.*

Remark 1.2.14. Let $(M, \{U_\alpha, \phi_\alpha\}_{\alpha\in A})$ be a manifold with boundary.

(i) The domains $\Omega_{\alpha\beta} := \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{H}^m$ of the transition maps $\phi_{\beta\alpha}$ in Definition 1.2.13 need not be open subsets of $\mathbb{R}^m$ and the transition maps $\phi_{\beta\alpha}$ are understood to be smooth in the sense of Definition 1.2.3.

(ii) Let $p \in M$ and let $\alpha, \beta \in A$ such that $p \in U_\alpha \cap U_\beta$. Then

$$\phi_\alpha(p) \in \partial\mathbb{H}^m \qquad \iff \qquad \phi_\beta(p) \in \partial\mathbb{H}^m. \tag{1.2.10}$$

Hence the definition of a boundary point in (1.2.8) is independent of the choice of the index $\alpha \in A$ with $p \in U_\alpha$. To verify (1.2.10) we assume, by contradiction, that $\bar{x} := \phi_\alpha(p) \in \Omega_{\alpha\beta} \setminus \partial\mathbb{H}^m$ and $\phi_\beta(p) \in \partial\mathbb{H}^m$. Then the mth coordinate $\phi_{\beta\alpha,m} : \Omega_{\alpha\beta} \to \mathbb{R}$ has a local minimum at $\bar{x}$ and hence the Jacobi matrix $d\phi_{\beta\alpha}(\bar{x})$ is not invertible, a contradiction.

(iii) The boundary ∂M admits the natural structure of an $(m-1)$-manifold without boundary. (**Exercise:** Prove this.)

(iv) Let $p \in M$. The **tangent space** of M at p is defined as the quotient

$$T_pM := \bigcup_{p\in U_\alpha} \{\alpha\} \times \mathbb{R}^m/\!\sim \tag{1.2.11}$$

under the equivalence relation

$$(\alpha, \xi) \sim (\beta, \eta) \qquad \overset{\text{def}}{\iff} \qquad \eta = d\phi_{\beta\alpha}(\phi_\alpha(p))\xi.$$

Thus T_pM is a whole vector space, even if p is a boundary point. If $p \in U_\alpha$, then the derivative of the coordinate chart ϕ_α at p is the canonical vector space isomorphism

$$d\phi_\alpha(p) : T_pM \to \mathbb{R}^m$$

given by $d\phi_\alpha(p)v := \xi$ for $\xi \in \mathbb{R}^m$, where $v = [\alpha, \xi]_p \in T_pM$ is the equivalence class of (α, ξ) (Remark 1.1.17).

(v) Let $p \in \partial M$ and let $v \in T_pM$. The tangent vector v is called **outward pointing** iff

$$d\phi_\alpha(p)v \in \mathbb{R}^m \setminus \mathbb{H}^m$$

for some, and hence every, $\alpha \in A$ such that $p \in U_\alpha$. It is called **inward pointing** iff

$$d\phi_\alpha(p)v \in \mathbb{H}^m \setminus \partial\mathbb{H}^m$$

for some, and hence every, $\alpha \in A$ such that $p \in U_\alpha$. (**Exercise:** Prove that these conditions are independent of the choice of α.)

Definition 1.2.15 (Submanifold with Boundary). *Let M be a smooth m-manifold with boundary (possibly empty) and let d be an an integer such that $0 < d \leq m$. A subset $P \subset M$ is called a* **d-dimensional submanifold with boundary** *iff every point $p_0 \in P$ has an open neighborhood $U \subset M$ such that $U \cap P$ is diffeomorphic to an open subset of $\mathbb{H}^d$. In this situation a diffeomorphism $\psi : \Omega \to U \cap P$ is called a parametrization of $U \cap P$.*

An element $p \in P$ is called a **boundary point of** P *iff it is mapped to the boundary of $\mathbb{H}^d$ under some (and hence every) diffeomorphism from a neighborhood of p in P to an open subset of $\mathbb{H}^d$. The set of all boundary points of P is called the* **boundary of** P *and will be denoted by ∂P.*

The tangent spaces of a submanifold with boundary are defined exactly as in the case of submanifolds without boundary. Namely Lemma 1.2.5 and its proof carry over verbatim to submanifolds with boundary.

Definition 1.2.16 (Tangent Spaces). *Let $P \subset M$ be a d-dimensional submanifold with boundary of a smooth m-manifold M with boundary and let $\psi : \Omega \to U \cap P$ be parametrization of the intersection of P with an open set $U \subset M$, defined on an open set $\Omega \subset \mathbb{H}^d$. Let $x \in \Omega$ and $p := \psi(x)$. Then*

$$\begin{aligned} T_pP &:= \left\{ \dot\gamma(0) \;\middle|\; \begin{array}{l} \gamma : I \to M \text{ is a smooth curve} \\ \text{on an interval } I \subset \mathbb{R} \text{ such that} \\ 0 \in I,\ \#I > 1,\ \gamma(I) \subset P,\ \gamma(0) = p \end{array} \right\} \\ &= \operatorname{im} d\psi(x) \end{aligned} \tag{1.2.12}$$

is a d-dimensional subspace of T_pM, called the **tangent space of P at p.**

If $p \in \partial P$, then in the formula (1.2.12) an interval of the form $I = [-\varepsilon, 0]$ (with 0 as a right boundary point) must be used for outward pointing tangent vectors and an interval of the form $I = [0, \varepsilon]$ (with 0 as a left boundary point) for inward pointing tangent vectors.

Definition 1.2.17. *Let M be a smooth m-manifold with boundary and let d be an an integer such that $0 < d \leq m$. A submanifold $P \subset M$ with boundary is called* **transverse to the boundary** *iff $\partial P = P \cap \partial M$ and*

$$T_pM = T_pP + T_p\partial M \qquad \text{for all } p \in \partial P.$$

With this terminology in place we are ready to prove two lemmas in [26] which show how submanifolds with boundary can appear. In Lemma 1.2.18 they are sublevel sets of a smooth real valued function on a manifold without boundary, and in Lemma 1.2.20 they are preimages of regular values for maps that are defined on manifolds with boundary and satisfy the requirements of Definition 1.2.17.

Lemma 1.2.18. *Let M be a smooth m-manifold without boundary and let $g : M \to \mathbb{R}$ be a smooth function that has 0 as a regular value. Then the set*

$$M_0 := \{p \in M \,|\, g(p) \geq 0\}$$

(see Figure 1.4) is a smooth m-dimensional submanifold of M with boundary

$$\partial M_0 := \{p \in M \,|\, g(p) = 0\}.$$

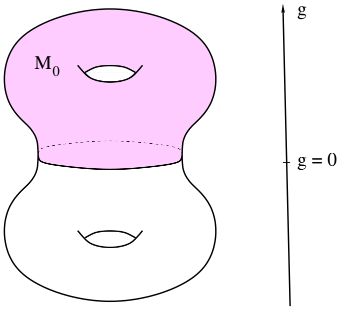

Figure 1.4: The set $\{g \geq 0\}$ as a submanifold with boundary.

Proof. This is [26, §2, Lemma 3]. We argue as in the proof of Lemma 1.2.7. Let $p_0 \in M$ such that $g(p_0) = 0$. Then, since zero is a regular value of g, its derivative $dg(p_0)$ is nonzero, so the kernel of $dg(p_0)$ is an $(m-1)$-dimensional subspace of $T_{p_0}M$. Choose a linear map $\Lambda : T_{p_0}M \to \mathbb{R}^{m-1}$ which restricts to an isomorphism from $\ker dg(p_0)$ to $\mathbb{R}^{m-1}$. By Exercise 1.7.3 choose a smoth map $h : M \to \mathbb{R}^{m-1}$ such that $dh(p_0) = \Lambda$ and define $G : M \to \mathbb{R}^m$ by

$$G(p) := \big(h(p), g(p)\big)$$

for $p \in M$. Then the derivative $dG(p_0) = \Lambda \times dg(p_0) : T_{p_0}M \to \mathbb{R}^{m-1} \times \mathbb{R}$ is a vector space isomorphism. Hence, by the Inverse Function Theorem 1.1.19, there exists an open neighborhood $U \subset M$ of p_0 and a real number $r > 0$ such that G restricts to a diffeomorphism from U to $B_r := \{x \in \mathbb{R}^m \,|\, |x| < r\}$. By definition of G, every $p \in U$ satisfies $g(p) \geq 0$ if and only if $G(p) \in B_r \cap \mathbb{H}^m$. Hence $\phi := G|_{U \cap M_0}$ is the required diffeomorphism from $U \cap M_0$ to the open subset $\Omega := B_r \cap \mathbb{H}^m$ of $\mathbb{H}^m$. This proves Lemma 1.2.18. □

Example 1.2.19. The closed unit disk

$$\mathbb{D}^m := \{x \in \mathbb{R}^m \,|\, |x| \leq 1\}$$

is a smooth manifold with boundary $\partial \mathbb{D}^m = S^{m-1} = \{x \in \mathbb{R}^m \,|\, |x| = 1\}$. This follows from Lemma 1.2.18 with $M = \mathbb{R}^m$ and $g(x) = 1 - \sum_{i=1}^m x_i^2$.

Lemma 1.2.20. *Let M be a smooth m-manifold with boundary and let N be a smooth n-manifold without boundary, where $n < m$. Let $f : M \to N$ be a smooth map and let $q \in N$ be a common regular value of f and $f|_{\partial M}$. Then the set*

$$P := f^{-1}(q) = \big\{ p \in M \,\big|\, f(p) = q \big\} \subset M$$

is an $(m-n)$-dimensional submanifold with boundary $\partial P = P \cap \partial M$ and P is transvers to ∂M (see Definition 1.2.15).

Proof. This is [26, §2, Lemma 4]. Since it is a local statement, it suffices to consider the special case of a smooth map $f : \mathbb{H}^m \to \mathbb{R}^n$ with a common regular value $y \in \mathbb{R}^n$ of f and $f|_{\partial\mathbb{H}^m}$. Let $\bar{x} \in f^{-1}(y)$. If $\bar{x} \in \mathbb{H}^m \setminus \partial\mathbb{H}^m$ is an interior point, then $f^{-1}(y)$ is a smooth submanifold in a neighborhood of $\bar{x}$ by Lemma 1.2.7.

Suppose that $\bar{x} \in \partial\mathbb{H}^m$ and choose a smooth map $F : U \to \mathbb{R}^n$ that is defined in a neighborhood $U \subset \mathbb{R}^m$ of $\bar{x}$ and agrees with f on $U \cap \mathbb{H}^m$. Since y is a regular value of f, the derivative $dF(\bar{x}) = df(\bar{x}) : \mathbb{R}^m \to \mathbb{R}^n$ is surjective. Hence, replacing U by a smaller neighborhood if necessary, we may assume that F has no critical points. By Lemma 1.2.7 this implies that the set $F^{-1}(y)$ is a smooth $(m-n)$-dimensional submanifold of the open set $U \subset \mathbb{R}^m$.

Let $\pi : F^{-1}(y) \to \mathbb{R}$ be the coordinate projection

$$\pi(x_1, \ldots, x_m) := x_m \qquad \text{for } x = (x_1, \ldots, x_m) \in F^{-1}(y). \tag{1.2.13}$$

We claim that zero is a regular value of π. To see this, observe that the tangent space of $F^{-1}(y)$ at $x \in \pi^{-1}(0)$ is the kernel of the linear map

$$dF(x) = df(x) : \mathbb{R}^m \to \mathbb{R}^n$$

Since $x \in \partial\mathbb{H}^m$ and y is a regular value of $f|_{\partial\mathbb{H}^m}$ by assumption, it follows that the restriction of $df(x)$ to $\partial\mathbb{H}^m = \mathbb{R}^{m-1} \times \{0\}$ is surjective. Hence the $(m-n)$-dimensional kernel of $df(x)$ cannot be contained in $\mathbb{R}^{m-1} \times \{0\}$ and so the linear functional $d\pi(x) : \ker df(x) \to \mathbb{R}$ is surjective. Thus zero is a regular value of the function $\pi : F^{-1}(y) \to \mathbb{R}$ in (1.2.13) as claimed.

With this understood, it follows from Lemma 1.2.18 that the set

$$U \cap f^{-1}(y) = \big\{ x \in F^{-1}(y) \,|\, \pi(x) \geq 0 \big\} = F^{-1}(y) \cap \mathbb{H}^m$$

is a smooth $(m-n)$-dimensional submanifold of $\mathbb{H}^m$ with boundary

$$\partial\big(U \cap f^{-1}(y)\big) = \big\{ x \in F^{-1}(y) \,|\, \pi(x) = 0 \big\} = U \cap f^{-1}(y) \cap \partial\mathbb{H}^m.$$

Moreover, if $x \in U \cap f^{-1}(y) \cap \partial\mathbb{H}^m$, then $T_x(U \cap f^{-1}(y)) = \ker df(x) \not\subset \partial\mathbb{H}^m$ and hence $U \cap f^{-1}(y)$ is transverse to $\partial\mathbb{H}^m$. This proves Lemma 1.2.20. □

Exercise 1.2.21. Let $\bar{x}$ and F be as in the proof of Lemma 1.2.20. Reorder the coordinates $x_1, \dots, x_{m-1}$ such that $\mathbb{R}^n \times \{0\}$ is transverse to the kernel of the linear map $dF(\bar{x})$. Define the map $\Phi : U \to \mathbb{R}^m$ by

$$\Phi(x) := (F(x), x_{n+1}, \dots, x_m)$$

for $x = (x_1, \dots, x_m) \in U$. After shrinking U, if necessary, prove that the map $\phi := \Phi|_{U \cap \mathbb{H}^m}$ is a diffeomorphism from $U \cap \mathbb{H}^m$ onto its image

$$\Omega := \phi(U \cap \mathbb{H}^m) \subset \mathbb{H}^m$$

that satisfies

$$\phi(U \cap f^{-1}(y)) = \Omega \cap (\{0\} \times \mathbb{H}^{m-n}).$$

(See Figure 1.5.)

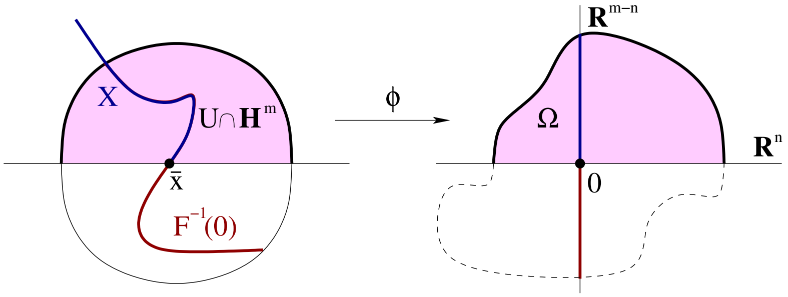

Figure 1.5: A submanifold transverse to the boundary.

Exercise 1.2.22. The concept of an embedding in Definition 1.2.8 extends verbatim to manifolds with boundary. Let Q be a smooth d-manifold with boundary and let M be a smooth m-manifold with (possibly empty) boundary. A smooth map $f : Q \to M$ is called an **embedding** iff it is a proper injective immersion. If $f : Q \to M$ is an embedding, prove that its image

$$P := f(Q) \subset M$$

is a d-dimensional submanifold with boundary.

In Theorem A.3.1 it is shown that every smooth m-dimensional manifold with boundary whose topology is second countable and Haudorff admits an embedding into $\mathbb{R}^{2m+1}$ with a closed image.

The next lemma shows that a submanifold that is transverse to the boundary has a normal form in a suitable local coordinate chart as depicted in Figure 1.5. It carries over Lemma 1.2.11 to submanifolds with boundary.

Lemma 1.2.23. *Let M be a smooth m-manifold with boundary, let d be an integer such that $0 < d \le m$, and let $P \subset M$. Then the following are equivalent.*

(i) *P is a d-dimensional submanifold of M with boundary $\partial P = P \cap \partial M$ and P is transverse to the boundary of M.*

(ii) *For every $p_0 \in P$ there exists an open neighborhood $U \subset M$ of p_0 and a coordinate chart $\phi : U \to \Omega$ onto an open set $\Omega \subset \mathbb{H}^m$ such that*

$$\phi(U \cap P) = \Omega \cap (\{0\} \times \mathbb{H}^d). \tag{1.2.14}$$

Proof. That (ii) implies (i) follows directly from the definitions. The proof of the converse is an exercise with hints. Since (ii) is a local statement, it suffices to consider the case $M = \mathbb{H}^m$. Let $P \subset \mathbb{H}^m$ be a d-dimensional submanifold with boundary $\partial P = P \cap \partial\mathbb{H}^m$ that is transverse to $\partial\mathbb{H}^m$. Let $\bar{x} \in P$. If $\bar{x} \in P \setminus \partial\mathbb{H}^m$, then part (ii) follows from Lemma 1.2.11. Assume $\bar{x} \in P \cap \partial\mathbb{H}^m = \partial P$. Then there exists an $r > 0$ and a smooth map

$$\psi_0 = (\psi_{0,1}, \dots, \psi_{0,m}) : B_r^d := \left\{ x \in \mathbb{R}^d \,\middle|\, |x| < r \right\} \to \mathbb{R}^m$$

such that $\psi_0(0) = \bar{x}$ and ψ_0 restricts to a diffeomorphism from $B_r^d \cap \mathbb{H}^d$ to an open neighborhood of $\bar{x}$ in P. Prove the following three assertions.

Claim 1. *$\psi_0(B_r^d \cap \partial\mathbb{H}^d) \subset \partial P \subset \partial\mathbb{H}^m$ and every $x \in B_r^d \cap \partial\mathbb{H}^d$ satisfies*

$$\partial_{x_i}\psi_{0,m}(x) = 0 \qquad \text{for } i = 1, \dots, d-1, \qquad \partial_{x_d}\psi_{0,m}(x) > 0.$$

Claim 2. *After shrinking r if necessary, we have $\psi_{0,m}(x) \ge 0 \iff x_d \ge 0$ and $\psi_{0,m}(x) \le 0 \iff x_d \le 0$ for every $x \in B_r^d$.*

Claim 3. *Let $e_1, \dots, e_{m-d} \in \partial\mathbb{H}^m$ such that* $\operatorname{span}\{e_i\} \oplus \operatorname{im}(d\psi_0(0)) = \mathbb{R}^m$ *and define the map $\psi : B_r^m \to \mathbb{R}^m$ by*

$$\psi(x) := x_1 e_1 + \dots + x_{m-d} e_{m-d} + \psi_0(x_{m-d+1}, \dots, x_m) \qquad \text{for } x \in B_r^m.$$

If $r > 0$ is chosen sufficiently small, then $\psi(B_r^m)$ is an open subset of $\mathbb{R}^m$ and $\psi : B_r^m \to \psi(B_r^m)$ is a diffeomorphism that satisfies

$$\psi(B_r^m) \cap \mathbb{H}^m = \psi(B_r^m \cap \mathbb{H}^m)$$

and

$$\psi(B_r^m) \cap P = \psi_0(B_r^d \cap \mathbb{H}^d) = \psi(B_r^m \cap (\{0\} \times \mathbb{H}^d)).$$

The inverse $\phi := \psi^{-1}$ of the diffeomorphism in Claim 3 satisfies (1.2.14) with $\Omega := B_r^m \cap \mathbb{H}^m$ and $U := \psi(B_r^m) \cap \mathbb{H}^m$. This proves Lemma 1.2.23. □

1.2.3 The Brouwer Fixed Point Theorem

Recall from Example 1.2.19 that the closed unit disk

$$\mathbb{D}^m := \left\{x \in \mathbb{R}^m \,|\, x_1^2 + x_2^2 + \cdots + x_m^2 \leq 1\right\}$$

in $\mathbb{R}^m$ is a smooth manifold with boundary $\partial\mathbb{D}^m = S^{m-1}$. The following fixed point theorem was proved by L.E.J. Brouwer [4] in 1910.

Theorem 1.2.24 (**Brouwer Fixed Point Theorem**). *Every continuous map $g : \mathbb{D}^m \to \mathbb{D}^m$ has a fixed point.*

Brouwer's Fixed Point Theorem extends to continuous maps from any nonempty compact convex subset of $\mathbb{R}^m$ to itself. An infinite-dimensional variant of this result is the **Tychonoff Fixed Point Theorem** [41] which asserts that, if C is a nonempty compact convex subset of a locally convex topological vector space, then every continuous map $g : C \to C$ has a fixed point. Another generalization of Brouwer's Fixed Point Theorem is the Lefschetz Fixed Point Theorem in Corollary 4.4.13.

Following Milnor [26] we will first prove Theorem 1.2.24 for smooth map and then use an approximation argument to establish the result for all continuous maps. In the smooth case the proof is based on the following key lemma which uses Sard's Theorem 1.2.1 about the existence of regular values and Lemma 1.2.20 about the preimages of regular values.

Lemma 1.2.25. *Let M be a compact smooth manifold with boundary. There does not exist a smooth map $f : M \to \partial M$ that restricts to the identity map on the boundary.*

Proof. This is [26, §2, Lemma 5]. Suppose that there exists a smooth map

$$f : M \to \partial M$$

such that

$$f(p) = p \qquad \text{for all } p \in \partial M.$$

By Corollary 1.2.2 there exists a regular value $q \in \partial M$ of f. Since q is also a regular value of the identity map $\mathrm{id} = f|_{\partial M}$, it follows from Lemma 1.2.20 that the set $X := f^{-1}(q)$ is a compact smooth 1-dimensional manifold with a single boundary point

$$\partial X = f^{-1}(q) \cap \partial M = \{q\}.$$

However, Theorem A.4.1 asserts that X is a finite union of circles and arcs and hence must have an even number of boundary points. This contradiction proves Lemma 1.2.25. □

Lemma 1.2.26. *Let $g : \mathbb{D}^m \to \mathbb{D}^m$ be a smooth map. Then there exists an element $x \in \mathbb{D}^m$ such that $g(x) = x$.*

Proof. This is [26, §2, Lemma 6]. Suppose g has no fixed point. For $x \in \mathbb{D}^m$ let $f(x) \in S^{m-1}$ be the unique intersection point of the straight line through the points x and $g(x)$ that is closer to x than to $g(x)$ (see Figure 1.6). Then $f(x) = x$ for all $x \in S^{m-1}$. An explicit formula for $f(x)$ is

$$f(x) = x + tu, \qquad u := \frac{x - g(x)}{|x - g(x)|}, \qquad t := \sqrt{1 - |x|^2 + \langle x, u\rangle^2} - \langle x, u\rangle.$$

This formula shows that the map $f : \mathbb{D}^m \to S^{m-1}$ is smooth. Such a map does not exist by Lemma 1.2.25. Hence our assumption that g does not have a fixed point must have been wrong, and this proves Lemma 1.2.26. □

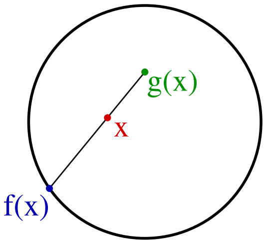

Figure 1.6: Proof of Brouwer's Fixed Point Theorem.

Proof of Theorem 1.2.24. Let $g : \mathbb{D}^m \to \mathbb{D}^m$ be a continuous map and assume that $g(x) \neq x$ for all $x \in \mathbb{D}^m$. Then, since $\mathbb{D}^m$ is a compact subset of $\mathbb{R}^m$, there exists a constant $\varepsilon > 0$ such that $|g(x) - x| \geq 2\varepsilon$ for all $x \in \mathbb{D}^m$. By the Weierstraß Approximation Theorem (see for example [7, Thm 5.4.5] with $M = \mathbb{D}^m$ and $\mathcal{A}$ the set of polynomials in m variables with real coefficients), there exists a polynomial map $p : \mathbb{D}^m \to \mathbb{R}^m$ such that

$$|p(x) - g(x)| < \varepsilon \qquad \text{for all } x \in \mathbb{D}^m.$$

Define the map $q : \mathbb{D}^m \to \mathbb{R}^m$ by

$$q(x) := \frac{p(x)}{1 + \varepsilon} \qquad \text{for } x \in \mathbb{D}^m.$$

Then $|q(x)| \leq 1$ and

$$|q(x) - g(x)| = \frac{|p(x) - g(x) - \varepsilon g(x)|}{1 + \varepsilon} \leq \frac{|p(x) - g(x)|}{1 + \varepsilon} + \frac{\varepsilon |g(x)|}{1 + \varepsilon} < 2\varepsilon$$

for all $x \in \mathbb{D}^m$. Thus $q : \mathbb{D}^m \to \mathbb{D}^m$ is a smooth map without fixed points, in contradiction to Lemma 1.2.26. This proves Theorem 1.2.24. □

1.3 Proof of Sard's Theorem

The proof given below follows closely the argument in Milnor [26].

Proof of Theorem 1.2.1. Let $U \subset \mathbb{R}^m$ be an open set, let $f : U \to \mathbb{R}^n$ be a smooth map, and denote the set of critical points of f by

$$\mathcal{C} := \left\{ x \in U \,\middle|\, df(x) : \mathbb{R}^m \to \mathbb{R}^n \text{ is not surjective} \right\}.$$

We prove by induction on m that the set $f(\mathcal{C}) \subset \mathbb{R}^n$ of critical values of f has Lebesgue measure zero.

Assume first that $m = 0$. If $n = 0$, then $\mathcal{C} = \emptyset$ and so $f(\mathcal{C}) = \emptyset$ has Lebesgue measure zero. If $n \geq 1$ then either $\mathcal{C} = U = \emptyset$ or $\mathcal{C} = U = \mathbb{R}^0$ is a singleton, and in both cases the set $f(\mathcal{C})$ has Lebesgue measure zero.

Now let $m \in \mathbb{N}$ be a positive integer and assume by induction that the assertion holds with m replaced by $m - 1$. For $k \in \mathbb{N}$ define

$$\mathcal{C}_k := \left\{ x \in \mathcal{C} \,\middle|\, \partial^\alpha f(x) = 0 \begin{array}{l} \text{for all } \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m \\ \text{such that } |\alpha| = \alpha_1 + \cdots + \alpha_m \leq k \end{array} \right\}.$$

Thus the $\mathcal{C}_k$ form a descending sequence of relatively closed sets

$$\mathcal{C} \supset \mathcal{C}_1 \supset \mathcal{C}_2 \supset \mathcal{C}_3 \supset \cdots.$$

The proof that the set $f(\mathcal{C})$ of critical values of f has Lebesgue measure zero will consist of the following three steps.

Step 1. *The set $f(\mathcal{C} \setminus \mathcal{C}_1)$ has Lebesgue measure zero.*

Step 2. *The set $f(\mathcal{C}_k \setminus \mathcal{C}_{k+1})$ has Lebesgue measure zero for each $k \in \mathbb{N}$.*

Step 3. *The set $f(\mathcal{C}_k)$ has Lebesgue measure zero whenever $k > \frac{m}{n} - 1$.*

It follows from these steps with $k > \frac{m}{n} - 1$ that the set

$$f(\mathcal{C}) = f(\mathcal{C} \setminus \mathcal{C}_1) \cup \bigcup_{i=1}^{k-1} f(\mathcal{C}_i \setminus \mathcal{C}_{i+1}) \cup f(\mathcal{C}_k)$$

has Lebesgue measure zero. We also remark that, if f is a nonconstant real analytic function and U is connected, then $\bigcap_{i \in \mathbb{N}} \mathcal{C}_i = \emptyset$. In this situation only Steps 1 and 2 are needed to deduce that the set

$$f(\mathcal{C}) = f(\mathcal{C} \setminus \mathcal{C}_1) \cup \bigcup_{i=1}^{\infty} f(\mathcal{C}_i \setminus \mathcal{C}_{i+1})$$

has Lebesgue measure zero.

Proof of Step 1. The set

$$\mathcal{C} \setminus \mathcal{C}_1 = \left\{x \in U \,\middle|\, df(x) \text{ is not surjective and } df(x) \neq 0\right\}$$

is empty for $n = 0$ and $n = 1$. Thus assume $n \geq 2$. Under this assumption we prove the following.

Claim. *Every element $\overline{x} \in \mathcal{C} \setminus \mathcal{C}_1$ has an open neighborhood $V \subset U$ such that the set $f(V \cap \mathcal{C})$ has Lebesgue measure zero.*

We show first that the claim implies Step 1. To see this, note that $U \setminus \mathcal{C}_1$ is an open subset of $\mathbb{R}^m$ and hence can be expressed as a countable union of compact sets $K_i \subset U \setminus \mathcal{C}_1$, i.e. $U \setminus \mathcal{C}_1 = \bigcup_{i=1}^{\infty} K_i$. Thus

$$\mathcal{C} \setminus \mathcal{C}_1 = \bigcup_{i=1}^{\infty} (K_i \cap \mathcal{C}).$$

Since each set $K_i \cap \mathcal{C}$ is compact it can be covered by finitely many open sets V as in the claim. Hence there exist countable many sets $V_1, V_2, V_3, \dots$ as in the claim such that

$$\mathcal{C} \setminus \mathcal{C}_1 \subset \bigcup_{j=1}^{\infty} (V_j \cap \mathcal{C}).$$

Thus

$$f(\mathcal{C} \setminus \mathcal{C}_1) \subset \bigcup_{j=1}^{\infty} f(V_j \cap \mathcal{C})$$

and so by the claim the set $f(\mathcal{C} \setminus \mathcal{C}_1)$ has Lebesgue measure zero. Thus it remains to prove the claim. The proof makes use of the following version of Fubini's Theorem. Denote by μ_n the Lebesgue measure on $\mathbb{R}^n$.

Fubini's Theorem. *Let $A \subset \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ be a Lebesgue measurable set and, for $t \in \mathbb{R}$, define*

$$A_t := \left\{(y_2, \dots, y_n) \in \mathbb{R}^{n-1} \,\middle|\, (t, y_2, \dots, y_n) \in A\right\}.$$

If $\mu_{n-1}(A_t) = 0$ for all $t \in \mathbb{R}$, then $\mu_n(A) = 0$.

When A is a Borel set, this assertion follows directly from [37, Thm 7.28] with $k = 1$ and f the characteristic function of A. In general, choose a Borel set $B \subset A$ such that $\mu_n(A \setminus B) = 0$ (see [37, Thms 1.55 & 2.14]) and apply [37, Thm 7.28] to the set B.

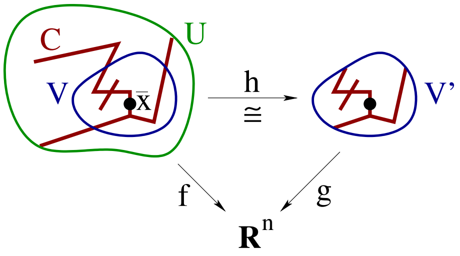

Figure 1.7: The ciritcal set of f.

With these preparations we are ready to prove the claim. Thus fix an element $\overline{x} \in \mathcal{C} \setminus \mathcal{C}_1$. Then $df(\overline{x}) \neq 0$ and so some partial derivative of f does not vanish at $\overline{x}$. Reordering the coordinates of $\mathbb{R}^m$ and $\mathbb{R}^n$ if necessary, we may assume without loss of generality that

$$\frac{\partial f_1}{\partial x_1}(\overline{x}) \neq 0.$$

Now define the map $h : U \to \mathbb{R}^m$ by

$$f(x) := (f_1(x), x_2, \ldots, x_m)$$

for $x = (x_1, x_2, \ldots, x_m) \in U$. Then

$$dh(\overline{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\overline{x}) & * & \cdots & \cdots & * \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and so $\det(dh(\overline{x})) \neq 0$. Thus it follows from the Inverse Function Theorem 1.1.19 that there exists an open neighborhood $V \subset U$ of $\overline{x}$ such that the set $V' := h(V) \subset \mathbb{R}^m$ is open and $h|_V : V \to V'$ is a diffeomorphism. Define

$$g := f \circ (h|_V)^{-1} : V' \to \mathbb{R}^n.$$

Then the set of critical points of g is given by

$$\mathcal{C}' := \left\{ x' \in V' \,\middle|\, dg(x') \text{ is not surjective} \right\} = h(V \cap \mathcal{C}).$$

Thus $g(\mathcal{C}') = g \circ h(V \cap \mathcal{C}) = f(V \cap \mathcal{C})$ (see Figure 1.7).

Next observe that, if $(t, x_2, \dots, x_m) \in V'$, then

$$h^{-1}(t, x_2, \dots, x_m) = (x_1, x_2, \dots, x_m) \in V,$$

where

$$t = f_1(x_1, \dots, x_m),$$

and hence

$$g(t, x_2, \dots, x_m) = f(x_1, \dots, x_m) \in \{t\} \times \mathbb{R}^{m-1}.$$

For $t \in \mathbb{R}$ define the open set $V'_t \subset \mathbb{R}^{m-1}$ by

$$V'_t := \left\{ (x_2, \dots, x_m) \in \mathbb{R}^{m-1} \,\middle|\, (t, x_2, \dots, x_m) \in V' \right\}$$

and the smooth map $g_t : V'_t \to \mathbb{R}^{n-1}$ by

$$(t, g_t(x_2, \dots, x_m)) := g(t, x_2, \dots, x_m)$$

for $(x_2, \dots, x_m) \in V'_t$. Then

$$dg(t, x_2, \dots, x_m) = \begin{pmatrix} 1 & 0 \\ * & dg_t(x_2, \dots, x_m) \end{pmatrix}$$

for $x_2, \dots, x_m \in V_t$. Thus the derivative $dg(t, x_2, \dots, x_m)$ is not surjective if and only if the derivative $dg_t(x_2, \dots, x_m)$ is not surjective. This means that

$$\begin{aligned} \mathcal{C}'_t &:= \left\{ (x_2, \dots, x_m) \in V'_t \,\middle|\, dg_t(_2, \dots, x_m) \text{ is not surjective} \right\} \\ &= \left\{ (x_2, \dots, x_m) \in \mathbb{R}^{m-1} \,\middle|\, (t, x_2, \dots, x_m) \in \mathcal{C}' \right\}. \end{aligned}$$

Thus it follows from the induction hypothesis that the set $g_t(\mathcal{C}'_t) \subset \mathbb{R}^{n-1}$ has Lebesgue measure zero for each $t \in \mathbb{R}$. Since

$$g_t(\mathcal{C}'_t) = \left\{ (y_2, \dots, y_n) \in \mathbb{R}^{n-1} \,|\, (t, y_2, \dots, y_n) \in g(\mathcal{C}') \right\} = g(\mathcal{C}')_t$$

for all t, it follows from Fubini's Theorem that the set

$$g(\mathcal{C}') = f(V \cap \mathcal{C}) \subset \mathbb{R}^n$$

has Lebesgue measure zero. This proves the claim and Step 1.

Proof of Step 2. Fix a positive integer k and an element $\overline{x} \in \mathcal{C}_k \setminus \mathcal{C}_{k+1}$. We will prove that there exists an open neighborhood $V \subset U$ of $\overline{x}$ such that the set $f(V \cap \mathcal{C}_k)$ has Lebesgue measure zero. Since the set $f(\mathcal{C}_k \setminus \mathcal{C}_{k+1})$ can be covered by countably many such neighborhoods, this implies that the set $f(\mathcal{C}_k \setminus \mathcal{C}_{k+1})$ has Lebesgue measure zero.

By assumption, there exist indices $i_1, i_2, \dots, i_{k+1} \in \{1, \dots, m\}$ such that

$$\frac{\partial^{k+1} f}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3} \cdots \partial x_{i_{k+1}}}(\overline{x}) \neq 0.$$

Assume without loss of generality that $i_1 = 1$ and consider the function

$$w := \frac{\partial^k f}{\partial x_{i_2} \partial x_{i_3} \cdots \partial x_{i_{k+1}}} : U \to \mathbb{R}.$$

Then

$$w|_{\mathcal{C}_k} = 0, \qquad \frac{\partial w}{\partial x_1}(\overline{x}) \neq 0.$$

Now define the map $h : U \to \mathbb{R}^m$ by

$$h(x) := (w(x), x_2, \dots, x_m)$$

for $x = (x_1, x_2, \dots, x_m) \in U$. Then $\det(dh(\overline{x})) \neq 0$ and so the Inverse Function Theorem 1.1.19 asserts that there exists an open neighborhood $V \subset U$ of $\overline{x}$ such that $V' := h(V)$ is an open subset of $\mathbb{R}^m$ and $h|_V : V \to V'$ is a diffeomorphism. Moreover, the following holds.

(a) $h(V \cap \mathcal{C}_k) \subset \{0\} \times \mathbb{R}^{m-1}$.

(b) $x \in V \cap \mathcal{C}_k \implies df(x) = 0$.

Again consider the map

$$g := f \circ (h|_V)^{-1} : V' \to \mathbb{R}^n$$

and define

$$V_0' := (\{0\} \times \mathbb{R}^{m-1}) \cap V', \qquad g_0 := g|_{V_0'} : V_0' \to \mathbb{R}^n.$$

Then by (a) and (b) the set $h(V \cap \mathcal{C}_k) \subset V_0'$ is contained in the set of critical points of g_0. Hence it follows from the induction hypothesis that the set

$$g_0(h(V \cap \mathcal{C}_k)) = g(h(V \cap \mathcal{C}_k)) = f(V \cap \mathcal{C}_k)$$

has Lebesgue measure zero. This proves Step 2.

Proof of Step 3. Assume

$$k > \frac{m}{n} - 1 \tag{1.3.1}$$

and fix any closed cube $Q \subset U$ of sidelength $\delta > 0$. Thus Q is a set of the form

$$Q = [a_1, b_1] \times \cdots \times [a_m, b_m], \qquad b_i - a_i = \delta.$$

We will prove that the set $f(\mathcal{C}_k \cap Q)$ has Lebesgue measure zero. Since $f(\mathcal{C}_k)$ can be covered by countably many such sets, this will imply Step 3. To prove the assertion, observe that by Taylor's Theorem there exists a constant $c > 0$ such that, for all $x \in \mathcal{C}_k \cap Q$ and all $h \in \mathbb{R}^m$ with $x + h \in Q$, we have

$$|f(x+h) - f(x)| \le c|h|^{k+1}. \tag{1.3.2}$$

For each $r \in \mathbb{N}$ subdivide the cube Q into r^m subcubes of sidelength δ/r and then consider the limit $r \to \infty$. For a fixed value of r let Q_1 be one of the cubes in this subdivision containing a point $x \in \mathcal{C}_k \cap Q_1$. Then every element of Q_1 has the form

$$x + h, \qquad |h| \le \frac{\sqrt{m}\delta}{r}. \tag{1.3.3}$$

In this situation it follows from (1.3.2) and (1.3.3) that

$$|f(x+h) - f(x)| \le c|h|^{k+1} \le c\left(\frac{\sqrt{m}\delta}{r}\right)^{k+1}. \tag{1.3.4}$$

This shows that $f(Q_1)$ is contained in a cube with sidelength

$$2c\left(\frac{\sqrt{m}\delta}{r}\right)^{k+1} = \frac{a}{r^{k+1}}, \qquad a := 2c\left(\sqrt{m}\delta\right)^{k+1}. \tag{1.3.5}$$

Hence

$$\mu_n(f(Q_1)) \le \frac{a^n}{r^{(k+1)n}}. \tag{1.3.6}$$

Since the set $\mathcal{C}_k \cap Q$ is contained in the union of at most r^m such cubes, it follows that

$$\mu_n(f(\mathcal{C}_k \cap Q)) \le \frac{a^n r^m}{r^{(k+1)n}} = a^n r^{m-(k+1)n}. \tag{1.3.7}$$

Since $(k+1)n > m$ by (1.3.1), the term on the right in (1.3.7) tends to zero as r tends to infinity, and hence $\mu_n(f(\mathcal{C}_k \cap Q)) = 0$. This proves Step 3 and Theorem 1.2.1. □

1.4 The Degree Modulo Two of a Smooth Map

Let M be a compact smooth m-manifold without boundary and let N be connected smooth manifold of the same dimension m. If $q \in N$ is a regular value of f, recall that $\#f^{-1}(q)$ denotes the number of solutions $p \in M$ to the equation $f(p) = q$. By Lemma 1.1.21 the set $\mathcal{R}_f \subset N$ of regular values is open and that the function

$$\mathcal{R}_f \to \mathbb{N}_0 : q \mapsto \#f^{-1}(q)$$

is locally constant. We will prove that the *residue class modulo* 2 *of* $\#f^{-1}(q)$ does not depend on the choice of the regular value q, i.e. either $\#f^{-1}(q)$ is even for every regular value q or $\#f^{-1}(q)$ is odd for every regular value q. This residue class is called the mod 2 degree of f.

1.4.1 Smooth Homotopy and Smooth Isotopy

Let M and N ge smooth manifolds, possibly with boundary. Then the product $[0,1] \times M$ of the unit interval $[0,1] \subset \mathbb{R}$ with M need not be a manifold with boundary; if the boundary of M is nonempty, this product is what is called a *manifold with corrners*; namely, boundary points $p \in \partial M$ give rise to *corner points* $(0,p), (1,p) \in [0,1] \times M$. Howver, $[0,1] \times M$ is a subset of the product $\mathbb{R} \times M$, and since $\mathbb{R} \times M$ is a manifold with boundary, smooth maps on $[0,1] \times M$ are understood as in Definition 1.2.3.

Definition 1.4.1 (Smooth Homotopy). *Two smooth maps*

$$f_0, f_1 : M \to N$$

are called **smoothly homotopic** *iff there exists a smooth map*

$$F : [0,1] \times M \to N$$

such that

$$F(0,p) = f_0(p), \qquad f(1,p) = f_1(p)$$

for all $p \in M$*. The map* F *is called a* **smooth homotopy from** f_0 **to** f_1*. The notation* $f_0 \sim f_1$ *will mean that* f_0 *and* f_1 *are smoothly homotopic.*

A smooth homotopy $F : [0,1] \times M \to N$ from f_0 to f_1 can be understood as a family of smooth maps $f_t : M \to N$ for $0 \le t \le 1$, defined by

$$f_t(p) := F(t,p) \qquad \text{for } p \in M.$$

In this notation we call $\{f_t\}_{0 \le t \le 1}$ a smooth homotopy from f_0 to f_1, and it is then understood that the map $[0,1] \times M \to N : (t,p) \mapsto f_t(p)$ is smooth.

Remark 1.4.2. The relation of smooth homotopy is an equivalence relation. To prove that it is transitive, choose a smooth function

$$\beta : [0,1] \to [0,1]$$

such that

$$\beta(t) = \begin{cases} 0, & \text{for } 0 \le t \le 1/3, \\ 1, & \text{for } 2/3 \le t \le 1. \end{cases} \tag{1.4.1}$$

For example, take

$$\beta(t) := \frac{\lambda(t-1/3)}{\lambda(t-1/3)+\lambda(2/3-t)},$$

where the $\lambda : \mathbb{R} \to \mathbb{R}$ is the smooth function defined by $\lambda(t) := 0$ for $t \le 0$ and by $\lambda(t) := e^{-1/t}$ for $t > 0$. Given a smooth homotopy $F : [0,1] \times M \to N$ from f_0 to f_1, the formula

$$G(t,p) := F(\beta(t), p)$$

defines another smooth homotopy from f_0 to f_1 such that

$$G(t,p) = \begin{cases} f_0(p), & \text{for } 0 \le t \le 1/3, \\ f_1(p), & \text{for } 2/3 \le t \le 1. \end{cases}$$

With the aid of this construction it is easy to verify that $f_0 \sim f_1$ and $f_1 \sim f_2$ imply $f_0 \sim f_2$.

If f_0 and f_1 are diffeomorphism from M to N, one can define the concept of a *smooth isotopy* from f_0 to f_1. This will again be an equivalence relation.

Definition 1.4.3 (Smooth Isotopy). *Two diffeomorphisms*

$$f_0, f_1 : M \to N$$

are called **smoothly isotopic** *iff there exists a smooth homotopy* $\{f_t\}_{0 \le t \le 1}$ *from* f_0 *to* f_1 *such that* $f_t : M \to N$ *is a diffeomorphism for every* $t \in [0,1]$. *In this case* $\{f_t\}_{0 \le t \le 1}$ *is called a* **smooth isotopy** *from* f_0 *to* f_1.

It is, in general, much more difficult to prove that two diffeomorphisms are smoothly isotopic than to prove that they are smoothly homotopic. For example, it is an open question whether every diffeomorphism of $\mathbb{C}\mathrm{P}^2$ that is smoothly homotopic to the identity is smoothly isotopic to the identity.

It will turn out that the mod 2 degree of a smooth map depends only on its homotopy class.

Lemma 1.4.4 (**Homotopy Lemma**). *Let M and N be smooth manifollds of the same dimension m and assume that M is compact and without boundary. Let $f_0, f_1 : M \to N$ be smooth maps and let $q \in N$ be a regular value of both f_0 and f_1. Then*

$$\#f_0^{-1}(q) \equiv \#f_1^{-1}(q) \ \ (\textit{modulo } 2). \tag{1.4.2}$$

Proof. Let $F : [0,1] \times M \to N$ be a smooth homotopy from f_0 to f_1. Assume first that q is a regular value of F. Then by Lemma 1.2.20 the set

$$X := F^{-1}(q) \subset [0,1] \times M$$

is a 1-dimensional submanifold with boundary

$$\begin{aligned}\partial X &= X \cap \partial\bigl([0,1] \times M\bigr)\\ &= X \cap \bigl(\{0,1\} \times M\bigr)\\ &= \{0\} \times f_0^{-1}(q) \cup \{1\} \times f_1^{1}(q).\end{aligned}$$

(See Figure 1.8.) Hence $\#\partial X = \#f_0^{-1}(q) + \#f_1^{-1}(q)$. Since X is compact, it follows from Theorem A.4.1 that $\#\partial X$ is an even number and this proves (1.4.2) under the assumption that q is a regular value of F.

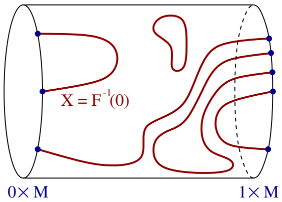

Figure 1.8: The number of boundary points on the left is congruent to the number of boundary points on the right modulo 2.

Now suppose that q is not a regular value of F. Recal from Lemma 1.1.21 that the set $\mathcal{R}_{f_0}$ of regular values of f_0 is open and that $\#f_0^{-1}(q')$ is a locally locally constant function of $q' \in \mathcal{R}_{f_0}$. Hence, for $i = 0, 1$, there exists an open neighborhood $V_i \subset N$ of q such that $V_i \subset \mathcal{R}_{f_i}$ and

$$\#f_i^{-1}(q) = \#f_i^{-1}(q') \qquad \text{for all } q' \in V_i.$$

By Corollary 1.2.2 we may choose a regular value q' of F within $V_0 \cap V_1$. Then, by the first part of the proof,

$$\#f_1^{-1}(q) - \#f_0^{-1}(q) = \#f_1^{-1}(q') - \#f_0^{-1}(q') \in 2\mathbb{Z},$$

This proves Lemma 1.4.4. □

We also need the following result.

Lemma 1.4.5 (**Homogeneity Lemma).** *Let N be a connected smooth manifold without boundary and let $q_0, q_1 \in N$. Then there exists a diffeomorphism $h : N \to N$ that is smoothly isotopic to the identity and satisfies*

$$h(q_0) = q_1.$$

If N is not compact, then the isotopy $\{h_t\}_{0 \le t \le 1}$ from the identity $h_0 = \mathrm{id}$ to $h_1 = h$ can be chosen with **uniform compact support***, i,e. there exists a compact set $K \subset N$ such that*

$$h_t(q) = q \qquad \text{for all } (t, q) \in [0, 1] \times (N \setminus K). \tag{1.4.3}$$

Remark 1.4.6. In the special case $N = S^m$ the diffeomorphism h can be chosen as a rotation which carries q_0 to q_1 and fixes all vectors orthogonal to the plane through q_0 and q_1.

Proof of Lemma 1.4.5. The proof has two steps. The first step deals with the special case $N = \mathbb{R}^m$. Denote the open unit ball in $\mathbb{R}^m$ by

$$B^m := \left\{ x \in \mathbb{R}^m \,\middle|\, |x| < 1 \right\}.$$

Step 1. *Let $\xi \in \mathbb{R}^m$ such that $0 < |\bar{x}| < 1$. Then there exists a smooth isotopy $\{\psi_t\}_{0 \le t \le T}$ of $\mathbb{R}^m$ such that*

$$\psi_0 = \mathrm{id}, \qquad \psi_T(0) = \bar{x} \tag{1.4.4}$$

and

$$\psi_t(x) = x \qquad \text{for all } t \in [0, T] \text{ and all } x \in \mathbb{R}^m \setminus B^m. \tag{1.4.5}$$

(See Figure 1.9.)

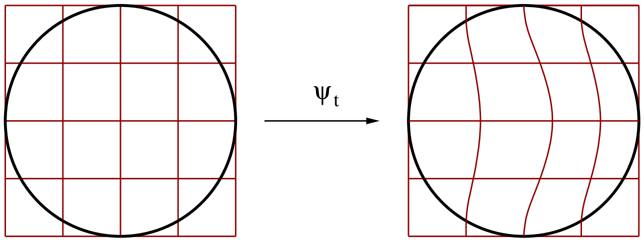

Figure 1.9: Deformation of the unit ball.

Assume $\bar{x} \neq 0$ and define $\lambda := |\bar{x}|$ and $c := \bar{x}/\lambda \in S^{m-1}$ so that

$$\bar{x} = \lambda c.$$

Choose a smooth function $\beta : \mathbb{R}^m \to [0,1]$ such that

$$\begin{aligned} \beta(x) > 0 \quad & \text{for } x \in B^m, \\ \beta(x) = 0 \quad & \text{for } x \in \mathbb{R}^m \setminus B^m. \end{aligned}$$

For example, take $\beta(x) = e^{-1/(1-|x|^2)}$ for $x \in B^m$. Now let $\{\psi_t\}_{t\in\mathbb{R}}$ be tthe flow of the differential equation

$$\dot{x} = \beta(x)c. \tag{1.4.6}$$

(See Figure 1.9). Then

$$\psi_0 = \mathrm{id}, \qquad \psi_t|_{\mathbb{R}^m\setminus B^m} = \mathrm{id} \qquad \text{for all } t,$$

and $\psi_t(0) = \lambda(t)c$ for a smooth function $\lambda : \mathbb{R} \to (-1,1)$ with positive derivative such that $\lambda(0) = 0$ and $\lim_{t\to\pm\infty} = \pm 1$. Hence there exists a unique real number $T > 0$ such that $\lambda(T) = \lambda$ and hence $\psi_T(0) = \lambda c = \bar{x}$. This proves Step 1.

Step 2. *We prove the lemma.*

Call two points $q_0 \in N$ and $q_1 \in N$ **isotopic** iff there exists a smooth isotopy $\{h_t\}_{0\le t\le 1}$ of N with uniform compact support such that $h_0 = \mathrm{id}$ and $h_1(q_0) = q_1$. This is an equivalence relation by the argument in Remark 1.4.2. By Step 1 each equivalence class is open. Namely, given $q_0 \in N$ choose an open neighborhood $U_0 \subset N$ of q_0 that is diffeomorphic to $\mathbb{R}^m$ and choose a diffeomorphism $\phi_0 : U_0 \to \mathbb{R}^m$ such that $\psi_0(q_0) = 0$. Given an element $q_1 \in V_0 := \phi_0^{-1}(B^m) \subset U_0$, define the isotopy $\{h_t\}_{0\le t\le 1}$ of N by $h_t|_{U_0} := \phi_0^{-1} \circ \psi_{tT} \circ \phi_0$ and $h_t|_{N\setminus U0} = \mathrm{id}$, where $\{\psi_t\}_{0\le t\le T}$ is the isotopy in Step 1 with $\bar{x} := \phi_0(q_1)$. Then the isotopy $\{h_t\}_{0\le t\le 1}$ has uniform compact support in $K := \phi_0^{-1}(\overline{B^m})$ and satisfies $h_1(q_0) = \phi_0^{-1}(\psi_T(0)) = \phi_0^{-1}(\bar{x}) = q_1$. Thus the equivalence class of q_0 contains the set V_0.

Since every equivalence class is open, it follows that every equivalence class is also closed. Since N is connected, this implies that there is only one equivalence class. Hence any two points in N are isotopic, and this proves Lemma 1.4.5. □

1.4.2 The Degree Modulo Two

With these preparations we are ready to prove the main result of this section.

Theorem 1.4.7. *Let M and N be smooth m-manifolds without boundary such that M is compact and N is connected.*

(i) *Let $f : M \to N$ be a smooth map. If q, q' are regular values of f, then*

$$\#f^{-1}(q) \equiv \#f^{-1}(q') \qquad (\textit{modulo } 2).$$

The common residue class

$$\deg_2(f) := \begin{cases} 0, & \textit{if } \#f^{-1}(q) \textit{ is even for all } q \in \mathcal{R}_f, \\ 1, & \textit{if } \#f^{-1}(q) \textit{ is odd for all } q \in \mathcal{R}_f \end{cases} \tag{1.4.7}$$

is called the **mod 2 degree of** *f.*

(ii) *The mod 2 degree has the following properties.*

(Homotopy) *If $f_0, f_1 : M \to N$ are smoothly homotopic, then*

$$\deg_2(f_0) = \deg_2(f_1).$$

(Identity) *The identity map on M has mod 2 degree $\deg_2(\mathrm{id}_M) = 1$.*

(Composition) *If N is compact, P is a connected m-manifold without boundary, and $f : M \to N$ and $g : N \to P$ are smooth maps, then*

$$\deg_2(g \circ f) = \deg_2(f) \deg_2(g).$$

(Zero) *If W is a compact $(m+1)$-manifold with boundary and $F : W \to N$ is a smooth map, then $\deg_2(F|_{\partial W}) = 0$.*

Proof. Let $q, q' \in \mathcal{R}_f$ be regular values of f. By the Homogeneity Lemma choose a smooth isotopy $\{h_t\}_{0 \le t \le 1}$ of N such that $h_0 = \mathrm{id}$ and $h_1(q') = q$. Then q is a regular value of $f_1 := h_1 \circ f$. Namely, if $p \in M$ satisfies the equation $f_1(p) = q$, then $f(p) = h_1^{-1}(f_1(p)) = h_1^{-1}(q) = q'$, thus $df(p)$ is surjective because q' is a regular value of f, and so $df_1(p) = dh_1(q') \circ df(p)$ is surjective. Thus q is a common regular value of f and f_1. Moreover,

$$f_1^{-1}(q) = f^{-1}(q').$$

Since f and f_1 are smoothly homotopic via $\{h_t \circ f\}_{0 \le t \le 1}$, it follows from the Homotopy Lemma that $\#f^{-1}(q') = \#f_1^{-1}(q) \equiv \#f^{-1}(q)$ (modulo 2). This proves part (i).

In part (ii) the (Homotopy) axiom follows from the Homotopy Lemma and the (Identity) axiom is obvious. To prove the (Composition) axiom, choose any regular value $z \in \mathcal{R}_{g\circ f}$ of $g \circ f$ and denote its preimages under g by $q_1, \dots, q_k \in N$. Then

$$k \equiv \deg_2(g) \quad \text{(modulo 2)}$$

and each q_j is a regular value of f. Hence

$$\begin{aligned} \#(g \circ f)^{-1}(z) &= \sum_{j=1}^{k} \# f^{-1}(q_j) \\ &\equiv \deg_2(f)k \quad \text{(modulo 2)}. \\ &\equiv \deg_2(f)\deg_2(g) \quad \text{(modulo 2)}. \end{aligned}$$

This implies $\deg_2(g \circ f) = \deg_2(f)\deg_2(g)$.

To prove the (Zero) axiom, choose a common regular value q of F and $F|_{\partial W}$. Then by Lemma 1.2.20 the set $F^{-1}(q) \subset W$ is a 1-dimensional submanifold with boundary

$$\partial F^{-1}(q) = F^{-1}(q) \cap \partial W = (F|_{\partial W})^{-1}(q).$$

Hence it follows from Theorem A.4.1.that the number $\#(F|_{\partial W})^{-1}(q)$ is even. This proves Theorem 1.4.7. □

Example 1.4.8. Let M be a compact connected smooth manifold without boundary. Then we observe the following.

(i) Every constant map $f : M \to M$ has mod 2 degree $\deg_2(f) = 0$,

(ii) The identity map $\mathrm{id} : M \to M$ has mod 2 degree $\deg_2(\mathrm{id}) = 1$. Hence, by Theorem 1.4.7 the identity map of M is not smoothly homotopic to a constant map.

(iii) In the case $M = S^m$ part (ii) implies that there does not exist a smooth map $F : \mathbb{D}^{m+1} \to S^m$ which restricts to the identity map on the sphere, because such a map would give rise to a smooth homotopy

$$f_t : S^m \to S^m, \qquad f_t(x) := F(tx)$$

from a constant map f_0 to the identity map f_1 (see Lemma 1.2.25). This follows also from the (Zero) axiom in Theorem 1.4.7.

(iv) Using an approximation argument, one can prove that the identity map of S^m is not continuously homotopic to a constant map (Exercise 1.7.5). This can be used to prove the following consequence of Brouwer's Invariance of Domain Theorem: *If two open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$* (Exercise 1.7.8).

1.5 The Borsuk–Ulam Theorem

Call a map $f : S^m \to \mathbb{R}^n$ **odd** iff it satisfies

$$f(-x) = -f(x) \tag{1.5.1}$$

for all $x \in S^m$. The same notion is used for a map from S^m to $S^{n-1} \subset \mathbb{R}^n$. The Borsuk–Ulam Theorem asserts that an odd map from a sphere to itself is of odd degree. This was proved in 1933 by Karol Borsuk [2]. According to Borsuk, Corollary 1.5.2 had been conjectured earlier by Stanislaw Ulam.

Theorem 1.5.1 (Borsuk–Ulam). *Let $f : S^m \to S^m$ be a smooth map. If f is odd, then $\deg_2(f) = 1$.*

Corollary 1.5.2. *If $f : S^m \to \mathbb{R}^m$ is a continuous map, then there exists an element $x \in S^m$ such that $f(x) = f(-x)$.*

Proof. We first prove the assertion for smooth maps. Let $f : S^m \to \mathbb{R}^m$ be a smooth map and assume $f(x) \neq f(-x)$ for all $x \in S^m$. Then the map

$$S^m \to S^{m-1} : x \mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

is smooth and odd. Its composition with the canonical inclusion of S^{m-1} into S^m has mod 2 dgree zero and is odd, in contradiction to Theorem 1.5.1. This proves the corollary for smooth maps.

Now let $f : S^m \to \mathbb{R}^m$ be a continuous map and assume $f(x) \neq f(-x)$ for all $x \in S^m$. Then there exists a constant $\delta > 0$ such that

$$|f(x) - f(-x)| \geq 2\delta \qquad \text{for all } x \in S^m.$$

By Weierstraß Approximation (see for example [7, Theorem 5.4.5]) choose a polynomial map $P : \mathbb{R}^{m+1} \to \mathbb{R}^m$ such that

$$|P(x) - f(x)| < \delta \qquad \text{for all } x \in S^m.$$

Then, by the triangle inequality, the smooth map $p := P|_{S^m} : S^m \to \mathbb{R}^m$ satisfies $p(x) \neq p(-x)$ for all $x \in S^m$, in contradiction to the first part of the proof. This proves Corollary 1.5.2. □

Corollary 1.5.2 can be illustrated by the example that there is always a pair of antipodal points on the equator with the same temperature ($m = 1$) and a pair of antipodal points on the surface of the earth with the same temperature and pressure ($m = 2$).

Corollary 1.5.3. *Let* $A_0, A_1, \dots, A_m \subset S^m$ *be compact sets such that*

$$S^m = \bigcup_{i=0}^{m} A_i.$$

Then at least one of the sets A_i *contains a pair of antipodal points.*

Proof. Define the map $f = (f_1, \dots, f_m) : S^m \to \mathbb{R}^m$ by

$$\begin{aligned} f_1(x) &:= d(x, A_1) - d(x, A_0), \\ f_2(x) &:= d(x, A_2) - d(x, A_0 \cup A_1), \\ f_3(x) &:= d(x, A_3) - d(x, A_0 \cup A_1 \cup A_2), \\ &\vdots \\ f_m(x) &:= d(x, A_m) - d(x, A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_{m-1}) \end{aligned}$$

for $x \in S^m$, where $d(x, A) := \inf_{a \in A} |x - a|$. Then f is continuous. For every vector $y \in \mathbb{R}^m$ define

$$i(y) := \min \left\{ i \in \{0, 1, \dots, m\} \;\middle|\; \begin{array}{l} y_j > 0 \text{ for all } j \in \{1, \dots, m\} \\ \text{such that } j > i \end{array} \right\}. \tag{1.5.2}$$

Thus $i(y) = m$ when $y_m \le 0$ and $i(y) = 0$ when $y_j > 0$ for $j = 1, \dots, m$.

Claim. $f^{-1}(y) \subset A_{i(y)}$ *for all* $y \in \mathbb{R}^m$.

Let $x \in S^m$ and $i := i(f(x))$. Then $f_j(x) > 0$ for all $j \in \{1, \dots, m\}$ such that $j > i$. Hence $d(x, A_j) > 0$ and so $x \notin A_j$ for all $j > i$. This implies

$$x \in A_0 \cup A_1 \cup \cdots \cup A_i. \tag{1.5.3}$$

For $i = 0$ this proves the claim. Thus assume $i > 0$. Then $f_i(x) \le 0$ and so

$$d(x, A_i) \le d(x, A_0 \cup \cdots \cup A_{i-1}).$$

Hence $d(x, A_i) = 0$, because otherwise $x \notin A_i$ and $x \notin A_0 \cup \cdots \cup A_{i-1}$, contradicting (1.5.3). Hence $x \in A_i$ and this proves the claim.

Now Corollary 1.5.2 asserts that there exists an element $x \in S^m$ such that $f(x) = f(-x) =: y$. With $i := i(y)$ it follows from the claim that A_i contains the antipodal pair $\{x, -x\}$. This proves Corollary 1.5.3. □

In [2] it is pointed out that Corollary 1.5.3 had been proved with different methods in a 1930 paper by Ljusternik–Shnirelman [23].

As a warmup for the proof of the Borsuk–Ulam Theorem consider the following example.

Example 1.5.4. Every invertible $m \times m$-matrix $A \in \mathrm{GL}(m,\mathbb{R})$ determines a smooth map $\phi_A : S^{m-1} \to S^{m-1}$ defined by

$$\phi_A(x) := \frac{Ax}{|Ax|} \qquad \text{for } x \in S^{m-1}.$$

These maps are all odd and they satisfy

$$\phi_{AB} = \phi_A \circ \phi_B, \qquad \phi_{\mathbb{1}} = \mathrm{id}_{S^{m-1}}$$

for all $A, B \in \mathrm{GL}(m,\mathbb{R})$. Hence $\phi_A : S^{m-1} \to S^{m-1}$ is a diffeomorphism and hence $\deg_2(\phi_A) = 1$ for every $A \in \mathrm{GL}(m,\mathbb{R})$.

Lemma 1.5.5. *Let $U \subset \mathbb{R}^m$ be an open set, let $\bar{x} \in U$, and let $f : U \to \mathbb{R}^m$ be a smoth map such that $\det(df(\bar{x})) \neq 0$. Choose $r > 0$ such that*

$$0 < |x - \bar{x}| < r \qquad \Longrightarrow \qquad x \in U \text{ and } f(x) \neq f(\bar{x})$$

for all $x \in \mathbb{R}^m$. For $0 < \varepsilon < r$ define the map $g_\varepsilon : S^{m-1} \to S^{m-1}$ by

$$g_\varepsilon(x) := \frac{f(\bar{x} + \varepsilon x) - f(\bar{x})}{|f(\bar{x} + \varepsilon x) - f(\bar{x})|} \qquad \text{for } x \in S^{m-1}.$$

Then $\deg_2(g_\varepsilon) = 1$ for $0 < \varepsilon < r$.

Proof. We will prove that g_ε is homotopic to the map $g_0 : S^{m-1} \to S^{m-1}$ given by $g_0(x) := |df(\bar{x})x|^{-1} df(\bar{x})x$. To see this, define

$$h_i(x) := \int_0^1 \partial_i f(\bar{x} + sx)\, ds \qquad \text{for } x \in B_r^m \text{ and } i = 1, \dots, m.$$

Then $h_i : B_r^m \to \mathbb{R}^m$ is smooth and $h_i(0) = \partial_i f(\bar{x})$ for each i. Moreover,

$$f(\bar{x} + x) - f(\bar{x}) = \sum_{i=1}^m x_i h_i(x) \qquad \text{for all } x \in B_r^m.$$

Hence

$$F(t,x) := \sum_{i=1}^m x_i h_i(tx) = \begin{cases} t^{-1}\big(f(\bar{x} + tx) - f(\bar{x})\big), & \text{if } 0 < t < r, \\ df(\bar{x})x, & \text{if } t = 0, \end{cases}$$

for $0 \leq t < r$ and $x \in S^{m-1}$. This map is smooth and has no zeros and hence gives tise to a smooth homotopy

$$[0,\varepsilon] \times S^{m-1} \to S^{m-1} : (t,x) \mapsto \frac{F(t,x)}{|F(t,x)|} = g_t(x)$$

joining g_0 to g_ε. Hence it follows from Theorem 1.4.7 and Example 1.5.4 that $\deg_2(g_\varepsilon) = \deg_2(g_0) = 1$ for $0 < \varepsilon < r$. This proves Lemma 1.5.5. □

The proof of Theorem 1.5.1 is based on a lemma by Heinz Hopf.

Lemma 1.5.6 (Hopf). *Let M be a compact m-manifold with boundary, let $f : M \to \mathbb{R}^m$ be a smooth map, let $y \in \mathbb{R}^m \setminus f(\partial M)$ be a regular value of f, and define the map $g : \partial M \to S^{m-1}$ by*

$$g(p) := \frac{f(p) - y}{|f(p) - y|} \qquad \text{for } p \in \partial M. \tag{1.5.4}$$

Then $\# f^{-1}(y) \equiv \deg_2(g)$ modulo 2.

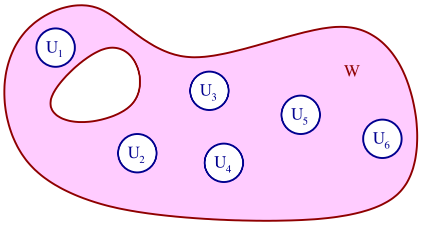

Figure 1.10: Removing neighborhoods of the preimages of a regular value.

Proof. The set $f^{-1}(y) \subset M$ is compact and we have $f^{-1}(y) \cap \partial M = \emptyset$, because $y \notin f(\partial M)$. Moreover, $f^{-1}(y)$ is discrete because y is a regular value of M (see Lemma 1.1.21). Hence $f^{-1}(y)$ is a finite subset of $M \setminus \partial M$. If $f^{-1}(y)$ is empty, then g extends to a smooth map from M to S^{m-1} and hence $\deg_2(g) = 0$ by Theorem 1.4.7. Thus assume $k := \# f^{-1}(y) > 0$ and choose $p_1., \dots, p_k \in M \setminus \partial M$ such that $f^{-1}(y) = \{p_1, \dots, p_k\}$. Choose open balls $U_i \subset M$, centered at p_i such that $\overline{U}_i \cap \partial M = \emptyset$ and $\overline{U}_i \cap \overline{U}_j = \emptyset$ for $i, j = 1, \dots, k$ such that $i \neq j$. Then

$$W := M \setminus \bigl(U_1 \cup \cdots \cup U_k\bigr)$$

is a smooth m-manifold with boundary $\partial W = \partial M \cup \partial U_1 \cup \cdots \cup \partial U_k$ (see Figure 1.10) and $f(p) \neq y$ for all $p \in W$. Define the map $G : W \to S^{m-1}$ by

$$G(p) := \frac{f(p) - y}{|f(p) - y|} \qquad \text{for } p \in W.$$

Then $G|_{\partial M} = g$ and $\deg_2(G|_{\partial U_i}) = 1$ for $i = 1, \dots, k$ by Lemma 1.5.5. Hence it follows from the (Zero) axiom in Theorem 1.4.7 that

$$0 = \deg_2(G|_{\partial W}) \equiv \deg_2(G|_{\partial M}) - \sum_{i=1}^{k} \deg_2(G|_{\partial U_i}) \equiv \deg_2(g) - k \pmod 2.$$

This proves Lemma 1.5.6. □

In the Borsuk–Ulam Theorem we tacitly assume $m \geq 1$. For $m = 0$ the unit "sphere" $S^0 = \{-1, +1\} \subset \mathbb{R}$ is not connected and so, strictly speaking, the maps $f : S^0 \to S^0$ do not fit into the framework of Theorem 1.4.7. Nevertheless, one could still define the mod 2 degree to be zero for the two constant maps (where the count of pre-images is zero or two) and to be one for plus or minus the identity (where the count of pre-images is one). Then the odd maps have degree one, in accordance with Theorem 1.5.1. A similar issue arises with Lemma 1.5.6 in the case $m = 1$, however, by Theorem A.4.1 the boundary of M is then a finite set consisting of an even number of points, and so the residue class modulo two of the number of pre-images for the map $g : \partial M \to S^0$ is again independent of the choice of the element of S^0. The reader may verify that with these conventions the induction argument in the proof of Theorem 1.5.1 below remains valid in the case $m = 1$. We will instead give a direct proof of Theorem 1.5.1 for $m = 1$ and proceed with the induction argument for $m \geq 2$.

Proof of Theorem 1.5.1. The proof is by induction on the dimension m. For $m = 1$ take $S^1 \subset \mathbb{C}$ to be the unit circle in the complex plane and let $f : S^1 \to S^1$ be a smooth map. Then, by Exercise 1.7.10, there exists a smooth map $\theta : \mathbb{R} \to \mathbb{R}$ and an integer $k \in \mathbb{Z}$ such that

$$f\big(\exp(2\pi\mathbf{i}t)\big) = \exp\big(2\pi\mathbf{i}\theta(t)\big), \qquad \theta(t+1) = \theta(t) + k \tag{1.5.5}$$

for all $t \in \mathbb{R}$. Given θ and k as in (1.5.5), we prove the following.

Claim 1. $\deg_2(f) \equiv k \ (mod\, 2)$.

Claim 2. *If $f(-z) = -f(z)$ for all $z \in S^1$, then k is odd.*

To prove Claim 1, define the map $f_\lambda : S^1 \to S^1$ by

$$f_\lambda\big(\exp(2\pi\mathbf{i}t)\big) := \exp\big(2\pi\mathbf{i}\big((1-\lambda)kt + \lambda\theta(t)\big)\big)$$

for $0 \leq \lambda \leq 1$ and $t \in \mathbb{R}$. Then the map $[0,1] \times S^1 \to S^1 : (\lambda, z) \mapsto f_\lambda(z)$ is a smooth homotopy from the map $f_0(z) = z^k$ fo $f_1 = f$. Thus $\# f_0^{-1}(w) = k$ for all $w \in S^1$ and so $\deg_2(f) = \deg_2(f_0) \equiv k (\mathrm{mod}\, 2)$. This proves Claim 1.

Now assume $f(-z) = -f(z)$ for all $z \in S^1$. Then, for all $t \in \mathbb{R}$,

$$\begin{aligned}\exp\big(2\pi\mathbf{i}\theta(t+1/2)\big) &= f\big(\exp(2\pi\mathbf{i}(t+1/2))\big) = f\big(-\exp(2\pi\mathbf{i}t)\big) \\ &= -f\big(\exp(2\pi\mathbf{i}t)\big) = \exp\big(2\pi\mathbf{i}(\theta(t)+1/2)\big).\end{aligned}$$

Thus there is an integer ℓ such that $\theta(t+1/2) - \theta(t) - 1/2 = \ell$ for all t. Hence, by (1.5.5), $k - 1 = \theta(1) - \theta(1/2) - 1/2 + \theta(1/2) - \theta(0) - 1/2 = 2\ell$. Thus $k = 2\ell + 1$ is odd. This proves Claim 2 and Theorem 1.5.1 for $m = 1$.

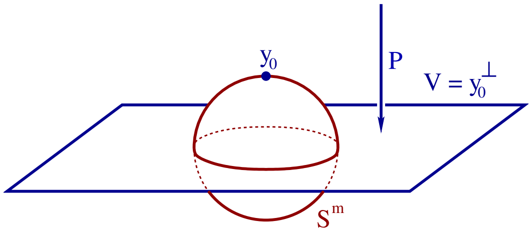

Figure 1.11: The orthogonal projection onto V $= y_0^\perp$.

Now let $m \geq 2$ and assume, by induction, that the result has been established for $m-1$. Let $f : S^m \to S^m$ be a smooth map such that

$$f(-x) = -f(x) \qquad \text{for all } x \in S^m,$$

and define $g := f \circ \iota : S^{m-1} \to S^m$, where $\iota : S^{m-1} \to S^m$ denotes the canonical inclusion given by $\iota(x) := (x, 0)$ for $x \in S^{m-1}$. By the Theorem of Sard and Brown (Corollary 1.2.2) there exists an element $y_0 \in S^m$ such that y_0 is a regular value of both f and g. Then $-y_0$ is also a regular value of both f and g and hence

$$f(x) \notin \{y_0, -y_0\} \qquad \text{for all } x \in \iota(S^{m-1}).$$

Define $V := y_0^\perp \subset \mathbb{R}^{m+1}$ and let $P : \mathbb{R}^{m+1} \to V$ be the orthogonal projection. Thus $Py = y - \langle y, y_0 \rangle y_0 \in V$ for all $y \in \mathbb{R}^{m+1}$ (see Figure 1.11). Define

$$\begin{aligned} S &:= \left\{ y \in V \,\middle|\, |y| = 1 \right\}, \\ M &:= \left\{ x = (x_1, \dots, x_{m+1}) \in S^m \,\middle|\, x_{m+1} \geq 0 \right\}. \end{aligned}$$

Then M is a manifold with boundary $\partial M = \iota(S^{m-1})$ (Lemma 1.2.18). Define the map $F : M \to V$ by

$$F(x) := Pf(x) \qquad \text{for } x \in M.$$

Then, for $x \in \partial M = \iota(S^{m-1})$, we have $f(x) \neq \pm y_0$ and hence $F(x) \neq 0$. Thus $0 \notin F(\partial M)$. Second, we observe that 0 is a regular value of F. Namely, if $F(x) = Pf(x) = 0$, then $f(x) = \pm y_0$, hence $df(x) : T_x S^m \to T_{f(x)} S^m = V$ is surjective and so is $dF(x) = df(x)$. Hence, by the induction hypothesis,

$$1 = \deg_2 \left(\frac{F}{|F|} : \partial M \to S \right) \equiv \#F^{-1}(0) \equiv \#f^{-1}(y_0) \equiv \deg_2(f) \pmod 2.$$

Here the second step uses Lemma 1.5.6 and the third step holds because $F(x) = 0$ if and only if $f(x) \in \{y_0, -y_0\}$ if and only if $y_0 \in \{f(x), f(-x)\}$. This proves Theorem 1.5.1 □

1.6 The Brouwer Invariance of Domain Theorem

The purpose of this section is to give a proof of Brouwer's *Invariance of Domain Theorem* [5].

Theorem 1.6.1 (Invariance of Domain Theorem). *Let* $U \subset \mathbb{R}^m$ *be an open set and let* $f : U \to \mathbb{R}^m$ *be a continuous injective map. Then* $f(U)$ *is an open subset of* $\mathbb{R}^m$*.*

It is a consequence of Theorem 1.6.1 that every continuous injective map from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^m$ is a homeomorphism onto its image, i.e. its inverse is necessarily continuous. Another consequence is the following.

Corollary 1.6.2. *Let* $U \subset \mathbb{R}^m$ *and* $V \subset \mathbb{R}^n$ *be nonempty open subsets and suppose that* $\phi : U \to V$ *is a homeomorphism. Then* $m = n$*.*

Proof. If $n < m$ and $\iota : \mathbb{R}^n \to \mathbb{R}^m$ is a linear inclusion, then $\iota \circ \phi : U \to \mathbb{R}^m$ is a continuous injective map whose image is not open, in contradiction to Theorem 1.6.1. Thus $m \le n$. The same argument with ϕ replaced by ϕ^{-1} shows that $m \ge n$ and this proves Corollary 1.6.2. □

Another proof of Corollary 1.6.2 that circumvents the rather technical argument in the proof of Theorem 1.6.1 is outlined in Exercise 1.7.8.

We remark that there do exist continuous surjective maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ for $m < n$ by the Peano curve construction.

The proof of Theorem 1.6.1 given below is taken from a 2011 blog entry by Terence Tao [39], who attributes the argument to a paper by Kulpa [22]. The proof uses the Brouwer Fixed Point Theorem, the Theorem of Sard, the Tietze Extension Theorem, and the Weierstraß Approximation Theorem.

Proof of Theorem 1.6.1. Let $f : \mathbb{D}^m \to \mathbb{R}^m$ be a continuous injective map. We must prove that $f(0) \in \mathrm{int}(f(\mathbb{D}^m))$. Assume, by contradiction, that

$$f(0) \notin \mathrm{int}(f(\mathbb{D}^m)). \tag{1.6.1}$$

Since f is a continuous injection from a compact space to a Hausdorff space, its inverse $f^{-1} : f(\mathbb{D}^m) \to \mathbb{D}^m$ is continuous. The overall strategy of the proof will be to construct a continuous map $h : f(\mathbb{D}^m) \to \mathbb{R}^m$ that satisfies

$$h(y) \ne 0, \qquad \left|h(y) - f^{-1}(y)\right| \le 1 \qquad \text{for all } y \in f(\mathbb{D}^m). \tag{1.6.2}$$

Once h has been found, the formula $\phi(x) := x - h(f(x))$ for $x \in \mathbb{D}^m$ defines a continuous map $\phi : \mathbb{D}^m \to \mathbb{D}^m$ without fixed points, in contradiction to the Brouwer Fixed Point Theorem. 1.2.24.

1. To begin the construction of h, we first recall that $f(\mathbb{D}^m)$ is a closed subset of $\mathbb{R}^m$ and that the map $f^{-1} : f(\mathbb{D}^m) \to \mathbb{R}^m$ is continuous. Hence, by the Tietze Extension Theorem, the map f^{-1} extends to a continuous map on all of $\mathbb{R}^m$. Thus there exists a continuous map $g : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$g(f(x)) = x \qquad \text{for all } x \in \mathbb{D}^m. \tag{1.6.3}$$

Choose $\varepsilon > 0$ such that every $y \in \mathbb{R}^m$ satisfies

$$|y - f(0)| \le 2\varepsilon \qquad \Longrightarrow \qquad |g(y)| \le \frac{1}{3}. \tag{1.6.4}$$

By (1.6.1) there exists a vector $c \in \mathbb{R}^m$ such that

$$|c - f(0)| < \varepsilon, \qquad c \notin f(\mathbb{D}^m). \tag{1.6.5}$$

2. Define the set $\Sigma \subset \mathbb{R}^m$ by

$$\begin{aligned} \Sigma &:= \Sigma_1 \cup \Sigma_2, \\ \Sigma_1 &:= \left\{ y \in f(\mathbb{D}^m) \,\big|\, |y - c| \ge \varepsilon \right\}, \\ \Sigma_2 &:= \left\{ y \in \mathbb{R}^m \,\big|\, |y - c| = \varepsilon \right\}. \end{aligned} \tag{1.6.6}$$

Then Σ is compact and the map $\Phi : f(\mathbb{D}^m) \to \Sigma$, defined by

$$\Phi(y) := \begin{cases} y, & \text{if } |y - c| \ge \varepsilon, \\ c + \varepsilon |y - c|^{-1}(y - c), & \text{if } |y - c| \le \varepsilon, \end{cases} \tag{1.6.7}$$

is continuous.

3. It follows from (1.6.5) and (1.6.6) that $f(0) \notin \Sigma_1$, and so by (1.6.3) we have $g(y) \neq 0$ for every $y \in \Sigma_1$. Hence there exists a $\delta > 0$ such that

$$2\delta \le \frac{1}{3}, \qquad |g(y)| \ge 2\delta \quad \text{for all } y \in \Sigma_1. \tag{1.6.8}$$

Now the Weierstraß Approximation Theorem (see e.g. [7, Theorem 5.4.5]) asserts that there exists a polynomial map $p : \mathbb{R}^m \to \mathbb{R}^m$ that satisfies

$$|p(y) - g(y)| < \delta \qquad \text{for all } y \in \Sigma. \tag{1.6.9}$$

By (1.6.8) and (1.6.9) the polynomial map p does not have any zeros in Σ_1, however it may vanish somewhere on Σ_2.

4. Since Σ_2 is a submanifold of $\mathbb{R}^m$, it follows from Sard's Theorem 1.2.1 that there exists a regular value $z \in \mathbb{R}^m$ of $p|_{\Sigma_2}$ such that $|z| < \delta$. Hence $z \notin p(\Sigma_2)$ and hence, by (1.6.8) and (1.6.9), we have

$$|p(y) - z - g(y)| < 2\delta \quad \text{and} \quad p(y) - z \neq 0 \qquad \text{for all } y \in \Sigma. \tag{1.6.10}$$

5. Define the map $h : f(\mathbb{D}^m) \to \mathbb{R}^m$ by

$$h(y) := p(\Phi(y)) - z \qquad \text{for } y \in f(\mathbb{D}^m). \tag{1.6.11}$$

Then it follows from (1.6.10) that

$$h(y) \neq 0 \qquad \text{for all } y \in f(\mathbb{D}^m).$$

6. If $y \in f(\mathbb{D}^m)$ satsfies $|y - c| \geq \varepsilon$, then $\Phi(y) = y$ by (1.6.7), and hence it follows from (1.6.10) and (1.6.11) that

$$|h(y) - g(y)| = |p(y) - z - g(y)| \leq 2\delta \leq \frac{1}{3}.$$

Here the last step uses (1.6.8).

7. Suppose an element $y \in f(\mathbb{D}^m)$ satisfies

$$|y - c| < \varepsilon.$$

Then $|\Phi(y) - c| = \varepsilon$ by (1.6.7). Hence it follows from (1.6.5) that

$$|y - f(0)| \leq 2\varepsilon, \qquad |\Phi(y) - f(0)| \leq 2\varepsilon,$$

and hence, by (1.6.4), we have

$$|g(y)| \leq \frac{1}{3}, \qquad |g(\Phi(y))| \leq \frac{1}{3}.$$

Now use (1.6.10) and (1.6.11) to obtain the estimate

$$\begin{aligned} |h(y) - g(y)| &= |p(\Phi(y)) - z - g(y)| \\ &= |p(\Phi(y)) - z - g(\Phi(y)) + g(\Phi(y)) - g(y)| \\ &\leq |p(\Phi(y)) - z - g(\Phi(y))| + |g(\Phi(y))| + |g(y)| \\ &\leq 2\delta + 2/3 \\ &\leq 1. \end{aligned}$$

Here the last step follows again from (1.6.8).

It follows from Steps 5, 6, and 7 that the map $h : f(\mathbb{D}^m) \to \mathbb{R}^m$ in (1.6.11) satisfies (1.6.2) as claimed, and so gives rise to a map $\phi : \mathbb{D}^m \to \mathbb{D}^m$ that contradicts the Brouwer Fixed Point Theorem. This contradiction shows that our assumption $f(0) \notin \mathrm{int}(f(\mathbb{D}^m))$ must have been wrong. Hence

$$f(0) \in \mathrm{int}(f(\mathbb{D}^m))$$

as claimed. This proves Theorem 1.6.1. □

1.7 Examples and Exercises

Exercise 1.7.1 (Stereographic projection). Prove that the map

$$\phi_y : S^m \setminus \{y\} \to y^\perp, \qquad \phi_y(x) := \frac{x - \langle x, y\rangle y}{1 - \langle x, y\rangle}, \tag{1.7.1}$$

is a diffeomorphism for every $y \in S^m$. Deduce that the complement of a point in S^m is diffeomorphic to $\mathbb{R}^m$. The diffeomorphism ϕ_y in (1.7.1) is called the **stereographic projection from** y (see Figure 1.12).

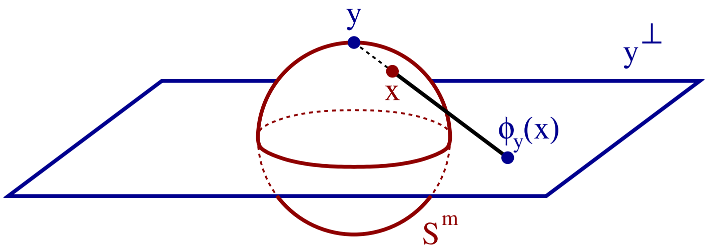

Figure 1.12: The stereographic projection from y.

Exercise 1.7.2. Let M be a smooth manifold and let $f_0, f_1 : M \to S^n$ be two continuous maps such that

$$\sup_{p\in M} |f_0(p) - f_1(p)| < 2.$$

Prove that f_0 is homotopic to f_1 and that the homotopy can be chosen smooth whenever f_0, f_1 are smooth. **Hint:** Use the stereographic projection in Exercise 1.7.1 to construct a homotopy. Verify the explicit formula

$$\begin{aligned} f_t = {} & \frac{2t}{1 + \langle f_0, f_1\rangle + t^2(1 - \langle f_0, f_1\rangle)} \Big(f_1 - \langle f_0, f_1\rangle f_0 \Big) \\ & + \frac{1 + \langle f_0, f_1\rangle - t^2(1 - \langle f_0, f_1\rangle)}{1 + \langle f_0, f_1\rangle + t^2(1 - \langle f_0, f_1\rangle)} f_0 \end{aligned}$$

for $0 \le t \le 1$.

Exercise 1.7.3. For every $p \in M$ and every linear map $\Lambda : T_pM \to \mathbb{R}$ there exists a smooth function $h : M \to \mathbb{R}$ such that $h(p) = 0$ and $dh(p) = \Lambda$.

Exercise 1.7.4. Let M be a compact smooth manifold (possibly with boundary). Prove that every continuous map $f : M \to S^m$ can be uniformly approximated by a smooth map. If two smooth maps $f_0, f_1 : M \to S^m$ are continuously homotopic, prove that they are smoothly homotopic.

Exercise 1.7.5. Prove that the identity map on S^m is not continuously homotopic to a constant map. **Hint:** Theorem 1.4.7 and Exercise 1.7.4.

Exercise 1.7.6. Reprove the Brouwer Fixed Point Theorem 1.2.24. **Hint:** If $g : \mathbb{D}^m \to \mathbb{D}^m$ is a continuous map without fixed points, use the map f in the proof of Lemma 1.2.26 to find a continuous homotopy from the identity map on S^{m-1} to a constant map, in contradiction to Exercise 1.7.5.

Exercise 1.7.7. Let M be a compact smooth m-manifold and let $n > m$. Prove that every continuous map $f : M \to S^n$ is continuously homotopic to a constant map. **Hint:** Approximate f by a smooth map (Exercise 1.7.2). Use the stereographic projection in Exercise 1.7.1 and the fact that a smooth map $f : M \to S^n$ cannot be surjective to construct the required homotopy.

Exercise 1.7.8 (Invariance of Domain). Prove Corollary 1.6.2 directly: *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets containing zero and let $\phi : U \to V$ be a homeomorphism such that $\phi(0) = 0$. Then $m = n$.*

Hint: For $r > 0$ define $B_r^m := \{x \in \mathbb{R}^m \,|\, |x| < r\}$. Choose positive real numbers $\varepsilon > 0$ and $\delta > 0$ such that

$$\overline{B_\varepsilon^n} \subset V, \qquad \overline{B_\delta^m} \subset U, \qquad \phi(\overline{B_\delta^m}) \subset B_\varepsilon^n.$$

Let $\psi := \phi^{-1}$ and define $\phi_\delta : S^{m-1} \to S^{n-1}$ and $\psi_\varepsilon : S^{n-1} \to S^{m-1}$ by

$$\phi_\delta(x) := \frac{\phi(\delta x)}{|\phi(\delta x)|}, \qquad \psi_\varepsilon(y) := \frac{\psi(\varepsilon y)}{|\psi(\varepsilon y)|}.$$

Claim: *$\psi_\varepsilon \circ \phi_\delta$ is continuously homotopic to the identity map of S^{m-1}.*
For $0 \le t \le 1$ define the map $f_t : S^{m-1} \to S^{m-1}$ by

$$f_t(x) := \left|\psi\Big(\frac{\varepsilon\phi(\delta x)}{t|\phi(\delta x)| + (1-t)\varepsilon}\Big)\right|^{-1} \psi\Big(\frac{\varepsilon\phi(\delta x)}{t|\phi(\delta x)| + (1-t)\varepsilon}\Big).$$

Show that $\{f_t\}_{0\le t\le 1}$ is a continuous homotopy from $f_0 = \mathrm{id}$ to $f_1 = \psi_\varepsilon \circ \phi_\delta$.

If $n > m$, then by Exercise 1.7.7 the map $\phi_\delta : S^{m-1} \to S^{n-1}$ is homotopic to a constant map. Hence $\psi_\varepsilon \circ \phi_\delta : S^{m-1} \to S^{m-1}$ is continuously homotopic to a constant map, and hence by the Claim the identity map of S^{m-1} is continuously homotopic to a constant map in contradiction to Exercise 1.7.5. Thus $n \le m$. Deduce that $n = m$ by reversing the roles of ϕ and ψ.

Exercise 1.7.9. Let M be an m-manifold without boundary. Prove that a closed subset $M_0 \subset M$ is an m-dimensional submanifold with boundary if and only if its boundar $\partial M_0 = M_0 \setminus \mathrm{int}(M_0)$ agrees with the boundary of its interior and is an $(m-1)$-dimensional submanifold of M.

Exercise 1.7.10. Let $f : S^1 \to S^1$ be a smooth map. Prove that there exists a smooth map $\theta : \mathbb{R} \to \mathbb{R}$ and an integer $k \in \mathbb{Z}$ such that

$$f(e^{2\pi\mathbf{i}t}) = e^{2\pi\mathbf{i}\theta(t)}, \qquad \theta(t+1) = \theta(t) + k$$

for all $t \in \mathbb{R}$.

Exercise 1.7.11. Let $X \subset \mathbb{R}^m$ be any subset, let $f : X \to \mathbb{R}^d$ be a smooth map (as in Definition 1.2.3), and let $\Omega \subset \mathbb{R}^d$ be an open set. Prove that there exists an open set $U \subset \mathbb{R}^m$ such that $U \cap X = f^{-1}(\Omega)$.

Chapter 2

The Brouwer Degree

This chapter introduces oriented manifolds, defines the Brouwer degree, and proves the Poincaré–Hopf Theorem, following Milnor's book [26, §5 and §6].

2.1 Oriented Manifolds and the Brouwer Degree

In order to define the degree as an integer (rather than an integer modulo 2) we must introduce orientations. An orientation for a finite-dimensional vector space is an equivalence class of ordered bases as follows.

Definition 2.1.1 (**Oriented Vector Space).** *Let V be an m-dimensional real vector space. Assume first that $m > 0$ and denote by $\mathcal{B}(V) \subset V^m$ the set of ordered bases of V. Two ordered bases $v_1, \dots, v_m$ and $v'_1, \dots, v'_m$, related by*

$$v_i = \sum_j a_{ij} v'_j,$$

are said to **determine the same orientation** *iff* $\det(a_{ij}) > 0$ *and to determine* **opposite orientations** *iff* $\det(a_{ij}) < 0$. *This defines an equivalence relation on $\mathcal{B}(V)$ with precisely two equivalence classes. Denote by* $\mathrm{Or}(V)$ *the set of equivalence classes. An* **orientation of** V *is the choice of an equivalence class $\mathfrak{o} \in \mathrm{Or}(V)$, and the pair $(V, \mathfrak{o})$ is then called an* **oriented vector space**. *Thus every positive-dimensional vector space has precisely two orientations. When $m > 0$ the* **standard orientation of** $\mathbb{R}^m$ *is the equivalence class of the standard basis $e_1, \dots, e_m$, where $e_i := (0, \dots, 0, 1, 0, \dots, 0)$ with the number 1 in the ith place.*

When $m = 0$ it is convenient to define $\mathrm{Or}(V) := \{-1, +1\}$, *so an orientation of a zero-dimensional vector space is the choice of a sign $+1$ or -1.*

Given an oriented vector space, it is often unnecessary to introduce a notation for the preferred equivalence class of ordered bases. An ordered basis is then called **positive** or **positively oriented** iff it belongs to the preferred equivalence class, and is called **negative** or **negatively oriented** otherwise.

Definition 2.1.2 (Oriented Manifold). *Let M be a smooth m-manifold. An* **orientation of** M *is a collection of orientations $\mathfrak{o}_p \in \mathrm{Or}(T_pM)$ of the tangent spaces, one for each $p \in M$, satisfying the following condition. For every $p_0 \in M$ there exists a coordinate chart $\phi : U \to \Omega$ on an open neighborhood $U \subset M$ of p_0 with values in an open set $\Omega \subset \mathbb{R}^m$ (or $\Omega \subset \mathbb{H}^m$), such that for every $p \in U$ the derivative*

$$d\phi(p) : T_pM \to \mathbb{R}^m$$

sends the orientation $\mathfrak{o}_p \in \mathrm{Or}(T_pM)$ to the standard orientation of $\mathbb{R}^m$. If an orientation of M is given, such a coordinate chart $\phi : U \to \Omega$ is called **orientation preserving**.

An **oriented manifold** *is a smooth m-manifold M equipped with an orientation $\mathfrak{o} = \{\mathfrak{o}_p \in \mathrm{Or}(T_pM)\}_{p\in M}$. A smooth manifold M is called* **orientable** *iff it admits an orientation.*

Remark 2.1.3. A manifold M is orientable if and only if it admits an atlas $\mathcal{A} = \{U_\alpha, \phi_\alpha\}_{\alpha\in A}$ such that $\det\big((d\phi_\beta \circ \phi_\alpha^{-1})(x)\big) > 0$ for all $\alpha, \beta \in A$ and all $x \in \phi_\alpha(U_\alpha \cap U_\beta)$. If M is oriented, an atlas is called **compatible with the orientation** iff each coordinate chart ϕ_α is orientation preserving.

Remark 2.1.4. If M is a connected orientable manifold, then M admits precisely two orientations.

If M is an m-manifold with boundary, one can distinguish three kinds of vectors in the tangent space T_pM at a boundary point $p \in \partial M$.

- There are the vectors tangent to the boundary forming an $(m-1)$-dimensional subspace $T_p\partial M \subset T_pM$.
- There are the *outward pointing* tangent vectors, forming an open half space bounded by $T_p\partial M$ (see part (v) of Remark 1.2.14).
- There are the *inward pointing* tangent vectors, forming a complementary open half space.

Given an orientation of M, this leads to an orientation of the boundary of M as follows. The rule for orienting the boundary can be summarized in the phrase: "*The outward pointing tangent vector comes first*".

Definition 2.1.5 (Orientation of the Boundary). *Let M be an oriented m-manifold with boundary. Then the boundary of M is* **oriented** *as follows. For $p \in \partial M$ choose a positively oriented basis $v_1, v_2, \dots, v_m$ of T_pM such that $v_2, \dots, v_m$ are tangent to the boundary (assuming $m \geq 2$) and v_1 is an outward pointing tangent vector. Then the ordered basis $v_2, \dots, v_m$ of $T_p\partial M$ determines the required orientation of $T_p\partial M$.*

In the case $m = 1$ a boundary point is assigned the orientation $+1$ iff the outward pointing vectors are positively oriented, and the orientation -1 iff the outward pointing tangent vectors are negatively oriented (see Figure 2.1).

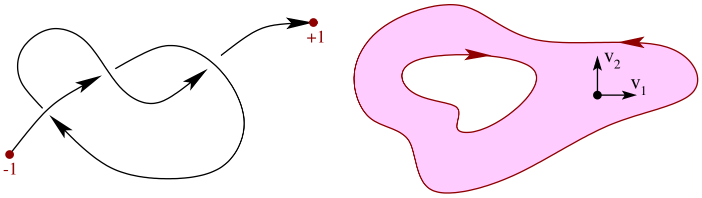

Figure 2.1: How to orient the boundary.

Example 2.1.6. The sphere S^m and the torus $\mathbb{T}^m$ are orientable. The standard orientation of the sphere is as the boundary of the closed unit ball $\mathbb{D}^{m+1|} \subset \mathbb{R}^{m+1}$, and the standard orientation of the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ is induced by the standard orientation of $\mathbb{R}^m$.

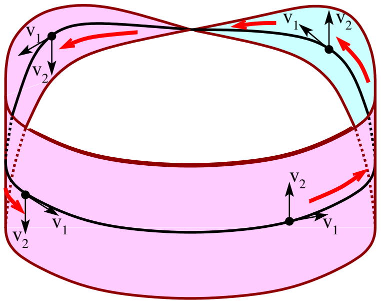

Figure 2.2: The Möbius band.

Example 2.1.7. The simplest non-orientable manifold is the **Möbius band**

$$M := \mathbb{R} \times [-1, 1]/\sim,$$

where $(s, t) \sim (s', t')$ if and only if $(s', t') = (s+k, (-1)^k t)$ for some integer k. (See Figure 2.2 for an embedding of the Möbius band into $\mathbb{R}^3$.)

Example 2.1.8. Other non-orientable examples are the real projective plane $\mathbb{R}\mathrm{P}^2$ (Example 1.1.7) and the **Klein bottle** $K := \mathbb{R}^2/{\sim}$ with the equivalence relation $(s,t) \sim (s+k, (-1)^k t + \ell)$ for $s,t \in \mathbb{R}$ and $k, \ell \in \mathbb{Z}$. The real projective space $\mathbb{R}\mathrm{P}^3$ is orientable.

Let M and N be oriented smooth m-manifolds without boundary such that M is compact and N is connected, and let $f : M \to N$ be a smooth map. Then the degree of f is defined as follows.

Definition 2.1.9 (Brouwer Degree). *If $p \in M$ is a regular point of f, then the derivative $df(p) : T_pM \to T_{f(p)}N$ is a vector space isomorphism and the* **sign of** $df(p)$ *is defined by*

$$\mathrm{sign}\big(df(p)\big) := \begin{cases} +1, & \text{if } df(p) \text{ is orientation preserving,} \\ -1, & \text{if } df(p) \text{ is orientation reversing.} \end{cases} \tag{2.1.1}$$

If $q \in N$ is a regular value of f, then the **degree of** f **at** q *is the integer*

$$\deg(f,q) := \sum_{p \in f^{-1}(q)} \mathrm{sign}\big(df(p)\big) \in \mathbb{Z}. \tag{2.1.2}$$

This number is well defined, because $f^{-1}(q)$ is a finite set. Moreover, the integer $\deg(f,q)$ is a locally constant function of q on the dense open subset $\mathcal{R}_f \subset N$ of regular values of f (Exercise 2.4.5).

Theorem 2.1.10. *Let M and N be oriented m-manifolds without boundary such that M is compact and N is connected.*

(i) *Let $f : M \to N$ be a smooth map. Then the integer $\deg(f,q) \in \mathbb{Z}$ is independent of the choice of the regular value q of f. It is called the* **degree** *of f and is denoted by $\deg(f)$.*

(ii) *The degree has the following properties.*

(Homotopy) *If $f_0, f_1 : M \to N$ are smoothly homotopic, then*

$$\deg(f_0) = \deg(f_1).$$

(Identity) *The identity map on M has degree $\deg(\mathrm{id}_M) = 1$.*

(Composition) *If N is compact, P is a connected oriented m-manifold without boundary, and $f : M \to N$ and $g : N \to P$ are smooth maps, then*

$$\deg(g \circ f) = \deg(f)\deg(g).$$

(Zero) *Let W be a compact oriented $(m+1)$-manifold with boundary and let $F : W \to N$ be a smooth map. Then $\deg(F|_{\partial W}) = 0$, where the degree is defined with respect to the boundary orientation of ∂W.*

The proof is essentially the same as that of Theorem 1.4.7 in §1.4. It is only necessary to carefully keep track of the orientations and signs. We first consider the situation of the (Zero) axiom, where M is the boundary of a compact oriented $(m+1)$-manifold W, the map $f : M \to N$ extends to a smooth map $F : W \to N$, and M is oriented as the boundary of W. This choice of orientation is especially important when $M = \partial W$ is disconnected. The degree can change significantly when the orientation is reversed on one of the connected components of M but not on the others.

Lemma 2.1.11. *Let W be a compact oriented smooth $(m+1)$-manifold with boundary, let $F : W \to N$ be a smooth map, and define $f := F|_{\partial W}$. Orient the boundary $M := \partial W$ as in Definition 2.1.5. Then* $\deg(f, q) = 0$ *for every regular value $q \in N$ of f.*

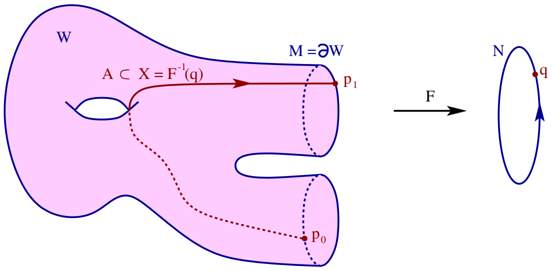

Figure 2.3: An arc in the 1-dimensional submanifold $X = F^{-1}(q) \subset W$.

Proof. This is [26, §5, Lemma 1]. Assume first that q is a regular value of F as well as of $f = F|_{\partial W}$. Then, by Lemma 1.2.20, the set

$$X := F^{-1}(q) \subset W$$

is a 1-dimensional submanifold with boundary $\partial X = F^{-1}(q) \cap \partial W = f^{-1}(q)$. Since X is compact, it follows from Theorem A.4.1 that X is a finite union of circles and arcs, with the boundary points of the arcs all lying on $M = \partial W$. Let $A \subset X = F^{-1}(q)$ be one of these arcs with the boundary

$$\partial A = \{p_0, p_1\} = f^{-1}(q) \cap A \subset M.$$

(see Figure 2.3.) We will prove that

$$\operatorname{sign}\big(df(p_0)\big) + \operatorname{sign}\big(df(p_1)\big) = 0. \tag{2.1.3}$$

Hence, by summing over all such arcs, we obtain $\deg(f, q) = 0$.

To prove (2.1.3) we observe that the orientations of W and N determine an orientation of A as follows. Fix an element $p \in A$ and a nonzero tangent vector $v \in T_pA = \ker dF(p)$. Call the vector v **positive** iff for some (and hence every) positively oriented basis $v_1, \dots, v_{m+1}$ of T_pW with $v_1 = v$ the vectors $dF(p)v_2, \dots, dF(p)v_{m+1}$ form a positively oriented basis of T_qN. Then either v or $-v$ is positive.

Now choose a diffeomorphism $\gamma : [0,1] \to A$ such that $\dot\gamma(t)$ is a positive vector in $T_{\gamma(t)}A$ for some (and hence every) number t such that $0 \le t \le 1$. Then $\partial A = \{\gamma(0), \gamma(1)\}$ and we may assume without loss of generality that

$$\gamma(0) = p_0, \qquad \gamma(1) = p_1.$$

Moreover, the vector $\dot\gamma(1)$ is outward pointing and $\dot\gamma(0)$ is inward pointing. Choose a positiv basis $v_1, \dots, v_{m+1}$ of $T_{p_1}W$ such that $v_1 = \dot\gamma(1)$ and the vectors $v_2, \dots, v_{m+1}$ form a basis of $T_{p_1}\partial W$. Since v_1 is outward pointing, Definition 2.1.5 asserts that the vectors $v_2, \dots, v_{m+1}$ form a positive basis of $T_{p_1}\partial W$. Moreover,, since v_1 is a positive tangent vector in $T_{p_1}A$, it follows that the vectors $dF(p)v_2, \dots, dF(p)v_{m+1}$ form a positive basis of T_qN. Thus the derivative $df(p_1) : T_{p_1}\partial W \to T_qN$ is orientation preserving and so

$$\operatorname{sign}\bigl(df(p_1)\bigr) = +1.$$

The same argument for $t = 0$ with $v_1 = \dot\gamma(0)$ inward pointing shows that the vectors $v_2, \dots, v_{m+1}$ form a negative basis of $T_{p_0}\partial W$ and hence

$$\operatorname{sign}\bigl(df(p_0)\bigr) = -1.$$

This proves (2.1.3). Thus we have proved the equation $\deg(f,q) = 0$ under the assumption that $q \in N$ is a common regular value of F and $f = F|_{\partial W}$.

Now assume that q is a regular vallue of f, but not of F. Then, since the function $\mathcal{R}_f \to \mathbb{Z} : q \mapsto \deg(f,q)$ is locally constant (Exercise 2.4.5), there exists an open neighborhood $U \subset \mathcal{R}_f$ of q such that

$$\deg(f,q) = \deg(f,q')$$

for all $q' \in U$. By Corollary 1.2.2 of Sard's Theorem 1.2.1 there exists a regular value $q' \in U$ of F. Hence

$$\deg(f,q) = \deg(f,q') = 0$$

and this proves Lemma 2.1.11. □

Exercise 2.1.12. Verify the usage of the phrase "and hence every" in the above proof of Lemma 2.1.11

Lemma 2.1.13 (**Homotopy Lemma with Signs).** *If $f_0, f_1 : M \to N$ are smoothly homotopic and $q \in N$ is a common regular value of f_0 and f_1, then* $\deg(f_0, q) = \deg(f_1, q)$.

Proof. This is [26, §5, Lemma 2]. Choose a smooth homotopy

$$F : [0, 1] \times M \to N$$

such that $F(0, p) = f_0(p)$ and $F(1, p) = f_1(p)$ for all $p \in M$. Suppose that the manifold $W := [0, 1] \times M$ is equipped with the product orientation. Then according to Definition 2.1.5 the induced orientation of the boundary

$$\partial W = (\{0\} \times M) \cup (\{1\} \times M)$$

coincides with the orientation of M on $\{1\} \times M$ and with the opposite orientation on $\{0\} \times M$. Thus it follows from Lemma 2.1.11 that

$$0 = \deg(F|_{\partial W}) = \deg(f_1, q) - \deg(f_0, q)$$

for every regular value q of $F|_{\partial W}$, i.e. for every common regular value $q \in N$ of f_0 and f_1. This proves Lemma 2.1.13 □

Proof of Theorem 2.1.10. With these preparations the proof is analogous to that of Theorem 1.4.7. Let $q, q' \in N$ be regular values of f and choose a diffeomorphism $h : N \to N$ that is smoothly isotopic to the identity and satisfies $h(q') = q$ (see Lemma 1.4.5). Then q is a regular value of $h \circ f$, and $h \circ f$ is smoothly homotopic to f. Hence, by Lemma 2.1.13,

$$\deg(f, q) = \deg(h \circ f, q) = \deg(f, q').$$

Here the last equality uses the fact that the vector space isomorphism

$$dh(q') : T_{q'}N \to T_qN$$

is orientation preserving. This proves (i).

In part (ii) the (Homotopy) axiom follows from Lemma 2.1.13, the (Zero) axiom follows from Lemma 2.1.11, and the (Identity) axiom is obvious. To prove the (Composition) axiom, choose a regular value $z \in P$ of $g \circ f$ and choose $q_1, \dots, q_k \in N$ such that $g^{-1}(z) = \{q_1, \dots, q_k\}$. Then each q_i is a regular value of f, and $\operatorname{sign}(d(g \circ f)(p)) = \operatorname{sign}(dg(f(p))) \cdot \operatorname{sign}(df(p))$ for every regular point $p \in M$ of $g \circ f$ by the chain rule. Hence, by part (i),

$$\deg(g \circ f, z) = \sum_{i=1}^{k} \operatorname{sign}\bigl(dg(q_i)\bigr) \left(\sum_{p \in f^{-1}(q_i)} \operatorname{sign}\bigl(df(p)\bigr) \right) = \deg(g) \deg(f).$$

This proves Theorem 2.1.10. □

Example 2.1.14. For $k \in \mathbb{Z}$ the map $f : S^1 \to S^1$ defined by $f(z) := z^k$ for $z \in S^1 \subset \mathbb{C}$ has degree $\deg(f) = k$.

Example 2.1.15. If $m \geq 1$ and $f : M \to N$ is a constant map between compact connected m-manifolds without boundary, then $\deg(f) = 0$.

Example 2.1.16. A diffeomorphism $f : M \to M$ from a compact connected oriented m-manifold M without boundary to itself has degree $+1$ or -1, depending on whether it preserves or reverses the orientation. Thus an orientation reversing diffeomorphism is not homotopic to the identity.

For $i = 1, \dots, m+1$ consider the reflection $r_i : S^m \to S^m$, defined by

$$r_i(x) := (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_{m+1}) \qquad \text{for } x \in S^m.$$

This is an example of an orientation reversing diffeomorphism of S^m and hence $\deg(r_i) = -1$. Now the antipodal map $\tau_{S^m} : S^m \to S^m$, defined by

$$\tau_{S^m}(x) := -x \qquad \text{for } x \in S^m, \tag{2.1.4}$$

can be expressed as the composition $\tau_{S^m} = r_1 \circ r_2 \circ \cdots \circ r_{m+1}$ and hence by Theorem 2.1.10 has the degree $\deg(\tau_{S^m}) = (-1)^{m+1}$. Thus the antipodal map is not homotopic to the identity whenever m is even. This observation cannot be detected by the degree modulo two.

Following Brouwer, one can deduce that the m-sphere admits a nowhere vanishing vector field if and only if m is odd. To make this precise, recall that a vector field on a submanifold $M \subset \mathbb{R}^k$ can be thought of as a smooth map $v : M \to \mathbb{R}^k$ such that $v(x) \in T_xM$ for every $x \in M$ (see [35, §2.4]).

Corollary 2.1.17. *For every positive integer m the following are equivalent.*

(i) *There exists a vector field on S^m without zeros.*

(ii) *The antipodal map on S^m is smoothly homotopic to the identity.*

(iii) *m is odd.*

Proof. Let v be a vector field on S^m without zeros. Thus $v : S^m \to \mathbb{R}^{m+1}$ is a smooth map such that $v(x) \perp x$ and $v(x) \neq 0$ for all $x \in S^m$. For $0 \leq t \leq 1$ define the map $f_t : S^m \to S^m$ by $f_t(x) := \cos(\pi t)x + \sin(\pi t)|v(x)|^{-1}v(x)$. Then the map $[0,1] \times S^m \to S^m : (t,x) \mapsto f_t(x)$ is a smooth homotopy from $f_0 = \mathrm{id}_{S^m}$ to the antipodal map $f_1 = \tau_{S^m}$. Thus (i) implies (ii).

If (ii) holds, then we have $1 = \deg(\mathrm{id}_{S^m}) = \deg(\tau_{S^m}) = (-1)^{m+1}$ by Theorem 2.1.10, and hence m is odd. Thus (ii) implies (iii).

If m is odd, then the formula $v(x) := (x_2, -x_1, x_4, -x_3, \dots, x_{m+1}, -x_m)$ defines a vector field on S^m without zeros. Thus (iii) implies (i).

This proves Corollary 2.1.17. □

Example 2.1.18. By Corollary 2.1.17 every vector field on the 2-sphere must have a zero. Some examples of vector fields on the 2-sphere are depicted in Figure 2.4. For $m = 1$ the vector field $v(x_1, x_2) = (-x_2, x_1)$ on the unit circle $S^1 \subset \mathbb{R}^2$ in the proof of Corollary 2.1.17 is illustrated in Figure 2.5.

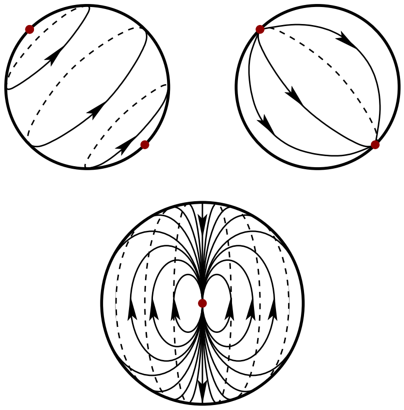

Figure 2.4: Vector fields on the 2-sphere.

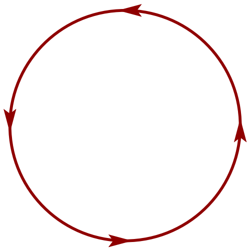

Figure 2.5: A nonvanishing vector field on the circle.

By Corollary 2.1.17 the antipodal map

$$S^m \to S^m : x \mapsto -x$$

is homotopic to the identity if and only if m is odd. We will prove later the much more general assertion that, if M is a compact connected oriented smooth m-manifold without boundary, then two maps $f_0, f_1 : M \to S^m$ are smoothly homotopic if and only if they have the same degree. This is the content of the Hopf Degree Theorem 3.3.1.

2.2 Zeros of a Vector Field

As another application of the Brouwer degree we examine vector fields on general compact manifolds without boundary. The present section introduces the index of an isolated zero of a vector field in preparation for the Poincaré–Hopf Theorem in §2.3, which asserts that the sum of the indices of the zeros of a vector field with only isolated zeros is a topological invariant of the underlying manifold. This invariant is called the **Euler characteristic**.

2.2.1 Vector Fields

We begin with some recollections about vector fields in the extrinsic and intrinsic settings. In the extrinsic setting used in [26] the manifold M is assumed to be an m-dimensional submanifold of the Euclidean space $\mathbb{R}^k$ for some positive integer k. If M has no boundary, then the tangent space of M at $p \in M$ is the m-dimensional linear subspace of $\mathbb{R}^k$ given by

$$T_pM = \left\{ \dot\gamma(0) \,\middle|\, \begin{array}{l} \gamma : \mathbb{R} \to \mathbb{R}^k \text{ is a smooth curve such that} \\ \gamma(\mathbb{R}) \subset M \text{ and } \gamma(0) = p \end{array} \right\}. \tag{2.2.1}$$

(See Lemma 1.2.5 with the inclusion $P \subset M$ replaced by $M \subset \mathbb{R}^k$.) In this situation a **vector field on** M is a smooth map $X : M \to \mathbb{R}^k$ such that $X(p) \in T_pM$ for all $p \in M$. In particular, when $k = m$ and $M = \Omega$ is an open subset of $\mathbb{R}^m$, a vector field on Ω is simply a smooth map $\xi : \Omega \to \mathbb{R}^m$.

Definition 2.2.1 (Vector Field). *Let $(M, \{U_\alpha, \phi_\alpha\}_{\alpha \in A})$ be an m-manifold. A* **vector field** *on M is a collection of tangent vectors $X(p) \in T_pM$, one for each $p \in M$, such that, for every $\alpha \in A$, the map*

$$\xi_\alpha : \Omega_\alpha = \phi_\alpha(U_\alpha) \to \mathbb{R}^m,$$

defined by

$$d\phi_\alpha(p)X(p) = \xi_\alpha(\phi_\alpha(p)) \qquad \text{for } p \in U_\alpha, \tag{2.2.2}$$

is smooth. The space of all vector fields on M will be denoted by $\mathrm{Vect}(M)$.

Let N be another m-manifold, let $\phi : M \to N$ be a diffeomorphism, and let $Y \in \mathrm{Vect}(N)$. The **pullback of Y under** ϕ *is the vector field*

$$X = \phi^*Y \in \mathrm{Vect}(M)$$

defined by

$$(\phi^*Y)(p) := d\phi(p)^{-1}Y(\phi(p)) \in T_pM \tag{2.2.3}$$

for $p \in M$.

Remark 2.2.2. **(i)** Let $X \in \mathrm{Vect}(M)$ be a vector field on M and for $\alpha \in A$ define the vector field $\xi_\alpha : \Omega_\alpha \to \mathbb{R}^m$ by (2.2.2). Then, for all $\alpha, \beta \in A$ and all $x \in \Omega_{\alpha\beta} = \phi_\alpha(U_\alpha \cap U_\beta)$ we have

$$d\phi_{\beta\alpha}(x)\xi_\alpha(x) = \xi_\beta(\phi_{\beta\alpha}(x)), \tag{2.2.4}$$

where

$$\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1} : \Omega_{\alpha\beta} \to \Omega_{\beta\alpha}$$

denotes the transition map. Conversely, every collection of smooth vector fields $\{\xi_\alpha : \Omega_\alpha \to \mathbb{R}^m\}_{\alpha\in A}$ satisfying (2.2.4) determines a unique smooth vector field $X \in \mathrm{Vect}(M)$ via (2.2.2).

(ii) In terms of the pullback construction equation (2.2.2) takes the form

$$X|_{U_\alpha} = \phi_\alpha^*\xi_\alpha$$

and equation (2.2.4) can be expressed as

$$\xi_\alpha|_{\Omega_{\alpha\beta}} = \phi_{\beta\alpha}^*(\xi_\beta|_{\Omega_{\beta\alpha}}).$$

(iii) The tangent bundle

$$TM := \{(p, v) \,|\, p \in M \,|\, v \in T_pM\}$$

is a smooth $2m$-manifold with the coordinate charts

$$TU_\alpha \to \Omega_\alpha \times \mathbb{R}^m : (p, v) \mapsto (\phi_\alpha(p), d\phi_\alpha(p)v).$$

Thus the condition (2.2.2) can be expressed in the form that the map

$$M \to TM : p \mapsto (p, X(p))$$

is smooth. This map is a section of the tangent bundle. It is often also denoted by $X : M \to TM$ and the composition of this section with the canonical projection $\pi : TM \to M$ is the identity

$$\pi \circ X = \mathrm{id}_M.$$

Note the slight abuse of notation in that the same letter X denotes on the one hand the collection of tangent vectors $\{X(p) \in T_pM\}_{p\in M}$ and on the other hand the resulting map from M to TM.

(iv) While Definition 2.2.1 allows for manifolds with boundary, we will in the present section often deal with manifolds without boundary, so the Ω_α in Definition 2.2.1 may be open subsets of $\mathbb{H}^m$ or of $\mathbb{R}^m$. When extending the definition of the index of an isolated zero of a vector field to manifolds with boundary, one must assume that the zero does not lie on the boundary.

2.2.2 Isolated Zeros

Let M be a smooth manifold without boundary and let $X \in \mathrm{Vect}(M)$.

Definition 2.2.3 (**Isolated Zero**). *A point $p_0 \in M$ is called an* **isolated zero of** X *iff $X(p_0) = 0$ and there is an open set $U \subset M$ such that $p_0 \in U$ and $X(p) \neq 0$. for all $p \in U \setminus \{p_0\}$.*

The goal is to assign an index $\iota(p_0, X) \in \mathbb{Z}$ to each isolated zero of X. As a first step we consider vector fields on open subsets of $\mathbb{R}^m$.

Definition 2.2.4 (**Index**). *Let $\Omega \subset \mathbb{R}^m$ be an open set, let $\xi : \Omega \to \mathbb{R}^m$ be a smooth vector field, and let $x_0 \in \Omega$ be an isolated zero of ξ. Choose $\varepsilon > 0$ such that, for all $x \in \mathbb{R}^m$,*

$$0 < |x - x_0| \leq \varepsilon \qquad \Longrightarrow \qquad x \in \Omega \text{ and } \xi(x) \neq 0. \tag{2.2.5}$$

Then the integer

$$\iota(x_0, \xi) := \deg\left(S^{m-1} \to S^{m-1} : x \mapsto \frac{\xi(x_0 + \varepsilon x)}{|\xi(x_0 + \varepsilon x)|}\right) \tag{2.2.6}$$

is independent of the choice of ε and is called the **index of** ξ **at** x_0.

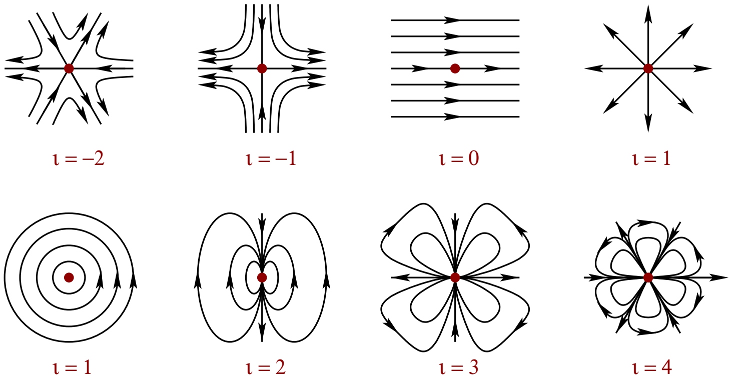

Figure 2.6: Examples of vector fields in dimension two.

Some examples of isolated zeros of vector fields in dimension two are illustrated in Figure 2.6. For $k \in \mathbb{N}$ the vector field $\xi(z) = z^k$ on the complex plane has an isolated zero of index k at the origin and the vector field $\xi(z) = \bar{z}^k$ has an isolated zero of index $-k$ at the origin; the vector field $\xi(z) = |z|^2$ has an isolated zero of index zero at the origin.

The first key observation is that the index of an isolated zero is preserved under a change of coordinates.

Lemma 2.2.5. *Let* $\Omega \subset \mathbb{R}^m$ *and* $\Omega' \subset \mathbb{R}^m$ *be open subsets, let* $f : \Omega \to \Omega'$ *be a diffeomorphism, and let* $\xi : \Omega \to \mathbb{R}^m$ *and* $\xi' : \Omega' \to \mathbb{R}^m$ *be smooth vector fields such that* $\xi = f^*\xi'$*, i.e.*

$$\xi(x) = df(x)^{-1}\xi'(f(x)) \qquad \textit{for all } x \in \Omega. \tag{2.2.7}$$

Suppose that $x_0 \in \Omega$ *is an isolated zero of* ξ*. Then* $f(x_0) \in \Omega'$ *is an isolated zero of* ξ' *and* $\iota(x_0, \xi) = \iota(f(x_0), \xi')$*.*

Assuming Lemma 2.2.5 we can define the index of a vector field at an isolated zero on any manifold as follows.

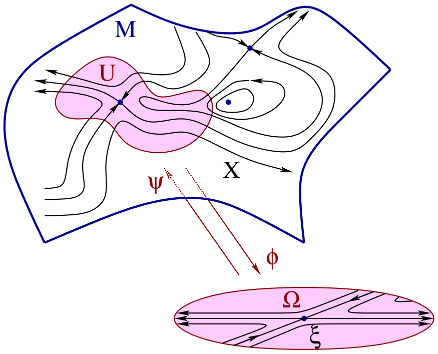

Figure 2.7: Defining the index of an isolated zero of a vector field.

Definition 2.2.6 (Index of an isolated Zero). *Let* M *be a smooth* m*-manifold without boundary and let* $X \in \mathrm{Vect}(M)$ *be a vector field on* M *with an isolated zero at* $p_0 \in M$*. Choose a parametrization* $\psi : \Omega \to U$ *of an open neighborhood* $U \subset M$ *of* p_0*, defined on an open set* $\Omega \subset \mathbb{R}^m$ *(see Figure 2.7), and define the vector field* $\xi : \Omega \to \mathbb{R}^m$ *as the pullback of* $X|_U$ *under* ψ*, i.e.*

$$\xi(x) := (\psi^*X)(x) = d\psi(x)^{-1}X(\psi(x)) \qquad \textit{for } x \in \Omega. \tag{2.2.8}$$

Then $\xi = \psi^*X$ *has an isolated zero at the point* $x_0 := \psi^{-1}(p_0) \in \Omega$ *and the* **index of the vector field** X **at** p_0 *is defined as the integer*

$$\iota(p_0, X) := \iota(\psi^{-1}(p_0), \psi^*X) \in \mathbb{Z}. \tag{2.2.9}$$

By Lemma 2.2.5 the integer $\iota(p_0, X)$ *is independent of the choice of the parametrization* ψ *used to define it.*

The proof of Lemma 2.2.5 will be based on the following lemma.

Lemma 2.2.7. *Every orientation preserving diffeomorphism* $f : \mathbb{R}^m \to \mathbb{R}^m$ *is smoothly isotopic to the identity.*

In contrast, for many values of m there exist orientation preserving diffeomorphisms of S^m that are not smoothly isotopy to the identity. For $m = 6$ this is closely related to Milnor's exotic 7-spheres [24, 19].

Proof of Lemma 2.2.7. This is [26, §6, Lemma 2]. Assume $f(0) = 0$. As in Lemma 1.5.5 define the maps $h_i : \mathbb{R}^m \to \mathbb{R}^m$ by $h_i(x) := \int_0^1 \partial_i f(sx)\,ds$ for $i = 1, \dots, m$ and define the map $F : [0,1] \times \mathbb{R}^m \to \mathbb{R}^m$ by

$$F(t,x) := f_t(x) := \sum_{i=1}^m x_i h_i(tx) = \left\{ \begin{array}{rl} t^{-1}f(tx), & \text{if } t > 0, \\ df(0)x, & \text{if } t = 0 \end{array} \right. \tag{2.2.10}$$

for $0 \le t \le 1$ and $x \in \mathbb{R}^m$. Then F is smooth and f_t is an orientation preserving diffeomorphism of $\mathbb{R}^m$ for every t. Thus F is a smooth isotopy from the linear map $f_0(x) = \Phi x := df(0)x$ to $f_1 = f$. The invertible linear map Φ is isotopic to the identity because $\det(\Phi) > 0$ and $\mathrm{GL}^+(m,\mathbb{R})$ is connected (Exercise 2.4.11). This proves Lemma 2.2.7. □

Remark 2.2.8. The proof of Lemma 2.2.7 shows more. If $f : \Omega \to \mathbb{R}^m$ is an embedding of a convex open neighborhood $\Omega \subset \mathbb{R}^m$ of the origin, then (2.2.10) defines a smooth isotopy of embeddings

$$f_t : \Omega \to \mathbb{R}^m, \qquad \Omega_t := f_t(\Omega),$$

from $f_0 = df(0) =: \Phi$ to $f_1 = f$ that satisfy the following.

Claim. *For every open neighborhood* $U \subset \Omega$ *of the origin there exists a constant* $\varepsilon > 0$ *such that, for all* $(t,y) \in [0,1] \times \mathbb{R}^m$,

$$|y| \le \varepsilon \qquad \Longrightarrow \qquad y \in f_t(U) \subset \Omega_t. \tag{2.2.11}$$

This is a direct consequence of the Lipshitz continuity of $f^{-1} : f(\Omega) \to \Omega$. Indeed, there exist constants $r > 0$ and $c \ge 1$ such that, for all $x, y \in \mathbb{R}^m$,

$$|y| \le r \quad \Longrightarrow \quad y \in f(\Omega) \text{ and } |f^{-1}(y)| \le c|y|, \tag{2.2.12}$$

$$|x| \le r \quad \Longrightarrow \quad x \in U. \tag{2.2.13}$$

Then (2.2.11) holds with $\varepsilon = r/c$. Namely, let $y \in \mathbb{R}^m$ such that $|y| \le \varepsilon$. Then for $0 < t \le 1$ we have $|ty| \le \varepsilon \le r$, hence it follows from (2.2.12) that

$$ty \in f(\Omega), \qquad |t^{-1}f^{-1}(ty)| \le c|y| \le r,$$

and hence by (2.2.13) we have $x_t := t^{-1}f^{-1}(ty) \in U$. Now take the limit to obtain $|\Phi^{-1}y| = \lim_{t\to 0}|x_t| \le r$, and hence $x_0 := \Phi^{-1}y \in U$ by (2.2.13). Thus $y = f_t(x_t) \in f_t(U)$ for $0 \le t \le 1$. This proves (2.2.11).

Proof of Lemma 2.2.5. This is [26, §6, Lemma 1]. Assume w.l.o.g. that Ω is convex, $x_0 = 0 \in \Omega$, and $f(0) = 0 \in \Omega'$. Then (2.2.10) defines a smooth homotopy of embeddings $f_t : \Omega \to \mathbb{R}^m$ such that $f_0 = \Phi$ and $f_1 = f$. Define

$$\Omega_t := f_t(\Omega), \qquad \xi_t(y) := df_t\big(f_t^{-1}(y)\big)\xi\big(f_t^{-1}(y)\big) \quad \text{for } y \in \Omega_t. \tag{2.2.14}$$

Then $0 \in \Omega_t$ and $\xi_t(0) = 0$ for all t. Moreover, by the Claim in Remark 2.2.8 (with $U \subset \Omega$ a neighborhood of the origin that contains no other zeros of ξ) there exists a constant $\varepsilon > 0$ such that, for all $t \in [0,1]$ and all $y \in \mathbb{R}^m$,

$$0 < |y| \le \varepsilon \qquad \Longrightarrow \qquad y \in \Omega_t \text{ and } \xi_t(y) \ne 0. \tag{2.2.15}$$

By (2.2.15) there is a smooth homotopy $g_t : S^{m-1} \to S^{m-1}$, defined by

$$g_t(x) := \frac{\xi_t(\varepsilon x)}{|\xi_t(\varepsilon x)|} \qquad \text{for } 0 \le t \le 1 \text{ and } x \in S^{m-1}.$$

Since $f_1 = f$, it follows from (2.2.7) and (2.2.14) that $\xi_1 = \xi'$ and hence

$$\deg(g_1) = \iota(0, \xi'). \tag{2.2.16}$$

Since $f_0 = \Phi$, we have $g_0(x) = |\Phi\xi(\varepsilon\Phi^{-1}x)|^{-1}\Phi\xi(\varepsilon\Phi^{-1}x)$. Now use the polar decomposition $\Phi = P\Phi_0$, where $P := (\Phi\Phi^T)^{1/2}$ is a positive definite symmetric matrix and $\Phi_0 := P^{-1}\Phi \in \mathrm{O}(m)$, and define

$$\Phi_\lambda := \big(\lambda P + (1-\lambda)\mathbb{1}\big)\Phi_0 \in \mathrm{GL}(m, \mathbb{R}), \qquad 0 \le \lambda \le 1.$$

Choose $\varepsilon > 0$ so small that, for all $\lambda \in [0,1]$ and all $y \in \mathbb{R}^m$, we have

$$0 < |y| \le \varepsilon \qquad \Longrightarrow \qquad \Phi_\lambda^{-1}y \in \Omega \text{ and } \xi(\Phi_\lambda^{-1}y) \ne 0,$$

and define the homotopy $h_\lambda : S^{m-1} \to S^{m-1}$ by

$$h_\lambda(x) := \frac{\Phi_\lambda\xi(\varepsilon\Phi_\lambda^{-1}x)}{\big|\Phi_\lambda\xi(\varepsilon\Phi_\lambda^{-1}x)\big|}$$

for $0 \le \lambda \le 1$ and $x \in S^{m-1}$. Since $\Phi_1 = \Phi$, we have $h_1 = g_0$. Since Φ_0 is an orthogonal matrix, it restricts to a diffeomorphism from S^{m-1} to itself, still denoted by Φ_0, and so the map h_0 can be expressed as the composition

$$h_0 = \Phi_0 \circ h \circ \Phi_0^{-1}, \qquad h(x) := \frac{\xi(\varepsilon x)}{|\xi(\varepsilon x)|} \qquad \text{for } x \in S^{m-1}.$$

Since Φ_0 is a diffeomorphism, it satisfies $\deg(\Phi_0) = \deg(\Phi_0^{-1}) \in \{-1, +1\}$ (see Example 2.1.16). Hence, by (2.2.16) and Theorem 2.1.10,

$$\iota(0, \xi') = \deg(g_1) = \deg(g_0) = \deg(h_1) = \deg(h_0) = \deg(h) = \iota(0, \xi).$$

This proves Lemma 2.2.5. $\square$

2.2.3 Nondegenerate Zeros

The next goal to examine the special case of nondegenerate zeros of a vector field. Such zeros are isolated and have indices plus or minus one, given by an especially simple formula. To introduce the concept, we must first define the *vertical derivative* of a vector field at a zero.

Definition 2.2.9 (Vertical Derivative). *Let* $X \in \mathrm{Vect}(M)$ *be a vector field on a smooth* m*-manifold* M *without boundary and let* $p \in M$ *such that*

$$X(p) = 0.$$

Then there exists a unique linear map

$$DX(p) : T_pM \to T_pM, \tag{2.2.17}$$

such that every coordinate chart $\phi : U \to \Omega$ *on an open neighborhood* $U \subset M$ *of* p *with values in an open set* $\Omega \subset \mathbb{R}^m$ *satisfies the equation*

$$d\phi(p) \circ DX(p) = d\xi(\phi(p)) \circ d\phi(p) : T_pM \to \mathbb{R}^m. \tag{2.2.18}$$

Here the vector field $\xi : \Omega \to \mathbb{R}^m$ *is defined by* $\xi(x) = d\phi(\phi^{-1}(x))X(\phi^{-1}(x))$*. The linear map* (2.2.17) *is called the* **vertical derivative of** X **at** p

Remark 2.2.10. Suppose $\phi' : U \to \Omega'$ is another coordinate chart (on the same neighborhood of p), let $\xi' : \Omega' \to \mathbb{R}^m$ be the induced vector field, let

$$\psi := \phi' \circ \phi^{-1} : \Omega \to \Omega'$$

be the transition map, and define $\bar{x} := \phi(p) \in \Omega$ and $\bar{x}' := \phi'(p) \in \Omega'$, Then

$$\xi = \psi^*\xi', \qquad \psi(\bar{x}) = \bar{x}', \qquad \xi(\bar{x}) = 0, \qquad \xi'(\bar{x}') = 0.$$

Differentiate the equation $\xi'(\psi(x)) = d\psi(x)\xi(x)$ for $x \in \Omega$ at $x = \bar{x}$ and use the chain rule to obtain $d\xi'(\bar{x}') \circ d\psi(\bar{x}) = d\psi(\bar{x}) \circ d\xi(\bar{x})$. Hence by (2.2.18) there is a commutativ diagram

$$\begin{tikzcd}
T_pM \arrow[r, "DX(p)"] \arrow[d, "d\phi(p)"'] \arrow[dd, bend right=60, "d\phi'(p)"'] & T_rM \arrow[d, "d\phi(p)"] \arrow[dd, bend left=60, "d\phi'(p)"] & . \\
\mathbb{R}^m \arrow[r, "d\xi(\phi(p))"] \arrow[d, "d\psi(\phi(p))"'] & \mathbb{R}^m \arrow[d, "d\psi(\phi(p))"] & \\
\mathbb{R}^m \arrow[r, "d\xi'(\phi'(p))"] & \mathbb{R}^m &
\end{tikzcd}$$

This shows that the linear map (2.2.17) satisfies (2.2.18) for some coordinate chart on a neighborhood of p if and only if it satisfies (2.2.18) for every such coordinate chart. Thus the vertcal derivative at a zero is well defined.

The next lemnma examines the vertical derivative in the extrinsic setting. Thus we assume that $M \subset \mathbb{R}^n$ is a smooth m-dimensional submanifold without boundary and that $X \in \mathrm{Vect}(M)$ is a smooth vector field on M. This means that

$$X : M \to \mathbb{R}^n$$

is a smooth map such that $X(p) \in T_pM$ for all $p \in M$. Hence the derivative of X at each point $p \in M$ is a linear map

$$dX(p) : T_pM \to \mathbb{R}^n$$

which in general does not take values in the tangent space T_pM. However, if p is a zero of X, then the derivative $dX(p)$ does take values in the tangent space and agrees with the vertical derivative of X at p.

Lemma 2.2.11 (Vertical Derivative). *Let $\bar{p} \in M$ such that*

$$X(\bar{p}) = 0.$$

Then $\mathrm{im}\, dX(\bar{p}) \subset T_{\bar{p}}M$ *and the derivative $dX(\bar{p}) : T_{\bar{p}}M \to T_{\bar{p}}M$ agrees with the vertical derivative $DX(\bar{p})$ of X at $\bar{p}$.*

Proof. This is [26, §6, Lemma 5]. Choose an open neighborhood $U \subset \mathbb{R}^n$ of p such that $U \cap M$ is diffeomorphic to an open set $\Omega \subset \mathbb{R}^m$. and choose a diffeomorphism $\phi : U \cap M \to \Omega$. Denote its inverse by

$$\psi := \phi^{-1} : \Omega \to U \cap M$$

and let $\xi := \psi^* X : \Omega \to \mathbb{R}^m$ be the induced vector field on Ω, so that

$$\xi(x) = d\psi(x)^{-1} X\big(\psi(x)\big) \qquad \text{for } x \in \Omega.$$

Define $\bar{x} := \phi(\bar{p}) \in \Omega$. Then $\xi(\bar{x}) = 0$ and hence, by differentiating the equation $X(\psi(x)) = d\psi(x)\xi(x)$ at $x = \bar{x}$, we obtain

$$dX\big(\psi(\bar{x})\big) \circ d\psi(\bar{x}) = d\psi(\bar{x}) \circ d\xi(\bar{x}) : \mathbb{R}^m \to \mathbb{R}^n. \tag{2.2.19}$$

Since $\psi(\bar{x}) = \bar{p}$ and $T_{\bar{p}}M = \mathrm{im}\, d\psi(\bar{x})$, this implies that the image of the linear map $dX(\bar{p}) : T_{\bar{p}}M \to \mathbb{R}^n$ is contained in the tangent space $T_{\bar{p}}M$. Since $d\phi(\bar{p}) = d\psi(\bar{x})^{-1} : T_{\bar{p}}M \to \mathbb{R}^m$, it also follows from (2.2.19) that

$$d\phi(\bar{p}) \circ dX(\bar{p}) = d\xi\big(\phi(\bar{p})\big) \circ d\phi(\bar{p}) : T_{\bar{p}}M \to \mathbb{R}^m.$$

Hence $dX(\bar{p}) = DX(\bar{p})$ and this proves Lemma 2.2.11. □

Definition 2.2.12 (Nondegenerate Zero). *Let M be a smooth m-manifold and let $X \in \mathrm{Vect}(M)$. A zero $p \in M$ of X is called* **nondegenerate** *iff the vertical derivative $DX(p) : T_pM \to T_pM$ is a vector space isomorphism.*

Lemma 2.2.13. *Let $X \in \mathrm{Vect}(M)$ and let $p \in M$ be a nondgenerate zero of X. Then p is an isolated zero of X and*

$$\begin{aligned} \iota(p,X) &= \mathrm{sign}\big(\det(DX(p))\big) \\ &= \left\{ \begin{array}{rl} +1, & \textit{if } DX(p) \textit{ is orientation preserving,} \\ -1, & \textit{if } DX(p) \textit{ is orientation reversing.} \end{array} \right. \end{aligned} \tag{2.2.20}$$

Proof. This is [26, §6, Lemma 4]. It suffices to prove the assertion for a vector field $\xi : \Omega \to \mathbb{R}^m$ on an open set $\Omega \subset \mathbb{R}^m$ with a nongenerate zero at $\bar{x}$. Assume without loss of generality that Ω is convex and $\bar{x} = 0$. Then

$$\xi(0) = 0, \qquad A := d\xi(0) \in \mathrm{GL}(m, \mathbb{R}).$$

Hence, by the Inverse Function Theorem 1.1.1, there exists an $\varepsilon > 0$ such that $\overline{B}_\varepsilon := \{x \in \mathbb{R}^m \,|\, |x| \le \varepsilon\} \subset \Omega$ and $\xi(x) \ne 0$ for all $x \in \overline{B}_\varepsilon \setminus \{0\}$. Repeating again the argument in the proof of Lemma 1.5.5 we define

$$\xi_t(x) := \left\{ \begin{array}{rl} t^{-1}\xi(tx), & \text{if } t > 0, \\ Ax, & \text{if } t = 0, \end{array} \right. \qquad \text{for } 0 \le t \le 1 \text{ and } x \in \Omega.$$

Then the map $[0,1] \times \Omega \to \mathbb{R}^m : (t,x) \mapsto \xi_t(x)$ is smooth and and hence determines a smooth homotopy $g_t : S^{m-1} \to S^{m-1}$ via

$$g_t(x) := \frac{\xi_t(\varepsilon x)}{|\xi_t(\varepsilon x)|} \qquad \text{for } 0 \le t \le 1 \text{ and } x \in S^{m-1}.$$

Since $\xi_1 = \xi$, it follows from Definition 2.2.4 that

$$\iota(0,\xi) = \deg(g_1) = \deg(g_0) = \deg(\phi_A) = \mathrm{sign}(\det(A)). \tag{2.2.21}$$

Here the second equality follows from Theorem 2.1.10 and the third equality holds because

$$g_0(x) = \frac{Ax}{|Ax|} = \phi_A(x)$$

is the diffeomorphism in Example 1.5.4. By Exercise 2.4.11, ϕ_A can be joined to the identity by a smooth isotopy whenever $\det(A) > 0$, and hence can be joined to the reflection $S^{m-1} \to S^{m-1} : (x_1, \dots, x_m) \mapsto (-x_1, x_2, \dots, x_m)$ by a smooth isotopy whenever $\det(A) < 0$. This proves the last equality in (2.2.21) and Lemma 2.2.13. □

2.3 The Poincaré–Hopf Theorem

The purpose of this section is to prove the following classical result.

Theorem 2.3.1 (Poinaré–Hopf). *Let M be a compact smooth m-manifold with boundary and let $X \in \mathrm{Vect}(M)$ be a smooth vector field on M that points out on the boundary. Assume that X has only isolated zeros. Then*

$$\sum_{p\in M,\, X(p)=0} \iota(p,X) = \sum_{k=0}^{m} (-1)^k \dim(H^k(M)), \tag{2.3.1}$$

where $H^*(M)$ denotes the de Rham cohomology of M. In particular, the left hand side is independent of the choice of the vector field X. It is called the **Euler characteristic of** *M and is denoted by*

$$\chi(M) := \sum_{p\in M,\, X(p)=0} \iota(p,X). \tag{2.3.2}$$

Theorem 2.3.1 was proved in 1885 by Poincaré in the case $\dim(M) = 2$. After partial results by Brouwer and Hadamard, the theorem was established in full generality in 1926 by Heinz Hopf.

In this section we will only prove that the sum of the indices of the zeros of a vector field with with only isolated zeros that points out on the boundary is independent of the choice of the vector field. The formula (2.3.1) for the de Rham cohomology groups will be established in Theorem 6.4.9.

Definition 2.3.2 (Gauß Map). *Let $N \subset \mathbb{R}^n$ be a compact smooth n-dimensional submanifold with boundary, i.e. N is compact and its boundary agrees with the boundary of its interior and is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. The* **Gauß map of** *N is the map $g : \partial N \to S^{n-1}$ which assigns to each boundary point $x \in \partial N$ the unit vector $g(x) \in S^{n-1}$ that is orthogonal to $T_x\partial N$ and points out of N. The Gauß map is smooth.*

The Hopf Lemma is a special case of Theorem 2.3.1.

Lemma 2.3.3 (Hopf). *Let $N \subset \mathbb{R}^n$ be as in Definition 2.3.2 and suppose that $Y : N \to \mathbb{R}^n$ is a vector field with only isolated zeros such that $Y(x) \neq 0$ for all $x \in \partial N$. Then*

$$\sum_{x\in N,\, Y(x)=0} \iota(x,Y) = \deg\left(\frac{Y}{|Y|} : \partial N \to S^{n-1}\right). \tag{2.3.3}$$

If Y points out on the boundary, then $\deg(|Y|^{-1}Y : \partial N \to S^{n-1}) = \deg(g)$, where $g : \partial N \to S^{n-1}$ is the Gauß map.

Note the similarity to Lemma 1.5.6 with the map $f : M \to \mathbb{R}^m$ replaced by the vector field $Y : N \to \mathbb{R}^n$. One difference is that the manifold N is now assumed to be embedded in $\mathbb{R}^n$ so that the map is a vector field. Second, the regular value in Lemma 1.5.6 is replaced by zero so that its preimages are now the zeros of the vector field Y. A third difference is that 0 is not assumed to be a regular value of Y so that the zeros need not be nondegenerate but are only isolated, and this extension is only possible because the zeros are counted with their indices.

Proof of Lemma 2.3.3. This is [26, §6, Lemma 3]. We proceed precisely as in the proof of Lemma 1.5.6. Let $x_1, \dots, x_k \in N$ be the zeros of Y and choose a number $\varepsilon > 0$ such that $|x_i - x_j| > 2\varepsilon$ for $i \neq j$ and $|x_i - x| > \varepsilon$ for all i and all $x \in \partial N$. For $i = 1, \dots, k$ define

$$U_i := \left\{ x \in \mathbb{R}^n \,\middle|\, |x - x_i| < \varepsilon \right\}.$$

Then the closures of the open balls U_i are pairwise disjoint and are all contained in $N \setminus \partial N$. (See Figure 1.10.) Hence the set

$$W := N \setminus \big(U_1 \cup \dots \cup U_k\big)$$

is an n-manifold with boundary $\partial W = \partial N \cup \partial U_1 \cup \dots \cup \partial U_k$ and the vector field Y has no zeros in W. Define the map $F : W \to S^{n-1}$ by

$$F(x) := \frac{Y(x)}{|Y(x)|} \qquad \text{for } x \in W.$$

Then it follows from Theorem 2.1.10 that

$$\begin{aligned} 0 = \deg(F|_{\partial W}) &= \deg(F|_{\partial N}) - \sum_{i=1}^{k} \deg(F|_{\partial U_i}) \\ &= \deg(F|_{\partial N}) - \sum_{i=1}^{k} \iota(y_i, Y). \end{aligned}$$

Here the minus sign in the second equality follows from the fact that the orientation of ∂U_i as a boundary component of W is opposite to the orientation of ∂U_i as the boundary of U_i. The last equality follows from the definition of the index of an isolated zero of a vector field. This proves equation (2.3.3).

Now suppose that Y points out on the boundary of N. Then the formula

$$g_\lambda(x) := \frac{\lambda g(x) + (1-\lambda)F(x)}{|\lambda g(x) + (1-\lambda)F(x)|} \in S^{n-1} \qquad \text{for } y \in \partial N$$

defines a smooth homotopy from $g_0 = F|_{\partial N}$ to the Gauß map $g_1 = g$. Hence $\deg(F|_{\partial N}) = \deg(g)$ and this proves Lemma 2.3.3. □

Example 2.3.4. Let $Y : \mathbb{D}^n \to \mathbb{R}^n$ be a vector field on the closed unit disk in $\mathbb{R}^n$ that points out on the boundary and has only isolated zeros. Then

$$\sum_x \iota(x, Y) = 1,$$

where the sum runs over all $x \in \mathbb{D}^n$ such that $Y(x) = 0$ (see Figure 2.8).

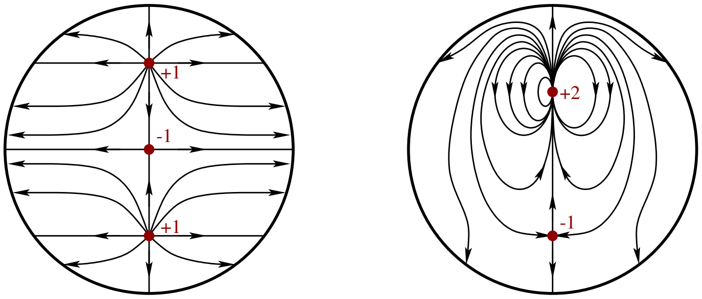

Figure 2.8: Two examples with index sum +1.

Lemma 2.3.5 (Perturbation Lemma). *Let M be an m-manifold with boundary, let X be a vector field on M, let $p_0 \in M \setminus \partial M$ be an isolated zero of X, and let $U \subset M \setminus \partial M$ be an open neighborhood of p_0 such that p_0 is the only zero of X in U. Then there exists a vector field X' on M such that X' agress with X on $M \setminus U$, the zeros of X' in U are all nondegenerate, and*

$$\sum_{p \in U,\, X'(p)=0} \iota(p, X') = \iota(p_0, X). \tag{2.3.4}$$

Proof. This is [26, page 40, Step 2]. Shrinking U, if necessary, we may assume that there exists a local coordinate chart $\phi : U \to \Omega$ with values in an open set $\Omega \subset \mathbb{R}^m$. Assume without loff of generality that $\phi(p_0) = 0 \in \Omega$, and let $\xi : \Omega \to \mathbb{R}^m$ be the pushforward vector field of X so that

$$\xi(\phi(p)) = d\phi(p)X(p) \qquad \text{for all } p \in U. \tag{2.3.5}$$

Then $\xi(0) = 0$ and $\xi(x) \neq 0$ for all $x \in \Omega \setminus \{0\}$. Choose $\varepsilon > 0$ such that

$$\bar{B}_\varepsilon = \left\{x \in \mathbb{R}^m \,\middle|\, |x| \leq 2\varepsilon\right\} \subset \Omega,$$

and by Sard's Theorem 1.2.1 choose a regular value $y \in \mathbb{R}^m$ of ξ such that

$$|y| < \inf_{\varepsilon \leq |x| \leq 2\varepsilon} |\xi(x)|. \tag{2.3.6}$$

Let $\lambda : \Omega \to [0, 1]$ be a smooth function such that

$$\lambda(x) = \begin{cases} 1, & \text{if } |x| \le \varepsilon, \\ 0, & \text{if } |x| \ge 2\varepsilon, \end{cases} \qquad \text{for } x \in \Omega, \tag{2.3.7}$$

and define the vector field $\xi' : \Omega \to \mathbb{R}^m$ by

$$\xi'(x) := \xi(x) - \lambda(x)y \qquad \text{for } x \in \Omega. \tag{2.3.8}$$

By (2.3.6) and (2.3.7) the zeros of the vector field ξ' in (2.3.8) lie all in the open ball B_ε of radius ε and they are all nondegenerate, because y is a regular value of ξ. Hence, by (2.3.5) and (2.3.6), the vector field

$$X' \in \mathrm{Vect}(M)$$

defined by

$$X'(p) := \begin{cases} d\phi(p)^{-1}\xi'(\phi(p)), & \text{for } p \in U, \\ X(p), & \text{for } p \in M \setminus U, \end{cases}$$

is smooth and has only nondegenerate zeros in U. Moreover, it follows from the definition of the index and Lemma 2.3.3 that

$$\begin{aligned} \sum_{p\in U,\, X'(p)=0} \iota(p, X') &= \sum_{x\in\Omega,\, \xi'(x)=0} \iota(x, \xi') \\ &= \sum_{x\in B_{2\varepsilon},\, \xi'(x)=0} \iota(x, \xi') \\ &= \deg\left(S^{m-1} \to S^{m-1} : x \mapsto \frac{\xi(2\varepsilon x)}{|\xi(2\varepsilon x)|}\right) \\ &= \iota(0, \xi) \\ &= \iota(p_0, X). \end{aligned}$$

This proves equation (2.3.4) and Lemma 2.3.5. □

The proof of Theorem 2.3.1 makes use of the Tubular Neighborhood Theorem for submanifolds of Euclidean spaces. Let $M \subset \mathbb{R}^n$ be a compact m-dimensional submanifold without boundary. Then the normal bundle

$$TM^\perp := \{(p, v) \,|\, p \in M\, v \in \mathbb{R}^n,\, v \perp T_pM\}$$

is a smooth n-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ (see Exercise 2.4.12). Define the map $f : TM^\perp \to \mathbb{R}^n$ by $f(p, v) := p + v$ for $(p, v) \in TM^\perp$.

Theorem 2.3.6 (Tubular Neighborhood Theorem). *For* $\varepsilon > 0$ *define*

$$V_\varepsilon := \left\{(p,v) \in TM^\perp \,\middle|\, |v| < \varepsilon\right\}, \qquad U_\varepsilon := \left\{x \in \mathbb{R}^n \,\middle|\, \min_{p\in M}|x-p| < \varepsilon\right\}.$$

Then, for $\varepsilon > 0$ *sufficiently small, the map* f *restricts to a diffeomorphism*

$$f_\varepsilon := f|_{V_\varepsilon} : V_\varepsilon \to U_\varepsilon, \qquad f_\varepsilon(p,v) = p + v.$$

Proof. The proof has three steps.

Step 1. $f(V_\varepsilon) = U_\varepsilon$ *for every* $\varepsilon > 0$.

Let $\bar{x} \in U_\varepsilon$ and choose $\bar{p} \in M$ such that

$$|\bar{x} - \bar{p}| = \min_{p\in M}|\bar{x} - p|.$$

Then the function $M \to \mathbb{R} : p \mapsto |\bar{x} - p|^2$ takes on its minimum at $p = \bar{p}$. Hence its derivative vanishes at $p = \bar{p}$ and this implies

$$\langle \bar{x} - \bar{p}, v\rangle = 0 \qquad \text{for all } v \in T_pM.$$

Thus $\bar{v} := \bar{x} - \bar{p} \in T_pM^\perp$ and $|\bar{v}| < \varepsilon$. Hence $(\bar{p}, \bar{v}) \in V_\varepsilon$ and $f(\bar{p}, \bar{v}) = \bar{x}$.

Step 2. *If* $\varepsilon > 0$ *is sufficiently small, then* f_ε *is a local diffeomorphism, i.e. its derivative is bijective everywhere.*

For every $p \in M$ the derivative of f at $(p,0)$ is given by

$$df(p,0)(\widehat{p}, \widehat{v}) = \widehat{p} + \widehat{v} \qquad \text{for } (\widehat{p}, \widehat{v}) \in T_{(p,0)}TM^\perp = T_pM \times T_pM^\perp$$

and so is bijective. Hence the derivative of f is bijective in some open neigborhood of $M \times \{0\}$ in $TM^\perp$. Since M is compact, this proves Step 2.

Step 3. *If* $\varepsilon > 0$ *is sufficiently small, then* f_ε *is injective.*

Suppose, by contradiction, that f_ε is not injective for any $\varepsilon > 0$. Then there exist sequences $(p_i, v_i), (p_i', v_i') \in TM^\perp$ such that

$$\lim_{i\to\infty}|v_i| = \lim_{i\to\infty}|v_i'| = 0, \qquad p_i + v_i = p_i' + v_i', \qquad (p_i, v_i) \neq (p_i', v_i'). \quad (2.3.9)$$

Passing to a subsequence, we may assume that $(p_i)_{i\in\mathbb{N}}$ and $(p_i')_{i\in\mathbb{N}}$ converge. Hence

$$p := \lim_{i\to\infty} p_i = \lim_{i\to\infty}(p_i + v_i) = \lim_{i\to\infty}(p_i' + v_i') = \lim_{i\to\infty} p_i' \in M.$$

Since $df(p,0)$ is bijective, the Inverse Function Theorem 1.1.19 asserts that the restriction of f to a neighborhood of $(p,0)$ is injective, and hence we deduce that $(p_i, v_i) = (p_i', v_i')$ for i sufficiently large, contradicting (2.3.9). This proves Step 3 and Theorem 2.3.6. □

We will first prove Theorem 2.3.1 for manifolds without boundary.

Proof of Theorem 2.3.1 for manifolds without boundary. By Theorem A.3.1 we may assume without loss of generality that $M \subset \mathbb{R}^n$ is a compact m-dimensional submanifold without boundary. Choose $\varepsilon_0 > 0$ such that the map $f_{\varepsilon_0} : V_{\varepsilon_0} \to U_{\varepsilon_0}$ in Theorem 2.3.6 is a diffeomorphism.

1. Denote by $\pi : TM^\perp \to M$ the cononical projection and consider the map $r := \pi \circ f_{\varepsilon_0}^{-1} : U_{\varepsilon_0} \to M$. For all $x \in U_{\varepsilon_0}$ it satisfies

$$|x - r(x)| = \min_{p\in M}|x - p|, \qquad x - r(x) \perp T_{r(x)}M. \tag{2.3.10}$$

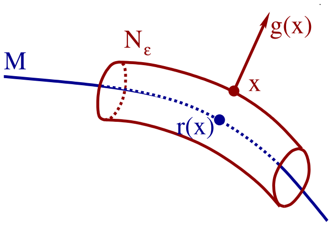

Figure 2.9: A tubular neighborhood of M.

2. The function $h : U_{\varepsilon_0} \to \mathbb{R}$, defined by

$$h(x) := \min_{p\in M}|x - p|^2 = |x - r(x)|^2 \qquad \text{for } x \in U_{\varepsilon_0}, \tag{2.3.11}$$

is smooth and its gradient is given by

$$\nabla h(x) = 2(x - r(x))$$

for all $x \in U_{\varepsilon_0}$. Hence every nonzero real number is a regular value of h. By Lemma 1.2.18 this implies that for $0 < \varepsilon < \varepsilon_0$ the set

$$\begin{aligned} N_\varepsilon &:= \left\{ x \in \mathbb{R}^n \,\middle|\, \min_{p\in M}|x - p| \le \varepsilon \right\} \\ &= \left\{ x \in U_{\varepsilon_0} \,\middle|\, h(x) \le \varepsilon^2 \right\} \end{aligned} \tag{2.3.12}$$

is a compact n-dimensional submanifold of $\mathbb{R}^n$ with boundary

$$\partial N_\varepsilon = h^{-1}(\varepsilon^2) = \left\{ x \in \mathbb{R}^n \,\middle|\, \min_{p\in M}|x - p| = \varepsilon \right\}$$

(see Figure 2.9).

3. The Gauß map $g : \partial N_\varepsilon \to S^{n-1}$ is given by

$$g(x) = \frac{\nabla h(x)}{|\nabla h(x)|} = \frac{x - r(x)}{\varepsilon} \qquad \text{for } x \in \partial N_\varepsilon.$$

To see this, let $x \in \partial N_\varepsilon$. Then, by Lemma 1.2.7,

$$T_x \partial N_\varepsilon = \ker dh(x) = \nabla h(x)^\perp$$

and $\nabla h(x)$ point out of N_ε because $dh(x)\nabla h(x) = |\nabla h(x)|^2 > 0$.

4. Define the vector field $Y : N_\varepsilon \to \mathbb{R}^n$ by

$$Y(x) := X(r(x)) + x - r(x) \qquad \text{for } x \in N_\varepsilon. \tag{2.3.13}$$

Then Y points out on the boundary, becuase for $x \in \partial N_\varepsilon$ we have

$$\begin{aligned} \langle Y(x), g(x) \rangle &= \frac{\langle X(r(x)) + x - r(x), x - r(x) \rangle}{\varepsilon} \\ &= \frac{|x - r(x)|^2}{\varepsilon} \\ &= \varepsilon > 0. \end{aligned}$$

5. The zeros of $Y : N_\varepsilon \to \mathbb{R}^n$ all lie in M. Namely,

$$x - r(x) \perp X(r(x))$$

for all $x \in N_\varepsilon$, and so $Y(x) = 0$ implies $x = r(x) \in M$ and $X(x) = Y(x) = 0$.

6. Let $x \in M$ be a zero of X. Then by Lemma 2.2.11 the vertical derivative of X at x is the usual derivative $dX(x) : T_xM \to \mathbb{R}^m$ which actually takes values in T_xM. Moreover, for all $\widehat{x} \in \mathbb{R}^n$,

$$dY(x)\widehat{x} = \left\{ \begin{array}{rl} dX(x)\widehat{x}, & \text{if } \widehat{x} \in T_xM, \\ \widehat{x}, & \text{if } \widehat{x} \perp T_xM. \end{array} \right. \tag{2.3.14}$$

For the first term assume $\widehat{x} \in T_xM$, choose a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = \widehat{x}$, and note that $Y(\gamma(t)) = X(\gamma(t))$ for all t. For the second term assume $\widehat{x} \perp T_xM$ and note that $r(x + t\widehat{x}) = x$ and hence $Y(x + t\widehat{x}) = t\widehat{x}$. This proves (2.3.14). Equation (2.3.14) shows that, if x is a nondegenerate zero of X, then it is also a nondegenerate zero of Y and their indices agree, so $\iota(x, Y) = \iota(x, X)$.

7. If X has only nondegenerate zeros, so does Y and so, by Lemma 2.3.3,

$$\sum_{x \in M,\, X(x)=0} \iota(x, X) = \sum_{x \in N_\varepsilon,\, Y(x)=0} \iota(x, Y) = \deg(g : \partial N_\varepsilon \to S^{n-1}).$$

8. If the vector field $X \in \mathrm{Vect}(M)$ has only isolated zeros, it follows from Lemma 2.3.5 that there exists another vector field $X' \in \mathrm{Vect}(M)$ with only nondegenerate zeros whose index sum agrees with the one of X. Thus

$$\sum_{x \in M,\, X(x)=0} \iota(x, X) = \sum_{x \in M,\, X'(x)=0} \iota(x, X') = \deg(g : \partial N_\varepsilon \to S^{n-1}).$$

Here the term on the left is independent of the choice of the embedding of M into $\mathbb{R}^n$ and the term on the right is independent of the choice of X. This shows that the sum of the indices of the zeros of a vector field with only isolated zeros is independent of the choice of the vector field and hence is a topological invariant of the manifold M. This concludes the proof of Theorem 2.3.1 for compact manifolds without boundary. □

Remark 2.3.7. Let M be a compact m-manifold without boundary. The topological invariant in the above proof is called the **Euler characteristic** of M and will be denoted by $\chi(M)$. Thus

$$\chi(M) := \sum_{x \in M,\, X(x)=0} \iota(x, X) = \deg(g : \partial N_\varepsilon \to S^{n-1})$$

for every vector field X on M with only isolated zeros, and for g the Gauß map of a tubular neighborhood associated to an embedding of M into $\mathbb{R}^n$.

Remark 2.3.8. If M is a compact connected m-manifold without boundary such that $\chi(M) = 0$ (i.e. if there exists a vector field on M with only isolated zeros whose index sum is zero), then a theorem of Hopf asserts that there exists a vector field on M without zeros (see Theorem 3.3.8).

Remark 2.3.9. The above proof does not include the identification of the Euler characteristic with the alternating sum of the Betti numbers. For the deRham cohomology groups this will be established in Theorem 6.4.9. Another method is to consider the special case where $X = -\nabla f$ is the negative gradient vector field of a Morse function $f : M \to \mathbb{R}$ with respect to some Riemannian metric. The zeros are all nondegenerate, and one can relate the index sum to the alternating sum of the Betti numbers by examining the change in the singular homology of the sublevel sets

$$M^c := \{p \in M \mid f(p) \le c\}$$

as c passes through a critical value of f. The details of this argument can be found in the book by Milnor [25, p 29–36]. Another reference is the paper by Floer [10] on the Morse–Witten complex. For a Conley index approach to this story see [36].

Proof of Theorem 2.3.1 for manifolds with boundary. The starting point for adapting the proof to manifolds with boundary is to choose an embedding with a product structure near the boundary.

1. By Corollary A.3.3 we may assume that M is a compact m-dimensional submanifold of $\mathbb{H}^n$ such that $\partial M = M \cap \partial\mathbb{H}^n =: Q \times \{0\}$ and

$$M \cap (\mathbb{R}^{n-1} \times [0,1]) = Q \times [0,1]. \tag{2.3.15}$$

Under this assumption M extends to a noncompact manifold

$$\widehat{M} := M \cup \bigl(Q \times (-\infty, 0)\bigr).$$

2. The Tubular Neighborhood Theorem 2.3.6 extends to $\widehat{M}$. Thus there exists a smooth map $r : U \to \widehat{M}$ on an open neighborhood $U \subset \mathbb{R}^n$ of $\widehat{M}$ such that, for all $x \in U$,

$$|x - r(x)| = \min_{p \in \widehat{M}} |x - p| < 1, \qquad x - r(x) \perp T_{r(x)}\widehat{M}. \tag{2.3.16}$$

Then, for all $x \in U$, we have

$$\begin{aligned} x_n \geq 0 \qquad &\Longrightarrow \qquad r_n(x) \geq 0, \\ x_n \leq 0 \qquad &\Longrightarrow \qquad r_n(x) = x_n. \end{aligned} \tag{2.3.17}$$

Namely, if $x_n \leq 0$, then we must have $r_n(x) < 1$ because $|x - r(x)| < 1$, hence $r(x) \in Q \times (-\infty, 1)$ by (2.3.15), and so $x - r(x) \perp e_n$ by (2.3.16).

3. Choose a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ that satisfies $\rho(t) = 0$ for $t \geq 0$ and $\rho(t) > 0$, $\dot\rho(t) < 0$ for $t < 0$. (For example take $\rho(t) := e^{1/t}$ for $t < 0$.) Then the function $h : U \to \mathbb{R}$ defined by

$$h(x) := \min_{p \in \widehat{M}} |x - p|^2 + \rho(x_n) = |x - r(x)|^2 + \rho(x_n)$$

is smooth and has the gradient $\nabla h(x) = 2(x - r(x)) + \dot\rho(x_n)e_n$ for $x \in U$. It satisfies $h > 0$ on $U \setminus M$ and equation (2.3.17) shows that the only critical points of h are the elements of M, so every positive number is a regular value of h. Hence, by Lemma 1.2.18, for $\varepsilon > 0$ sufficiently small the set

$$N_\varepsilon := \bigl\{x \in U \bigm| h(x) \leq \varepsilon^2\bigr\} \subset \mathbb{R}^n$$

is a compact manifold with boundary $\partial N_\varepsilon = h^{-1}(\varepsilon^2)$ and the Gauß map

$$g(x) = \frac{\nabla h(x)}{|\nabla h(x)|} = \frac{2(x - r(x)) + \dot\rho(x_n)e_n}{|2(x - r(x)) + \dot\rho(x_n)e_n|} \qquad \text{for } x \in \partial N_\varepsilon.$$

We will now fix such a constant $\varepsilon > 0$.

4. Let X be a vector field on M that points out on the boundary and has only nondegenerate zeros. Assume also that X is translation invariant near the boundary, i.e. there exists a constant $0 < \delta < 1$ such that

$$\partial_n X|_{Q\times[0,\delta]} = 0. \tag{2.3.18}$$

Then X extends uniquely to a smooth vector field $\widehat{X} \in \mathrm{Vect}(\widehat{M})$ that agrees with X on $M \subset \widehat{M}$ and satisfies $\partial_n \widehat{X}(x) = 0$ whenever $x_n \le 0$. Now define the vector field $Y : N_\varepsilon \to \mathbb{R}^n$ as before by

$$Y(x) := \widehat{X}(r(x)) + x - r(x) \qquad \text{for } x \in N_\varepsilon.$$

Since $x - r(x) \perp \widehat{X}(r(x))$ we obtain again that the zeros of Y are the zeros of X and have the same indices. Moreover, for $x \in \partial N_\varepsilon$ we have

$$\begin{aligned}\langle Y(x), \nabla h(x)\rangle &= \langle \widehat{X}(r(x)) + x - r(x), 2(x - r(x)) + \dot\rho(x_n)e_n\rangle \\ &= 2|x - r(x)|^2 + \big(\widehat{X}_n(x) + x_n - r_n(x)\big)\dot\rho(x_n) > 0.\end{aligned}$$

Indeed, if $x_n \ge 0$, then $\dot\rho(x_n) = 0$ and $2|x - r(x)^2| = 2\varepsilon^2 > 0$. It $x_n \le 0$, then we have $x_n = r_n(x)$ by (2.3.17) and $\widehat{X}_n(x) < 0$ by the outward pointing property of X, and hence $\langle Y(x), \nabla h(x)\rangle \ge \widehat{X}_n(x)\dot\rho(x_n) > 0$. Thus Y is outward pointing on the boundary of N_ε. Since it has the same zeros with the same indices as X, it follows again from the Hopf Lemma 2.3.3 that

$$\sum_{x\in M,\, X(x)=0} \iota(x, X) = \deg(g : \partial N_\varepsilon \to S^{n-1}). \tag{2.3.19}$$

5. Assume X points out on the boundary and has only nondegenerate zeros. Choose $0 < \delta < 1/3$ such that X has no zeros in $Q \times [0, 3\delta]$ and choose a smooth function $\beta : [0, 3\delta] \to [0, 3\delta]$ such that $\dot\beta \ge 0$ and $\beta|_{[0,\delta]} \equiv 0$ and $\beta(t) = t$ for $2\delta \le t \le 3\delta$. Define the vector field $X' \in \mathrm{Vect}(M)$ by $X' := X$ on $M \setminus (Q \times [0, 3\delta])$ and

$$X'(q, x_n) := X(q, \beta(x_n)) \qquad \text{for } q \in Q \text{ and } 0 \le x_n \le 3\delta.$$

Then X' satisfies (2.3.18) and has the same zeros with the same indices as X. Hence X satisfies (2.3.19) by Step 4. This proves (2.3.19) for all vector fields on M with only nondegenerate zeros that point out on the boundary. The case of isolated zeros reduces to the nondegenerate case as before by an appeal to Lemma 2.3.5. This shows that the sum of the indices of the zeros of a vector field with only isolated zeros that points out on the boundary is independent of the choice of the vector field and hence is a topological invariant of the manifold M. This concludes the proof of Theorem 2.3.1 for compact manifolds with boundary. □

Remark 2.3.10. An alternative approach to Theorem 2.3.1 in the boundary case, as suggested by Milnor [26, p 40], would be to define the function h by (2.3.11) and the neighborhood $N_\varepsilon \subset \mathbb{R}^m$ by (2.3.12) as before, so that the number $h(x)$ is the square of the distance to M, and N_ε is the set of all points in $\mathbb{R}^n$ whose distance to M is at most ε. Then h is continuously differentiable in a neighborhood of M, however, it is not C^∞ smooth near the boundary of M. Thus the neighborhood N_ε will only be a C^1-manifold and the map r which assigns to each $x \in N_\varepsilon$ the nearest point $r(x) \in M$ will only be continuous. Thus the extended vector field $Y(x) := X(r(x)) + x - r(x)$ is only continuous, and one has to argue that the Hopf Lemma 2.3.3 remains valid for continuous vector fields on C^1 manifolds.

A third approach to Theorem 2.3.1 would be to prove that any two vector fields X_0, X_1 on M with only nondegenerate zeros can be joined by a smooth homotopy of vector fields X_t that satisfies the transversality condition $\mathrm{im}\, DX_t(p) + \mathbb{R}\partial_t X_t(p) = T_pM$ for all t and all zeros $p \in M$ of X_t. In this case it follows that the collective zero set

$$Z := \left\{(t,p) \in [0,1] \times M \,\middle|\, X_t(p) = 0\right\}$$

of the X_t is a smooth 1-dimensional submanifold of $[0,1] \times M$ with boundary

$$\partial Z = \big(\{0\} \times Z_0\big) \cup \big(\{1\} \times Z_1\big), \qquad Z_i := \left\{p \in M \,\middle|\, X_i(p) = 0\right\}.$$

Then one can deduce directly that the index sums of X_0 and X_1 agree. This argument is similar in spirit to the proof of Lemma 2.1.11.

At this point we have not proved equation (2.3.1), but we have only shown that the left hand side of the equation is independent of the choice of the vector field. Thus the index sum of a vector field with only isolated zeros that points out on the boundary is a topological invariant of M. As Theorem 2.3.1 asserts, this invariant turns out to be the Euler characteristic in its standard homological definition as the alternating sum of the Betti number. Until we prove this in Chapter 6, it will be convenient to instead use the index sum as the definition of the Euler characteristic (see Remark 2.3.7).

Definition 2.3.11. *Let M be a compact m-manifold with boundary. The* **Euler characteristic** *of M is the index sum*

$$\chi(M) := \sum_{p \in M,\, X(p)=0} \iota(p, X) \tag{2.3.20}$$

of a vector field X on M with only isolated zeros that points out on the boundary. Such a vector field exists by Exercise 2.4.15 and the index sum is independent of the choice of X by equation (2.3.19).

2.4 Examples and Exercises

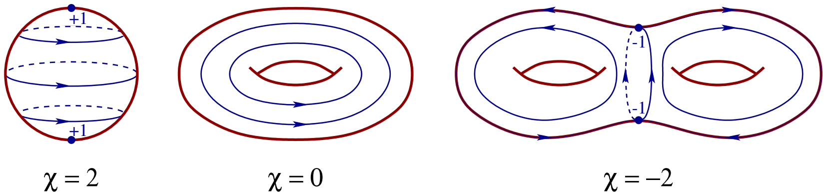

Figure 2.10: Examples of compact oriented surfaces without boundary.

Example 2.4.1 (**Surfaces**). The Euler characteristic of a compact oriented 2-manifold Σ without boundary is even and is less than or equal to 2 (see Example 6.4.15). Hence this number can be written as $\chi(\Sigma) = 2 - 2g$, where the integer $g \geq 0$ is called the **genus of** Σ. Thus the 2-sphere has genus $g = 0$ because $\chi(S^2) = 2$, and the 2-torus has genus $g = 1$ because $\chi(\mathbb{T}^2) = 0$ (see Figure 2.10). One can in fact show that two compact oriented 2-manifolds without boundary are diffeomorphic if and only if they have the same genus. A proof can be found in the book by Hirsch [13].

Exercise 2.4.2. **(a)** Prove that $\chi(\mathbb{T}^m) = 0$ in all dimensions.

(b) Prove that $\chi(S^m) = 2$ when m is even and $\chi(S^m) = 0$ when m is odd.

(c) Prove that $\chi(\mathbb{R}\mathrm{P}^m) = \frac{1}{2}\chi(S^m)$.

(d) Prove that $\chi(\mathbb{C}\mathrm{P}^n) = n + 1$ in all dimensions.

Exercise 2.4.3. Prove that $\mathbb{R}\mathrm{P}^m$ is orientable if and only if m is odd. Prove that $\mathbb{C}\mathrm{P}^n$ is orientable for every n.

Exercise 2.4.4. Prove that every simply connected manifold is orientable. (Recall from [35, §6.1] that a manifold M is called **simply connected** iff any two paths $\gamma_0, \gamma_1 : [0,1] \to M$ with the same endpoints $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$ are homotopic with fixed endpoints.)

Exercise 2.4.5. Let M and N be oriented m-manifolds without boundary such that M is compact and let $f : M \to N$ be a smooth map. Prove that the integer $\deg(f, q)$ in (2.1.2) is a locally constant function of q on the open set $\mathcal{R}_f \subset N$ of regular values of f. **Hint:** Show that $\mathrm{sign}(df(p)) \in \{-1, +1\}$ is a locally constant function of p on the open set of regular points of f. Then argue as in the proof of Lemma 1.1.21.

Exercise 2.4.6 (Hopf Lemma over the Integers). *Let M be a compact oriented m-manifold with boundary, let $f : M \to \mathbb{R}^m$ be a smooth map, let $y \in \mathbb{R}^m \setminus f(\partial M)$ be a regular value of f, and define $g : \partial M \to S^{m-1}$ by*

$$g(p) := \frac{f(p) - y}{|f(p) - y|} \qquad \text{for } p \in \partial M.$$

Then

$$\deg(g) = \deg(f, y) = \sum_{p \in f^{-1}(y)} \mathrm{sign}(df(p)). \tag{2.4.1}$$

Hint: See Lemma 1.5.6 and Lemma 2.3.3.

Exercise 2.4.7 (Brouwer). Prove that every smooth map $f : S^m \to S^m$ of degree $\deg(f) \neq (-1)^{m+1}$ must have a fixed point.

Exercise 2.4.8 (Brouwer). Let $f : S^m \to S^m$ be a smooth map of odd degree. Prove that there exists a point $x \in S^m$ such that $f(-x) = -f(x)$.

Exercise 2.4.9. Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$ be compact smooth manifolds. Prove that every continuous map $f : N \to M$ can be uniformly appeoximated by a smooth map. **Hint:** Use Weierstraß Apprximation and the Tubular Neighborhood Theorem 2.3.6.

Exercise 2.4.10. Let $M \subset \mathbb{R}^{m+1}$ be a compact connected m-dimensional submanifold without boundary. For $x \in \mathbb{R}^{m+1} \setminus M$ define $f_x : M \to S^m$ by

$$f_x(p) := \frac{p - x}{|p - x|} \qquad \text{for } p \in M.$$

Prove the following.

(a) The sets

$$\begin{aligned} U_0 &:= \left\{ x \in \mathbb{R}^{m+1} \setminus M \,\middle|\, \deg_2(f_x) = 0 \right\}, \\ U_1 &:= \left\{ x \in \mathbb{R}^{m+1} \setminus M \,\middle|\, \deg_2(f_x) = 1 \right\} \end{aligned}$$

are nonempty and connected.

(b) There exists a smooth map

$$\nu : M \to S^m$$

such that $\nu(p) \perp T_pM$ for all $p \in M$.

(c) The set

$$W := U_1 \cup M \subset \mathbb{R}^{m+1}$$

is a compact $(m+1)$-manifold with boundary $\partial W = M$.

(d) M is orientable. (Compare Exercise 4.2.4 for another proof.)

Exercise 2.4.11. Prove that the group $\mathrm{GL}^+(m,\mathbb{R})$ is connected.

Hint 1: Let $\Phi \in \mathrm{GL}^+(m,\mathbb{R})$ and define

$$\Phi_0 := P^{-1}\Phi, \qquad P := (\Phi\Phi^T)^{1/2}.$$

Then $\Phi_0 \in \mathrm{SO}(m)$ and $\Phi_\lambda := (\lambda P + (1-\lambda)\mathbb{1})\Phi_0 \in \mathrm{GL}(m,\mathbb{R})$ for $0 \le \lambda \le 1$.

Hint 2: Let $x : [0,1] \to S^m$ be a smooth curve and let $v_0 \in x(0)^\perp$. Then the unique solution of the differential equation

$$\dot v(t) + \langle v(t), \dot x(t)\rangle x(t) = 0, \qquad v(0) = v_0,$$

satisfies $\langle v(t), x(t)\rangle = 0$ and $|v(t)| = |v_0|$ for all t. (Levi-Civita on TS^m.)

Hint 3: Let $\pi : \mathrm{SO}(m+1) \to S^m$ be the map which assigns to each matrix in $\mathrm{SO}(m+1)$ its first column. Let $x : [0,1] \to S^m$ be a smooth curve and let $\Phi_0 \in \mathrm{SO}(m+1)$ such that

$$\pi(\Phi_0) = x(0).$$

Then there exists a smooth curve $\Phi : [0,1] \to \mathrm{SO}(m+1)$ such that

$$\Phi(0) = \Phi_0, \qquad \pi(\Phi(t)) = x(t) \qquad \text{for all } t.$$

Hint 4: Prove by induction that $\mathrm{SO}(m)$ is connected for every m.

Exercise 2.4.12. Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold Define the **tangent bundle** and the **normal bundle** of M as subsets of $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\begin{aligned} TM &:= \{(p,v)\,|\,p \in M,\ v \in T_pM\}, \\ TM^\perp &:= \{(p,v)\,|\,p \in M,\ v \in \mathbb{R}^n,\ v \perp T_pM\}. \end{aligned}$$

Prove that the tangent bundle is a $2m$-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ and that the normal bundle is an n-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. Prove that, for all $p \in M$,

$$T_{p,0}TM = T_pM \times T_pM, \qquad T_{p,0}TM^\perp = T_pM \times T_pM^\perp.$$

If M has a nonempty boundary, prove that

$$\partial TM = TM \cap (\partial M \times \mathbb{R}^n), \qquad \partial TM^\perp = TM^\perp \cap (\partial M \times \mathbb{R}^n).$$

Exercise 2.4.13. Let $M \subset \mathbb{R}^n$ be a compact m-dimensional submanifold without boundary and consider the tubular neighborhood

$$U_\varepsilon = \left\{ p + v \,\middle|\, p \in M,\, v \in T_pM^\perp,\, |v| < \varepsilon \right\}$$

of M in Theorem 2.3.6 equipped with the map $r : U_\varepsilon \to M$ defined by

$$r(p+v) := p \qquad \text{for } p \in M \text{ and } v \in T_pM^\perp \text{ with } |v| < \varepsilon.$$

Prove that r is a homotopy equivalence and that the obvious inclusion

$$\iota : M \to U_\varepsilon$$

is a homotopy inverse of r. Prove that the closed tubular neighborhod

$$N_\varepsilon = \left\{ p + v \,\middle|\, p \in M,\, v \in T_pM^\perp,\, |v| \le \varepsilon \right\}$$

is also homotopy equivalent to M when $\varepsilon > 0$ is chosen sufficiently small. **Hint:** Consider the homotopy $f_t(x) := r(x) + t(x - r(x))$ for $x \in U_\varepsilon$.

Exercise 2.4.14. Let $M \subset \mathbb{R}^n$ be a compact m-dimensional submanifold without boundary and consider the map $\Pi : M \to \mathbb{R}^{n\times n}$ which assigns to each $p \in M$ the orthogonal projection $\Pi(p) : \mathbb{R}^n \to T_pM$. Thus

$$\Pi(p)^2 = \Pi(p) = \Pi(p)^T, \qquad \operatorname{im}\Pi(p) = T_pM$$

for all $p \in M$. Prove the following.

(a) The map $\Pi : M \to \mathbb{R}^{n\times n}$ is smooth

(b) If $\widehat{p} \in T_pM$ and $x \in T_pM^\perp$, then $(d\Pi(p)\widehat{p})x \in T_pM$.

(c) The set $\mathscr{N} := \{(p,x) \in M \times S^{n-1} \,|\, x \perp T_pM\}$ is a compact smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ with the tangent spaces

$$T_{(p,x)}\mathscr{N} = \left\{ (\widehat{p}, \widehat{x}) \in T_pM \times T_xS^{n-1} \,\middle|\, \big(d\Pi(p)\widehat{p}\big)x + \Pi(p)\widehat{x} = 0 \right\}.$$

(The manifold $\mathscr{N}$ is called the **unit normal bundle of** M.)

(d) Let $x \in S^{n-1}$ be a regular value of the canonical projection

$$\pi : \mathscr{N} \to S^{n-1}, \qquad \pi(p,x) := x.$$

Then the vector field X on M, defined by $X(p) := \Pi(p)x \in T_pM$ for $p \in M$, has only nondegenerate zeros.

(e) Let $x \in S^{n-1}$ be a regular value of the projection $\pi : \mathscr{N} \to S^{n-1}$ as in (d). Then the function $f : M \to \mathbb{R}$, defined by $f(p) := \langle p, x\rangle$ for $p \in M$, is a Morse function.

Exercise 2.4.15. Prove that every compact manifold with boundary admits a vector field that points out on the boundary and has only nondegenerate zeros. **Hint:** Assume (without loss of generality by Corollary A.3.3) that

$$M \subset \mathbb{H}^n$$

is a submanifold such that $\partial M = M \cap \partial \mathbb{H}^n$ and

$$M \cap \left(\mathbb{R}^{n-1} \times [0,1]\right) = Q \times [0,1]$$

for some $(m-1)$-manifold $Q \subset \mathbb{R}^{n-1}$ without boundary. As in Exercise 2.4.14 denote by $\mathscr{N}$ the unit normal bundle of M and choose a regular value x of the projection $\pi : \mathscr{N} \to S^{n-1}$ such that $x_n < 0$.

Chapter 3

Homotopy and Framed Cobordisms

The degree of a map $f : M \to M'$ is defined only when the manifolds M and M' have the same dimension. The purpose of the present chapter is to study an extension, due to Pontryagin, for smooth maps from a compact boundaryless manifold to a sphere of smaller dimension, following [26, §7]. We begin with some basic definitions. It will be convenient to formulate these in the extrinsic setting, where the notion of a normal vector field is more readily accessible. We emphasize that in §3.1 and §3.2 none of the manifolds is required to be orientable.

3.1 The Pontryagin Construction

In the following definitions we assume that $M \subset \mathbb{R}^k$ is an m-dimensional submanifold without boundary and N, $N' \subset M$ are compact n-dimensional submanifolds without boundary. The difference in dimensions $d = m - n$ is called the **codimension** of the submanifolds.

Definition 3.1.1 (Cobordism). *N is called* **cobordant to N' (within M)** *iff there exists a compact $(n+1)$-dimensional submanifold $W \subset [0,1] \times M$ with boundary $\partial W = \bigl(\{0\} \times N\bigr) \cup \bigl(\{1\} \times N'\bigr)$ such that*

$$\begin{aligned} W \cap \bigl([0,\varepsilon) \times M\bigr) &= [0,\varepsilon) \times N, \\ W \cap \bigl((1-\varepsilon,1] \times M\bigr) &= (1-\varepsilon,1] \times N. \end{aligned}$$

for some $\varepsilon > 0$. In particular, $\partial W = W \cap \bigl(\{0,1\} \times M\bigr) = W \cap \partial\bigl([0,1] \times M\bigr)$. This is an equivalence relation (see Figure 3.1). The submanifold W is called a **cobordism from N to N'**.

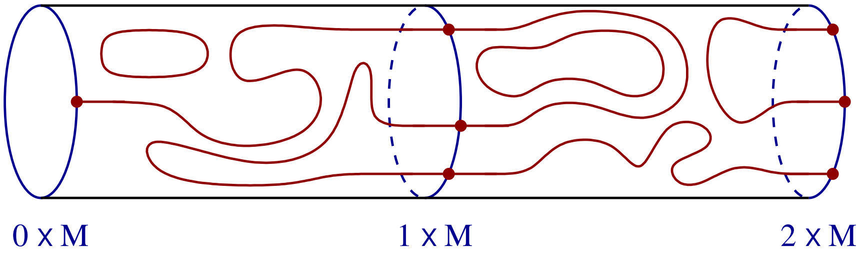

Figure 3.1: Composition of two cobordisms.

Definition 3.1.2 (Framing). (i) *A* **normal vector field on** N *is a smooth map*

$$X : N \to \mathbb{R}^k$$

which assigns to each $p \in M$ *a normal vector to the tangent space of* N *in the tangent space of* M *at* p*, i.e.*

$$X(p) \in T_pN^\perp := \left\{ v \in T_pM \,\middle|\, v \perp T_pN \right\} \qquad \textit{for all } p \in N. \tag{3.1.1}$$

The space of all normal vector fields on N *will be denoted by* $\mathrm{Vect}^\perp(N)$*.*

(ii) *A* **framing** *of* $N \subset M$ *is an* $(m-n)$*-tuple of normal vector fields*

$$X = \big(X_1, \ldots, X_{m-n}\big) \in \mathrm{Vect}^\perp(N)^{m-n}$$

such that the vectors $X_1(p), \ldots, X_{m-n}(p)$ *form a basis of* $T_pN^\perp$ *for each element* $p \in N$ *(see Figure 3.2). The pair* (N, X) *is called a* **framed submanifold** *of* M*.*

(iii) *Two* n*-dimensional framed submanifolds* (N, X) *and* (N', X') *of* M *are called* **framed cobordant** *iff there exists a cobordism*

$$W \subset [0,1] \times M$$

from N *to* N' *and a framing* $Y = (Y_1, \ldots, Y_{m-n})$ *of* W *that satisfies*

$$Y_i(t,p) = \begin{cases} (0, X_i(p)), & \textit{if } 0 \le t < \varepsilon \textit{ and } p \in N, \\ (0, X_i'(p)), & \textit{if } 1-\varepsilon < t \le 1 \textit{ and } p \in N', \end{cases}$$

for $i = 1, \ldots, m-n$ *and some* $\varepsilon > 0$*. This is again an equivalence relation. The pair* (W, Y) *is called a* **framed cobordism from** (N, X) **to** (N', X')*.*

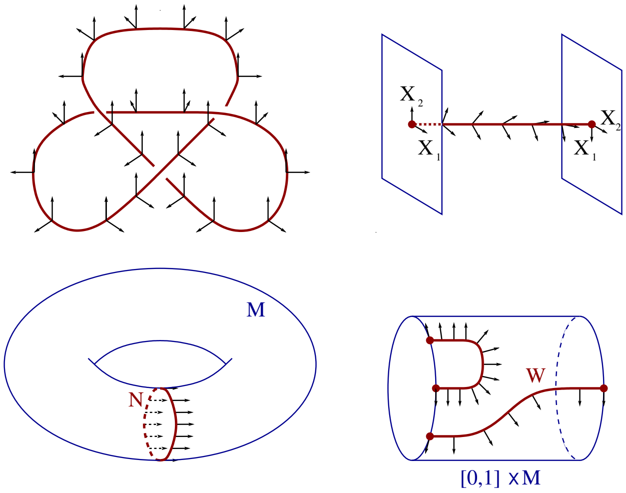

Figure 3.2: Framed submanifolds and framed cobordisms.

Now let $f : M \to S^d$ be a smooth map, let $y \in S^d$ be a regular value, and let $v_1, \dots, v_d$ be a positive basis of the tangent space $T_y S^d$. These data determine a framed submanifold of M as follows. By Lemma 1.2.7 the set

$$N := f^{-1}(y) \subset M$$

is a submanifold of M of dimension $n = m - d$ and its tangent space at each point $p \in N$ is the kernel of the derivative $df(p) : T_p N \to T_y S^d$. Hence, for each $p \in N$, this derivative restricts to a vector space isomorphism

$$df(p)|_{T_p N^\perp} : T_p N^\perp \to T_y S^d$$

and thus determines a unique basis $X_1(p), \dots, X_d(p)$ of $T_p N^\perp$ such that

$$df(p) X_i(p) = v_i \qquad \text{for } i = 1, \dots, d. \tag{3.1.2}$$

The resulting normal vector fields $X_1, \dots, X_d \in \mathrm{Vect}^\perp(N)$ define a framing of the submanifold $f^{-1}(y) \subset M$, denoted by

$$f^* v := (X_1, \dots, X_d).$$

Definition 3.1.3. *The framed submanifold* $(f^{-1}(y), f^* v)$ *defined by* (3.1.2) *is called the* **Pontryagin manifold** *associated to* f *and* $(y, v_1, \dots, v_d)$.

Of course f has many Pontryagin manifolds, corresponding to different choices of y and v. They all belong to the same framed cobordism class.

Theorem 3.1.4. *If y' is another regular value of f and $v' = (v'_1, \dots, v'_d)$ is a positive basis of $T_{y'}S^d$, then the Pontryagin manifold $(f^{-1}(y'), f^*v')$ is framed cobordant to $(f^{-1}(y), f^*v)$.*

Theorem 3.1.5. *Two smooth maps from M to S^d are smoothly homotopic if and only if the associated Pontryagin manifolds are framed cobordant.*

Theorem 3.1.6. *Every framed submanifold (N, X) of codimension d in M occurs as the Pontryagin manifold of some smooth map $f : M \to S^d$.*

The proof of Theorem 3.1.4 is very similar to the proofs in §1.4 and §2.1. It is based on three lemmas. We assume throughout that $M \subset \mathbb{R}^k$ is a compact m-dimensional submanifold without boundary, that $f : M \to S^d$ is a smooth map, and that $y \in S^d$ is a regular value of f.

Lemma 3.1.7. *If $v = (v_1, \dots, v_d)$ and $v' = (v'_1, \dots, v'_d)$ are two different positively oriented bases of T_yS^d, then the Pontryagin manifold $(f^{-1}(y), f^*v)$ is framed cobordant to $(f^{-1}(y), f^*v')$.*

Proof. This is [26, §7, Lemma 1]. Since the bases v and v' are positively oriented, they are related by a matrix with positive determinant, i.e.

$$v'_i = \sum_{j=1}^{d} a_{ij} v_j, \qquad A = (a_{ij})_{i,j=1}^{d} \in \mathrm{GL}^+(d, \mathbb{R}).$$

Since the group $\mathrm{GL}^+(d, \mathbb{R})$ is connected (Exercise 2.4.11), this implies that there exists a smooth curve $[0, 1] \to \mathrm{GL}^+(d, \mathbb{R}) : t \mapsto A(t)$ such that $A(0) = \mathbb{1}$ and $A(1) = A$. Moreover, this curve can be chosen such that $A(t)$ is locally constant near $t = 0$ and near $t = 1$. Thus v and v' are joined by the smooth path of positively oriented bases $v(t) = (v_1(t), \dots, v_d(t))$ of T_yS^d, given by

$$v_i(t) := \sum_j a_{ij}(t) v_j, \qquad (a_{ij}(t))_{i,j=1}^{d} := A(t).$$

They satisfy $v(t) = v$ for t sufficiently close to zero and $v(t) = v'$ for t sufficiently close to 1. Hence the trivial cobordism

$$W := [0, 1] \times N$$

with the framing $Y = (Y_1, \dots, Y_d)$, defined by

$$Y_i(t, p) := (0, X_i(t, p)), \qquad X_i(t, p) \in T_pN^\perp, \qquad df(p)X_i(t, p) = v_i(t)$$

for $i = 1, \dots, d$ and $(t, p) \in W$ is a framed cobordism from $(f^{-1}(y), f^*v)$ to $(f^{-1}(y), f^*v')$. This proves Lemma 3.1.7. □

Lemma 3.1.8. *If $y' \in S^d$ is sufficiently close to y, then $f^{-1}(y)$ is framed cobordant to $f^{-1}(y')$.*

Proof. This is [26, §7, Lemma 2]. Since M is compact, so is the set of critical values of f. Hence there exists an $\varepsilon \in (0, 1/2)$ such that every $y' \in S^d$ with $|y - y'| < \varepsilon$ is a regular value of f. Given $y' \in S^d$ with $|y - y'| < \varepsilon$, choose a smooth family of rotations $r_t : S^d \to S^d$ that satisfies the following.

(a) r_t is the identity for $0 \le t \le \varepsilon$.

(b) $r_t = r_1$ for $1 - \varepsilon \le t \le 1$ and $r_1(y') = y$.

(c) The point $y(t) := r_t^{-1}(y) \in S^d$ satisfies $|y(t) - y| < \varepsilon$ for $0 \le t \le 1$.

Define the homotopy $F : [0, 1] \times M \to S^d$ by

$$F(t, p) := r_t\big(f(p)\big) \qquad \text{for } 0 \le t \le 1 \text{ and } p \in M.$$

For each t note that y is a regular value of the composition

$$r_t \circ f : M \to S^d,$$

because $y(t) = r_t^{-1}(y)$ is a regular value of f. This implies that y is a regular value of F as well as of $F|_{\partial([0,1]\times M)} = F|_{\{0,1\}\times M}$. Hence

$$W := F^{-1}(y) \subset [0, 1] \times M$$

is a cobordism from $f^{-1}(y)$ to $(r_1 \circ f)^{-1}(y) = f^{-1}(y')$. Moreover, for any positive basis $v = (v_1, \dots, v_d)$ of T_yS^d, the vectors $v_i' := r_1^{-1}(v_i)$ form a positive basis of $T_{y'}S^d$, and the pair $(F^{-1}(y), F^*v)$ is a framed cobordism from $(f^{-1}(y), f^*v)$ to $(f^{-1}(y'), f^*v')$. This proves Lemma 3.1.8. □

Exercise 3.1.9. Here is an explicit formula for the family of rotations r_t satisfying (a), (b), (c) in the above proof. Define

$$y(t) := c(t)y + s(t)y'', \qquad y'' := \frac{y' - \langle y, y'\rangle y}{\sqrt{1 - \langle y, y'\rangle^2}} \perp y,$$

where

$$c(t) := \sqrt{1 - \lambda(t)^2 + \lambda(t)^2\langle y, y'\rangle^2}, \qquad s(t) := \lambda(t)\sqrt{1 - \langle y, y'\rangle^2},$$

and $\lambda : [0, 1] \to [0, 1]$ is a smooth function such that $\lambda(t) = 0$ for $0 \le t \le \varepsilon$ and $\lambda(t) = 1$ for $1 - \varepsilon \le t \le 1$. Now define $r_t : S^d \to S^d$ by

$$\begin{aligned} r_t(x) &:= x - \langle x, y\rangle y - \langle x, y''\rangle y'' \\ &\quad + \Big(c(t)\langle x, y\rangle + s(t)\langle x, y''\rangle\Big)y + \Big(-s(t)\langle x, y\rangle + c(t)\langle x, y''\rangle\Big)y''. \end{aligned}$$

Verify that $y(t) \in S^d$ and $|y(t) - y| < \varepsilon$ for all t. Verify that r_t is a rotation and satisfies $r_t(y(t)) = y$ for all t.

Lemma 3.1.10. *If $g : M \to S^d$ is smoothly homotopic to f and $y \in S^d$ is a regular value of both f and g, then $f^{-1}(y)$ is framed cobordant to $g^{-1}(y)$.*

Proof. This is [26, §7, Lemma 3]. Choose a smooth homotopy

$$F : [0,1] \times M \to S^d$$

from f to g satisfying the condition

$$F(t,p) = \begin{cases} f(p), & \text{if } 0 \le t \le \varepsilon, \\ g(p), & \text{if } 1-\varepsilon \le t \le 1. \end{cases}$$

for some $\varepsilon > 0$. Choose a regular value $z \in S^d$ of F that is sufficiently close to y so that $f^{-1}(z)$ is framed cobordant to $f^{-1}(y)$ and $g^{-1}(z)$ is framed cobordant to $g^{-1}(y)$. Such a regular value exists by Lemma 3.1.8 and Sard's Theorem. Choose a positive basis $w = (w_1, \dots, w_d)$ of T_zS^d. Then the pair

$$(W,Y) := (F^{-1}(z), F^*w)$$

is a framed cobordism from the Pontryagin manifold $(f^{-1}(z), f^*w)$ of f to the Pontryagin manifold $(g^{-1}(z), g^*w)$ of g. This proves Lemma 3.1.10. □

Proof of Theorem 3.1.4. This is [26, §7, Theorem A]. Let y and y' be regular values of f. Then there exists a smooth family of rotations

$$r_t : S^d \to S^d$$

such that

$$r_0 = \mathrm{id}, \qquad r_1(y') = y.$$

(For example given by the formulas in Exercise 3.1.9.) Hence f is smoothly homotopic to $r_1 \circ f$ and y is a regular value of both f and $r_1 \circ f$. By Lemma 3.1.7 and Lemma 3.1.10 this implies that $f^{-1}(y)$ is framed cobordant to the Pontryagin manifold

$$(r_1 \circ f)^{-1}(y) = f^{-1}\bigl(r_1^{-1}(y)\bigr) = f^{-1}(y').$$

This proves Theorem 3.1.4. □

The proofs of Theorem 3.1.5 and Theorem 3.1.6 require the Product Neighborhood Theorem and will be given in the next section.

3.2 The Product Neighborhood Theorem

Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional submanifold without boundary and let (N, X) be a compact framed submanifold of M without boundary and of codimension d in M. Denote by $x_1, \dots, x_d$ the coordinates in $\mathbb{R}^d$.

Theorem 3.2.1 (Product Neighborhood Theorem). *There exists an open neighborhood $U \subset M$ of N and diffeomorphism $\psi : N \times \mathbb{R}^d \to U$ such that, for $i = 1, \dots, d$ and all $p \in N$,*

$$\psi(p, 0) = p, \qquad \frac{\partial \psi}{\partial x_i}(p, 0) = X_i(p). \tag{3.2.1}$$

Example 3.2.2. Product neighborhoods do not exist for arbitrary submanifolds. An example of an unframable submanifold is the equator N in the Möbius band M (Figure 2.2). Other examples are the 2-sphere $N = S^2 \times \{0\}$ as a submanifold of its tangent bundle

$$M = \left\{ (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \,\middle|\, |x| = 1,\ \langle x, y \rangle = 0 \right\} = TS^2,$$

and the complex projective line $N = \mathbb{C}\mathrm{P}^1$ as a submanifold of the complex projective plane $M = \mathbb{C}\mathrm{P}^2$. Examples where product neighborhoods do exist are compact codimension-1 submanifolds without boundary of $M = \mathbb{R}^m$ (see Exercise 2.4.10) and preimages of regular values.

Proof of Theorem 3.2.1. In the case $M = \mathbb{R}^m$ the result is a corollary of the Tubular Neighborhood Theorem 2.3.6. Namely, consider the composition

$$\phi := f \circ \Phi : N \times \mathbb{R}^d \to \mathbb{R}^m, \qquad \phi(p, x) = p + \sum_{i=1}^{d} x_i X_i(p), \tag{3.2.2}$$

of the diffeomorphism $\Phi : N \times \mathbb{R}^d \to TN^\perp : (p, x) \mapsto (p, \sum_i x_i X_i(p))$ with the map $TN^\perp \to \mathbb{R}^m : (p, v) \mapsto f(p, v) := p + v$ in Theorem 2.3.6. Choose a number $\varepsilon_0 > 0$ such that f maps the open set $V_0 = \{(p, v) \in TN^\perp \,|\, |v| < \varepsilon_0\}$ diffeomorphically onto an open neighborhood $U_0 \subset \mathbb{R}^m$ of M. If $\varepsilon > 0$ is chosen such that $\Phi(N \times B^d_\varepsilon) \subset V_0$, then ϕ maps $N \times B^d_\varepsilon$ diffeomorphically onto an open neigborhood $U \subset U_0$ of M. Since B^d_ε is diffeomorphic to $\mathbb{R}^d$, this gives rise to the required map ψ. For example, the diffeomorphism

$$\psi : N \times \mathbb{R}^d \to U, \qquad \psi(p, x) := \phi\left(p, \frac{\varepsilon x}{\sqrt{\varepsilon^2 + |x|^2}} \right), \tag{3.2.3}$$

satisfies (3.2.1).

To prove the result in general, we use the expoential map $\exp_p : V_p \to M$ for $p \in M$. It is a smooth map, defined on an open neighborhood $V_p \subset T_pM$ of the origin, such that

$$\exp_p(0) = p, \qquad d\exp_p(0) = \mathrm{id}_{T_pM}. \tag{3.2.4}$$

The other key properties of the exponential map that are needed here are that the set $V := \{(p, v) \,|\, p \in M,\, v \in V_p\}$ is an open subset of the tangent bundle TM and that the map $V \to M : (p, v) \to \exp_p(v)$ is smooth. (For more details see [35, §4.3] and for a summary see Appendix A.6.) With this understood, we must replace the map ϕ in (3.2.2) by the map

$$\phi : N \times B_r^d \to M, \qquad \phi(p, x) := \exp_p\left(\sum_{i=1}^{d} x_i X_i(p)\right). \tag{3.2.5}$$

Here the number $r > 0$ is chosen so small that $\sum_i x_i X_i(p) \in V_p$ for all $p \in N$ and all $x \in \mathbb{R}^d$ such that $|x| < r$. By (3.2.4) we have

$$d\phi(p, 0)(\widehat{p}, \widehat{x}) = \widehat{p} + \sum_{i=1}^{d} \widehat{x}_i X_i(p) \tag{3.2.6}$$

for all $\widehat{p} \in T_pN$ and all $\widehat{x} \in \mathbb{R}^d$. Since the vectors $X_1(p), \dots, X_d(p)$ form a basis of $T_pN^\perp$, it follows that the derivative $d\phi(p, 0) : T_pN \times \mathbb{R}^d \to T_pM$ is bijective for all $p \in N$. The proof of the Product Neighborhood Theorem now rests on the following two assertions.

Claim 1. *If $\varepsilon > 0$ is sufficiently small, then $d\phi(p, x) : T_pN \times \mathbb{R}^d \to T_{\phi(p,x)}M$ is bijective for every $p \in N$ and every $x \in B_\varepsilon^d$.*

Claim 2. *If $\varepsilon > 0$ is sufficiently small, then $\phi|_{N \times B_\varepsilon^d}$ is injective.*

Claim 1 follows from the fact that the set of all points at which the derivative of ψ is bijective is an open subset of $N \times B_r^d$ containing $N \times \{0\}$, and that N is compact. The proof of Claim 2 is verbatim the same as that of Step 3 in the proof of the Tubular Neighborhood Theorem 2.3.6.

If $\varepsilon > 0$ is sufficiently small, then it follows from Claim 1 and Claim 2 that ϕ maps the set $N \times B_\varepsilon^d$ diffeomorphically is onto an open neighborhood

$$U = \phi(N \times B_\varepsilon^d) \subset M$$

of N. Hence the formula (3.2.3) defines a diffeomorphism

$$\psi : N \times \mathbb{R}^d \to U.$$

Moreover, by (3.2.6) this diffeomorphism satisfies the equations in (3.2.1). This proves Theorem 3.2.1. □

Proof of Theorem 3.1.6. This is [26, §7, Theorem C]. Let N be a compact boundaryless codimension-d submanifold of M with framing $X_1, \dots, X_d$. Choose a product neigborhood diffeomorphism $\psi : N \times \mathbb{R}^d \to U \subset M$ as in Theorem 2.3.6 and define the projection $\pi : U \to \mathbb{R}^d$ by $\pi(\psi(p, x)) := x$ for $(p, x) \in N \times \mathbb{R}^d$ (see Figure 3.3). Then it follows from (3.2.1) that

$$\pi(p) = 0, \qquad d\pi(p)X_i(p) = e_i \qquad \text{for } p \in N \text{ and } i = 1, \dots, d. \tag{3.2.7}$$

Hence 0 is a regular value of π and its preimage $\pi^{-1}(0)$ is the manifold N with its given framing.

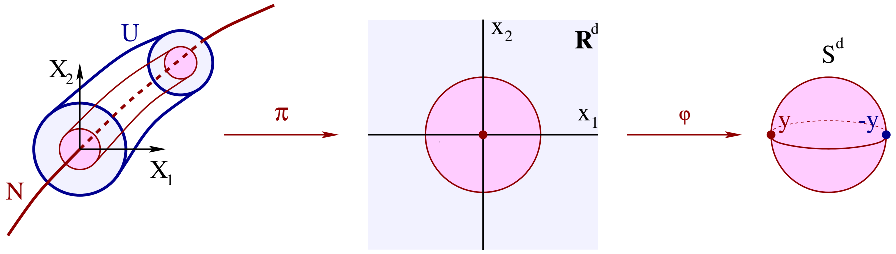

Figure 3.3: Constructing a map with a Pontryagin manifold.

Now choose a smooth map $\varphi : \mathbb{R}^d \to S^d$ which maps the origin to a point y, maps every vector $x \in \mathbb{R}^d$ with $|x| \geq 1$ to the antipodal point $-y$, and maps the open unit ball in $\mathbb{R}^d$ onto $S^d \setminus \{-y\}$ by an orientation preserving diffeomorphism. An explicit example of such a map with

$$\varphi(0) = y := (1, 0, \dots, 0), \qquad d\varphi(0)\widehat{x} = (0, \widehat{x}) \qquad \text{for } \widehat{x} \in \mathbb{R}^d, \tag{3.2.8}$$

is given by the formula

$$\varphi(x) = \left(\frac{\lambda(|x|^2)^2 - |x|^2}{\lambda(|x|^2)^2 + |x|^2}, \frac{2\lambda(|x|^2)x_1}{\lambda(|x|^2)^2 + |x|^2}, \dots, \frac{2\lambda(|x|^2)x_d}{\lambda(|x|^2)^2 + |x|^2} \right),$$

where $\lambda : \mathbb{R} \to [0, 1]$ is a smooth cutoff function such that $\lambda(0) = 2$, $\lambda(t) > 0$ for $0 \leq t < 1$, $\lambda(t) = 0$ for $t \geq 1$, and $\lambda'(t) \leq 0$ for all t. (For example take $\lambda(t) = 2e^{t/(t-1)}$ for $0 \leq t < 1$.)

Define the map $f : M \to S^d$ by

$$f(p) := \begin{cases} \varphi(\pi(p)), & \text{for } p \in U, \\ -y, & \text{for } p \in M \setminus U. \end{cases} \tag{3.2.9}$$

This map is well defined and smooth, it has the point $y = \varphi(0)$ as a regular value with the preimage $f^{-1}(y) = \pi^{-1}(0) = N$, and by (3.2.7) and (3.2.8) the framing associated to the basis $v = (v_1, \dots, v_d)$ of T_yS^d with $v_i := e_{i+1}$ is the given framing $f^*v = (X_1, \dots, X_d)$. This proves Theorem 3.1.6. □

In order to prove Theorem 3.1.5 we must first show that the Pontryagin manifold determines the homotopy class.

Lemma 3.2.3. *Let $f : M \to S^d$ and $g : M \to S^d$ be smooth maps with a common regular value y and choose a positive basis $v = (v_1, \dots, v_d)$ of T_yS^d. If the framed submanifold $(f^{-1}(y), f^*v)$ is equal to $(g^{-1}(y), g^*v)$, then f is smoothly homotopic to g.*

Proof. This is [26, §7, Lemma 4].

1. Assume first that the maps f and g agree in some neighborhood $U \subset M$ of N. Let $h : S^d \setminus \{y\} \to y^\perp$ be the stereographic projection, given by

$$\begin{aligned} h(x) &= \frac{x - \langle x, y\rangle y}{1 - \langle x, y\rangle} = \eta, & |\eta|^2 &= \frac{1 + \langle x, y\rangle}{1 - \langle x, y\rangle}, \\ h^{-1}(\eta) &= \frac{2\eta + (|\eta|^2 - 1)y}{|\eta|^2 + 1} = x, & \langle x, y\rangle &= \frac{|\eta|^2 - 1}{|\eta|^2 + 1}, \end{aligned}$$

for $x \in S^d \setminus \{y\}$ and $\eta \in y^\perp$. Then the formula

$$H(t, p) := H_t(p) := \begin{cases} f(p), & \text{for } p \in U, \\ h^{-1}\big((1 - t)h(f(p)) + th(g(p))\big), & \text{for } p \in M \setminus N, \end{cases}$$

defines a smooth homotopy $H : [0, 1] \times M \to S^d$ from $H_0 = f$ to $H_1 = g$.

2. Consider the general case and define $X = (X_1, \dots, X_d) := f^*v = g^*v$ so that the normal vector fields $X_i \in \mathrm{Vect}^\perp(N)$ satisfy

$$df(p)X_i(p) = dg(p)X_i(p) = v_i \qquad \text{for } p \in N \text{ and } i = 1, \dots, d. \tag{3.2.10}$$

Choose a product neighborhood diffeomorphism $\psi : N \times \mathbb{R}^d \to U$ as in Theorem 3.2.1. Then $y \in f(U) \cap g(U)$ and by shrinking U, if necessary, we may assume without loss of generality that $-y \notin f(U) \cup g(U)$. Now choose a stereographic projection $h : S^d \setminus \{-y\} \to \mathbb{R}^d$ such that

$$h(y) = 0, \qquad dh(y)v_i = e_i \qquad \text{for } i = 1, \dots, d, \tag{3.2.11}$$

and consider the composition

$$\begin{tikzcd} N \times \mathbb{R}^d \arrow[r, "\psi"] \arrow[rrr, bend left, "F"] & U \arrow[r, "f"] & S^d \setminus \{-y\} \arrow[r, "h"] & \mathbb{R}^d . \end{tikzcd}$$

This composition $F := h \circ f \circ \psi : N \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies $F^{-1}(0) = N \times \{0\}$.

3. Let $p \in N$ and define $F_p : \mathbb{R}^d \to \mathbb{R}^d$ by

$$F_p(x) := F(p,x) = h\bigl(f(\psi(p,x))\bigr)$$

for $x \in \mathbb{R}^d$. Then $F_p(0) = 0$ and we claim that $dF_p(0) = \mathrm{id}_{\mathbb{R}^d}$. Indeed it follows from (3.2.1), (3.2.10), and (3.2.11) that

$$\begin{aligned}
dF_p(0)\widehat{x} &= dh(y)df(p)d\psi(p,0)(0,\widehat{x}) \\
&= dh(y)df(p)\left(\sum_{i=1}^d \widehat{x}_i \frac{\partial\psi}{\partial x_i}(p,0)\right) \\
&= dh(y)df(p)\left(\sum_{i=1}^d \widehat{x}_i X_i(p)\right) \\
&= dh(y)\left(\sum_{i=1}^d \widehat{x}_i v_i\right) \\
&= \sum_{i=1}^d \widehat{x}_i e_i \\
&= \widehat{x}
\end{aligned}$$

for $\widehat{x} \in \mathbb{R}^d$.

4. There exists a constant $\delta > 0$ such that all $(p,x) \in N \times \mathbb{R}^d$ satisfy

$$0 < |x| < \delta \qquad \Longrightarrow \qquad \langle F(p,x), x\rangle > 0.$$

To see this, choose $\delta > 0$ such that all $p \in N$ and all $x, \widehat{x} \in \mathbb{R}^d$ satisfy

$$|x| \le \delta \qquad \Longrightarrow \qquad |dF_p(x)\widehat{x} - dF_p(0)\widehat{x}| \le \tfrac{1}{2}|\widehat{x}|.$$

For $p \in N$ and $x \in \mathbb{R}^d$ with $0 < |x| < \delta$ this implies, by Step 3, that

$$\begin{aligned}
\langle F(p,x), x\rangle &= \int_0^1 \frac{d}{dt}\langle F_p(tx), x\rangle\,dt = \int_0^1 \langle dF_p(tx)x, x\rangle\,dt \\
&= \int_0^1 \langle dF_p(0)x, x\rangle\,dt + \int_0^1 \langle dF_p(tx)x - dF_p(0)x, x\rangle\,dt \\
&\ge |x|^2 - \int_0^1 |dF_p(tx)x - dF_p(0)x||x|\,dt \\
&\ge \tfrac{1}{2}|x|^2 > 0.
\end{aligned}$$

5. Apply the same argument to the map $G := h \circ g \circ \psi : N \times \mathbb{R}^d \to \mathbb{R}^d$. to obtain a constant $\delta > 0$ such that, for all $p \in N$ and all $x \in \mathbb{R}^d$,

$$0 < |x| < \delta \qquad \Longrightarrow \qquad \langle F(p,x), x\rangle > 0 \quad \text{and} \quad \langle G(p,x), x\rangle > 0. \tag{3.2.12}$$

By (3.2.12) we have $(1-t)F(p,x)+tG(p,x) \neq 0$ for all pairs $(p,x) \in N \times \mathbb{R}^d$ such that $0 < |x| < \delta$ and all all $t \in [0,1]$. Now choose a smooth cutoff function $\lambda : \mathbb{R}^d \to [0,1]$ such that

$$\lambda(x) = \begin{cases} 1, & \text{if } |x| \le \delta/3, \\ 0, & \text{if } |x| \ge 2\delta/3, \end{cases}$$

and define the maps $F_t : M \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$F_t(p,x) := (1 - \lambda(x)t)F(p,x) + \lambda(x)tG(p,x)$$

for $0 \le t \le 1$ and $(p,x) \in N \times \mathbb{R}^d$. They have the following properties.

(a) $F_0 = F$

(b) If $(p,x) \in N \times \mathbb{R}^d$ such that $|x| < \delta$, then $F_1(p) = G(p)$.

(c) $F_t(p,x) \neq 0$ for all t and all $(p,x) \in N \times \mathbb{R}^d$ with $x \neq 0$.

(d) $F_t(p,x) = F(p,x)$ for all $(p,x) \in N \times \mathbb{R}^d$ such that $|x| \ge 2\delta/3$.

6. The functions F_t give rise to a homotopy $f_t : M \to S^d$ defined by

$$f_t(p) := \begin{cases} f(p), & \text{for } p \in M \setminus U, \\ h^{-1} \circ F_t \circ \psi^{-1}(p), & \text{for } p \in U. \end{cases}$$

By (d) this homotopy is smooth, by (a) it satisfies $f_0 = f$, by (b) the map f_1 agrees with g on a neighborhood of N, and by (c) we have $f_1^{-1}(y) = N$. Hence $f_1 \sim g$ by Step 1, and hence $f \sim g$. This proves Lemma 3.2.3. □

Proof of Theorem 3.1.5. This is [26, §7, Theorem B]. If $f, g : M \to S^d$ are smoothly homotopic, then by Lemma 3.1.10 their Pontryagin manifolds are framed cobordant. Conversely, suppose that the Pontryagin manifolds of f and g are framed cobordant. Choose a common regular value $y \in S^d$ of f and g and a positive basis $v = (v_1, \dots, v_d)$ of T_yS^d, and let (W, Y) be a framed cobordism from $(f^{-1}(y), f^*v)$ to $(g^{-1}(y), g^*v)$. Then an argument completely analogous to the one in the proof of Theorem 3.1.6 constructs a smooth homotopy

$$F : [0,1] \times M \to S^d$$

whose Pontryagin manifold $(F^{-1}(y), F^*v)$ is precisely the given pair (W, Y). For $0 \le t \le 1$ define the map $F_t := F(t, \cdot) : M \to S^d$. Then F_0 and f have exactly the same Pontryagin manifold. Hence $F_0 \sim f$ by Lemma 3.2.3, and likewise $F_1 \sim g$. Therefore $f \sim g$, and this proves Theorem 3.1.5. □

3.3 The Hopf Degree Theorem

As an application of the Pontryagin Construction we prove the following theorem by Heinz Hopf [15].

Theorem 3.3.1 (**Hopf Degree Theorem).** *Let M be a compact connected oriented m-manifold without boundary and let $f, g : M \to S^m$ be smooth maps. Then f is smoothly homotopic to g if and only if* $\deg(f) = \deg(g)$.

Since M is compact, a framed 0-dimensional submanifold of M is a finite set $N \subset M$ equipped with a basis $X_1(p), \dots, X_m(p)$ of T_pM for each $p \in N$. Since M is oriented, we can attach to each $p \in N$ the sign

$$\operatorname{sign}(p, X) := \begin{cases} +1, & \text{if the basis } X_1(p_i), \dots, X_m(p_i) \text{ is positive}, \\ -1, & \text{if the basis } X_1(p_i), \dots, X_m(p_i) \text{ is negative}, \end{cases}$$

and define the **degree of** (N, X) as the sum of these signs

$$\deg(N, X) := \sum_{p \in N} \operatorname{sign}(p, X) \in \mathbb{Z}. \tag{3.3.1}$$

Denote the equivalence relation of framed cobordism by $\sim$.

Theorem 3.3.2. *Let $M \subset \mathbb{R}^n$ be a compact connected oriented m-manifold without boundary and let (N, X) and (N', X') be framed 0-dimensional submanifolds of M. Then (N, X) is framed cobordant to (N', X') if and only if* $\deg(N, X) = \deg(N', X')$.

The *"only if"* part of Theorem 3.3.2 follows directly from the Pontryagin construction. Namely, if (N, X) and (N', X') are framed cobordant, then by Theorem 3.1.6 there exist smooth maps $f, f' : M \to S^m$ such that (N, X) is the Pontryagin manifold of f associated to a regular value y and (N', X') is the Pontryagin manifold of f' associated to a regular value y'. Since (N, X) is framed cobordant to (N', X'), it then follows from Theorem 3.1.5 that f is smoothly homotic to f', hence by Theorem 2.1.10 the maps f and f' have the same degree, and hence it follows from the definitions that

$$\deg(N, X) = \deg(f) = \deg(f') = \deg(N', X').$$

Of course, the *"only if"* part of Theorem 3.3.2 can also be proved directly by the same arguments that were used in Lemma 2.1.11. The proof of the *"if"* part of Thoerem 3.3.2 relies on the following three lemmas.

Lemma 3.3.3. *Let $M \subset \mathbb{R}^n$ be an oriented m-manifold without boundary, let $N \subset M$ be a finite set, and let X and X' be two framings of N such that* $\mathrm{sign}(p, X) = \mathrm{sign}(p, X')$ *for all $p \in N$. Then $(N, X) \sim (N, X')$.*

Proof. This is essentially the same as Lemma 3.1.7 with the same proof. Two bases of the same sign can be joined by a smooth curve of bases, giving rise to a framing Y of the trivial cobordism $W = [0, 1] \times N$ such that (W, Y) is a framed cobordism from (N, X) to (N, X'). This proves Lemma 3.3.3. □

Lemma 3.3.4 (Replacement Lemma). *Let $m \geq 2$ and let $M \subset \mathbb{R}^n$ be a connected oriented m-manifold without boundary. Let (N, X) and (N', X') be compact 0-dimensional framed submanifolds of M such that there exists a pair of points $q \in N$ and $q' \in N'$ satisfying*

$$\mathrm{sign}(q, X) = \mathrm{sign}(q', X'), \quad X|_P = X'|_P, \quad P := N \setminus \{q\} = N' \setminus \{q'\}.$$

Then $(N, X) \sim (N', X')$.

Proof. Since $m \geq 2$ and P is a finite set, the complement $M \setminus P$ is connected. Hence, by Lemma 1.4.5, there exists a smooth isotopy $h_t : M \setminus P \to M \setminus P$ with uniform compact support such that $h_0 = \mathrm{id}$ and $h_1(q) = q'$. After reparametrization we may also assume that

$$h_t = \begin{cases} \mathrm{id}, & \text{for } 0 \leq t \leq 1/3, \\ h_1, & \text{for } 2/3 \leq 1 \leq 1. \end{cases}$$

Then the set

$$W := A \cup ([0, 1] \times P), \qquad A := \left\{(t, h_t(q)) \,\middle|\, 0 \leq t \leq 1\right\},$$

is a cobordism from N to N', Define the framing $Y = (Y_1, \dots, Y_m)$ of W by

$$\begin{aligned} Y_i(t, p) &:= (0, X_i(p)), \\ Y_i(t, h_t(q)) &:= \Pi(t, h_t(q))(0, dh_t(q)X_i(q)) \\ &= (0, dh_t(q)X_i(q)) - \frac{\langle dh_t(q)X_i(q), \partial_t h_t(q)\rangle}{1 + |\partial_t h_t(q)|^2}(1, \partial_t h_t(q)) \end{aligned}$$

for $0 \leq t \leq 1$ and $p \in P$. Here $\Pi(t, h_t(q)) : \mathbb{R} \times T_{h_t(q)}M \to T_{(t,h_t(q))}W^\perp$ is the orthogonal projection and its restriction to the subspace $\{0\} \times T_{h_t(q)}M$ is a vector space isomorphism. Hence (W, Y) is a framed cobordism from (N, X) to (N', X''), where $X_i''|_P = X_i|_P$ and $X_i''(q') = dh_1(q)X_i(q)$ for $i = 1, \dots, m$. Since $dh_1(q) : T_qM \to T_{q'}M$ is an orientation preserving isomorphism, we find that $\mathrm{sign}(q', X'') = \mathrm{sign}(q, X) = \mathrm{sign}(q', X')$. Thus (N', X'') is framed cobordant to (N', X') by Lemma 3.3.3. This proves Lemma 3.3.4. □

Lemma 3.3.5 (Removal Lemma). *Let $m \geq 2$ and let $M \subset \mathbb{R}^n$ be a connected oriented m-manifold without boundary. Let (N, X) be a compact 0-dimensional framed submanifold of M, let $p_0, p_1 \in N$ such that*

$$\operatorname{sign}(p_0, X) = -1, \qquad \operatorname{sign}(p_1, X) = +1, \tag{3.3.2}$$

and define $N' := N \setminus \{p_0, p_1\}$. Then $(N, X) \sim (N', X|_{N'})$.

Proof. By Lemma 3.3.4 we may assume that p_1 is arbitrarily close to p_0. Under this assumption there exists a parametrization $\psi : \mathbb{R}^m \to U$ of an open neighorhood $U \subset M$ of p_0 such that $U \cap N = \{p_0, p_1\}$ and $\psi(0) = p_0$. Choose a smooth function $\lambda : [0, 1] \to [0, 1]$ satisfying

$$\begin{aligned} \lambda(t) &= 0, \quad \text{for } 0 \leq t \leq 1/3, \\ \dot{\lambda}(t) &> 0, \quad \text{for } 1/3 < t < 2/3, \\ \lambda(t) &= 1, \quad \text{for } 2/3 \leq t \leq 1, \end{aligned}$$

and define the curve $\gamma : [0, 1] \to M$ by $\gamma(t) := \psi\big(\lambda(t)\psi^{-1}(p_1)\big)$ for $0 \leq t \leq 1$. Then $\gamma(t) = p_0$ for $0 \leq t \leq 1/3$, $\gamma(t) = p_1$ for $2/3 \leq t \leq 1$, and the restriction of γ to the interval $(1/3, 2/3)$ is an embedding. Thus the set

$$W := A \cup \big([0, 1] \times N'\big), \qquad A := \big\{\big(t(1-t), \gamma(t)\big) \,\big|\, 0 \leq t \leq 1\big\},$$

is a cobordism from N to N'. For $p \in N'$ define $Y_i(t, p) := (0, X_i(p))$. For A use parallel transport to find a framing such that $Y_i(t, p_0) = (0, X_i(p_0))$ for t near zero. Then (W, Y) is a framed cobordism from (N, X'') to (N', X') with $X''|_{N\setminus\{p_1\}} = X|_{N\setminus\{p_1\}}$ and $\operatorname{sign}(p_1, X'') = -\operatorname{sign}(p_0, X) = \operatorname{sign}(p_1, X)$. Hence $(N, X'') \sim (N, X)$ by Lemma 3.3.3, and this proves Lemma 3.3.5. □

Proof of Theorem 3.3.2 for $m \geq 2$. By Lemma 3.3.5 we may assume that all points in N have the same sign without changing the framed cobordism class, and likewise for N'. Assume in addition that $\deg(N.X) = \deg(N', X')$. Then N and N' have the same cardinality and all points have the same index. Hence $(N, X) \sim (N', X')$ by Lemma 3.3.4. This proves Theorem 3.3.2. □

Proof of Theorem 3.3.1. Assume $m \geq 2$. (For $m = 1$ see Exercise 3.4.1.) By Theorem A.3.1 we may assume that M is embedded in $\mathbb{R}^n$. Suppose that $\deg(f) = \deg(g)$ and choose a common regular value y of f and g and a positive basis v of T_yS^d. Then it follows from the definitions that

$$\deg(f^{-1}(y), f^*v) = \deg(f) = \deg(g). = \deg(g^{-1}(y), g^*v).$$

Hence Theorem 3.3.2 asserts that the Pontryagin manifolds $(f^{-1}(y), f^*v)$ and $(g^{-1}(y), g^*v)$ are framed cobordant, and hence by Theorem 3.1.5 the map f is smoothly homotopic to g. This proves Theorem 3.3.1 for $m \geq 2$. □

In [15] Heinz Hopf proved a similar result for nonorientable manifolds.

Theorem 3.3.6 (**Nonorientable Hopf Degree Theorem**). *Let M be a compact connected m-manifold without boundary that is nonorientable, and let $f, g : M \to S^m$ be smooth maps. Then f is smoothly homotopic to g if and only if* $\deg_2(f) = \deg_2(g)$.

The key to the proof of Theorem 3.3.6 is the observation that the framed cobordism class of a 0-dimensional framed submanifold of M is independent of the choice of the framing in the nonorientable case. The reason is that by nonorientability there must exist loops in M along which a frame reverses orientation.

Lemma 3.3.7. *Let $M \subset \mathbb{R}^n$ be a connected m-manifold without boundary that is nonorientable. Let $N \subset M$ be a finite set with two framings X, X'. Then $(N, X) \sim (N, X')$.*

Proof. See Exercise 3.4.3. □

Proof of Theorem 3.3.6. **1.** Assume that $M \subset \mathbb{R}^n$ is a compact connected nonorientable m-manifold without boundary and let $f, f' : M \to S^m$ be smooth maps such that

$$\deg_2(f) = \deg_2(f') \tag{3.3.3}$$

Denote their Pontryagin manifolds (for some choice of a common regular value y and a positive basis of T_yS^m) by (N, X) and (N', X'), respectively.

2. The proof of the Replacement Lemma 3.3.4 carries over verbatim to nonorientable manifolds and shows that (N, X) is framed cobordant to any finite subset $P \subset M$ of the same cardinality as N with some choice of framing and hence, by Lemma 3.3.7, with every choice of framing. If $\#N = \#N'$, this shows that $(N, X) \sim (N', X')$.

3. If $\#N \neq \#N'$, assume without loss of generality that $\#N > \#N'$. Then by (3.3.3) there exists a positive integer k such that

$$\#N = \#N' + 2k.$$

The proof of the Removal Lemma 3.3.5 also carries over to nonorientable manifolds and (together with Lemma 3.3.7) shows that (N, X) is framed cobordant to $(P, X|_P)$ for any subset $P \subset N$ such that $\#P = \#N - 2$. Use this argument k times to deduce that (N, X) is framed cobordant to $(P, X|_P)$ for any subset $P \subset N$ such that $\#P = \#N - 2k$. Now use Step 2 to deduce that $(P, X|_P)$ is framed cobordant to (N', X'). Hence (N, X) is framed cobordant to (N', X') and hence, by Theorem 3.1.5, the map f is smoothly homotopic to f'. This proves Theorem 3.3.6. □

As an application of the Hopf Degree Theorem we prove the following theorem of Hopf [16] which asserts the existence of vector fields without any zeros at all on manifolds with Euler characteristic zero. Recall that at this point we have defined the Euler characteristic of a compact manifold with boundary not in terms of the homology, but as the index sum of a vector field with only isolated zeros that point out on the boundary (Definition 2.3.11). Recall also that the existence of vector fields with only nondegenerate zeros that point out on the boundary is established in Exercise 2.4.15.

Theorem 3.3.8 (Hopf). *Let M be a compact connected m-manifold with boundary such that $\chi(M) = 0$. Then there exists a vector field on M that points out on the boundary and has no zeros.*

Proof. For $m = 1$ the only compact 1-manifolds up to diffeomorphism are the circle and the closed unit interval (Theorem A.4.1) and the existence of vector fields on the circle without zeros was exhibited in Corollary 2.1.17. Thus assume $m \geq 2$ and let $X \in \mathrm{Vect}(M)$ be a vector field with only isolated zeros that points out on the boundary. Denote the zeros of X by

$$p_1, \dots, p_k \in M.$$

Choose a parametrization $\psi : \mathbb{R}^m \to U$ of an open set $U \subset M \setminus \{p_1, \dots, p_k\}$, define

$$U_0 = \psi(B_1), \qquad B_1 := \left\{x \in \mathbb{R}^m \,\middle|\, |x| < 1\right\},$$

and choose k distinct points $q_1, \dots, q_k \in U_0$. Since $m \geq 2$, the set

$$M_i := M \setminus \Big(\partial M \cup \left\{q_1, \dots, q_{i-1}, p_{i+1}, \dots, p_k\right\}\Big)$$

is a connected m-manifold without boundary. Hence, by Lemma 1.4.5, there exists, for each i, a diffeomorphism $h_i : M_i \to M_i$ with compact support (i.e. equal to the identity near the boundary and the removed points) such that

$$h_i(q_i) = p_i, \qquad \text{for } i = 1, \dots, k.$$

Extending h_i smoothly over all of M we also have $h_i(q_j) = q_j$ for $j < i$ and $h_i(p_j) = p_j$ for $j > i$. Hence the composition

$$h := h_1 \circ h_2 \circ h_3 \circ \cdots \circ h_{k-1} \circ h_k : M \to M$$

satisfies

$$h(q_i) = p_i \qquad \text{for } i = 1, \dots, k.$$

This implies that the vector field $Y(q) := h^*X(q) = dh(q)^{-1}X(h(q))$ has its zeros at the points $q_1, \dots, q_k \in U_0$ and points out on the boundary. Hence the vector field

$$\eta := \psi^*Y : \mathbb{R}^m \to \mathbb{R}^m, \qquad \eta(x) = d\psi(x)^{-1}Y(\psi(x)),$$

has its zeros in the open unit ball $B_1 = \psi^{-1}(U_0)$. Since the index sum is zero, by assumption, it follows from the Hopf Lemma 2.3.3 that

$$\deg\left(\frac{\eta}{|\eta|} : S^{m-1} \to S^{m-1}\right) = 0. \tag{3.3.4}$$

Hence, by the Hopf Degree Theorem 3.3.1, the map $|\eta|^{-1}\eta : S^{m-1} \to S^{m-1}$ is smoothly homotopic to a constant map. Choose an element $\bar{x} \in S^{m-1}$ and a smooth homotopy

$$[0,1] \times S^{m-1} \to S^{m-1} : (t,x) \mapsto F_t(x)$$

such that

$$F_t(x) = \begin{cases} \bar{x}, & \text{for } 0 \le t \le 1/3 \text{ and } x \in S^{m-1}, \\ |\eta(x)|^{-1}\eta(x), & \text{for } 2/3 \le t \le 1 \text{ and } x \in S^{m-1}. \end{cases}$$

Now define the map $\zeta : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\begin{aligned} \zeta(tx) &:= (1-\lambda(t))F_t(x) + \lambda(t)\eta(tx), && \text{for } 0 \le t \le 1 \text{ and } x \in S^{m-1}, \\ \zeta(tx) &:= \eta(tx), && \text{for } t \ge 1 \text{ and } x \in S^{m-1}, \end{aligned}$$

where $\lambda : [0,1] \to [0,1]$ is a smooth function that vanishes for $0 \le t \le 1-2\varepsilon$ and satisfies $\lambda(t) = 1$ for $t \ge 1-\varepsilon$. If $0 < 2\varepsilon \le 1/3$, then

$$\zeta(tx) = \bigl(1-\lambda(t)\bigr)\frac{\eta(x)}{|\eta(x)|} + \lambda(t)\eta(tx) \qquad \text{for } 1-2\varepsilon \le t \le 1 \text{ and } x \in S^{m-1},$$

and therefore

$$|\zeta(tx)|^2 = \bigl(1-\lambda(t)\bigr)^2 + \lambda(t)^2|\eta(tx)|^2 + 2\frac{\bigl(1-\lambda(t)\bigr)\lambda(t)}{|\eta(x)|}\langle \eta(x), \eta(tx)\rangle.$$

Thus, if $\varepsilon > 0$ is chosen so small that $\langle \eta(x), \eta(tx)\rangle > 0$ for all $t \in [1-2\varepsilon, 1]$ and all $x \in S^{m-1}$, we deduce that $\zeta(x) \ne 0$ for all $x \in \mathbb{R}^m$. Since $\zeta(x) = \eta(x)$ for all $x \in \mathbb{R}^m$ such that $|x| \ge 1$, the formula

$$Z(p) := \begin{cases} d\psi(\psi^{-1}(p))\zeta(\psi^{-1}(p)), & \text{for } x \in U, \\ Y(p), & \text{for } p \in M \setminus U, \end{cases}$$

defines a vector field on M without zeros that points out on the boundary. This proves Theorem 3.3.8. □

The theory of framed cobordisms was introduced by Pontryagin [33, 34] in order to study homotopy classes of maps from S^m to S^d for $d < m$. For example, if $m \geq 4$, then there are precisely two homotopy classes of smooth maps from S^m to S^{m-1}. Pontryagin proved this result by classifying the framed cobordism classes of 1-manifolds in S^m (see Example 3.4.8). He was also able to prove that for $m \geq 4$ there are precisely two homotopy classes of maps from S^m to S^{m-2} by studying framed 2-dimensional submanifolds of S^m. However, this is much more difficult, and for $m - d \geq 3$ this method appears to "run into manifold difficulties" in the words of Milnor [26]. He points out that the Pontryagin construction can also be used in the opposite direction, namely to get information about framed cobordism classes from results in homotopy theory obtained with other methods.

3.4 Examples and Exercises

Exercise 3.4.1. Prove Theorem 3.3.1 for $m = 1$. **Hint:** By Theorem A.4.1 every compact connected 1-manifold without boundary is diffeomorphic to the circle $M = S^1$. For a smooth map $f : S^1 \to S^1$ show that $\deg(f) = k$ in the notation of Exercise 1.7.10.

Exercise 3.4.2. Prove Theorem 3.3.2 for $m = 1$. **Hint:** Use Exercise 3.4.1 and the Pontryagin construction.

Exercise 3.4.3. Let $M \subset \mathbb{R}^n$ be a connected m-manifold without boundary.

(a) Examine the parallel transport of bases of the tangent spaces along a curve $\gamma : [0, 1] \to M$ using the Levi-Civita connection as in Theorem A.5.6.

(b) Prove that M is nonorientable if and only if for some, and hence every, element $p_0 \in M$ there exists a smooth curve $\gamma : [0, 1] \to M$ such that

$$\gamma(0) = \gamma(1) = p_0$$

and the parallel transport isomorphism

$$\Phi_\gamma(1, 0) : T_{p_0}M \to T_{p_0}M$$

is orientation reversing.

(c) Use part (b) to prove Lemma 3.3.7.

Exercise 3.4.4. Verify that Theorem 3.3.8 extends to compact manifolds with boundary and vector fields that point out on the boundary.

Exercise 3.4.5. Let $M, N \subset \mathbb{R}^{k+1}$ be disjoint smooth manifolds. Define the **linking map** $\lambda : M \times N \to S^k$ by

$$\lambda(p, q) := \frac{p - q}{|p - q|} \qquad \text{or } (p, q) \in M \times N. \tag{3.4.1}$$

If M and N are compact, oriented, and boundaryless, and

$$m + n = k, \qquad \dim(M) = m, \qquad \dim(N) = n,$$

define the **linking number** of M and N by

$$\ell(M, N) := \deg(\lambda). \tag{3.4.2}$$

(a) Prove that

$$\ell(N, M) = (-1)^{(m+1)(n+1)}\ell(M, N).$$

(b) If $W \subset \mathbb{R}^{k+1}$ is a compact oriented $(m + 1)$-dimensional submanifold with boundary $\partial W = M$ and $W \cap N = \emptyset$, prove that $\ell(M, N) = 0$.

(c) Define the linking number for disjoint submanifolds of S^{m+n+1}. **Hint:** Use stereographic projection to get a map from $M \times N$ to S^{m+n}.

Exercise 3.4.6 (Hopf Invariant). Let $f : S^{2d-1} \to S^d$ be a smooth map. Choose two distinct regular values $y, z \in S^d$ and orient their preimages as in §2.1 (see Lemma 2.1.11). So the linking number $\ell\bigl(f^{-1}(y), f^{-1}(z)\bigr)$ is defined.

(a) Prove that the linking number is locally constant as a function of y.

(b) Let $g : S^{2d-1} \to S^d$ be another smooth map such that

$$\sup_{x \in S^{2d-1}} |f(x) - g(x)| < |y - z|$$

and such that y and z are also regular values of g. Prove that

$$\ell\bigl(f^{-1}(y), f^{-1}(z)\bigr) = \ell\bigl(g^{-1}(y), f^{-1}(z)\bigr) = \ell\bigl(g^{-1}(y), g^{-1}(z)\bigr).$$

(c) Prove that the linking number depends only on the homotopy class of f and does not depend on the choice of y and z.

The number $H(f) := \ell\bigl(f^{-1}(y), f^{-1}(z)\bigr)$ is called the **Hopf invariant of** f. It was introduced in [17].

Exercise 3.4.7 (Hopf Fibration).
(a) If d is odd, prove that $H(f) = 0$ for every smooth map $f : S^{2d-1} \to S^d$.

(b) For a composition $S^{2d-1} \xrightarrow{f} S^d \xrightarrow{g} S^d$ prove that

$$H(g \circ f) = H(f) \deg(g)^2.$$

(c) The **Hopf fibration** $f : S^3 \to S^2$ is the composition of the map

$$S^3 \to \mathbb{C}\mathrm{P}^1 : (z, w) \mapsto [z : w]$$

(where S^3 is understood as the unit sphere in $\mathbb{C}^2$) with the diffeomorphism from $\mathbb{C}\mathrm{P}^1$ to S^2 in Example 1.1.13, i.e.

$$f(z, w) = \left(2\mathrm{Re}(\bar{z}w), 2\mathrm{Im}(\bar{z}w), |z|^2 - |w|^2\right).$$

Prove that $H(f) = 1$. Thus f is not homotopic to a constant map.

(d) Prove that there are infinitely many homotopy classes of smooth mapst from S^3 to S^2.

Example 3.4.8 (Pontryagin). Fix an integer $m \geq 4$. Then a theorem of Pontryagin [33, 34] asserts that there are precisely two homotopy classes of smooth maps from S^m to S^{m-1}. Pontryagin proved this result by classifying framed cobordism classes of 1-manifolds in S^m. By stereographis projection it suffices to classify framed cobordism classes of 1-manifolds in $\mathbb{R}^m$. Here is a sketch of the argument.

(a) One can prove that two embedded circles $C_0, C_1 \subset \mathbb{R}^m$ are cobordant by a cylinder $W \cong [0, 1] \times S^1$. For this one can choose parametrizations of the circles and join these parametrizations by a smooth homotopy (e.g. via convex combination). Since $m \geq 4$, one can then use a transversality argument, similar to Step 2 in the proof of Theorem A.3.1, to obtain a homotopy by embedded circles. A further transversality argument as in §4.1 can then be used to make the homotopy transverse to, and hence disjoint from, a given 1-manifold $N \subset \mathbb{R}^m$ that does not intersect $C_0 \cup C_1$. This argument shows, in particular, that there exists a cobordism from an embedded circle C to itself that reverses the orientation of C.

(b) Every disjoint union of two circles in $\mathbb{R}^m$ is cobordant to a circle by an orientable cobordism. By (a) it suffices to construct a single example of such a cobordism, for instance by using an embedding of the 2-sphere into $\mathbb{R}^m$ and a Morse function $f : S^2 \to \mathbb{R}$. (Explicitly, the function $f(x) = 1 - x_1^2 + \varepsilon x_2$ with $0 < \varepsilon < 1$ has two minima and one maximum and $S := f^{-1}([0, 1])$ is a *"pair-of-pants"* with three boundary components.)

(c) A framed cobordism in $\mathbb{R}^m$ is necessarily orientable. Moreover, if W is a "pair-of-pants" as in (b), then any framing of two boundary components of W extends to a framing of W. This reduces the classification problem to that of framed cobordism classes of embedded circles.

(d) Any two framings of an embeded circle $C \subset \mathbb{R}^m$ are related by a smooth map from C to $\mathrm{GL}(m-1, \mathbb{R})$. Moreover, by (a) any framing of C is cobordant to one that induces the opposite orientation of the normal bundle, and any two framings that induce the same orientation of the normal bundle are related by a loop in $\mathrm{GL}^+(m-1, \mathbb{R})$. Since $m \geq 4$, we have

$$\pi_1(\mathrm{GL}^+(m-1, \mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}. \tag{3.4.3}$$

This is proved by induction. One first shows that the inclusion of the special orthogonal group $\mathrm{SO}(m-1)$ into $\mathrm{GL}^+(m-1, \mathbb{R})$ is a homotopy equivalence. Second, $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{R}\mathrm{P}^3$ and $\pi_1(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}/2\mathbb{Z}$, because every loop in $\mathbb{R}\mathrm{P}^3$ is homotopic to a loop in $\mathbb{R}\mathrm{P}^1$, and a loop in $\mathbb{R}\mathrm{P}^1$ is contractible in $\mathbb{R}\mathrm{P}^2$ if and only if it has even degree. Third, one proves (3.4.3) for $m > 4$ by an induction and homotopy lifting argument as in Exercise 2.4.11.

(e) Since homotopic framings are cobordant, it follows from (d) that there are at most two framed cobordism classes of embedded circles in $\mathbb{R}^m$. To prove that there are precisely two such framed cobordism classes, one must show that two framings of an embedded circle $C \subset \mathbb{R}^m$ that induce the same orientation of the normal bundle are framed cobordant if and only if they are related by a contractible loop in $\mathrm{GL}^+(m-1, \mathbb{R})$. This can be done by classifying orientable real vector bundle of rank at least three over a given closed orientable 2-manifold. Namely, up to isomorphism there are precisely two such bundles of a given rank, one trivial and one nontrivial. This observation rests on the fact that a loop in a connected Lie group is homologous to zero if and only if it is homotopic to a constant loop.

Let X, Y be cobordant framings of C and suppose, by contradiction, that they are related by a noncontractible loop in $\mathrm{GL}^+(m-1, \mathbb{R})$. Then one of the framings extends over a cobordism from C to the empty set, and hence so does the other. Thus there exist two framed cobordisms $W_X \subset [-1, 0] \times \mathbb{R}^m$ and $W_Y \subset [0, 1] \times \mathbb{R}^m$ with boundaries $\partial W_X = \partial W_Y = \{0\} \times C$ such that the framing of W_X restricts to X and the framing of W_Y restricts to Y on their common boundary C. Hence the normal bundle $T\Sigma^\perp$ of the oriented 2-manifold $\Sigma := W_X \cup W_Y \subset \mathbb{R}^{m+1}$ is nontrivial. However, since the direct sum $T\Sigma \oplus T\Sigma^\perp = \Sigma \times \mathbb{R}^{m+1}$ is trivial, this contradicts the general observation that the direct sum of a nontrivial rank-$(m-1)$ bundle and a rank-2 bundle with an even Euler number (§8.3.3) is a nontrivial rank-$(m+1)$ bundle.

Chapter 4

Intersection Theory

The purpose of the present chapter is to extend the degree theory developed in Chapters 1 and 2 to smooth maps between manifolds of different dimensions, with the dimension of the source being smaller than the dimension of the target and regular values replaced by submanifolds. The relevant transversality theory is the subject of §4.1, orientation and intersection numbers are introduced in §4.2, self-intersection numbers are examined in §4.3, and §4.4 defines the Lefschetz number of a smooth map from a compact manifold to itself and establishes the Lefschetz–Hopf Theorem and the Lefschetz Fixed Point Theorem.

4.1 Transversality

Thie section begins with the basic definitions and some examples in §4.1.1 and establishes the main transversality theorem in §4.1.2.

4.1.1 Transversal Maps

Let m, n, k be nonnegative integers such that $k \le n$, let M be a smooth m-manifold, let N be a smooth n-manifold, and let $Q \subset N$ be a smooth submanifold of dimension $n - k$. The number k is called the **codimension** of Q and will be denoted by $\mathrm{codim}(Q) := \dim(N) - \dim(Q)$. A smooth map $f : M \to N$ is said to be **transverse to Q at** $p \in f^{-1}(Q)$ iff

$$T_{f(p)}N = \mathrm{im}\,(df(p)) + T_{f(p)}Q. \tag{4.1.1}$$

It is called **transverse to** Q iff it is transverse to Q at every $p \in f^{-1}(Q)$. The notation $f \pitchfork Q$ signifies that the map f is transverse to the submanifold Q.

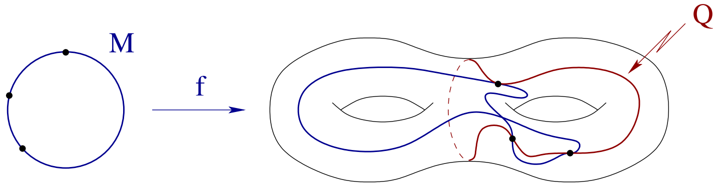

Figure 4.1: Transverse and nontransverse intersections.

Example 4.1.1. **(i)** If $Q = N$, then every smooth map $f : M \to N$ is transverse to Q.

(ii) If $Q = \{q\}$ is a single point in N, then a smooth map $f : M \to N$ is transverse to Q if and only if q is a regular value of f.

(iii) If $f : M \to N$ is an embedding, then its image $P := f(M)$ is a smooth submanifold of N (see [35, Theorem 2.3.4]). In this situation f is transverse to Q if and only if

$$T_qN = T_qP + T_qQ \qquad \text{for all } q \in P \cap Q. \tag{4.1.2}$$

If (4.1.2) holds, we say that P **is transverse to** Q and write $P \pitchfork Q$.

(iv) Assume $\partial M = \emptyset$, let $TM = \{(p, v) \,|\, p \in M,\ v \in T_pM\}$ be the tangent bundle, and let $Z = \{(p, v) \in TM \,|\, v = 0\}$ be the zero section in TM. Identify a vector field $X \in \mathrm{Vect}(M)$ with the map $M \to TM : p \mapsto (p, X(p))$. This map is transverse to the zero section if and only if the vector field X has only nondegenerate zeros. (**Exercise:** Prove this).

(v) Assume $\partial M = \emptyset$. Then the graph of a smooth map $f : M \to M$ is transverse to the diagonal $\Delta = \{(p, p) \,|\, p \in M\} \subset M \times M$ if and only if every fixed point $p = f(p) \in M$ is nondegenerate, i.e. $\det(\mathbb{1} - df(p)) \neq 0$. (**Exercise:** Prove this).

The next lemma generalizes the observation that the preimage of a regular value is a smooth submanifold (see Lemma 1.2.20).

Lemma 4.1.2. *Let M be an m-manifold with boundary, let N be an n-manifold without boundary, and let $Q \subset N$ be a codimension-k submanifold without boundary. Assume f and $f|_{\partial M}$ are transverse to Q. Then the set*

$$P := f^{-1}(Q) = \left\{p \in M \,\middle|\, f(p) \in Q\right\}$$

is a codimension-k submanifold of M with boundary $\partial P = P \cap \partial M$ and its tangent space at $p \in P$ is the linear subspace

$$T_pP = \left\{v \in T_pM \,\middle|\, df(p)v \in T_{f(p)}Q\right\}.$$

Proof. Let $p_0 \in P = f^{-1}(Q)$ and define $q_0 := f(p_0) \in Q$. Then it follows from [35, Theorem 2.3.4] that there exists an open neighborhood $V \subset N$ of q_0 and a smooth map $g : V \to \mathbb{R}^k$ such that the origin $0 \in \mathbb{R}^k$ is a regular value of g and $V \cap Q = g^{-1}(0)$. We prove the following.

Claim: *Zero is a regular value of the map* $g \circ f : U := f^{-1}(V) \to \mathbb{R}^k$ *and also of the map* $g \circ f|_{U \cap \partial M} : U \cap \partial M \to \mathbb{R}^k$.

To see this, fix an element $p \in U$ such that $g(f(p)) = 0$ and let $\eta \in \mathbb{R}^k$. Then

$$q := f(p) \in V \cap Q, \qquad g(q) = 0.$$

Since zero is a regular value of g, there exists a vector $w \in T_qN$ such that

$$dg(q)w = \eta.$$

Since f is transverse to Q, there exists a vector $v \in T_pM$ such that

$$w - df(p)v \in T_qQ.$$

Since $T_qQ = \ker dg(q)$, this implies $d(g \circ f)(p)v = dg(q)df(p)v = dg(q)w = \eta$. Thus zero is a regular value of $g \circ f : U \to \mathbb{R}^k$, and the same argument shows that zero is also a regular value of the restriction of $g \circ f$ to $U \cap \partial M$.

By Lemma 1.2.20 and the claim the set $P \cap U = f^{-1}(Q) \cap U = (g \circ f)^{-1}(0)$ is a smooth $(m-k)$-dimensional submanifold of M with boundary

$$\partial(P \cap U) = P \cap U \cap \partial M$$

and the tangent spaces

$$\begin{aligned} T_pP &= \ker d(g \circ f)(p) \\ &= \ker dg(q)df(p) \\ &= \left\{ v \in T_pM \,\middle|\, df(p) \in \ker dg(q) = T_qQ \right\} \end{aligned}$$

for $p \in U$ with $q := f(p) \in Q$. This proves Lemma 4.1.2. □

The next goal is to show that, given a compact submanifold $Q \subset N$ without boundary, every smooth map $f : M \to N$ is smoothly homotopic to a map that is transverse to Q. This is in contrast to Sard's theorem in Chapter 1 which asserts, in the case where $Q = \{q\}$ is a singleton, that almost every element $q \in N$ is a regular value of f. Instead, the results of the present section imply that, given an element $q \in N$, every smooth map $f : M \to N$ is homotopic to one that has q as a regular value.

4.1.2 Thom–Smale Transversality

Assume throughout that M is a smooth m-manifold with boundary, that N is a smooth n-manifold without boundary, and that $Q \subset N$ is a codimension-k submanifold without boundary that is closed as a subset of N.

Definition 4.1.3 (Relative Homotopy). *Let $A \subset M$ be any subset and let $f, g : M \to N$ be smooth maps such that $f(p) = g(p)$ for all $p \in A$. A smooth map $F : [0,1] \times M \to N$ is called a* **homotopy from** f **to** g **relative to** A *if*

$$F(0,p) = f(p), \qquad F(1,p) = g(p) \qquad \text{for all } p \in M, \tag{4.1.3}$$

$$F(t,p) = f(p) = g(p) \qquad \text{for all } t \in [0,1] \text{ and all } p \in A. \tag{4.1.4}$$

The maps f and g are called **homotopic relative to** A *if there exists a smooth homotopy from f to g relative to A. We write*

$$f \overset{A}{\sim} g$$

to mean that f is homotopic to g relative to A. That relative homotopy is an equivalence relation is shown as in §1.4.

Theorem 4.1.4 (Local Transversality). *Let $f : M \to N$ be a smooth map and let $U \subset M$ be an open set with compact closure such that*

$$f\left(\overline{U} \setminus U\right) \cap Q = \emptyset.$$

Then the following holds.

(i) *There exists a smooth map $g : M \to N$ such that g is homotopic to f relative to $M \setminus U$ and both $g|_U$ and $g|_{U \cap \partial M}$ are transverse to Q.*

(ii) *If $f|_{U \cap \partial M}$ is transverse to Q, then there exists a smooth map $g : M \to N$ such that g is homotopic to f relative to $\partial M \cup (M \setminus U)$ and $g|_U$ is transverse to Q.*

Corollary 4.1.5 (Global Transversality). *Assume M is compact. Then every smooth map $f : M \to N$ is homotopic to a smooth map $g : M \to N$ such that both g and $g|_{\partial M}$ are transverse to Q, and the homotopy can be chosen relative to the boundary whenever the restriction of f to the boundary is transverse to Q.*

Proof. Theorem 4.1.4 with $U = M$. □

The proof of Theorem 4.1.4 relies on the following lemma.

Lemma 4.1.6. *Let N be an n-manifold without boundary, let $Q \subset N$ be a closed set, let $K \subset N$ be a compact set, and let $V \subset N$ be an open neighborhood of $K \cap Q$ with compact closure. Then there exists an integer $\ell \geq 0$ and a smooth map $G : \mathbb{R}^\ell \times N \to N$ such that*

$$G(0, q) = q \quad \text{for all } q \in N, \tag{4.1.5}$$

$$G(\lambda, q) \notin Q \quad \text{for all } (\lambda, q) \in \mathbb{R}^\ell \times (K \setminus V) \tag{4.1.6}$$

$$T_{G(\lambda,q)}N = \operatorname{span}\left\{ \frac{\partial G}{\partial \lambda_i}(\lambda, q) \,\middle|\, i = 1, \dots, \ell \right\} \quad \text{for all } (\lambda, q) \in \mathbb{R}^\ell \times \overline{V}. \tag{4.1.7}$$

Moreover, if $W \subset N$ is an open neighborhood of $\overline{V}$, then G can be chosen such that $G(\lambda, q) = q$ for all $\lambda \in \mathbb{R}^\ell$ and all $q \in N \setminus W$.

Proof. The proof has three steps.

Step 1. *Let $W \subset N$ be an open neighborhood of $\overline{V}$ with compact closure. Then there are vector fields $X_1, \dots, X_\ell \in \mathrm{Vect}(N)$ such that $\mathrm{supp}(X_i) \subset W$ for all i and $T_qN = \operatorname{span}\{X_1(q), \dots, X_\ell(q)\}$ for all $q \in \overline{V}$.*

Assume without loss of generality that $N \subset \mathbb{R}^\ell$ is a smooth submanifold of the Euclidean space $\mathbb{R}^\ell$ for some integer ℓ and that N is a closed subset of $\mathbb{R}^\ell$ (see Theorem A.3.1). By Theorem A.2.2 there exists a partition of unity subordinate to the open cover $M = W \cup (M \setminus \overline{V})$ and hence there exists a smooth cutoff function $\rho : M \to [0, 1]$ such that $\mathrm{supp}(\rho) \subset W$ and $\rho|_{\overline{V}} \equiv 1$. Define the vector fields $X_1, \dots, X_\ell \in \mathrm{Vect}(N)$ by

$$X_i(q) := \rho(q)\Pi(q)e_i$$

for $i = 1, \dots, \ell$ and $q \in N$, where the e_i are the standard basis vectors of $\mathbb{R}^\ell$ and $\Pi(q) \in \mathbb{R}^{\ell \times \ell}$ denotes the orthogonal projection onto T_qN. These vector fields are supported in W and the vectors $X_1(q), \dots, X_\ell(q)$ span the tangent space T_qN for every $q \in \overline{V}$. This proves Step 1.

Step 2. *Let W and $X_1, \dots, X_\ell$ be as in Step 1, for each i let $\phi_i^t \in \mathrm{Diff}(M)$ be the flow of X_i, and define the map $\psi : \mathbb{R}^\ell \times N \to N$ by*

$$\psi(t_1, \dots, t_\ell, q) := \phi_1^{t_1} \circ \phi_2^{t_2} \circ \cdots \circ \phi_\ell^{t_\ell}(q)$$

for $t_i \in \mathbb{R}$ and $q \in N$. Then $\psi(0, q) = q$ for all $q \in N$ and there exists a constant $\varepsilon > 0$ such that the following holds.

(I) *If $q \in \overline{V}$ and $t \in \mathbb{R}^\ell$ satisfies $\max_i |t_i| < \varepsilon$, then*

$$T_{\psi(t,q)}N = \operatorname{span}\left\{ \frac{\partial \psi}{\partial t_i}(t, q) \,\middle|\, i = 1, \dots, \ell \right\}. \tag{4.1.8}$$

(II) *If $q \in K$ and $t \in \mathbb{R}^\ell$ satisfies $\max_i |t_i| < \varepsilon$ and $\psi(t, q) \in Q$, then $q \in V$.*

The vector fields X_i have compact support and hence are complete. Thus the map $\psi : \mathbb{R}^\ell \times N \to N$ is well defined. It satisfies

$$\psi(0,q) = q, \qquad \frac{\partial \psi}{\partial t_i}(0,q) = X_i(q)$$

for all $q \in N$ and all $i \in \{1,\dots,\ell\}$. Hence (4.1.8) holds for $t = 0$ by Step 1 and so assertion (I) follows from the fact that $\overline{V}$ is compact and the set of all pairs $(t,q) \in \mathbb{R}^\ell \times N$ that satisfy (4.1.8) is open.

To prove (II) we argue by contradition and assume that (II) is wrong for every constant $\varepsilon > 0$. Then there exist sequences $t^\nu \in \mathbb{R}^\ell$ and $q^\nu \in K \setminus V$ such that $\lim_{\nu\to\infty} t^\nu = 0$ and $\psi(t^\nu, q^\nu) \in Q$ for all ν. Since K is compact, there exists a subsequence (still denoted by q^ν) that converges to an element $q \in K$. Moreover, since G is continuous and Q is a closed subset of N, we have $q = \psi(0,q) = \lim_{\nu\to\infty} \psi(t^\nu, q^\nu) \in Q$. Thus $q \in K \cap Q \subset V$. Since V is an open subset of N, this implies $q^\nu \in V$ for ν sufficiently large, a contradiction. Thus (II) must hold for some $\varepsilon > 0$ and this proves Step 2.

Step 3. *We prove Lemma 4.1.6.*

Let ψ be as in Step 2 and define the map $G : \mathbb{R}^\ell \times N \to N$ by

$$G(\lambda_1,\dots,\lambda_\ell,q) := \psi\left(\frac{\varepsilon\lambda_1}{\sqrt{\varepsilon^2+\lambda_1^2}},\dots,\frac{\varepsilon\lambda_\ell}{\sqrt{\varepsilon^2+\lambda_\ell^2}},q\right) \tag{4.1.9}$$

for $\lambda_i \in \mathbb{R}$ and $q \in N$. Then $G(0,q) = q$ for all $q \in N$ and so G satisfies (4.1.5). Moreover, G satisfies (4.1.6) by (II) and satisfies (4.1.7) by (I). This proves Lemma 4.1.6. □

Remark 4.1.7. The assertion of Lemma 4.1.6 holds with $\ell \le 2n$. To see this, suppose that the vector fields $X_1,\dots,X_\ell$ satisfy the requirements of Step 1 in the proof of Lemma 4.1.6 with $\ell > 2n$. Choose a Riemannian metric on N and define the map $f : TN \to \mathbb{R}^\ell$ by

$$f(q,w) := \big(\langle w, X_1(q)\rangle,\dots,\langle w, X_\ell(q)\rangle\big) \qquad \text{for } q \in N \text{ and } w \in T_qN.$$

This map has a regular value $\xi = (\xi_1,\dots,\xi_\ell) \in \mathbb{R}^\ell$ by Sard's theorem. Since $\ell > 2n = \dim(TN)$, we have $\xi \notin f(TN)$ and, in particular, $\xi \ne 0$. Assume without loss of generality that $\xi_\ell \ne 0$ and define $Y_i \in \mathrm{Vect}(N)$ by

$$Y_i(q) := X_i(q) - \frac{\xi_i}{\xi_\ell} X_\ell(q) \qquad \text{for } q \in N \text{ and } i = 1,\dots \ell-1.$$

Then, since $\xi \notin f(TN)$, it follows that $T_qN = \mathrm{span}\,\{Y_1(q),\dots,Y_{\ell-1}(q)\}$ for all $q \in K$. (**Exercise:** Verify the details.)

We also need the following lemma. Let $Q \subset N$ be a codimension-k submanifold without boundary and let $F : \mathbb{R}^\ell \times M \to N$ be a smooth map such that F and $F|_{\mathbb{R}^\ell \times \partial M}$ are transverse to Q. Then, by Lemma 4.1.2, the set

$$\mathscr{M} := F^{-1}(Q) = \left\{ (\lambda, p) \in \mathbb{R}^\ell \times M \,\middle|\, F(\lambda, p) \in Q \right\}$$

is a smooth submanifold of $\mathbb{R}^\ell \times M$ with boundary $\partial\mathscr{M} = \mathscr{M} \cap (\mathbb{R}^\ell \times \partial M)$. Denote by $\pi : \mathscr{M} \to \mathbb{R}^\ell$ the obvious projection.

Lemma 4.1.8. *Fix an element* $\lambda \in \mathbb{R}^\ell$ *and define the map* $F_\lambda : M \to N$ *by* $F_\lambda(p) := F(\lambda, p)$ *for* $p \in M$*. Then the following holds.*

(i) λ *is a regular value of* π *if and only if* F_λ *is transverse to* Q*.*

(ii) λ *is a regular value of* $\pi|_{\partial\mathscr{M}}$ *if and only if* $F_\lambda|_{\partial M}$ *is transverse to* Q*.*

Proof. Choose an element $p \in M$ such that $q := F_\lambda(p) = F(\lambda, p) \in Q$. Then $(\lambda, p) \in \mathscr{M}$, the tangent space of $\mathscr{M}$ at (λ, p) is given by

$$T_{(\lambda,p)}\mathscr{M} = \left\{ (\widehat{\lambda}, v) \in \mathbb{R}^\ell \times M \,\middle|\, dF(\lambda, p)(\widehat{\lambda}, v) \in T_qQ \right\},$$

and $d\pi(\lambda, p)(\widehat{\lambda}, v) = \widehat{\lambda}$ for $(\widehat{\lambda}, v) \in T_{(\lambda,p)}\mathscr{M}$. The following are equivalent.

(A) The differential $d\pi(\lambda, p) : T_{(\lambda,p)}\mathscr{M} \to \mathbb{R}^\ell$ is surjective.

(B) $T_qN = \mathrm{im}\,(dF_\lambda(p)) + T_qQ$.

Assume first that (B) holds and fix an element $\widehat{\lambda} \in \mathbb{R}^\ell$. Define

$$w := -\sum_{i=1}^{\ell} \widehat{\lambda}_i \frac{\partial F}{\partial \lambda_i}(\lambda, p) \in T_qN.$$

By (B) there exists a vector $v \in T_pM$ such that $w - dF_\lambda(p)v \in T_qQ$. Hence

$$dF(\lambda, p)(\widehat{\lambda}, v) = dF_\lambda(p)v + \sum_{i=1}^{\ell} \widehat{\lambda}_i \frac{\partial F}{\partial \lambda_i}(\lambda, p) = dF_\lambda(p)v - w \in T_qQ.$$

Hence $(\widehat{\lambda}, v) \in T_{(\lambda,p)}\mathscr{M}$ and $d\pi(\lambda, p)(\widehat{\lambda}, v) = \widehat{\lambda}$, and so (A) holds. Conversely, assume (A) and fix an element $w \in T_qN$. Then, since F is transverse to Q, there exists a pair $(\widehat{\lambda}, v) \in \mathbb{R}^\ell \times T_pM$ such that $w - dF(\lambda, p)(\widehat{\lambda}, v) \in T_qQ$. Now it follows from (A) that there exists a tangent vector $v_0 \in T_pM$ such that $(\widehat{\lambda}, v_0) \in T_{(\lambda,p)}\mathscr{M}$ and so $dF(\lambda, p)(\widehat{\lambda}, v_0) \in T_qQ$. This implies

$$w - dF_\lambda(p)(v - v_0) = w - dF(\lambda, p)(\widehat{\lambda}, v) - dF(\lambda, p)(\widehat{\lambda}, v_0) \in T_qQ$$

and so (B) holds. This shows that (A) is equivalent to (B) and this proves (i). The proof of (ii) is analogous and this proves Lemma 4.1.8. □

Proof of Theorem 4.1.4. We prove part (i). Since $\overline{U}$ is compact, so is

$$K := f(\overline{U}) \subset N.$$

Moreover, $f(\overline{U} \setminus U) \cap Q = \emptyset$ and this implies $K \cap Q \subset N \setminus f(\overline{U} \setminus U)$. Since the set $N \setminus f(\overline{U} \setminus U)$ is open, Lemma A.1.2 asserts that there exists an open set $V \subset N$ with compact closure such that

$$K \cap Q \subset V \subset \overline{V} \subset N \setminus f(\overline{U} \setminus U).$$

Hence $f(\overline{U} \setminus U) \cap \overline{V} = \emptyset$ and so the set

$$B := U \cap f^{-1}(\overline{V}) = \overline{U} \cap f^{-1}(\overline{V})$$

is compact. Hence there exists a smooth function $\beta : M \to [0, 1]$ such that

$$\mathrm{supp}(\beta) \subset U, \qquad \beta|_B = 1. \tag{4.1.10}$$

(See Theorem A.2.2.) Choose a map $G : \mathbb{R}^\ell \times N \to N$ as in Lemma 4.1.6 and define $F : R^\ell \times M \to N$ by

$$F(\lambda, p) := F_\lambda(p) := G(\beta(p)\lambda, f(p)) \qquad \text{for } (\lambda, p) \in \mathbb{R}^\ell \times M. \tag{4.1.11}$$

Then

$$F_0 = f, \qquad F_\lambda|_{M \setminus U} = f|_{M \setminus U}$$

for all λ by (4.1.5) in Lemma 4.1.6. We prove that $F|_{\mathbb{R}^\ell \times U}$ and $F|_{\mathbb{R}^\ell \times (U \cap \partial M)}$ are transverse to Q. Fix an element $(\lambda, p) \in \mathbb{R}^\ell \times U$ with $F(\lambda, p) \in Q$. Then $G(\beta(p)\lambda, f(p)) = F(\lambda, p) \in Q$ by definition of F, and so it follows from (4.1.6) with $q := f(p) \in K$ and λ replaced by $\beta(p)\lambda$ that $f(p) \in V$. This implies $p \in U \cap f^{-1}(\overline{V}) = B$, and hence the vectors

$$\frac{\partial F}{\partial \lambda_i}(\lambda, p) = \beta(p)\frac{\partial G}{\partial \lambda_i}(\beta(p)\lambda, f(p)) = \frac{\partial G}{\partial \lambda_i}(\lambda, f(p))$$

span the tangent space $T_{F(\lambda,p)}N$ by (4.1.7) in Lemma 4.1.6. This shows that $F|_{\mathbb{R}^\ell \times U}$ and $F|_{\mathbb{R}^\ell \times (U \cap \partial M)}$ are transverse to Q as claimed. Hence, by Lemma 4.1.2, the set

$$\mathscr{M} := (\mathbb{R}^\ell \times U) \cap F^{-1}(Q)$$

is a smooth submanifold of $\mathbb{R}^\ell \times U$ with boundary $\partial\mathscr{M} = \mathbb{R}^\ell \times (U \cap \partial M)$. By Sard's theorem there exists a common regular value $\lambda \in \mathbb{R}^\ell$ of the projection $\pi : \mathscr{M} \to \mathbb{R}^\ell$ and of $\pi|_{\partial\mathscr{M}} : \partial\mathscr{M} \to \mathbb{R}^\ell$. Hence, by Lemma 4.1.8, the homotopy $f_t(p) := F(t\lambda, p)$ satisfies the requirements of part (i).

We prove part (ii). Thus assume that $f|_{U\cap\partial M}$ is transverse to Q. As in the proof of (i), consider the compact set

$$K := f(\overline{U}) \subset N$$

and recall that $f(\overline{U}\setminus U)\cap Q=\emptyset$. Hence there exists an open set $V\subset N$ with compact closure such that

$$K\cap Q\subset V, \qquad f(\overline{U}\setminus U)\cap \overline{V}=\emptyset,$$

and then the set

$$B := U\cap f^{-1}(\overline{V}) = \overline{U}\cap f^{-1}(\overline{V})\subset M$$

is compact.

We claim that there exists a smooth function $\beta : M\to\mathbb{R}$ such that

$$\operatorname{supp}(\beta)\subset U, \qquad \beta|_{U\cap\partial M}=0, \qquad \beta_{B\setminus\partial M}>0. \tag{4.1.12}$$

To see this choose a smooth function $\beta_1 : M\to[0,1]$ with

$$\operatorname{supp}(\beta_1)\subset U, \qquad \beta_1|_B = 1$$

as in (4.1.10). Choose an atlas $\{U_\alpha,\phi_\alpha\}_{\alpha\in A}$ on M and let $\rho_\alpha : M\to[0,1]$ be a partition of unity subordinate to the cover, i.e. each point in M has an open neighborhood on which only finitely many of the ρ_α do not vanish and

$$\operatorname{supp}(\rho_\alpha)\subset U_\alpha, \qquad \sum_\alpha \rho_\alpha = 1.$$

(See Theorem A.2.2.) For $\alpha\in A$ define $\beta_\alpha : U_\alpha\to\mathbb{R}$ by

$$\beta_\alpha\circ\phi_\alpha^{-1}(x) := x_m$$

for $x\in\phi_\alpha(U_\alpha)\subset\mathbb{H}^m$. Then the function $\rho_\alpha\beta_\alpha : U_\alpha\to\mathbb{R}$ extends uniquely to a smooth function on M that vanishes on $M\setminus U_\alpha$, the function

$$\beta_0 := \sum_\alpha \rho_\alpha\beta_\alpha : M\to\mathbb{R}$$

vanishes on the boundary and is positive in the interior, and so the product function $\beta := \beta_0\beta_1$ satisfies (4.1.12).

With this understood, the proof of part (ii) proceeds exactly as the proof of (i). The key observation is that the function $F : \mathbb{R}^\ell\times M\to N$ in (4.1.11) still has the property that $F|_{\mathbb{R}^\ell\times U}$ and $F|_{\mathbb{R}^\ell\times(U\cap\partial M)}$ are transverse to Q, because $F(\lambda,\cdot)|_{\partial M} = f|_{\partial M}$ for all $\lambda\in\mathbb{R}^\ell$ and $f|_{U\cap\partial M}$ is transverse to Q by assumption. This proves Theorem 4.1.4. $\square$

4.2 Intersection Numbers

This section introduces intersection numbers modulo two in §4.2.1 and as integers when the manifolds are oriented in §4.2.2. It then moves on to examine intersection indices of isolated intersections in §4.2.3.

4.2.1 Intersection Numbers Modulo Two

Let N be a n-manifold without boundary, let $Q \subset N$ be a codimension-m submanifold without boundary that is closed as a subset of N, and let M be a compact m-manifold with boundary. If $f : M \to N$ is a smooth map that is transverse to Q and satisfies

$$f(\partial M) \cap Q = \emptyset, \tag{4.2.1}$$

then the set $f^{-1}(Q) \subset M \setminus \partial M$ is a compact zero-dimensional submanifold by Lemma 4.1.2 and hence is a finite set (see Figure 4.2).

Theorem 4.2.1 (Intersection Number Modulo Two). *Let $f : M \to N$ be a smooth map satisfying* (4.2.1). *Then the following holds.*

(i) *There exists a smooth map $g : M \to N$ that is transverse to Q and homotopic to f relative to the boundary.*

(ii) *Let g be as in (i). Then the number $\#g^{-1}(Q)$ is finite and its residue class modulo two is independent of the choice of g. It is called the* **intersection number of f and Q modulo two** *and is denoted by*

$$I_2(f, Q) := \begin{cases} 0, & \text{if } \#g^{-1}(Q) \text{ is even,} \\ 1, & \text{if } \#g^{-1}(Q) \text{ is odd,} \end{cases} \qquad \text{for } g \overset{\partial M}{\sim} f \text{ with } g \pitchfork Q. \tag{4.2.2}$$

(iii) *Let $f_0, f_1 : M \to N$ be smooth maps satisfying the condition* (4.2.1) *and let $F : [0,1] \times M \to N$ be a smooth homotopy from f_0 to f_1 such that*

$$F([0,1] \times \partial M) \cap Q = \emptyset. \tag{4.2.3}$$

Then

$$I_2(f_0, Q) = I_2(f_1, Q).$$

(iv) *Let W be a compact $(m+1)$-manifold with boundary and let $F : W \to N$ be a smooth map. Then $I_2(F|_{\partial W}, Q) = 0$.*

The proof relies on the following lemma.

Lemma 4.2.2. *Let $f_0, f_1 : M \to N$ be smooth maps that satisfy* (4.2.1) *and are transverse to Q. Let $F : [0,1] \times M \to N$ be a smooth homotopy from f_0 to f_1 that satisfies* (4.2.3). *Then there exists a smooth homotopy $G : [0,1] \times M \to N$ from f_0 to f_1 such that G is transverse to Q and*

$$G(t,p) = F(t,p) \qquad \text{for all } t \in [0,1] \text{ and all } p \in \partial M.$$

Moreover, $\#f_0^{-1}(Q) \equiv \#f_1^{-1}(Q)$ (*modulo* 2).

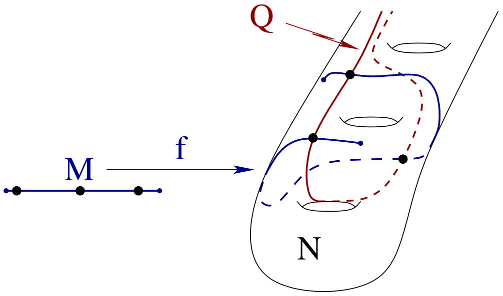

Figure 4.2: The intersection number modulo two.

Proof. Since $A := F^{-1}(Q)$ is a compact subset of $W := [0,1] \times (M \setminus \partial M)$, there exists an open subset $U \subset [0,1] \times M$ such that $A \subset U \subset \overline{U} \subset W$. Now W is a noncompact manifold with boundary $\partial W = \{0,1\} \times (M \setminus \partial M)$ and the homotopy F restricts to a smooth map $F : W \to N$ such that $F|_{\partial W}$ is transverse to Q. Hence it follows from part (ii) of Theorem 4.1.4 that there exists a smooth map $G : W \to N$ such that G is transverse to Q and

$$G|_{\partial W \cup (W \setminus U)} = F|_{\partial W \cup (W \setminus U)}.$$

This map G extends to a smooth homotopy from f_0 to f_1 on all of $[0,1] \times M$ that satisfies $G(t,p) = F(t,p)$ for all $(t,p) \in [0,1] \times \partial M$.

Since G is continuous, the set $X := G^{-1}(Q) \subset [0,1] \times M$ is compact. Since $G([0,1] \times \partial M) \cap Q = \emptyset$, we have

$$X = G^{-1}(Q) \subset [0,1] \times (M \setminus \partial M) = W.$$

Since $G|_W$ and $G|_{\partial W}$ are transverse to Q, it follows from Lemma 4.1.2 that X is a 1-dimensional submanifold of W with boundary

$$\partial X = X \cap \partial W = \big(\{0\} \times f_0^{-1}(Q)\big) \cup \big(\{1\} \times f_1^{-1}(Q)\big).$$

Hence $\#f_0^{-1}(Q) + \#f_1^{-1}(Q) = \#\partial X \in 2\mathbb{Z}$ by Theorem A.4.1 and this proves Lemma 4.2.2. □

Proof of Theorem 4.2.1. Part (i) follows directly from Corollary 4.1.5.

We prove part (ii). Assume that $g, h : M \to N$ are both transverse to Q and homotopic to f relative to the boundary. Then g is homotopic to h relative to the boundary and hence $\# g^{-1}(Q) \equiv \# h^{-1}(Q)$ (modulo 2) by Lemma 4.2.2. This proves (ii).

We prove part (iii). For $i = 0, 1$ it follows from (i) that there exists a smooth map $g_i : M \to N$ such that g_i is transverse to Q and homotopic to f_i relative to the boundary. Compose the homotopies to obtain a smooth homotopy $G : [0,1] \times M \to N$ from g_0 to g_1 with

$$G([0,1] \times \partial M) \cap Q = \emptyset.$$

Then

$$\# g_0^{-1}(Q) \equiv \# g_1^{-1}(Q) \text{ (modulo 2)}$$

by Lemma 4.2.2 and this proves (iii).

We prove part (iv). Corollary 4.1.5 asserts that there exists a smooth map $G : W \to N$ such that G is homotopic to F and both G and $G|_{\partial W}$ are transverse to Q. By Lemma 4.1.2 the set

$$X := G^{-1}(Q) \subset W$$

is a compact 1-dimensional submanifold with boundary

$$\partial X = X \cap \partial W = (G|_\partial W)^{-1}(Q).$$

Hence $\#(G|_\partial W)^{-1}(Q)$ is an even number by Theorem A.4.1. Since $F|_{\partial W}$ is smoothly homotopic to $G|_{\partial W}$ it follows that $I_2(F|_{\partial W}, Q) = 0$. This proves Theorem 4.2.1. □

Example 4.2.3. Let $N = \mathbb{R}\mathrm{P}^n$ be the real projective space and fix an integer $0 < m < n$. Define the inclusion $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ by

$$f([x_0 : \cdots : x_m]) := ([x_0 : \cdots : x_m : 0 : \cdots : 0])$$

for $[x_0 : \cdots : x_m] \in \mathbb{R}\mathrm{P}^m$ and consider the submanifold

$$Q := \left\{ [x_0 : x_1 : \cdots : x_n] \in \mathbb{R}\mathrm{P}^n \,\middle|\, x_0 = \cdots = x_{m-1} = 0 \right\}.$$

Then f is transverse to Q and $I_2(f, Q) = 1$. Hence f is not homotopic to a constant map. With $m = 1$ this shows that $\mathbb{R}\mathrm{P}^n$ is not simply connected.

Exercise 4.2.4. *Let $M \subset \mathbb{R}^n$ be a compact connected smooth codimension-1 submanifold without boundary. Then $\mathbb{R}^n \setminus M$ has two connected components and M is orientable.* (Compare Exercise 2.4.10.)

Step 1. *There exists a constant $\varepsilon > 0$ such that $p + v \notin M$ for all $p \in M$ and all $v \in T_pM^\perp$ with $0 < |v| \le \varepsilon$, and the set*

$$U_\varepsilon := \{p + v \,|\, p \in M,\ v \in T_pM^\perp,\ |v| < \varepsilon\}$$

is an open neighborhood of M. **Hint:** This is a special case of the Tubular Neighborhood Theorem 4.3.8 below. It can be proved directly as follows. Let $V \subset \mathbb{R}^n$ be an open set and let $f : V \to \mathbb{R}$ be a smooth function such that zero is a regular value of f and

$$f^{-1}(0) = V \cap M =: W.$$

Define the normal vector field $X : W \to \mathbb{R}^n$ by

$$X := \frac{\nabla f}{|\nabla f|}.$$

Show the map $W \times \mathbb{R} \to \mathbb{R}^n : (p, t) \mapsto p + tX(p)$ restricts to a diffeomorphism from $W \times (-\varepsilon, \varepsilon)$ onto an open subset of $\mathbb{R}^n$ for some $\varepsilon > 0$ (after shrinking W if necessary). Cover M by finitely many such open sets V.

Step 2. *Let $p \in M$, let $v \in T_pM^\perp \cap S^{n-1}$, and let $\varepsilon > 0$ be as in Step 1. Define the curve $\gamma : [-\varepsilon, \varepsilon] \to \mathbb{R}^n$ by $\gamma(t) = p + tv$. Then $I_2(\gamma, M) = 1$ and hence $p + \varepsilon v$ and $p - \varepsilon v$ cannot be joined by a curve in $\mathbb{R}^n \setminus M$.*

Step 3. *Let $p_0, p_1 \in M$. Then there exist smooth curves*

$$\gamma : [0, 1] \to M, \qquad v : [0, 1] \to S^{n-1}$$

such that $\gamma(0) = p_0$, $\gamma(1) = p_1$, and $v(t) \perp T_{\gamma(t)}M$ for $0 \le t \le 1$. **Hint:** Use parallel transport in the normal bundle (see [35, §3.3]).

Step 4. *Let U_ε be as in Step 1. Then $U_\varepsilon \setminus M$ has precisely two connected components.* **Hint:** By Step 2 the set $U_\varepsilon \setminus M$ has at least two connected components and by Step 3 it has at most two connected components.

Step 5. *The set $\mathbb{R}^n \setminus M$ has precisely two connected components.* **Hint:** Every element of $\mathbb{R}^n \setminus M$ can be joined to $U_\varepsilon \setminus M$ by a curve in $\mathbb{R}^n \setminus M$.

Step 6. *There exists a smooth map $X : M \to S^{n-1}$ such that $X(p) \perp T_pM$ for all $p \in M$. Hence M is orientable.*

Exercise 4.2.5. Let N be a connected manifold without boundary and let $M \subset N$ be a compact connected codimension-1 submanifold without boundary. Find an example where $N \setminus M$ is connected. If N is simply connected, show that $N \setminus M$ has two connected components.

4.2.2 Orientation and Intersection Numbers

Let M and N be oriented smooth manifolds and let $Q \subset N$ be an oriented submanifold with $\dim(M) = m$, $\dim(N) = n$, and $\dim(Q) = n - k$. The next definition shows how the orientations of M, Q, N induce an orientation of the manifold $f^{-1}(Q)$ whenever $f : M \to N$ is tranverse to Q.

Definition 4.2.6 (Orientation). *Let $f : M \to N$ be a smooth map that is transverse to Q. The manifold $P := f^{-1}(Q) \subset M$ is* **oriented** *by a map which assigns to every basis of every tangent space of P a sign $\nu \in \{\pm 1\}$. Let $p \in P$ and fix a basis $v_1, \dots, v_{m-k}$ of T_pP. The sign*

$$\nu(p; v_1, \dots, v_{m-k}) \in \{\pm 1\}$$

is defined as follows. Choose tangent vectors $v_{m-k+1}, \dots, v_m \in T_pM$ such that the vectors $v_1, \dots, v_m$ form a positive basis of T_pM and choose a positive basis $w_{k+1}, \dots, w_n$ of $T_{f(p)}Q$. Then define

$$\nu(p; v_1, \dots, v_{m-k}) := \begin{cases} +1, & \begin{array}{l}\text{if the vectors } w_1, \dots, w_n, \text{ with} \\ w_i := df(p)v_{m-k+i} \text{ for } 1 \le i \le k, \\ \text{form a positive basis of } T_{f(p)}N, \end{array} \\ -1, & \text{otherwise.} \end{cases} \tag{4.2.4}$$

If $k = 0$, then $Q \subset N$ and $P \subset M$ are open sets and the sign is determined by the orientation of T_pM. If $k \in \{m, n\}$, the sign is understood as follows.

Case 1: *$k = m < n$. In this case P is a zero-dimensional submanifold of M, there is only the 'empty basis' of $T_pP = \{0\}$, and the sign is denoted by $\nu(p)$. Thus $\nu(p) = +1$ if and only if signs match in $T_{f(p)}N = \mathrm{im}\,(df(p)) \oplus T_{f(p)}Q$.*

Case 2: *$k = m = n$. In this case $Q \subset N$ and $P \subset M$ are zero-dimensional submanifolds, the orientation of Q is a function $\varepsilon : Q \to \{\pm 1\}$, the derivative $df(p) : T_pM \to T_{f(p)}N$ is a vector space isomorphism, and*

$$\nu(p) := \begin{cases} +\varepsilon(f(p)), & \begin{array}{l}\text{if } df(p) : T_pM \to T_{f(p)}N \\ \text{is orientation preserving,}\end{array} \\ -\varepsilon(f(p)), & \text{otherwise.} \end{cases} \tag{4.2.5}$$

Note that this formula is consitent with Case 1 and equation (4.2.4).

Case 3: *$k = n < m$. In this case Q has dimension zero and the orientation is a map $\varepsilon : Q \to \{\pm 1\}$. Now choose $v_{m-n+1}, \dots, v_m \in T_pM$ such that $v_1, \dots, v_m$ form a positive basis of T_pM. Then*

$$\nu(p, v_1, \dots, v_{m-k}) := \begin{cases} +\varepsilon(f(p)), & \begin{array}{l}\text{if } df(p)v_{m-n+1}, \dots, df(p)v_m \\ \text{is a positive basis of } T_{f(p)}N,\end{array} \\ -\varepsilon(f(p)), & \text{otherwise.} \end{cases} \tag{4.2.6}$$

The next definition introduces the intersection index of a transverse intersection in the case of complementary dimensions.

Definition 4.2.7 (Intersection Index). *Let M be a compact oriented m-manifold with boundary, let N be an oriented n-manifold without boundary, and let $Q \subset N$ be oriented $(n-m)$-dimensional submanifold without boundary that is closed as a subset of N. Let $f : M \to N$ be a smooth map that satisfies $f(\partial M) \cap Q = \emptyset$ and is transverse to Q. Fix an element $p \in f^{-1}(Q) \subset M \setminus \partial M$. Then*

$$T_{f(p)}N = \mathrm{im}\,(df(p)) \oplus T_{f(p)}Q.$$

and the **intersection index of f and Q at** *p is defined as the sign $\nu(p; f, Q)$ obtained by comparing orientations in this decomposition. Thus*

$$\nu(p; f, Q) := \begin{cases} +1, & \text{if } df(p)v_1, \dots, df(p)v_m, w_{m+1}, \dots w_n \\ & \text{is a positive basis of } T_{f(p)}N \\ & \text{for every positive basis } v_1, \dots, v_m \text{ of } T_pM \\ & \text{and every positive basis } w_{m+1}, \dots, w_n \text{ of } T_{f(p)}Q, \\ -1, & \text{otherwise.} \end{cases}$$

This corresponds to Case 1 in Definition 4.2.6.

Theorem 4.2.8 (Intersection Number). *Let M and $Q \subset N$ be as in Definition 4.2.7 and let $f : M \to N$ be a smooth map with $f(\partial M) \cap Q = \emptyset$. Then the following holds.*

(i) *There exists a smooth map $g : M \to N$ that is transverse to Q and homotopic to f relative to the boundary.*

(ii) *Let g be as in (i). Then the integer $I(g, Q) := \sum_{p \in g^{-1}(Q)} \nu(p; g, Q)$ is independent of the choice of g. It is called the* **intersection number of f and Q** *and is denoted by*

$$I(f, Q) := f \cdot Q := \sum_{p \in g^{-1}(Q)} \nu(p; g, Q) \qquad \text{for } g \overset{\partial M}{\sim} f \text{ with } g \pitchfork Q. \tag{4.2.7}$$

(iii) *Let $f_0, f_1 : M \to N$ be smooth maps satisfying $f_i(\partial M) \cap Q = \emptyset$ and let $F : [0,1] \times M \to N$ be a smooth homotopy from f_0 to f_1 such that $F([0,1] \times \partial M) \cap Q = \emptyset$. Then $I(f_0, Q) = I(f_1, Q)$.*

(iv) *Let W be a compact oriented $(m+1)$-manifold with boundary and let $F : W \to N$ be a smooth map. Then $I(F|_{\partial W}, Q) = 0$.*

The proof relies on the following two lemmas.

Lemma 4.2.9 (Vanishing). *Let* W *be an oriented smooth* $(m+1)$*-manifold with boundary and let* $F : W \to N$ *be a smooth map such that* F *and* $F|_{\partial W}$ *are transverse to* Q*. Assume that the set* $F^{-1}(Q) \subset W$ *is compact. Then the intersection* $F^{-1}(Q) \cap \partial W$ *is a finite set and*

$$\sum_{p \in F^{-1}(Q)\cap\partial W} \nu(p; F|_{\partial W}, Q) = 0.$$

Here we assume that ∂W *is oriented a the boundary of* W*.*

Proof. By Lemma 4.1.2, the set

$$X := F^{-1}(Q) \subset W$$

is a compact oriented smooth 1-manifold with boundary

$$\partial X = X \cap \partial W = (F|_{\partial W})^{-1}(Q).$$

Thus X is a finite union of circles and arcs by Theorem A.4.1. Let $A \subset X$ be an arc and choose an orientation preserving diffeomorphism $\gamma : [0,1] \to A$. Then $\gamma(0), \gamma(1) \in \partial W$, the vector $\dot\gamma(0)$ points into W, and $\dot\gamma(1)$ points out of W. Let $v_1, \dots, v_m$ be a positive basis of $T_{\gamma(1)}\partial W$ and let $w_{m+1}, \dots, w_n$ be a positive basis of $T_{F(\gamma(1))}Q$. Since $\dot\gamma(1)$ is outward pointing, it follows from the definition of the boundary orientation that $\dot\gamma(1), v_1, \dots, v_m$ is a positive basis of $T_{\gamma(1)}W$. Since $\dot\gamma(1)$ is a positive tangent vector in $T_{\gamma(t)}X$ it follows from the sign convention in Definition 4.2.6 that the vectors

$$dF(\gamma(1))v_1, \dots, dF(\gamma(1))v_m, w_{m+1}, \dots, w_n$$

form a positive basis of $T_{F(\gamma(1))}N$. Hence it follows from the definition of the intersection index in Definition 4.2.7 that

$$\nu(\gamma(1); F|_{\partial W}, Q) = +1.$$

Since $\dot\gamma(0)$ points in to W, the same argument shows that

$$\nu(\gamma(0); F|_{\partial W}, Q) = -1.$$

Thus $\nu(\gamma(0); F|_{\partial W}, Q) + \nu(\gamma(1); F|_{\partial W}, Q) = 0$. Since this holds for the endpoints of every arc $A \subset X$, we obtain

$$\sum_{p \in F^{-1}(Q)\cap\partial W} \nu(p; F|_{\partial W}, Q) = 0.$$

This proves Lemma 4.2.9. □

Lemma 4.2.10 (Homotopy). *Let M and $Q \subset N$ be as in Definition 4.2.7 and let $f_0, f_1 : M \to N$ be smooth maps that satisfy* (4.2.1)*, are transverse to Q, and are smoothly homotopic by a homotopy that satisfies* (4.2.3)*. Then*

$$\sum_{p \in f_0^{-1}(Q)} \nu(p; f_0, Q) = \sum_{p \in f_1^{-1}(Q)} \nu(p; f_1, Q).$$

Proof. By Lemma 4.2.2 there exists a smooth homotopy $F : [0,1] \times M \to N$ from f_0 to f_1 that satisfies (4.2.3) and is transverse to Q. Thus $F^{-1}(Q)$ is compact and contained in the set $W := [0,1] \times (M \setminus \partial M)$. This set is an oriented $(m+1)$-manifold with boundary $\partial W = \{0,1\} \times M$. The boundary orientation of ∂W agrees with the orientation of M at $t = 1$ and is opposite to the orientation of M at $t = 0$. Moreover, $F|_{\partial W}$ is transverse to Q by assumption. Hence it follows from Lemma 4.2.9 that

$$\begin{aligned} 0 &= \sum_{(t,p) \in F^{-1}(Q) \cap \partial W} \nu((t,p); F|_{\partial W}, Q) \\ &= \sum_{p \in f_1^{-1}(Q)} \nu(p; f_1, Q) - \sum_{p \in f_0^{-1}(Q)} \nu(p; f_0, Q). \end{aligned}$$

This proves Lemma 4.2.10. □

Proof of Theorems 4.2.8. Part (i) follows directly from Corollary 4.1.5.

We prove part (ii). Assume that $g, h : M \to N$ are both transverse to Q and homotopic to f relative to the boundary. Then g is homotopic to h relative to the boundary and hence

$$\sum_{p \in g^{-1}(Q)} \nu(p; g, Q) = \sum_{p \in h^{-1}(Q)} \nu(p; h, Q).$$

by Lemma 4.2.10. This proves (ii).

We prove part (iii). For $i = 0, 1$ it follows from (i) that there exists a smooth map $g_i : M \to N$ such that g_i is transverse to Q and homotopic to f_i relative to the boundary. Compose the homotopies to obtain a smooth homotopy $G : [0,1] \times M \to N$ from g_0 to g_1 with $G([0,1] \times \partial M) \cap Q = \emptyset$. Then $I(f_0, Q) = I(g_0, Q) = I(g_1, Q) = I(f_1, Q)$ by Lemma 4.2.10 and this proves (iii).

We prove part (iv). Corollary 4.1.5 asserts that there exists a smooth map $G : W \to N$ such that G and $G|_{\partial W}$ are transverse to Q and G is homotopic to F. Then $F|_{\partial W}$ is homotopic to $G|_{\partial W}$ and $G^{-1}(Q)$ is compact because W is compact. Hence $I(F|_{\partial W}, Q) = I(G|_{\partial W}, Q) = 0$ by Lemma 4.2.9. This proves Theorem 4.2.8. □

Exercise 4.2.11. Let P, Q, N be compact oriented smooth manifolds without boundary such that

$$\dim(P) + \dim(Q) = \dim(N)$$

and let $f : P \to N$ and $g : Q \to N$ be smooth maps. The map f is called **transverse to** g if every pair $(p, q) \in P \times Q$ with $f(p) = g(q)$ satisfies

$$T_{f(p)}N = \operatorname{im}(df(p)) \oplus \operatorname{im}(dg(q)). \tag{4.2.8}$$

In the transverse case the intersection index $\nu(p, q; f, g) \in \{\pm 1\}$ is defined to be ± 1 according to whether or not the orientations match in the direct sum (4.2.8), and the **intersection number of** f **and** g is defined by

$$I(f, g) := f \cdot g := \sum_{f(p)=g(q)} \nu(p, q; f, g). \tag{4.2.9}$$

(i) Prove that every smooth map $f : P \to N$ is smoothly homotopic to a map $f' : P \to N$ that is transverse to g.

(ii) If $f_0, f_1 : P \to N$ are transverse to g, prove that $I(f_0, g) = I(f_1, g)$. Deduce that the intersection number $I(f, g)$ is well defined for every pair of smooth maps $f : P \to N$ and $g : Q \to N$, transverse or not.

(iii) Prove that

$$I(g, f) = (-1)^{\dim(P)\dim(Q)} I(f, g). \tag{4.2.10}$$

(iv) Define the map $f \times g : P \times Q \to N \times N$ by $(f \times g)(p, q) := (f(p), g(q))$ for $p \in P$ and $q \in Q$ and let $\Delta \subset N \times N$ be the diagonal. Prove that

$$I(f, g) = (-1)^{\dim(Q)} I(f \times g, \Delta). \tag{4.2.11}$$

Exercise 4.2.12. Let $N := \mathbb{C}\mathrm{P}^2$. A smooth map $f : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ is called a **polynomial map of degree** $\deg(f) = d$ iff it has the form

$$f([z_0 : z_1]) = [f_0(z_0, z_1) : f_1(z_0, z_2) : f_2(z_0, z_1)],$$
$$f_i(z_0, z_1) = \sum_{j=0}^{d} a_{ij} z_0^j z_1^{d-j},$$

with $a_{ij} \in \mathbb{C}$ and the homogeneous polynomials $f_i : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$ have no common zeros. Let $f, g : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ be polynomial maps. Prove that

$$f \cdot g = \deg(f)\deg(g).$$

Hint: Show that any two polynomial maps from $\mathbb{C}\mathrm{P}^1$ to $\mathbb{C}\mathrm{P}^2$ of degree d are smoothly homotopic. Consider the examples $f([z_0 : z_1]) = [z_0^d - z_1^d : 0 : z_1^d]$ and $g([z_0 : z_1]) = [0 : z_0^{d'} - z_1^{d'} : z_1^{d'}]$ and show that f is transverse to g.

4.2.3 Isolated Intersections

In this subsection we assign an intersection index to each isolated intersection which agrees with the index in Definition 4.2.7 in the transverse case.

Definition 4.2.13 (The Index of an Isolated Intersection). *Let M be a compact oriented m-manifold with boundary, let N be an oriented n-manifold without boundary, let $Q \subset N$ be an oriented codimension-m submanifold without boundary that is closed as a subset of N, and let $f : M \to N$ be a smooth map such that $f(\partial M) \cap Q = \emptyset$. An element $p_0 \in M \setminus \partial M$ is called an* **isolated intersection of f and Q** *iff $f(p_0) \in Q$ and there exists an open neighborhood U of p_0 such that $f(p) \notin Q$ for all $p \in U \setminus \{p_0\}$.*

Let $p_0 \in M$ be an isolated intersection. The intersection index of f and Q at p_0 is defined as follows. Choose an orientation preserving diffeomorphism $\psi : V \to \mathbb{R}^n$, defined on an open neighborhood $V \subset N$ of $f(p_0)$ such that $\psi(V \cap Q) = \{0\} \times \mathbb{R}^{n-m}$ and the map

$$V \cap Q \to \mathbb{R}^{n-m} : q \mapsto (\psi_{m+1}(q), \dots, \psi_n(q))$$

is an orientation preserving diffeomorphism. Choose an orientation preserving diffeomorphism $\phi : U \to \mathbb{R}^m$, defined on an open neighborhood $U \subset M$ of p_0 such that $f(U) \subset V$ and $U \cap f^{-1}(Q) = \{p_0\}$. Then the integer

$$\begin{aligned} \nu(p_0; f, Q) &:= \deg\left(S^{m-1} \to S^{m-1} : x \mapsto \frac{\xi(x_0 + \varepsilon x)}{|\xi(x_0 + \varepsilon x)|}\right), \\ \xi &:= (\psi_1, \dots, \psi_m) \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^m, \\ x_0 &:= \phi(p_0), \qquad \varepsilon > 0, \end{aligned} \tag{4.2.12}$$

is called the **intersection index of f and Q at** *p_0 (see Figure 4.3).*

Theorem 4.2.14. *Let M, Q, N, and $f : M \to N$ be as in Definition 4.2.13. Then the following holds.*

(i) *The intersection index of f and Q at an isolated intersection p_0 is independent of the choice of the coordinate charts ϕ and ψ used to define it.*

(ii) *If f and Q intersect transversally at p_0, then the intersection index in Definition 4.2.13 agrees with the intersection index in Definition 4.2.7.*

(iii) *If f and Q have only isolated intersections, then*

$$\sum_{p \in f^{-1}(Q)} \nu(p; f, Q) = I(f, Q). \tag{4.2.13}$$

The proof will be based on the following perturbation result.

Lemma 4.2.15 (Perturbation). *Let M, Q, N, f be as in Definition 4.2.13. Let $p_0 \in M \setminus \partial M$ be an isolated intersection of f and Q and let $U \subset M$ be an open neighborhood of p_0 such that $\overline{U} \cap f^{-1}(Q) = \{p_0\}$ and $\overline{U} \cap \partial M = \emptyset$. Let $\nu(p_0; f, Q)$ be the integer in (4.2.12) associated to coordinate charts ϕ and ψ as in Definition 4.2.13. Then there exists a smooth map $g : M \to N$, homotopic to f relative to $M \setminus U$, such that $g|_U$ is transverse to Q and*

$$\nu(p_0; f, Q) = \sum_{p \in U \cap g^{-1}(Q)} \nu(p; g, Q). \tag{4.2.14}$$

Here the summands on the right are the indices in Definition 4.2.7.

Proof. Shrinking U, if necessary, we may assume that there exist coordinate charts $\phi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^n$ as in Definition 4.2.13. The resulting map $\xi := (\psi_1, \dots, \psi_m) \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^m$ is a vector field on $\mathbb{R}^m$ with an isolated zero at $x_0 = \phi(p_0)$ and no other zeros. Also, the index in (4.2.12) agrees with the index of the isolated zero x_0 of ξ in Definition 2.2.4, i.e.

$$\nu(p_0; f, Q) = \iota(x_0, \xi). \tag{4.2.15}$$

We prove the following.

Claim 1: *p_0 is a transverse intersection of f and Q if and only if the Jacobi matrix $d\xi(x_0) \in \mathbb{R}^{m \times m}$ is nonsingular.*

Claim 2: *If p_0 is a transverse intersection of f and Q, then the intersection index in Definition 4.2.13 is given by $\nu(p_0; f, Q) = \operatorname{sign}(\det(d\xi(x_0)))$ and agrees with the intersection index in Definition 4.2.7.*

To see this, observe that the transversality condition

$$\operatorname{im}(df(p_0)) \oplus T_{f(p_0)}Q = T_{f(p_0)}N \tag{4.2.16}$$

in local coordinates takes the form

$$\operatorname{im}\big(d(\psi \circ f \circ \phi^{-1})(x_0)\big) \oplus \big(\{0\} \times \mathbb{R}^{n-m}\big) = \mathbb{R}^n.$$

This holds if and only if the linear map $d\xi(x_0) : \mathbb{R}^m \to \mathbb{R}^m$ is bijective, which proves Claim 1. To prove Claim 2, assume (4.2.16). Then it follows from (4.2.15) and Lemma 2.2.13 that

$$\begin{aligned} \nu(p_0; f, Q) &= \iota(x_0, \xi) = \operatorname{sign}\big(\det(d\xi(x_0))\big) \\ &= \begin{cases} +1, & \text{if } d\xi(x_0) \text{ is orientation preserving,} \\ -1, & \text{if } d\xi(x_0) \text{ is orientation reversing.} \end{cases} \end{aligned}$$

This sign is +1 if and only if the orientations match in the direct sum decomposition (4.2.16) and this proves Claim 2.

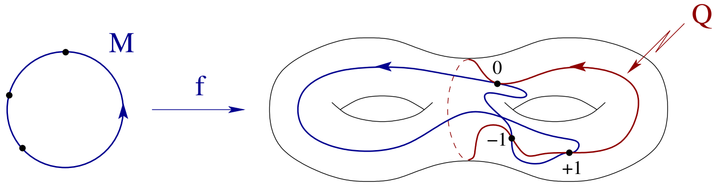

Figure 4.3: The intersection index at isolated intersections.

By Lemma 2.3.5 there exists a vector field $\xi' : \mathbb{R}^m \to \mathbb{R}^m$ with only nondegenerate zeros such that $\xi'(x) = \xi(x)$ for all $x \in \mathbb{R}^m$ with $|x - x_0| \geq 1$ and

$$\iota(x_0, \xi) = \sum_{\xi'(x)=0} \operatorname{sign}\bigl(\det(d\xi'(x))\bigr). \qquad (4.2.17)$$

Let $\eta := (\psi_{m+1}, \dots, \psi_n) \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^{n-m}$ and define $f_t : M \to N$ by

$$f_t|_{M \setminus U} = f|_{M \setminus U}, \qquad f_t|_U := \psi^{-1} \circ ((1-t)\xi + t\xi', \eta) \circ \phi$$

for $0 \leq t \leq 1$. Then $(\psi_1, \dots, \psi_m) \circ f_1 \circ \phi^{-1} = \xi' : \mathbb{R}^m \to \mathbb{R}^m$. Hence $f_1|_U$ intersects Q transversally by Claim 1. Thus by (4.2.15), (4.2.17), and Claim 2,

$$\nu(p_0; f, Q) = \iota(x_0, \xi) = \sum_{\xi'(x)=0} \operatorname{sign}(\det(d\xi'(x))) = \sum_{p \in U \cap f_1^{-1}(Q)} \nu(p; f_1, Q).$$

This proves Lemma 4.2.15 with $g = f_1$. □

Proof of Theorem 4.2.14. Let p_0 be an isolated intersection of f and Q and choose an open neighborhood $U \subset M$ of p_0 such that $\overline{U}$ is diffeomorphic to a closed ball, $\overline{U} \cap f^{-1}(Q) = \{p_0\}$, and $\overline{U} \cap \partial M = \emptyset$. By Lemma 4.2.15 any two coordinate charts ϕ and ψ as in Definition 4.2.13 give rise to map $g = g_{\phi,\psi}$ that is homotopic to f relative to $M \setminus U$ such that $g|_U$ is transverse to Q and satisfies (4.2.14). By part (ii) of Theorem 4.2.8 with M replaced by $\overline{U}$ the right hand side of equation (4.2.14) is independent of the choice of g. Hence the left hand side of (4.2.14) is independent of the choice of the local coordinate charts ϕ and ψ used to define it. That it agrees with the intersection index in Definition 4.2.7 in the transverse case follows by taking $g = f$. Now assume that f and Q have only isolated intersections. Then by Lemma 4.2.15 there exists a smooth map $g : M \to N$ that is transverse to Q and homotopic to f relative to the boundary such that

$$\sum_{p \in f^{-1}(Q)} \nu(p; f, Q) = \sum_{p \in g^{-1}(Q)} \nu(p; g, Q) = I(g, Q) = I(f, Q).$$

This proves Theorem 4.2.14. □

4.3 Self-Intersection Numbers

In §4.2.2 we have defined the intersection number $I(f, Q) \in \mathbb{Z}$ of a smooth map $f : P \to N$ with a smooth submanifold $Q \subset N$ A special case arises when P is a submanifold of N and $f : P \to N$ is the inclusion.

Definition 4.3.1 (**Intersection Number of two Submanifolds**). *Let N be an oriented n-manifold without boundary and let $P, Q \subset N$ be compact oriented submanifolds without boundary satisfying the dimension condition*

$$\dim(P) + \dim(Q) = \dim(N). \tag{4.3.1}$$

The **intersection number of** P **and** Q *is the integer*

$$P \cdot Q := I(P, Q) := I(\iota_P, Q) \in \mathbb{Z}, \tag{4.3.2}$$

where $\iota_P : P \to N$ denotes the canonical inclusion.

If P is transverse to Q (see Example 4.1.1), then $P \cap Q$ is a finite set. In this case the **intersection index of** P **and** Q **at** $q \in P \cap Q$ is the number $\nu(q; P, Q) \in \{\pm 1\}$, defined by

$$\nu(q; P, Q) := \begin{cases} +1, & \text{if } w_1, \dots w_n \text{ is a positive basis of } T_qN \\ & \text{whenever } w_1, \dots, w_m \text{ is a positive basis of } T_qP \\ & \text{and } w_{m+1}, \dots, w_n \text{ is a positive basis of } T_qQ, \\ -1, & \text{otherwise.} \end{cases}$$

Here $m := \dim(P)$. In the transverse case the intersection number is the sum of the intersection indices of the intersection points of P and Q, i.e.

$$I(P, Q) = \sum_{q \in P \cap Q} \nu(q; P, Q). \tag{4.3.3}$$

However, the intersection number is also well defined when P and Q do not intersect transversally. In this case it is given by $I(P, Q) = I(f, Q)$, where $f : P \to N$ is any smooth map that is transverse to Q and smoothly homotopic to the canonical inclusion $\iota_P : P \to N$. Such a map exists by Corollary 4.1.5 and the intersection number is independent of the choice of f by Theorem 4.2.8. In particular, the intersection number is well-defined in the case $P = Q$.

Definition 4.3.2 (**Self-Intersection Number**). *Let N be a compact oriented $2m$-dimensional manifold without boundary and let $Q \subset N$ be a compact oriented m-dimensional submanifold without boundary. The* **self-intersection number of** Q *is the integer $Q \cdot Q = I(Q, Q) \in \mathbb{Z}$.*

4.3.1 Self-Intersections and the Normal Bundle

It follows from equation (4.3.3) that the intersection numbers satisfy the symmetry condition

$$Q\cdot P = (-1)^{\dim(P)\dim(Q)}P\cdot Q \tag{4.3.4}$$

in the situation of Definition 4.3.1. Hence the self-intersection number $Q\cdot Q$ vanishes whenever the dimension $\dim(Q)=\frac{1}{2}\dim(N)$ is odd.

The next goal is to show that the self-intersection number of Q is the algebraic count of the zeros of a section of the normal bundle, in analogy with the Poincaré–Hopf theorem. To make this precise, we first consider the general case where N is a smooth n-manifold without boundary and $Q\subset N$ is a smooth m-dimensional submanifold without boundary. Choose a Riemannian metric on N and define the **normal bundle of** Q by

$$\begin{aligned} TQ^\perp &:= \left\{(q,w)\,\middle|\,q\in Q,\ w\in T_qQ^\perp\right\},\\ T_qQ^\perp &:= \left\{w\in T_qN\,\middle|\,\langle w,v\rangle = 0 \text{ for all } v\in T_qQ\right\}. \end{aligned} \tag{4.3.5}$$

Denote by $\pi: TQ^\perp\to Q$ the canonical projection given by $\pi(q,w):=q$ for $(q,w)\in TQ^\perp$. The normal bundle is a smooth submanifold of the tangent bundle TN and is a vector bundle over Q (see Exercise 4.3.4). A **normal vector field on** Q is a section of the normal bundle, i.e. it is a smooth map $Y:Q\to TQ^\perp$ whose composition with the projection $\pi:TQ^\perp\to Q$ is the identity. Denote the space of normal vector fields on Q by

$$\mathrm{Vect}^\perp(Q) := \{Y:Q\to TN\,|\,Y \text{ is smooth and } \pi\circ Y=\mathrm{id}\}.$$

Thus a normal vector field $Y\in\mathrm{Vect}^\perp(Q)$ assigns to an element $q\in Q$ a pair $Y(q)=(q,w)$ with $w\in T_qQ^\perp$. Slightly abusing notation, it is often convenient to discard the first component and write $Y(q)=w\in T_qQ^\perp$. In this notation a normal vector field is a *natural transformation* which assigns to each element $q\in Q$ a normal vector $Y(q)\in T_qQ^\perp$ such that the map $Q\to TQ^\perp: q\mapsto(q,Y(q))$ is smooth. If $N\subset\mathbb{R}^k$ is an embedded submanifold of the Euclidean space $\mathbb{R}^k$ for some k and the Riemannian metric is determined by the inner product on $\mathbb{R}^k$, then a normal vector field on Q is a smooth map $Y:Q\to\mathbb{R}^k$ such that $Y(q)\in T_qN\cap T_qQ^\perp$ for all $q\in Q$. (In the embedded case the notation $T_qQ^\perp$ refers to the orthogonal complement in the ambient space $\mathbb{R}^k$ and so has a different meaning than in (4.3.5).)

In the following we denote by ∇ the Levi-Civita connection of the Riemannian metric on N (see [35, Chapter 3]).

Lemma 4.3.3 (Vertical Derivative). *Let $Y \in \mathrm{Vect}^\perp(Q)$ and let $q_0 \in Q$ such that $Y(q_0) = 0$. Then there exists a unique linear map*

$$DY(q_0) : T_{q_0}Q \to T_{q_0}Q^\perp,$$

called the **vertical derivative of** Y **at** q_0*, such that every $v \in T_{q_0}Q$ and every smooth curve $\gamma : \mathbb{R} \to Q$ with $\gamma(0) = q_0$ and $\dot\gamma(q_0) = v$ satisfies*

$$DY(q_0)v = \nabla_t(Y \circ \gamma)(0). \tag{4.3.6}$$

Proof. Choose a coordinate chart $\psi : U \to \Omega \subset \mathbb{R}^n$ on an open neighborhood $U \subset N$ of q_0 such that $\psi(U \cap Q) = \Omega \cap (\mathbb{R}^m \times \{0\})$. Let $g : \Omega \to \mathbb{R}^{n\times n}$ be the metric tensor and write it in the form

$$g(x) = \begin{pmatrix} a(x) & b(x) \\ b(x)^T & d(x) \end{pmatrix} \qquad \text{for } x \in \Omega, \tag{4.3.7}$$

where $a(x) \in \mathbb{R}^{m\times m}$, $b(x) \in \mathbb{R}^{m\times(n-m)}$, and $d(x) \in \mathbb{R}^{(n-m)\times(n-m)}$. Define

$$\Omega' := \{x \in \mathbb{R}^m \,|\, (x, 0) \in \Omega\}.$$

Then, for $x \in \Omega'$ and $q := \psi^{-1}(x, 0) \in U \cap Q$, we have

$$d\psi(q)T_qQ^\perp = \left\{ \begin{pmatrix} -a(x,0)^{-1}b(x,0)\eta \\ \eta \end{pmatrix} \,\middle|\, \eta \in \mathbb{R}^{n-m} \right\}. \tag{4.3.8}$$

Hence there exists a smooth map $\eta : \Omega' \to \mathbb{R}^{n-m}$ such that, for all $x \in \Omega'$,

$$d\psi(q)Y(q) = \begin{pmatrix} -a(x,0)^{-1}b(x,0)\eta(x) \\ \eta(x) \end{pmatrix}, \qquad q := \psi^{-1}(x, 0). \tag{4.3.9}$$

Let $x_0 \in \Omega'$ such that $(x_0, 0) := \psi(q_0)$. Then $\eta(x_0) = 0$ and so, for $v \in T_{q_0}Q$ and $\xi \in \mathbb{R}^m$ with $(\xi, 0) := d\psi(q_0)v$, equation (4.3.6) takes the form

$$d\psi(q_0)DY(q_0)v = \begin{pmatrix} -a(x_0,0)^{-1}b(x_0,0)d\eta(x_0)\xi \\ d\eta(x_0)\xi \end{pmatrix}. \tag{4.3.10}$$

Hence the right hand side of (4.3.6) defines an element $DY(q_0)v \in T_{q_0}Q^\perp$ that is independent of the choice of γ and the map $DY(q_0) : T_{q_0}Q \to T_{q_0}Q^\perp$ is linear. This proves Lemma 4.3.3. □

Exercise 4.3.4. Verify the formula (4.3.8) for the normal bundle in the proof of Lemma 4.3.3. Deduce that $TQ^\perp$ is a smooth submanifold of TN and a vector bundle over Q.

Let us now return to the special case where $\dim(N) = 2\dim(Q)$.

Definition 4.3.5 (**The Index of a Zero of a Normal Vector Field).** *Let N be an oriented Riemannian $2m$-manifold without boundary, let $Q \subset N$ be a compact oriented m-dimensional submanifold without boundary, and let $Y \in \mathrm{Vect}^{\perp}(Q)$ be a normal vector field on Q.*

(i) *An element $q_0 \in Q$ is called a* **nondegenerate zero of** Y *iff $Y(q_0) = 0$ and the vertical derivative $DY(q_0) : T_{q_0}Q \to T_{q_0}Q^{\perp}$ is bijective. The* **index of** Y **at a nondegenerate zero** q_0 *is the number*

$$\iota(q_0, Y) := \begin{cases} +1, & \textit{if every positive basis } v_1, \dots, v_m \textit{ of } T_{q_0}Q \\ & \textit{gives rise to a positive basis} \\ & v_1, \dots, v_m, DY(q_0)v_1, \dots, DY(q_0)v_m \textit{ of } T_{q_0}N, \\ -1, & \textit{otherwise.} \end{cases} \tag{4.3.11}$$

(ii) *An element $q_0 \in Q$ is called an* **isolated zero of** Y *iff $Y(q_0) = 0$ and there exists an open neighborhood $V \subset N$ of q_0 such that*

$$Y(q) \neq 0 \qquad \textit{for all } q \in V \cap Q \setminus \{q_0\}. \tag{4.3.12}$$

Let $q_0 \in Q$ be an isolated zero of Y. To define the index of Y at q_0, choose an open neighborhood $V \subset N$ of q_0 that satisfies (4.3.12) *and an orientation preserving diffeomorphism $\psi : V \to \mathbb{R}^{2m}$ such that $\psi(V \cap Q) = \mathbb{R}^m \times \{0\}$ and the diffeomorphism $(\psi_1, \dots, \psi_m) : V \cap Q \to \mathbb{R}^m$ is orientation preserving. Define $\eta : \mathbb{R}^m \to \mathbb{R}^m$ by* (4.3.9), *and define $x_0 \in \mathbb{R}^m$ by $(x_0, 0) := \psi(q_0)$. Then $\eta(x) \neq 0$ for all $x \in \mathbb{R}^m \setminus \{x_0\}$. The* **index of** Y **at** q_0 *is the integer*

$$\iota(q_0, Y) := \deg\left(S^{m-1} \to S^{m-1} : x \mapsto \frac{\eta(x_0 + x)}{|\eta(x_0 + x)|}\right) \in \mathbb{Z}. \tag{4.3.13}$$

Lemma 4.3.6. *Let $Q \subset N$ and $Y \in \mathrm{Vect}^{\perp}(Q)$ be as in Definition 4.3.5. Then the index of an isolated zero of Y is independent of the choice of the coordinate chart used to define it, every nondegenerate zero is isolated, and in the nondegenerate case the indices in* (4.3.11) *and* (4.3.13) *agree.*

Proof. The index of Y at q_0 agrees by definition with the intersection index of the zero section $Z := \{(q, w) \in TQ^{\perp} \mid w = 0\} \subset TQ^{\perp}$ and the smooth map $Q \to TQ^{\perp} : q \mapsto (q, Y(q))$ at every isolated intersection point q_0, as defined in Definition 4.2.13. (Note the change in the ordering between the map and the submanifold.) Hence by Theorem 4.2.14 it is independent of the coordinate chart used to define it. That the indices in (4.3.11) and (4.3.13) agree in the nondegenerate case, follows directly from Lemma 2.2.13. □

Theorem 4.3.7. *Let N be an oriented Riemannian $2m$-manifold without boundary, let $Q \subset N$ be a compact oriented m-dimensional submanifold without boundary, and let $Y \in \mathrm{Vect}^{\perp}(Q)$ be a normal vector field on Q with only isolated zeros. Then*

$$\sum_{q \in Q,\, Y(q)=0} \iota(q, Y) = Q \cdot Q. \tag{4.3.14}$$

The proof will rely on the Tubular Neighborhood Theorem. Note that Theorems 2.3.6 and 3.2.1 are special cases of Theorem 4.3.8, and that the proofs of all three theorems are minor modifications of each other.

Theorem 4.3.8 (Tubular Neighborhood Theorem). *Let N be a Riemannian n-manifold without boundary, let $Q \subset N$ be a compact m-dimensional submanifold without boundary, and let $\varepsilon_Q := \inf_{q \in Q} \mathrm{inj}(q, N) > 0$. For $0 < \varepsilon < \varepsilon_Q$ define*

$$V_\varepsilon := \left\{ (q, w) \in TQ^{\perp} \,\middle|\, |w| < \varepsilon \right\}, \qquad U_\varepsilon := \left\{ p \in N \,\middle|\, \inf_{q \in Q} d(p, q) < \varepsilon \right\}.$$

Then there exists a constant $0 < \varepsilon_0 \le \varepsilon_Q$ such that the map

$$V_\varepsilon \to U_\varepsilon : (q, w) \mapsto \psi_\varepsilon(q, w) := \exp_q(w) \tag{4.3.15}$$

is a diffeomorphism for $0 < \varepsilon < \varepsilon_0$.

Proof. The proof has three steps.

Step 1. *The map $\psi_\varepsilon : V_\varepsilon \to U_\varepsilon$ is a local diffeomorphism for $\varepsilon > 0$ sufficiently small.*

The set $V_\varepsilon \subset TQ^{\perp}$ is an open neighborhood of the zero section and, for every $q \in Q$, we have

$$T_{(q,0)} TQ^{\perp} = T_q Q \oplus T_q Q^{\perp}.$$

By Lemma A.6.2 the map $\psi_\varepsilon : V_\varepsilon \to U_\varepsilon$ is smooth and its derivative at $(q, 0)$ is the map $d\psi_\varepsilon(q, 0) : T_q Q \oplus T_q Q^{\perp} \to T_q N$ given by

$$d\psi_\varepsilon(q, 0)(\widehat{q}, \widehat{w}) = \widehat{q} + \widehat{w}$$

for $\widehat{q} \in T_q Q$ and $\widehat{w} \in T_q Q^{\perp}$. Hence the derivative of ψ_ε is bijective at every point $(q, w) \in TQ^{\perp}$ with $w = 0$. Since Q is compact, this implies that the derivative is bijective at every point $(q, w) \in V_\varepsilon$ for $\varepsilon > 0$ sufficently small. This proves Step 1.

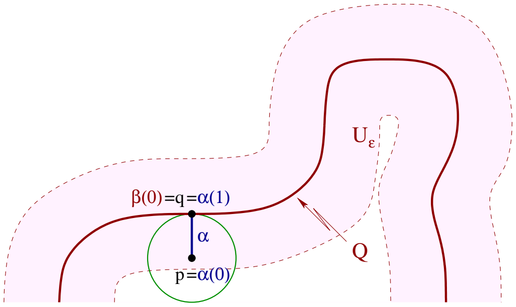

Figure 4.4: A Tubular Neighborhood.

Step 2. *The map* $\psi_\varepsilon : V_\varepsilon \to U_\varepsilon$ *is surjective for* $0 < \varepsilon < \varepsilon_Q$.

Let $p \in U_\varepsilon$. Since Q is compact, there exists an element $q \in Q$ such that

$$d(p,q) = \inf_{q' \in Q} d(p,q') < \varepsilon < \varepsilon_Q.$$

By Theorem A.6.4 there is a unique tangent vector $w \in T_qN$ such that

$$\exp_q(w) = p, \qquad |w| = d(p,q) < \varepsilon.$$

We must prove that $w \perp T_qQ$. Assume first that $|w| < \mathrm{inj}(p,N)$, let $v \in T_qQ$, and choose a curve $\beta : \mathbb{R} \to Q$ such that

$$\beta(0) = q, \qquad \dot\beta(0) = v, \qquad d(p,\beta(t)) < \mathrm{inj}(p,N)$$

for all t. Then there exists a unique smooth curve $u : \mathbb{R} \to T_pN$ such that

$$\beta(t) = \exp_p(u(t)), \qquad |u(t)| = d(p,\beta(t))$$

for all t. Since $d(p,q) \le d(p,\beta(t))$, there is a unique function $\lambda : \mathbb{R} \to \mathbb{R}$ such that $0 < \lambda(t) \le 1$ and $d(p,\exp_p(\lambda(t)u(t))) = d(p,q)$ for all t and $\lambda(0) = 1$. Define

$$\alpha(s) := \exp_p(su(0)) = \exp_q((1-s)w), \qquad \gamma(t) := \exp_p(\lambda(t)u(t)).$$

Then $\alpha(1) = \gamma(0) = q$ (see Figure 4.4) and $\dot\alpha(1)$ is orthogonal to $\dot\gamma(0)$ by the Gauß Lemma A.6.5. Moreover, $\lambda(0) = 1 = \max_t \lambda(t)$, thus $\dot\lambda(0) = 0$, and therefore

$$\dot\alpha(1) = -w, \qquad \dot\gamma(0) = \dot\beta(0) = v.$$

Hence $\langle v, w\rangle = 0$. Thus we have $w \perp T_qQ$ whenever $|w| < \mathrm{inj}(p,N)$. If $|w| \ge \mathrm{inj}(p,N)$, repeat this argument with p replaced by $p_\varepsilon := \exp_q(\varepsilon w)$ for $\varepsilon > 0$ sufficently small to obtain $w \perp T_qQ$. This proves Step 2.

Step 3. *The map $\psi_\varepsilon : V_\varepsilon \to U_\varepsilon$ is a injective for $\varepsilon > 0$ sufficiently small.*

Suppose this is wrong. Then there exist sequences $q_i, q_i' \in Q$, and $w_i \in T_{q_i}Q^\perp$ and $w_i' \in T_{q_i'}Q^\perp$ such that

$$\lim_{i\to\infty} |w_i| = \lim_{i\to\infty} |w_i'| = 0, \quad \exp_{q_i}(w_i) = \exp_{q_i'}(w_i'), \quad (q_i, w_i) \neq (q_i', w_i').$$

Since Q is compact, we may assume without loss of generality that the limits

$$q := \lim_{i\to\infty} q_i, \qquad q' := \lim_{i\to\infty} q_i'$$

exist. Since $\exp_{q_i}(w_i) = \exp_{q_i'}(w_i')$, the distance

$$d(q_i, q_i') \le |w_i| + |w_i'|$$

converges to zero and so $q = q'$. However, by Step 2 and the inverse function theorem, the restriction of the map ψ_ε to a neighborhood of the point $(q, 0)$ is injective, a contradiction. This proves Step 3 and Theorem 4.3.8. □

Proof of Theorem 4.3.7. Choose $0 < \varepsilon < \varepsilon_Q$ such that the map $\psi_\varepsilon : V_\varepsilon \to U_\varepsilon$ in Theorem 4.3.8 is a diffeomorphism, and assume without loss of generality that $|Y(q)| < \varepsilon$ for all $q \in Q$. Define the map $f : Q \to N$ by

$$f(q) := \exp_q(-Y(q)) \qquad \text{for } q \in Q.$$

Then $f(q) \in Q$ if and only if $Y(q) = 0$ and so f and Q have only isolated intersections. We prove that

$$\iota(q, Y) = \nu(q; f, Q) \qquad \text{for all } q \in f^{-1}(Q). \tag{4.3.16}$$

To see this, fix an element $q_0 \in Q$ with $Y(q_0) = 0$, choose an open neighborhood $U \subset Q$ that is diffeomorphic to $\mathbb{R}^m$ and contains no other zeros of Y, and choose a positive orthonormal frame of the normal bundle $TQ^\perp$ over U. Write this frame as a smooth family of isometric vector space isomorphisms

$$\Phi_q : T_qQ^\perp \to \mathbb{R}^m \qquad \text{for } q \in U.$$

Then the vector space isomorphism

$$T_qQ \times \mathbb{R}^m \to T_qN : (v, y) \mapsto v + \Phi_q^{-1}(y)$$

is orientation preserving for each $q \in U$.

Now denote

$$V := \left\{ \exp_q(w) \,\middle|\, q \in U,\, w \in T_qQ^\perp,\, |w| < \varepsilon \right\}, \qquad B_\varepsilon := \{ y \in \mathbb{R}^m \,|\, |y| < \varepsilon \},$$

choose an orientation preserving diffeomorphism $\phi : U \to \mathbb{R}^m$, and define the coordinate chart $\psi : V \to \mathbb{R}^m \times B_\varepsilon$ by

$$\psi(\exp_q(w)) := (\phi(q), \Phi_q(w))$$

for $q \in U$ and $w \in T_qQ^\perp$ with $|w| < \varepsilon$. Then

$$\psi(V \cap Q) = \mathbb{R}^m \times \{0\}$$

and

$$\psi\big(f(q)\big) = \big(\phi(q), -\Phi_q(Y(q))\big), \qquad d\psi(q)w = \big(0, \Phi_q(w)\big) \tag{4.3.17}$$

for all $q \in U$ and all $w \in T_qQ^\perp$. Define the map $\eta : \mathbb{R}^m \to B_\varepsilon$ by

$$\eta(x) := \Phi_q\big(Y(q)\big), \qquad q := \phi^{-1}(x) = \psi^{-1}(x, 0), \qquad \text{for } x \in \mathbb{R}^m.$$

Then it follows from (4.3.17) that

$$(\psi_{m+1}, \dots, \psi_{2m}) \circ f \circ \phi^{-1} = -\eta, \qquad d\psi(q)Y(q) = \big(0, \eta(\phi(q))\big),$$

for all $q \in U$ and so η satisfies (4.3.9) (with $b = 0$). Hence, with $x_0 := \phi(q_0)$, it follows from Definition 4.2.13 and Definition 4.3.5 that

$$\begin{aligned}
\iota(q_0; f, Q) &= (-1)^m \deg\left(S^{m-1} \to S^{m-1} : x \mapsto -\frac{\eta(x_0 + x)}{|\eta(x_0 + x)|} \right) \\
&= \deg\left(S^{m-1} \to S^{m-1} : x \mapsto \frac{\eta(x_0 + x)}{|\eta(x_0 + x)|} \right) \\
&= \iota(q_0, Y).
\end{aligned}$$

Here the sign $(-1)^m$ is required by the sign convention in Definition 4.2.13. This proves (4.3.16). It follows from (4.3.16) and Theorem 4.2.14 that

$$\sum_{q \in Q,\, Y(q) = 0} \iota(q, Y) = \sum_{q \in f^{-1}(Q)} \nu(q; f, Q) = f \cdot Q = Q \cdot Q.$$

This proves Theorem 4.3.7 . □

4.3.2 Examples and Exercises

Exercise 4.3.9. Let $Q \subset N$, $Y \in \mathrm{Vect}^{\perp}(Q)$, and $f : Q \to N$ be as in the proof of Theroem 4.3.7 so that $f(q) = \exp_q(-Y(q))$ for $q \in Q$. Let $q \in Q$ such that $Y(q) = 0$. Prove that

$$df(q)w = w - DY(q)w. \tag{4.3.18}$$

Deduce that q is a nondegenerate zero of Y if and only if it is a transverse intersection of f and Q. Verify equation (4.3.16) in the transverse case.

Exercise 4.3.10. Let M be a compact oriented manifold without boundary and consider the zero section in the tangent bundle, i.e.

$$N = TM, \qquad Q = \left\{(p, v) \in TM \,\middle|\, v = 0\right\}.$$

Prove that $Q \cdot Q = \chi(M)$ is the Euler characteristic of M. Prove that the Euler characteristic of every odd-dimensional compact manifold without boundary vanishes. The Poincaré–Hopf theorem does not require the manifold M to be orientable. How do you explain this?

Exercise 4.3.11. Let M be a compact oriented manifold without boundary and consider the diagonal $\Delta \subset M \times M$. Prove that $\Delta \cdot \Delta = \chi(M)$.

Exercise 4.3.12. Let N be a $2m$-manifold without boundary and let $Q \subset N$ be a compact m-dimensional submanifold without boundary. Define the **self-intersection number modulo two** $I_2(Q, Q) \in \{0, 1\}$ and extend Theorem 4.3.7 to the nonorientable case. Find an example where Q is odd-dimensional and $I_2(Q, Q) = 1$. **Hint:** Consider the Möbius band.

Exercise 4.3.13. Define the submanifolds $C, T, Q \subset N := \mathbb{C}\mathrm{P}^2$ by

$$\begin{aligned}
C &:= \left\{[z_0 : z_1 : z_2] \in \mathbb{C}\mathrm{P}^2 \,|\, z_2 = 0\right\} \cong \mathbb{C}\mathrm{P}^1, \\
T &:= \left\{[z_0 : z_1 : z_2] \in \mathbb{C}\mathrm{P}^2 \,|\, |z_0| = |z_1| = |z_2|\right\} \cong \mathbb{T}^2, \\
Q &:= \left\{[z_0 : z_1 : z_2] \in \mathbb{C}\mathrm{P}^2 \,|\, z_0, z_1, z_2 \in \mathbb{R}\right\} \cong \mathbb{R}\mathrm{P}^2.
\end{aligned}$$

What is meant by the *complex orientation* of N? Note that C and T are orientable while Q is not orientable. The submanifold C is canonically oriented as a complex submanifold of $\mathbb{C}\mathrm{P}^2$ and the orientation of T is a matter of choice. The submanifold $T \subset \mathbb{C}\mathrm{P}^2$ is called the **Clifford torus**. Prove that

$$C \cdot C = 1, \qquad C \cdot T = T \cdot T = 0.$$

Prove that $I_2(Q, Q) = 1$ and $I_2(Q, C) = I_2(Q, T) = 0$. Prove that $\mathbb{C}\mathrm{P}^2$ does not admit an orientation reversing diffeomorphism.

Exercise 4.3.14. Define the set $N \subset \mathbb{C}^2 \times \mathbb{C}\mathrm{P}^1$ by

$$N := \left\{ (x, y, [a : b]) \in \mathbb{C}^2 \times \mathbb{C}\mathrm{P}^1 \,\middle|\, ay = bx \right\}$$

Prove that N is a complex submanifold of $\mathbb{C}^2 \times \mathbb{C}\mathrm{P}^1$ of real dimension four and that $E := \{0\} \times \mathbb{C}\mathrm{P}^1$ is a complex submanifold of N. Prove that

$$E \cdot E = -1$$

with respect to the complex orientation.

Exercise 4.3.15. The tangent bundle of the 2-sphere is the 4-manifold

$$TS^2 = \left\{ (x, y) \in \mathbb{R}^3 \,|\, |x| = 1,\, \langle x, y \rangle = 0 \right\}.$$

Define the set $N \subset \mathbb{C}^3 \times \mathbb{C}\mathrm{P}^1$ by

$$N := \left\{ (z, [a : b]) \in \mathbb{C}^3 \times \mathbb{C}\mathrm{P}^1 \,\middle|\, \begin{array}{l} z_1^2 + z_2^2 + z_3^2 = 0, \\ b(z_1 + \mathbf{i}z_2) - az_3 = 0, \\ a(z_1 - \mathbf{i}z_2) + bz_3 = 0 \end{array} \right\}$$

and let $E := \{0\} \times \mathbb{C}\mathrm{P}^1$. Show that N is a complex submanifold of $\mathbb{C}^3 \times \mathbb{C}\mathrm{P}^1$ and that E is a complex submanifold of N. Prove that the formula

$$\phi(x, y) := \left(-x \times y + \mathbf{i}y, [x_1 + \mathbf{i}x_2 : 1 + x_3] \right)$$

defines an orientation reversing diffeomorphism $\phi : TS^2 \to N$ that sends the zero section to E. Deduce that

$$E \cdot E = -2.$$

Prove that there does not exist an orientation preserving diffeomorphism from TS^2 to N.

Exercise 4.3.16. **(i)** In the situation of Theorem 4.3.7, prove the existence of a normal vector field $Y \in \mathrm{Vect}^{\perp}(Q)$ with only nondegenerate zeros. **Hint:** Use Corollary 4.1.5 and Theorem 4.3.8. Alternatively, see Exercise 7.3.5.

(ii) If $Q \cdot Q = 0$, prove the existence of a normal vector field $Y \in \mathrm{Vect}^{\perp}(Q)$ without zeros. **Hint:** Combine the Homogeneity Lemma with parallel transport to find a normal vector field whose zeros are all contained in an arbitrarily small ball. Then use the Hopf Degree Theorem.

(iii) If $Q \cdot Q = 0$, prove the existence of a diffeomorphism $\phi : N \to N$ that is smoothly isotopic to the identity and satisfies $Q \cap \phi(Q) = \emptyset$. **Hint:** Use the Tubular Neighborhood Theorem 4.3.8 to extend the normal vector field Y in (ii) to a vector field $X \in \mathrm{Vect}(N)$ on all of N and use the flow of X.

Remark 4.3.17 (Whitney's Theorem). Let N be a simply connected smooth manifold without boundary and let $P, Q \subset N$ be compact connected submanifolds without boundary such that

$$\begin{aligned} \dim(P) + \dim(Q) &= \dim(N), \\ \dim(P) = \operatorname{codim}(Q) &\geq 3, \\ \dim(Q) = \operatorname{codim}(P) &\geq 3. \end{aligned} \tag{4.3.19}$$

Denote by $\mathrm{Diff}_0(N)$ the group of diffeomorphisms of N that are smoothly isotopic to the identity.

(i) If P, Q, N are oriented and $I(P, Q) = 0$, then a theorem of Whitney [27] asserts that there exists a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ with $\phi(P) \cap Q = \emptyset$.

(ii) Whitney's theorem continues to hold when at least one of the submanifolds P or Q is not orientable and $I_2(P, Q) = 0$.

(iii) If $P = Q$ is not orientable and $I_2(Q, Q) = 0$, then it follows from (ii) that there exists a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ with $\phi(Q) \cap Q = \emptyset$.

(iv) The manifold N is simply connected and hence orientable. Choose an orientation of N, assume $P = Q$ is not orientable, and let $Y \in \mathrm{Vect}^\perp(Q)$ be a normal vector field on Q with only nondegenerate zeros. Then every zero q of Y has a well-defined index $\iota(q, Y) \in \{\pm 1\}$ (see Definition 4.3.5). Moreover, it follows as in the Poincaré–Hopf Theorem 2.3.1 that the integer

$$\text{Euler number}(TQ^\perp) := \sum_{q \in Q,\, Y(q)=0} \iota(q, Y) \in \mathbb{Z}$$

(called the **Euler number of the normal bundle**) is independent of the choice of Y, and it follows from Theorem 4.3.7 that

$$\text{Euler number}(TQ^\perp) \equiv I_2(Q, Q) \pmod{2}.$$

Thus, if the Euler number of $TQ^\perp$ is even, it follows from (iii) that there exists a diffeomorphism $\phi \in \mathrm{Diff}_0(M)$ with $\phi(Q) \cap Q = \emptyset$. If in addition the Euler number is nonzero, then there is no normal vector field on Q without zeros as in Exercise 4.3.16, and the proof requires Whitney's theorem.

(v) An example of a nonorientable middle-dimensional submanifold Q of a simply connected manifold N can be obtained by blowing up two points on the Clifford torus $T \subset \mathbb{C}\mathrm{P}^2$ (see Exercise 4.3.13). This gives rise to a nonorientable submanifold $L \subset M := \mathbb{C}\mathrm{P}^2 \# 2\overline{\mathbb{C}\mathrm{P}}^2$ with Euler number 2. Take the products $N := M \times M$ and $Q := L \times L$ to obtain an example of codimension 4 with Euler number 4. Then by (iv) there exists a $\phi \in \mathrm{Diff}_0(N)$ such that $\phi(Q) \cap Q = \emptyset$. This diffeomorphism cannot be supported in a small neighborhood of Q. The details go beyond the scope of this book.

4.4 The Lefschetz Number of a Smooth Map

In this section we introduce the Lefschetz number of a smooth map f from a compact manifold M to itself as the algebraic count of the fixed point indices. If the manifold is oriented, the Lefschetz number can also be defined as the intersection number of the graph of f with the diagonal. However, orientability is not required and the Lefschetz number is always a homotopy invariant. The Lefschetz–Hopf theorem asserts that the Lefschetz number is the sum of the fixed point indices whenever the fixed points are all isolated.

4.4.1 Isolated Fixed Points

Assume throughout that M is a compact smooth m-manifold with boundary, not necessarily orientable, and that $f : M \to M$ is a smooth map.

Definition 4.4.1 (Fixed Point Index). *An element $p \in M$ is called a* **fixed point of** f *iff $f(p) = p$. The set of all fixed points of f is denoted by*

$$\mathrm{Fix}(f) := \left\{p \in M \,\middle|\, f(p) = p\right\}.$$

A fixed point $p_0 \in \mathrm{Fix}(f) \setminus \partial M$ is called **isolated** *iff there exists an open neighborhood $U \subset M$ of p_0 such that $U \cap \mathrm{Fix}(f) = \{p_0\}$. It is called* **nondegenerate** *iff the linear map $\mathbb{1} - df(p_0) : T_{p_0}M \to T_{p_0}M$ is a vector space isomorphism. The map f is called a* **Lefschetz map** *iff it has no fixed point on the boundary and its fixed points are all nondegenerate.*

Let $p_0 \in M \setminus \partial M$ be an isolated fixed point of f and let $U \subset M \setminus \partial M$ be an open neighborhood of p_0 with $U \cap \mathrm{Fix}(f) = \{p_0\}$ such that there exists a diffeomorphism $\phi : U \to \phi(U) \subset \mathbb{R}^m$. Given such a coordinate chart ϕ, define the open set $\Omega \subset \mathbb{R}^m$ and the smooth map $\eta : \Omega \to \phi(U)$ by

$$\Omega := \phi(U \cap f^{-1}(U)) \subset \mathbb{R}^m, \qquad \eta := \phi \circ f \circ \phi^{-1} : \Omega \to \phi(U). \tag{4.4.1}$$

Let $x_0 := \phi(p_0)$ and choose $\varepsilon > 0$ such that $\overline{B_\varepsilon(x_0)} \subset \Omega$. Then the integer

$$\begin{aligned} \iota(p_0, f) &:= \iota(x_0, \eta) := \iota_{\mathrm{FP}}(x_0, \eta) \\ &:= \deg\left(S^{m-1} \to S^{m-1} : x \mapsto \frac{x_0 + \varepsilon x - \eta(x_0 + \varepsilon x)}{|x_0 + \varepsilon x - \eta(x_0 + \varepsilon x)|}\right) \end{aligned} \tag{4.4.2}$$

is called the **fixed point index of** f **at** p_0.

It will sometimes be convenient to use the notation $\iota_{\mathrm{FP}}(x_0, \eta)$ for the fixed point index in (4.4.2) in local coordinates to distinguish it from the index $\iota_{\mathrm{VF}}(x_0, \xi)$ of x_0 as a zero of a vector field $\xi : \Omega \to \mathbb{R}^m$. With this notation we have $\iota_{\mathrm{FP}}(x_0, \eta) = \iota_{\mathrm{VF}}(x_0, \mathrm{id}_\Omega - \eta)$ (see Definition 2.2.4).

Lemma 4.4.2. *If $p_0 \in M \setminus \partial M$ is an isolated fixed point of f, then its fixed point index is independent of the choice of the coordinate chart ϕ.*

Proof. The proof is essentially the same as that of Lemma 2.2.5. Assume without loss of generality that ϕ and ϕ' are two coordinate charts defined on the same open neighborhood $U \subset M$ of p_0 with $U \cap \mathrm{Fix}(f) = \{p_0\}$, that they satisfy $\phi(p_0) = \phi'(p_0) = 0$, and that $\phi(U)$ is convex. Define

$$\Omega' := \phi'(U \cap f^{-1}(U)), \qquad \eta' := \phi' \circ f \circ \phi'^{-1} : \Omega' \to \phi'(U),$$

as before. Denote the transition map by $\psi := \phi' \circ \phi^{-1} : \phi(U) \to \phi'(U)$. Then ψ restricts to a diffeomorphism from Ω to Ω', satisfies $\psi(0) = 0$, and

$$\eta' = \psi \circ \eta \circ \psi^{-1} : \Omega' \to \phi'(U). \tag{4.4.3}$$

Since $\phi(U)$ is convex, we conclude as in Lemma 2.2.7 that the formula

$$\psi_t(x) := \left\{ \begin{array}{rl} t^{-1}\psi(tx), & \text{for } 0 < t \leq 1 \text{ and } x \in \phi(U), \\ \Psi x := d\psi(0)x, & \text{for } t = 0 \text{ and } x \in \phi(U), \end{array} \right. \tag{4.4.4}$$

defines a smooth isotopy of embeddings $\psi_t : \phi(U) \to \mathbb{R}^n$. Define

$$\eta_t := \psi_t \circ \eta \circ \psi_t^{-1} : \psi_t(\Omega) \to \psi_t(\phi(U)) \qquad \text{for } 0 \leq t \leq 1. \tag{4.4.5}$$

Then $x = 0$ is the unique fixed point of η_t and, by (4.4.3) and (4.4.4),

$$\eta_1 = \eta', \qquad \eta_0(x) = \Psi\eta\big(\Psi^{-1}x\big) \quad \text{for } x \in \Psi(\Omega).$$

By Remark 2.2.8 there exists an $\varepsilon > 0$ such that $\bar{B}_\varepsilon \subset \psi_t(\Omega)$ for $0 \leq t \leq 1$. Define the maps $g, g_t, g' : S^{m-1} \to S^{m-1}$ by

$$g(x) := \frac{\varepsilon x - \eta(\varepsilon x)}{|\varepsilon x - \eta(\varepsilon x)|}, \quad g_t(x) := \frac{\varepsilon x - \eta_t(\varepsilon x)}{|\varepsilon x - \eta_t(\varepsilon x)|}, \quad g'(x) := \frac{\varepsilon x - \eta'(\varepsilon x)}{|\varepsilon x - \eta'(\varepsilon x)|}$$

for $x \in S^{m-1}$ and $0 \leq t \leq 1$. Then $g_1 = g'$ and

$$g_0(x) = \frac{\varepsilon x - \Psi\eta(\varepsilon\Psi^{-1}x)}{|\varepsilon x - \Psi\eta(\varepsilon\Psi^{-1}x)|} = \frac{\Psi\xi(\varepsilon\Psi^{-1}x)}{|\Psi\xi(\varepsilon\Psi^{-1}x)|}. \qquad \text{for } x \in S^{m-1}.$$

Here the map $\xi := \mathrm{id} - \eta : \Omega \to \mathbb{R}^m$ is a vector field with an isolated zero at the origin. Hence it follows from Lemma 2.2.5, with Ω' and ξ' replaced by $\Omega_0 = \Psi(\Omega)$ and the vector field $\xi_0 := \mathrm{id} - \eta_0 = \Psi \circ \xi \circ \Psi^{-1} : \Omega_0 \to \mathbb{R}^m$, that $x = 0$ has the same index as a zero of ξ and as a zero of ξ_0. Thus

$$\begin{aligned} \iota_{\mathrm{FP}}(0, \eta) &= \deg(g) = \iota_{\mathrm{VF}}(0, \xi) = \iota_{\mathrm{VF}}(0, \xi_0) = \deg(g_0) \\ &= \deg(g_1) = \deg(g') = \iota_{\mathrm{FP}}(0, \eta'). \end{aligned}$$

This proves Lemma 4.4.2. □

Lemma 4.4.3. *If $p_0 \in M \setminus \partial M$ is a nondegenerate fixed point of f, then p_0 is an isolated fixed point of f and its fixed point index is given by*

$$\begin{aligned}\iota(p_0, f) &= \operatorname{sign}\bigl(\det(\mathbb{1} - df(p_0))\bigr) \\ &= \begin{cases} +1, & \text{if } \mathbb{1} - df(p_0) \text{ is orientation preserving,} \\ -1, & \text{if } \mathbb{1} - df(p_0) \text{ is orientation reversing.} \end{cases}\end{aligned} \tag{4.4.6}$$

Proof. Choose a local coordinate chart $\phi : U \to \phi(U) \subset \mathbb{R}^m$ on an open neighborhood $U \subset M$ of p_0, let $\Omega := \phi(U \cap f^{-1}(U))$ and $\eta := \phi \circ f \circ \phi^{-1}$ be as in (4.4.1) and define $x_0 := \phi(p_0) \in \Omega$. Then the linear map

$$\mathbb{1} - d\eta(x_0) = d\phi(p_0)(\mathbb{1} - df(p_0))d\phi(p_0)^{-1} : \mathbb{R}^m \to \mathbb{R}^m$$

is invertible and so x_0 is a nondegenerate zero of the vector field $\xi := \mathrm{id} - \eta$. By Lemma 2.2.13 this implies that x_0 is an isolated zero of ξ and

$$\iota_{\mathrm{VF}}(x_0, \xi) = \operatorname{sign}\bigl(\det(d\xi(x_0))\bigr).$$

Hence x_0 is an isolated fixed point of the map $\eta = \mathrm{id} + \xi$. By Definition 2.2.4 and Definition 4.4.1, the index of x_0 as a zero of ξ agrees with the index of x_0 as a fixed point of η. Hence

$$\begin{aligned}\iota(p_0, f) = \iota_{\mathrm{FP}}(x_0, \eta) = \iota_{\mathrm{VF}}(x_0, \xi) &= \operatorname{sign}\bigl(\det(d\xi(x_0))\bigr) \\ = \operatorname{sign}\bigl(\det(\mathbb{1} - d\eta(x_0))\bigr) &= \operatorname{sign}\bigl(\det(\mathbb{1} - df(p_0))\bigr).\end{aligned}$$

This proves Lemma 4.4.3. □

Define the diagonal in $M \times M$ and the graph of f by

$$\Delta := \{(p, p) \,|\, p \in M\}, \qquad \mathrm{graph}(f) := \{(p, f(p)) \,|\, p \in M\}.$$

The fixed points of f are in one-to-one correspondence with the intersection points of the graph of f and the diagonal, and the nondegeneracy condition on the fixed point translates into transversality for the intersection point.

Lemma 4.4.4. *A point $p \in M \setminus \partial M$ is a nondegenerate fixed point of f if and only if the graph of f and the diagonal intersect transversally at (p, p).*

Proof. The graph of f and the diagonal intersect transversally at (p, p) if and only if $T_pM \times T_pM = T_{(p,p)}\mathrm{graph}(f) \oplus T_{(p,p)}\Delta$. This is equivalent to the condition $T_{(p,p)}\mathrm{graph}(f) \cap T_{(p,p)}\Delta = \{(0, 0)\}$ and hence to the implication

$$v \in T_pM, \qquad df(p)v = v \qquad \Longrightarrow \qquad v = 0. \tag{4.4.7}$$

The implication (4.4.7) holds if and only if $\mathbb{1} - df(p)$ is bijective which means that p is a nondegenerate fixed point of f. This proves Lemma 4.4.4 □

The next lemma shows that the fixed point index of a nondegenerate fixed point of f agrees with the intersection index of the graph of f with the diagonal whenever M is oriented.

Lemma 4.4.5. *Assume M is oriented and let $p \in M \setminus \partial M$ be a nondegenerate fixed poijnt of f. Then*

$$\operatorname{sign}\bigl(\det(\mathbb{1} - df(p))\bigr) = \nu\bigl((p,p); \operatorname{graph}(f), \Delta\bigr). \tag{4.4.8}$$

Proof. Fix a positive basis $v_1, \dots, v_m$ of T_pM and consider the basis

$$(v_1, df(p)v_1), \dots, (v_m, df(p)v_m), (v_1, v_1), \dots, (v_m, v_m),$$

of $T_pM \times T_pM$. Subtracting the ith vector from the $(m+i)$th vector in this basis we obtain the basis

$$(v_1, df(p)v_1), \dots, (v_m, df(p)v_m), (0, v_1 - df(p)v_1), \dots, (0, v_m - df(p)v_m).$$

Now subtract a suitable linear combination of the last m vectors from each of the first m vectors to obtain the basis

$$(v_1, 0), \dots, (v_m, 0), (0, v_1 - df(p)v_1), \dots, (0, v_m - df(p)v_m)$$

of $T_pM \times T_pM$. This basis is related to the original basis of $T_pM \times T_pM$ by a matrix of determinant one and it is a positive basis of $T_pM \times T_pM$ if and only if $\det(\mathbb{1} - df(p)) > 0$. This proves Lemma 4.4.5. □

Assume M is an oriented manifold without boundary. Then it follows from Lemma 4.4.5 and equation (4.3.3) that the sum of the fixed point indices of a Lefschetz map f is the intersection number

$$\sum_{p \in \operatorname{Fix}(f)} \iota(p, f) = \operatorname{graph}(f) \cdot \Delta. \tag{4.4.9}$$

By Theorem 4.2.8 this implies that the left hand side of equation (4.4.9) is a homotopy invariant for Lefschetz maps. This homotopy invariant is called the **Lefschetz number of** f and will be denoted by

$$L(f) := \sum_{p \in \operatorname{Fix}(f)} \iota(p, f). \tag{4.4.10}$$

We will prove that every smooth map is smoothly homotopic to a Lefschetz map (Lemma 4.4.8) and hence the Lefschetz number is well defined for every smooth map $f : M \to M$. In contrast to intersection theory it turns out that the Lefschetz number continues to be a homotopy invariant when M is nonorientable or has nonempty boundary. This is the content of the Lefschetz–Hopf Theorem which can be viewed as a natural extension of the Poincaré–Hopf Teorem for maps instead of vector fields.

4.4.2 The Lefschetz–Hopf Theorem

Our goal will be to prove the following result.

Theorem 4.4.6 (Lefschetz–Hopf). *Let M be a compact manifold with boundary and let $f : M \to M$ be a smooth map such that*

$$\mathrm{Fix}(f) \cap \partial M = \emptyset.$$

If f has only isolated fixed points, then

$$\sum_{p \in \mathrm{Fix}(f)} \iota(p, f) = \sum_{k=0}^{m} (-1)^k \mathrm{trace}\big(f^* : H^k(M) \to H^k(M)\big). \tag{4.4.11}$$

Here $H^*(M)$ denotes the de Rham cohomology of M.

The full version of equation (4.4.11) will be established in Theorem 6.4.9. In the present section we prove a weaker result, which asserts that the sum of the fixed point indices of a smooth map f with only isolated fixed points and no fixed point on the boundary is a homotopy invariant, called the Lefschetz number of f.

The strategy for the proof will be to show that every smooth map with only isolated fixed points and no fixed points on the boundary is homotopic to a Lefschetz map with the same sum of the fixed point indices (Lemma 4.4.7) and then to show that the sum of the fixed point indices is a homotopy invariant for Lefschetz maps (Lemma 4.4.9). To prove that the Lefschetz number is defined for every smooth map, we must also show that every smooth map is homotopic to a Lefschetz map (Lemma 4.4.8).

Lemma 4.4.7 actually asserts the existence of a local perturbation of a map f near an isolated fixed point p_0 such that the perturbed map has only nondegenerate fixed points near p_0, the sum of whose indices is the fixed point index of f at p_0. This is analogous to Lemma 2.3.5 for isolated zeros of vector fields and the proof is essentially the same.

We assume throughout that M is a compact m-manifold with boundary.

Lemma 4.4.7 (Local Perturbation). *Let $f : M \to M$ be a smooth map, let $p_0 \in \mathrm{Fix}(f) \setminus \partial M$ be an isolated fixed point, and let $U \subset M$ be an open neighborhood of p_0 such that*

$$\mathrm{Fix}(f) \cap \overline{U} = \{p_0\}, \qquad \overline{U} \cap \partial M = \emptyset. \tag{4.4.12}$$

Then there exists a smooth map $g : M \to M$ that has only nondegenerate fixed points in U, is smoothly homotopic to f relative to $M \setminus U$, and satisfies

$$\iota(p_0, f) = \sum_{p \in U \cap \mathrm{Fix}(g)} \mathrm{sign}\big(\det(\mathbb{1} - dg(p))\big). \tag{4.4.13}$$

Proof. After shrinking U, if necessary, we may assume that there exists a diffeomorphism $\phi : U \to \mathbb{R}^m$. Define the map $\eta : \Omega \to \phi(U)$ by (4.4.1), so

$$\Omega := \phi(U \cap f^{-1}(U)) \subset \mathbb{R}^m, \qquad \eta := \phi \circ f \circ \phi^{-1} : \Omega \to \phi(U).$$

Let $x_0 := \phi(p_0)$ and choose a constant $\varepsilon > 0$ such that

$$\overline{B_\varepsilon(x_0)} \subset \Omega.$$

Then, as in the proof of Lemma 4.4.3, the map $\xi : \Omega \to \mathbb{R}^m$, defined by

$$\xi(x) := x - \eta(x) \qquad \text{for } x \in \Omega,$$

is a smooth vector field with x_0 as its only zero and Defintion 4.4.1 shows that the fixed point index of p_0 agrees with the index of x_0 as a zero of the vector field ξ in Definition 2.2.4, i.e.

$$\iota(p_0, f) = \deg\left(S^{m-1} \to S^{m-1} : x \mapsto \frac{\xi(x_0 + \varepsilon x)}{|\xi(x_0 + \varepsilon x)|}\right) = \iota_{\mathrm{VF}}(x_0, \xi). \quad (4.4.14)$$

By Lemma 2.3.5 there exists a smooth vector field $\xi' : \Omega \to \mathbb{R}^m$ with only nondegenerate zeros such that $\xi'(x) = \xi(x)$ for all $x \in \Omega \setminus B_\varepsilon(x_0)$ and

$$\iota_{\mathrm{VF}}(x_0, \xi) = \sum_{\xi'(x)=0} \operatorname{sign}\bigl(\det(d\xi'(x))\bigr). \quad (4.4.15)$$

Moreover, the proof of Lemma 2.3.5 shows that the perturbation ξ' can be cosen arbitrarily close to ξ. Thus we may choose ξ' auch that, in addition,

$$\eta_t(x) := x + (1-t)\xi(x) + t\xi'(x) \in \phi(U) \quad (4.4.16)$$

for all $(t, x) \in [0, 1] \times \Omega$. Then the map $\eta' := \mathrm{id} + \xi' = \eta_1 : \Omega \to \phi(U)$ has only nondegenerate fixed points and agrees with $\eta = \eta_0$ on $\Omega \setminus B_\varepsilon(x_0)$. Hence the map $g : M \to M$, defined by

$$g(p) := \begin{cases} f(p), & \text{for } p \in M \setminus (U \cap f^{-1}(U)), \\ \phi^{-1} \circ \eta' \circ \phi(p), & \text{for } p \in U \cap f^{-1}(U), \end{cases}$$

has only nondegenerate fixed points in U and, by (4.4.14) and (4.4.15), it satisfies (4.4.13). Moreover, it follows from (4.4.16) that f is smoothly homotopic to g via the homotopy

$$f_t(p) := \begin{cases} f(p), & \text{for } p \in M \setminus (U \cap f^{-1}(U)), \\ \phi^{-1} \circ \eta_t \circ \phi(p), & \text{for } p \in U \cap f^{-1}(U), \end{cases}$$

from $f_0 = f$ to $f_1 = g$. This proves Lemma 4.4.7. □

Lemma 4.4.8 (Local Transversality). *Let $U \subset M \setminus \partial M$ be an open set, and let $f : M \to M$ be a smooth map such that*

$$\mathrm{Fix}(f) \cap \overline{U} \setminus U = \emptyset. \tag{4.4.17}$$

Then there exists a smooth map $g : M \to M$ that has only nondegenerate fixed points in U and is smoothly homotopic to f relative to $M \setminus U$.

Proof. Shrinking U if necessary, we may assume that $\overline{U} \cap \partial M = \emptyset$ and U still satisfies (4.4.17). By (4.4.17) the set $\mathrm{Fix}(f) \cap U = \mathrm{Fix}(f) \cap \overline{U}$ is compact. Hence there exists an open neighborhood V of $\mathrm{Fix}(f) \cap U$ such that $\overline{V} \subset U$ and a smooth function $\beta : M \to [0,1]$ such that $\mathrm{supp}(\beta) \subset U$ and $\beta|_{\overline{V}} \equiv 1$. Now Lemma 4.1.6 (with $N = M \setminus \partial M$ and $K = \overline{U}$) asserts that there exists a smooth map $G : \mathbb{R}^\ell \times M \to M$ such that

(A) $G(0,p) = p$ for all $p \in M$,

(B) $T_{G(\lambda,p)}M = \mathrm{span}\{\partial_{\lambda_i} G(\lambda,p) \,|\, i = 1,\dots,\ell\}$ for all $p \in \overline{V}$ and all $\lambda \in \mathbb{R}^\ell$,

(C) $G(\lambda, f(p)) \neq p$ for all $\lambda \in \mathbb{R}^\ell$ and all $p \in \overline{U} \setminus V$.

Here (A) and (B) follow directly from Lemma 4.1.6, while (C) only holds for $|\lambda|$ sufficiently small by (A) and the compactness of $\overline{U} \setminus V$. To achieve condition (C) for all $\lambda \in \mathbb{R}^\ell$, replace the map G by $G(\delta\lambda/\sqrt{\delta^2 + |\lambda|^2}, p)$ for $\delta > 0$ sufficiently small.

Define the maps $f_\lambda : M \to M$ by

$$f_\lambda(p) := G(\beta(p)\lambda, f(p)) \qquad \text{for } \lambda \in \mathbb{R}^\ell \text{ and } p \in M,$$

and define the map $\mathcal{F} : \mathbb{R}^\ell \times U \to M \times M$ by

$$\mathcal{F}(\lambda, p) := (p, f_\lambda(p)) \qquad \text{for } \lambda \in \mathbb{R}^\ell \text{ and } p \in U.$$

Then $\mathcal{F}$ is transverse to Δ. Namely, if $\lambda \in \mathbb{R}^\ell$ and $p \in U$ satisfy $\mathcal{F}(\lambda,p) \in \Delta$, then $G(\beta(p)\lambda, f(p)) = f_\lambda(p) = p$, hence $p \in V$ by (C), therefore $\beta(p) = 1$, hence $T_pM = \mathrm{span}\{\partial_{\lambda_i} f_\lambda(p) \,|\, i = 1,\dots,\ell\}$ by (B), and hence we obtain the equation $T_pM \times T_pM = \mathrm{im}\, d\mathcal{F}(\lambda,p) + T_{(p,p)}\Delta$. This shows that the set

$$\mathscr{M} := \mathcal{F}^{-1}(\Delta) = \left\{ (\lambda, p) \in \mathbb{R}^\ell \times U \,\middle|\, f_\lambda(p) = p \right\}$$

is a smooth submanifold of $\mathbb{R}^\ell \times U$, by Lemma 4.1.2. By Sard's Theorem there exists a regular value $\lambda \in \mathbb{R}^\ell$ of the canonical projection $\pi : \mathscr{M} \to \mathbb{R}^\ell$. Then, by Lemma 4.1.8, the map $U \to M \times M : p \mapsto (p, f_\lambda(p))$ is transverse to Δ. Thus $g := f_\lambda$ has only nondegenerate fixed points in U by Lemma 4.4.4 and is homotopic to f via $t \mapsto f_{t\lambda}$ by (A). This proves Lemma 4.4.8. □

Lemma 4.4.9 (**Local Homotopy Invariance**). *Let $U \subset M \setminus \partial M$ be an open set. Let $f_0, f_1 : M \to M$ be smooth maps that satisfy* (4.4.17) *and have only nondegenerate fixed points in U, and suppose there exists a smooth homotopy $[0,1] \times M \to M : (t,p) \mapsto f_t(p)$ from f_0 to f_1 such that f_t satisfies* (4.4.17) *for all t. Then*

$$\sum_{p \in U \cap \mathrm{Fix}(f_0)} \mathrm{sign}(\det(\mathbb{1} - df_0(p))) = \sum_{p \in U \cap \mathrm{Fix}(f_1)} \mathrm{sign}(\det(\mathbb{1} - df_1(p))). \quad (4.4.18)$$

Proof. The proof has two steps. The first step shows that the homotopy can be chosen transverse to the diagonal and is analogous to Lemma 4.4.8.

Step 1. *There exists a smooth homotopy $[0,1] \times M \to M : (t,p) \mapsto f_t(p)$ from f_0 to f_1 such that f_t satisfies* (4.4.17) *for all t and the map*

$$F : [0,1] \times U \to M \times M, \qquad F(t,p) := \big(p, f_t(p)\big),$$

is transverse to Δ.

Let $\{f_t\}_{0 \le t \le 1}$ be the homotopy in the assumptions of the lemma. Then the set $K := \bigcup_t (\mathrm{Fix}(f_t) \cap U) \subset U$ is compact by (4.4.17). Hence, shrinking U if necessary, we may assume that $\overline{U} \cap \partial M = \emptyset$ and U still satisfies (4.4.17) and the set K remains unchanged. Moreover, there exists a an open neighborhood V of K such that $\overline{V} \subset U$. and a smooth function $\beta : M \to [0,1]$ such that $\mathrm{supp}(\beta) \subset U$ and $\beta|_K \equiv 1$. As in the proof of Lemma 4.4.8 it follows from Lemma 4.1.6 that exists a smooth map $G : \mathbb{R}^\ell \times M \to M$ satisfying the conditions

(A) $G(0,p) = p$ for all $p \in M$,

(B) $T_{G(\lambda,p)}M = \mathrm{span}\{\partial_{\lambda_i} G(\lambda,p) \,|\, i = 1, \dots, \ell\}$ for all $p \in \overline{V}$ and all $\lambda \in \mathbb{R}^\ell$,

(C) $G(\lambda, f_t(p)) \ne p$ for all $\lambda \in \mathbb{R}^\ell$, all $t \in [0,1]$, and all $p \in \overline{U} \setminus V$.

Define the map $\mathcal{F} : \mathbb{R}^\ell \times [0,1] \times U \to M \times M$ by

$$\mathcal{F}(\lambda,t,p) := \Big(p, f_{\lambda,t}(p)\Big), \qquad f_{\lambda,t}(p) := G\Big(t(1-t)\beta(p)\lambda, f_t(p)\Big).$$

This map and its restriction to $\mathbb{R}^\ell \times \{0,1\} \times U$ are transverse to Δ. Thus by Lemma 4.1.2 the set

$$\mathscr{M} := \mathcal{F}^{-1}(\Delta)$$

is a submanifold of $\mathbb{R}^\ell \times [0,1] \times U$ with boundary. Choose a regular value λ of the projection $\mathscr{M} \to \mathbb{R}^\ell : (\lambda,t,p) \mapsto \lambda$. Then the map $F'(t,p) := \mathcal{F}(\lambda,t,p)$ is transverse to Δ by Lemma 4.1.8. Hence the homotopy $f'_t(p) := f_{\lambda,t}(p)$ (defined by $f'_t(p) := f_t(p)$ for $p \in M \setminus U$) satisfies the requirments of Step 1.

Step 2. *We prove* (4.4.18).

Let F be as in Step 1. Then F is transverse to Δ and $F|_{\{0,1\}\times U}$ is transverse to Δ by Lemma 4.4.4. Hence Lemma 4.1.2 asserts that the set

$$X := F^{-1}(\Delta) = \big\{(t,p) \in [0,1]\times U \,\big|\, f_t(p) = p\big\}$$

is a 1-manifold with boundary $\partial X = (\{0\}\times \mathrm{Fix}(f_0|_U)) \cup (\{1\}\times \mathrm{Fix}(f_1|_U))$ (see Figure 4.5). It is compact because $\mathrm{Fix}(f_t) \cap \overline{U}\setminus U = \emptyset$ for all t.

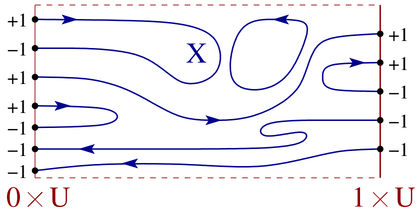

Figure 4.5: The local Lefschetz number.

For each element $(t,p)\in X$ we have $T_pM = \mathrm{im}\,(\mathbb{1} - df_t(p)) + \mathbb{R}\partial_t f_t(p)$ and $T_{(t,p)}X = \{(\tau,v)\in\mathbb{R}\times T_pM \mid df_t(p)v + \tau\partial_t f_t(p) = v\}$. Given a metric, this implies that the linear map $\Phi_{\tau_0,v_0} : \mathbb{R}\times T_pM \to \mathbb{R}\times T_pM$, defined by

$$\Phi_{\tau_0,v_0}(\tau,v) := (\tau_0\tau + \langle v_0, v\rangle, v - df_t(p)v - \tau\partial_t f_t(p)), \tag{4.4.19}$$

is bijective for each nonzero tangent vector $(\tau_0,v_0)\in T_{(t,p)}X$. Orient X by calling this vector **positive** iff Φ_{τ_0,v_0} is orientation preserving.

By Theorem A.4.1 the set X is a finite union of circles and arcs, oriented as above. Let $A\subset X$ be an arc and choose an orientation preserving diffeomorphism $\gamma : [0,1]\to A$. We examine the boundary points.

Case 1: $\gamma(0) = (0,p)$**.** Then $f_0(p) = p$ and $\dot\gamma(0) = (\tau_0,v_0)$ with $\tau_0 > 0$. Since $\det(\Phi_{\tau_0,v_0}) > 0$, it follows from (4.4.19) that $\det(\mathbb{1} - df_0(p)) > 0$.

Case 2: $\gamma(1) = (1,p)$**.** Then $f_1(p) = p$ and $\dot\gamma(1) = (\tau_0,v_0)$ with $\tau_0 > 0$. Since $\det(\Phi_{\tau_0,v_0}) > 0$, it follows from (4.4.19) that $\det(\mathbb{1} - df_1(p)) > 0$.

Case 3: $\gamma(0) = (1,p)$**.** Then $f_1(p) = p$ and $\dot\gamma(0) = (\tau_0,v_0)$ with $\tau_0 < 0$. Since $\det(\Phi_{\tau_0,v_0}) > 0$, it follows from (4.4.19) that $\det(\mathbb{1} - df_1(p)) < 0$.

Case 4: $\gamma(1) = (0,p)$**.** Then $f_0(p) = p$ and $\dot\gamma(1) = (\tau_0,v_0)$ with $\tau_0 < 0$. Since $\det(\Phi_{\tau_0,v_0}) > 0$, it follows from (4.4.19) that $\det(\mathbb{1} - df_0(p)) < 0$.

To verify these assertion, it is convenient to choose a basis of $\mathbb{R}\times T_pM$ of the form $(\tau_0,v_0), (0,v_1),\dots,(0,v_m)$. The four cases show that the signs of two fixed points of f_0 (respectively f_1) in U that are joined by an arc cancel and that the signs of a fixed point of f_0 and a fixed point of f_1 that are joined by an arc agree (see Figure 4.5). This proves Step 4 and Lemma 4.4.9. □

The next result is the weaker version of Theorem 4.4.6 announced in the beginning of this section.

Theorem 4.4.10 (Lefschetz–Hopf). (i) *Every smooth map $f : M \to M$ is smoothly homotopic to a Lefschetz map.*

(ii) *If $f, g : M \to M$ are smoothly homotopic Lefschetz maps, then*

$$\sum_{p\in\mathrm{Fix}(f)} \iota(p, f) = \sum_{p\in\mathrm{Fix}(g)} \iota(p, g). \tag{4.4.20}$$

(iii) *Let $f : M \to M$ be a smooth map with only isolated fixed points and no fixed points on the boundary and let $g : M \to M$ be a Lefschetz map. If f is smoothly homotopic to g, then* (4.4.20) *holds.*

Proof. We prove part (i). Let $f : M \to M$ be any smooth map. Choose a vector field X that points in on the boundary and let $\{\phi_t\}_{t\geq 0}$ be the semiflow of X. Then $\phi_1 \circ f$ is homotopic to f and has no fixed points on the boundary. Hence it follows from Lemma 4.4.8 with $U = M \setminus \partial M$ that $\phi_1 \circ f$ is homotopic to a Lefschetz map. This proves part (i).

We prove part (ii). Let $F : [0, 1] \times M \to M$ be a smooth homotopy from f to g, choose a vector field X that points in on the boundary, and denote its semiflow by $\{\phi_t\}_{t\geq 0}$. Then the homotopy $f_t(p) := \phi_{t(1-t)}(F(t, p))$ from $f_0 = f$ to $f_1 = g$ satisfies $\mathrm{Fix}(f_t) \cap \partial M = \emptyset$ for all t. Hence part (ii) follows from Lemma 4.4.9 with $U = M \setminus \partial M$.

We prove part (iii). Apply the Perturbation Lemma 4.4.7 to the map f in a neighborhood of each fixed point p to obtain a Lefschetz map g that satisfies (4.4.20) and is homotopic to f. Then, if g' is any other Lefschetz map homotopic to f, it also satisfies (4.4.20) by part (ii). This proves part (iii) and Theorem 4.4.10. □

With these preparations we are ready to give a formal definition of the Lefschetz number of a smooth map.

Definition 4.4.11 (Lefschetz Number). *Let $f : M \to M$ be a smooth map. By part (i) of Theoram 4.4.10 there exists a Lefschetz map $g : M \to M$ that is smoothly homotopic to f. By part (ii) of Theorem 4.4.10 the sum of its fixed point indices is independent of the choice of g. It is called the* **Lefschetz number of** f *and is denoted by*

$$L(f) := \sum_{p\in\mathrm{Fix}(g)} \mathrm{sign}\big(\det(\mathbb{1} - dg(p))\big) \qquad \begin{array}{l} \textit{for } g \sim f \\ \textit{with } \mathrm{Fix}(g) \cap \partial M = \emptyset \\ \textit{and } \mathrm{graph}(g) \pitchfork \Delta. \end{array} \tag{4.4.21}$$

The definition of the Lefschetz number can be extended to continuous maps by using the Weierstrass Approximation Theorem.

Exercise 4.4.12. Let M be a compact m-manifold with boundary.

(i) Prove that every continuous map $f : M \to M$ is continuously homotopic to a smooth map $g : M \to M$ (compare Exercise 1.7.4 and Exercise 2.4.9).

(ii) It two smooth maps $f, g : M \to M$ are continuously homotopic, prove that they are smoothly homotopic.

(iii) If $f : M \to M$ is a continuous map without fixed points, prove that there exists a smooth map $g : M \to M$ without fixed points

Hint: Assume M is a submanifold of $\mathbb{H}^n$ as constructed in Corollary A.3.3 so that $\partial M = M \cap \partial \mathbb{H}^n$ and $M \cap (\mathbb{R}^{n-1} \times [0,1]) = Q \times [0,1]$. For $\varepsilon > 0$ define

$$U_\varepsilon := \left\{ p + v \,\middle|\, p \in M \setminus \partial M,\ v \in \mathbb{R}^n,\ v \perp T_pM,\ |v| < \varepsilon \right\}.$$

Prove that for $\varepsilon > 0$ sufficiently small there exists a smooth map

$$r : U_\varepsilon \to M \setminus \partial M$$

such that $r(p + v) = p$ for $p \in M$ and $v \in T_pM^\perp$ with $|v| < \varepsilon$. In (i) assume $f(M) \subset M \backslash \partial M$ and use Weierstraß Approximation to find a smooth map $h : M \to U_\varepsilon$ such that $\sup_{p \in M} |h(p) - f(p)| < \varepsilon$. Define

$$f_t(p) := r((1-t)f(p) + th(p)).$$

If f has no fixed points, choose $\varepsilon < \inf_{p \in M} |p - f(p)|$ to prove (iii).

The **Lefschetz number** $L(f)$ **of a continuous map** $f : M \to M$ is defined as the Lefschetz number of any smooth map that is continuously homotopic to f. By Theorem 4.4.10 and parts (i) and (ii) of Exercise 4.4.12 the integer $L(f)$ is well defined and is a homotopy invariant of f. If f has no fixed points, then $L(f) = 0$ by part (iii) of Exercise 4.4.12. This implies the following classical result.

Corollary 4.4.13 (**Lefschetz Fixed Point Theorem**). *Let* $f : M \to M$ *be a continuous map such that* $L(f) \neq 0$. *Then* f *has a fixed point.*

For example, every continuous map $f : \mathbb{D}^m \to \mathbb{D}^m$ has the Lefschetz number $L(f) = 1$ and hence the Brouwer Fixed Point Theorem follows from the Lefschetz Fixed Point Theorem. More generally, the Lefschetz Fixed Point Theorem is particularly useful in combination with the formula

$$L(f) = \sum_{k=0}^{m} (-1)^k \mathrm{trace}\big(f^* : H^k(M;\mathbb{R}) \to H^k(M;\mathbb{R})\big). \tag{4.4.22}$$

As mentioned before, this formula will be proved in Theorem 6.4.9.

4.4.3 The Lefschetz Number and the Euler Number

The goal in this section is to relate the index of an isolated zero of a vector field X to the fixed point index of the same point for its flow. This will enable us to relate the Euler number of the vector field to the Lefschetz number of its flow. While this relation is a tautology when one uses the homological definitions of these numbers, the relation between the indices of zeros of vector fields and fixed points of their flows requires proof, although not particularly difficult. We will in fact use this relation in Chapter 6 to establish the homological formula for the Euler characteristic.

We first prove that a smooth vector field on a compact manifold cannot have nonconstant periodic orbits of arbitrarily small periods, following an argument by Hofer–Zehnder [14, page 185]. Let $M \subset \mathbb{R}^n$ be a compact smooth m-dimensional submanifold with boundary and let $X \in \mathrm{Vect}(M)$. Thus $X : M \to \mathbb{R}^n$ is a smooth map such that $X(p) \in T_pM$ for every $p \in M$. Define

$$C := \max_{p \in M} \max_{0 \neq v \in T_pM} \frac{|dX(p)v|}{|v|}. \tag{4.4.23}$$

The next lemma shows that nonconstant periodic orbits of X cannot have periods less than $2\pi/C$. This estimate is sharp. The vector field $X(z) = \mathbf{i}z$ on $S^1 \subset \mathbb{C}$ with $C = 1$ has a nonconstant periodic orbit of period 2π.

Lemma 4.4.14 (Hofer–Zehnder). *Let $\gamma : \mathbb{R} \to M$ be a smooth curve that satisfies the differential equation*

$$\dot{\gamma}(t) = X(\gamma(t)), \qquad \gamma(t+T) = \gamma(t) \qquad \text{for all } t \in \mathbb{R}, \tag{4.4.24}$$

with $0 < T < 2\pi/C$. Then $\dot{\gamma}(t) \equiv 0$.

Lemma 4.4.15. *Let $\xi : [0,1] \to \mathbb{R}^n$ be a smooth curve such that $\xi(0) = \xi(1)$ and $\int_0^1 \xi(t)\,dt = 0$. Then*

$$\int_0^1 |\xi(t)|^2\,dt \le \frac{1}{4\pi^2} \int_0^1 |\dot{\xi}(t)|^2\,dt. \tag{4.4.25}$$

The estimate (4.4.25) is again sharp. Equality holds for all curves of the form $\xi(t) = \cos(2\pi t)x + \sin(2\pi t)y$ with $x, y \in \mathbb{R}^n$ (and only for those).

Proof of Lemma 4.4.15. Write ξ as a Fourier series $\xi(t) = \sum_k \xi_k \exp(2\pi \mathbf{i} k t)$ with $\xi_k \in \mathbb{C}^n$ and $\xi_0 = 0$. Then $\dot{\xi}(t) = \sum_k 2\pi \mathbf{i} k \xi_k \exp(2\pi \mathbf{i} k t)$ and hence

$$\int_0^1 |\dot{\xi}(t)|^2\,dt = 4\pi^2 \sum_{k \neq 0} k^2 |\xi_k|^2 \ge 4\pi^2 \sum_{k \neq 0} |\xi_k|^2 = 4\pi^2 \int_0^1 |\xi(t)|^2\,dt.$$

This proves Lemma 4.4.15. □

Proof of Lemma 4.4.14. Define the curve $\xi : \mathbb{R} \to \mathbb{R}^n$ by

$$\xi(t) := X\bigl(\gamma(Tt)\bigr) = \dot{\gamma}(Tt), \qquad \dot{\xi}(t) = TdX\bigl(\gamma(Tt)\bigr)\dot{\gamma}(Tt).$$

It is 1-periodic, has mean value zero, and satisfies $|\dot{\xi}(t)| \le TC|\xi(t)|$. Hence the estimate (4.4.25) in Lemma 4.4.15 implies

$$\int_0^1 |\xi(t)|^2\,dt \le \frac{1}{4\pi^2}\int_0^1 |\dot{\xi}(t)|^2\,dt \le \frac{T^2C^2}{4\pi^2}\int_0^1 |\xi(t)|^2\,dt.$$

Since $TC < 2\pi$ it follows that $\xi(t) \equiv 0$ and hence the curve $\gamma : \mathbb{R} \to M$ is constant. This proves Lemma 4.4.14. □

Now let $\phi_t : M \to M$ be the flow of X defined by

$$\partial_t\phi_t = X \circ \phi_t, \qquad \phi_0 = \mathrm{id}_M.$$

The map ϕ_t is well defined on all of M and for all $t \in \mathbb{R}$ when M has no boundary. If the boundary of M is nonempty, then the flow of X is well defined on all of M in forward time when X points in on the boundary, and in backward time when X points out on the boundary. These are the three cases of interest to us. In all three cases the map $\phi_t : M \to M$ is smooth on all of M and, if $0 < |t| < 2\pi/C$, then it follows from Lemma 4.4.14 that every fixed point of ϕ_t is a zero of the vector field X. In particular, every isolated zero of X is an isolated fixed point of ϕ_t for $0 < |t| < 2\pi/C$ and vice versa. With this understood, we can now compare the indices.

Lemma 4.4.16. *Let X, C, ϕ_t be as above and assume*

$$0 < |t| < \delta := \frac{2\pi}{C}.$$

Then the following holds.

(i) *A point $p \in M \setminus \partial M$ is an isolated zero of X if and only if it is an isolated fixed point of ϕ_t.*

(ii) *A point $p \in M \setminus \partial M$ is a nondegenerate zero of X if and only if it is a nondegenerate fixed point of ϕ_t.*

(iii) *If $p \in M \setminus \partial M$ is an isolated zero of X, then*

$$\iota(p, \phi_t) = \begin{cases} (-1)^m \iota(p, X), & \text{if } 0 < t < \delta, \\ \iota(p, X), & \text{if } -\delta < t < 0. \end{cases} \tag{4.4.26}$$

Proof. We have already noted that part (i) follows from Lemma 4.4.14. To prove part (ii), differentiate the equation $\partial_t\phi_t = X\circ\phi_t$ at p to obtain the formula $\partial_t d\phi_t(p) = DX(p)d\phi_t(p)$ and hence $d\phi_t(p) = \exp(tDX(p))$ for all t. Since $|DX(p)v| \le C|v|$ for all $v\in T_pM$, it follows from Lemma 4.4.14 with M replaced by T_pM and X replaced by $DX(p)$ that

$$\ker\bigl(\mathbb{1} - d\phi_t(p)\bigr) = \ker\bigl(DX(p)\bigr) \qquad \text{for } 0<|t|<\delta.$$

Hence $\mathbb{1} - d\phi_t(p)$ is invertible if and only if $DX(p)$ is invertible, and this proves part (ii).

To prove part (iii), assume first that p is a nondegenerate zero of X. Then, by what we have just observed, $\lim_{t\to0} t^{-1}\bigl(d\phi_t(p) - \mathbb{1}\bigr) = DX(p)$. Since $DX(p)$ is invertible, this implies

$$\mathrm{sign}\bigl(\det(t^{-1}(d\phi_t(p) - \mathbb{1}))\bigr) = \mathrm{sign}\bigl(\det(DX(p))\bigr)$$

for small nonzero t and hence for $0<|t|<\delta$. Thus

$$\mathrm{sign}\bigl(\det(\mathbb{1} - d\phi_t(p))\bigr) = \begin{cases} (-1)^m\mathrm{sign}\bigl(\det(DX(p))\bigr), & \text{if } 0<t<\delta,\\ \mathrm{sign}\bigl(\det(DX(p))\bigr), & \text{if } -\delta<t<0.\end{cases}$$

This proves (4.4.26) in the nondegenerate case. For general isolated zeros the formula reduces to the nondegenerate case by the Perturbation Lemma 2.3.5 for vector fields, the Perturbation Lemma 4.4.7 for maps, and the Local Homotopy Invariance Lemma 4.4.9.. This proves Lemma 4.4.16. □

The next lemma is a consequence of Lemme 4.4.16 and the definitions of the Euler Characteristic (Definition 2.3.11) and the Lefschetz Number (Definition 4.4.11) in terms of index counts.

Lemma 4.4.17. *Let M be a smooth manifold with boundary. Then*

$$L(\mathrm{id}_M) = \chi(M). \tag{4.4.27}$$

Proof. Choose a vector field X on M that points out on the boundary and has only nondegenerate zeros. Denote by $\{\phi_t\}_{t\le0}$ the flow of X, let $C>0$ be the constant in (4.4.23), and choose a number $-2\pi/C<t<0$. Then ϕ_t is isotopic to the identity and, by Lemma 4.4.16,, the fixed points of ϕ_t coincide with the zeros of X and have the same indices. Hence

$$L(\mathrm{id}_M) = L(\phi_t) = \sum_{p\in\mathrm{Fix}(\phi_t)} \iota(p,\phi_t) = \sum_{X(p)=0} \iota(p,X) = \chi(M).$$

This proves Lemma 4.4.17. □

The next theorem sums up what we have proved in this section and lists the properties of the Lefschetz number established so far. These properties characterize the Lefschetz number axiomatically. For a smooth manifold M denote by $\mathrm{Map}(M,M)$ the space of all smooth maps $f : M \to M$.

Theorem 4.4.18. *Let M be a compact manifold with boundary. Then there exists a function*

$$\mathrm{Map}(M,M) \to \mathbb{Z} : f \mapsto L(f), \tag{4.4.28}$$

called the **Lefschetz number***, that satisfies the following axioms for all smooth maps $f, g : M \to M$.*

(Homotopy) *If f is smoothly homotopic to g, then*

$$L(f) = L(g).$$

(Lefschetz) *If f is a Lefschetz map, then*

$$L(f) = \sum_{p \in \mathrm{Fix}(f)} \mathrm{sign}\big(\det(\mathbb{1} - df(p))\big).$$

(Hopf) *If $\mathrm{Fix}(f) \cap \partial M = \emptyset$ and f has only isolated fixed points, then*

$$L(f) = \sum_{p \in \mathrm{Fix}(f)} \iota(p,f).$$

(Euler) *$L(\mathrm{id}_M) = \chi(M)$ is the Euler characteristic of M.*

(Fixed Point) *If $L(f) \neq 0$, then*

$$\mathrm{Fix}(f) \neq \emptyset.$$

(Conjugacy) *If $\phi : M \to M$ is a diffeomorphism, then*

$$L(\phi \circ f \circ \phi^{-1}) = L(f).$$

(Graph) *If M is oriented and $\partial M = \emptyset$, then*

$$L(f) = \mathrm{graph}(f) \cdot \Delta.$$

The map (4.4.28) *is uniquely determined by the (Homotopy) and (Lefschetz) axioms.*

Proof. The uniqueness statement follows from the fact that every smooth map $f : M \to M$ is smoothly homotopic to a Lefschetz map by part (i) of Theorem 4.4.10. That a map (4.4.28) exists that satisfies the (Homotopy) and (Lefschetz) axioms follows from parts (i) and (ii) of Theorem 4.4.10. That this map also satisfies the (Hopf) axiom follows from part (iii) of Theorem 4.4.10. The (Euler) axiom was established in Lemma 4.4.17, and the (Fixed Point) axiom follows directky from the (Lefschetz) axiom, because a smooth mao $f : M \to M$ without fixed points is trivially a Lefschetz map and satisfies $L(f) = 0$. The (Conjugacy) axiom holds because

$$\mathrm{Fix}(\phi \circ f \circ \phi^{-1}) = \phi\big(\mathrm{Fix}(f)\big)$$

and the map

$$\mathrm{Fix}(f) \to \mathrm{Fix}(\phi \circ f \circ \phi^{-1}) : p \mapsto \phi(p)$$

preserves the fixed point indices whenever f is a Lefschetz map. The (Graph) axiom follows from Lemma 4.4.5, and this proves Theorem 4.4.18. $\square$

The axiom that is notably missing in Theorem 4.4.18 is the homological formula that expresses the Lefschetz number as the alternating sum of the traces on the de Rham cohomology groups. This formula will be established in Theorem 6.4.9.

4.4.4 Examples and Exercises

Exercise 4.4.19. Prove that every square matrix A satisfies

$$\det\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ A & \mathbb{1} \end{pmatrix} = \det(\mathbb{1} - A).$$

Relate this formula to the proof of Lemma 4.4.5.

Exercise 4.4.20. If f is homotopic to a constant map, then $L(f) = 1$.

Exercise 4.4.21. Deduce the Brouwer Fixed Point Theorem from the Lefschetz Fixed Point Theorem (Corollary 4.4.13). **Hint:** Show that every continuous map $f : \mathbb{D}^m \to \mathbb{D}^m$ has the Lefschetz number $L(f) = 1$.

Exercise 4.4.22. A smooth map $f : S^1 \to S^1$ has the Lefschetz number

$$L(f) = 1 - \deg(f).$$

Find a smooth map $f : S^1 \to S^1$ of degree 1 without fixed points.

Exercise 4.4.23. A smooth map $f : S^2 \to S^2$ has the Lefschetz number

$$L(f) = 1 + \deg(f).$$

Find a smooth map $f : S^2 \to S^2$ of degree -1 without fixed points.

Exercise 4.4.24. Prove that, for every integer $m \geq 0$, the m-sphere S^m admits a diffeomorphism without fixed points. What is the degree of such a diffeomorphism?

Exercise 4.4.25. Let $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be the map whose lift to $\mathbb{R}^2$ is given by

$$f(x, y) = (ax + by, cx + dy)$$

for $(x, y) \in \mathbb{R}^2$, where $a, b, c, d \in \mathbb{Z}$. Then $\deg(f) = ad - bc$ and

$$L(f) = 1 - a - d + ad - bc = \det \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}. \tag{4.4.29}$$

Each of the maps $f : \mathbb{T}^2 \to \mathbb{T}^2$ in this example has a fixed point. If $L(f) = 0$, prove that f is homotopic to a smooth map without fixed points.

Exercise 4.4.26. Let $A \in \mathbb{Z}^{n\times n}$ be an integer matrix. Prove that the Lefschetz number of the induced map $f : \mathbb{T}^n \to \mathbb{T}^n$ is $L(f) = \det(\mathbb{1} - A)$.

Example 4.4.27. Use Theorem 4.4.18 to show that every compact Lie group of positive dimension has Euler characteristic zero. **Hint:** Find a smooth map without fixed points that is homotopic to the identity.

Exercise 4.4.28. Let $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ be a smooth map. Prove that there exists an integer d such that

$$L(f) = 1 + d + d^2 + \cdots + d^n.$$

If n is even, deduce that every smooth map $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ has a fixed point. If n is odd, find a smooth map $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ without fixed points. **Hint:** Use Theorem 7.3.19 to prove the formula for the Lefschetz number. See also Corollary 7.3.20. If $n = 1$, consider the antipodal map of the 2-sphere.

Exercise 4.4.29. Use the Lefschetz Fixed Point Theorem to prove that every matrix $A \in \mathbb{C}^{n\times n}$ has an eigenvector. **Hint:** Assume $\det(A) \neq 0$ and consider the induced map $\phi_A : \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^{n-1}$. Show that ϕ_A is homotopic to the identity. Deduce that $L(\phi_A) = \chi(\mathbb{C}\mathrm{P}^{n-1}) = n$, so ϕ_A has a fixed point.

Exercise 4.4.30. If n is odd, prove that every matrix $A \in \mathbb{R}^{n\times n}$ has a real eigenvector. **Hint:** Exercise 4.4.29 with $\mathbb{R}\mathrm{P}^{n-1}$ instead of $\mathbb{C}\mathrm{P}^{n-1}$.

Exercise 4.4.31. Deduce the **Fundamental Theorem of Algebra** from Exercise 4.4.29. Use Exercise 4.4.30 to show that every polynomial of odd degree with real coefficients has a real root.

Exercise 4.4.32. Give an alternative proof of the formula

$$L(\mathrm{id}_M) = \chi(M)$$

in Lemma 4.4.17 by examining maps of the form

$$f(p) = \exp_p\bigl(-\varepsilon X(p)\bigr),$$

where X is a vector field that points out on the boundary and has only nondegenerate zeros.

Exercise 4.4.33. Prove that every manifold with boundary admits a vector field that points in on the boundary. **Hint:** In the intrinsic setting consider the function $g : M \to \mathbb{R}$ in Step 6 of the proof of Theorem A.3.1. In the extrinsic setting of Corollary A.3.3 consider the vector field $X(p) = \Pi(p)e_n$ (see Exercise 2.4.14 for the notation $\Pi(p)$).

Exercise 4.4.34. Let M be a compact manifold with boundary.

(i) Let $f : M \to M$ be a smooth map. Prove that f is homotopic to a smooth map $g : M \to M$ such that

$$g(M) \cap \partial M = \emptyset,$$

and hence $\mathrm{Fix}(g) \cap \partial M = \emptyset$. **Hint:** Construct a vector field $X \in \mathrm{Vect}(M)$ that points in on the boundary and compose f with the semi-flow of X.

(ii) Let $f_0, f_1 : M \to M$ be smooth maps such that

$$\mathrm{Fix}(f_0) \cap \partial M = \mathrm{Fix}(f_1) \cap \partial M = \emptyset.$$

Suppose that f_0 and f_1 are smoothly homotopic. Prove that there exists a smooth homotopy $[0,1] \times M \to M : (t.p) \mapsto f_t(p)$ from f_0 to f_1 such that

$$\mathrm{Fix}(f_t) \cap \partial M = \emptyset \qquad \text{for } 0 \le t \le 1.$$

Hint: Given any smooth homotopy $\{f_t\}_{0\le t\le 1}$ from f_0 to f_1 and a vector field $X \in \mathrm{Vect}(M)$ that points in on the boundary, consider the homotopy $g_t := \phi_{t(1-t)} \circ f_t$, where $\{\phi_t\}_{t\ge 0}$ is the semi-flow of X.

Chapter 5

Differential Forms

This chapter begins with an elementary discussion of differential forms on manifolds. §5.1 explains the exterior algebra of a real vector space and its relation to the determinant of a square matrix and introduces differential forms on manifolds. In §5.2 we introduce the exterior differential, define the integral of a compactly supported differential form of top degree over an oriented manifold, and prove Stokes' Theorem. That section includes a brief discussion of de Rham cohomology. In §5.3 we prove Cartan's formula for the Lie derivative of a differential form in the direction of a vector field and use it to show that a differential form of top degree on a compact connected oriented smooth manifold without boundary is exact if and only if its integral vanishes. §5.4 discusses several applications including the Gauß–Bonnet formula and Moser isotopy for volume forms.

5.1 Exterior Algebra

We introduce alternating forms on a vector space (§5.1.1), exterior product and pullback (§5.1.2), and differential forms on manifolds (§5.1.3).

5.1.1 Alternating Forms

We assume throughout that V is an m-dimensional real vector space and fix a positive integer $k \in \mathbb{N}$. Let S_k denote the permutation group on k elements, i.e. the group of all bijective maps $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$. The group operation is given by composition and there is a group homomorphism $\varepsilon : S_k \to \{\pm 1\}$ defined by

$$\varepsilon(\sigma) := (-1)^{\nu(\sigma)}, \qquad \nu(\sigma) := \# \left\{ (i,j) \in \{1, \dots, k\}^2 \,|\, i < j,\ \sigma(i) > \sigma(j) \right\}.$$

Definition 5.1.1. *An* **alternating k-form** *on V is a multi-linear map*

$$\omega : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$

satisfying

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \varepsilon(\sigma)\omega(v_1, \ldots, v_k)$$

for all $v_1, \ldots, v_k \in V$ and all $\sigma \in S_k$. An **alternating 0-form** *is by definition a real number. The vector space of all alternating k-forms on V will be denoted by*

$$\Lambda^k V^* := \left\{ \omega : V^k \to \mathbb{R} \,|\, \omega \text{ is an alternating } k\text{-form} \right\}.$$

For $\omega \in \Lambda^k V^*$ the integer $k =: \deg(\omega)$ is called the **degree** *of ω.*

Example 5.1.2. An alternating 0-form on V is a real number and so

$$\Lambda^0 V^* = \mathbb{R}.$$

Example 5.1.3. An alternating 1-form on V is a linear functional and so

$$\Lambda^1 V^* = V^* := \mathrm{Hom}(V, \mathbb{R}).$$

In the case $V = \mathbb{R}^m$ denote by $dx^i : \mathbb{R}^m \to \mathbb{R}$ the projection onto the ith coordinate, i.e.

$$dx^i(\xi) := \xi^i$$

for $\xi = (\xi^1, \ldots, \xi^m) \in \mathbb{R}^m$ and $i = 1, \ldots, m$. Then the linear functionals $dx^1, \ldots, dx^m$ form a basis of the dual space $(\mathbb{R}^m)^* = \Lambda^1(\mathbb{R}^m)^*$.

Example 5.1.4. An alternating 2-form on V is a skew-symmetric bilinear map $\omega : V \times V \to \mathbb{R}$ so that

$$\omega(v, w) = -\omega(w, v)$$

for all $v, w \in V$. In the case $V = \mathbb{R}^m$ an alternating 2-form can be written in the form

$$\omega(\xi, \eta) = \langle \xi, A\eta \rangle$$

for $\xi, \eta \in \mathbb{R}^m$, where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^m$ and $A = -A^T \in \mathbb{R}^{m \times m}$ is a skew-symmetric matrix. Thus

$$\dim(\Lambda^2 V^*) = \frac{m(m-1)}{2}.$$

for every m-dimensional real vector space V.

Definition 5.1.5. *Let* $\mathcal{I}_k = \mathcal{I}_k(m)$ *denote the set of ordered* k*-tuples*

$$I = (i_1, \dots, i_k) \in \mathbb{N}^k, \qquad 1 \le i_1 < i_2 < \cdots < i_k \le m.$$

For $I = (i_1, \dots, i_k) \in \mathcal{I}_k$ *the alternating* k*-form*

$$dx^I : \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k \text{ times}} \to \mathbb{R}$$

is defined by

$$\begin{aligned} dx^I(\xi_1, \dots, \xi_k) &:= \det \begin{pmatrix} \xi_1^{i_1} & \xi_2^{i_1} & \cdots & \xi_k^{i_1} \\ \xi_1^{i_2} & \xi_2^{i_2} & \cdots & \xi_k^{i_2} \\ \vdots & \vdots & & \vdots \\ \xi_1^{i_k} & \xi_2^{i_k} & \cdots & \xi_k^{i_k} \end{pmatrix} \\ &= \sum_{\sigma \in S_k} \varepsilon(\sigma) \xi_{\sigma(1)}^{i_1} \xi_{\sigma(2)}^{i_2} \cdots \xi_{\sigma(k)}^{i_k} \end{aligned} \tag{5.1.1}$$

for $\xi_j = (\xi_j^1, \dots, \xi_j^m) \in \mathbb{R}^m$, $j = 1, \dots, k$.

Lemma 5.1.6. *The elements* dx^I *for* $I \in \mathcal{I}_k$ *form a basis of* $\Lambda^k(\mathbb{R}^m)^*$. *Thus, for every* m*-dimensional real vector space* V, *we have*

$$\dim(\Lambda^k V^*) = \binom{m}{k} \qquad \text{for } k = 0, 1, \dots, m,$$

and $\Lambda^k V^* = 0$ *for* $k > m$.

Proof. The proof relies on the following three observations.

(1) Let $e_1, \dots, e_m$ be the standard basis of $\mathbb{R}^m$ and let $J = (j_1, \dots, j_k) \in \mathcal{I}_k$. Then, for every $I \in \mathcal{I}_k$, we have

$$dx^I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \ne J. \end{cases}$$

(2) For every $\omega \in \Lambda^k(\mathbb{R}^m)^*$ we have

$$\omega = 0 \qquad \Longleftrightarrow \qquad \omega(e_{i_1}, \dots, e_{i_k}) = 0 \quad \forall I = (i_1, \dots, i_k) \in \mathcal{I}_k.$$

(3) Every $\omega \in \Lambda^k(\mathbb{R}^m)^*$ can be written as

$$\omega = \sum_{I \in \mathcal{I}_k} \omega_I dx^I, \qquad \omega_I := \omega(e_{i_1}, \dots, e_{i_k}).$$

Assertions (1) and (2) follow directly from the definitions and assertion (3) follows from (1) and (2). That the dx^I span the space $\Lambda^k(\mathbb{R}^m)^*$ follows immediately from (3). To prove that the dx^I are linearly independent, assume that the ω_I for $I \in \mathcal{I}_k$ are real numbers such that

$$\omega := \sum_I \omega_I dx^I = 0.$$

Then, by (1), we have $\omega(e_{j_1}, \dots, e_{j_k}) = \omega_J$ for $J = (j_1, \dots, j_k) \in \mathcal{I}_k$ and so

$$\omega_J = 0 \qquad \text{for all } J \in \mathcal{I}_k.$$

This proves Lemma 5.1.6. □

5.1.2 Exterior Product and Pullback

Let $k, \ell \in \mathbb{N}$ be positive integers. The set $S_{k,\ell} \subset S_{k+\ell}$ of (k,ℓ)-**shuffles** is the set of all permutations in $S_{k+\ell}$ that leave the order of the first k and of the last ℓ elements unchanged:

$$S_{k,\ell} := \{\sigma \in S_{k+\ell} \,|\, \sigma(1) < \cdots < \sigma(k),\, \sigma(k+1) < \cdots < \sigma(k+\ell)\}.$$

The terminology arises from *shuffling a card deck* with $k + \ell$ cards.

Definition 5.1.7. *The* **exterior product** *of $\omega \in \Lambda^k V^*$ and $\tau \in \Lambda^\ell V^*$ is the alternating $(k+\ell)$-form $\omega \wedge \tau \in \Lambda^{k+\ell} V^*$ defined by*

$$(\omega \wedge \tau)(v_1, \dots, v_{k+\ell}) := \sum_{\sigma \in S_{k,\ell}} \varepsilon(\sigma) \omega\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right) \tau\left(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}\right)$$

for $v_1, \dots, v_{k+\ell} \in V$.

Exercise 5.1.8. Show that the multi-linear map $\omega \wedge \tau : V^{k+\ell} \to \mathbb{R}$ in Definition 5.1.7 is alternating. **Hint:** Prove the formula

$$(\omega \wedge \tau)(v_2, v_1, v_3, \dots, v_{k+\ell}) = -(\omega \wedge \tau)(v_1, v_2, v_3, \dots, v_{k+\ell})$$

by examining the shuffles $\sigma \in S_{k,\ell}$ in the three cases (a) $\sigma(1) = 1, \sigma(2) = 2$, (b) $\sigma(k+1) = 1, \sigma(k+2) = 2$, (c) $\{\sigma(1), \sigma(k+1)\} = \{1, 2\}$. More generally, prove that swapping v_i and v_{i+1} results in a sign change.

Example 5.1.9. The exterior product of $\alpha, \beta \in \Lambda^1 V^* = V^*$ is the 2-form

$$(\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v).$$

The exterior product of $\alpha \in \Lambda^1 V^* = V^*$ and $\omega \in \Lambda^2 V^*$ is the 3-form

$$(\alpha \wedge \omega)(u, v, w) = \alpha(u)\omega(v, w) + \alpha(v)\omega(w, u) + \alpha(w)\omega(u, v).$$

Lemma 5.1.10. **(i)** *The exterior product is* **distributive***:*

$$\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$$

for $\omega_1, \omega_2, \omega_3 \in \Lambda^* V^*$.

(ii) *The exterior product is* **associative***:*

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$$

for $\omega_1, \omega_2, \omega_3 \in \Lambda^* V^*$.

(iii) *The exterior product is* **super-commutative***:*

$$\omega \wedge \tau = (-1)^{\deg(\omega)\deg(\tau)} \tau \wedge \omega$$

for $\omega, \tau \in \Lambda^* V^*$.

Proof. Part (i) follows directly from the definitons. To prove part (ii), choose alternating forms $\omega_i \in \Lambda^{k_i} V^*$, denote $k := k_1 + k_2 + k_3$, and define

$$S_{k_1,k_2,k_3} := \left\{ \sigma \in S_k \;\middle|\; \begin{array}{l} \sigma(1) < \cdots < \sigma(k_1), \\ \sigma(k_1+1) < \cdots < \sigma(k_1+k_2), \\ \sigma(k_1+k_2+1) < \cdots < \sigma(k) \end{array} \right\},$$

Let $\omega \in \Lambda^k V^*$ be the alternating k-form

$$\begin{aligned} \omega(v_1, \dots, v_k) := \sum_{\sigma \in S_{k_1,k_2,k_3}} & \varepsilon(\sigma) \omega_1 \left(v_{\sigma(1)}, \dots, v_{\sigma(k_1)} \right) \cdot \\ & \cdot \omega_2 \left(v_{\sigma(k_1+1)}, \dots, v_{\sigma(k_1+k_2)} \right) \omega_3 \left(v_{\sigma(k_1+k_2+1)}, \dots, v_{\sigma(k)} \right). \end{aligned}$$

Then by Definition 5.1.7 we have $\omega_1 \wedge (\omega_2 \wedge \omega_3) = \omega = (\omega_1 \wedge \omega_2) \wedge \omega_3$, and this proves (ii).

To prove assertion (iii) we define the bijection $S_{k,\ell} \to S_{\ell,k} : \sigma \mapsto \widetilde{\sigma}$ by

$$\widetilde{\sigma}(i) := \begin{cases} \sigma(k+i), & \text{for } i = 1, \dots, \ell, \\ \sigma(i-\ell), & \text{for } i = \ell+1, \dots, \ell+k. \end{cases}$$

Then $\varepsilon(\widetilde{\sigma}) = (-1)^{k\ell} \varepsilon(\sigma)$ and hence

$$\begin{aligned} & (\omega \wedge \tau)(v_1, \dots, v_{k+\ell}) \\ & = \sum_{\sigma \in S_{k,\ell}} \varepsilon(\sigma) \omega \left(v_{\sigma(1)}, \dots, v_{\sigma(k)} \right) \tau \left(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)} \right) \\ & = (-1)^{k\ell} \sum_{\widetilde{\sigma} \in S_{\ell,k}} \varepsilon(\widetilde{\sigma}) \omega \left(v_{\widetilde{\sigma}(\ell+1)}, \dots, v_{\widetilde{\sigma}(\ell+k)} \right) \tau \left(v_{\widetilde{\sigma}(1)}, \dots, v_{\widetilde{\sigma}(\ell)} \right) \\ & = (-1)^{k\ell} (\tau \wedge \omega)(v_1, \dots, v_{k+\ell}). \end{aligned}$$

for $\omega \in \Lambda^k V^*$, $\tau \in \Lambda^\ell V^*$, and $v_i \in V$. This proves Lemma 5.1.10. □

Exercise 5.1.11. The **Determinant Theorem** asserts that

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, \dots, v_k) = \det \begin{pmatrix} \alpha_1(v_1) & \alpha_1(v_2) & \cdots & \alpha_1(v_k) \\ \alpha_2(v_1) & \alpha_2(v_2) & \cdots & \alpha_2(v_k) \\ \vdots & \vdots & & \vdots \\ \alpha_k(v_1) & \alpha_k(v_2) & \cdots & \alpha_k(v_k) \end{pmatrix} \tag{5.1.2}$$

for all $\alpha_1, \dots, \alpha_k \in V^*$ and $v_1, \dots, v_k \in V$. Prove this and deduce that

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for $I = (i_1, \dots, i_k) \in \mathcal{I}_k$, where $dx^I \in \Lambda^k(\mathbb{R}^m)^*$ is given by (5.1.1).

Exercise 5.1.12. An alternating k-form $\theta \in \Lambda^k V^*$ is called **decomposable** iff there exist linear functionals $\alpha_1, \dots, \alpha_k \in V^*$ such that $\theta = \alpha_1 \wedge \cdots \wedge \alpha_k$. This notion extends to complex valued alternating k-forms $\theta \in \Lambda^k V^* \otimes_{\mathbb{R}} \mathbb{C}$. Now suppose V has real dimension $2n$ and let $\theta \in \Lambda^n V^* \otimes_{\mathbb{R}} \mathbb{C}$ be a nonzero complex valued alternating n-form. Prove that θ is decomposable if and only if there exists a **linear complex structure** $J : V \to V$ (i.e. a linear map $J : V \to V$ with $J \circ J = -1\!\!1$) such that θ is complex multi-linear with respect to J. Prove that, in this situation, J is uniquely determined by θ.

Definition 5.1.13 (**Pullback**). *Let $\Phi : V \to W$ be a linear map between real vector spaces. The* **pullback** *of an alternating k-form $\omega \in \Lambda^k W^*$ under Φ is the alternating k-form $\Phi^*\omega \in \Lambda^k V^*$ defined by*

$$(\Phi^*\omega)(v_1, \dots, v_k) := \omega(\Phi v_1, \dots, \Phi v_k)$$

for $v_1, \dots, v_k \in V$.

Lemma 5.1.14. **(i)** *The map $\Lambda^* W \to \Lambda^* V : \omega \mapsto \Phi^*\omega$ is linear and preserves the exterior product, i.e. $\Phi^*(\omega \wedge \tau) = \Phi^*\omega \wedge \Phi^*\tau$ for all $\omega \in \Lambda^k W^*$ and all $\tau \in \Lambda^\ell W^*$.*

(ii) *If $\Psi : W \to Z$ is another linear map with values in a real vector space Z, then $(\Psi \circ \Phi)^*\omega = \Phi^*\Psi^*\omega$ for every $\omega \in \Lambda^k Z^*$. Moreover, if $\mathrm{id} : V \to V$ denotes the identity map, then $\mathrm{id}^*\omega = \omega$ for all $\omega \in \Lambda^k V^*$.*

(iii) *If $\Phi : V \to V$ is an endomorphism of an m-dimensional real vector space V, then $\Phi^*\omega = \det(\Phi)\omega$ for all $\omega \in \Lambda^m V^*$.*

Proof. Assertions (i) and (ii) follow directly from the definitions. By (ii) it suffices to prove (iii) for $V = \mathbb{R}^m$. In this case assertion (iii) can be written in the form $\Phi^*\left(dx^1 \wedge \cdot \wedge dx^m\right) = \det(\Phi) dx^1 \wedge \cdot \wedge dx^m$ for $\Phi \in \mathbb{R}^{m \times m}$, and this follows from (5.1.1) and the product formula for the determinant. This proves Lemma 5.1.14. □

5.1.3 Differential Forms on Manifolds

Let M be a smooth m-manifold and let k be a nonnegative integer.

Definition 5.1.15 (**Differential Form).** *A* **differential** k**-form** *on M is a collection of alternating k-forms*

$$\omega_p : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ times}} \to \mathbb{R},$$

one for each element $p \in M$, such that, for every k-tuple of smooth vector fields $X_1, \ldots, X_k \in \mathrm{Vect}(M)$, the function

$$M \to \mathbb{R} : p \mapsto \omega_p\big(X_1(p), \ldots, X_k(p)\big)$$

is smooth. The set of differential k-forms on M will be denoted by $\Omega^k(M)$. A differential form $\omega \in \Omega^k(M)$ is said to have **compact support** *iff the set*

$$\mathrm{supp}(\omega) := \overline{\{p \in M \,|\, \omega_p \neq 0\}}$$

(called the **support** *of ω) is compact. The set of compactly supported k-forms on M will be denoted by $\Omega^k_c(M) \subset \Omega^k(M)$. As before we call the integer $k =: \deg(\omega)$ the* **degree** *of $\omega \in \Omega^k(M)$.*

Remark 5.1.16. The set

$$\Lambda^k T^*M := \left\{(p, \omega) \,\middle|\, p \in M,\, \omega \in \Lambda^k T^*M\right\}$$

is a vector bundle over M. This concept will be discussed in detail in §7.1. We remark here that $\Lambda^k T^*M$ admits the structure of a smooth manifold, the obvious projection $\pi : \Lambda^k T^*M \to M$ is a smooth submersion, each fiber $\Lambda^k T_p^*M$ is a vector space, and addition and scalar multiplication define smooth maps. The manifold structure is uniquely determined by the fact that each differential k-form $\omega \in \Omega^k(M)$ defines a smooth map

$$M \to \Lambda^k T^*M : p \mapsto (p, \omega_p),$$

still denoted by ω. Its composition with π is the identity on M and such a map is called a smooth section of the vector bundle. Thus $\Omega^k(M)$ can be identified the space of smooth sections of $\Lambda^k T^*M$. It is a vector space and is infinite-dimensional (unless M is a finite set or $k > \dim(M)$). In particular, for $k = 0$ we have $\Lambda^0 T^*M = M \times \mathbb{R}$ and the space

$$\Omega^0(M) = \{f : M \to \mathbb{R} \,|\, f \text{ is smooth}\}$$

is the set of smooth real valued functions on M, also denoted by $\mathcal{F}(M)$ or $C^\infty(M, \mathbb{R})$ or simply $C^\infty(M)$.

Definition 5.1.17 (Exterior Product and Pullback). *The (pointwise)* **exterior product** *of $\omega \in \Omega^k(M)$ and $\tau \in \Omega^\ell(M)$ is the differential $(k+\ell)$-form $\omega \wedge \tau \in \Omega^{k+\ell}(M)$ given by*

$$(\omega \wedge \tau)_p := \omega_p \wedge \tau_p \tag{5.1.3}$$

for $p \in M$. If $f : M \to N$ is a smooth map between smooth manifolds and $\omega \in \Omega^k(N)$ is a differential k-form on N, its **pullback** *under f is the differential k-form $f^*\omega \in \Omega^k(M)$ defined by*

$$(f^*\omega)_p(v_1, \dots, v_k) := \omega_{f(p)}(df(p)v_1, \dots, df(p)v_k) \tag{5.1.4}$$

for $p \in M$ and $v_1, \dots, v_k \in T_pM$.

The next lemma summarizes the basic properties of the exterior product and pullback of differential forms.

Lemma 5.1.18. *Let M, N, P be smooth manifolds.*

(i) *The exterior product is distributive, i.e.*

$$\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$$

for all $\omega_1 \in \Omega^k(M)$ and all $\omega_2, \omega_3 \in \Omega^\ell(M)$.

(ii) *The exterior product is associative, i.e.*

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$$

for all $\omega_1, \omega_2, \omega_3 \in \Omega^*(M)$.

(iii) *The exterior product is graded commutative, i.e.*

$$\omega \wedge \tau = (-1)^{\deg(\omega)\deg(\tau)} \tau \wedge \omega$$

for all $\omega, \tau \in \Omega^*(M)$.

(iv) *Pullback is linear and preserves the exterior product, i.e.*

$$f^*(\omega \wedge \tau) = f^*\omega \wedge f^*\tau$$

for all $\omega, \tau \in \Omega^*(N)$ and all smooth maps $f : M \to N$.

(v) *Pullback is contravariant, i.e. $(g \circ f)^*\omega = f^*g^*\omega$ for all $\omega \in \Omega^k(P)$ and all smooth maps $f : M \to N$ and $g : N \to P$. Moreover, $\mathrm{id}^*\omega = \omega$ for all $\omega \in \Omega^k(M)$, where $\mathrm{id} : M \to M$ denotes the identity map.*

(vi) *Pullback satisfies the following naturality condition. If $\phi : M \to N$ is a diffeomorphism and $\omega \in \Omega^k(N)$ and $X_1, \dots, X_k \in \mathrm{Vect}(N)$, then*

$$(\phi^*\omega)(\phi^*X_1, \dots, \phi^*X_k) = \omega(X_1, \dots, X_k) \circ \phi.$$

Proof. Assertions (i), (ii) and (iii) follow from Lemma 5.1.10, assertion (iv) follows from Lemma 5.1.14, (v) follows from Lemma 5.1.14 and the chain rule, and (vi) follows directly from the definitions. □

Remark 5.1.19 (Differential forms in local coordinates). Let M be an m-manifold equipped with an atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$. Thus the U_α form an open cover of M and each map $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism onto an open subset of $\mathbb{R}^m$ (or of the upper half space $\mathbb{H}^m$ in case M has a nonempty boundary) such that the transition maps

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are smooth. In this situation every differential k-form $\omega \in \Omega^k(M)$ determines a family of differential k-forms $\omega_\alpha \in \Omega^k(\phi_\alpha(U_\alpha))$, one for each $\alpha \in A$, such that the restriction of ω to U_α (denoted by $\omega|_{U_\alpha}$ and defined as the pullback of ω under the inclusion of U_α into M) is given by

$$\omega|_{U_\alpha} = \phi_\alpha^* \omega_\alpha \tag{5.1.5}$$

for every $\alpha \in A$. Explicitly, if

$$p \in U_\alpha, \qquad v_i \in T_pM, \qquad x := \phi_\alpha(p), \qquad \xi_i := d\phi_\alpha(p)v_i$$

for $i = 1, \dots, k$, then

$$\omega_\alpha(x; \xi_1, \dots, \xi_k) = \omega_p(v_1, \dots, v_k). \tag{5.1.6}$$

Recall that $v_i \in T_pM$ and $\xi_i \in \mathbb{R}^m$ are related by $v_i = [\alpha, \xi_i]_p$ in the tangent space model $T_pM = \bigcup_{p \in U_\alpha} \{\alpha\} \times \mathbb{R}^m / \sim$. Now let $e_1, \dots e_m$ denote the standard basis of $\mathbb{R}^m$ and define the functions $\omega_{\alpha,I} : U_\alpha \to \mathbb{R}$ by

$$\omega_{\alpha,I}(x) := \omega_\alpha\left(x; e_{i_1}, \dots, e_{i_k}\right) = \omega_p\left([\alpha, e_{i_1}]_p, \dots, [\alpha, e_{i_k}]_p\right)$$

for $x \in \phi_\alpha(U_\alpha)$, $p := \phi_\alpha^{-1}(x) \in U_\alpha$, and $I = (i_1, \dots, i_k) \in \mathcal{I}_k$. Then the differential form $\omega_\alpha \in \Omega^k(\phi_\alpha(U_\alpha))$ can be written in the form

$$\omega_\alpha = \sum_{I \in \mathcal{I}_k} \omega_{\alpha,I} dx^I. \tag{5.1.7}$$

Remark 5.1.20. The differential forms $\omega_\alpha \in \Omega^k(\phi_\alpha(U_\alpha))$ in local coordinates satisfy the equation

$$\omega_\alpha|_{\phi_\alpha(U_\alpha \cap U_\beta)} = \left(\phi_\beta \circ \phi_\alpha^{-1}\right)^* \omega_\beta|_{\phi_\beta(U_\alpha \cap U_\beta)} \tag{5.1.8}$$

for all $\alpha, \beta \in A$. Conversely, every family of k-forms $\phi_\alpha \in \Omega^k(\phi_\alpha(U_\alpha))$ that satisfy (5.1.8) for all $\alpha, \beta \in A$ determine a unique k-form $\omega \in \Omega^k(M)$ such that (5.1.5) holds for every $\alpha \in A$.

5.2 The Exterior Differential and Integration

In this Section we first introduce the exterior differential of a differential form on an open set in $\mathbb{R}^m$ and establish its basic properties (§5.2.1). The definition of the exterior differential of a differential form on a manifold is then a straight forward construction in local coordinates. We will explain the integral of a compactly supported m-form on an oriented m-manifold in §5.2.2 and prove the Theorem of Stokes in §5.2.3.

5.2.1 The Exterior Differential

Let $U \subset \mathbb{R}^m$ be an open set. The exterior differential on U is a linear operator $d : \Omega^k(U) \to \Omega^{k+1}(U)$.

Definition 5.2.1. *Let $\omega \in \Omega^k(U)$. Then ω is a smooth map*

$$\omega : U \times \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k \text{ times}} \to \mathbb{R}$$

such that, for every $x \in U$, the map

$$\underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k \text{ times}} \to \mathbb{R} : (\xi_1, \dots, \xi_k) \mapsto \omega(x; \xi_1, \dots, \xi_k)$$

is an alternating k-form on $\mathbb{R}^m$. The **exterior differential of** *ω is the $(k+1)$-form $d\omega \in \Omega^{k+1}(U)$ defined by*

$$\begin{aligned} &d\omega\,(x; \xi_1, \dots, \xi_{k+1}) \\ &:= \sum_{j=1}^{k+1} (-1)^{j-1} \left.\frac{d}{dt}\right|_{t=0} \omega\left(x + t\xi_j; \xi_1, \dots, \widehat{\xi_j}, \dots, \xi_{k+1}\right) \end{aligned} \tag{5.2.1}$$

for $x \in U$ and $\xi_1, \dots, \xi_{k+1} \in \mathbb{R}^m$. Here the hat indicates that the jth term is deleted.

Example 5.2.2. Let $f \in \Omega^0(U)$ be a smooth real valued function on U. Then $df \in \Omega^1(U)$ is the usual differential of f, which assigns to each element $x \in U$ the derivative $df(x) : \mathbb{R}^m \to \mathbb{R}$, given by

$$df(x; \xi) = df(x)\xi = \lim_{t \to 0} \frac{f(x + t\xi) - f(x)}{t} = \sum_{\nu=1}^m \frac{\partial f}{\partial x^\nu}(x)\xi^\nu$$

for $\xi = (\xi^1, \dots, \xi^m) \in \mathbb{R}^m$. Here the last equality asserts that the derivative of f at x is given by multiplication with the Jacobi matrix. Thus

$$df = \sum_{\nu=1}^m \frac{\partial f}{\partial x^\nu} dx^\nu.$$

In local coordinates a differential form can be written in the form (5.1.6) or (5.1.7). The formula in Definition 5.2.1 corresponds to (5.1.6) and the next lemma expresses the differential in terms of the functions ω_I in (5.1.7).

Lemma 5.2.3. *Let* $\omega \in \Omega^k(U)$ *and, for* $I = (i_1, \dots, i_k) \in \mathcal{I}_k$*, define the function* $\omega_I : U \to \mathbb{R}$ *by*

$$\omega_I(x) := \omega(x; e_{i_1}, \dots, e_{i_k})$$

for $x \in U$*. Then*

$$\omega = \sum_{I \in \mathcal{I}_k} \omega_I dx^I$$

and its exterior differential is the $(k+1)$*-form*

$$d\omega = \sum_{I \in \mathcal{I}_k} d\omega_I \wedge dx^I, \qquad d\omega_I := \sum_{\nu=1}^m \frac{\partial \omega_I}{\partial x^\nu} dx^\nu. \tag{5.2.2}$$

Proof. For $x \in U$ and $\xi_1, \dots, \xi_k \in \mathbb{R}^m$ we have

$$\omega(x; \xi_1, \dots, \xi_k) = \sum_{I \in \mathcal{I}_k} \omega_I(x) dx^I(\xi_1, \dots, \xi_k)$$

Hence, by (5.2.1), we have

$$\begin{aligned}
& d\omega(x; \xi_1, \dots, \xi_{k+1}) \\
&= \sum_{I \in \mathcal{I}_k} \sum_{j=1}^{k+1} (-1)^{j-1} \left.\frac{d}{dt}\right|_{t=0} \omega_I(x + t\xi_j) dx^I \left(\xi_1, \dots, \widehat{\xi_j}, \dots, \xi_{k+1}\right) \\
&= \sum_{I \in \mathcal{I}_k} \sum_{j=1}^{k+1} (-1)^{j-1} \left(\sum_{\nu=1}^m \frac{\partial \omega_I}{\partial x^\nu}(x) \xi_j^\nu\right) dx^I \left(\xi_1, \dots, \widehat{\xi_j}, \dots, \xi_{k+1}\right) \\
&= \sum_{I \in \mathcal{I}_k} \sum_{j=1}^{k+1} (-1)^{j-1} d\omega_I(x; \xi_j) dx^I \left(\xi_1, \dots, \widehat{\xi_j}, \dots, \xi_{k+1}\right) \\
&= \sum_{I \in \mathcal{I}_k} \left(d\omega_I \wedge dx^I\right)(x; \xi_1, \dots, \xi_{k+1})
\end{aligned}$$

for all $x \in U$ and $\xi_1, \dots, \xi_{k+1} \in \mathbb{R}^m$. The last term agrees with the right hand side of (5.2.2). This proves Lemma 5.2.3 □

Lemma 5.2.4. *Let $U \subset \mathbb{R}^m$ be an open set.*

(i) *The exterior differential $d : \Omega^k(U) \to \Omega^{k+1}(U)$ is a linear operator.*

(ii) *If $\omega \in \Omega^k(U)$ and $\tau \in \Omega^\ell(U)$, then*

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg(\omega)}\omega \wedge d\tau.$$

(iii) *The exterior differential satisfies $d \circ d = 0$.*

(iv) *The exterior differential commutes with pullback: If $f : U \to V$ is a smooth map to an open subset $V \subset \mathbb{R}^n$, then for every $\omega \in \Omega^k(V)$ we have*

$$f^*d\omega = df^*\omega.$$

Part (ii) of Lemma 5.2.4 follows from the Leibniz rule, part (iii) follows from Schwarz's Theorem which asserts that the second partial derivatives commute, and part (iv) follows from the chain rule.

Proof of Lemma 5.2.4. Assertion (i) follows directly from the definition. To prove part (ii) it suffices to consider two differential forms

$$\omega = f dx^I, \qquad \tau = g dx^J$$

with $I = (i_1, \dots, i_k) \in \mathcal{I}_k$, $J = (j_1, \dots, j_\ell) \in \mathcal{I}_\ell$, and $f, g : U \to \mathbb{R}$. Then it follows from Lemma 5.2.3 that

$$\begin{aligned} d(\omega \wedge \tau) &= d(fg dx^I \wedge dx^J) = d(fg) \wedge dx^I \wedge dx^J \\ &= \bigl(g df + f dg\bigr) \wedge dx^I \wedge dx^J \\ &= (df \wedge dx^I) \wedge (g dx^J) + (-1)^k (f dx^I) \wedge (dg \wedge dx^J) \\ &= d\omega \wedge \tau + (-1)^k \omega \wedge d\tau. \end{aligned}$$

For general differential forms part (ii) follows from the special case and (i).

We prove part (iii). For $f \in \Omega^0(U)$ we have

$$ddf = d\left(\sum_{j=1}^m \frac{\partial f}{\partial x_j} dx^j\right) = \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j = 0.$$

Hence, for every smooth function $f : U \to \mathbb{R}$ and every $I = (i_1, \dots, i_k) \in \mathcal{I}_k$,

$$dd(f dx^I) = d(df \wedge dx^I) = ddf \wedge dx^I - df \wedge ddx^I = 0.$$

Here the second equality follows from (ii) and the last equality holds because $ddf = 0$ and $ddx^I = 0$. This proves (iii).

We prove part (iv). Denote the elements of U by $x = (x^1, \dots, x^m)$, the elements of V by $y = (y^1, \dots, y^n)$, and the coordinates of $f(x)$ by

$$f(x) =: (f^1(x), \dots, f^n(x))$$

for $x \in U$. Thus each f^j s a smooth real valued function on U and the value of f^*dy^j at $(x, \xi) \in U \times \mathbb{R}^m$ is the jth coordinate of $df(x)\xi$. Thus

$$f^*dy^j = df^j = \sum_{i=1}^m \frac{df^j}{dx^i} dx^i. \tag{5.2.3}$$

Moreover, if $g \in \Omega^0(V)$ is a smooth real valued function on V, then

$$f^*g = g \circ f, \qquad dg = \sum_{j=1}^n \frac{\partial g}{\partial y^j} dy^j.$$

Hence

$$\begin{aligned} f^*dg &= f^*\left(\sum_{j=1}^n \frac{\partial g}{\partial y^j} dy^j\right) \\ &= \sum_{j=1}^n \left(f^*\frac{\partial g}{\partial y^j}\right) f^*dy^j \\ &= \sum_{i=1}^m \sum_{j=1}^n \left(\frac{\partial g}{\partial y^j} \circ f\right) \frac{\partial f^j}{\partial x^i} dx^i \\ &= \sum_{i=1}^m \frac{\partial (g \circ f)}{\partial x^i} dx^i \\ &= d(f^*g) \end{aligned} \tag{5.2.4}$$

Here the third equality uses (5.2.3) and the fourth equality follows from the chain rule. For $J = (j_1, \dots, j_k) \in \mathcal{I}_k$ we have

$$d(f^*dy^J) = d(f^*dy^{j_1} \wedge \cdots \wedge f^*dy^{j_k}) = d(df^{j_1} \wedge \cdots \wedge df^{j_k}) = 0. \tag{5.2.5}$$

Here the first equality follows from Lemma 5.1.18 and the determinant theorem in Exercise 5.1.11, the second equality follows from (5.2.3), and the last equality follows from the Leibniz rule in (ii) and the fact that $ddf^j = 0$ for every j by part (iii). Combining (5.2.4) and (5.2.5) we obtain

$$\begin{aligned} df^*(g dy^J) &= d\big((f^*g) f^*dy^J\big) = d(f^*g) \wedge f^*dy^J = f^*dg \wedge f^*dy^J \\ &= f^*\big(dg \wedge dy^J\big) = f^*d(g dy^J) \end{aligned}$$

for all $g \in \Omega^0(V)$ and all $J \in \mathcal{I}_k$. This proves (iv) and Lemma 5.2.4. □

Definition 5.2.5 (The Exterior Differential on Manifolds). *Let M be a smooth m-manifold with an atlas $\{U_\alpha, \phi_\alpha\}_{\alpha\in A}$ and let $\omega \in \Omega^k(M)$. Denote by $\omega_\alpha \in \Omega^k(\phi_\alpha(U_\alpha))$ the differential forms in local coordinates so that*

$$\omega|_{U_\alpha} = \phi_\alpha^* \omega_\alpha \tag{5.2.6}$$

for every $\alpha \in A$. The **exterior differential** *of ω is defined as the unique $(k+1)$-form $d\omega \in \Omega^{k+1}(M)$ that satisfies*

$$d\omega|_{U_\alpha} = \phi_\alpha^* d\omega_\alpha \tag{5.2.7}$$

for every $\alpha \in A$.

To see that such a $(k+1)$-form exists, recall that the ω_α satisfy (5.1.8) for all $\alpha, \beta \in A$. Since the exterior differential commutes with pullback, by part (iv) of Lemma 5.2.4, we have

$$d\omega_\alpha|_{\phi_\alpha(U_\alpha\cap U_\beta)} = (\phi_\beta \circ \phi_\alpha^{-1})^* d\omega_\beta|_{\phi_\beta(U_\alpha\cap U_\beta)}$$

for all $\alpha, \beta \in A$. Hence the existence and uniqueness of the $(k+1)$-form $d\omega$ satisfying equation (5.2.7) follows from Remark 5.1.20.

Lemma 5.2.6. *Let M be a smooth manifold.*

(i) *The exterior differential*

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

is a linear operator.

(ii) *The exterior differential satisfies the* **Leibniz rule**

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg(\omega)} \omega \wedge d\tau.$$

(iii) *The exterior differential satisfies $d \circ d = 0$.*

(iv) *The exterior differential commutes with pullback: If $f : M \to N$ is a smooth map between manifolds, then for every $\omega \in \Omega^k(N)$ we have*

$$f^* d\omega = df^*\omega.$$

Proof. This follows immediately from Lemma 5.2.4 and the definitions. □

Definition 5.2.7 (De Rham Cohomology). *By Lemma 5.2.6 there is a cochain complex*

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M),$$

called the **de Rham complex**. *A differential form $\omega \in \Omega^k(M)$ is called* **closed** *iff $d\omega = 0$ and is called* **exact** *iff there exists a $\tau \in \Omega^{k-1}(M)$ such that $d\tau = \omega$. Lemma 5.2.6 (iii) asserts that every exact k-form is closed and the quotient space*

$$\begin{aligned} H^k(M) &:= \frac{\ker d : \Omega^k(M) \to \Omega^{k+1}(M)}{\mathrm{im}\ d : \Omega^{k-1}(M) \to \Omega^k(M)} \\ &= \frac{\{\textit{closed } k\textit{-forms on } M\}}{\{\textit{exact } k\textit{-forms on } M\}} \end{aligned}$$

is called the kth **de Rham cohomology group** *of M. By Lemma 5.2.6 (i) the kth de Rham cohomology group $H^k(M)$ is a real vector space and, by Lemma 5.2.6 (ii), the exterior product defines a bilinear map*

$$H^k(M) \times H^\ell(M) \to H^{k+\ell}(M) : ([\omega], [\tau]) \mapsto [\omega] \cup [\tau] := [\omega \wedge \tau]$$

called the **cup product**. *By Lemma 5.1.18 (iv) every smooth map*

$$f : M \to N$$

induces a **pullback homomorphism**

$$f^* : H^k(N) \to H^k(M).$$

By Lemma 5.1.18 this map is linear and preserves the cup product.

Example 5.2.8. The de Rham cohomology group $H^0(M)$ is the space of smooth functions $f : M \to \mathbb{R}$ whose differential vanishes everywhere. Thus $H^0(M)$ is the space of locally constant real valued functions on M. If M is connected, this shows that $H^0(M) \cong \mathbb{R}$.

To gain a better understanding of the de Rham cohomology groups we introduce the integral of a differential form of maximal degree over a compact oriented manifold (§5.2.2), prove the theorem of Stokes (§5.2.3), and prove the formula of Cartan for the Lie derivative of a differential form in the direction of a vector field (§5.3.1).

5.2.2 Integration

Let M be an oriented m-manifold, with or without boundary and not necessarily compact. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ be an oriented atlas on M. Thus the sets U_α form an open cover of M and the maps

$$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$$

are homeomorphisms onto open subsets $\phi_\alpha(U_\alpha) \subset \mathbb{H}^m$ of the upper half space

$$\mathbb{H}^m := \{x \in \mathbb{R}^m \,|\, x^m \geq 0\}$$

such that the transition maps

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are smooth and

$$\det\left(d\phi_{\beta\alpha}(x)\right) > 0$$

for all $\alpha, \beta \in A$ and all $x \in \phi_\alpha(U_\alpha \cap U_\beta)$. Choose a partition of unity

$$\rho_\alpha : M \to [0,1], \qquad \alpha \in A,$$

subordinate to the open cover $\{U_\alpha\}_{\alpha \in A}$. Thus each point $p \in M$ has a neighborhood on which only finitely many of the ρ_α do not vanish and

$$\mathrm{supp}(\rho_\alpha) \subset U_\alpha, \qquad \sum_\alpha \rho_\alpha \equiv 1.$$

Definition 5.2.9. *Let $\omega \in \Omega_c^m(M)$ be a differential form with compact support and, for $\alpha \in A$, let*

$$\omega_\alpha \in \Omega^m(\phi_\alpha(U_\alpha)), \qquad g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}$$

be given by

$$\omega|_{U_\alpha} =: \phi_\alpha^* \omega_\alpha, \qquad \omega_\alpha =: g_\alpha(x) dx^1 \wedge \cdots \wedge dx^m.$$

The **integral of** ω **over** M *is the real number*

$$\int_M \omega := \sum_{\alpha \in A} \int_{\phi_\alpha(U_\alpha)} \rho_\alpha(\phi_\alpha^{-1}(x)) g_\alpha(x) dx^1 \cdots dx^m \tag{5.2.8}$$

The sum on the right is finite because only finitely many of the products $\rho_\alpha \omega$ are nonzero. (Prove this!)

Lemma 5.2.10. *The integral of ω over M is independent of the choice of the oriented atlas and the partition of unity used to define it.*

Proof. Choose another atlas $\{V_\beta, \psi_\beta\}_{\beta \in B}$ on M and a partition of unity $\theta_\beta : M \to [0,1]$ subordinate to the cover $\{V_\beta\}_{\beta\in B}$. For $\beta \in B$ define

$$\omega_\beta \in \Omega^m(\psi_\beta(V_\beta)), \qquad h_\beta : \psi_\beta(V_\beta) \to \mathbb{R}$$

by

$$\omega|_{V_\beta} =: \psi_\beta^*\omega_\beta, \qquad \omega_\beta =: h_\beta(y)dy^1 \wedge \cdots \wedge dy^m.$$

Then it follows from Lemma 5.1.14 (iv) that

$$g_\alpha(x) = h_\beta\left(\psi_\beta \circ \phi_\alpha^{-1}(x)\right) \underbrace{\det\left(d(\psi_\beta \circ \phi_\alpha^{-1})(x)\right)}_{>0} \tag{5.2.9}$$

for every $x \in \phi_\alpha(U_\alpha \cap V_\beta)$. Hence

$$\begin{aligned}
\int_M \omega &= \sum_{\alpha\in A} \int_{\phi_\alpha(U_\alpha)} (\rho_\alpha \circ \phi_\alpha^{-1}) g_\alpha dx^1 \cdots dx^m \\
&= \sum_\alpha \sum_\beta \int_{\phi_\alpha(U_\alpha \cap V_\beta)} (\rho_\alpha \circ \phi_\alpha^{-1})(\theta_\beta \circ \phi_\alpha^{-1}) g_\alpha dx^1 \cdots dx^m \\
&= \sum_\alpha \sum_\beta \int_{\psi_\beta(U_\alpha \cap V_\beta)} (\rho_\alpha \circ \psi_\beta^{-1})(\theta_\beta \circ \psi_\beta^{-1}) h_\beta dy^1 \cdots dy^m \\
&= \sum_\beta \int_{\psi_\beta(V_\beta)} (\theta_\beta \circ \psi_\beta^{-1}) h_\beta dy^1 \cdots dy^m.
\end{aligned}$$

Here the first equation is the definition of the integral, the second equation follows from the fact that the θ_β form a partition of unity, the third equation follows from (5.2.9) and the change of variables formula, and the last equation follows from the fact that the ρ_α form a partition of unity. This proves Lemma 5.2.10. □

One can think of the integral as a functional

$$\Omega_c^m(M) \to \mathbb{R} : \omega \mapsto \int_M \omega.$$

It follows directly from the definition that this functional is linear.

Exercise 5.2.11. If $f : M \to N$ is an orientation preserving diffeomorphism between oriented m-manifolds, then $\int_M f^*\omega = \int_N \omega$ for every $\omega \in \Omega_c^m(N)$. If $f : M \to N$ is an orientation reversing diffeomorphism between oriented m-manifolds, then $\int_M f^*\omega = -\int_N \omega$ for every $\omega \in \Omega_c^m(N)$.

5.2.3 The Theorem of Stokes

In this section we prove the following fundamental theorem about the integrals of differential forms. It is the *Fundamental Theorem of Calculus* for manifolds.

Theorem 5.2.12 (**Stokes**). *Let M be an oriented m-manifold with boundary and let $\omega \in \Omega_c^{m-1}(M)$. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. The proof has three steps.

Step 1. *The theorem holds for $M = \mathbb{H}^m$.*

The boundary of the upper half space

$$\mathbb{H}^m = \left\{ x = (x^1, \ldots, x^m) \in \mathbb{R}^m \,|\, x^m \geq 0 \right\}$$

is the subset

$$\partial\mathbb{H}^m = \left\{ x = (x^1, \ldots, x^m) \in \mathbb{R}^m \,|\, x^m = 0 \right\},$$

diffeomorphic to $\mathbb{R}^{m-1}$. Consider the differential $(m-1)$-form

$$\omega = \sum_{i=1}^m g_i(x) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m$$

where the

$$g_i : \mathbb{H}^m \to \mathbb{R}$$

are smooth functions with compact support (in the closed upper half space) and the hat indicates that the ith term is deleted in the ith summand. Then the differential of ω is the m-form

$$\begin{aligned} d\omega &= \sum_{i=1}^m \frac{\partial g_i}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m \\ &= \sum_{i=1}^m (-1)^{i-1} \frac{\partial g_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^m. \end{aligned}$$

Choose $R > 0$ so large that the support of each coordinate g_i is contained in the set $[-R, R]^{m-1} \times [0, R]$. Then

$$\begin{aligned}
\int_{\mathbb{H}^m} d\omega &= \sum_{i=1}^{m} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial g_i}{\partial x^i}(x^1, \ldots, x^m) dx^1 \cdots dx^m \\
&= (-1)^{m-1} \int_{-R}^R \cdots \int_{-R}^R \int_0^R \frac{\partial g_m}{\partial x^m}(x^1, \ldots, x^m) dx^m dx^1 \cdots dx^{m-1} \\
&= (-1)^m \int_{-R}^R \cdots \int_{-R}^R g_m(x^1, \ldots, x^{m-1}, 0) dx^1 \cdots dx^{m-1} \\
&= \int_{\partial \mathbb{H}^m} \omega
\end{aligned}$$

Here the first equation follows from the Fundamental Theorem of Calculius, the second equation follows from Fubini's Theorem, and the third equation follows again from the Fundamental Theorem of Calculus. To understand the last equation we observe that the restriction of ω to the boundary is

$$\omega|_{\partial \mathbb{H}^m} = g_m(x^1, \ldots, x^{m-1}, 0)\, dx^1 \wedge \cdots \wedge dx^{m-1}.$$

Moreover, the orientation of $\mathbb{R}^{m-1}$ as the boundary of $\mathbb{H}^m$ is $(-1)^m$ times the standard orientation of $\mathbb{R}^{m-1}$ because the outward pointing unit normal vector at any boundary point is

$$\nu = (0, \ldots, 0, -1).$$

This proves the last equation above and completes the proof of Step 1.

Step 2. *We prove Theorem 5.2.12 for every differential $(m-1)$-form whose support is compact and contained in a coordinate chart.*

Let $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{H}^m$ be a coordinate chart and let $\omega \in \Omega_c^{m-1}(M)$ be a compactly supported differential form with

$$\mathrm{supp}(\omega) \subset U_\alpha.$$

Define $\omega_\alpha \in \Omega^{m-1}(\phi_\alpha(U_\alpha))$ by

$$\omega|_{U_\alpha} =: \phi_\alpha^* \omega_\alpha$$

and extend ω_α to all of $\mathbb{H}^m$ by setting ω_α equal to zero on $\mathbb{H}^m \setminus \phi_\alpha(U_\alpha)$.

Since $\phi_\alpha(U_\alpha \cap \partial M) = \phi_\alpha(U_\alpha) \cap \partial\mathbb{H}^m$ we obtain, using Step 1, that

$$\begin{aligned}\int_M d\omega &= \int_{U_\alpha} d\phi_\alpha^*\omega_\alpha \\ &= \int_{U_\alpha} \phi_\alpha^* d\omega_\alpha \\ &= \int_{\phi_\alpha(U_\alpha)} d\omega_\alpha \\ &= \int_{\phi_\alpha(U_\alpha)\cap\partial\mathbb{H}^m} \omega_\alpha \\ &= \int_{U_\alpha\cap\partial M} \phi_\alpha^*\omega_\alpha \\ &= \int_{\partial M} \omega.\end{aligned}$$

This proves Step 2.

Step 3. *We prove Theorem 5.2.12.*

Choose an atlas $\{U_\alpha, \phi_\alpha\}_\alpha$ and a partition of unity $\rho_\alpha : M \to [0,1]$ subordinate to the cover $\{U_\alpha\}_\alpha$. Then, by Step 2, we have

$$\int_M d\omega = \sum_\alpha \int_M d(\rho_\alpha\omega) = \sum_\alpha \int_{\partial M} \rho_\alpha\omega = \int_{\partial M} \omega.$$

This proves Step 3 and Theorem 5.2.12. □

Example 5.2.13. Let $U \subset \mathbb{R}^2$ be a bounded open set with connected smooth boundary $\Gamma := \partial U$. Orient Γ as the boundary of U and choose an oriented parametrization of Γ by an embedded loop $\mathbb{R}/\mathbb{Z} \to \Gamma : t \mapsto (x(t), y(t))$. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions and define $\omega := f dx + g dy \in \Omega^1(\mathbb{R}^2)$. Then

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

and hence, by Stokes' theorem, we have

$$\begin{aligned}\int_U \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy &= \int_\Gamma (f dx + g dy) \\ &= \int_0^1 \left(f(x(t), y(t))\dot{x}(t) + g(x(t), y(t))\dot{y}(t)\right) dt.\end{aligned}$$

Example 5.2.14. Let

$$\Sigma \subset \mathbb{R}^3$$

be a 2-dimensional embedded surface and let

$$\nu : \Sigma \to S^2$$

be a Gauß map. Thus $\nu(x) \perp T_x\Sigma$ for every $x \in \Sigma$. Define the 2-form

$$\mathrm{dvol}_\Sigma \in \Omega^2(\Sigma)$$

by

$$\mathrm{dvol}_\Sigma(x; v, w) := \det(\nu(x), v, w)$$

for $x \in \Sigma$ and $v, w \in T_x\Sigma$. In other words

$$\mathrm{dvol}_\Sigma = \nu^1 dx^2 \wedge dx^3 + \nu^2 dx^3 \wedge dx^1 + \nu^3 dx^1 \wedge dx^2,$$

$$\nu^1 \mathrm{dvol}_\Sigma = dx^2 \wedge dx^3, \quad \nu^2 \mathrm{dvol}_\Sigma = dx^3 \wedge dx^1, \quad \nu^3 \mathrm{dvol}_\Sigma = dx^1 \wedge dx^2.$$

Let $u = (u_1, u_2, u_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth map and consider the 1-form

$$\omega = u_1 dx^1 + u_2 dx^2 + u_3 dx^3 \in \Omega^1(\Sigma).$$

Its exterior differential is

$$d\omega = \langle \mathrm{curl}(u), \nu \rangle \, \mathrm{dvol}_\Sigma, \qquad \mathrm{curl}(u) := \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix},$$

and hence Stokes' theorem gives the identity

$$\int_\Sigma \langle \mathrm{curl}(u), \nu \rangle \mathrm{dvol}_\Sigma = \int_{\partial\Sigma} \sum_{i=1}^3 u_i dx^i.$$

Example 5.2.15. Let M be an oriented m-manifold without boundary and let $\tau \in \Omega_c^{m-1}(M)$ be a compactly supported $(m-1)$-form. Then

$$\int_M d\tau = 0$$

by Stokes' theorem. We prove in the next section that, when M is connected, the converse holds as well, i.e. if $\omega \in \Omega_c^m(M)$ satisfies

$$\int_M \omega = 0,$$

then there exists a compactly supported $(m-1)$-form $\tau \in \Omega_c^{m-1}(M)$ such that $d\tau = \omega$.

5.3 The Lie Derivative

This section introduces the Lie derivative of a differential form in the direction of a vector field and establishes Cartan's formula (§5.3.1). As an application of this formula we prove that a differential form of top degree on a compact connected oriented manifold without boundary is exact if and only if its integral vanishes (§5.3.2). For the Lie bracket of two vector fields we will use the sign convention in [35, §2.4.3].

5.3.1 Cartan's Formula

Assume throughout that M is a smooth m-manifold without boundary. The *Lie derivative* of any object on M (such as a vector field or a differential form or a Riemannian metric or an endomorphism of the tangent bundle) in the direction of a vector field is defined as the derivative at time zero of the pullback of the object under the flow of the vector field. For differential forms this leads to the following definition.

Definition 5.3.1 (Lie Derivative). *Let $\omega \in \Omega^k(M)$ and let $X \in \mathrm{Vect}(M)$.*
(i) *If X is complete and $\phi_t \in \mathrm{Diff}(M)$ denotes the flow of X, then the* **Lie derivative of ω in the direction of** *X is defined by*

$$\mathcal{L}_X\omega := \frac{d}{dt}\bigg|_{t=0} \phi_t^*\omega.$$

This formula continues to be meaningful pointwise even if X is not complete.
(ii) *The* **interior product** *(or* **contraction***) of the vector field X with ω is the $(k-1)$-form $\iota(X)\omega \in \Omega^{k-1}(M)$ defined by*

$$(\iota(X)\omega)_p(v_1, \dots, v_{k-1}) := \omega_p(X(p), v_1, \dots, v_{k-1})$$

for $p \in M$ and $v_1, \dots, v_{k-1} \in T_pM$.

Cartan's formula for the Lie derivative is the key identity for many computations with differential forms.

Theorem 5.3.2 (Cartan). *The Lie derivative of a differential form ω in the direction of a vector field X is given by*

$$\mathcal{L}_X\omega = d\iota(X)\omega + \iota(X)d\omega. \tag{5.3.1}$$

We will deduce Theorem 5.3.2 from the following more general result.

Theorem 5.3.3 (Cartan). *Let M and N be smooth manifolds, let $I \subset \mathbb{R}$ be an interval, and let $I \times M \to N : (t,p) \mapsto f_t(p)$ be a smooth map. For $t \in I$ define the operator $h_t : \Omega^k(N) \to \Omega^{k-1}(M)$ by*

$$(h_t\omega)_p(v_1,\dots,v_{k-1}) := \omega_{f_t(p)}\left(\partial_t f_t(p), df_t(p)v_1,\dots,df_t(p)v_{k-1}\right) \tag{5.3.2}$$

for $\omega \in \Omega^k(N)$ and $v_1,\dots,v_{k-1} \in T_pM$. Then

$$\frac{d}{dt}f_t^*\omega = dh_t\omega + h_t d\omega \tag{5.3.3}$$

for all $\omega \in \Omega^k(N)$ and all $t \in I$.

Proof. The proof has four steps.

Step 1. *Equation* (5.3.3) *holds for $k = 0$.*

Let $g : N \to \mathbb{R}$ be a smooth function. Then

$$\frac{d}{dt}(f_t^*g)(p) = \frac{d}{dt}g(f_t(p)) = dg(f_t(p))\frac{\partial f_t}{\partial t}(p) = h_t dg(p)$$

as claimed.

Step 2. *Equation* (5.3.3) *holds for $k = 1$.*

Assume first that $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. Let

$$I \times \mathbb{R}^m \to \mathbb{R}^n : (t,x) \mapsto f_t(x) = (f_t^1(x),\dots,f_t^n(x))$$

be a smooth map, let $g_\nu : \mathbb{R}^n \to \mathbb{R}$ be a smooth function for $\nu = 1,\dots,n$, and define

$$\beta = \sum_{\nu=1}^n g_\nu dy^\nu \in \Omega^1(\mathbb{R}^n).$$

Then

$$d\beta = \sum_{\mu,\nu=1}^n \frac{\partial g_\nu}{\partial y^\mu} dy^\mu \wedge dy^\nu, \qquad h_t\beta = \sum_{\nu=1}^n (g_\nu \circ f_t)\frac{\partial f_t^\nu}{\partial t},$$

$$h_t d\beta = \sum_{\mu,\nu=1}^n \left(\frac{\partial g_\nu}{\partial y^\mu} \circ f_t\right)\left(\frac{\partial f_t^\mu}{\partial t}df_t^\nu - \frac{\partial f_t^\nu}{\partial t}df_t^\mu\right),$$

$$dh_t\beta = \sum_{\nu=1}^n (g_\nu \circ f_t) d\frac{\partial f_t^\nu}{\partial t} + \sum_{\mu,\nu=1}^n \frac{\partial f_t^\nu}{\partial t} d(g_\nu \circ f_t).$$

Moreover, $f_t^*\beta = \sum_{\nu=1}^n (g_\nu \circ f_t) df_t^\nu$ and hence

$$\begin{aligned}
\frac{d}{dt} f_t^*\beta &= \sum_{\nu=1}^n \frac{\partial}{\partial t}\bigl((g_\nu \circ f_t) df_t^\nu\bigr) \\
&= \sum_{\mu,\nu=1}^n \left(\frac{\partial g_\nu}{\partial y^\mu} \circ f_t\right) \frac{\partial f_t^\mu}{\partial t} df_t^\nu + \sum_{\nu=1}^n (g_\nu \circ f_t) \frac{\partial}{\partial t} df_t^\nu \\
&= dh_t\beta + h_t d\beta
\end{aligned}$$

as claimed. This proves Step 2 for $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. The general case follows from this special case via local coordinates.

Step 3. *The operator* $h_t : \Omega^*(M) \to \Omega^{*-1}(M)$ *is linear and satisfies*

$$h_t(\omega \wedge \tau) = h_t\omega \wedge f_t^*\tau + (-1)^{\deg(\omega)} f_t^*\omega \wedge h_t\tau$$

for all $\omega, \tau \in \Omega^*(M)$.

This follows directly from the definitions.

Step 4. *Equation* (5.3.3) *holds for every* $\omega \in \Omega^k(M)$ *and every* $k \geq 0$.

We prove this by induction on k. For $k = 0$ and $k = 1$ the assertion was proved in Step 1 and Step 2. Thus let $k \geq 2$ and assume that the assertion has been established for $k - 1$. Since every k-form $\omega \in \Omega^k(N)$ can be written as a finite sum of exterior products of a 1-form and a $(k-1)$-form it suffices to assume that $\omega = \beta \wedge \tau$, where $\beta \in \Omega^1(N)$ and $\tau \in \Omega^{k-1}(N)$. Then

$$\begin{aligned}
\frac{d}{dt} f_t^*\omega &= \left(\frac{d}{dt} f_t^*\beta\right) \wedge f_t^*\tau + f_t^*\beta \wedge \left(\frac{d}{dt} f_t^*\tau\right) \\
&= \bigl(dh_t\beta + h_t d\beta\bigr) \wedge f_t^*\tau + f_t^*\beta \wedge \bigl(dh_t\tau + h_t d\tau\bigr) \\
&= d(h_t\beta \wedge f_t^*\tau) - h_t\beta \wedge df_t^*\tau \\
&\quad + h_t(d\beta \wedge \tau) - f_t^* d\beta \wedge h_t\tau \\
&\quad - d(f_t^*\beta \wedge h_t\tau) + df_t^*\beta \wedge h_t\tau \\
&\quad - h_t(\beta \wedge d\tau) + h_t\beta \wedge f_t^* d\tau \\
&= d(h_t\beta \wedge f_t^*\tau - f_t^*\beta \wedge h_t\tau) + h_t(d\beta \wedge \tau - \beta \wedge d\tau) \\
&= dh_t\omega + h_t d\omega.
\end{aligned}$$

Here the first equality follows from Lemma 5.1.18 and the Leibniz rule, the second equality follows from Step 2 and the induction hypothesis, the third equality follows from Step 3 and the Leibniz rule for the exterior derivative, the fourth equality follows from the fact that the exterior derivative commutes with pullback, and the last equality follows again from Step 3 and the Leibniz rule for the exterior derivative. This proves Theorem 5.3.3. □

Proof of Theorem 5.3.2. Assume for simplicity that X is complete and let ϕ_t be the flow of X. Then the operator $h_t : \Omega^k(M) \to \Omega^{k-1}(M)$ in (5.3.2) is given by $h_t\omega = \phi_t^*\iota(X)\omega$. In particular, $h_0\omega = \iota(X)\omega$ and hence (5.3.1) follows from (5.3.3) with $f_t = \phi_t$ and $t = 0$. This proves Theorem 5.3.2. □

Corollary 5.3.4. *Let $\omega \in \Omega^k(M)$ and $X_1, \dots, X_{k+1} \in \mathrm{Vect}(M)$. Then*

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} \mathcal{L}_{X_i} \Big(\omega\big(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}\big) \Big) \\ &\quad + \sum_{i<j} (-1)^{i+j-1} \omega\big([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}\big) \end{aligned} \tag{5.3.4}$$

Proof. For $k = 0$ the equation is a tautology. Let $k \geq 1$ and assume by induction that the assertion holds with k replaced by $k - 1$. Let $\omega \in \Omega^k(M)$ and let $X_1, \dots, X_{k+1} \in \mathrm{Vect}(M)$. and define $\tau := (-1)^{k-1}\iota(X_{k+1})\omega \in \Omega^{k-1}(M)$. Then it follows from the induction hypothesis that

$$\begin{aligned} d\tau(X_1, \dots, X_k) &= \sum_{i=1}^{k} (-1)^{i-1} \mathcal{L}_{X_i} \big(\omega\big(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}\big) \big) \\ &\quad + \sum_{1 \leq i < j \leq k} (-1)^{i+j-1} \omega\big([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}\big) \end{aligned} \tag{5.3.5}$$

Now assume that X_{k+1} is complete and denote by ϕ_t the flow of X_{k+1}. Then $\omega(X_1, \dots, X_k) \circ \phi_t = (\phi_t^*\omega)(\phi_t^*X_1, \dots, \phi_t^*X_k)$ for all t by part (vi) of Lemma 5.1.18. Differentiate this equation and use Theorem 5.3.2 to obtain

$$\begin{aligned} \mathcal{L}_{X_{k+1}}\big(\omega\big(X_1, \dots, X_k\big)\big) &= (\mathcal{L}_{X_{k+1}}\omega)\big(X_1, \dots, X_k\big) \\ &\quad + \sum_{i=1}^{k} (-1)^{i-1} \omega\big([X_i, X_{k+1}], X_1, \dots, \widehat{X_i}, \dots, X_k\big) \\ &= (-1)^k d\omega\big(X_1, \dots, X_{k+1}\big) - (-1)^k d\tau\big(X_1, \dots, X_k\big) \\ &\quad + \sum_{i=1}^{k} (-1)^{i-1} \omega\big([X_i, X_{k+1}], X_1, \dots, \widehat{X_i}, \dots, X_k\big). \end{aligned}$$

Hence

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= d\tau(X_1, \dots, X_k) + (-1)^k \mathcal{L}_{X_{k+1}}\big(\omega\big(X_1, \dots, X_k\big)\big) \\ &\quad + \sum_{i=1}^{k} (-1)^{i+k} \omega\big([X_i, X_{k+1}], X_1, \dots, \widehat{X_i}, \dots, X_k\big) \end{aligned}$$

Insert (5.3.5) into this formula to obtain (5.3.4) and Corollary 5.3.4. □

Exercise 5.3.5. Prove the formula (5.3.4) directly in local coordinates.

Example 5.3.6. For $\beta \in \Omega^1(M)$ and $X, Y \in \mathrm{Vect}(M)$ equation (5.3.4) takes the form

$$d\beta(X,Y) = \mathcal{L}_X\big(\beta(Y)\big) - \mathcal{L}_Y\big(\beta(X)\big) + \beta([X,Y]). \tag{5.3.6}$$

For $\omega \in \Omega^2(M)$ and $X, Y, Z \in \mathrm{Vect}(M)$ equation (5.3.4) takes the form

$$\begin{aligned} d\omega(X,Y,Z) &= \mathcal{L}_X\big(\omega(Y,Z)\big) + \mathcal{L}_Y\big(\omega(Z,X)\big) + \mathcal{L}_Z\big(\omega(X,Y)\big) \\ &\quad + \omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y). \end{aligned} \tag{5.3.7}$$

Exercise 5.3.7. Deduce the formula (5.3.1) in Theorem 5.3.2 from (5.3.4) by an induction argument, starting with $k = 1$.

Exercise 5.3.8. Deduce the formula (5.3.3) in Theorem 5.3.3 from (5.3.1). **Hint:** Assume first that the map $f_t : M \to N$ is an embedding for every t. Then prove that there exists a smooth family of vector fields $Y_t \in \mathrm{Vect}(N)$ such that $Y_t \circ f_t = \partial_t f_t$ for all t. For example, choose a Riemannian metric on N and take $Y_t(\exp_{f_t(p)}(w)) := \rho(|w|) d\exp_{f_t(p)}(w)\partial_t f_t(p)$ for $w \in T_{f_t(p)}N$ and a suitable cutoff function ρ. Let ψ_t be the isotopy generated by Y_t via

$$\partial_t \psi_t = Y_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}.$$

Show that $f_t = \psi_t \circ f_0$ for all t. Now deduce (5.3.3) from (5.3.1) for $\mathcal{L}_{Y_t}\omega$. To prove (5.3.3) in general, replace the map $f_t : M \to N$ by the embedding $M \to M \times N : p \mapsto F_t(p) := (p, f_t(p))$ and argue as above.

Corollary 5.3.9. *Let M^m and N^n be oriented manifolds without boundary and let $f_t : M \to N$, $0 \le t \le 1$, be a proper smooth homotopy, so that*

$$K \subset N \text{ is compact} \qquad \Longrightarrow \qquad \bigcup_t f_t^{-1}(K) \subset M \text{ is compact.}$$

Let $\omega \in \Omega^k_c(N)$ be closed k-form with compact support. Then there exists a $(k-1)$-form $\tau \in \Omega^{k-1}_c(M)$ with compact support such that

$$d\tau = f_1^*\omega - f_0^*\omega.$$

Proof. By Theorem 5.3.3, we have

$$f_1^*\omega - f_0^*\omega = \int_0^1 \frac{d}{dt} f_t^*\omega \, dt = \int_0^1 dh_t\omega \, dt = d\tau,$$

where $\tau := \int_0^1 h_t\omega \, dt$ and $h_t\omega \in \Omega^{k-1}(M)$ is given by (5.3.2). Moreover,

$$\mathrm{supp}(\tau) \subset \bigcup_{0 \le t \le 1} f_t^{-1}(\mathrm{supp}(\omega))$$

and so τ has compact support. This proves Corollary 5.3.9. □

5.3.2 Integration and Exactness

The integral of an exact top degree form over a compact oriented manifold without boundary vanishes by Stokes' Theorem 5.2.12. If M is connected, the converse holds as well, so if the integral vanishes, the form is exact. This result extends to compactly supported differential forms on noncompact manifolds. The proof uses Corollary 5.3.9.

Theorem 5.3.10. *Let M be a connected oriented m-dimensional manifold without boundary and let $\omega \in \Omega_c^m(M)$ be an m-form with compact support. Then the the following are equivalent.*

(i) *The integral of ω over M vanishes.*

(ii) *There is an $(m-1)$-form τ on M with compact support such that $d\tau = \omega$.*

Proof. That (ii) implies (i) follows from Stokes' Theorem 5.2.12. We prove in two steps that (i) implies (ii).

Step 1. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function whose support is contained in the set $(a,b)^m$ where $a < b$ and assume that*

$$\int_{\mathbb{R}^m} f = 0.$$

Then there exist smooth functions $u_i : \mathbb{R}^m \to \mathbb{R}$ for $i = 1, \dots, m$, whose support is contained in the open set $(a,b)^m$, such that

$$f = \sum_{i=1}^m \frac{\partial u_i}{\partial x^i}.$$

Thus

$$f dx^1 \wedge \cdots \wedge dx^m = d\left(\sum_{i=1}^m (-1)^{i-1} u_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m\right).$$

To see this, choose a smooth function $\rho : \mathbb{R} \to [0,1]$ such that

$$\rho(t) = \begin{cases} 0, & \text{for } t \leq a + \varepsilon, \\ 1, & \text{for } t \geq b - \varepsilon, \end{cases} \qquad \text{for some } \varepsilon > 0.$$

Define $f_i : \mathbb{R}^m \to \mathbb{R}$ by $f_0 := 0$, $f_m := f$, and, for $i = 1, \dots, m-1$, by

$$f_i(x) := \int_a^b \cdots \int_a^b f(x^1, \dots, x^i, \xi^{i+1}, \dots, \xi^m) \dot\rho(x^{i+1}) \cdots \dot\rho(x^m) d\xi^{i+1} \cdots d\xi^m.$$

Then each f_i is supported in $(a,b)^m$. For $i=1,\dots,m$ and $x\in\mathbb{R}^n$ define

$$\begin{aligned} u_i(x) &:= \int_a^{x_i} (f_i - f_{i-1})(x^1,\dots,x^{i-1},\xi,x^{i+1},\dots,x^m)\,d\xi \\ &= \int_a^{x_i} f_i(x^1,\dots,x^{i-1},\xi,x^{i+1},\dots,x^m)\,d\xi \\ &\quad - \rho(x_i)\int_a^b f_i(x^1,\dots,x^{i-1},\xi,x^{i+1},\dots,x^m)\,d\xi. \end{aligned}$$

Here the second equality holds for $i\geq 2$ by definition of f_i and it holds for $i=1$ because $\int_{\mathbb{R}^m} f = 0$. Thus each u_i is supported in $(a,b)^m$ and

$$\frac{\partial u_i}{\partial x^i} = f_i - f_{i-1}.$$

This proves Step 1.

Step 2. *We prove that (i) implies (ii).*

Choose a point $p_0\in M$, an open neighborhood $U_0\subset M$ of p_0, and an orientation preserving coordinate chart $\phi_0 : U_0\to\mathbb{R}^m$ such that

$$\phi_0(U_0)\subset(0,1)^m.$$

Since M is connected and has no boundary there exists, for every $p\in M$, a diffeomorphism $\psi_p : M\to M$, isotopic to the identity, such that

$$\psi_p(p_0) = p.$$

(Note the use of Lemma 1.4.5 and the axiom of choice.) Thus the sets

$$U_p := \psi_p(U_0)$$

form an open cover of M. Choose a partition of unity $\rho_p : M\to[0,1]$ that is subordinate to this cover. Since the set $K := \mathrm{supp}(\omega)$ is compact there are only finitely many points $p\in M$ such that the function ρ_p does not vanish on K. Number these points as $p_1,\dots,p_n$ and abbreviate

$$\rho_i := \rho_{p_i},\qquad \psi_i := \psi_{p_i},\qquad U_i := U_{p_i}\qquad \text{for } i=1,\dots,n.$$

Then $\mathrm{supp}(\rho_i)\subset U_i = \psi_i(U_0)$ for all i and $\sum_{i=1}^n \rho_i|_K \equiv 1$. Hence

$$\mathrm{supp}(\rho_i\omega)\subset U_i,\qquad \mathrm{supp}(\psi_i^*(\rho_i\omega))\subset U_0.$$

Since ψ_i is smoothly isotopic to the identity and $\rho_i\omega$ has compact support, it follows from Corollary 5.3.9 that there exists a compactly supported $(m-1)$-form $\tau_i \in \Omega_c^{m-1}(M)$ such that

$$d\tau_i = \psi_i^*(\rho_i\omega) - \rho_i\omega.$$

Hence it follows from Stokes' theorem 5.2.12 that

$$\int_M \sum_{i=1}^n \psi_i^*(\rho_i\omega) = \int_M \sum_{i=1}^n \rho_i\omega = \int_M \omega = 0.$$

Now $\psi_i^*(\rho_i\omega)$ is supported in $\psi_i^{-1}(U_i) = U_0$ and so is $\sum_{i=1}^n \psi_i^*(\rho_i\omega)$. Hence the pushforward of this sum under the chart $\phi_0 : U_0 \to \mathbb{R}^m$ has support contained in the set $(0,1)^m = \phi_0(U_0)$ and thus can be smoothly extended to all of $\mathbb{R}^m$ by setting it equal to zero on $\mathbb{R}^m \setminus (0,1)^m$. Moreover,

$$\int_{\mathbb{R}^m} (\phi_0)_* \sum_{i=1}^n \psi_i^*(\rho_i\omega) = \int_M \sum_{i=1}^n \psi_i^*(\rho_i\omega) = 0.$$

Hence, by Step 1 there exists an $(m-1)$-form $\tau_0 \in \Omega_c^{m-1}(\mathbb{R}^m)$ with support in $(0,1)^m$ such that

$$d\tau_0 = (\phi_0)_* \sum_{i=1}^n \psi_i^*(\rho_i\omega).$$

Thus $\phi_0^*\tau_0 \in \Omega_c^{m-1}(U_0)$ has compact support in U_0 and therefore extends to all of M by setting it equal to zero on $M \setminus U_0$. This extension satisfies

$$d\phi_0^*\tau_0 = \sum_{i=1}^n \psi_i^*(\rho_i\omega).$$

Hence the $(m-1)$-form

$$\tau := \phi_0^*\tau_0 - \sum_{i=1}^n \tau_i \in \Omega_c^{m-1}(M)$$

satisfies

$$\begin{aligned} d\tau &= d\phi_0^*\tau_0 - \sum_{i=1}^n d\tau_i \\ &= \sum_{i=1}^n \psi_i^*(\rho_i\omega) - \sum_{i=1}^n \Big(\psi_i^*(\rho_i\omega) - \rho_i\omega\Big) \\ &= \omega. \end{aligned}$$

This proves Theorem 5.3.10. □

Exercise 5.3.11. Assume M is a compact connected oriented m-manifold without boundary. Let Λ be a manifold and let

$$\Lambda \to \Omega^m(M) : \lambda \mapsto \omega_\lambda$$

be a smooth family of m-forms on M such that $\int_M \omega_\lambda = 0$ for every $\lambda \in \Lambda$. Prove that there exists a smooth family of $(m-1)$-forms

$$\Lambda \to \Omega^{m-1}(M) : \lambda \mapsto \tau_\lambda$$

such that $d\tau_\lambda = \omega_\lambda$ for all $\lambda \in \Lambda$. **Hint:** Use the argument in the proof of Theorem 5.3.10 to construct a linear operator

$$h : \left\{\omega \in \Omega^m(M) \,\Big|\, \int_M \omega = 0\right\} \to \Omega^{m-1}(M)$$

such that

$$\int_M \omega = 0 \qquad \Longrightarrow \qquad dh\omega = \omega$$

for every $\omega \in \Omega^m(M)$. Find an explicit formula for the operator h. Note that U_i, ρ_i, ψ_i can be chosen once and for all, independently of ω.

Corollary 5.3.12. *Let M be a compact connected oriented m-manifold without boundary. Then the map*

$$\Omega^m(M) \to \mathbb{R} : \omega \mapsto \int_M \omega$$

induces an ismorphism $H^m(M) \cong \mathbb{R}$.

Proof. The kernel of this map is the space of exact forms, by Theorem 5.3.10. Hence the induced homomorphism on de Rham cohomology is bijective. $\square$

Exercise 5.3.13. Let M be a compact connected nonorientable m-manifold without boundary. Prove that every m-form on M is exact and hence

$$H^m(M) = 0.$$

Hint: Let $\pi : \widetilde{M} \to M$ be the oriented double cover of M. More precisely, a point in $\widetilde{M}$ is a pair (p, o) consisting of a point $p \in M$ and an orientation o of T_pM. Prove that $\widetilde{M}$ is a compact connected oriented m-dimensional manifold without boundary and that $\pi : \widetilde{M} \to M$ is a local diffeomorphism. Prove that the integral of $\pi^*\omega$ vanishes over $\widetilde{M}$ for every $\omega \in \Omega^m(M)$.

5.4 Volume Forms

A volume form on a smooth manifold is a nowhere vanishing differential form of top degree. The existence of a volume form is equivalent to orientability. For smooth maps between closed connected oriented manifolds of the same dimension the degree theorem asserts that the integral of the pullback of a volume form is the product of the degree with the integral of the original volume form (§5.4.1). A corollary of this result and the Poincaré–Hopf Theorem 2.3.1 is the Gauß–Bonnet formula (§5.4.2). In §5.4.3 we introduce Moser isotopy for volume forms.

5.4.1 Integration and Degree

Let M and N be compact oriented smooth m-manifolds without boundary and suppose that N is connected.

Theorem 5.4.1 (**Degree Formula**). *For every smooth map $f : M \to N$ and every $\omega \in \Omega^m(N)$, we have*

$$\int_M f^*\omega = \deg(f) \int_N \omega$$

Proof. Let $q \in N$ be a regular value of f. Then $f^{-1}(q)$ is a finite subset of M. Denote its elements by $p_1, \dots, p_n$ and define $\varepsilon_i = \pm 1$ according to whether or not $df(p_i)$ is orientation preserving or orientation reversing. Thus

$$f^{-1}(q) = \{p_1, \dots, p_n\}, \quad \varepsilon_i = \operatorname{sign}\det(df(p_i)), \quad \deg(f) = \sum_{i=1}^n \varepsilon_i. \tag{5.4.1}$$

Next we observe that there are open neighborhoods $V \subset N$ of q and $U_i \subset M$ of p_i for $i = 1, \dots, n$ satisfying the following conditions.

(a) f restricts to a diffeomorphism from U_i to V for every i; it is orientation preserving when $\varepsilon_i = 1$ and orientation reversing when $\varepsilon_i = -1$.

(b) The sets U_i are pairwise disjoint.

(c) $f^{-1}(V) = U_1 \cup \cdots \cup U_n$.

In fact, since $df(p_i) : T_{p_i}M \to T_qN$ is a vector space isomorphism, it follows from the implicit function theorem that there are connected open neighborhoods U_i of p_i and V_i of q such that $f|_{U_i} : U_i \to V_i$ is a diffeomorphism. Shrinking the sets U_i, if necessary, we may assume $U_i \cap U_j = \emptyset$ for $i \neq j$. Now take $V := V_1 \cap \cdots \cap V_n \setminus f(M \setminus (U_1 \cup \cdots \cup U_n))$ and replace U_i by the set $U_i \cap f^{-1}(V)$. These sets satisfy (a), (b), and (c).

If $\omega \in \Omega^m(N)$ is supported in V, then

$$\begin{aligned}\int_M f^*\omega &= \sum_{i=1}^n \int_{U_i} f^*\omega \\ &= \sum_{i=1}^n \varepsilon_i \int_V \omega \\ &= \deg(f) \int_N \omega.\end{aligned}$$

Here the first equality follows from (b) and (c), the second equality follows from (a) and Exercise 5.2.11, and the last equality follows from (5.4.1).

Now let $\omega \in \Omega^m(N)$ be any m-form and choose $\omega' \in \Omega^m(N)$ such that

$$\operatorname{supp}(\omega') \subset V, \qquad \int_N \omega' = \int_N \omega.$$

Then, by Theorem 5.3.10, there exists an $(m-1)$-form

$$\tau \in \Omega^{m-1}(N)$$

such that

$$d\tau = \omega - \omega'.$$

Hence

$$\begin{aligned}\int_M f^*\omega &= \int_M f^*(\omega' + d\tau) \\ &= \int_M f^*\omega' \\ &= \deg(f) \int_N \omega' \\ &= \deg(f) \int_N \omega.\end{aligned}$$

Here the last but one equality follows from the fact that ω' is supported in V. This proves Theorem 5.4.1. □

Theorem 5.4.1 allows us to express the integrals of certain differential forms of top degree in terms of topological data, such as the degree of a smooth map or the Euler characteristic. A case in point is the Gauß–Bonnet formula in the next section.

5.4.2 The Gauß–Bonnet Formula

Let M be an oriented m-dimensional Riemannian manifold. Then there exists a unique m-form $\mathrm{dvol}_M \in \Omega^m(M)$, called the **volume form of** M, that satisfies the condition

$$(\mathrm{dvol}_M)_p(e_1, \ldots, e_m) = 1$$

for every $p \in M$ and every positive orthonormal basis $e_1, \ldots, e_m$ of T_pM.

Exercise 5.4.2. Let M be an oriented m-dimensional Riemannian manifold equipped with an oriented atlas $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^m$ and a metric tensor $g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}$. Prove that the volume form dvol_M is in local coordinates given by

$$(\mathrm{dvol}_M)_\alpha = \sqrt{\det(g_\alpha(x))}\, dx^1 \wedge \cdots \wedge dx^m.$$

Let $M \subset \mathbb{R}^{m+1}$ be a compact m-dimensional manifold without boundary. Then M inherits a Riemannian metric from the standard Euclidean inner product on $\mathbb{R}^{m+1}$ and it carries a **Gauß map**

$$\nu : M \to S^m$$

defined as follows. The complement of M in $\mathbb{R}^{m+1}$ has two connected components, one bounded and one unbounded (See Exercise 4.2.4). These connected components can be distinguished by the mod-2 degree of the map $f_x : M \to S^m$ defined by $f_x(p) := |p-x|^{-1}(p-x)$ for $p \in M$. The bounded component is the set of all $x \in \mathbb{R}^{m+1} \setminus M$ that satisfy $\deg_2(f_x) = 1$ and its closure will be denoted by W. Thus $W \subset \mathbb{R}^{m+1}$ is a compact connected oriented manifold with boundary $\partial W = M$ and we orient M as the boundary of W. The Gauß map $\nu : M \to S^m$ is characterized by the condition that $\nu(p) \in S^m$ is the unique unit vector that is orthogonal to T_pM and points out of W. The volume form $\mathrm{dvol}_M \in \Omega^m(M)$ associated to the metric and orientation of M is then given by the explicit formula

$$(\mathrm{dvol}_M)_p(v_1, \ldots, v_m) = \det(\nu(p), v_1, \ldots, v_m).$$

Moreover, the derivative of the Gauß map at $p \in M$ is a linear map from the tangent space T_pM to itself because $T_{\nu(p)}S^m = \nu(p)^\perp = T_pM$. The **Gaußian curvature** of M is the function $K : M \to \mathbb{R}$ defined by

$$K(p) := \det\bigl(d\nu(p) : T_pM \to T_pM\bigr)$$

for $p \in M$. When M is even-dimensional, this function is independent of the choice of the Gauß map. In m is odd then replacing ν by $-\nu$ changes the sign of K.

Theorem 5.4.3 (Gauß–Bonnet). *Let m be an even positive integer and let $M \subset \mathbb{R}^{m+1}$ be a compact m-dimensional submanifold without boundary. Then*

$$\int_M K \mathrm{dvol}_M = \frac{\mathrm{Vol}(S^m)}{2}\chi(M), \tag{5.4.2}$$

where $\chi(M)$ denotes the Euler characteristic of M.

Remark 5.4.4. When m is odd the Euler characteristic of M is zero. When $m = 2n$ we have

$$\frac{\mathrm{Vol}(S^{2n})}{2} = \frac{2^{2n}n!}{(2n)!}\pi^n.$$

Proof of Theorem 5.4.3. The Gauß map of S^m is the identity. Hence the volume form on S^m is given by

$$\mathrm{dvol}_{S^m} = \sum_{i=1}^{m+1}(-1)^{i-1}x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{m+1}$$

or, equivalently, $(\mathrm{dvol}_{S^m})_x(\xi_1, \dots, \xi_m) = \det(x, \xi_1, \dots, \xi_m)$ for all $x \in S^m$ and all $\xi_1, \dots, \xi_m \in T_xS^m = x^\perp$. Hence the pullback of dvol_{S^2} under the Gauß map is given by

$$\begin{aligned}
(\nu^*\mathrm{dvol}_{S^m})_p(v_1, \dots, v_m) &= (\mathrm{dvol}_{S^m})_{\nu(p)}(d\nu(p)v_1, \dots, d\nu(p)v_m) \\
&= \det(\nu(p), d\nu(p)v_1, \dots, d\nu(p)v_m) \\
&= K(p)\det(\nu(p), v_1, \dots, v_m) \\
&= K(p)(\mathrm{dvol}_M)_p(v_1, \dots, v_m)
\end{aligned}$$

for $p \in M$ and $v_1, \dots, v_m \in T_pM = \nu(p)^\perp$. Thus

$$K\mathrm{dvol}_M = \nu^*\mathrm{dvol}_{S^m}.$$

Since m is even, the Poincaré–Hopf Theorem 2.3.1 shows that the degree of the Gauß map is half the Euler characteristic of M. (Exercise: Verify this!) Hence it follows from Theorem 5.4.1 that

$$\int_M K\mathrm{dvol}_M = \int_M \nu^*\mathrm{dvol}_{S^m} = \deg(\nu)\int_{S^m}\mathrm{dvol}_{S^m} = \frac{\chi(M)}{2}\mathrm{Vol}(S^m).$$

This proves Theorem 5.4.3. □

Remark 5.4.5. We shall prove in §6.2 that the de Rham cohomology of a compact m-manifold M (with or without boundary) is finite-dimensional and in §6.4 that its Euler characteristic is the alternating sum of the **Betti numbers** $b_i := \dim(H^i(M))$, i.e. $\chi(M) = \sum_{i=0}^m (-1)^i \dim(H^i(M))$.

5.4.3 Moser Isotopy

We begin with the definition of a volume form.

Definition 5.4.6. *Let M be a smooth m-manifold. A* **volume form** *on M is a nowhere vanishing differential m-form on M. If M is oriented, a volume form $\omega \in \Omega^m(M)$ is called* **compatible with the orientation** *iff*

$$\omega_p(v_1, \dots, v_m) > 0 \tag{5.4.3}$$

for every $p \in M$ and every positively oriented basis $v_1, \dots, v_m$ of T_pM. If a volume form ω on an oriented m-manifold M is compatible with the orientation, we write $\omega > 0$.

The first observation is an existence statement for volume forms.

Lemma 5.4.7. *A manifold M admits a volume form if and only if it is orientable.*

Proof. If $\omega \in \Omega^m(M)$ is a volume form, then $\omega_p(v_1, \dots, v_m) \neq 0$ for every element $p \in M$ and every basis $v_1, \dots, v_m$ of T_pM. Hence a volume form on M determines an orientation of each tangent space T_pM. Namely, a basis $v_1, \dots, v_m$ is called *positively oriented* iff (5.4.3) holds. These orientation fit together smoothly. To see this, fix a point $p_0 \in M$ and a positive basis $v_1, \dots, v_m$ of $T_{p_0}M$ and choose vector fields $X_1, \dots, X_m \in \mathrm{Vect}(M)$ such that $X_i(p_0) = v_i$ for $i = 1, \dots, m$. Then there exists a connected open neighborhood $U \subset M$ of p_0 such that the vectors $X_1(p), \dots X_m(p)$ form a basis of T_pM for every $p \in U$. Hence the function

$$U \to \mathbb{R} : p \mapsto \omega_p(X_1(p), \dots, X_m(p))$$

is everywhere nonzero and hence is everywhere positive, because it is positive at $p = p_0$. Thus the vectors $X_1(p), \dots, X_m(p)$ form a positive basis of T_pM for every $p \in U$.

Here is a different argument. Given a volume form $\omega \in \Omega^m(M)$, choose an atlas $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ such that the m-forms

$$\omega_\alpha := (\phi_\alpha)_* \omega \in \Omega^m(\phi_\alpha(U_\alpha))$$

in local coordinates have the form

$$\omega_\alpha = f_\alpha dx^1 \wedge \dots \wedge dx^m$$

with

$$f_\alpha > 0.$$

Then, for all α, β and all $x \in \phi_\alpha(U_\alpha \cap U_\beta)$, we have

$$d(\phi_\beta \circ \phi_\alpha^{-1})(x) = \frac{f_\alpha(x)}{f_\beta(\phi_\beta \circ \phi_\alpha^{-1})(x)} > 0$$

Hence the atlas $\{U_\alpha, \phi_\alpha\}_\alpha$ is oriented.

Conversely, suppose M is oriented. Then one can choose a Riemannian metric and take $\omega = \mathrm{dvol}_M$ to be the volume form associated to the metric and orientation. Alternatively, choose an atlas $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^m$ on M such that the transition maps

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are orientation preserving diffeomorphisms for all α and β. Let

$$\rho_\alpha : M \to [0, 1]$$

be a partition of unity subordinate to the cover $\{U_\alpha\}_\alpha$ so that

$$\mathrm{supp}(\rho_\alpha) \subset U_\alpha, \qquad \sum_\alpha \rho_\alpha \equiv 1.$$

Define $\omega \in \Omega^m(M)$ by

$$\omega := \sum_\alpha \rho_\alpha \phi_\alpha^* dx^1 \wedge \cdots \wedge dx^m,$$

where $\rho_\alpha \phi_\alpha^* dx^1 \wedge \cdots \wedge dx^m \in \Omega_c^m(U_\alpha)$ is extended to all of M by setting it equal to zero on $M \setminus U_\alpha$. Then we have

$$\omega_p(v_1, \ldots, v_m) := \sum_{p \in U_\alpha} \rho_\alpha(p) \det(d\phi_\alpha(p)v_1, \ldots, d\phi_\alpha(p)v_m)$$

for $p \in M$ and $v_1, \ldots, v_m \in T_pM$. Here the sum is understood over all α such that $p \in U_\alpha$. For each $p \in M$ and each basis $v_1, \ldots, v_m$ of T_pM all the summands have the same sign and at least one summand is nonzero. Hence ω is a volume form on M and is compatible with the orientation determined by the atlas. This proves Lemma 5.4.7. □

The next theorem is the main result of this section. It asserts that on a compact connected oriented m-manifold M without boundary any two volume forms with the same total volume are related by a diffeomorphism that is isotopic to the identity.

Theorem 5.4.8 (Moser Isotopy). *Let M be a compact connected oriented m-manifold without boundary and let $\omega_0, \omega_1 \in \Omega^m(M)$ be volume forms such that*

$$\int_M \omega_0 = \int_M \omega_1.$$

Then there exists a diffeomorphism $\psi : M \to M$, isotopic to the identity, such that $\psi^*\omega_1 = \omega_0$.

Proof. We prove that ω_0 and ω_1 have the same sign on each basis of each tangent space. Let $U \subset M$ be the set of all $p \in M$ such that the real numbers $(\omega_0)_p(v_1, \dots, v_m)$ and $(\omega_1)_p(v_1, \dots, v_m)$ have the same sign for some (and hence every) basis $v_1, \dots, v_m$ of T_pM. Then U and $M \setminus U$ are open sets because ω_0 and ω_1 are volume forms, $U \neq \emptyset$ because the integral of ω_0 and ω_1 agree, and hence $U = M$ because M is connected. Thus ω_0 and ω_1 determine the same orientation of M. Hence the convex combinations

$$\omega_t := (1-t)\omega_0 + t\omega_1, \qquad 0 \le t \le 1,$$

are all volume forms on M.

The idea of the proof is to find a smooth isotopy $\psi_t \in \mathrm{Diff}(M)$, $0 \le t \le 1$, starting at the identity, such that

$$\psi_t^*\omega_t = \omega_0 \tag{5.4.4}$$

for all t. Every isotopy starting at the identity determines, and is determined by, a smooth family of vector fields $X_t \in \mathrm{Vect}(M)$, $0 \le t \le 1$, via

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}. \tag{5.4.5}$$

By assumption the integral of $\omega_1 - \omega_0$ vanishes. Hence, by Theorem 5.3.10, there exists an $(m-1)$-form $\tau \in \Omega^{m-1}(M)$ such that

$$d\tau = \omega_1 - \omega_0 = \partial_t\omega_t.$$

If ψ_t and X_t satisfy (5.4.5), then by Cartan's formula in Theorem 5.3.2

$$\frac{d}{dt}\psi_t^*\omega_t = \psi_t^*(\mathcal{L}_{X_t}\omega_t + \partial_t\omega_t) = \psi_t^* d\big(\iota(X_t)\omega_t + \tau\big). \tag{5.4.6}$$

By Exercise 5.4.9 there exists a smooth family of vector fields

$$X_t := -I_{\omega_t}^{-1}(\tau) \in \mathrm{Vect}(M), \qquad \iota(X_t)\omega_t + \tau = 0.$$

Let $\psi_t \in \mathrm{Diff}(M)$, $0 \le t \le 1$, be the isotopy of M determined by the vector fields X_t via equation (5.4.5). Then it follows from (5.4.6) that the volume form $\psi_t^*\omega_t$ is independent of t and therefore satisfies (5.4.4). Hence the diffeomorphism $\psi := \psi_1$ satisfies the requirements of Theorem 5.4.8. □

5.4.4 Examples and Exercises

Exercise 5.4.9. Let M be an m-manifold, let $\omega \in \Omega^m(M)$ be a volume form, and define the map $I_\omega : \mathrm{Vect}(M) \to \Omega^{m-1}(M)$ by $I_\omega(X) := \iota(X)\omega$. Prove that I_ω is a vector space isomorphism.

Definition 5.4.10. *Let M be a compact m-manifold without boundary and let $\omega \in \Omega^m(M)$ be a volume form. A diiffeomorphism $\phi : M \to M$ is called* **volume preserving** *iff $\phi^*\omega = \omega$. (This condition means that ϕ preserves not only the Borel measure determined by ω but also the orientation of M.) The volume preserving diffeomorphisms form a group denoted by*

$$\mathrm{Diff}(M, \omega) := \{\phi \in \mathrm{Diff}(M) \,|\, \phi^*\omega = \omega\}.$$

Remark 5.4.11. Let M be a compact m-manifold without boundary and let $\omega_0 \in \Omega^m(M)$ be a volume form. One can use Moser isotopy to prove that the inclusion of $\mathrm{Diff}(M, \omega_0) \to \mathrm{Diff}(M)$ is a homotopy equivalence. This is to be understood with respect to the C^∞-topology on the group of diffeomorphisms. A sequence ψ_ν converges in this topology, by definition, iff it converges uniformly with all derivatives.

To prove the assertion, consider the set

$$\mathcal{V}(M) := \left\{\omega \in \Omega^m(M) \,\middle|\, \omega \text{ is a volume form and } \int_M \omega = 1\right\}$$

of all volume forms on M with volume one and assume $\omega_0 \in \mathcal{V}(M)$. The group $\mathrm{Diff}(M)$ acts on $\mathcal{V}(M)$ and the isotropy subgroup of ω_0 is $\mathrm{Diff}(M, \omega_0)$. By Theorem 5.4.8 the map $\mathrm{Diff}(M) \to \mathcal{V}(M) : \psi \mapsto \psi^*\omega_0$ is surjective. Also, there exists a continuous map $\mathcal{V}(M) \to \mathrm{Diff}(M) : \omega \mapsto \psi_\omega$ such that

$$\psi_{\omega_0} = \mathrm{id}, \qquad \psi_\omega^*\omega = \omega_0 \qquad \text{for all } \omega \in \mathscr{V}(M).$$

To see this, construct an affine map $\mathcal{V}(M) \to \Omega^{m-1}(M) : \omega \mapsto \tau_\omega$ such that

$$d\tau_\omega = \omega - \omega_0 \qquad \text{for all } \omega \in \mathcal{V}(M),$$

following Exercise 5.3.11, and then use the proof of Theorem 5.4.8 to find ψ_ω. It follows that the map

$$\mathrm{Diff}(M) \to \mathcal{V}(M) \times \mathrm{Diff}(M, \omega_0) : \psi \mapsto (\psi^*\omega_0, \psi \circ \psi_{\psi^*\omega_0}) \tag{5.4.7}$$

is a homeomophism with inverse $(\omega, \phi) \mapsto \phi \circ \psi_\omega^{-1}$. Since $\mathcal{V}(M)$ is convex, this shows that the inclusion of $\mathrm{Diff}(M, \omega_0)$ into $\mathrm{Diff}(M)$ is a homotopy equivalence. Explicitly, the formula $\mathcal{F}_t(\psi) := \psi \circ \psi_{(1-t)\psi^*\omega_0 + t\omega_0}$ for $0 \le t \le 1$ and $\psi \in \mathrm{Diff}(M)$ defines a continuous homotopy $\mathcal{F}_t : \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ such that $\mathcal{F}_0 = \mathrm{id}$, $\mathcal{F}_t|_{\mathrm{Diff}(M,\omega_0)} = \mathrm{id}$ for all t, and $\mathrm{im}\,(\mathcal{F}_1) \subset \mathrm{Diff}(M, \omega_0)$.

Exercise 5.4.12. Prove that there are metrics on $\mathrm{Diff}(M)$ and $\Omega^m(M)$ that induce the C^∞-topology on these spaces. Prove that the map (5.4.7) is a homeomorphism. **Hint:** If $d : X \times X \to \mathbb{R}$ is a metric, so is $d/(1+d)$.

Remark 5.4.13. Let M be compact oriented m-manifold without boundary and let $\omega \in \Omega^m(M)$ be a positive volume form.

(a) The group $\mathrm{Diff}(M,\omega)$ of volume preserving diffeomorphisms can be viewed as as an infinite-dimensional analogue of a Lie group, whose Lie algebra is the space

$$\mathrm{Vect}(M,\omega) := \bigl\{X \in \mathrm{Vect}(M) \,\big|\, \iota(X)\omega \text{ is closed}\bigr\} \tag{5.4.8}$$

of **divergence free vector fields** on M.

(b) The formula $\iota([X,Y])\omega = d\iota(Y)\iota(X)\omega$ for $X, Y \in \mathrm{Vect}(M,\omega)$ shows that the subspace

$$\mathrm{Vect}^{\mathrm{ex}}(M,\omega) := \bigl\{X \in \mathrm{Vect}(M) \,\big|\, \iota(X)\omega \text{ is exact}\bigr\} \tag{5.4.9}$$

of **exact divergence free vector fields** is an ideal in $\mathrm{Vect}(M,\omega)$.

(c) There is a corresponding normal subgroup of $\mathrm{Diff}(M,\omega)$ consisting of the socalled **exact volume preserving diffeomorphisms** that can be joined to the identity by an isotopy that is generated by exact divergence free vector fields. This subgroup will be denoted by

$$\mathrm{Diff}^{\mathrm{ex}}(M,\omega) := \left\{ \phi_1 \,\middle|\, \begin{array}{l} \{\phi_t\}_{0\le t\le 1} \text{ is an isotopy such that } \phi_0 = \mathrm{id} \\ \text{and } \partial_t\phi_t \circ \phi_t^{-1} \in \mathrm{Vect}^{\mathrm{ex}}(M,\omega) \text{ for all } t \end{array} \right\}.$$

If M has dimension 2, then ω is a symplectic form and $\mathrm{Diff}^{\mathrm{ex}}(M,\omega)$ is the group of Hamiltonian symplectomorphisms.

(d) The **Flux homomorphism** assigns to every smooth isotopy $\{\phi_t\}_{0\le t\le 1}$ of volume preserving diffeomorphisms the de Rham cohomology class

$$\mathrm{Flux}\bigl(\{\psi_t\}_{0\le t\le 1}\bigr) := \left[\int_0^2 \iota(\partial_t\phi_t \circ \phi_t^{-1})\omega\, dt\right] \in H^{m-1}(M,\mathbb{R}). \tag{5.4.10}$$

(e) Define $\Gamma \subset H^{m-1}(M)$ as the subgroup of all classes $\mathrm{Flux}\bigl(\{\phi_t\}_{0\le t\le 1}\bigr)$, where the isotopy $\{\phi_t\}_{0\le t\le 1}$ in $\mathrm{Diff}(M,\omega)$ satisfies $\phi_0 = \phi_1 = \mathrm{id}$. Then the Flux homomorphism in (5.4.10) descends to a homomorphism

$$\mathrm{Flux} : \mathrm{Diff}_0(M,\omega) \to H^{m-1}(M)/\Gamma, \tag{5.4.11}$$

defined on the identity component $\mathrm{Diff}_0(M,\omega)$ of $\mathrm{Diff}(M,\omega)$, whose kernel is the subgroup $\mathrm{Diff}^{\mathrm{ex}}(M,\omega) \subset \mathrm{Diff}_0(M,\omega)$.

(f) Γ is a discrete subgroup of $H^{m-1}(M)$ and this implies that $\mathrm{Diff}^{\mathrm{ex}}(M,\omega)$ is a closed subgroup of $\mathrm{Diff}_0(M,\omega)$ with respect to the C^∞ topology.

Exercise 5.4.14. Let M be compact oriented boundaryless m-manifold and let $\omega \in \Omega^m(M)$ be a positive volume form.

(a) Prove that $\mathrm{Diff}^{\mathrm{ex}}(M,\omega)$ is the kernel of the homomorphism (5.4.11)

(b) If $\tau \in \Omega^{m-1}(M)$ is a closed $(m-1)$-form whose cohomology class belongs to Γ and $\alpha \in \Omega^1(M)$ is a closed 1-form whose integral over every loop is an integer, prove that

$$\frac{\int_M \tau \wedge \alpha}{\int_M \omega} \in \mathbb{Z}.$$

Deduce that Γ is a discrete subgroup of $H^{m-1}(M)$. **Hint:** The proof requires Poinceré duality (see §6.4).

(c) Prove that $\mathrm{Diff}^{\mathrm{ex}}(M,\omega)$ is a closed subgroup of $\mathrm{Diff}(M,\omega)$ with respect to the C^∞ topology.

Exercise 5.4.15. Consider the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ with the standard volume form $\omega = dx_1 \wedge \cdots \wedge dx_m$. Then any translation of $\mathbb{T}^m$ is volume preserving, but not exact volume preserving unless it is the identity.

Chapter 6

De Rham Cohomology

In this chapter we take a closer look at the de Rham cohomology groups of a smooth manifold that were introduced in §5.2.1. Here we follow closely the classical textbook of Bott and Tu [3]. An immediate consequence of Cartan's formula in Theorem 5.3.3 is the observation that smoothly homotopic maps induce the same homomorphism on de Rham cohomology, that homotopy equivalent manifolds have isomorphic de Rham cohomology groups, and that the de Rham cohomology of a contractible space vanishes in positive degrees. In the case of Euclidean space this is a consequence of the Poincaré Lemma which follows directly from Cartan's formula. These observations are discussed in §6.1, which closes with the computation of the de Rham cohomology of a sphere. This computation is a special case of the Mayer–Vietoris argument, the subject of §6.2. It is a powerful tool in differential and algebraic topology and can be used, for example, to prove that the de Rham cohomology groups are finite-dimensional and to establish the Künneth formula for the de Rham cohomology of a product manifold. §6.3 extends the previous discussion to compactly supported de Rham cohomology and §6.4 is devoted to Poincaré duality, which again can be proved with the Mayer–Vietoris argument. Using Poincaré duality and the Künneth formula one can then show that the Euler characteristic of a compact oriented manifold without boundary, originally defined as the algebraic number of zeros of a generic vector field, is indeed equal to the alternating sum of the Betti-numbers. A natural generalization of the Mayer–Vietoris sequence is the Čech–de Rham complex which will be discussed in §6.5. In particular, we show that the de Rham cohomology of a manifold is, under suitable hypotheses, isomorphic to the Čech cohomology.

6.1 The Poincaré Lemma

Let M be an m-manifold, let N be an n-manifold, and let $f : M \to N$ be a smooth map. By Lemma 5.2.6 the pullback of differential forms under f commutes with the exterior differential, i.e.

$$f^* \circ d = d \circ f^*. \tag{6.1.1}$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \Omega^2(M) \arrow[r, "d"] & \cdots . \\
\Omega^0(N) \arrow[r, "d"] \arrow[u, "f^*"'] & \Omega^1(N) \arrow[r, "d"] \arrow[u, "f^*"'] & \Omega^2(N) \arrow[r, "d"] \arrow[u, "f^*"'] & \cdots
\end{tikzcd}$$

Thus $f^* : \Omega^k(N) \to \Omega^k(M)$ is a linear map which assigns closed forms to closed forms and exact forms to exact forms. Hence it descends to a homomorphism

$$H^k(N) \to H^k(M) : [\omega] \mapsto f^*[\omega] := [f^*\omega]$$

on de Rham cohomology, still denoted by f^*. If $g : N \to Q$ is another smooth map between smooth manifolds then, by Lemma 5.1.18, we have

$$(g \circ f)^* = f^* \circ g^* : H^k(Q) \to H^k(M).$$

Moreover, it follows from Lemmas 5.1.18 and 5.2.6 that de Rham cohomology is equipped with a **cup product structure**

$$H^k(M) \times H^\ell(M) \to H^{k+\ell}(M) : ([\omega], [\tau]) \mapsto [\omega] \cup [\tau] := [\omega \wedge \tau]$$

and that the cup product is preserved by pullback.

Theorem 6.1.1. *If $f_0, f_1 : M \to N$ are smoothly homotopic, then there exists a collection of linear maps $h : \Omega^k(N) \to \Omega^{k-1}(M)$, one for every nonnegative integer k, such that*

$$f_1^* - f_0^* = d \circ h + h \circ d : \Omega^k(N) \to \Omega^k(M) \tag{6.1.2}$$

for every nonnegative integer k. In particular, the homomorphisms induced by f_0 and f_1 on de Rham cohomology agree, i.e.

$$f_0^* = f_1^* : H^*(N) \to H^*(M).$$

Proof. Choose a smooth homotopy $F : [0, 1] \times M \to N$ satisfying

$$F(0, p) = f_0(p), \qquad F(1, p) = f_1(p)$$

for every $p \in M$, and for $0 \le t \le 1$, define $f_t : M \to N$ by

$$f_t(p) := F(t, p).$$

By Theorem 5.3.3, we have

$$\frac{d}{dt} f_t^* \omega = dh_t\omega + h_t d\omega$$

for $\omega \in \Omega^k(N)$, where $h_t : \Omega^k(N) \to \Omega^{k-1}(M)$ is defined by

$$(h_t\omega)_p(v_1, \ldots, v_{k-1}) := \omega_{f_t(p)}(\partial_t f_t(p), df_t(p)v_1, \ldots, df_t(p)v_{k-1})$$

for $p \in M$ and $v_i \in T_pM$. Integrating over t we find

$$f_1^*\omega - f_0^*\omega = \int_0^1 \frac{d}{dt} f_t^*\omega \, dt = dh\omega + hd\omega$$

where $h : \Omega^k(N) \to \Omega^{k-1}(M)$ is defined by

$$\begin{aligned} &(h\omega)_p(v_1, \ldots, v_{k-1}) \\ &:= \int_0^1 \omega_{f_t(p)}(\partial_t f_t(p), df_t(p)v_1, \ldots, df_t(p)v_{k-1}) \, dt \end{aligned} \tag{6.1.3}$$

for $p \in M$ and $v_i \in T_pM$. This proves Theorem 6.1.1. □

Remark 6.1.2. In homological algebra equation (6.1.1) says that

$$f^* : \Omega^*(N) \to \Omega^*(M)$$

is a **chain map**. Equation (6.1.2) says that the chain maps f_0^* and f_1^* are **chain homotopy equivalent** and the map

$$h : \Omega^*(N) \to \Omega^{*-1}(M)$$

is called a **chain homotopy equivalence** from f_0^* to f_1^*. In other words, smoothly homotopic maps between manifold induce chain homotopy equivalent chain maps between the associated de Rham cochain complexes. Chain homotopy equivalent chain maps always descend to the same homorphism on (co)homology.

Definition 6.1.3. *Two manifolds M and N are called* **homotopy equivalent** *iff there exist smooth maps $f : M \to N$ and $g : N \to M$ such that the compositions*

$$g \circ f : M \to M, \qquad f \circ g : N \to N$$

are both homotopic to the respective identity maps. If this holds, the maps f and g are called **homotopy equivalences** *and g is called a* **homotopy inverse** *of f.*

Exercise 6.1.4. The closed unit disk in $\mathbb{R}^m$ (an m-manifold with boundary) is homotopy equivalent to a point (a 0-manifold without boundary).

Corollary 6.1.5. *Homotopy equivalent manifolds have isomorphic de Rham cohomology (including the product structures).*

Proof. Let $f : M \to N$ be a homotopy equivalence and $g : N \to M$ be a homotopy inverse of f. Then it follows from Theorem 6.1.1 that

$$f^* \circ g^* = (g \circ f)^* = \mathrm{id} : H^*(M) \to H^*(M)$$

and

$$g^* \circ f^* = (f \circ g)^* = \mathrm{id} : H^*(N) \to H^*(N).$$

Hence $f^* : H^*(N) \to H^*(M)$ is a vector space isomorphism and

$$(f^*)^{-1} = g^* : H^*(M) \to H^*(N).$$

This proves Corollary 6.1.5. □

Example 6.1.6. For every smooth manifold M we have

$$H^*(M) \cong H^*(\mathbb{R} \times M).$$

To see this, define $\pi : \mathbb{R} \times M \to M$ and $\iota : M \to \mathbb{R} \times M$ by

$$\pi(s, p) := p, \qquad \iota(p) := (0, p)$$

for $s \in \mathbb{R}$ and $p \in M$. Then $\pi \circ \iota = \mathrm{id} : M \to M$ and $\iota \circ \pi : \mathbb{R} \times M \to \mathbb{R} \times M$ is homotopic to the identity. An explicit homotopy is given by

$$f_t : \mathbb{R} \times M \to \mathbb{R} \times M, \qquad f_t(s, p) := (st, p), \qquad f_0 = \iota \circ \pi, \qquad f_1 = \mathrm{id}.$$

Hence M and $\mathbb{R} \times M$ are homotopy equivalent and so the assertion follows from Corollary 6.1.5. Explicitly, the map $\pi^* : H^*(M) \to H^*(\mathbb{R} \times M)$ is an isomorphism with the inverse $\iota^* : H^*(\mathbb{R} \times M) \to H^*(M)$.

Definition 6.1.7. *A smooth manifold M is called* **contractible** *iff the identity map on M is homotopic to a constant map.*

Exercise 6.1.8. Every contractible manifold is nonempty and connected.

Exercise 6.1.9. A manifold is contractible if and only if it is homotopy equivalent to a point.

Exercise 6.1.10. Every nonempty geodesically convex open subset of a Riemannian m-manifold without boundary is contractible.

Corollary 6.1.11 (Poincaré Lemma). *Let M be a contractible manifold. Then there is a collection of linear maps $h : \Omega^k(M) \to \Omega^{k-1}(M)$, one for every nonnegative integer k, such that*

$$d \circ h + h \circ d = \mathrm{id} : \Omega^k(M) \to \Omega^k(M), \qquad k \geq 1. \tag{6.1.4}$$

Hence $H^0(M) = \mathbb{R}$ and $H^k(M) = 0$ for $k \geq 1$.

Proof. Let $p_0 \in M$ and let $[0,1] \times M \to M : (t,p) \mapsto f_t(p)$ be a smooth homotopy such that $f_0(p) = p_0$ and $f_1(p) = p$ for all $p \in M$. Define the linear map $h : \Omega^k(M) \to \Omega^{k-1}(M)$ by (6.1.3). Then, for every k-form $\omega \in \Omega^k(M)$ with $k \geq 1$, it follows from Theorem 6.1.1 that

$$\omega = f_1^*\omega - f_0^*\omega = dh\omega - hd\omega.$$

(The assumption $k \geq 1$ is needed in the first equation.) Hence, for $k \geq 1$, every closed k-form on M is exact and so $H^k(M) \cong 0$. Since M is connected we have $H^0(M) = \mathbb{R}$. This proves Corollary 6.1.11. □

Example 6.1.12. The Euclidean space $\mathbb{R}^m$ is contractible. An explicit homotopy from a constant map to the identity is given by $f_t(x) := tx$ for $0 \leq t \leq 1$ and $x \in \mathbb{R}^m$. Hence

$$H^k(\mathbb{R}^m) = \begin{cases} \mathbb{R}, & \text{for } k = 0, \\ 0, & \text{for } k \geq 1. \end{cases}$$

The chain homotopy equivalence $h : \Omega^k(\mathbb{R}^m) \to \Omega^{k-1}(\mathbb{R}^m)$ associated to the above homotopy f_t via (6.1.3) is given by

$$(h\omega)(x; \xi_1, \ldots, \xi_{k-1}) = \int_0^1 t^{k-1}\omega(x; tx, \xi_1, \ldots, \xi_{k-1})\, dt \tag{6.1.5}$$

for $\omega \in \Omega^k(\mathbb{R}^m)$ and $x, \xi_1, \ldots, \xi_{k-1} \in \mathbb{R}^m$. By Corollary 6.1.5 it satisfies

$$d \circ h + h \circ d = \mathrm{id} : \Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^m)$$

for $k \geq 1$. This is the **Poincaré Lemma** in its original form.

Example 6.1.13. For $m \ge 1$ the de Rham cohomology of the unit sphere

$$S^m \subset \mathbb{R}^{m+1}$$

is given by

$$H^k(S^m) = \begin{cases} \mathbb{R}^m, & \text{for } k = 0 \text{ and } k = m, \\ 0, & \text{for } 1 \le k \le m-1. \end{cases}$$

That $H^0(S^m) = \mathbb{R}$ follows from Example 5.2.8 because S^m is connected (whenever $m \ge 1$). That $H^m(S^m) = \mathbb{R}$ follows from Corollary 5.3.12 because S^m is a compact connected oriented manifold without boundary.

We prove that

$$H^1(S^m) = 0$$

for every $m \ge 2$. To see this consider the open sets

$$U^{\pm} := S^m \setminus \{(0, \dots, 0, \mp 1)\}.$$

Their union is S^m, each set U^+ and U^- is diffeomorphic to $\mathbb{R}^m$ via stereographic projection, and their intersection $U^+ \cap U^-$ is diffeomorphic to $\mathbb{R}^m \setminus \{0\}$ and hence to $\mathbb{R} \times S^{m-1}$:

$$U^+ \cong U^- \cong \mathbb{R}^m, \qquad U^+ \cap U^- \cong \mathbb{R} \times S^{m-1}.$$

In particular, the intersection $U^+ \cap U^-$ is connected because $m \ge 2$. Now let $\alpha \in \Omega^1(S^m)$ be a closed 1-form. Then it follows from Example 6.1.12 that the restrictions of α to U^+ and U^- are exact. Hence there are smooth functions $f^{\pm} : U^{\pm} \to \mathbb{R}$ such that

$$\alpha|_{U^+} = df^+, \qquad \alpha|_{U^-} = df^-.$$

The differential of the difference $f^+ - f^- : U^+ \cap U^- \to \mathbb{R}$ vanishes. Since $U^+ \cap U^-$ is connected there is a constant $c \in \mathbb{R}$ such that

$$f^+(x) - f^-(x) = c \qquad \forall\, x \in U^+ \cap U^-.$$

Define $f : S^m \to \mathbb{R}$ by

$$f(x) := \begin{cases} f^-(x) + c, & \text{for } x \in U^-, \\ f^+(x), & \text{for } x \in U^+. \end{cases}$$

This function is well defined and smooth and satisfies $df = \alpha$. Thus we have proved that every closed 1-form on S^m is exact, when $m \ge 2$, and thus $H^1(S^m) = 0$, as claimed.

We prove by induction on m that $H^k(S^m) = 0$ for $1 \le k \le m-1$ and $m \ge 2$. We have just seen that this holds for $m = 2$. Thus let $m \ge 3$ and assume, by induction, that the assertion holds for $m-1$. We have already shown that $H^1(S^m) = 0$. Thus we fix an integer

$$2 \le k \le m-1$$

and prove that

$$H^k(S^m) = 0.$$

Let $\omega \in \Omega^k(S^m)$ be a closed k-form. By Example 6.1.12, the restrictions of ω to U^+ and U^- are both exact. Hence there are smooth $(k-1)$-forms $\tau^\pm \in \Omega^{k-1}(U^\pm)$ such that

$$\omega|_{U^+} = d\tau^+, \qquad \omega|_{U^-} = d\tau^-.$$

Hence the $(k-1)$-form

$$\tau^+|_{U^+\cap U^-} - \tau^-|_{U^+\cap U^-} \in \Omega^{k-1}(U^+ \cap U^-)$$

is closed. By Example 6.1.6 and the induction hypothesis, we have

$$H^{k-1}(U^+ \cap U^-) \cong H^{k-1}(\mathbb{R} \times S^{m-1}) \cong H^{k-1}(S^{m-1}) = 0.$$

Hence there is a $(k-2)$-form $\beta \in \Omega^{k-2}(U^+ \cap U^-)$ such that

$$d\beta = \tau^+|_{U^+\cap U^-} - \tau^-|_{U^+\cap U^-}.$$

Now choose a smooth cutoff function $\rho : S^m \to [0,1]$ such that

$$\rho(x) = \begin{cases} 0, & \text{for } x \text{ near } (0,\dots,0,-1), \\ 1, & \text{for } x \text{ near } (0,\dots,0,1), \end{cases}$$

and define $\tau \in \Omega^{k-1}(S^m)$ by

$$\tau := \begin{cases} \tau^- + d(\rho\beta) & \text{on } U^-, \\ \tau^+ - d((1-\rho)\beta) & \text{on } U^+. \end{cases}$$

Then $d\tau = \omega$. Thus we have proved that every closed k-form on S^m is exact and hence $H^k(S^m) = 0$, as claimed.

The computation of the de Rham cohomology of S^m in Example 6.1.13 is an archetypal example of a Mayer–Vietoris argument. More generally, if we have a cover of a manifold by two well chosen open sets U and V, the computation of the de Rham cohomology of M can be reduced to the computation of the de Rham cohomology of the manifolds U, V, and $U \cap V$ by means of the Mayer–Vietoris sequence. We shall see that this exact sequence is a powerful tool for understanding de Rham cohomology.

6.2 The Mayer–Vietoris Sequence

The purpose of this section is to introduce the Mayer–Vietoris sequence and show that it is exact (§6.2.1), to show that manifolds with finite good covers have finite-dimensional de Rham cohomology groups (§6.2.2), and to prove the Künneth formula (§6.2.3).

6.2.1 Long Exact Sequences

Let M be a smooth m-dimensional manifold (not necessarily compact or connected and with or without boundary). Let $U, V \subset M$ be open sets such that $M = U \cup V$. The **Mayer–Vietoris sequence** associated to this open cover by two sets is the sequence of homomorphisms

$$0 \longrightarrow \Omega^k(M) \xrightarrow{i^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j^*} \Omega^k(U \cap V) \longrightarrow 0, \tag{6.2.1}$$

where $i^* : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V)$ and $j^* : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$ are defined by

$$i^*\omega := (\omega|_U, \omega|_V), \qquad j^*(\omega_U, \omega_V) := \omega_V|_{U\cap V} - \omega_U|_{U\cap V}$$

for $\omega \in \Omega^k(M)$ and $\omega_U \in \Omega^k(U)$, $\omega_V \in \Omega^k(V)$. Thus i^* is given by restriction and j^* by restriction followed by subtraction.

Lemma 6.2.1. *The Mayer–Vietoris sequence* (6.2.1) *is exact.*

Proof. That i^* is injective, is obvious: if $\omega \in \Omega^k(M)$ vanishes on U and on V then it vanishes on all of M. That the image of i^* agrees with the kernel of j^* is also obvious: if $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ agree on the intersection $U \cap V$, then they determine a unique global k-form $\omega \in \Omega^k(M)$ such that $\omega|_U = \omega_U$ and $\omega|_V = \omega_V$.

We prove that j^* is surjective. Choose a partition of unity subordinate to the open cover $M = U \cup V$. It consists of two smooth functions $\rho_U : M \to [0,1]$ and $\rho_V : M \to [0,1]$ satisfying

$$\mathrm{supp}(\rho_U) \subset U, \qquad \mathrm{supp}(\rho_V) \subset V, \qquad \rho_U + \rho_V \equiv 1. \tag{6.2.2}$$

Now let $\omega \in \Omega^k(U \cap V)$ and define $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ by

$$\omega_U := \begin{cases} -\rho_V\omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus V, \end{cases} \qquad \omega_V := \begin{cases} \rho_U\omega & \text{on } U \cap V, \\ 0 & \text{on } V \setminus U. \end{cases}$$

Then

$$j^*(\omega_U, \omega_V) = \omega_V|_{U\cap V} - \omega_U|_{U\cap V} = \rho_U\omega + \rho_V\omega = \omega$$

as claimed. This proves Lemma 6.2.1. □

The Mayer–Vietoris sequence (6.2.1) is an example of what is called a **short exact sequence** in homological algebra in that it is short (five terms starting and ending with zero), it is exact, and it consists of chain homomorphisms. Thus the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{k+1}(M) \arrow[r, "i^*"] & \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) \arrow[r, "j^*"] & \Omega^{k+1}(U \cap V) \arrow[r] & 0 . \\
0 \arrow[r] & \Omega^{k}(M) \arrow[r, "i^*"] \arrow[u, "d"] & \Omega^{k}(U) \oplus \Omega^{k}(V) \arrow[r, "j^*"] \arrow[u, "d"] & \Omega^{k}(U \cap V) \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

Any such short exact sequence gives rise to a **long exact sequence** in cohomology. The relevant boundary operator will be denoted by

$$d^* : H^k(U \cap V) \to H^{k+1}(M)$$

and it is defined as follows. Let $\omega \in \Omega^k(U \cap V)$ be a closed k-form and choose a pair $(\omega_U, \omega_V) \in \Omega^k(U) \oplus \Omega^k(V)$ whose image under j^* is ω. Then the pair $(d\omega_U, d\omega_V)$ belongs to the kernel of j^* because ω is closed, and hence belongs to the image of i^* by exactness. hence there exists a unique $(k+1)$-form $d^*\omega \in \Omega^{k+1}(M)$ whose image under i^* is the pair $(d\omega_U, d\omega_V)$. Since i^* is injective and $i^* d(d^*\omega) = di^*(d^*\omega) = d(d\omega_U, d\omega_V) = 0$, it follows that $d^*\omega$ is closed. Moreover, one can check that the cohomology class of $d^*\omega$ is independent of the choice of the pair (ω_U, ω_V) used in this construction.

Here is an explicit formula for the operator d^* coming from the proof of Lemma 6.2.1. Namely, choose smooth functions $\rho_U, \rho_V : M \to [0,1]$ that satisfy (6.2.2) and define the operator $d^* : \Omega^k(U \cap V) \to \Omega^{k+1}(M)$ by

$$d^*\omega := \begin{cases} d\rho_U \wedge \omega & \text{on } U \cap V, \\ 0 & \text{on } M \setminus (U \cap V). \end{cases} \tag{6.2.3}$$

This operator is well defined because the 1-form $d\rho_U = -d\rho_V$ is supported in $U \cap V$. Moreover, we have

$$d \circ d^* + d^* \circ d = 0 \tag{6.2.4}$$

and hence d^* assigns closed forms to closed forms and exact forms to exact forms. Thus d^* descends to a homomorphism on cohomology.

Exercise 6.2.2. Prove that the linear map $d^* : \Omega^k(U \cap V) \to \Omega^{k+1}(M)$ defined by (6.2.3) satisfies equation (6.2.4) and hence descends to a homomorphism $d^* : H^k(U \cap V) \to H^{k+1}(M)$. Prove that the induced homomorphism on cohomology is independent of the choice of the partition of unity ρ_U, ρ_V and agrees with the homomorphism defined by *diagram chasing* as above.

The homomorphisms on de Rham cohomology induced by i^*, j^*, d^* give rise to a long exact sequence

$$\cdots H^k(M) \xrightarrow{i^*} H^k(U) \oplus H^k(V) \xrightarrow{j^*} H^k(U \cap V) \xrightarrow{d^*} H^{k+1}(M) \cdots \qquad (6.2.5)$$

which is also called the **Mayer–Vietoris sequence**.

Theorem 6.2.3. *The Mayer–Vietoris sequence* (6.2.5) *is exact.*

Proof. The equation $j^* \circ i^* = 0$ follows directly from the definitions.

We prove that $d^* \circ j^* = 0$. Let $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ be closed and define $\omega \in \Omega^k(M)$ by

$$\omega := \begin{cases} \rho_U \omega_U + \rho_V \omega_V & \text{on } U \cap V, \\ \rho_U \omega_U & \text{on } U \setminus V, \\ \rho_V \omega_V & \text{on } V \setminus U. \end{cases}$$

Then

$$\begin{aligned} d^* j^*(\omega_U, \omega_V) &= d^*(\omega_V|_{U\cap V} - \omega_U|_{U\cap V}) \\ &= d\rho_U \wedge (\omega_V|_{U\cap V} - \omega_U|_{U\cap V}) \\ &= -d\omega \end{aligned}$$

and hence

$$d^* j^*([\omega_U], [\omega_V]) = 0.$$

Thus $d^* \circ j^* = 0$.

We prove that $i^* \circ d^* = 0$. Let $\omega \in \Omega^k(U \cap V)$ be closed and define the k-forms $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ by

$$\omega_U := \begin{cases} -\rho_V \omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus V, \end{cases} \qquad \omega_V := \begin{cases} \rho_U \omega & \text{on } U \cap V, \\ 0 & \text{on } V \setminus U. \end{cases}$$

as in the proof of Lemma 6.2.1. Then

$$d\omega_U|_{U\cap V} = -d\rho_V \wedge \omega = d\rho_U \wedge \omega = d\omega_V|_{U\cap V} = (d^*\omega)|_{U\cap V}.$$

Hence $d\omega_U = (d^*\omega)|_U$ and $d\omega_V = (d^*\omega)|_V$, and so

$$i^* d^*[\omega] = ([(d^*\omega)|_U], [(d^*\omega)|_V]) = 0.$$

Thus $i^* \circ d^* = 0$.

We prove that $\ker d^* = \operatorname{im} j^*$. Let $\omega \in \Omega^k(U \cap V)$ be a closed k-form such that $d^*[\omega] = [d^*\omega] = 0$. Then the k-form $d^*\omega \in \Omega^{k+1}(M)$ is exact. Thus there exists a k-form $\tau \in \Omega^k(M)$ such that

$$d\tau = d^*\omega$$

or, equivalently,

$$d\tau|_{U\cap V} = d\rho_U \wedge \omega, \qquad d\tau|_{M\setminus(U\cap V)} = 0.$$

Define $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ by

$$\omega_U := -\rho_V\omega - \tau|_U, \qquad \omega_V := \rho_U\omega - \tau|_V.$$

Here it is understood that the k-form $-\rho_V\omega$ on $U \cap V$ is extended to all of U by setting it equal to zero on $U \setminus V$ and the k-form $\rho_U\omega$ on $U \cap V$ is extended to all of V by setting it equal to zero on $V \setminus U$. The k-forms ω_U and ω_V are closed and hence determine cohomology classes $[\omega_U] \in H^k(U)$ and $[\omega_V] \in H^k(V)$. Moreover,

$$\omega_V|_{U\cap V} - \omega_U|_{U\cap V} = \rho_U\omega + \rho_V\omega = \omega$$

and hence

$$j^*([\omega_U], [\omega_V]) = [\omega].$$

Thus we have proved that $\ker d^* = \operatorname{im} j^*$.

We prove that $\ker j^* = \operatorname{im} i^*$. Let $\omega_U \in \Omega^k(U)$ and $\omega_V \in \Omega^k(V)$ be closed k-forms such that $j_*([\omega_U], [\omega_V]) = 0$. Then the k-form $j_*(\omega_U, \omega_V)$ on $U \cap V$ is exact. Thus there exists a $(k-1)$-form $\tau \in \Omega^{k-1}(U \cap V)$ such that

$$\omega_V|_{U\cap V} - \omega_U|_{U\cap V} = d\tau.$$

By Lemma 6.2.1 there exist $(k-1)$-forms $\tau_U \in \Omega^{k-1}(U)$ and $\tau_V \in \Omega^{k-1}(V)$ such that

$$\tau_V|_{U\cap V} - \tau_U|_{U\cap V} = \tau.$$

Combining the last two equations we find that $\omega_U - d\tau_U$ agrees with $\omega_V - d\tau_V$ on $U \cap V$. Hence there is a global k-form $\omega \in \Omega^k(M)$ such that

$$\omega|_U = \omega_U - d\tau_U, \qquad \omega|_V = \omega_V - d\tau_V.$$

This form is obviously closed, its restriction to U is cohomologous to ω_U, and its restriction to V is cohomologous to ω_V. Hence $i^*[\omega] = ([\omega_U], [\omega_V])$. Thus we have proved that $\ker j^* = \operatorname{im} i^*$.

We prove that $\ker i^* = \operatorname{im} d^*$. Let $\omega \in \Omega^k(M)$ be a closed k-form such that $i^*[\omega] = 0$. Then the restricted k-forms $\omega|_U$ and $\omega|_V$ are exact. Thus there exist $(k-1)$-forms $\tau_U \in \Omega^{k-1}(U)$ and $\tau_V \in \Omega^{k-1}(V)$ such that

$$d\tau_U = \omega|_U, \qquad d\tau_V = \omega|_V.$$

Hence the $(k-1)$-form

$$\tau := \tau_V|_{U\cap V} - \tau_U|_{U\cap V} \in \Omega^{k-1}(U\cap V)$$

is closed. We prove that $d^*[\tau] = [\omega]$. To see this, define $\sigma \in \Omega^{k-1}(M)$ by

$$\sigma := \begin{cases} \rho_U\tau_U + \rho_V\tau_V & \text{on } U\cap V, \\ \rho_U\tau_U & \text{on } U\setminus V, \\ \rho_V\tau_V & \text{on } V\setminus U. \end{cases}$$

Then

$$\tau_U = -\rho_V\tau + \sigma|_U, \qquad \tau_V = \rho_U\tau + \sigma|_V.$$

Here the $(k-1)$-form $\rho_V\tau$ on $U\cap V$ is understood to be extended to all of U by setting it equal to zero on $U\setminus V$ and the $(k-1)$-form $\rho_U\tau$ on $U\cap V$ is understood to be extended to all of V by setting it equal to zero on $V\setminus U$. Since τ is closed we obtain

$$d^*\tau = \left\{ \begin{array}{ll} -d(\rho_V\tau) & \text{on } U \\ d(\rho_U\tau) & \text{on } V \end{array} \right\} = \left\{ \begin{array}{ll} d\tau_U - d\sigma|_U & \text{on } U \\ d\tau_V - d\sigma|_V & \text{on } V \end{array} \right\} = \omega - d\sigma.$$

Hence $d^*[\tau] = [\omega]$ as claimed. Thus we have proved that $\ker i^* = \operatorname{im} d^*$ and this completes the proof of Theorem 6.2.3. □

Corollary 6.2.4. *If $M = U\cup V$ is the union of two open sets such that the de Rham cohomology of U, V, $U\cap V$ is finite-dimensional, then so is the de Rham cohomology of M.*

Proof. By Theorem 6.2.3 the vector space $H^k(M)$ is isomorphic to the direct sum of the image of the homomorphism

$$d^* : H^{k-1}(U\cap V) \to H^k(M)$$

and the image of the homomorphism

$$i^* : H^k(M) \to H^k(U) \oplus H^k(V).$$

As both summands are finite-dimensional so is $H^k(M)$. This proves Corollary 6.2.4. □

6.2.2 Finite Good Covers

The previous result can be used to prove finite-dimensionality of the de Rham cohomology for a large class of manifolds. A collection $\mathscr{U} = \{U_i\}_{i\in I}$ of nonempty open subsets $U_i \subset M$ is called a **good cover** iff $M = \bigcup_{i\in I} U_i$ and each intersection $U_{i_0} \cap \cdots \cap U_{i_k}$ is either empty or diffeomorphic to $\mathbb{R}^m$; it is called a **finite good cover** iff it is a good cover and I is a finite set. Note that the existence of a good cover implies that M has no boundary.

Exercise 6.2.5. Prove that every compact m-manifold without boundary has a finite good cover. **Hint:** Choose a Riemannian metric and cover M by finitely many geodesic balls of radius at most half the injectivity radius. Show that the intersections are all geodesically convex and use Exercise 6.2.6.

Exercise 6.2.6. Every nonempty geodesically convex open subset of a Riemannian m-manifold M without boundary is diffeomorphic to $\mathbb{R}^m$.

Hint 1: It is diffeomorphic to a bounded **star shaped** open set $U \subset \mathbb{R}^m$ centered at the origin so that, if $x \in U$, then $tx \in U$ for $0 \le t \le 1$.

Hint 2: Prove that there exists a smooth function $g : \mathbb{R}^m \to \mathbb{R}$ such that $g(x) > 0$ for every $x \in U$, $g(x) = 1$ for $|x|$ sufficiently small, and $g(x) = 0$ for $x \in \mathbb{R}^n \setminus U$. Define the nonnegative function $h : U \to \mathbb{R}$ by

$$h(x) := \int_0^1 \frac{dt}{g(tx)}.$$

Prove that the map $\phi : U \to \mathbb{R}^m$, $\phi(x) := h(x)x$, is a diffeomorphism.

Hint 3: There is a **lower semicontinuous** function $f : S^{m-1} \to \mathbb{R} \cup \{\infty\}$ such that $f > 0$ and $U = U_f := \left\{rx \,|\, x \in S^{m-1},\ 0 \le r < f(x)\right\}$. (Lower semicontinuity is characterized by the fact that the set U_f is open.) The **Moreau envelopes** of f are the functions

$$(e_n f)(x) := \inf_{y\in S^{m-1}} \Big(f(y) + \frac{n}{2} |x-y|^2 \Big).$$

They are continuous and real valued (unless $f \equiv \infty$) and they approximate f pointwise from below. Use this to prove that there exists a sequence of smooth functions $f_n : S^{m-1} \to \mathbb{R}$ satisfying $0 < f_n < f_{n+1} < f$ for every n and $\lim_{n\to\infty} f_n(x) = f(x)$ for every x. Construct a diffeomorphism from $\mathbb{R}^m$ to U_f that maps the open ball of radius n diffeomorphically onto the set U_{f_n}.

Exercise 6.2.7. Let M be a compact manifold with boundary. Prove that $M \setminus \partial M$ has a finite good cover. **Hint:** Choose a Riemannian metric on M that restricts to a product metric in a tubular neighborhood of the boundary.

Corollary 6.2.8. *If M admits a finite good cover, then its de Rham cohomology is finite-dimensional.*

Proof. The proof is by induction on the number of elements in the good cover. If M has a good cover consisting of precisely one open set, then M is diffeomorphic to $\mathbb{R}^m$ and hence its de Rham cohomology is one-dimensional by Example 6.1.12. Now fix an integer $n \geq 2$ and suppose, by induction, that every smooth manifold that admits a good cover by at most $n-1$ open sets has finite-dimensional de Rham cohomology. Let $M = U_1 \cup U_2 \cup \cdots \cup U_n$ be a good cover and denote

$$U := U_1 \cup \cdots \cup U_{n-1}, \qquad V := U_n.$$

Then the open set $U \cap V$ has a good cover consisting of the open sets $U_i \cap U_n$ for $i = 1, \dots, n-1$. Hence it follows from the induction hypothesis that the manifolds U, V, $U \cap V$ have finite-dimensional de Rham cohomology. Thus, by Corollary 6.2.4, the de Rham cohomology of M is finite-dimensional as well. This proves Corollary 6.2.8. □

Corollary 6.2.9. *Every compact manifold M has finite-dimensional de Rham cohomology.*

Proof. The manifold $M \setminus \partial M$ has a finite good cover by Exercise 6.2.7 and is homotopy equivalent to M. (Prove this.) Hence the assertion follows from Corollary 6.1.5 and Corollary 6.2.8. □

Corollary 6.2.10. *Let M be a smooth m-manifold, let $U \subset M$ be an open subset, and let $f : M \to M$ be a smooth map such that $\overline{f(M)} \subset U$. Assume that the de Rham cohomology groups of both M and U are finite-dimensional. Then, for $k = 0, 1, \dots, m$, we have*

$$\operatorname{trace}\Big(f^* : H^k(M) \to H^k(M)\Big) = \operatorname{trace}\Big((f|_U)^* : H^k(U) \to H^k(U)\Big)$$

Proof. Define $V := M \setminus \overline{f(M)}$. Then the Mayer–Vietoris sequence associated to the cover $M = U \cup V$ gives rise to a commutative diagram

$$\begin{tikzcd}
H^{k-1}(U \cap V) \arrow[r, "d^*"] \arrow[d, "0"] & H^k(M) \arrow[r, "i^*"] \arrow[d, "f^*"] & H^k(U) \oplus H^k(V) \arrow[r, "j^*"] \arrow[d, "f^*"] & H^k(U \cap V) \arrow[d, "0"] \\
H^{k-1}(U \cap V) \arrow[r, "d^*"] & H^k(M) \arrow[r, "i^*"] & H^k(U) \oplus H^k(V) \arrow[r, "j^*"] & H^k(U \cap V)
\end{tikzcd}$$

where the second vertical map sends $([\omega_U], [\omega_V])$ to $([(f|_U)^*\omega_U], [(f|_V)^*\omega_U])$. Since the horizontal sequences are exact, this proves Corollary 6.2.10. □

6.2.3 The Künneth Formula

Let M and N be smooth manifolds and consider the projections

$$\begin{tikzcd} & M \times N \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_N"] & \\ M & & N \end{tikzcd}.$$

They induce a linear map

$$\Omega^k(M) \otimes \Omega^\ell(N) \to \Omega^{k+\ell}(M \times N) : \omega \otimes \tau \mapsto \pi_M^*\omega \wedge \pi_N^*\tau. \qquad (6.2.6)$$

If ω and τ are closed, then so is $\pi_M^*\omega \wedge \pi_N^*\tau$ and if, in addition, one of the forms is exact so is $\pi_M^*\omega \wedge \pi_N^*\tau$. Hence the map (6.2.6) induces a homomorphism

$$\kappa : H^*(M) \otimes H^*(N) \to H^*(M \times N)$$

on de Rham cohomology, given by

$$\kappa([\omega] \otimes [\tau]) := [\pi_M^*\omega \wedge \pi_N^*\tau] \qquad (6.2.7)$$

for two closed forms $\omega \in \Omega^*(M)$ and $\tau \in \Omega^*(N)$.

Theorem 6.2.11 (Künneth Formula). *If M and N admit finite good covers, then κ is an isomorphism; thus*

$$H^\ell(M \times N) \cong \bigoplus_{k=0}^{\ell} H^k(M) \otimes H^{\ell-k}(N)$$

for every integer $\ell \geq 0$ and

$$\dim(H^*(M \times N)) = \dim(H^*(M)) \cdot \dim(H^*(N)).$$

Proof. The proof is by induction on the number n of elements in a good cover of M. If $n = 1$, then M is diffeomorphic to $\mathbb{R}^m$. In this case it follows from Example 6.1.6 that the projection $\pi_N : M \times N \to N$ induces an isomorphism

$$\pi_N^* : H^*(N) \to H^*(M \times N)$$

on de Rham cohomology. Moreover, $H^0(\mathbb{R}^m) = \mathbb{R}$ and $H^k(\mathbb{R}^m) = 0$ for $k > 0$ by Example 6.1.12, and hence κ is an isomorphism, as claimed.

Now fix an integer $n \geq 2$ and assume, by induction, that the Küenneth formula holds for $M \times N$ whenever M admits a good cover by at most $n-1$ open sets. Suppose that

$$M = U_1 \cup U_2 \cup \cdots \cup U_n$$

is a good cover and denote

$$U := U_1 \cup \cdots \cup U_{n-1}, \qquad V := U_n.$$

Then the induction hypothesis asserts that the Künneth formula holds for the product manifolds

$$U \times N, \qquad V \times N, \qquad (U \cap V) \times N.$$

We abbreviate

$$\widetilde{H}^\ell(M) := \bigoplus_{k=0}^{\ell} H^k(M) \otimes H^{\ell-k}(N), \qquad \widehat{H}^\ell(M) := H^\ell(M \times N),$$

so that κ is a homomorphism from $\widetilde{H}^\ell(M)$ to $\widehat{H}^\ell(M)$. Then the Mayer–Vietoris sequence gives rise to the following commutative diagram:

$$\begin{tikzcd}
\widetilde{H}^\ell(M) \arrow[r, "i^*"] \arrow[d, "\kappa"'] & \widetilde{H}^\ell(U) \oplus \widetilde{H}^\ell(V) \arrow[r, "j^*"] \arrow[d, "\kappa"'] & \widetilde{H}^\ell(U \cap V) \arrow[r, "d^*"] \arrow[d, "\kappa"'] & \widetilde{H}^{\ell+1}(M) \arrow[d, "\kappa"'] \\
\widehat{H}^\ell(M) \arrow[r, "i^*"] & \widehat{H}^\ell(U) \oplus \widehat{H}^\ell(V) \arrow[r, "j^*"] & \widehat{H}^\ell(U \cap V) \arrow[r, "d^*"] & \widehat{H}^{\ell+1}(M)
\end{tikzcd}$$

That the first two squares in this diagram commute is obvious from the definitions. We examine the third square. It has the form

$$\begin{tikzcd}
\bigoplus_{k=0}^{\ell} H^k(U \cap V) \otimes H^{\ell-k}(N) \arrow[r, "d^*"] \arrow[d, "\kappa"'] & \bigoplus_{k=0}^{\ell} H^{k+1}(M) \otimes H^{\ell-k}(N) \arrow[d, "\kappa"'] \\
H^\ell((U \cap V) \times N) \arrow[r, "d^*"] & H^{\ell+1}(M \times N)
\end{tikzcd}$$

If $\omega \in \Omega^k(U \cap V)$ and $\tau \in \Omega^{\ell-k}(N)$ are closed forms, we have

$$\begin{aligned}
\kappa d^*(\omega \otimes \tau) &= \pi_M^* d^*\omega \wedge \pi_N^* \tau \\
d^*\kappa(\omega \otimes \tau) &= d^*(\pi_M^* \omega \wedge \pi_N^* \tau).
\end{aligned}$$

Recall that $d^*\omega \in \Omega^{k+1}(M)$ is given by $d\rho_U \wedge \omega$ on $U \cap V$ and vanishes on the set $M \setminus (U \cap V)$, where $\rho_U, \rho_V : M \to [0,1]$ are as in the proof of Lemma 6.2.1. These functions give rise to a partition of unity on $M \times N$, subordinate to the cover by the open sets $U \times N$ and $V \times N$, and defined by

$$\begin{aligned} \pi_M^*\rho_U &= \rho_U \circ \pi_M : M \times N \to [0,1], \\ \pi_M^*\rho_V &= \rho_V \circ \pi_M : M \times N \to [0,1]. \end{aligned}$$

Using this partition of unity for the definition of the boundary operator

$$d^* : \Omega^\ell((U \cap V) \times N) \to \Omega^{\ell+1}(M \times N)$$

in the Mayer–Vietoris sequence for $M \times N$, we obtain the equation

$$\begin{aligned} d^*\kappa(\omega \otimes \tau) &= d^*(\pi_M^*\omega \wedge \pi_N^*\tau) \\ &= d(\pi_M^*\rho_U) \wedge \pi_M^*\omega \wedge \pi_N^*\tau \\ &= \pi_M^*(d\rho_U \wedge \omega) \wedge \pi_N^*\tau \\ &= \pi_M^*d^*\omega \wedge \pi_N^*\tau \\ &= \kappa d^*(\omega \otimes \tau). \end{aligned}$$

on the open set $(U \cap V) \times N$. Since both sides of this equation vanish on the set $(M \setminus (U \cap V)) \times N$, we have proved that

$$d^* \circ \kappa = \kappa \circ d^*.$$

Thus the homomorphism

$$\kappa : \widetilde{H}^* \to \widehat{H}^*$$

in (6.2.7) induces a commuting diagram of the Mayer–Vietoris sequences for $\widetilde{H}^*$ and $\widehat{H}^*$. The induction hypothesis asserts that κ is an isomorphism for each of the manifolds U, V, and $U \cap V$. Hence it follows from the Five Lemma 6.2.12 below that it also is an isomorphism for M. This completes the induction argument and the proof of Theorem 6.2.11. □

Lemma 6.2.12 (Five Lemma). *Let*

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "\alpha"'] & B \arrow[r, "f_2"] \arrow[d, "\beta"'] & C \arrow[r, "f_3"] \arrow[d, "\gamma"'] & D \arrow[r, "f_4"] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\ A' \arrow[r, "f_1'"] & B' \arrow[r, "f_2'"] & C' \arrow[r, "f_3'"] & D' \arrow[r, "f_4'"] & E' \end{tikzcd}$$

be a commutative diagram of homomorphisms of abelian groups such that the horizontal sequences are exact. If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is γ.

Proof. Exercise. □

6.3 Compactly Supported Differential Forms

This section introduces compactly supported de Rham cohomology groups, establishes the Mayer–Vietoris sequence in this setting, and derives various consequences such as finite-dimensionality and the Künneth formula.

6.3.1 Definition and Basic Properties

Let M be an m-dimensional smooth manifold (possibly with boundary) and, for every integer $k \geq 0$, denote by $\Omega_c^k(M)$ the space of compactly supported k-forms on M (see §5.1.3). Consider the cochain complex

$$\Omega_c^0(M) \xrightarrow{d} \Omega_c^1(M) \xrightarrow{d} \Omega_c^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_c^m(M).$$

The cohomology of this complex is called the **compactly supported de Rham cohomology** of M and will be denoted by

$$H_c^k(M) := \frac{\ker d : \Omega_c^k(M) \to \Omega_c^{k+1}(M)}{\operatorname{im} d : \Omega_c^{k-1}(M) \to \Omega_c^k(M)}$$

for $k = 0, 1, \dots, m$.

Remark 6.3.1. If M is compact, then every differential form on M has compact support and hence $\Omega_c^*(M) = \Omega^*(M)$ and $H_c^*(M) = H^*(M)$.

Remark 6.3.2. The compactly supported de Rham cohomology of a manifold is not functorial. If $f : M \to N$ is a smooth map (between noncompact manifolds) and $\omega \in \Omega_c^k(N)$ is a compactly supported differential form on N, then

$$\operatorname{supp}(f^*\omega) \subset f^{-1}(\operatorname{supp}(\omega)).$$

Thus $f^*\omega$ may not have compact support.

Remark 6.3.3. If $f : M \to N$ is **proper** in the sense that

$$K \subset N \text{ is compact} \qquad \Longrightarrow \qquad f^{-1}(K) \subset M \text{ is compact},$$

then pullback under f is a cochain map

$$f^* : \Omega_c^*(N) \to \Omega_c^*(M)$$

and thus induces a homomorphism on compactly supported de Rham cohomology. By Corollary 5.3.9 the induced map on cohomology is invariant under proper homotopies. Here it is not enough to assume that each map f_t in a homotopy is proper; one needs the condition that the homotopy $[0,1] \times M \to N : (t,p) \mapsto f_t(p)$ itself is proper.

Remark 6.3.4. If $\iota : U \to M$ is the inclusion of an open set, then every compactly supported differential form on U can be extended to a smooth differential form on all of M by setting it equal to zero on $M \setminus U$. Thus there is an inclusion induced cochain map

$$\iota_* : \Omega_c^*(U) \to \Omega_c^*(M)$$

and a homomorphism on compactly supported de Rham cohomology.

These remarks show that the compactly supported de Rham cohomology of a noncompact manifold behaves rather differently from the usual de Rham cohomology. This is also illustrated by the following examples.

Example 6.3.5. The compactly supported de Rham cohomology of the 1-manifold $M = \mathbb{R}$ is given by

$$H_c^0(\mathbb{R}) = 0, \qquad H_c^1(\mathbb{R}) = \mathbb{R}.$$

That $H_c^0(\mathbb{R}) = 0$ follows from the fact that every compactly supported function $f : \mathbb{R} \to \mathbb{R}$ with $df = 0$ vanishes identically. To prove $H_c^1(\mathbb{R}) = \mathbb{R}$ we observe that a 1-form $\omega \in \Omega_c^1(\mathbb{R})$ can be written in the form

$$\omega = g(x)\, dx,$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth function with compact support. Thus $\omega = df$ where $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) := \int_{-\infty}^x g(t)\, dt$. This function has compact support if and only if the integral of g over $\mathbb{R}$ vanishes. Thus ω belongs to the image of the operator $d : \Omega_c^0(\mathbb{R}) \to \Omega_c^1(\mathbb{R})$ if and only if its integral is zero. This is a special case of Theorem 5.3.10.

Example 6.3.6. If M is connected and not compact, then every compactly supported locally constant function on M vanishes and hence

$$H_c^0(M) = 0.$$

Example 6.3.7. If M is a nonempty connected oriented smooth m-dimensional manifold without boundary, then

$$H_c^m(M) \cong \mathbb{R}.$$

An explicit isomorphism from $H_c^m(M)$ to the reals is given by

$$H_c^m(M) \to \mathbb{R} : [\omega] \to \int_M \omega.$$

This map is surjective, because M is nonempty, and it is injective by Theorem 5.3.10.

Theorem 6.3.8. *For every smooth m-manifold M we have*

$$H_c^{k+1}(M \times \mathbb{R}) \cong H_c^k(M), \qquad k = 0, 1, \dots, m.$$

Corollary 6.3.9. *The compactly supported de Rham cohomology of $\mathbb{R}^m$ is given by*

$$H_c^k(\mathbb{R}^m) = \begin{cases} \mathbb{R}, & \text{for } k = m, \\ 0, & \text{for } k < m. \end{cases}$$

Proof. This follows from Example 6.3.5 by induction. The induction step uses Example 6.3.6 for $k = 0$ and Theorem 6.3.8 for $k > 0$. □

Proof of Theorem 6.3.8. As a warmup we consider the case $M = \mathbb{R}^m$ and use the coordinates $(x^1, \dots, x^m, t)$ on $\mathbb{R}^m \times \mathbb{R}$. Then a (compactly supported) k-form on $\mathbb{R}^m \times \mathbb{R}$ has the form

$$\omega = \sum_{|I|=k-1} \alpha_I(x,t) dx^I \wedge dt + \sum_{|J|=k} \beta_J(x,t) dx^J,$$

where the α_I and β_J are smooth real valued functions on $\mathbb{R}^m \times \mathbb{R}$ (with compact support). Fixing a real number $t \in \mathbb{R}$ we obtain differential forms

$$\alpha_t := \sum_{|I|=k-1} \alpha_I(x,t) dx^I \in \Omega_c^{k-1}(\mathbb{R}^m),$$
$$\beta_t := \sum_{|J|=k} \beta_J(x,t) dx^J \in \Omega_c^k(\mathbb{R}^m).$$

Going to the general case, we see that a compactly supported differential form $\omega \in \Omega_c^k(M \times \mathbb{R})$ can be written as

$$\omega = \alpha_t \wedge dt + \beta_t, \tag{6.3.1}$$

where $\mathbb{R} \to \Omega_c^{k-1}(M) : t \mapsto \alpha_t$ and $\mathbb{R} \to \Omega_c^k(M) : t \mapsto \beta_t$ are smooth families of differential forms on M such that the set

$$\mathrm{supp}(\omega) = \overline{\bigcup_{t \in \mathbb{R}} \{t\} \times \big(\mathrm{supp}(\alpha_t) \cup \mathrm{supp}(\beta_t)\big)}$$

is compact. The formula in local coordinates shows that the exterior differential of $\omega \in \Omega_c^k(M \times \mathbb{R})$ is given by

$$d\omega = d^{M \times \mathbb{R}} \omega = \Big(d^M \alpha_t + (-1)^k \partial_t \beta_t\Big) \wedge dt + d^M \beta_t. \tag{6.3.2}$$

Choose a smooth function $e : \mathbb{R} \to \mathbb{R}$ with compact support such that

$$\int_{-\infty}^{\infty} e(t)\, dt = 1$$

and define the operators

$$\pi_* : \Omega_c^{k+1}(M \times \mathbb{R}) \to \Omega_c^k(M), \qquad e_* : \Omega_c^k(M) \to \Omega_c^{k+1}(M \times \mathbb{R}),$$

by

$$\pi_*\omega := \int_{-\infty}^{\infty} \alpha_t\, dt, \qquad e_*\alpha := e(t)\alpha \wedge dt. \tag{6.3.3}$$

for $\omega = \alpha_t \wedge dt + \beta_t \in \Omega_c^{k+1}(M \times \mathbb{R})$ and $\alpha \in \Omega_c^k(M)$. Then it follows from equation (6.3.2) that

$$\pi_* \circ d = d^M \circ \pi_*, \qquad d \circ e_* = e_* \circ d^M. \tag{6.3.4}$$

Hence π_* and e_* induce homomorphisms on compactly supported de Rham cohomology, still denoted by π_* and e_*. We have the identity

$$\pi_* \circ e_* = \mathrm{id}$$

both on $\Omega_c^k(M)$ and on $H_c^k(M)$. We prove that the composition $e_* \circ \pi_*$ is chain homotopy equivalent to the identity, i.e. there exists a collection of linear operators $K : \Omega_c^{k+1}(M \times \mathbb{R}) \to \Omega_c^k(\mathbb{R} \times M)$, one for each k, such that

$$\mathrm{id} - e_* \circ \pi_* = d \circ K + K \circ d. \tag{6.3.5}$$

Given $\omega = \alpha_t \wedge dt + \beta_t \in \Omega_c^{k+1}(M \times \mathbb{R})$ define the k-form $K\omega \in \Omega_c^k(M \times \mathbb{R})$ by $K\omega := \widetilde{\alpha}_t \wedge dt + \widetilde{\beta}_t$, where

$$\widetilde{\alpha}_t := 0, \qquad \widetilde{\beta}_t := (-1)^k \int_{-\infty}^{t} \Big(\alpha_s - e(s)\pi_*\omega\Big)\, ds. \tag{6.3.6}$$

Combining (6.3.2) and (6.3.6) we find

$$\begin{aligned} dK\omega &= \Big(\alpha_t - e(t)\pi_*\omega\Big) \wedge dt + (-1)^k d^M \int_{-\infty}^{t} \Big(\alpha_s - e(s)\pi_*\omega\Big)\, ds, \\ Kd\omega &= (-1)^{k+1} \int_{-\infty}^{t} \Big(d^M\alpha_s + (-1)^{k+1}\partial_s\beta_s - e(s)\pi_* d\omega\Big)\, ds \\ &= \beta_t + (-1)^{k+1} d^M \int_{-\infty}^{t} \Big(\alpha_s - e(s)\pi_*\omega\Big)\, ds. \end{aligned}$$

Here the last equality uses (6.3.4). Take the sum to obtain

$$dK\omega + Kd\omega = \alpha_t \wedge dt - e(t)\pi_*\omega \wedge dt + \beta_t = \omega - e_*\pi_*\omega.$$

This proves (6.3.5) and Theorem 6.3.8. □

6.3.2 The Mayer–Vietoris Sequence for H_c^*

Let M be a smooth m-manifold and let $U, V \subset M$ be two open sets such that $U \cup V = M$. The **Mayer–Vietoris sequence** in this setting has the form

$$0 \longleftarrow \Omega_c^k(M) \xleftarrow{i_*} \Omega_c^k(U) \oplus \Omega_c^k(V) \xleftarrow{j_*} \Omega_c^k(U \cap V) \longleftarrow 0, \tag{6.3.7}$$

where the homomorphisms

$$i_* : \Omega_c^k(U) \oplus \Omega_c^k(V) \to \Omega_c^k(M), \qquad j_* : \Omega_c^k(U \cap V) \to \Omega_c^k(U) \oplus \Omega_c^k(V)$$

are defined by

$$i_*(\omega_U, \omega_V) := \omega_U + \omega_V, \qquad j_*\omega := (-\omega, \omega)$$

for $\omega_U \in \Omega_c^k(U)$, $\omega_V \in \Omega_c^k(V)$, and $\omega \in \Omega_c^k(U \cap V)$. Here the first summand in the pair $(-\omega, \omega) \in \Omega_c^k(U) \oplus \Omega_c^k(V)$ is understood in the first component as the extension of $-\omega$ to all of U by setting it zero on $U \setminus V$ and in the second component as the extension of ω to all of V by setting it zero on $V \setminus U$. Likewise, the k-form $\omega_U + \omega_V \in \Omega_c^k(M)$ is understood as the sum after extending ω_U to all of M by setting it zero on $V \setminus U$ and extending ω_V to all of M by setting it zero on $U \setminus V$.

Lemma 6.3.10. *The Mayer–Vietoris sequence* (6.3.7) *is exact.*

Proof. That j_* is injective is obvious. That the image of j_* agrees with the kernel of i_* follows from the fact that if the sum of the compactly supported differential form $\omega_U \in \Omega_c^k(U)$ and $\omega_V \in \Omega^k(V)$ vanishes on all of M, then the compact set $\mathrm{supp}(\omega_V) = \mathrm{supp}(\omega_U)$ is contained in $U \cap V$.

We prove that i_* is surjective. As in the proof of Lemma 6.2.1 we choose a partition of unity subordinate to the cover $M = U \cup V$, consisting of two smooth functions $\rho_U : M \to [0,1]$ and $\rho_V : M \to [0,1]$ satisfying

$$\mathrm{supp}(\rho_U) \subset U, \qquad \mathrm{supp}(\rho_V) \subset V, \qquad \rho_U + \rho_V \equiv 1.$$

Let $\omega \in \Omega_c^k(M)$ and define $\omega_U \in \Omega_c^k(U)$ and $\omega_V \in \Omega_c^k(V)$ by

$$\omega_U := \rho_U \omega|_U, \qquad \omega_V := \rho_V \omega|_V.$$

Then

$$i_*(\omega_U, \omega_V) = \omega_U + \omega_V = \omega.$$

This proves Lemma 6.3.10. □

As in §6.2 we have that i_* and j_* are cochain maps so that the following diagram commutes

$$\begin{tikzcd}
0 & \Omega_c^{k+1}(M) \arrow[l] & \Omega_c^{k+1}(U) \oplus \Omega_c^{k+1}(V) \arrow[l, "i_*"'] & \Omega_c^{k+1}(U \cap V) \arrow[l, "j_*"'] & 0 \arrow[l] \; . \\
0 & \Omega_c^{k}(M) \arrow[l] \arrow[u, "d"'] & \Omega_c^{k}(U) \oplus \Omega_c^{k}(V) \arrow[l, "i_*"'] \arrow[u, "d"'] & \Omega_c^{k}(U \cap V) \arrow[l, "j_*"'] \arrow[u, "d"'] & 0 \arrow[l]
\end{tikzcd}$$

The boundary operator

$$d_* : H_c^k(M) \to H_c^{k+1}(U \cap V)$$

for the long exact sequence is is defined as follows. Let $\omega \in \Omega_c^k(M)$ be a closed k-form with compact support and choose a pair

$$(\omega_U, \omega_V) \in \Omega_c^k(U) \oplus \Omega_c^k(V)$$

whose image under i_* is ω. Then the pair $(d\omega_U, d\omega_V)$ belongs to the kernel of i_* because ω is closed, and hence belongs to the image of j_* by exactness. Hence there exists a unique $(k+1)$-form $d_*\omega \in \Omega_c^{k+1}(U \cap V)$ with compact support whose image under j_* is the given pair $(d\omega_U, d\omega_V)$. As before, this form is closed and its cohomology class in $H_c^{k+1}(U \cap V)$ is independent of the choice of the pair (ω_U, ω_V) used in this construction.

Again, there is an explicit formula for the operator d_* coming from the proof of Lemma 6.3.10. Define the map $d_* : \Omega_c^k(M) \to \Omega_c^{k+1}(U \cap V)$ by

$$d_*\omega := d\rho_V \wedge \omega|_{U \cap V}. \tag{6.3.8}$$

This operator is well defined because the 1-form $d\rho_V = -d\rho_U$ is supported in $U \cap V$. Moreover, we have

$$d \circ d_* + d_* \circ d = 0 \tag{6.3.9}$$

and hence d_* assigns closed forms to closed forms and exact forms to exact forms. Thus d_* descends to a homomorphism on cohomology.

Exercise 6.3.11. Prove that the linear map $d_* : \Omega_c^k(M) \to \Omega_c^{k+1}(U \cap V)$ defined by (6.3.8) satisfies equation (6.3.9) and hence descends to a homomorphism $d_* : H_c^k(M) \to H_c^{k+1}(U \cap V)$. Prove that the induced homomorphism on cohomology is independent of the choice of the partition of unity ρ_U, ρ_V and agrees with the homomorphism defined by *diagram chasing* as above.

The homomorphisms on compactly supported de Rham cohomology induced by i_*, j_*, d_* give rise to a long exact sequence

$$\cdots H_c^k(M) \xleftarrow{i_*} H_c^k(U) \oplus H_c^k(V) \xleftarrow{j_*} H_c^k(U \cap V) \xleftarrow{d_*} H_c^{k-1}(M) \cdots \qquad (6.3.10)$$

which is also called the **Mayer–Vietoris sequence**.

Theorem 6.3.12. *The Mayer–Vietoris sequence* (6.3.10) *is exact.*

Proof. That the composition of any two successive homomorphisms is zero follows directly from the definitions.

We prove that $\ker d_* = \operatorname{im} i_*$. Let $\omega \in \Omega_c^k(M)$ be a closed compactly supported k-form on M such that $d_*[\omega] = 0$. Then there exists a compactly supported k-form $\tau \in \Omega_c^k(U \cap V)$ such that

$$d\tau = d(\rho_V \omega)|_{U\cap V} = -d(\rho_U \omega)|_{U\cap V}.$$

Define $\omega_U \in \Omega_c^k(U)$ and $\omega_V | in \Omega_c^k(V)$ by

$$\omega_U := \begin{cases} \rho_U\omega + \tau & \text{on } U \cap V, \\ \rho_U \omega & \text{on } U \setminus V, \end{cases} \qquad \omega_V := \begin{cases} \rho_V\omega + \tau & \text{on } U \cap V, \\ \rho_V \omega & \text{on } V \setminus U. \end{cases}$$

These forms are closed and have compact support. Moreover, $\omega_U + \omega_V = \omega$ and hence $i_*([\omega_U], [\omega_V]) = [\omega]$. Thus we have proved that $\ker d_* = \operatorname{im} i_*$.

We prove that $\ker i_* = \operatorname{im} j_*$. Let $\omega_U \in \Omega_c^k(U)$ and $\omega_V \in \Omega_c^k(V)$ be compactly supported closed k-forms such that $i_*([\omega_U], [\omega_V]) = 0$. Then there exists a compactly supported $(k-1)$-form $\tau \in \Omega_c^{k-1}(M)$ such that

$$d\tau = \begin{cases} \omega_U + \omega_V & \text{on } U \cap V, \\ \omega_U & \text{on } U \setminus V, \\ \omega_V & \text{on } V \setminus U. \end{cases}$$

It follows that the k-form

$$\omega := \omega_V|_{U\cap V} - d(\rho_V\tau)|_{U\cap V} = -\omega_U|_{U\cap V} + d(\rho_U\tau)|_{U\cap V} \in \Omega_c^k(U \cap V)$$

has compact support in $U \cap V$. Moreover, ω is closed and the pair

$$j_*\omega = \left(\left\{ \begin{array}{ll} -\omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus V \end{array} \right\}, \left\{ \begin{array}{ll} \omega & \text{on } U \cap V, \\ 0 & \text{on } V \setminus U \end{array} \right\} \right) \in \Omega_c^k(U) \oplus \Omega_c^k(V)$$

is cohomologous to (ω_U, ω_V). Hence $j_*[\omega] = ([\omega_U], [\omega_V])$. Thus we have proved that $\ker i_* = \operatorname{im} j_*$.

We prove that $\ker j_* = \operatorname{im} d_*$. Let $\omega \in \Omega_c^k(U \cap V)$ be a compactly supported closed k-form such that $j_*[\omega] = 0$. Then there exist compactly supported $(k-1)$-forms $\tau_U \in \Omega_c^{k-1}(U)$ and $\tau_V \in \Omega_c^{k-1}(V)$ such that

$$d\tau_U := \begin{cases} -\omega & \text{on } U \cap V, \\ 0 & \text{on } U \setminus V, \end{cases} \qquad d\tau_V := \begin{cases} \omega & \text{on } U \cap V, \\ 0 & \text{on } V \setminus U. \end{cases}$$

Define $\tau \in \Omega_c^{k-1}(M)$ and $\sigma \in \Omega_c^{k-1}(U \cap V)$ by

$$\tau := \begin{cases} \tau_U + \tau_V & \text{on } U \cap V, \\ \tau_U & \text{on } U \setminus V, \\ \tau_V & \text{on } V \setminus U, \end{cases} \qquad \sigma := \rho_V \tau_U|_{U\cap V} - \rho_U \tau_V|_{U\cap V}.$$

Note that the set $\operatorname{supp}(\tau) \subset \operatorname{supp}(\tau_U) \cup \operatorname{supp}(\tau_V)$ is a compact subset of M and the set $\operatorname{supp}(\sigma) \subset (\operatorname{supp}(\rho_V) \cap \operatorname{supp}(\tau_U)) \cup (\operatorname{supp}(\rho_U) \cap \operatorname{supp}(\tau_V))$ is a compact subset of $U \cap V$. Moreover, τ is closed and

$$\rho_V \tau|_{U\cap V} = \tau_V|_{U\cap V} + \sigma.$$

Hence

$$\begin{aligned} d_*[\tau] &= [d_*\tau] \\ &= [d\rho_V \wedge \tau|_{U\cap V}] \\ &= [d(\rho_V\tau)|_{U\cap V}] \\ &= [d\tau_V|_{U\cap V} + d\sigma] \\ &= [d\tau_V|_{U\cap V}] \\ &= [\omega]. \end{aligned}$$

Thus $\ker j_* = \operatorname{im} d_*$ and this proves Theorem 6.3.12. □

The proof of Theorem 6.3.12 also follows from Lemma 6.3.10 and an abstract general principle in homological algebra, namely, that every short exact sequence of (co)chain complexes determines uniquely a long exact sequence in (co)homology. In the proof of Theorem 6.3.12 we have established exactness with the boundary map given by an explicit formula. The formulas for the boundary maps d^* and d_* in the Mayer–Vietoris sequences will be useful in the proof of Poincaré duality. The Mayer–Vietoris sequence for compactly supported de Rham cohomology can be used as before to establish finite-dimensionality and the Künneth formula. This is the content of the next three corollaries.

Corollary 6.3.13. *If $M = U \cup V$ is the union of two open sets such that the compactly supported de Rham cohomology of U, V, $U \cap V$ is finite-dimensional, then so is the compactly supported de Rham cohomology of M.*

Proof. By Theorem 6.3.12 the vector space $H_c^k(M)$ is isomorphic to the direct sum of the image of the homomorphism

$$i_* : H_c^k(U) \oplus H_c^k(V) \to H_c^k(M).$$

and the image of the homomorphism

$$d^* : H_c^k(M) \to H^{k+1}(U \cap V).$$

As both summands are finite-dimensional so is $H_c^k(M)$. This proves Corollary 6.3.13. □

Corollary 6.3.14. *If M admits a finite good cover, then its compactly supported de Rham cohomology is finite-dimensional.*

Proof. The proof is by induction on the number of elements in a good cover as in Corollary 6.2.8. Here one uses Corollary 6.3.9 instead of Example 6.1.12 and Corollary 6.3.13 instead of Corollary 6.2.4. □

Corollary 6.3.15 (Künneth Formula). *If M and N have finite good covers, then the map*

$$\Omega_c^k(M) \otimes \Omega_c^\ell(N) \to \Omega_c^{k+\ell}(M \times N) : \omega \otimes \tau \mapsto \pi_M^*\omega \wedge \pi_N^*\tau$$

induces an isomorphism

$$\kappa : H_c^*(M) \otimes H_c^*(N) \to H_c^*(M \times N).$$

Thus

$$\bigoplus_{k=0}^{\ell} H_c^k(M) \otimes H_c^{\ell-k}(N) \cong H_c^\ell(M \times N)$$

for every integer $\ell \geq 0$ and

$$\dim(H_c^*(M \times N)) = \dim(H_c^*(M)) \cdot \dim(H_c^*(N)).$$

Proof. The proof is exactly the same as that of Theorem 6.2.11. □

6.4 Poincaré Duality

In §6.4.1 we introduce Poincaré duality for oriented boundaryless manifolds that admit finite good covers. The proof will be given in §6.4.2. Poincaré duality is used in §6.4.3 to associate to a compact oriented submanifold without boundary a dual de Rham cohomology class. A key formula which relates the cup product of two such cohomology classes to the intersection number (Theorem 6.4.8) will be proved in §7.2.3. This result is used in §6.4.4 to prove the full version of the Poincaré–Hopf Theorem 2.3.1 and of the Lefschetz–Hopf Theorem 4.4.6. §6.4.5 uses Poincaré duality to compute the de Rham cohomology groups of some examples.

6.4.1 The Poincaré Pairing

Let M be an oriented smooth m-dimensional manifold without boundary. Then, for every integer $k \in \{0, 1, \dots, m\}$, there is a bilinear map

$$\Omega^k(M) \times \Omega_c^{m-k}(M) : (\omega, \tau) \mapsto \int_M \omega \wedge \tau. \tag{6.4.1}$$

If the differential forms ω and τ are closed and one of them is exact, then $\omega \wedge \tau$ is the exterior differential of a compactly supported $(m-1)$-form and so its integral vanishes by Theorem 5.2.12. Thus the pairing (6.4.1) descends to a bilinear form on de Rham cohomology, the **Poincaré pairing**

$$H^k(M) \times H_c^{m-k}(M) : ([\omega], [\tau]) \mapsto \int_M \omega \wedge \tau. \tag{6.4.2}$$

Theorem 6.4.1 (**Poincaré Duality**). *Let M be an oriented smooth m-dimensional manifold without boundary and suppose that M has a finite good cover. Then the Poincaré pairing* (6.4.2) *is nondegenerate. This is equivalent to the following two assertions.*

(a) *If $\omega \in \Omega^k(M)$ is closed and satisfies the condition*

$$\tau \in \Omega_c^{m-k}(M), \quad d\tau = 0 \qquad \Longrightarrow \qquad \int_M \omega \wedge \tau = 0,$$

then ω is exact.

(b) *If $\tau \in \Omega_c^{m-k}(M)$ is closed and satisfies the condition*

$$\omega \in \Omega^k(M), \quad d\omega = 0 \qquad \Longrightarrow \qquad \int_M \omega \wedge \tau = 0,$$

then there exists a differential form $\sigma \in \Omega_c^{m-k-1}(M)$ such that $d\sigma = \tau$.

The proof of Theorem 6.4.1 is deferred to §6.4.2.

Remark 6.4.2. The assumption that ω is closed is not needed in part (a) and the assumption that τ is closed is not needed in part (b). In fact, if $\int_M \omega \wedge d\sigma = 0$ for every $\sigma \in \Omega_c^{m-k-1}(M)$, then, by Stokes' Theorem 5.2.12, we have $\int_M d\omega \wedge \sigma = 0$ for every $\sigma \in \Omega_c^{m-k-1}(M)$ and hence $d\omega = 0$. Similarly for τ.

The Poincaré pairing (6.4.2) induces a homomorphism

$$\mathrm{PD} : H^k(M) \to H_c^{m-k}(M)^* = \mathrm{Hom}(H_c^{m-k}(M), \mathbb{R}) \tag{6.4.3}$$

which assigns to the cohomology class of a closed k-form $\omega \in \Omega^k(M)$ the homomorphism

$$H_c^{m-k}(M) \longrightarrow \mathbb{R} : [\tau] \mapsto \mathrm{PD}([\omega])([\tau]) := \int_M \omega \wedge \tau.$$

The Poincaré pairing (6.4.2) also induces a dual homomorphism

$$\mathrm{PD}^* : H_c^{m-k}(M) \to H^k(M)^* = \mathrm{Hom}(H^k(M), \mathbb{R}) \tag{6.4.4}$$

which sends a class $[\tau] \in H_c^{m-k}(M)$ to the homomorphism

$$H^k(M) \longrightarrow \mathbb{R} : [\omega] \mapsto \mathrm{PD}^*([\tau])([\omega]) := \int_M \omega \wedge \tau.$$

Corollary 6.4.3 (Poincaré Duality). Let M be an oriented m-manifold without boundary that admits a finite good cover. Then the homomorphisms (6.4.3) and (6.4.4) are bijective.

Proof. Since M admits a finite good cover, it follows from Corollary 6.2.8 and Corollary 6.3.14 that the cohomology groups $H^*(M)$ and $H_c^*(M)$ are finite-dimensional. Hence, by Exercise 6.4.7, the Poincaré pairing (6.4.2) is nondegenerate if and only if the homomorphism (6.4.3) is bijective, if and only if the homomorphism (6.4.4) is bijective. □

Remark 6.4.4. If $H^k(M)$ and $H_c^{m-k}(M)$ are finite-dimensional, then by Exercise 6.4.7 the homomorphism (6.4.3) is bijective if and only if (6.4.4) is bijective. However, in the infinite-dimensional case these two assertions are not only not equivalent but actually exclude one another. It turns out that the operator (6.4.3) is an isomorphism for every oriented m-manifold M without boundary while (6.4.4) is only an isomorphism when the de Rham cohomology is finite-dimensional (see [3, Remark 5.7]).

Remark 6.4.5. If M is compact without boundary, then $H_c^*(M) = H^*(M)$. In this case the homomorphisms $\mathrm{PD} : H^k(M) \to H^{m-k}(M)^*$ in (6.4.3) and $\mathrm{PD}^* : H^k(M) \to H^{m-k}(M)^*$ in (6.4.4) differ by a sign $(-1)^{k(m-k)}$.

Example 6.4.6. As a warmup we prove Poincaré duality for $M = \mathbb{R}^m$. That $\mathrm{PD} : H^k(\mathbb{R}^m) \to H_c^{m-k}(\mathbb{R}^m)^*$ is an isomorphism for $k > 0$ follows from the fact that both cohomology groups vanish. (See Example 6.1.12 and Corollary 6.3.9.) For $k = 0$ the Poincaré pairing has the form

$$\Omega^0(\mathbb{R}^m) \times \Omega_c^m(\mathbb{R}^m) : (f, \tau) \mapsto \int_{\mathbb{R}^m} f\tau.$$

If $f \in \Omega^0(\mathbb{R}^m)$ and $\int_M f\tau = 0$ for every compactly supported m-form on M, then f vanishes. To see this, suppose by contradiction that $f \neq 0$ on some nonempty open set $U \subset \mathbb{R}^m$. Then we can choose

$$\tau = f\rho dx^1 \wedge \cdots \wedge dx^m,$$

where $\rho : \mathbb{R}^m \to \mathbb{R}$ is a smooth function supported in U such that $\rho \geq 0$ and $\rho(x) > 0$ for some $x \in U$. With this choice of τ we have

$$\int_{\mathbb{R}^m} f\tau = \int_{\mathbb{R}^m} f^2(x)\rho(x)dx^1 \cdots dx^m > 0,$$

a contradiction. Conversely, if $\tau \in \Omega_c^m(\mathbb{R}^m)$ satisfies $\int_{\mathbb{R}^m} f\tau = 0$ for every constant function $f : M \to \mathbb{R}$, then $\int_{\mathbb{R}^m} \tau = 0$, and so by Theorem 5.3.10 there exists a $\sigma \in \Omega_c^{m-1}(\mathbb{R}^m)$ such that $d\sigma = \tau$.

Exercise 6.4.7. Let X and Y be real vector spaces, denote their dual spaces by $X^* = \mathrm{Hom}(X, \mathbb{R})$ and $Y^* = \mathrm{Hom}(Y, \mathbb{R})$, and let

$$B : X \times Y \to \mathbb{R}$$

be a bilinear form. Call B **nondegenerate** iff $B(x, \cdot) = 0$ implies $x = 0$ for all $x \in X$ and $B(\cdot, y) = 0$ implies $y = 0$ for all $y \in Y$. If X and Y are finite-dimensional, prove that the following are equivalent.

(a) B is nondegenerate.

(b) The linear map $X \to Y^* : x \mapsto B(x, \cdot)$ is bijective.

(c) The linear map $Y \to X^* : y \mapsto B(\cdot, y)$ is bijective.

If X and Y are infinite-dimensional, then (b) implies (a) and (c) implies (a), however, (a) no longer implies (b) or (c), and (b) and (c) exclude one another. **Hint:** Show that (a) implies $\dim(X) = \dim(Y)$. Consider the example where X is the space of all sequences $(x_i)_{i\in\mathbb{N}}$ of real numbers, Y is the space of all finite sequences $(y_1, \dots, y_n)$ of real numbers, and $B(x, y) = \sum_i x_i y_i$. Show that this example satisfies (a) and (b) but not (c). Relate this example to the Poincaré pairing of the m-manifold $M = \mathbb{N} \times \mathbb{R}^m \subset \mathbb{R}^{m+1}$.

6.4.2 Proof of Poincaré Duality

Proof of Theorem 6.4.1. The proof is by induction on the number n of sets in a good cover of M. If $n = 1$, then M is diffeomorphic to $\mathbb{R}^m$ and hence the assertion follows from Example 6.4.6. Now let $n \geq 2$, suppose that

$$M = U_1 \cup \cdots \cup U_n$$

is a good cover, and suppose that Poincaré duality has been established for every oriented m-manifold with a good cover by at most $n-1$ open sets. Denote by $U, V \subset M$ the open sets

$$U := U_1 \cup \cdots \cup U_{n-1}, \qquad V := U_n.$$

Then the induction hypothesis asserts that Poincaré duality holds for the manifolds U, V, and $U \cap V$. We shall prove that M satisfies Poincaré duality by considering simultaneously the Mayer–Vietoris sequences for H^* and H_c^* associated to the cover $M = U \cup V$.

We prove that the following diagram commutes

$$\begin{tikzcd}
H^k(M) \arrow[r, "i^*"] \arrow[d, "\mathrm{PD}"] & \begin{array}{c} H^k(U) \\ \oplus \\ H^k(V) \end{array} \arrow[r, "j^*"] \arrow[d, "\mathrm{PD}", "\cong"'] & H^k(U \cap V) \arrow[r, "d^*"] \arrow[d, "\mathrm{PD}", "\cong"'] & H^{k+1}(M) \arrow[d, "\mathrm{PD}"] \\
H_c^{m-k}(M)^* \arrow[r, "(i_*)^*"] & \begin{array}{c} H_c^{m-k}(U)^* \\ \oplus \\ H_c^{m-k}(V)^* \end{array} \arrow[r, "(j_*)^*"] & H_c^{m-k}(U \cap V)^* \arrow[r, "\pm(d_*)^*"] & H_c^{m-k-1}(M)^*
\end{tikzcd} \quad . \tag{6.4.5}$$

Commutativity of the first square in (6.4.5) asserts that all closed differential forms $\omega \in \Omega^k(M)$, $\tau_U \in \Omega_c^{m-k}(U)$, $\tau_V \in \Omega_c^{m-k}(V)$ satisfy

$$\int_M \omega \wedge i_*(\tau_U, \tau_V) = \int_U \omega|_U \wedge \tau_U + \int_V \omega|_V \wedge \tau_V.$$

This follows from the definition of the homomorphism

$$i_* : \Omega_c^{m-k}(U) \oplus \Omega_c^{m-k}(V) \to \Omega_c^{m-k}(M)$$

in (6.3.7). Commutativity of the second square in (6.4.5) asserts that all closed differential forms $\omega_U \in \Omega^k(U)$, $\omega_V \in \Omega^{m-k}(V)$, $\tau \in \Omega_c^{m-k}(U \cap V)$ satisfy

$$\int_U \omega_U \wedge (-\tau) + \int_V \omega_V \wedge \tau = \int_{U\cap V} j^*(\omega_U, \omega_V) \wedge \tau.$$

This follows from the definition of the homomorphism

$$j^* : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$$

in (6.2.1). Commutativity of the third square in (6.4.5) with the sign equal to $(-1)^{k+1}$ asserts that all closed differential forms $\omega \in \Omega^k(U \cap V)$ and $\tau \in \Omega_c^{m-k-1}(M)$ satisfy

$$\int_M d^*\omega \wedge \tau = (-1)^{k+1} \int_{U\cap V} \omega \wedge d_*\tau.$$

To see this, recall that

$$d^*\omega = d\rho_U \wedge \omega \in \Omega^{k+1}(M), \qquad d_*\tau = d\rho_V \wedge \tau \in \Omega_c^{m-k}(U \cap V).$$

Here $d\rho_U \wedge \omega$ is extended to all of M by setting it equal to zero on $M\backslash(U\cap V)$, and $d\rho_V \wedge \tau$ is restricted to $U \cap V$ where it still has compact support. Since $d\rho_U + d\rho_V = 0$ we obtain

$$\begin{aligned}\int_M d^*\omega \wedge \tau &= \int_{U\cap V} d\rho_U \wedge \omega \wedge \tau \\ &= (-1)^k \int_{U\cap V} \omega \wedge d\rho_U \wedge \tau \\ &= (-1)^{k+1} \int_{U\cap V} \omega \wedge d\rho_V \wedge \tau \\ &= (-1)^{k+1} \int_{U\cap V} \omega \wedge d_*\tau\end{aligned}$$

as claimed. This shows that the diagram (6.4.5) commutes. Since the horizontal sequences are exact and the Poincaré duality homomorphism

$$\mathrm{PD} : H^*(N) \to H_c^{m-*}(N)^*$$

is an isomorphisms for $N = U$, $N = V$, and $N = U \cap V$ by the induction hypothesis, it follows from the Five Lemma 6.2.12 that the homomorphism

$$\mathrm{PD} : H^*(M) \to H_c^{m-*}(M)^*$$

is an isomorphism as well. This proves Theorem 6.4.1. □

6.4.3 Poincaré Duality and Intersection Numbers

Let M be an oriented smooth m-manifold without boundary that admits a finite good cover. By Theorem 6.4.1 every linear map $\Lambda : H^{m-k}(M) \to \mathbb{R}$ determines a unique de Rham cohomology class $[\tau] \in H^k_c(M)$ with compact support such that $\Lambda([\omega]) = \int_M \omega \wedge \tau$ for every closed k-form $\omega \in \Omega^k(M)$. An important class of such homomorphisms arises from integration over submanifolds or, more genrally, from the integration of pullbacks. Thus, if P is a compact oriented ℓ-manifold without boundary and $f : P \to M$ is a smooth map, then there exists a closed k-form $\tau_f \in \Omega^{m-\ell}_c(M)$, unique up to an additive exact form, such that

$$\int_M \omega \wedge \tau_f = \int_P f^*\omega \tag{6.4.6}$$

for every closed ℓ-form $\omega \in \Omega^\ell(M)$. The cohomology class of τ_f in $H^{m-\ell}_c(M)$ is the inverse image of the linear functional $H^\ell(M) \to \mathbb{R} : [\omega] \mapsto \int_P f^*\omega$ under isomorphism $\mathrm{PD}^* : H^{m-\ell}_c(M) \to H^\ell(M)^*$ in (6.4.4). The unique de Rham cohomology class $[\tau_f] \in H^{m-\ell}_c(M)$ is called **(Poincaré) dual to** f. We also call each representative of this class **dual to** f. If $Q \subset M$ is a compact oriented codimension-ℓ submanifold without boundary, we use this construction for the obvious embedding of Q into M. Thus there exists a closed ℓ-form $\tau_Q \in \Omega^\ell_c(M)$, unique up to an additive exact form, such that

$$\int_M \omega \wedge \tau_Q = \int_Q \omega \tag{6.4.7}$$

for every closed $(m-\ell)$-form $\omega \in \Omega^{m-\ell}(M)$. The unique de Rham cohomology class $[\tau_Q] \in H^\ell_c(M)$ of such a form as well as the forms τ_Q themselves are called **(Poincaré) dual to** Q. The next theorem relates the cup product to intersection theory.

Theorem 6.4.8. *Let M be an oriented m-manifold without boundary that admits a finite good cover, let $Q \subset M$ be a compact oriented $(m-\ell)$-dimensional submanifold without boundary, let P be a compact oriented ℓ-manifold without boundary, let $f : P \to M$ be a smooth map, and let $\tau_f \in \Omega^{m-\ell}_c(M)$ and $\tau_Q \in \Omega^\ell_c(M)$ be closed forms dual to f and Q, respectively. Then the intersection number of f and Q is given by*

$$f \cdot Q = \int_M \tau_f \wedge \tau_Q = \int_Q \tau_f = (-1)^{\ell(m-\ell)} \int_P f^*\tau_Q. \tag{6.4.8}$$

The proof of Theorem 6.4.8 relies on the Thom Isomorphism Theorem and is deferred to §7.2.3. The material of §6.5 is not needed for this proof. The next section contains an important application of Theorem 6.4.8.

6.4.4 Euler Characteristic and Betti Numbers

Let M be a compact m-manifold. The **Betti numbers** of M are defined as the dimensions of the de Rham cohomology groups and are denoted by

$$b_i := \dim(H^i(M)), \qquad i = 0, \dots, m.$$

By Corollary 6.2.9 these numbers are finite. Recall that the **Euler characteristic** $\chi(M)$ is defined as the sum of the indices of the zeros of a vector field that points out on the boundary (Theorem 2.3.1). The next theorem shows that this invariant is the alternating sum of the Betti numbers. It shows also that the Lefschetz number of a smooth map from M to itself (as the algebraic count of the fixed points in Definition 4.4.11) is the alternating of the traces of the induced homomorphism on de Rham cohomology.

Theorem 6.4.9 (Euler Characteristic). *Let M be a a compact m-manifold with boundary and let $f : M \to M$ be a smooth map. Then the Euler characteristic of M and the Lefschetz number of f are given by*

$$\chi(M) = \sum_{i=0}^{m} (-1)^i \dim(H^i(M)), \tag{6.4.9}$$

$$L(f) = \sum_{i=0}^{m} (-1)^i \mathrm{trace}\left(f^* : H^i(M) \to H^i(M)\right). \tag{6.4.10}$$

Proof. By Lemma 4.4.17 the Euler characteristic is the Lefschetz number of the identity map, so (6.4.9) follows from (6.4.10) with $f = \mathrm{id}_M$. The starting point for the proof of (6.4.10) is the observation in Lemma 4.4.5 that for oriented boundaryless manifolds the Lefschetz number of f is the intersection number of the graph of f with the diagonal $\Delta \subset M \times M$, i.e.

$$L(f) = \mathrm{graph}(f) \cdot \Delta. \tag{6.4.11}$$

Now Theorem 6.4.8 (proved in §7.2.3) allows us to express this intersection number as an integral of a closed form τ_Δ, Poincaré dual to the diagonal. The master plan is to choose a basis of the de Rham cohomology of M and to use the Künneth formula to express the cohomology class of τ_Δ in terms of this basis (Step 1), and then to compute the integral to prove (6.4.10) for oriented manifolds without boundary (Step 2). The next tasks are to use a doubling argument to deal with the case of nonempty boundary (Steps 3 and 4) and to reduce the nonorientable case to the orientable case with the help of the Tubular Neighborhood Theorem (Step 5).

Step 1. *Assume that M is oriented and $\partial M = \emptyset$. Let $\tau_\Delta \in \Omega^m(M \times M)$ be a closed m-form whose cohomology class is Poincaré dual to the diagonal*

$$\Delta := \{(p,p) \,|\, p \in M\},$$

so (6.4.7) *holds with M replaced by $M \times M$ and $Q := \Delta$. Let*

$$\omega_i \in \Omega^{k_i}(M), \qquad i = 0, 1, \dots, n,$$

be closed forms whose cohomology classes $[\omega_i]$ form a basis of $H^*(M)$. Then there exist closed forms $\tau_j \in \Omega^{m-k_j}(M)$ for $j = 0, 1, \dots, n$ such that

$$\int_M \tau_j \wedge \omega_i = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{6.4.12}$$

Their cohomology classes also form a basis of $H^*(M)$ and

$$[\tau_\Delta] = \sum_{i=0}^{n} (-1)^{\deg(\tau_i)} [\pi_1^*\tau_i \wedge \pi_2^*\omega_i] \in H^m(M \times M). \tag{6.4.13}$$

Here $\pi_i : M \times M \to M$ denotes the projection onto the first factor for $i = 1$ and onto the second factor for $i = 2$.

The existence of the τ_j satisfying (6.4.12) and the fact that their cohomology classes form a basis of $H^*(M)$ follows directly from Theorem 6.4.1. By the Künneth formula in Theorem 6.2.11 the cohomology classes of the differential forms $\pi_1^*\omega_i \wedge \pi_2^*\tau_j$ form a basis of the de Rham cohomology of $M \times M$. Hence there exist real numbers $c_{ij} \in \mathbb{R}$ such that

$$[\tau_\Delta] = \sum_{i,j} c_{ij} [\pi_1^*\tau_j \wedge \pi_2^*\omega_i]. \tag{6.4.14}$$

We compute the coefficients c_{ij} by using equation (6.4.7), which asserts that

$$\int_\Delta \omega = \int_{M\times M} \omega \wedge \tau_\Delta, \qquad \omega := \pi_1^*\omega_k \wedge \pi_2^*\tau_\ell,$$

Define the map $\iota : M \to M \times M$ by $\iota(p) := (p,p)$ for $p \in M$. Then

$$\pi_1 \circ \iota = \pi_2 \circ \iota = \mathrm{id}$$

and hence

$$\int_\Delta \omega = \int_M \iota^*(\pi_1^*\omega_k \wedge \pi_2^*\tau_\ell) = \int_M \omega_k \wedge \tau_\ell = (-1)^{\deg(\omega_k)\deg(\tau_\ell)}\delta_{k\ell}.$$

Moreover, by (6.4.14), we have

$$\begin{aligned}\int_{M\times M}\omega\wedge\tau_\Delta &= \sum_{i,j} c_{ij}\int_{M\times M}\pi_1^*\omega_k\wedge\pi_2^*\tau_\ell\wedge\pi_1^*\tau_j\wedge\pi_2^*\omega_i \\ &= \sum_{i,j} c_{ij}(-1)^{\deg(\tau_j)\deg(\tau_\ell)}\int_{M\times M}\pi_1^*\omega_k\wedge\pi_1^*\tau_j\wedge\pi_2^*\tau_\ell\wedge\pi_2^*\omega_i \\ &= \sum_{i,j} c_{ij}(-1)^{\deg(\tau_j)\deg(\tau_\ell)}\int_M\omega_k\wedge\tau_j\int_M\tau_\ell\wedge\omega_i \\ &= \sum_{i,j} c_{ij}(-1)^{\deg(\tau_j)\deg(\tau_\ell)}(-1)^{\deg(\tau_j)\deg(\omega_k)}\delta_{jk}\delta_{i\ell} \\ &= (-1)^{\deg(\tau_k)\deg(\tau_\ell)}(-1)^{\deg(\tau_k)\deg(\omega_k)}c_{\ell k}\end{aligned}$$

Setting $k=\ell$ we find that $c_{k\ell}=(-1)^{\deg(\tau_k)}\delta_{k\ell}$ and this proves Step 1.

Step 2. *if M is oriented and $\partial M=\emptyset$, then* (6.4.10) *holds.*

Recall from equation (6.4.11) that $L(f)=\mathrm{graph}(f)\cdot\Delta$. Hence, by Step 1 and Theorem 6.4.8 with M,f,P,Q replaced by $M\times M,\mathrm{id}\times f,M,\Delta$,

$$\begin{aligned}L(f) &= \mathrm{graph}(f)\cdot\Delta \\ &= (-1)^m\int_M(\mathrm{id}\times f)^*\tau_\Delta \\ &= (-1)^m\sum_i(-1)^{\deg(\tau_i)}\int_M(\mathrm{id}\times f)^*(\pi_1^*\tau_i\wedge\pi_2^*\omega_i) \\ &= \sum_i(-1)^{\deg(\omega_i)}\int_M\tau_i\wedge f^*\omega_i.\end{aligned}\tag{6.4.15}$$

The last equality holds because $\deg(\omega_i)+\deg(\tau_i)=m$. By (6.4.12) we have

$$f^*\omega_i=\sum_{\deg(\omega_j)=k}a_{ij}\omega_j,\qquad a_{ij}:=\int_M\tau_j\wedge f^*\omega_i,$$

for all $i\in\{0,1,\dots,n\}$ with $\deg(\omega_i)=k$, and hence

$$\mathrm{trace}\big(f^*:H^k(M)\to H^k(M)\big)=\sum_{\deg(\omega_i)=k}a_{ii}=\sum_{\deg(\omega_i)=k}\int_M\tau_i\wedge f^*\omega_i.$$

Hence, by (6.4.15) we have

$$\begin{aligned}
L(f) &= \sum_i (-1)^{\deg(\omega_i)} \int_M \tau_i \wedge f^*\omega_i \\
&= \sum_{k=0}^m (-1)^k \sum_{\deg(\omega_i)=k} \int_M \tau_i \wedge f^*\omega_i \\
&= \sum_{k=0}^m (-1)^k \mathrm{trace}\big(f^* : H^k(M) \to H^k(M)\big).
\end{aligned}$$

This proves Step 2.

Step 3. *Let M be a compact m-manifold with boundary and let $f : M \to M$ be a smooth map such that $f(M) \cap \partial M = \emptyset$. Then there exists a compact m-manifold N without boundary, a smooth map $g : N \to N$, an open set $U \subset M \setminus \partial M$, and an embedding $\iota : M \to N$ such that*

$$g \circ \iota = \iota \circ f : U \to N, \qquad f(M) \subset U, \qquad g(N) \subset \iota(U), \tag{6.4.16}$$

and the inclusion of U into M is a homotopy equivalence (see Figure 6.1).

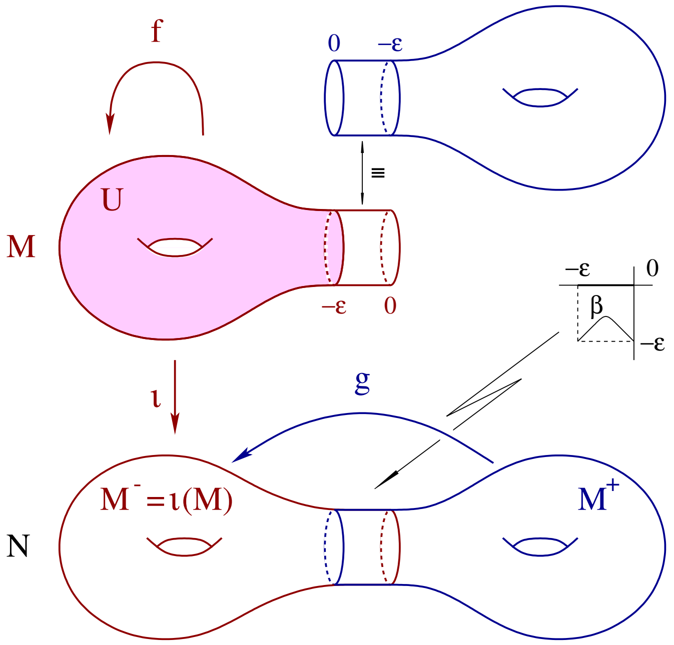

Figure 6.1: Doubling a manifold with boundary.

Choose a vector field $X \in \mathrm{Vect}(M)$ such that X points out on the boundary. Let $\{\phi_t : M \to M\}_{t \le 0}$ be the semi-flow of X, write $\phi(t,p) := \phi_t(p)$ for $t \le 0$ and $p \in M$, and define

$$V_\varepsilon := \{\phi(t,p) \mid -\varepsilon \le t \le 0,\, p \in \partial M\}.$$

Then V_ε is a compact neighborhood of the boundary and ϕ restricts to a diffeomorphism from $[-\varepsilon, 0] \times \partial M$ to V_ε for $\varepsilon > 0$ sufficiently small. Fix a constant $\varepsilon > 0$ so small that this holds and $f(M) \cap V_\varepsilon = \emptyset$. Define

$$N := \Big(M \times \{\pm 1\}\Big)/\equiv,$$

where the equivalence relation is given by

$$[p,-1] \equiv [p',+1] \qquad \overset{\text{def}}{\iff} \qquad \begin{array}{l} p, p' \in V_\varepsilon \text{ and there exist elements} \\ -\varepsilon \le t \le 0 \text{ and } q \in \partial M \text{ such that} \\ p = \phi(t,q) \text{ and } p' = \phi(-\varepsilon - t, q). \end{array}$$

Then N is a compact manifold without boundary, the map

$$M \to N : p \mapsto \iota(p) := [p, -1]$$

is an embedding, the set

$$U := M \setminus V_\varepsilon$$

is open, and the inclusion of U into M is a homotopy equivalence with a homotopy inverse given by $M \to U : p \mapsto \phi(2\varepsilon, p)$. Choose a smooth function

$$\beta : [-\varepsilon, 0] \to [-\varepsilon, 0]$$

such that

$$\begin{array}{lll} \beta(t) &= \beta(-\varepsilon - t) & \text{for } -\varepsilon \le t \le 0, \\ \beta(t) &= -\varepsilon - t & \text{for } -\varepsilon/3 \le t \le 0, \\ \beta(t) &= t & \text{for } -\varepsilon \le t \le -2\varepsilon/3. \end{array}$$

(See Figure 6.1.) Define the map $g : N \to N$ by

$$g\big([\phi(t,q), -1]\big) := g\big([\phi(t,q), +1]\big) := \big[f\left(\phi(\beta(t), q)\right), -1\big]$$

for $-\varepsilon \le t \le 0$ and $q \in \partial M$ and by

$$g\big([p,-1]\big) := g\big([p,+1]\big) := \big[f(p), -1\big] \qquad \text{for } p \in M \setminus V_\varepsilon.$$

This map is smooth and satisfies the requirements of Step 3.

Step 4. *Assume M is oriented. Then* (6.4.10) *holds.*

For oriented manifolds without boundary the formula has been established in Step 2. Thus assume $\partial M \neq \emptyset$. By Exercise 4.4.34 and Lemma 4.4.8 we may also assume that $f(M) \cap \partial M = \emptyset$ and that f has only nondegenerate fixed points. Choose the open set $U \subset M$ and the maps $\iota : M \to N, g : N \to N$ as in Step 3. Then $\mathrm{Fix}(g) = \iota(\mathrm{Fix}(f))$ and $\det(\mathbb{1} - dg(\iota(p))) = \det(\mathbb{1} - df(p))$ for all $p \in \mathrm{Fix}(f)$. Hence, by definition of the Lefschetz number as the sum of the fixed point indices, we have $L(f) = L(g)$ and thus, by Step 2,

$$\begin{aligned} L(f) &= \sum_{i=0}^{m} (-1)^i \mathrm{trace}\left(g^* : H^i(N) \to H^i(N)\right) \\ &= \sum_{i=0}^{m} (-1)^i \mathrm{trace}\left((f|_U)^* : H^i(U) \to H^i(U)\right) \\ &= \sum_{i=0}^{m} (-1)^i \mathrm{trace}\left(f^* : H^i(M) \to H^i(M)\right). \end{aligned}$$

Here the last two equalities follow from Corollary 6.2.10. This proves Step 4.

Step 5. *We prove* (6.4.10).

Assume that M is not orientable and $\partial M = \emptyset$. Assume also, without loss of generality, that M is a submanifold of $\mathbb{R}^n$ and that f has only nondegenerate fixed points. Then, for $\varepsilon > 0$ sufficiently small, it follows from Theorem 2.3.6 and Exercise 2.4.13 that $N := \{p + v \,|\, p \in M, v \in T_pM^\perp, |v| \leq \varepsilon\} \subset \mathbb{R}^n$ is an n-manifold with boundary, that the inclusion $\iota : M \to N$ is a homotopy equivalence, and that a homotopy inverse of ι is the map $r : N \to M$ defined by $r(p + v) := p$ for $p \in M$ and $v \in T_pM^\perp$ with $|v| < \varepsilon$. Define

$$g := \iota \circ f \circ r : N \to N.$$

Then $\mathrm{Fix}(g) = \mathrm{Fix}(f)$ and, for every $p \in \mathrm{Fix}(f)$, we have $dg(p)|_{T_pM} = df(p)$ and $dg(p)|_{T_pM^\perp} = 0$, and therefore $\det(\mathbb{1} - dg(p)) = \det(\mathbb{1} - df(p))$. This implies $L(f) = L(g)$. Since the inclusion ι is a homotopy equivalence with homotopy inverse r, we have

$$\mathrm{trace}\left(g^* : H^i(N) \to H^i(N)\right) = \mathrm{trace}\left(f^* : H^i(M) \to H^i(M)\right)$$

for $i = 0, 1, \dots, m$. Hence it follows from Step 4 that f satisfies (6.4.10). This proves equation (6.4.10) for nonorientable manifolds M without boundary. For nonorientable manifolds the case of nonempty boundary reduces to the case of empty boundary by the exact same argument that was used in Step 4. This proves Step 5 and Theorem 6.4.9. □

Remark 6.4.10. The **zeta function** of a smooth map

$$f : M \to M$$

on a compact oriented m-manifold M without boundary (thought of as a discrete-time dynamical system) is defined by

$$\zeta_f(t) := \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)t^n}{n}\right), \tag{6.4.17}$$

where

$$f^n := f \circ f \circ \cdots \circ f : M \to M$$

denotes the nth iterate of f. By definition of the Lefschetz numbers (in terms of an algebraic count of the fixed points) the zeta-function of f can be expressed in terms a count of the periodic points of f, provided that they are all isolated. If the periodic points of f are all nondegenerate, then the zeta-function of f can be written in the form

$$\zeta_f(t) = \prod_{n=1}^{\infty} \prod_{p \in \mathcal{P}_n(f)/\mathbb{Z}_n} \left(1 - \varepsilon(p, f^n)t^n\right)^{-\varepsilon(p,f^n)\iota(p,f^n)}, \tag{6.4.18}$$

where $\mathcal{P}_n(f)$ denotes the set of periodic points with minimal period n and

$$\begin{aligned} \iota(p, f^n) &:= \operatorname{sign}\det(\mathbb{1} - df^n(p)), \\ \varepsilon(p, f^n) &:= \operatorname{sign}\det(\mathbb{1} + df^n(p)) \end{aligned}$$

for $p \in \mathcal{P}_n(f)$. This formula is due to Ionel and Parker. One can use Theorem 6.4.9 to prove that

$$\begin{aligned} \zeta_f(t) &= \prod_{i=0}^{m} \det\left(\mathbb{1} - tf^* : H^i(M) \to H^i(M)\right)^{(-1)^{i+1}} \\ &= \frac{\det\left(\mathbb{1} - tf^* : H^{\mathrm{odd}}(M) \to H^{\mathrm{odd}}(M)\right)}{\det\left(\mathbb{1} - tf^* : H^{\mathrm{ev}}(M) \to H^{\mathrm{ev}}(M)\right)}. \end{aligned} \tag{6.4.19}$$

In particular, the zeta function is rational.

Exercise 6.4.11. Prove that the right hand side of (6.4.17) converges for t sufficiently small. Prove (6.4.18) and (6.4.19). **Hint:** Use the identities

$$\det(\mathbb{1} - tA)^{-1} = \exp\left(\operatorname{trace}\left(\sum_{n=1}^{\infty} \frac{t^n A^n}{n}\right)\right), \qquad \iota(p, f^n) = \iota(p, f)\varepsilon(p, f)^{n-1}$$

for a square matrix A and $t \in \mathbb{R}$ sufficiently small, and for a fixed point p of f that is nondegenerate for all iterates of f.

6.4.5 Examples and Exercises

Example 6.4.12 (The de Rham Cohomology of the Torus). It follows from the Künneth formula in Theorem 6.2.11 by induction that the de Rham cohomology of the m-torus

$$\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m \cong \underbrace{S^1 \times \cdots \times S^1}_{m \text{ times}}$$

has dimension

$$\dim(H^k(\mathbb{T}^m)) = \binom{m}{k}.$$

Hence every k-dimensional de Rham cohomology class can be represented uniquely by a k-form

$$\omega_c = \sum_{1 \le i_1 < \cdots < i_k \le m} c_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

with constant coefficients. Thus the map $c \mapsto [\omega_c]$ defines an isomorphism

$$\Lambda^*(\mathbb{R}^m)^* \to H^*(\mathbb{T}^m).$$

This is an isomorphism of algebras with the exterior product on the left and the cup product on the right.

Exercise 6.4.13. Show that a closed k-form $\omega \in \Omega^k(\mathbb{T}^m)$ is exact if and only if its integral vanishes over every compact oriented k-dimensional submanifold of $\mathbb{T}^m$. **Hint:** Given a closed k-form $\omega \in \Omega^k(\mathbb{T}^m)$ choose c such that $\omega - \omega_c$ is exact. Express the number $c_{i_1 \cdots i_k}$ as an integral of ω over a k-dimensional subtorus of $\mathbb{T}^m$.

Exercise 6.4.14. Prove that a 1-form $\omega \in \Omega^1(M)$ is exact if and only if its integral vanishes over every smooth loop in M. Show that every connected simply connected manifold M satisfies

$$H^1(M) = 0.$$

Hint: Assume that $\omega \in \Omega^1(M)$ satisfies the equation $\int_{S^1} \gamma^*\omega = 0$ for every smooth map $\gamma : S^1 \to M$. Fix an element $p_0 \in M$ and define the function $f : M \to \mathbb{R}$ as follows. Given an element $p \in M$ choose a smooth path $\gamma : [0,1] \to M$ jointing $\gamma(0) = p_0$ to $\gamma(1) = p$ and define

$$f(p) := \int_{[0,1]} \gamma^*\omega.$$

Prove that the value $f(p)$ does not depend on the choice of the path γ. Prove that f is smooth. Prove that $df = \omega$.

Example 6.4.15 (The Genus of a Surface). Let Σ be a compact connected oriented 2-manifold without boundary. Then Theorem 6.4.1 asserts that the Poincaré pairing

$$H^1(\Sigma)\times H^1(\Sigma)\to\mathbb{R}:([\alpha],[\beta])\mapsto\int_\Sigma\alpha\wedge\beta$$

is nondegenerate. Since this pairing is skew-symetric it follows that $H^1(\Sigma)$ is even-dimensional. Hence there is a nonnegative integer $g\in\mathbb{N}_0$, called the **genus of** Σ, such that

$$\dim(H^1(\Sigma))=2g.$$

Moreover, since Σ is connected, we have $H^0(\Sigma)=\mathbb{R}$ and $H^2(\Sigma)=\mathbb{R}$ (see Theorem 5.3.10 or Theorem 6.4.1). Hence, by Theorem 6.4.9, the Euler characteristic of Σ is given by

$$\chi(\Sigma)=2-2g.$$

Thus the Euler characteristic is even and less than or equal to two. Since the 2-sphere is simply connected we have $H^1(S^2)=0$, by Exercise 6.4.14, and hence the 2-sphere has genus zero and Euler characteristic two. This follows also from the Poincaré–Hopf Theorem. By Example 6.4.12 the 2-torus has genus one and Euler characteristic zero. This can again be derived from the Poincaré–Hopf theorem because there is a vector field on the torus without zeros. All higher genus surfaces have negative Euler characteristic. Examples of surfaces of genus zero, one, and two are depicted in Figure 6.2. By the Gauß–Bonnet formula only genus one surfaces can admit flat metrics. A fundamental result in two-dimensional differential topology is that two compact connected oriented 2-manifolds without boundary are diffeomorphic if and only if they have the same genus. A beautiful proof of this theorem, based on Morse theory, is contained in the book of Hirsch [13].

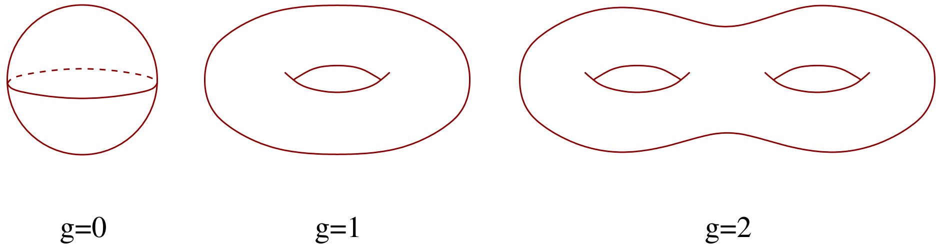

Figure 6.2: The genus of a surface.

Example 6.4.16 (The de Rham Cohomology of $\mathbb{C}\mathrm{P}^n$). The de Rham cohomology of $\mathbb{C}\mathrm{P}^n$ is given by

$$H^k(\mathbb{C}\mathrm{P}^n) = \left\{ \begin{array}{ll} \mathbb{R}, & \text{if } k \text{ is even}, \\ 0, & \text{if } k \text{ is odd}, \end{array} \right. \qquad \text{for } k = 0, 1, 2, \dots, 2n. \tag{6.4.20}$$

We explain the cup product structure on $H^*(\mathbb{C}\mathrm{P}^n)$ in Theorem 7.3.19.

For $\mathbb{C}\mathrm{P}^1 \cong S^2$ the formula (6.4.20) follows from Example 6.4.15. We prove the general formula by induction on n. Take $n \geq 2$ and suppose the assertion has been proved for $\mathbb{C}\mathrm{P}^{n-1}$. Consider the open subsets

$$\begin{aligned} U &:= \mathbb{C}\mathrm{P}^n \setminus \{[0 : \cdots : 0 : 1]\}, \\ V &:= \mathbb{C}\mathrm{P}^n \setminus \mathbb{C}\mathrm{P}^{n-1} = \{[z_0 : \cdots : z_{n-1} : z_n] \in \mathbb{C}\mathrm{P}^n \,|\, z_n \neq 0\}. \end{aligned}$$

These two sets cover $\mathbb{C}\mathrm{P}^n$, the set V is diffeomorphic to $\mathbb{C}^n$ and the obvious inclusion $\iota : \mathbb{C}\mathrm{P}^{n-1} \to U$ is a homotopy equivalence. A homotopy inverse of the inclusion is the projection $\pi : U \to \mathbb{C}\mathrm{P}^{n-1}$ given by

$$\pi([z_0 : \cdots : z_{n-1} : z_n]) := [z_0 : \cdots : z_{n-1}]$$

Then $\pi \circ \iota = \mathrm{id} : \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ and $\iota \circ \pi : U \to U$ is homotopic to the identity by the homotopy $f_t : U \to U$ given by

$$f_t([z_0 : \cdots : z_{n-1} : z_n]) := [z_0 : \cdots : z_{n-1} : t z_n]$$

with

$$f_0 = \iota \circ \pi, \qquad f_1 = \mathrm{id}.$$

Hence the inclusion $\iota : \mathbb{C}\mathrm{P}^{n-1} \to U$ induces an isomorphism on cohomology, by Corollary 6.1.5, and the cohomology of V is isomorphic to that of $\mathbb{C}^n$. Thus it follows from the induction hypothesis and Example 6.1.12 that

$$H^k(U) \cong \left\{ \begin{array}{ll} \mathbb{R}, & \text{if } k \text{ is even}, \\ 0, & \text{if } k \text{ is odd}, \end{array} \right. \qquad H^k(V) \cong \left\{ \begin{array}{ll} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{if } k > 0. \end{array} \right.$$

Moreover, the intersection $U \cap V$ is diffeomorphic to $\mathbb{C}^n \setminus \{0\}$ and therefore is homotopy equivalent to S^{2n-1}. Thus, by Example 6.1.13, we have

$$H^k(U \cap V) \cong \left\{ \begin{array}{ll} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{if } 1 \leq k \leq 2n-2, \\ \mathbb{R}, & \text{if } k = 2n-1. \end{array} \right.$$

Hence, for $2 \leq k \leq 2n-2$, the Mayer–Vietoris sequence takes the form

$$\begin{tikzcd}
H^{k-1}(U\cap V) \arrow[r, "d^*"] \arrow[d, "\cong"'] & H^k(\mathbb{C}\mathrm{P}^n) \arrow[r, "i^*"] \arrow[d, "="'] & H^k(U)\oplus H^k(V) \arrow[r, "j^*"] \arrow[d, "\cong"'] & H^k(U\cap V) \arrow[d, "\cong"'] \\
0 \arrow[r] & H^k(\mathbb{C}\mathrm{P}^n) \arrow[r] & H^k(\mathbb{C}\mathrm{P}^{n-1}) \arrow[r] & 0
\end{tikzcd}$$

This sequence is exact, by Theorem 6.2.3. Hence the inclusion induced homomorphism

$$\iota^* : H^k(\mathbb{C}\mathrm{P}^n) \to H^k(\mathbb{C}\mathrm{P}^{n-1}) \tag{6.4.21}$$

is an isomorphism for $2 \leq k \leq 2n-2$. Thus it follows from the induction hypothesis that equation (6.4.20) holds for $2 \leq k \leq 2n-2$. Moreover, since $\mathbb{C}\mathrm{P}^n$ is connected, we have $H^0(\mathbb{C}\mathrm{P}^n) = \mathbb{R}$ and, since $\mathbb{C}\mathrm{P}^n$ is simply connected by Exercise 6.4.17 below, it follows from Exercise 6.4.14 that $H^1(\mathbb{C}\mathrm{P}^n) = 0$. This last observation can also be deduced from the Mayer–Vietoris sequence. Since $\mathbb{C}\mathrm{P}^n$ is a complex manifold, it is oriented and therefore satisfies Poincaré duality. Hence, by Theorem 6.4.1, we have

$$H^{2n}(\mathbb{C}\mathrm{P}^n) \cong H^0(\mathbb{C}\mathrm{P}^n) = \mathbb{R}, \qquad H^{2n-1}(\mathbb{C}\mathrm{P}^n) \cong H^1(\mathbb{C}\mathrm{P}^n) = 0.$$

This proves (6.4.20) for all n. It also follows that the homomorphism (6.4.21) is an isomorphism for $0 \leq k \leq 2n-2$.

Exercise 6.4.17. Prove that $\mathbb{C}\mathrm{P}^n$ is simply connected.

Exercise 6.4.18 (The de Rham Cohomology of $\mathbb{R}\mathrm{P}^m$). Prove that the de Rham cohomology of $\mathbb{R}\mathrm{P}^m$ is

$$H^k(\mathbb{R}\mathrm{P}^m) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0, \\ 0, & \text{if } 1 \leq k \leq m-1, \\ 0, & \text{if } k = m \text{ is even}, \\ \mathbb{R}, & \text{if } k = m \text{ is odd}. \end{cases}$$

In particular, $\mathbb{R}\mathrm{P}^2$ has Euler characteristic one. **Hint:** $\mathbb{R}\mathrm{P}^m$ is oriented if and only if m is odd. Prove that, up to homotopy, there is only one noncontractible loop in $\mathbb{R}\mathrm{P}^m$, and hence its fundamental group is isomorphic to $\mathbb{Z}_2$. Use Exercise 6.4.14 to prove that $H^1(\mathbb{R}\mathrm{P}^m) \cong 0$ for $m \geq 2$. Use an induction argument and Mayer–Vietoris to prove that $H^k(\mathbb{R}\mathrm{P}^m) = 0$ for $2 \leq k \leq m-1$.

6.5 The Čech–de Rham Complex

In §6.2 on the Mayer–Vietoris sequence we have studied the de Rham cohomology of a smooth manifold M by restricting global differential forms on M to two open sets and differential forms on the two open sets to their intersection and examining the resulting combinatorics. We have seen that this technique is a powerful tool for understanding de Rham cohomology allowing us, for example, to prove finite-dimensionality, derive the Künneth formula, and establish Poincaré duality for compact manifolds in an elegant manner. The Mayer–Vietoris principle can be carried over to covers of M by arbitrarly many (or even infinitely many) open sets. Associated to any open cover (of any topological space) is the Čech cohomology. In general, this cohomology will depend on the choice of the cover. We shall prove that the Čech cohomology of a good cover of a smooth manifold is isomorphic to the de Rham cohomology and hence is independent of the choice of the good cover. This result is a key ingredient in the proof of de Rham's theorem which asserts that the de Rham cohomology of a manifold is isomorphic to the singular cohomology with real coefficients.

6.5.1 The Čech Complex

Let M be a smooth manifold and

$$\mathscr{U} = \{U_i\}_{i \in I}$$

be an open cover of of M, indexed by a set I, such that

$$U_i \neq \emptyset$$

for every $i \in I$. The combinatorics of the cover $\mathscr{U}$ is encoded in the sets of multi-indices associated to nonempty intersections, denoted by

$$\mathcal{I}_k(\mathscr{U}) := \Big\{ (i_0, \dots, i_k) \in I^k \,|\, U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset \Big\}$$

for every nonnegative integer k. The permutation group S_{k+1} of bijections of the set $\{0, 1, \dots, k\}$ acts on the set $\mathcal{I}_k(\mathscr{U})$ and the nonempty intersections of $k+1$ sets in $\mathscr{U}$ correspond to orbits under this action: reordering the indices doesn't change the intersection. We shall consider ordered nonempty intersections up to even permutations; the convention is that odd permutations act by a sign change on the data associated to an ordered nonempty intersection.

The simplest way of assigning a cochain complex to these data is to assign a real number to each ordered nonempty intersection of $k+1$ sets in $\mathscr{U}$. Thus real number $c_{i_0\cdots i_k}$ is assigned to each ordered tuple $(i_0,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})$ with the convention that the sign changes under every odd reordering of the indices. In particular, the number $c_{i_0\cdots i_k}$ is zero whenever there is any repetition among the indices and is undefined whenever $U_{i_0}\cap\cdots\cap U_{i_k}=\emptyset$. Let $C^k(\mathscr{U},\mathbb{R})$ denote the real vector space of all tuples

$$c=(c_{i_0\cdots i_k})_{(i_0,\dots,i_k)\in\mathcal{I}(\mathscr{U})}\in\mathbb{R}^{\mathcal{I}_k(\mathscr{U})}$$

that satisfy the condition

$$c_{i_{\sigma(0)}\cdots i_{\sigma(k)}}=\varepsilon(\sigma)c_{i_0\cdots i_k}$$

for $\sigma\in S_{k+1}$ and $(i_0,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})$. These spaces determine a cochain complex

$$C^0(\mathscr{U},\mathbb{R})\xrightarrow{\delta}C^1(\mathscr{U},\mathbb{R})\xrightarrow{\delta}C^2(\mathscr{U},\mathbb{R})\xrightarrow{\delta}C^3(\mathscr{U},\mathbb{R})\xrightarrow{\delta}\cdots. \tag{6.5.1}$$

called the **Čech complex of the open cover $\mathscr{U}$ with real coefficients**. The boundary operator $\delta:C^k(\mathscr{U},\mathbb{R})\to C^{k+1}(\mathscr{U},\mathbb{R})$ is defined by

$$(\delta c)_{i_0\cdots i_{k+1}}:=\sum_{\nu=0}^{k+1}(-1)^\nu c_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}} \tag{6.5.2}$$

for $c=(c_{i_0\cdots i_k})_{(i_0,\dots,i_k)\in\mathcal{I}(\mathscr{U})}\in C^k(\mathscr{U},\mathbb{R})$.

Example 6.5.1. A Čech 0-cochain $c\in C^0(\mathscr{U},\mathbb{R})$ assign a real number c_i to every open set U_i, a Čech 1-cochain $c\in C^1(\mathscr{U},\mathbb{R})$ assigns a real number c_{ij} to every nonempty ordered intersection $U_i\cap U_j$ such that

$$c_{ij}=-c_{ji},$$

and a Čech 2-cochain $c\in C^2(\mathscr{U},\mathbb{R})$ assigns a real number c_{ijk} to every nonempty ordered triple intersection $U_i\cap U_j\cap U_k$ such that

$$c_{ijk}=-c_{jik}=-c_{ikj}.$$

The boundary operator δ assigns to a 0-cochain $c=(c_i)_{i\in I}$ the 1-cochain

$$(\delta c)_{ij}=c_j-c_i,\qquad U_i\cap U_j\neq\emptyset,$$

and it assigns to every 1-cochain $c=(c_{ij})_{(i,j)\in\mathcal{I}_1(\mathscr{U})}$ the 2-cochain

$$(\delta c)_{ijk}=c_{jk}+c_{ki}+c_{ij},\qquad U_i\cap U_j\cap U_k\neq\emptyset.$$

One verifies immediately that $\delta\circ\delta=0$. This continues to hold in general as the next lemma shows.

Lemma 6.5.2. *The image of the linear map* $\delta : C^k(\mathscr{U},\mathbb{R}) \to \mathbb{R}^{\mathcal{I}_{k+1}(\mathscr{U})}$ *is contained in the subspace* $C^{k+1}(\mathscr{U},\mathbb{R})$ *and* $\delta\circ\delta = 0$.

Proof. The first assertion is left as an exercise for the reader. To prove the second assertion, let $c \in C^k(\mathscr{U},\mathbb{R})$, choose $(i_0,\dots,i_{k+2}) \in \mathcal{I}_{k+2}(\mathscr{U})$, and compute

$$\begin{aligned}
\delta(\delta c)_{i_0\cdots i_{k+2}} &= \sum_{\nu=0}^{k+2}(-1)^\nu(\delta c)_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}} \\
&= \sum_{0\le\mu<\nu\le k+2}(-1)^{\nu+\mu}c_{i_0\cdots\widehat{i_\mu}\cdots\cdots\widehat{i_\nu}\cdots i_{k+1}} \\
&\quad+ \sum_{0\le\nu<\mu\le k+2}(-1)^{\nu+\mu-1}c_{i_0\cdots\widehat{i_\nu}\cdots\cdots\widehat{i_\mu}\cdots i_{k+1}} \\
&= 0.
\end{aligned}$$

This proves Lemma 6.5.2. □

The cohomology of the Čech complex (6.5.1) is called the **Čech cohomology of $\mathscr{U}$ with real coefficients** and will be denoted by

$$H^k(\mathscr{U},\mathbb{R}) := \frac{\ker\delta : C^k(\mathscr{U},\mathbb{R}) \to C^{k+1}(\mathscr{U},\mathbb{R})}{\operatorname{im}\delta : C^{k-1}(\mathscr{U},\mathbb{R}) \to C^k(\mathscr{U},\mathbb{R})}. \tag{6.5.3}$$

This beautiful and elementary combinatorial construction works for every open cover of every topological space M and immediately gives rise to the following fundamental questions.

Question 1: *To what extent does the Čech cohomology* $H^*(\mathscr{U},\mathbb{R})$ *depend on the choice of the open cover?*

Question 2: *If* M *is a manifold, what is the relation between* $H^*(\mathscr{U},\mathbb{R})$ *and the de Rham cohomology* $H^*(M)$ *(or any other (co)homology theory)?*

Example 6.5.3. The Čech cohomology group $H^0(\mathscr{U},\mathbb{R})$ is the kernel of the operator $\delta : C^0(\mathscr{U},\mathbb{R}) \to C^1(\mathscr{U},\mathbb{R})$ and hence is the space of all tuples $c = (c_i)_{i\in I}$ that satisfy $c_i = c_j$ whenever $U_i \cap U_j \neq \emptyset$. This shows that, for every Čech 0-cocycle $c = (c_i)_{i\in I} \in H^0(\mathscr{U},\mathbb{R})$, there exists a locally constant function $f : M \to \mathbb{R}$ such that $f|_{U_i} \equiv c_i$ for every $i \in I$. If each open set U_i is connected, then $H^0(\mathscr{U},\mathbb{R})$ is isomorphic to the vector space of all locally constant real valued functions on M. Thus

$$H^0(\mathscr{U},\mathbb{R}) \cong \mathbb{R}^{\pi_0(M)} = H^0(M),$$

where $\pi_0(M)$ is the set of all connected components of M and $H^0(M)$ is the de Rham cohomology group. On the other hand, if $\mathscr{U}$ consists only of one open set $U = M$, then $H^0(\mathscr{U},\mathbb{R}) = \mathbb{R}$.

6.5.2 The Isomorphism

Let M be a smooth manifold and $\mathscr{U} = \{U_o\}_{i\in I}$ be an open cover of M. We show that there is a natural homomorphism from the Čech cohomology of $\mathscr{U}$ to the de Rham cohomology of M. The definition of the homomorphism on the cochain level depends on the choice of a partition of unity $\rho_i : M \to [0,1]$ subordinate to the cover $\mathscr{U} = \{U_i\}_{i\in I}$. Define the linear map

$$C^k(\mathscr{U},\mathbb{R}) \to \Omega^k(M) : c \mapsto \omega_c \tag{6.5.4}$$

by

$$\omega_c := \sum_{(i_0,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})} c_{i_0\cdots i_k}\rho_{i_0} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_k}. \tag{6.5.5}$$

for $c \in C^k(\mathscr{U},\mathbb{R})$.

Lemma 6.5.4. *The map* (6.5.4) *is a chain homomorphism and hence induces a homomorphism on cohomology*

$$H^*(\mathscr{U},\mathbb{R}) \to H^*(M) : [c] \mapsto [\omega_c]. \tag{6.5.6}$$

Proof. It will sometimes be convenient to set $c_{i_0\cdots i_k} := 0$ for $c \in C^k(\mathscr{U},\mathbb{R})$ and $(i_0,\dots,i_k) \in I^{k+1}\setminus\mathcal{I}_k(\mathscr{U})$. We prove that the map (6.5.4) is a chain homomorphism. For $c \in C^k(\mathscr{U},\mathbb{R})$ we compute

$$\begin{aligned}
\omega_{\delta c} &= \sum_{(i_0,\dots,i_{k+1})\in\mathcal{I}_{k+1}(\mathscr{U})} (\delta c)_{i_0\cdots i_{k+1}}\rho_{i_0} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_{k+1}} \\
&= \sum_{(i_0,\dots,i_{k+1})\in\mathcal{I}_{k+1}(\mathscr{U})} \sum_{\nu=0}^{k+1}(-1)^\nu c_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}}\rho_{i_0} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_{k+1}} \\
&= \sum_{(i_0,\dots,i_{k+1})\in I^{k+2}} c_{i_1\cdots i_{k+1}}\rho_{i_0} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_{k+1}} \\
&\quad + \sum_{\nu=1}^{k+1}(-1)^\nu \sum_{(i_0,\dots,i_{k+1})\in I^{k+2}} c_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}}\rho_{i_0} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_{k+1}} \\
&= \sum_{(i_1,\dots,i_{k+1})\in I^{k+1}} c_{i_1\cdots i_{k+1}} d\rho_{i_1}\wedge\cdots\wedge d\rho_{i_{k+1}} \\
&= d\omega_c.
\end{aligned}$$

Here we have used the fact that the respective summand vanishes whenever $(i_0,\dots,i_{k+1}) \notin \mathcal{I}_{k+1}(\mathscr{U})$ and that $\sum_{i\in I} d\rho_i = 0$ and $\sum_{i\in I}\rho_i = 1$. Thus (6.5.4) is a chain map and this proves Lemma 6.5.4. □

Remark 6.5.5. Let $c \in C^k(\mathscr{U}, \mathbb{R})$ such that $\delta c = 0$. Then, for all tuples $(i, j, i_1, \dots, i_k) \in \mathcal{I}_{k+1}(\mathscr{U})$, we have

$$c_{ii_1 \cdots i_k} = c_{ji_1 \cdots i_k} - \sum_{\nu=1}^{k} (-1)^\nu c_{iji_1 \cdots \widehat{i_\nu} \cdots i_k}$$

Multiply by $\rho_j d\rho_{i_1} \wedge \cdots d\rho_{i_k}$ and restrict to U_i. Since $\rho_j d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_k}$ vanishes on U_i whenever $(i, j, i_1, \dots, i_k) \notin \mathcal{I}_{k+1}(\mathscr{U})$, the resulting equation continues to hold for all tuples $(i, j, i_1, \dots, i_k) \in I^{k+2}$. Fixing i and taking the sum over all tuples $(j, i_1, \dots, i_k) \in I^{k+1}$ we find

$$\delta c = 0 \qquad \Longrightarrow \qquad \omega_c|_{U_i} = \sum_{(i_1, \dots, i_k) \in I^k} c_{ii_1 \cdots i_k} d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_k}. \tag{6.5.7}$$

This gives another proof that ω_c is closed whenever $\delta c = 0$.

The next theorem is the main result of this section. It answers the above questions under suitable assumptions on the cover $\mathscr{U}$.

Theorem 6.5.6. *If $\mathscr{U}$ is a good cover of M, then* (6.5.6) *is an isomorphism from the Čech cohomology of $\mathscr{U}$ to the de Rham cohomology of M*

The proof of Theorem 6.5.6 is deferred to §6.5.3. It will in fact show that, under the assumption that $\mathscr{U}$ is a good cover, the homomorphism (6.5.6) on cohomology is independent of the choice of the partition of unity used to define it. Moreover, we have the following immediate corollary.

Corollary 6.5.7. *The Čech cohomology groups with real coefficients associated to two good covers of a smooth manifold are isomorphic.*

If $\mathscr{U}$ is a finite good cover, the Čech complex $C^*(\mathscr{U}, \mathbb{R})$ is finite-dimensional and hence, so is its cohomology $H^*(\mathscr{U}, \mathbb{R})$. Combining this observation with Theorem 6.5.6, we obtain another proof that the de Rham cohomology is finite-dimensional as well.

Corollary 6.5.8. *If a smooth manifold admits a finite good cover, then its de Rham cohomology is finite-dimensional.*

Following Bott and Tu [3] we explain a proof of Theorem 6.5.6 that is based on a Mayer–Vietoris argument and involves differential forms of all degrees on the open sets in the cover and their intersections. Thus we build a cochain complex that contains both the de Rham complex and the Čech complex as subcomplexes.

6.5.3 The Čech–de Rham Complex

Associated to the open cover $\mathscr{U}=\{U_i\}_{i\in I}$ of our m-manifold M is a cochain complex defined as follows. Given two nonnegative integers k and p we introduce the vector space

$$C^k(\mathscr{U},\Omega^p)$$

of all tuples

$$\omega=(\omega_{i_0\cdots i_k})_{(i_0,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})},\qquad \omega_{i_0\cdots i_k}\in\Omega^p(U_{i_0}\cap\cdots\cap U_{i_k}),$$

that satisfy $\omega_{i_{\sigma(0)}\cdots i_{\sigma(k)}}=\varepsilon(\sigma)\omega_{i_0\cdots i_k}$ for $\sigma\in S_{k+1}$ and $(i_0,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})$. This complex carries two boundary operators

$$\delta:C^k(\mathscr{U},\Omega^p)\to C^{k+1}(\mathscr{U},\Omega^p),\qquad d:C^k(\mathscr{U},\Omega^p)\to C^k(\mathscr{U},\Omega^{p+1})$$

defined by

$$(\delta\omega)_{i_0\cdots i_{k+1}}:=\sum_{\nu=0}^{k+1}(-1)^\nu\omega_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}},\qquad (d\omega)_{i_0\cdots i_{k+1}}:=d\omega_{i_0\cdots i_{k+1}}.\tag{6.5.8}$$

They satisfy the equations

$$\delta\circ\delta=0,\qquad \delta\circ d=d\circ\delta,\qquad d\circ d=0.\tag{6.5.9}$$

Here the first equation is proved as in Lemma 6.5.2, the second equation is obvious, and the third equation follows from Lemma 5.2.6.

The complex is equipped with a **bigrading** by the integers k and p. The total grading is defined by

$$\deg(\omega):=k+p,\qquad \omega\in C^k(\mathscr{U},\Omega^p),$$

and the degree-n part of the complex will be denoted by

$$\check{C}^n(\mathscr{U}):=\bigoplus_{k+p=n}C^k(\mathscr{U},\Omega^p).$$

Let $\omega^{k,p}$ denote the projection of $\omega\in\check{C}^n(\mathscr{U})$ onto $C^k(\mathscr{U},\Omega^p)$. The bigraded complex carries a boundary operator $D:\check{C}^n(\mathscr{U})\to\check{C}^{n+1}(\mathscr{U})$, defined by

$$(D\omega)^{k,p}:=\delta\omega^{k-1,p}+(-1)^k d\omega^{k,p-1}\tag{6.5.10}$$

for $\omega\in\check{C}^n(\mathscr{U})$ and nonnegative integers k and p satisfying $k+p=n+1$. The sign $(-1)^k$ arises from the fact that d raises the second index in the bigrading by one and so is weighted by the parity of the first index k.

Lemma 6.5.9. *The operator* (6.5.10) *satisfies* $D \circ D = 0$.

Proof. Let $\omega \in \check{C}^n(\mathscr{U})$ and choose k and p such that $k + p = n + 2$. Then

$$\begin{aligned}
(D(D\omega))^{k,p} &= \delta(D\omega)^{k-1,p} + (-1)^k d(D\omega)^{k,p-1} \\
&= \delta\left(\delta\omega^{k-2,p} + (-1)^{k-1} d\omega^{k-1,p-1}\right) \\
&\quad + (-1)^k d\left(\delta\omega^{k-1,p-1} + (-1)^k d\omega^{k,p-2}\right) \\
&= \delta\delta\omega^{k-2,p} + (-1)^k (d\delta - \delta d)\omega^{k-1,p-1} + dd\omega^{k,p-2} \\
&= 0.
\end{aligned}$$

The last equation follows from (6.5.9) and this proves Lemma 6.5.9. □

The complex $(\check{C}^*(\mathscr{U}), D)$ is called the **Čech–de Rham complex** of the cover $\mathscr{U}$ and its cohomology

$$\check{H}^n(\mathscr{U}) := \frac{\ker\, D : \check{C}^n(\mathscr{U}) \to \check{C}^{n+1}(\mathscr{U})}{\operatorname{im}\, D : \check{C}^{n-1}(\mathscr{U}) \to \check{C}^n(\mathscr{U})}. \tag{6.5.11}$$

is called the **Čech–de Rham cohomology** of $\mathscr{U}$. There are natural cochain homomorphisms

$$\begin{aligned}
\iota &: C^k(\mathscr{U}, \mathbb{R}) \to C^k(\mathscr{U}, \Omega^0) \subset \check{C}^k(\mathscr{U}), \\
r &: \Omega^p(M) \to C^0(\mathscr{U}, \Omega^p) \subset \check{C}^p(\mathscr{U}).
\end{aligned} \tag{6.5.12}$$

The operator ι is the inclusion of the constant functions and r is the restriction defined by $(r\omega)_i := \omega|_{U_i}$ for $i \in I$. The maps r, δ, ι, d are depicted in the following diagram. We will prove that all rows except for the first and all columns except for the first are exact in the case of a good cover.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Omega^0(M) \arrow[r, "d"] \arrow[d, "r"] & \Omega^1(M) \arrow[r, "d"] \arrow[d, "r"] & \Omega^2(M) \arrow[r, "d"] \arrow[d, "r"] & \cdots. \\
C^0(\mathscr{U}, \mathbb{R}) \arrow[r, "\iota"] \arrow[d, "\delta"] & C^0(\mathscr{U}, \Omega^0) \arrow[r, "d"] \arrow[d, "\delta"] & C^0(\mathscr{U}, \Omega^1) \arrow[r, "d"] \arrow[d, "\delta"] & C^0(\mathscr{U}, \Omega^2) \arrow[r, "d"] \arrow[d, "\delta"] & \cdots \\
C^1(\mathscr{U}, \mathbb{R}) \arrow[r, "\iota"] \arrow[d, "\delta"] & C^1(\mathscr{U}, \Omega^0) \arrow[r, "d"] \arrow[d, "\delta"] & C^1(\mathscr{U}, \Omega^1) \arrow[r, "d"] \arrow[d, "\delta"] & C^1(\mathscr{U}, \Omega^2) \arrow[r, "d"] \arrow[d, "\delta"] & \cdots \\
C^2(\mathscr{U}, \mathbb{R}) \arrow[r, "\iota"] \arrow[d, "\delta"] & C^2(\mathscr{U}, \Omega^0) \arrow[r, "d"] \arrow[d, "\delta"] & C^2(\mathscr{U}, \Omega^1) \arrow[r, "d"] \arrow[d, "\delta"] & C^2(\mathscr{U}, \Omega^2) \arrow[r, "d"] \arrow[d, "\delta"] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Lemma 6.5.10. *The sequence*

$$0 \to \Omega^p(M) \xrightarrow{r} C^0(\mathscr{U},\Omega^p) \xrightarrow{\delta} C^1(\mathscr{U},\Omega^p) \xrightarrow{\delta} C^2(\mathscr{U},\Omega^p) \xrightarrow{\delta} \cdots \tag{6.5.13}$$

is exact for every integer $p \ge 0$*. If* $\mathscr{U}$ *is a good cover of* M*, then the sequence*

$$0 \to C^k(\mathscr{U},\mathbb{R}) \xrightarrow{\iota} C^k(\mathscr{U},\Omega^0) \xrightarrow{d} C^k(\mathscr{U},\Omega^1) \xrightarrow{d} C^k(\mathscr{U},\Omega^2) \xrightarrow{d} \cdots \tag{6.5.14}$$

is exact for every integer $k \ge 0$*.*

Proof. For the sequence (6.5.14) exactness follows immediately from Example 6.1.12 and the good cover condition. For the sequence (6.5.13) the good cover condition is not required. Exactness at $C^0(\mathscr{U},\Omega^p)$ follows directly from the definitions. To prove exactness at $C^k(\mathscr{U},\Omega^p)$ for $k \ge 1$ we choose a partition of unity $\rho_i : M \to [0,1]$ subordinate to the cover $\mathscr{U} = \{U_i\}_{i\in I}$. For $k \ge 1$ define the operator $h : C^k(\mathscr{U},\Omega^p) \to C^{k-1}(\mathscr{U},\Omega^p)$ by

$$(h\omega)_{i_0\cdots i_{k-1}} := \sum_{i\in I} \rho_i \omega_{ii_0\cdots i_{k-1}} \tag{6.5.15}$$

for $\omega \in C^k(\mathscr{U},\Omega^p)$ and $(i_0,\dots,i_{k-1}) \in I_{k-1}(\mathscr{U})$, where each term in the sum is understood as the extension to the open set $U_{i_0}\cap\cdots\cap U_{i_k}$ by setting it equal to zero on the complement of $U_i \cap U_{i_0}\cap\cdots\cap U_{i_k}$. We prove that

$$\delta\circ h + h\circ\delta = \mathrm{id} : C^k(\mathscr{U},\Omega^p) \to C^k(\mathscr{U},\Omega^p) \tag{6.5.16}$$

for $k \ge 1$. This shows that if $\omega \in C^k(\mathscr{U},\Omega^p)$ satisfies $\delta\omega = 0$, then $\omega = \delta h\omega$ belongs to the image of δ. To prove (6.5.16) we compute

$$\begin{aligned}
(h\delta\omega)_{i_0\cdots i_k} &= \sum_{i\in I} \rho_i (\delta\omega)_{ii_0\cdots i_k} \\
&= \sum_{i\in I} \rho_i \left(\omega_{i_0\cdots i_k} - \sum_{\nu=0}^{k} (-1)^\nu \omega_{ii_0\cdots\widehat{i_\nu}\cdots i_k} \right) \\
&= \omega_{i_0\cdots i_k} - \sum_{\nu=0}^{k} (-1)^\nu \sum_{i\in I} \rho_i \omega_{ii_0\cdots\widehat{i_\nu}\cdots i_k} \\
&= \omega_{i_0\cdots i_k} - \sum_{\nu=0}^{k} (-1)^\nu (h\omega)_{i_0\cdots\widehat{i_\nu}\cdots i_k} \\
&= \big(\omega - \delta h\omega\big)_{i_0\cdots i_k}
\end{aligned}$$

for $\omega \in C^k(\mathscr{U},\Omega^p)$ and $(i_0,\dots,i_k) \in \mathcal{I}_k(\mathscr{U})$. This proves (6.5.16) and Lemma 6.5.10. □

Theorem 6.5.11. *Let $\mathscr{U}$ be a good cover of M. Then the homorphism*

$$r : \Omega^*(M) \to \check{C}^*(\mathscr{U}), \qquad \iota : C^*(\mathscr{U}, \mathbb{R}) \to \check{C}^*(\mathscr{U})$$

induce isomorphism

$$r^* : H^*(M) \to \check{H}^*(\mathscr{U}), \qquad \iota^* : H^*(\mathscr{U}, \mathbb{R}) \to \check{H}^*(\mathscr{U})$$

on cohomology.

Proof. We prove that r is injective in cohomology. Let $\omega \in \Omega^p(M)$ be closed and assume that $\omega^{0,p} := r\omega = (\omega|_{U_i})_{i\in I} \in C^0(\mathscr{U}, \Omega^p) \subset \check{C}^p(\mathscr{U})$ is exact. Then there are elements $\tau^{k-1,p-k} \in C^{k-1}(\mathscr{U}, \omega^{p-k})$, $k = 1, \dots, p$, such that $r\omega = D\tau$:

$$\begin{aligned} \omega^{0,p} &= d\tau^{0,p-1}, \\ 0 &= \delta\tau^{k-1,p-k} + (-1)^k d\tau^{k,p-k-1}, \qquad k = 1, \dots, p-1, \\ 0 &= \delta\tau^{p-1,0}. \end{aligned} \tag{6.5.17}$$

We must prove that ω is exact. To see this we observe that there are elements $\sigma^{k-2,p-k} \in C^{k-2}(\mathscr{U}, \Omega^{p-k})$, $p \geq k \geq 2$, satisfying

$$\begin{aligned} \delta\sigma^{p-2,0} &= \tau^{p-1,0}, \\ \delta\sigma^{k-2,p-k} &= \tau^{k-1,p-k} + (-1)^k d\sigma^{k-1,p-k-1}, \qquad p-1 \geq k \geq 2. \end{aligned} \tag{6.5.18}$$

The existence of $\sigma^{p-2,0}$ follows immediately from the last equation in (6.5.17) and Lemma 6.5.10. If $2 \leq k \leq p-1$ and $\sigma^{k-1,p-k-1}$ has been found such that

$$\delta\sigma^{k-1,p-k-1} = \tau^{k,p-k-1} + (-1)^{k+1} d\sigma^{k,p-k-2},$$

then we have $d\delta\sigma^{k-1,p-k-1} = d\tau^{k,p-k-1}$ and hence

$$\delta\left(\tau^{k-1,p-k} + (-1)^k d\sigma^{k-1,p-k-1}\right) = \delta\tau^{k-1,p-k} + (-1)^k d\tau^{k,p-k-1} = 0.$$

Here the last equation follows from (6.5.17). Thus, by Lemma 6.5.10, there is an element $\sigma^{k-2,p-k}$ satisfying (6.5.18).

It follows from equation (6.5.17) with $k = 1$ that $\delta\tau^{0,p-1} = d\tau^{1,p-2}$ and from equation (6.5.18) with $k = 2$ that $\tau^{1,p-2} + d\sigma^{1,p-3} = \delta\sigma^{0,p-2}$. Hence

$$\begin{aligned} \delta\left(\tau^{0,p-1} - d\sigma^{0,p-2}\right) &= \delta\tau^{0,p-1} - d\tau^{1,p-2} = 0, \\ d\left(\tau^{0,p-1} - d\sigma^{0,p-2}\right) &= d\tau^{0,p-1} = \omega^{0,p}. \end{aligned} \tag{6.5.19}$$

The first equation in (6.5.19) shows that there is a global $(p-1)$-form $\widetilde{\tau}$ on M whose restriction to U_i agrees with the relevant component of the Čech–de Rham cochain $\tau^{0,p-1} - d\sigma^{0,p-2} \in C^0(\mathscr{U}, \Omega^{p-1})$. The second equation in (6.5.19) shows that $d\widetilde{\tau} = \omega$. Hence ω is exact, as claimed.

We prove that r is surjective in cohomology. Let $\omega^{k,p-k} \in C^k(\mathscr{U}, \Omega^{p-k})$ be given for $k = 0, \dots, p$ and suppose that $D\omega = 0$:

$$\begin{aligned} 0 &= d\omega^{0,p}, \\ 0 &= \delta\omega^{k,p-k} + (-1)^{k+1} d\omega^{k+1,p-k-1}, \qquad k = 0, \dots, p-1, \\ 0 &= \delta\omega^{p,0}. \end{aligned} \tag{6.5.20}$$

We construct elements $\tau^{k-1,p-k} \in C^{k-1}(\mathscr{U}, \Omega^{p-k})$, $k = 1, \dots, p$, satisfying

$$\begin{aligned} \delta\tau^{p-1,0} &= \omega^{p,0}, \\ \delta\tau^{k-1,p-k} &= \omega^{k,p-k} + (-1)^{k+1} d\tau^{k,p-k-1}, \qquad k = 1, \dots, p-1. \end{aligned} \tag{6.5.21}$$

The existence of $\tau^{p-1,0}$ follows immediately from the last equation in (6.5.20) and Lemma 6.5.10. If $1 \le k \le p-1$ and $\tau^{k,p-k-1}$ has been found such that

$$\delta\tau^{k,p-k-1} = \omega^{k+1,p-k-1} + (-1)^{k+2} d\tau^{k+1,p-k-1},$$

then we have $d\delta\tau^{k,p-k-1} = d\omega^{k+1,p-k-1}$ and hence

$$\delta\left(\omega^{k,p-k} + (-1)^{k+1} d\tau^{k,p-k-1}\right) = \delta\omega^{k,p-k} + (-1)^{k+1} d\omega^{k+1,p-k-1} = 0.$$

Here the last equation follows from (6.5.20). By exactness, this shows that there is an element $\tau^{k-1,p-k}$ satisfying (6.5.21). It follows from (6.5.21) that

$$\begin{aligned} (\omega - D\tau)^{0,p} &= \omega^{0,p} - d\tau^{0,p-1}, \\ (\omega - D\tau)^{k,p-k} &= \omega^{k,p-k} - \delta\tau^{k-1,p-k} - (-1)^k d\tau^{k,p-k-1} = 0, \\ (\omega - D\tau)^{p,0} &= \omega^{p,0} - \delta\tau^{p-1,0} = 0 \end{aligned} \tag{6.5.22}$$

for $k = 1, \dots, p-1$. Moreover, it follows from (6.5.20) with $k = 0$ that $\delta\omega^{0,p} = d\omega^{1,p-1}$ and from (6.5.21) with $k = 1$ that $\delta\tau^{0,p-1} = d\tau^{1,p-2}$. Hence

$$\begin{aligned} \delta(\omega - D\tau)^{0,p} &= \delta\left(\omega^{0,p} - d\tau^{0,p-1}\right) \\ &= d\left(\omega^{1,p-1} - \delta\tau^{0,p-1}\right) \\ &= d\left(-d\tau^{1,p-2}\right) = 0. \end{aligned}$$

This shows there is a global p-form $\widetilde{\omega}$ on M whose restriction to U_i agrees with the relevant component of $\omega^{0,p} - d\tau^{0,p-1} \in C^0(\mathscr{U}, \Omega^p)$. This form is closed and satisfies $r\widetilde{\omega} = \omega - D\tau$, by (6.5.22). Hence the cohomology class of ω in $\check{H}^p(\mathscr{U})$ belongs to the image of $r^* : H^p(M) \to \check{H}^p(\mathscr{U})$.

Thus we have proved that $r^* : H^*(M) \to \check{H}^*(\mathscr{U})$ is an isomorphism. The proof that $\iota^* : H^*(\mathscr{U}, \mathbb{R}) \to \check{H}^*(\mathscr{U})$ is an isomorphism as well follows by exactly the same argument with the rows and columns in our diagram interchanged. This proves Theorem 6.5.11. □

Proof of Theorem 6.5.6. Recall that the linear map

$$h : C^k(\mathscr{U}, \Omega^p) \to C^{k-1}(\mathscr{U}, \Omega^p)$$

in (6.5.15) has the form $(h\omega)_{i_0\cdots i_{k-1}} = \sum_{i\in I} \rho_i \omega_{ii_0\cdots i_{k-1}}$, and define the map

$$\Phi : C^k(\mathscr{U}, \Omega^p) \to C^{k-1}(\mathscr{U}, \Omega^{p+1})$$

by

$$(\Phi\omega)_{i_0\cdots i_{k-1}} := (-1)^k \sum_{i\in I} d\rho_i \wedge \omega_{ii_0\cdots i_{k-1}} = \sum_{i\in I} d\rho_i \wedge \omega_{i_0\cdots i_{k-1}i}$$

for $\omega \in C^k(\mathscr{U}, \Omega^{p-k})$. The product with $d\rho_i$ guarantees that each summand on the right extends smoothly to $U_{i_0\cdots i_{k-1}}$ by setting it equal to zero on the complement of the intersection with U_i. These two operators satisfy

$$\mathrm{id} = \delta \circ h + h \circ \delta, \qquad -\Phi = \big((-1)^{k-1} d\big) \circ h + h \circ \big((-1)^k d\big)$$

on $C^k(\mathscr{U}, \Omega^{p-k})$. Here the first equation is (6.5.16) and the second equation follows directly from the definitions. Combining these two equations we find

$$\mathrm{id} - \Phi = D \circ h + h \circ D.$$

Thus Φ induces the identity on $\check{H}^k(\mathscr{U})$.

Starting with $p = 0$ and iterating the operator k times we obtain a homomorphism

$$\Phi^k = \underbrace{\Phi \circ \Phi \circ \cdots \circ \Phi}_{k \text{ times}} : C^k(\mathscr{U}, \Omega^0) \to C^0(\mathscr{U}, \Omega^k),$$

inducing the identity on $\check{H}^k(\mathscr{U})$. This operator assigns to every element $f = (f_{i_0\cdots i_k})_{(i_0\cdots i_k)\in\mathcal{I}_k(\mathscr{U})} \in C^k(\mathscr{U}, \Omega^0)$ the tuple $\Phi^k f \in C^0(\mathscr{U}, \Omega^k)$ given by

$$(\Phi^k f)_i = \sum_{(i_1,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})} f_{ii_1\dots i_k} d\rho_{i_1} \wedge \cdots \wedge d\rho_{i_k} \in \Omega^k(U_i).$$

Hence, by Remark 6.5.5, the following diagram commutes on the kernel of δ:

$$\begin{tikzcd}
C^k(\mathscr{U}, \mathbb{R}) & \supset & \ker\,\delta \arrow[r, "c \mapsto \omega_c"] \arrow[d, "\iota"] & \Omega^k(M) \arrow[d, "r"] & . \\
 & & C^k(\mathscr{U}, \Omega^0) \arrow[r, "\Phi^k"] & C^k(\mathscr{U}, \Omega^k) &
\end{tikzcd}$$

Since Φ^k induces the identity on Čech–de Rham cohomology, we deduce that the composition of the homomorphism $H^k(\mathscr{U}, \mathbb{R}) \to H^*(M) : [c] \mapsto [\omega_c]$ in (6.5.6) with $r^* : H^*(M) \to \check{H}^k(\mathscr{U})$ is equal to $\iota^* : H^k(\mathscr{U}, \mathbb{R}) \to \check{H}^k(\mathscr{U})$. Hence it follows from Theorem 6.5.11 that the homomorphism (6.5.6) is an isomorphism. This proves Theorem 6.5.6. □

6.5.4 Product Structures

The Čech complex of an open cover $\mathscr{U} = \{U_i\}_{i\in I}$ is equipped with a **cup product**. The definition of this product structure is quite straight forward, however, it requires the choice of an order relation $\prec$ on the index set I. Given such an ordering, each cochain

$$\omega = (\omega_{i_0\cdots i_k})_{(i_0,\dots,i_k)\in\mathcal{I}_k(\mathscr{U})} \in C^k(\mathscr{U},\Omega^p)$$

is uniquely determined by the elements $\omega_{i_0\cdots i_k}$ for those tuples that satisfy $i_0 \prec i_1 \prec \cdots \prec i_k$. All the other elements are then determined by the equivariance condition under the action of the permutation group S_{k+1}.

Definition 6.5.12. *The* **cup product** *on $C^*(\mathscr{U},\Omega^*)$ is the bilinear map*

$$C^k(\mathscr{U},\Omega^p)\times C^\ell(\mathscr{U},\Omega^q)\to C^{k+\ell}(\mathscr{U},\Omega^{p+q}) : (\omega,\tau)\mapsto \omega\cup\tau$$

defined by

$$(\omega\cup\tau)_{i_0\cdots i_{k+\ell}} := (-1)^{\ell p}\omega_{i_0\cdots i_k}\wedge\tau_{i_k\cdots i_{k+\ell}} \tag{6.5.23}$$

for every $\omega\in\mathbb{C}^k(\mathscr{U},\Omega^p)$, every $\tau\in C^\ell(\mathscr{U},\Omega^q)$, and every $(k+\ell+1)$-tuple $(i_0,i_1,\dots,i_{k+\ell})\in\mathcal{I}_{k+\ell}(\mathscr{U})$ that satisfies

$$i_0\prec i_1\prec\cdots\prec i_{k+\ell}.$$

Here the right hand side in (6.5.23) is understood as the restriction of the differential form to the open subset $U_{i_0}\cap U_{i_1}\cap\cdots\cap U_{i_{k+\ell}}$.

Remark 6.5.13. The product structure on $C^*(\mathscr{U},\Omega^*)$ is sensitive to the choice of the ordering of the index set I and is not commutative in any way, shape, or form. In fact, the cup product $\tau\cup\omega$ associated to the reverse ordering agrees up to the usual sign $(-1)^{\deg(\omega)\deg(\tau)}$ with the cup product $\omega\cup\tau$ associated to the original ordering.

Remark 6.5.14. The sign in equation (6.5.23) is naturally associated to the interchanged indices p and ℓ.

Remark 6.5.15. The cup product on $C^*(\mathscr{U},\Omega^*)$ restricts to the product

$$(a\cup b)_{i_0\cdots i_{k+\ell}} = a_{i_0\cdots i_k}b_{i_k\cdots i_{k+\ell}},\qquad i_0\prec i_1\prec\cdots\prec i_{k+\ell}, \tag{6.5.24}$$

on $C^*(\mathscr{U},\mathbb{R})\subset C^*(\mathscr{U},\Omega^0)$.

Remark 6.5.16. The cup product on $C^*(\mathscr{U},\Omega^*)$ restricts to the exterior product for differential forms on $C^0(\mathscr{U},\Omega^*)$.

Lemma 6.5.17. *The cup product* (6.5.23) *on* $C^*(\mathscr{U},\Omega^*)$ *is associative and*

$$D(\omega\cup\tau)=(D\omega)\cup\tau+(-1)^{\deg(\omega)}\omega\cup(D\tau) \tag{6.5.25}$$

for $\omega\in C^k(\mathscr{U},\Omega^p)$ *and* $\tau\in C^\ell(\mathscr{U},\Omega^q)$*, where* $\deg(\omega)=k+p$*.*

Proof. The proof of associativity is left as an exercise. To prove (6.5.25) we compute

$$\begin{aligned}
\big(\delta(\omega\cup\tau)\big)_{i_0\cdots i_{k+\ell+1}} &= \sum_{\nu=0}^{k+\ell+1}(-1)^\nu(\omega\cup\tau)_{i_0\cdots\widehat{i_\nu}\cdots i_{k+\ell+1}} \\
&= \sum_{\nu=0}^{k}(-1)^\nu(-1)^{\ell p}\omega_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}}\wedge\tau_{i_{k+1}\cdots i_{k+\ell+1}} \\
&\quad+\sum_{\nu=k+1}^{k+\ell+1}(-1)^\nu(-1)^{\ell p}\omega_{i_0\cdots i_k}\wedge\tau_{i_k\cdots\widehat{i_\nu}\cdots i_{k+\ell+1}} \\
&= \sum_{\nu=0}^{k+1}(-1)^\nu(-1)^{\ell p}\omega_{i_0\cdots\widehat{i_\nu}\cdots i_{k+1}}\wedge\tau_{i_{k+1}\cdots i_{k+\ell+1}} \\
&\quad+\sum_{\nu=k}^{k+\ell+1}(-1)^\nu(-1)^{\ell p}\omega_{i_0\cdots i_k}\wedge\tau_{i_k\cdots\widehat{i_\nu}\cdots i_{k+\ell+1}} \\
&= (-1)^{\ell p}(\delta\omega)_{i_0\cdots i_{k+1}}\wedge\tau_{i_{k+1}\cdots i_{k+\ell+1}} \\
&\quad+(-1)^{\ell p+k}\omega_{i_0\cdots i_k}\wedge(\delta\tau)_{i_k\cdots i_{k+\ell+1}} \\
&= \big((\delta\omega)\cup\tau\big)_{i_0\cdots i_{k+\ell+1}}+(-1)^{k+p}\big(\omega\cup(\delta\tau)\big)_{i_0\cdots i_{k+\ell+1}}.
\end{aligned}$$

Thus we have proved that

$$\delta(\omega\cup\tau)=(\delta\omega)\cup\tau+(-1)^{\deg(\omega)}\omega\cup(\delta\tau). \tag{6.5.26}$$

Moreover,

$$\begin{aligned}
\big(d(\omega\cup\tau)\big)_{i_0\cdots i_{k+\ell+1}} &= (-1)^{\ell p}d\left(\omega_{i_0\cdots i_k}\wedge\tau_{i_k\cdots i_{k+\ell}}\right) \\
&= (-1)^{\ell p}d\omega_{i_0\cdots i_k}\wedge\tau_{i_k\cdots i_{k+\ell}} \\
&\quad+(-1)^{(\ell+1)p}\omega_{i_0\cdots i_k}\wedge d\tau_{i_k\cdots i_{k+\ell}}
\end{aligned}$$

Thus we have proved that

$$(-1)^{k+\ell}d(\omega\cup\tau)=\Big((-1)^k d\omega\Big)\cup\tau+(-1)^{\deg\omega}\omega\cup\Big((-1)^\ell d\tau\Big). \tag{6.5.27}$$

With this understood, equation (6.5.25) follows by taking the sum of the equations (6.5.26) and (6.5.27). This proves Lemma 6.5.17. □

The cochain homomorphisms r and ι intertwine the product structures on the cochain level. Hence the induced homomorphisms on cohomology

$$r^*: H^*(M) \to \check{H}^*(\mathscr{U}), \qquad \iota^*: H^*(\mathscr{U},\mathbb{R}) \to \check{H}^*(\mathscr{U})$$

also intertwine the product structures. If $\mathscr{U}$ is a good cover, then these are isomorphisms and hence, in this case, both cohomology groups $\check{H}^*(\mathscr{U})$ and $H^*(\mathscr{U},\mathbb{R})$ inherit the commutativity properties of the cup product on de Rham cohomology, although this is not at all obvious from the definitions.

6.5.5 Remarks on De Rham's Theorem

There is a natural homomorphism

$$H^*_{\mathrm{dR}}(M) \to H^*_{\mathrm{sing}}(M,\mathbb{R}) \tag{6.5.28}$$

from the de Rham cohomology of M to the singular cohomology with real coefficients, defined in terms of integration over smooth singular cycles. **De Rham's Theorem** asserts that this homomorphism is bijective. To prove this it suffices, in view of Theorem 6.5.6, to prove that the singular cohomology of M with real coefficients is isomorphic to the Čech cohomology group $H^*(\mathscr{U},\mathbb{R})$ associated to a good cover. The proof involves similar methods as that of Theorem 6.5.6 but will not be included in this book. Instead we restrict the discussion to some remarks and exercises. For more details an excellent reference is the book of Bott and Tu [3].

Remark 6.5.18. Let M be a compact oriented smooth m-manifold without boundary. It is a deep theorem in algebraic topology that a suitable integer multiple of any integral singular homology class on M can be represented by a compact oriented submanifold without boundary, in the sense that any triangulation of the submanifold gives rise to a singular cycle representing the homology class. The details of this are outside the scope of the present book. However, we mention without proof the following consequence of this result and de Rham's theorem:

There is a finite collection of compact oriented $(m-k_i)$-dimensional submanifolds without boundary

$$Q_i \subset M, \qquad i = 0,\dots,n,$$

such that the cohomology classes of the closed forms

$$\tau_i = \tau_{Q_i} \in \Omega^{k_i}(M),$$

dual to the submanifolds as in §6.4.3, form a basis of $H^*(M)$.

Remark 6.5.19. *It follows from the assertion in Remark 6.5.18 that every closed form $\omega \in \Omega^k(M)$ that satisfies*

$$\int_P f^*\omega = 0$$

for every compact oriented smooth k-manifold P without boundary and every smooth map $f : P \to M$ is exact. (This implies that the homomorphism (6.5.28) *is injective.)*

For $k = 1$ this follows from Exercise 6.4.14. To see this in general, let Q_i and τ_i be chosen as in Remark 6.5.18 and denote by $I_k \subset \{0, \dots, n\}$ the set of all indices i such that

$$\dim(Q_i) = m - k_i = k, \qquad \deg(\tau_i) = k_i = m - k.$$

If $\omega \in \Omega^k(M)$ satisfies our assumptions, then

$$\int_M \omega \wedge \tau_i = \int_{Q_i} \omega = 0$$

for every $i \in I_k$. Since the cohomology classes $[\tau_i]$ form a basis of $H^{m-k}(M)$ we have

$$\int_M \omega \wedge \tau = 0$$

for every closed $(m-k)$-form τ. Hence ω is exact, by Theorem 6.4.1.

Exercise 6.5.20. Define a homomorphism

$$H^1(M) \to \mathrm{Hom}(\pi_1(M, p_0), \mathbb{R}) : [\omega] \mapsto \rho_\omega \tag{6.5.29}$$

which assigns to every closed 1-form $\omega \in \Omega^1(M)$ the homomorphism

$$\rho_\omega : \pi_1(M, p_0) \to \mathbb{R}, \qquad \rho_\omega([\gamma]) := \int_{[0,1]} \gamma^*\omega,$$

for every smooth based loop $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = p_0$. By Theorem 6.1.1, ρ_ω depends only on the cohomology class of ω. By Exercise 6.4.14 the homomorphism $[\omega] \mapsto \rho_\omega$ is injective. Prove that it is surjective. **Hint:** Choose a good cover $\mathscr{U} = \{U_i\}_{i \in I}$ of M and, for each $i \in I$, choose an element $p_i \in U_i$ and a path $\gamma_i : [0,1] \to M$ such that $\gamma_i(0) = p_0$ and $\gamma_i(1) = p_i$. For $(i,j) \in \mathcal{I}_1(\mathscr{U})$ define the number $c_{ij} \in \mathbb{R}$ by

$$c_{ij} := \rho(\gamma), \qquad \left\{ \begin{array}{ll} \gamma(t) = \gamma_i(4t), & \text{for } 0 \le t \le 1/4, \\ \gamma(t) \in U_i, & \text{for } 1/4 \le t \le 1/2, \\ \gamma(t) \in U_j, & \text{for } 1/2 \le t \le 3/4, \\ \gamma(t) = \gamma_j(4(1-t)), & \text{for } 3/4 \le t \le 1. \end{array} \right.$$

Prove that any two such paths γ are homotopic with fixed endpoints. Prove that the numbers c_{ij} determine a 1-cocycle in the Čech complex $C^1(\mathscr{U},\mathbb{R})$. Prove that the 1-form

$$\omega_c := \sum_{(i,j)\in\mathcal{I}_1(\mathscr{U})} c_{ij}\rho_i d\rho_j$$

is closed and satisfies $\rho_{\omega_c} = \rho$. Note that the only conditions on $\mathscr{U}$ used in this proof are that the sets U_i are connected and simply connected, and that each nonempty intersection $U_i \cap U_j$ is connected.

Exercise 6.5.21. Consider the circle $M = S^1$ with its standard counterclockwise orientation and let

$$S^1 = U_1 \cup U_2 \cup U_3$$

be a good cover. Thus the sets U_1, U_2, U_3 are open intervals as are the intersections $U_1 \cap U_2$, $U_2 \cap U_3$, $U_3 \cap U_1$. Assume that in the counterclockwise ordering the endpoint of U_1 is contained in U_2 and the endpoint of U_2 in U_3. Prove that the composition of the isomorphism $H^1(\mathscr{U},\mathbb{R}) \to H^1(S^1)$ with the isomorphism $H^1(S^1) \to \mathbb{R}$, given by integration, is the map

$$H^1(\mathscr{U},\mathbb{R}) \to \mathbb{R} : [c_{23}, c_{13}, c_{12}] \mapsto c_{23} - c_{13} + c_{12}.$$

Deduce that the homomorphism

$$\rho_{\omega_c} : \pi_1(S^1) \to \mathbb{R}$$

associated to a cycle $c \in C^1(\mathscr{U},\mathbb{R})$ as in Exercise 6.5.20 maps the positive generator to the real number $c_{23} - c_{13} + c_{12}$.

Exercise 6.5.22. Choose a good cover $\mathscr{U}$ of the 2-sphere by four open hemispheres and compute its Čech complex. Find an explicit expression for the isomorphism $H^2(\mathscr{U},\mathbb{R}) \to \mathbb{R}$ associated to the standard orientation.

Chapter 7

Vector Bundles and the Euler Class

In this chapter we introduce smooth vector bundles over smooth manifolds in the intrinsic setting. Basic definitions and examples are discussed in §7.1. In §7.2 we define *Integration over the Fiber* for differential forms with *vertical compact support*, prove the *Thom Isomorphism Theorem*, and introduce the *Thom Class* and relate is to intersection theory. In §7.3 we introduce the *Euler Class* of an oriented vector bundle and show that, if the rank of the bundle agrees with the dimension of the base and the base is oriented, then its integral over the base, the *Euler Number*, is equal to the algebraic number of zeros of a section with only nondegenerate zeros. As an application we compute the product structure on the de Rham cohomology of complex projective space.

7.1 Vector Bundles

In [35] we have introduced the notion of a vector bundle

$$\pi : E \to M$$

over an (embedded) manifold M as a subbundle of the product $M \times \mathbb{R}^\ell$ for some integer $\ell \geq 0$. In this section we show how to carry the definitions of vector bundles, sections, and vector bundle homomorphisms over to the intrinsic setting. This is also the appropriate framework for introducing structure groups of vector bundles. In particular, we will discuss the notion of orientability, which specializes to orientability of a manifold in the case of the tangent bundle.

7.1.1 Definitions and Remarks

Let M be a smooth m-manifold and let n be a nonnegative integer.

Definition 7.1.1 (Vector Bundle). *A* **real vector bundle over** M **of rank** n *consists of a smooth manifold E of dimension $m+n$, a smooth map*

$$\pi : E \to M,$$

called the **projection**, *an open cover $\{U_\alpha\}_{\alpha\in A}$ of M, a real n-dimensional vector space V, a collection of diffeomorphisms*

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V, \qquad \alpha \in A,$$

called **local trivializations**, *that satisfy*

$$\mathrm{pr}_1 \circ \psi_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$$

so that the diagram

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\psi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times V \arrow[dl, "\mathrm{pr}_1"] \\
& U_\alpha &
\end{tikzcd} \tag{7.1.1}$$

commutes for every $\alpha \in A$, and a collection of smooth maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(V), \qquad \alpha, \beta \in A,$$

called **transition maps**, *that satisfy*

$$\psi_\beta \circ \psi_\alpha^{-1}(p, v) = (p, g_{\beta\alpha}(p)v) \tag{7.1.2}$$

for all $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$, and $v \in V$.

For $p \in M$ the set

$$E_p := \pi^{-1}(p)$$

is called the **fiber of** E **over** p. *If*

$$\mathrm{G} \subset \mathrm{GL}(V)$$

is a Lie subgroup and the transition maps $g_{\beta\alpha}$ all take values in G, *then we call E a* **vector bundle with structure group** G. *We say that the structure group of a vector bundle E* **can be reduced to** G *iff E can be covered by local trivializations whose transition maps all take values in* G.

It is sometimes convenient to write an element of a vector bundle E as a pair (p, e) consisting of a point $p \in M$ and an element $e \in E_p$ of the fiber of E over p. This notation suggests that we may think of a vector bundle over M as a *functor* which assigns to each element $p \in M$ a vector space E_p. The definition then requires that the disjoint union of the vector spaces E_p is equipped with the structure of a smooth manifold whose coordinate charts are compatible with the projection π and with the vector space structures on the fibers.

Remark 7.1.2. If $\pi : E \to M$ is a vector bundle, then the projection π is a surjective submersion because the diagram (7.1.1) commutes.

Remark 7.1.3. If $\pi : E \to M$ is a vector bundle, then for every $p \in M$ the fiber $E_p = \pi^{-1}(p)$ inherits a vector space structure from V via the bijection

$$\psi_\alpha(p) := \mathrm{pr}_2 \circ \psi_\alpha|_{E_p} : E_p \to V \tag{7.1.3}$$

for $\alpha \in A$ with $p \in U_\alpha$. In other words, for $\lambda \in \mathbb{R}$ and $e, e' \in E_p$ we define the sum $e + e' \in E_p$ and the product $\lambda e \in E_p$ by

$$e + e' := \psi_\alpha(p)^{-1}(\psi_\alpha(p)e + \psi_\alpha(p)e'), \qquad \lambda e := \psi_\alpha(p)^{-1}(\lambda\psi_\alpha(p)).$$

The vector space structure on E_p is independent of α because the map

$$\psi_\beta(p) \circ \psi_\alpha(p)^{-1} = g_{\beta\alpha}(p) : V \to V$$

is linear for all $\alpha, \beta \in A$ with $p \in U_\alpha \cap U_\beta$.

Remark 7.1.4. The transition maps of a vector bundle E satisfy the conditions

$$g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}, \qquad g_{\alpha\alpha} = \mathbb{1}, \tag{7.1.4}$$

for all $\alpha, \beta, \gamma \in A$. Here the first equation is understood on the intersection $U_\alpha \cap U_\beta \cap U_\gamma$ where all three transition maps are defined.

Conversely, every open cover $\{U_\alpha\}_{\alpha \in A}$ and every system of transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ satisfying (7.1.4) determines a vector bundle

$$\widetilde{E} := \bigcup_{\alpha \in A} \{\alpha\} \times U_\alpha \times V / \sim$$

where the equivalence relation is given by

$$[\alpha, p, v] \sim [\beta, p, g_{\beta\alpha}(p)v]$$

for $\alpha, \beta \in A$, $p \in U_\alpha \cap U_\beta$, and $v \in V$. The projection $\pi : E \to M$ is given by $[\alpha, p, v] \mapsto p$ and the local trivializations are given by $[\alpha, p, v] \mapsto (p, v)$. These local trivializations satisfy (7.1.2). This vector bundle is isomorphic to E (see Definition 7.1.18 below).

7.1.2 Examples and Exercises

Example 7.1.5 (Trivial Bundle). The simplest example of a vector bundle over M is the **trivial bundle**

$$E = M \times \mathbb{R}^n.$$

It has an obvious *global* trivialization. Every real rank-n vector bundle over M is locally isomorphic to the trivial bundle but there is not necessarily a global isomorphism. (See below for the definition of a vector bundle isomorphism.)

Example 7.1.6 (Möbius band). The simplest example of a nontrivial vector bundle is the real rank-1 vector bundle

$$E := \left\{(z,\zeta) \in S^1 \times \mathbb{C} \,\middle|\, z^2\zeta \in \mathbb{R}\right\}$$

over the circle

$$S^1 := \left\{z \in \mathbb{C} \,\middle|\, |z| = 1\right\},$$

called the **Möbius band**. **Exercise:** Prove that the Möbius band does not admit a global trivialization; it does not admit a global nonzero section. (See below for the definition of a section.) Prove that E is diffeomorphic to the manifold M in Example 2.1.7.

Example 7.1.7 (Tangent Bundle). Let M be a smooth m-manifold with an atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$. The tangent bundle

$$TM := \left\{(p,v) \,\middle|\, p \in M,\, v \in T_pM\right\}$$

is a vector bundle over M with the obvious projection $\pi : TM \to M$ and the local trivializations

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^m, \qquad \psi_\alpha(p,v) := (p, d\phi_\alpha(p)v).$$

The transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(m,\mathbb{R})$ are given by

$$g_{\beta\alpha}(p) = d(\phi_\beta \circ \phi_\alpha^{-1})(\phi_\alpha(p))$$

for $p \in U_\alpha \cap U_\beta$.

Exercise 7.1.8 (Dual bundle). Let $\pi : E \to M$ be a real vector bundle with local trivializations $\psi_\alpha(p) : E_p \to V$. Show that the **dual bundle**

$$E^* := \left\{(p,e^*) \,\middle|\, p \in M,\, e^* \in \mathrm{Hom}(E_p,\mathbb{R})\right\}$$

is a vector bundle with V replaced by V^* in the local trivializations and that the transition maps are related by $g_{\beta\alpha}^{E^*} = (g_{\alpha\beta}^{E})^* : U_\alpha \cap U_\beta \to \mathrm{GL}(V^*)$. Deduce that the **cotangent bundle** T^*M is a vector bundle over M.

Example 7.1.9 (Exterior Power). The **kth exterior power**

$$\Lambda^k T^*M := \Big\{(p,\omega)\,\Big|\,p \in M,\, \omega \in \Lambda^k T_p^*M\Big\}$$

of the cotangent bundle is a real vector bundle with the the local trivializations given by pushforward under the derivatives of the coordinate charts:

$$(d\phi_\alpha(p)^{-1})^* : \Lambda^k T_p^*M \to \Lambda_k(\mathbb{R}^m)^*.$$

The transition maps of $\Lambda^k T^*M$ are then given by

$$g_{\beta\alpha}^{\Lambda^k T^*M}(p) = (d(\phi_\alpha \circ \phi_\beta^{-1})(\phi_\beta(p)))^* \in \mathrm{GL}(\Lambda^k(\mathbb{R}^m)^*)$$

for $p \in U_\alpha \cap U_\beta$.

Example 7.1.10 (Pullback). Let $\pi^E : E \to M$ be a real vector bundle with local trivializations $\psi_\alpha^E(p) : E_p \to V$ and let $f : N \to M$ be a smooth map. Then the **pullback bundle**

$$f^*E := \big\{(q,e)\,\big|\,q \in N,\, e \in E,\, \pi^E(e) = f(q)\big\} \subset N \times E$$

is a submanifold of $N \times E$ and a vector bundle over N with the obvious projection $\pi^{f^*E} : f^*E \to N$ onto the first factor. The local trivializations are $\psi_\alpha^{f^*E}(q) = \psi_\alpha^E(f(q)) : (f^*E)_q = E_{f(q)} \to V$ for $q \in f^{-1}(U_\alpha)$ and the transition maps are given by

$$g_{\beta\alpha}^{f^*E} = g_{\beta\alpha}^E \circ f : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to \mathrm{GL}(V).$$

Example 7.1.11 (Whitney Sum). Let $\pi^E : E \to M$, $\pi^F : F \to M$ be vector bundles with local trivializations $\psi_\alpha^E(p) : E_p \to V$, $\psi_\alpha^F(p) : F_p \to V$ for $p \in U_\alpha$ (over the same open cover). The **Whitney sum**

$$E \oplus F := \bigcup_{p \in M} \{p\} \times (E_p \oplus F_p)\,,$$

is a vector bundle over M with the obvious projection $\pi : E \oplus F \to M$. The local trivializations are

$$\psi_\alpha^{E\oplus F}(p) := \psi_\alpha^E(p) \oplus \psi_\alpha^F(p) : E_p \oplus F_p \to V \oplus W, \qquad p \in U_\alpha,$$

and the transition maps are given by

$$g_{\beta\alpha}^{E\oplus F} = g_{\beta\alpha}^E \oplus g_{\beta\alpha}^F : U_\alpha \cap U_\beta \to \mathrm{GL}(V \oplus W).$$

Replacing everywhere $\oplus$ by $\otimes$ we obtain the **tensor product** of E and F.

Exercise 7.1.12 (Normal Bundle). Let M be a smooth m-manifold and let $Q \subset M$ be a k-dimensional submanifold. Choose a Riemannian metric on M. Prove that the **normal bundle**

$$TQ^\perp := \left\{(p,v)\,\middle|\,p \in Q,\, v \in T_pM,\, v \perp T_qQ\right\}$$

is a smooth vector bundle over Q of rank $m-k$. **Hint:** See Exercise 4.3.4. Alternatively, use geodesics to find coordinate charts $\phi_\alpha : U_\alpha \to \mathbb{R}^k \times \mathbb{R}^{m-k}$ such that $\phi_\alpha(U_\alpha \cap Q) = \phi_\alpha(U_\alpha) \cap (\mathbb{R}^k \times \{0\})$ and $v \perp T_qQ$ if and only if $d\phi_\alpha(q)v \in \{0\} \times \mathbb{R}^{m-k}$ for all $q \in Q$ and $v \in T_qM$. Another method is to identify the normal bundle with the quotient bundle $TM|_Q/TQ$ and use an arbitrary submanifold chart to find a local trivialization modelled on the quotient space $V = \mathbb{R}^m/\mathbb{R}^k$. If Q is totally geodesic, then one can use the Levi-Civita connection to construct local trivializations of the normal bundle.

7.1.3 Sections of vector bundles

Let $\pi : E \to M$ be a real vector bundle over a smooth manifold.

Definition 7.1.13 (Section of a Vector Bundle). *A* **section of** E *is a smooth map* $s : M \to E$ *such that* $\pi \circ s = \mathrm{id} : M \to M$.

The set of sections of E is a real vector space, denoted by

$$\Omega^0(M,E) := \left\{s : M \to E\,\middle|\,s \text{ is smooth and } \pi \circ s = \mathrm{id}\right\}.$$

If we write a point in E as a pair (p,e) with $p \in M$ and $e \in E_p$, then we can think of a section of E as a natural transformation which assigns to each element p of M and element $s(p)$ of the vector space E_p such that the map $M \to E : p \mapsto (p, s(p))$ is smooth. Slightly abusing notation we will switch between these two points of view whenever convenient and use the same letter s for the map $M \to E : p \mapsto (p, s(p))$ and for the assignment $p \mapsto s(p) \in E_p$.

Remark 7.1.14. In the local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ a section $s : M \to E$ is given by smooth maps $s_\alpha : U_\alpha \to V$ such that

$$\psi_\alpha(s(p)) := (p, s_\alpha(p)). \tag{7.1.5}$$

These maps satisfy the condition

$$s_\beta = g_{\beta\alpha} s_\alpha \tag{7.1.6}$$

on $U_\alpha \cap U_\beta$. Conversely, every collection of smooth maps $s_\alpha : U_\alpha \to V$ satisfying (7.1.6) determine a unique global section $s : M \to E$ via (7.1.5).

Example 7.1.15 (Zero Section). The zero section

$$\iota : M \to E, \qquad \iota(p) := 0_p \in E_p,$$

assigns to each $p \in M$ the zero element of the fiber $E_p = \pi^{-1}(E)$ with respect to the vector space structure of Remark 7.1.3. Its image is a submanifold

$$Z := \iota(M) = \{0_p \,|\, p \in M\} \subset E,$$

which will also be called the **zero section of** E.

Exercise 7.1.16. For every vector bundle $\pi : E \to M$, every $p \in M$, and every $e \in E_p$, there is a smooth section $s : M \to E$ such that $s(p) = e$.

Example 7.1.17. The space of sections of the tangent bundle is the space of vector fields, the space of sections of the cotangent bundle is the space of 1-forms, and the space of sections of the kth exterior power of the cotangent bundle is the space of k-forms on M:

$$\Omega^0(M, TM) = \mathrm{Vect}(M), \qquad \Omega^0(M, \Lambda^k T^*M) = \Omega^k(M).$$

If $Q \subset M$ is a submanifold of a Riemannian manifold, then the space of sections of the normal bundle of Q is the space $\Omega^0(Q, TQ^\perp) = \mathrm{Vect}^\perp(Q)$ of normal vector fields along Q.

7.1.4 Vector Bundle Homomorphisms

Definition 7.1.18 (Vector Bundle Homomorphism). *Let* $\pi^E : E \to M$ *and* $\pi^F : F \to M$ *be real vector bundles. A* **vector bundle homomorphism** *from* E *to* F *is a smooth map* $\Phi : E \to F$ *such that*

$$\pi^F \circ \Phi = \pi^E$$

and, for every $p \in M$*, the restriction* $\Phi_p := \Phi|_{E_p} : E_p \to F_p$ *is a linear map. A* **vector bundle isomorphism** *is a bijective vector bundle homomorphism. The vector bundles* E *and* F *are called* **isomorphic** *iff there exists a vector bundle isomorpism* $\Phi : E \to F$*.*

Exercise 7.1.19. (i) Every vector bundle isomorphism is a diffeomorphism.

(ii) Every injective vector bundle homomorphism is an embedding.

(iii) Every real vector bundle over a compact manifold M admits an injective vector bundle homomorphism $\Phi : E \to M \times \mathbb{R}^N$ for some integer N. **Hint:** Use a finite collection of local trivializations and a partition of unity.

Exercise 7.1.20. The Möbius band $\pi : E \to S^1$ in Example 7.1.6 is not isomorphic to the trivial bundle $F := S^1 \times \mathbb{R}$. The tangent bundle TM of any manifold M is isomorphic to the cotangent bundle T^*M.

Exercise 7.1.21. The set $\mathrm{Hom}(E, F) := \bigcup_{p\in M}\{p\} \times \mathrm{Hom}(E_p, F_p)$ is a vector bundle over M and the space of smooth sections of $\mathrm{Hom}(E, F)$ is the space of vector bundle homomorphisms from E to F. The vector bundle $E^* \otimes F$ is isomorphic to $\mathrm{Hom}(E, F)$.

7.1.5 Orientation

Definition 7.1.22 (Oriented Vector Bundle). *A vector bundle*

$$\pi : E \to M$$

is called **orientable** *iff its local trivializations can be chosen such that the transition maps take values in the group* $\mathrm{GL}^+(V)$ *of orientation preserving automorphisms of* V*, i.e. for all* $\alpha, \beta \in A$ *we have*

$$g_{\beta\alpha}(p) = \psi_\beta(p) \circ \psi_\alpha(p)^{-1} \in \mathrm{GL}^+(V), \qquad p \in U_\alpha \cap U_\beta. \tag{7.1.7}$$

It is called **oriented** *iff* V *is oriented and* (7.1.7) *holds.*

A vector bundle $\pi : E \to M$ is orientable if and only if its structure group can be reduced to $\mathrm{GL}^+(V)$. Care must be taken to distinguish between the orientability of E as a vector bundle and the orientability of E as a manifold. By definition, a manifold M is orientable if and only if its tangent bundle is orientable as a vector bundle. Thus E is orientable as a manifold if and only if its tangent bundle TE is orientable as a vector bundle. For example the trivial bundle $E = M \times \mathbb{R}^n$ is always orientable as a vector bundle but the manifold $M \times \mathbb{R}^n$ is only orientable if M is. Conversely, the tangent bundle of any manifold, orientable or not, is always an orientable manifold in the sense that its tangent bundle TTM is an orientable vector bundle.

Exercise 7.1.23. Let M be an orientable manifold and let $\pi : E \to M$ be a real vector bundle. Then E is orientable as a vector bundle if and only if the manifold E is orientable.

Exercise 7.1.24. The Möbius band in Example 7.1.6 is not orientable.

Exercise 7.1.25. A vector bundle $\pi : E \to M$ of rank n is oriented if and only if the fibers E_p are equipped with orientations that fit together smoothly in the following sense: for every element $p_0 \in M$ there exist an open neighborhood $U \subset M$ of p_0 and sections $s_1, \dots, s_n : U \to E$ such that the vectors $s_1(p), \dots, s_n(p)$ form a positive basis of E_p for every $p \in U$.

Exercise 7.1.26. The tangent bundle of the tangent bundle is orientable.

7.2 The Thom Class

We assume throughout that M is a smooth m-manifold (not necessarily compact and possibly with boundary) and that

$$\pi : E \to M$$

is an oriented real vector bundle of rank n. §7.2.1 introduces *integration over the fiber* for differential forms with vertical compact support. The *Thom Isomorphism Theorem* asserts that the induced homomorphism on de Rham cohomology is an isomorphism. A corollary is the existence of a *Thom form* $\tau \in \Omega^n_{\mathrm{vc}}(E)$, a closed n-form with vertical compact support whose integral over each fiber is one. In §7.2.2 we give two proofs of this result, one proof for bundles of finite type which is based on a Mayer–Vietoris argument, and another proof for compact oriented base manifolds M without boundary which is based on Poincaré duality and which first establishes the existence of Thom forms. §7.2.3 relates the Thom class to intersection theory and contains a proof of Theorem 6.4.8.

7.2.1 Integration over the Fiber

Integration over the fiber assigns to an $(n+k)$-form on the total space E of our vector bundle with *vertical compact support* a k-form on M. This homomorphism, also called *pushforward*, commutes with the differential and hence induces a homomorphism on de Rham cohomology.

Definition 7.2.1 (Vertical Compact Support). *A differential form*

$$\tau \in \Omega^\ell(E)$$

is said to have **vertical compact support** *iff the set*

$$\mathrm{supp}(\tau) \cap \pi^{-1}(K) \subset E$$

is compact for every compact subset $K \subset M$. The set of all ℓ-forms on E with vertical compact support will be denoted by

$$\Omega^\ell_{\mathrm{vc}}(E) := \Big\{ \tau \in \Omega^\ell(E) \,\Big|\, \tau \text{ has vertical compact support} \Big\}.$$

Differential forms with vertical compact support are preserved by the differential and the cohomology group

$$H^\ell_{\mathrm{vc}}(E) := \frac{\ker(d : \Omega^\ell_{\mathrm{vc}}(E) \to \Omega^{\ell+1}_{\mathrm{vc}}(E))}{\ker(d : \Omega^{\ell-1}_{\mathrm{vc}}(E) \to \Omega^\ell_{\mathrm{vc}}(E))}$$

is called the **de Rham cohomology with vertical compact support***.*

Definition 7.2.2 (Pushforward). *For $k = 0, 1, \ldots, m$ the* **pushforward under the projection** π *is the linear operator*

$$\pi_* : \Omega^{n+k}_{\mathrm{vc}}(E) \to \Omega^k(M),$$

defined as follows. Let $\tau \in \Omega^{n+k}_{\mathrm{vc}}(E)$ and choose $v_1, \ldots, v_k \in T_pM$. Associated to these data is a differential form $\tau^{p,v_1,\ldots,v_k} \in \Omega^n_c(E_p)$ defined as follows. Given $e \in E_p = \pi^{-1}(p)$ and $e_1, \ldots, e_n \in T_eE_p = \ker d\pi(e) \cong E_p$, choose lifts $\widetilde{v}_i \in T_eE$ so that $d\pi(e)\widetilde{v}_i = v_i$ for $i = 1, \ldots, k$, and define

$$\big(\tau^{p,v_1,\ldots,v_k}\big)_e(e_1, \ldots, e_n) := \tau_e\big(\widetilde{v}_1, \ldots, \widetilde{v}_k, e_1, \ldots, e_n\big). \tag{7.2.1}$$

The expression on the right is independent of the choice of the lifts $\widetilde{v}_i$; namely, if the e_j are linearly independent, then any two choices of lifts $\widetilde{v}_i$ differ by a linear combination of the e_j, and if the e_j are linearly dependent, then the right hand side of (7.2.1) vanishes for any choice of the $\widetilde{v}_i$. Now the **pushforward** *$\pi_*\tau \in \Omega^k(M)$ is defined by*

$$(\pi_*\tau)_p(v_1, \ldots, v_k) := \int_{E_p} \tau^{p,v_1,\ldots,v_k} \tag{7.2.2}$$

for $p \in M$ and $v_i \in T_pM$. The integral is well defined because $\tau^{p,v_1,\ldots,v_k}$ has compact support and E_p is an oriented n-dimensional manifold.

Exercise 7.2.3. Show that the map

$$(\pi_*\tau)_p : (T_pM)^k \to \mathbb{R}$$

in (7.2.2) is an alternating k-form for every p and that these alternating k-forms fit together smoothly. Show that the map $\tau \mapsto \pi_*\tau$ is linear.

Exercise 7.2.4. If $\tau \in \Omega^n_{\mathrm{vc}}(E)$, show that $\pi_*\tau \in \Omega^0(M)$ is the smooth real valued function on M defined by

$$(\pi_*\tau)(p) = \int_{E_p} \tau$$

for $p \in M$.

Exercise 7.2.5. If $\tau \in \Omega^{n+k}_c(E)$, show that $\pi_*\tau \in \Omega^k_c(M)$. Show that the map $\pi_* : \Omega^{k+1}_c(M \times \mathbb{R}) \to \Omega^k_c(M)$ in the proof of Theorem 6.3.8 is an example of integration over the fiber.

Lemma 7.2.6. *Let* $\pi : E \to M$ *be an oriented real rank-*n *vector bundle over a smooth* m*-manifold* M *with boundary and let* $\pi_* : \Omega^{n+*}_{\mathrm{vc}}(E) \to \Omega^*(M)$ *be the operator of Definition 7.2.2. Then*

$$\pi_*(\pi^*\omega \wedge \tau) = \omega \wedge \pi_*\tau, \tag{7.2.3}$$

$$\pi_* d\tau = d\pi_*\tau \tag{7.2.4}$$

for all $\omega \in \Omega^\ell(M)$ *and all* $\tau \in \Omega^{n+k}_{\mathrm{vc}}(E)$*. If* M *is oriented, then*

$$\int_M \omega \wedge \pi_*\tau = \int_E \pi^*\omega \wedge \tau \tag{7.2.5}$$

for all $\omega \in \Omega^{m-k}_c(M)$ *and all* $\tau \in \Omega^{n+k}_{\mathrm{vc}}(E)$*.*

Proof. The proof of equation (7.2.3) relies on the observation that the vectors $e_i \in T_eE_p = E_p$, used in the definition of the compactly supported n-form $(\pi^*\omega \wedge \tau)^{p,v_1,\dots,v_{k+\ell}}$ on E_p in Definition 7.2.2, can only lead to nonzero terms when they appear in τ. Thus

$$\begin{aligned}
&\bigl((\pi^*\omega \wedge \tau)^{p,v_1,\dots,v_{k+\ell}}\bigr)_e(e_1,\dots,e_n) \\
&= (\pi^*\omega \wedge \tau)_e\bigl(\widetilde{v}_1,\dots,\widetilde{v}_{k+\ell},e_1,\dots,e_n\bigr) \\
&= \sum_{\sigma\in S_{k,\ell}} \varepsilon(\sigma)\omega_p\bigl(v_{\sigma(1)},\dots,v_{\sigma(k)}\bigr)\tau_e\bigl(\widetilde{v}_{\sigma(k+1)},\dots,\widetilde{v}_{\sigma(k+\ell)},e_1,\dots,e_n\bigr) \\
&= \sum_{\sigma\in S_{k,\ell}} \varepsilon(\sigma)\omega_p\bigl(v_{\sigma(1)},\dots,v_{\sigma(k)}\bigr)\bigl(\tau^{p,v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}}\bigr)_e(e_1,\dots,e_n)
\end{aligned}$$

for $e_i \in T_eE_p$ and $\widetilde{v}_i \in T_eE$ with $d\pi(e)\widetilde{v}_i = v_i$. Integrate both sides of this equation over E_p to obtain

$$\begin{aligned}
&\bigl(\pi_*(\pi^*\omega \wedge \tau)\bigr)_p(v_1,\dots,v_{k+\ell}) \\
&= \int_{E_p} (\pi^*\omega \wedge \tau)^{p,v_1,\dots,v_{k+\ell}} \\
&= \sum_{\sigma\in S_{k,\ell}} \varepsilon(\sigma)\omega_p\bigl(v_{\sigma(1)},\dots,v_{\sigma(k)}\bigr)\int_{E_p} \tau^{p,v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}} \\
&= \sum_{\sigma\in S_{k,\ell}} \varepsilon(\sigma)\omega_p\bigl(v_{\sigma(1)},\dots,v_{\sigma(k)}\bigr)(\pi_*\tau)_p\bigl(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}\bigr) \\
&= (\omega \wedge \pi_*\tau)_p(v_1,\dots,v_{k+\ell}).
\end{aligned}$$

This proves (7.2.3).

To prove equation (7.2.4) we will work in a local trivialization of E followed by local coordinates on M. Thus we consider the vector bundle

$$U \times \mathbb{R}^n$$

over an open set $U \subset \mathbb{H}^m$. Denote the coordinates on U by $x^1, \dots, x^m$ and the coordinates on $\mathbb{R}^n$ by $t^1, \dots, t^n$. Then an $(n+k)$-form $\tau \in \Omega^{n+k}(U \times \mathbb{R}^n)$ can be written in the form

$$\tau = \sum_{|J|+|K|=n+k} \tau_{J,K}(x,t)\, dx^J \wedge dt^K. \tag{7.2.6}$$

The vertical compact support condition now translates into the assumption that the support of τ is contained in the product of U with a compact subset of $\mathbb{R}^n$. Integration over the fiber yields a k-form $\pi_*\tau \in \Omega^k(U)$ given by

$$\pi_*\tau = \sum_{|J|=k} \left(\int_{\mathbb{R}^n} \tau_{J,K_n}(x,t)\, dt^1 \cdots dt^n \right) dx^J, \tag{7.2.7}$$

where K_n denotes the maximal multi-index $K_n = (1, \dots, n)$. Next we apply the same operation to the form

$$\begin{aligned} d\tau = & \sum_{|J|+|K|=n+k} \sum_{i=1}^m \frac{\partial \tau_{J,K}}{\partial x^i}(x,t)\, dx^i \wedge dx^J \wedge dt^K \\ & + \sum_{|J|+|K|=n+k} \sum_{j=1}^n \frac{\partial \tau_{J,K}}{\partial t^j}(x,t)\, dt^j \wedge dx^J \wedge dt^K. \end{aligned}$$

The key observation is that, for every fixed element $x \in U$ and every fixed multi-index $J \in \mathbb{N}_0^m$ with $|J| = k+1$, the second summand belongs to the image of the operator $d : \Omega_c^{n-1}(\mathbb{R}^n) \to \Omega_c^n(\mathbb{R}^n)$ and hence its integral over $\mathbb{R}^n$ vanishes by Stokes' Theorem 5.2.12. Thus integration over the fiber yields the $(k+1)$-form

$$\begin{aligned} \pi_* d\tau &= \sum_{|J|=k} \sum_{i=1}^m \left(\int_{\mathbb{R}^n} \frac{\partial \tau_{J,K_n}}{\partial x^i}(x,t)\, dt^1 \cdots dt^n \right) dx^i \wedge dx^J \\ &= \sum_{i=1}^m \frac{\partial}{\partial x^i} \left(\sum_{|J|=k} \int_{\mathbb{R}^n} \tau_{J,K_n}(x,t)\, dt^1 \cdots dt^n \right) dx^i \wedge dx^J = d\pi_*\tau. \end{aligned}$$

Here the second equation follows by interchanging differentiation and integration and the last equation follows from (7.2.7). This proves (7.2.4).

We prove equation (7.2.5) under the assumption that M is oriented and ω has compact support. Using a partition of unity on M we may again reduce the identity to a computation in local coordinates. Thus we assume that $\tau \in \Omega^{n+k}(U \times \mathbb{R}^n)$ is given by (7.2.6) and has vertical compact support as before, and that $\omega \in \Omega_c^{m-k}(U)$ has the form

$$\omega = \sum_{|I|=m-k} \omega_I(x)\, dx^I$$

Then both forms $\pi^*\omega \wedge \tau \in \Omega_c^{m+n}(U \times \mathbb{R}^n)$ and $\omega \wedge \pi_*\tau \in \Omega_c^m(U)$ have compact support. To compare their integrals it is convenient to define a number

$$\varepsilon(I, J) \in \{\pm 1\}$$

by

$$dx^I \wedge dx^J =: \varepsilon(I,J) dx^1 \wedge \cdots \wedge dx^m$$

for multi-indices I and J with

$$|I| = m - k, \qquad |J| = k.$$

With this setup we obtain

$$\begin{aligned}
&\int_U \omega \wedge \pi_*\tau \\
&= \sum_{|I|=m-k} \sum_{|J|=k} \int_U \omega_I(x) \left(\int_{\mathbb{R}^n} \tau_{J,K_n}(x,t) dt^1 \cdots dt^n \right) dx^I \wedge dx^J \\
&= \sum_{|I|=m-k} \sum_{|J|=k} \varepsilon(I,J) \int_U \int_{\mathbb{R}^n} \omega_I(x) \tau_{J,K_n}(x,t) dt^1 \cdots dt^n dx^1 \cdots dx^m \\
&= \sum_{|I|=m-k} \sum_{|J|=k} \varepsilon(I,J) \int_{U \times \mathbb{R}^n} \omega_I(x) \tau_{J,K_n}(x,t) dx^1 \cdots dx^m dt^1 \cdots dt^n \\
&= \sum_{|I|=m-k} \sum_{|J|=k} \int_{U \times \mathbb{R}^n} \omega_I(x) \tau_{J,K_n}(x,t) dx^I \wedge dx^J \wedge dt^{K_n} \\
&= \int_{U \times \mathbb{R}^n} \pi^*\omega \wedge \tau.
\end{aligned}$$

Here the third equality follows from Fubini's thoerem. This proves (7.2.5) and Lemma 7.2.6. □

7.2.2 The Thom Isomorphism Theorem

Continue the standing assumption that M is a smooth m-manifold (possibly with boundary) and $\pi : E \to M$ is an oriented rank-n vector bundle. Equation (7.2.4) in Lemma 7.2.6 shows that the map $\pi_* : \Omega^{n+k}_{\mathrm{vc}}(E) \to \Omega^k(M)$ descends to de Rham cohomology.

Definition 7.2.7 (Finite Type). *The vector bundle E is said to have* **finite type** *iff there exists a finite open cover $M = U_1 \cup \dots \cup U_\ell$ such that E admits a trivialization over U_i for each i.*

Theorem 7.2.8 (Thom Isomorphism Theorem). *Let $\pi : E \to M$ be an oriented real rank-n vector bundle of finite type over a smooth m-manifold M (possibly with boundary). Then the homomorphism*

$$\pi_* : H^{n+k}_{\mathrm{vc}}(E) \to H^k(M) \tag{7.2.8}$$

is bijective for $k = 0, 1, \dots, m$. Moreover, $H^{n+k}_{\mathrm{vc}}(E) = 0$ for $k < 0$.

Proof. See page 276 and page 279. □

Definition 7.2.9 (Thom Form). *Let $\pi : E \to M$ be an oriented rank-n vector bundle over a smooth m-manifold M. A* **Thom form on** *E is a closed n-form $\tau \in \Omega^n_{\mathrm{vc}}(E)$ with vertical compact support such that $\pi_*\tau = 1$.*

By Theorem 7.2.8 every oriented vector bundle of finite type admits a Thom form and the difference of any two Thom forms is exact.

Corollary 7.2.10 (Thom Form). *Let $\pi : E \to M$ be an oriented real rank-n vector bundle of finite type over a smooth m-manifold M.*

(i) *Let $U \subset E$ be an open neighborhood of the zero section. Then there exists a compactly supported m-form $\tau \in \Omega^n_{\mathrm{vc}}(E)$ such that*

$$\mathrm{supp}(\tau) \subset U, \qquad d\tau = 0, \qquad \pi_*\tau = 1.$$

(ii) *Let $\tau \in \Omega^n_{\mathrm{vc}}(E)$ be closed. Then $\pi_*\tau = 0$ if and only if there exists an $(n-1)$-form form $\beta \in \Omega^{n-1}_{\mathrm{vc}}(E)$ such that $d\beta = \tau$.*

Proof. We prove part (ii). If $\beta \in \Omega^{n-1}_{\mathrm{vc}}(E)$, then the equation $\pi_* d\beta = 0$ follows directly from equation (7.2.4) in Lemma 7.2.6 with $k = -1$. (The proof shows that the equation continues to hold for $k < 0$.) Conversely, assume $\pi_*\tau = 0$. Then the existence of an $(n-1)$-form form $\beta \in \Omega^{n-1}_{\mathrm{vc}}(E)$ that satisfies $d\beta = \tau$ follows from the assertion in Theorem 7.2.8 that the homomorphism $\pi_* : H^n_{\mathrm{vc}}(E) \to H^0(M)$ is injective. This proves (ii).

We prove part (i). The existence of a Thom form $\tau \in \Omega^n_{\mathrm{vc}}(E)$ follows from the fact that the homomorphism $\pi_* : H^n_{\mathrm{vc}}(E) \to H^0(M)$ is surjective by Theorem 7.2.8. To obtain a Thom form with support in U, choose a smooth function $\lambda : M \to \mathbb{R}$ such that $\lambda \geq 1$ and

$$e \in \mathrm{supp}(\tau) \qquad \Longrightarrow \qquad \lambda(\pi(e))^{-1}e \in U.$$

Such a function can be constructed via a partition of unity subordinate to a suitable open cover. Define $f_t : E \to E$ by $f_t(e) := (t\lambda(\pi(e)) + 1 - t)e$ for $1 \leq t \leq \lambda$ and $e \in E$. Then $f_0 = \mathrm{id}$ and $\mathrm{supp}(f_1^*\tau) \subset U$. Moreover, the restriction of the homotopy to $E|_K$ is proper for every compact set $K \subset M$. Hence by Corollary 5.3.9 there exists an $(n-1)$-form $\beta \in \Omega^{n-1}_{\mathrm{vc}}(E)$ such that

$$f_1^*\tau - \tau = d\beta.$$

The n-form $f_1^*\tau \in \Omega^n_{\mathrm{vc}}(E)$ is closed and supported in U. Moreover, by (ii) it satisfies $\pi_*(f_1^*\tau) = 1$. This proves (i) and Corollary 7.2.10. □

Definition 7.2.11 (Thom Class). *Let $\pi : E \to M$ be an oriented rank-n vector bundle of finite type over a smooth m-manifold M. By Corollary 7.2.10 there exists a Thom form τ on E and its cohomology class is independent of the choice of τ. It is called the* **Thom class of** *E and will be denoted by*

$$\tau(E) := [\tau] \in H^n_{\mathrm{vc}}(E), \qquad \tau \in \Omega^n_{\mathrm{vc}}(E), \qquad d\tau = 0, \qquad \pi_*\tau = 1. \tag{7.2.9}$$

Corollary 7.2.12. *Let $\pi : E \to M$ be an oriented rank-n vector bundle of finite type over a smooth m-manifold M. Then the inverse of the isomorphism $\pi_* : H^{n+k}_{\mathrm{vc}}(E) \to H^k(M)$ is the map $\mathscr{T} : H^k(M) \to H^{n+k}_{\mathrm{vc}}(E)$ given by*

$$\mathscr{T}(a) := \pi^*a \cup \tau(E) \qquad \textit{for } a \in H^k(M). \tag{7.2.10}$$

Proof. Let $\tau \in \Omega^n_{\mathrm{vc}}(E)$ be a Thom form and let $\omega \in \Omega^k(M)$ be a closed k-form. Then $\mathscr{T}[\omega] = [\pi^*\omega \wedge \tau]$ and hence, by equation (7.2.3), we have

$$\pi_*\mathscr{T}[\omega] = [\pi_*(\pi^*\omega \wedge \tau)] = [\omega \wedge \pi_*\tau] = [\omega].$$

This shows that $\pi_* \circ \mathscr{T} = \mathrm{id}_{H^k(M)}$. The equation $\mathscr{T} \circ \pi_* = \mathrm{id}_{H^k_{\mathrm{vc}}(E)}$ then follows from the fact that π_* is injective. Tis proves Corollary 7.2.12. □

Exercise 7.2.13 (Pullback). Let $\pi : E \to M$ and $\pi' : E' \to M'$ be oriented rank-n vector bundles of finite type over smooth manifolds. Let $\phi : M' \to M$ and $\Phi : E' \to E$ be smooth maps such that $\pi' \circ \Phi = \phi \circ \pi$ and such that the map $\Phi_p := \Phi|_{E_p} : E_p \to E'_{\phi(p)}$ is an orientation preserving vector space isomorphism for every element $p \in M$. Prove that $\Phi^*\tau(E) = \tau(E') \in H^n_{\mathrm{vc}}(E')$.

We will give two proofs of Theorem 7.2.8. The first proof establishes the result in full generality and uses a Mayer–Vietoris argument. The second proof establishes the result in the special case where M is a compact oriented manifold without boundary. It circumvents the Mayer–Vietoris argument by using Poincaré duality.

First Proof of Theorem 7.2.8. Our first proof follows the argument given in Bott–Tu [3, Thm 6.17]. It has five steps. The second step is the Mayer–Vietoris sequence for de Rham cohomology with vertical compact support.

Step 1. *If E admits a trivialization, then $\pi_* : H^{n+k}_{\mathrm{vc}}(E) \to H^k(M)$ is bijective for every integer k.*

By Exercise 7.2.13 we may assume that $E = M \times \mathbb{R}^n$. For $i = 1, \dots, n$ integration over the fiber extends to a homomorphism

$$(\pi_i)_* : \Omega^{k+i}_{\mathrm{vc}}(M \times \mathbb{R}^i) \to \Omega^{k+i-1}_{\mathrm{vc}}(M \times \mathbb{R}^{i-1}).$$

Namely, let $t = (t_1, \dots, t_i)$ be the coordinates on $\mathbb{R}^i$ and write a differential form $\omega \in \Omega^{k+i}_{\mathrm{vc}}(M \times \mathbb{R}^i)$ as

$$\omega = \alpha_{t_i} \wedge dt_i + \beta_{t_i}$$

with $\alpha_{t_i} \in \Omega^{k+i-1}_{\mathrm{vc}}(M \times \mathbb{R}^{i-1})$ and $\beta_{t_i} \in \Omega^{k+i}_{\mathrm{vc}}(M \times \mathbb{R}^{i-1})$ and define

$$(\pi_i)_*\omega := \int_{-\infty}^{\infty} \alpha_{t_i}\, dt_i.$$

Then the proof of Theorem 6.3.8 carries over verbatim to the present setting and shows that the homomorphism $(\pi_i)_*$ descends to an isomorphism from $H^{k+i}_{\mathrm{vc}}(M \times \mathbb{R}^i)$ to $H^{k+i-1}_{\mathrm{vc}}(M \times \mathbb{R}^{i-1})$ for each i. Since

$$\pi_* = (\pi_1)_* \circ \cdots \circ (\pi_n)_* : H^{n+k}_{\mathrm{vc}}(M \times \mathbb{R}^n) \to H^k(M)$$

by Fubini's theorem, this proves Step 1.

Step 2. *Let U and V be open subsets of M such that $M = U \cup V$ and let $\rho_U, \rho_V : M \to [0,1]$ be a partition of unity subordinate to this cover. Then there is a long exact sequence*

$$\cdots H^\ell_{\mathrm{vc}}(E) \xrightarrow{i^*} H^\ell_{\mathrm{vc}}(E|_U) \oplus H^k(E|_V) \xrightarrow{j^*} H^\ell_{\mathrm{vc}}(E|_{U\cap V}) \xrightarrow{d^*} H^{\ell+1}_{\mathrm{vc}}(E) \cdots .$$

Here i^* and j^* are as in (6.2.1) *and the map $d^* : H^\ell_{\mathrm{vc}}(E|_{U\cap V}) \to H^{\ell+1}_{\mathrm{vc}}(E)$ is defined by $d^*[\omega] := [d^*\omega]$ for every closed ℓ-form $\omega \in \Omega^\ell_{\mathrm{vc}}(E|_{U\cap V})$, where $d^*\omega$ is given by $d^*\omega := (\pi^* d\rho_U) \wedge \omega$ on $E|_{U\cap V}$ and $d^*\omega := 0$ on $E|_{M\setminus(U\cap V)}$.*

This is proved verbatim as in Theorem 6.2.3.

Step 3. *Let $M = U \cup V$ as in Step 2. Then the following diagram commutes*

$$\begin{tikzcd}
H^{n+k}_{\mathrm{vc}}(E) \arrow[r, "i^*"] \arrow[d, "\pi_*"] & H^{n+k}_{\mathrm{vc}}(E|_U) \oplus H^{n+k}_{\mathrm{vc}}(E|_V) \arrow[r, "j^*"] \arrow[d, "\pi_*"] & H^{n+k}_{\mathrm{vc}}(E|_{U\cap V}) \arrow[r, "d^*"] \arrow[d, "\pi_*"] & H^{n+k+1}_{\mathrm{vc}}(E) . \arrow[d, "\pi_*"] \\
H^k(M) \arrow[r, "i^*"] & H^k(U) \oplus H^k(V) \arrow[r, "j^*"] & H^k(U \cap V) \arrow[r, "d^*"] & H^{k+1}(M)
\end{tikzcd}$$

That the first two squares commute follows directly from the definitions. To prove that the third square commutes, fix a k-form $\omega \in \Omega^{n+k}_{\mathrm{vc}}(E|_{U\cap V})$. Then

$$\pi_* d^*\omega = \pi_*\big((\pi^* d\rho_U) \wedge \omega\big) = (d\rho_U) \wedge \pi_*\omega = d^*\pi_*\omega$$

on $U \cap V$. Here the second equality follows from (7.2.3) in Lemma 7.2.6. Since both $\pi_* d^*\omega$ and $d^*\pi_*\omega$ vanish on $M \setminus (U \cap V)$, this proves Step 3.

Step 4. *Let $M = U \cup V$ as in Step 2. If the homomorphism*

$$\pi_* : H^{n+*}_{\mathrm{vc}}(E|_W) \to H^*(W)$$

is bijective for $W = U, V, U \cap V$, then it is bijective for $W = M$.

This follows directly from Step 3 and the Five Lemma 6.2.12.

Step 5. *We prove Theorem 7.2.8.*

Let $M = U_1 \cup \cdots \cup U_\ell$ be an open cover such that E admits a trivialization over U_i for each i. We prove the assertion by induction on ℓ. For $\ell = 1$ the assertion holds by Step 1. Thus assume $\ell \geq 2$ and assume by induction that the assertion holds with ℓ replaced by $\ell' \leq \ell - 1$. Define

$$U := U_1 \cup \cdots \cup U_{\ell-1}, \qquad V := U_\ell.$$

Then

$$U \cap V = (U_1 \cap U_\ell) \cup \cdots \cup (U_{\ell-1} \cap U_\ell)$$

admits a cover by at most $\ell - 1$ open sets over each of which the bundle E admits a trivialization. Hence it follows from the induction hypothesis that the homomorphism $\pi_* : H^{n+*}_{\mathrm{vc}}(E|_W) \to H^*(W)$ is bijective for $W = U, V, U \cap V$. Hence Step 4 asserts that it is bijective for $W = M$. This proves Step 5 and Theorem 7.2.8. □

Remark 7.2.14. The finite type hypothesis in Theorem 7.2.8 can be removed. The proof then requires sheaf theory and the Čech–de Rham complex. For details see Bott–Tu [3, Thm 12.2].

The second proof of Theorem 7.2.8 relies on the following lemma which characterizes Thom forms in the case where M is a compact oriented manifold without boundary (and so $\Omega^n_{\mathrm{vc}}(E) = \Omega^n_c(E)$).

Lemma 7.2.15. *Let $\pi : E \to M$ be an oriented real rank-n vector bundle over a compact oriented smooth m-manifold M without boundary. Denote by $\iota : M \to E$ the zero section, let $\lambda \in \mathbb{R}$, and let $\tau \in \Omega^n_c(E)$ be closed. Then the following are equivalent.*

(a) $\pi_*\tau = \lambda$.

(b) *Every m-form $\omega \in \Omega^m(M)$ satisfies $\int_E \pi^*\omega \wedge \tau = \lambda \int_M \omega$.*

(c) *Every closed m-form $\sigma \in \Omega^m(E)$ satisfies $\int_E \sigma \wedge \tau = \lambda \int_M \iota^*\sigma$.*

Proof. We prove that (a) is equivalent to (b). By Lemma 7.2.6 every m-form $\omega \in \Omega^m(M)$ satisfies the equation $\int_M \omega \wedge \pi_*\tau = \int_E \pi^*\omega \wedge \tau$. Condition (a) holds if and only if the term on the left is equal to $\lambda \int_M \omega$ for every ω, and (b) holds if and only if the term on the right is equal to $\lambda \int_M \omega$ for every ω. Thus (a) is equivalent to (b).

We prove that (b) is equivalent to (c). Since $\pi \circ \iota = \mathrm{id}_M$, every m-form $\omega \in \Omega^m(M)$ satisfies the equation $\iota^*\pi^*\omega = (\pi \circ \iota)^*\omega = \omega$. Hence (b) follows from (c) with $\sigma := \pi^*\omega$. Conversely, assume (b) and let $\sigma \in \Omega^m(E)$ be closed. Since the map $\iota \circ \pi : E \to E$ is the projection onto the zero section, it is homotopic to the identity via the homotopy $f_t(e) := te$ with $f_0 = \iota \circ \pi$ and $f_1 = \mathrm{id}$. Hence $\sigma - \pi^*\iota^*\sigma \in \Omega^m(E)$ is exact by Theorem 6.1.1. Since the n-form $\tau \in \Omega^n_c(E)$ is closed, this implies

$$\int_E \sigma \wedge \tau = \int_E \pi^*\iota^*\sigma \wedge \tau = \lambda \int_M \iota^*\sigma$$

by (b). Thus (b) implies (c) and this proves Lemma 7.2.15. □

Remark 7.2.16. A subset $U \subset E$ of a vector bundle is called **star shaped** iff it intersects each fiber of E in a star shaped set centered at zero, i.e.

$$e \in U, \quad 0 \le t \le 1 \qquad \Longrightarrow \qquad te \in U.$$

The proof of Lemma 7.2.15 shows that, if $U \subset E$ is a star shaped open neighborhood of the zero section and $\tau \in \Omega^n_c(E)$ satisfies

$$\mathrm{supp}(\tau) \subset U, \qquad d\tau = 0, \qquad \pi_*\tau = 1,$$

then (c) continues to hold for every closed m-form $\sigma \in \Omega^m(U)$. Namely, in this case the m-form $f_t^*\sigma$, with $f_t(e) = te$, is defined on all of U for $0 \le t \le 1$ and so $\sigma - \pi^*\iota^*\sigma = f_1^*\sigma - f_0^*\sigma$ is an exact m-form on U, by Theorem 6.1.1. Hence the integral of its exterior product with τ vanishes by Stokes' theorem.

Second Proof of Theorem 7.2.8. Assume M is a compact oriented smooth m-manifold without boundary and thus $H^{n+k}_{\mathrm{vc}}(E) = H^{n+k}_c(E)$. Then both manifolds M and E are oriented and have finite good covers and therefore satisfy Poincaré duality. With this understood, the proof has six Steps. The first three steps establish the existence and uniqueness of Thom forms.

Step 1. *Every $\beta \in \Omega^{n-1}_c(E)$ satisfies $\pi_* d\beta = 0$.*

By Stokes' Theorem 5.2.12, we have $\int_E \pi^*\omega \wedge d\beta = \int_E d(\pi^*\omega \wedge \beta) = 0$ for all $\omega \in \Omega^m(M)$. Hence $\pi_* d\beta = 0$ by Lemma 7.2.15 with $\lambda = 0$.

Step 2. *There exists a closed n-form $\tau \in \Omega^n_c(E)$ such that $\pi_*\tau = 1$.*

Let $\iota : M \to E$ be the inclusion of the zero section as in Example 7.1.15 and define the linear functional $\Lambda : H^m(E) \to \mathbb{R}$ by $\Lambda([\sigma]) := \int_M \iota^*\sigma$ for every closed m-form $\sigma \in \Omega^m(E)$. Since E is an oriented manifold and has a finite good cover it satisfies Poincaré duality, by Theorem 6.4.1. Hence there exists a closed n-form $\tau \in \Omega^n_c(E)$ such that $\int_E \sigma \wedge \tau = \Lambda([\sigma]) = \int_M \iota^*\sigma$ for every closed m-form $\sigma \in \Omega^m(E)$. By Lemma 7.2.15 this implies $\pi_*\tau = 1$.

Step 3. *If $\tau_0, \tau_1 \in \Omega^n_c(E)$ are closed and satisfy $\pi_*\tau_0 = \pi_*\tau_1 = 1$, then there exists a compactly supported form $\beta \in \Omega^{n-1}_c(E)$ such that $d\beta = \tau_1 - \tau_0$.*

Since $\pi_*(\tau_1 - \tau_0) = 0$ and $\tau_1 - \tau_0$ is closed, it follows from Lemma 7.2.15 with $\lambda = 0$ that $\int_E \sigma \wedge (\tau_1 - \tau_0) = 0$ for every closed m-form $\sigma \in \Omega^m(E)$. Hence Step 3 follows from Poincaré duality in Theorem 6.4.1.

Step 4. *Let $k \in \mathbb{Z}$. Then $H^{n+k}_c(E) \cong H^k(M)$ and $\dim(H^k(M)) < \infty$.*

Since M and E have finite good covers, the de Rham cohomology groups of M and E are finite-dimensional by Corollary 6.2.8 and Corollary 6.3.14. By Poincaré duality (Theorem 6.4.1) for E we have $H^{n+k}_c(E) \cong H^{m-k}(E)$. Moreover, the projection $\pi : E \to M$ is a homotopy equivalence and this implies $H^{m-k}(E) \cong H^{m-k}(M)$. This group vanishes for $k < 0$ and is isomorphic to $H^k(M)$ for $k \geq 0$ by Poncaré duality for M.

Step 5. *Let $k \in \{0, 1, \dots, m\}$ and let $\tau \in \Omega^n_c(E)$ be as in Step 2. Define the homomorphism $\mathscr{T} : H^k(M) \to H^{n+k}_c(E)$ by $\mathscr{T}[\omega] := [\pi^*\omega \wedge \tau]$ for every closed k-form $\omega \in \Omega^k(M)$. Then $\pi_* \circ \mathscr{T} = \mathrm{id}_{H^k(M)}$.*

By (7.2.3) we have $\pi_*\mathscr{T}[\omega] = [\pi_*(\pi^*\omega \wedge \tau)] = [\omega \wedge \pi_*\tau] = [\omega]$ for every closed k-form $\omega \in \Omega^k(M)$.

Step 6. *For $k = 0, 1, \dots, m$ the map $\pi_* : H^{n+k}_c(E) \to H^k(M)$ is bijective.*

The cohomology groups $H^{n+k}_c(E)$ and $H^k(M)$ are finite-dimensional and have the same dimension by Step 4. Moreover, it follows from Step 5 that the homomorphism $\pi_* : H^{n+k}_c(E) \to H^k(M)$ is surjective. Hence it is bijective. This completes the second proof of Theorem 7.2.8. □

7.2.3 Intersection Theory Revisited

It is interesting to review intersection theory in the light of the above results on the Thom class. We consider the following setting. Let M be an oriented (not necessarily compact) m-manifold without boundary and let

$$Q \subset M$$

be a compact oriented $(m-\ell)$-dimensional submanifold without boundary. Fix a Riemannian metric on M. For $\varepsilon > 0$ sufficiently small consider the ε-neighborhood $TQ_\varepsilon^\perp$ of the zero section in the normal bundle and the tubular ε-neighborhood $U_\varepsilon \subset M$ of Q. These sets are defined by

$$\begin{aligned} TQ_\varepsilon^\perp &:= \left\{ (q,v) \,\middle|\, \begin{array}{l} q \in Q,\, v \in T_qM, \\ v \perp T_qQ,\, |v| < \varepsilon \end{array} \right\}, \\ U_\varepsilon &:= \left\{ p \in M \,\middle|\, d(p,Q) = \min_{q\in Q} d(p,q) < \varepsilon \right\}. \end{aligned} \tag{7.2.11}$$

They are open and, for $\varepsilon > 0$ sufficiently small, the exponential map

$$\exp : TQ_\varepsilon^\perp \to U_\varepsilon$$

is a diffeomorphism by Theorem 4.3.8. We orient the normal bundle such that orientations match in the direct sum decomposition

$$T_qM = T_qQ \oplus T_qQ^\perp$$

for $q \in Q$. Choose a Thom form

$$\tau_\varepsilon \in \Omega_c^\ell(TQ^\perp)$$

such that

$$\operatorname{supp}(\tau_\varepsilon) \subset TQ_\varepsilon^\perp, \qquad d\tau_\varepsilon = 0, \qquad \pi_*\tau_\varepsilon = 1. \tag{7.2.12}$$

Such a form exists by Corollary 7.2.10. Now define the differential form

$$\tau_Q \in \Omega_c^\ell(M)$$

by

$$\tau_Q := \begin{cases} (\exp^{-1})^*\tau_\varepsilon & \text{on } U_\varepsilon, \\ 0 & \text{on } M \setminus U_\varepsilon. \end{cases} \tag{7.2.13}$$

This differential form is closed by definition. The next lemma shows that τ_Q is Poincaré dual to Q as explained in §6.4.3.

Lemma 7.2.17. *Let* $Q \subset M$ *and* $\tau_Q \in \Omega_c^\ell(M)$ *be as above. Then*

$$\int_M \omega \wedge \tau_Q = \int_Q \omega \tag{7.2.14}$$

for every closed $(m-\ell)$*-form* $\omega \in \Omega^{m-\ell}(M)$.

Proof. Denote the inclusion of the zero section in $TQ^\perp$ by

$$\iota_Q : Q \to TQ^\perp.$$

For every closed form $\omega \in \Omega^{m-\ell}(M)$ we compute

$$\begin{aligned} \int_M \omega \wedge \tau_Q &= \int_{U_\varepsilon} \omega \wedge \tau_Q \\ &= \int_{TQ_\varepsilon^\perp} \exp^* \omega \wedge \tau_\varepsilon \\ &= \int_Q \iota_Q^* \exp^* \omega \\ &= \int_Q (\exp \circ \iota_Q)^* \omega \\ &= \int_Q \omega. \end{aligned}$$

Here the third step follows from Lemma 7.2.15 and Remark 7.2.16, because the open set

$$TQ_\varepsilon^\perp \subset TQ^\perp$$

is a star shaped open neighborhood of the zero section. The last step follows from the fact that the map

$$\exp \circ \iota_Q : Q \to M$$

is just the inclusion of Q into M. This proves Lemma 7.2.17. □

When M has a finite good cover the existence of a closed ℓ-form τ_Q with compact support that is dual to Q, i.e. that satisfies equation (7.2.14) for every closed form $\omega \in \Omega^{m-\ell}(M)$, follows from Poincaré duality (§6.4.3). Lemma 7.2.17 gives us a geometrically explicit representative of this cohomology class that is supported in an arbitrarily small neighborhood of the submanifold Q. We will now show how this explicit representative can be used to relate the cup product in cohomology to the intersection numbers of submanifolds.

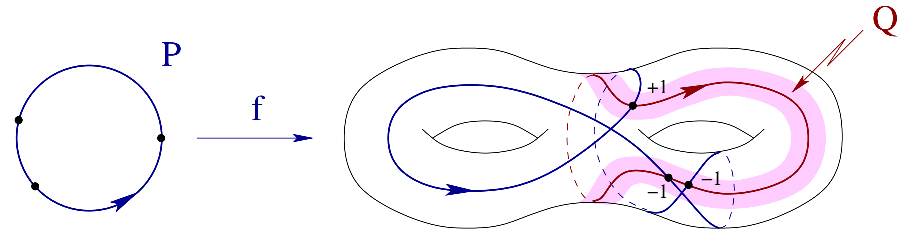

Figure 7.1: The intersection number of Q and f.

Theorem 7.2.18. *Let $Q \subset M$ and $\tau_Q \in \Omega_c^\ell(M)$ be as in Lemma 7.2.17. Let P be a compact oriented smooth ℓ-dimensional manifold without boundary and let $f : P \to M$ be a smooth map that is transverse to Q. Then*

$$Q \cdot f = \int_P f^* \tau_Q. \tag{7.2.15}$$

Proof. By assumption $f^{-1}(Q)$ is a finite set (see Figure 7.1). We denote it by $f^{-1}(Q) =: \{p_1, \dots, p_n\}$ and observe that

$$T_{f(p_i)}M = T_{f(p_i)}Q \oplus \operatorname{im} df(p_i), \qquad i = 1, \dots, n. \tag{7.2.16}$$

Since $\dim(P)+\dim(Q) = \dim(M)$, the derivative $df(p_i) : T_{p_i}P \to T_{f(p_i)}M$ is an injective linear map and hence its image inherits an orientation from $T_{p_i}P$. The intersection index $\nu(p_i; Q, f) \in \{\pm 1\}$ is obtained by comparing orientations in (7.2.16) (Definition 4.2.7). The intersection number of Q and f is the sum of the intersection indices $Q \cdot f = \sum_{i=1}^n \nu(p_i; Q, f)$ (Theorem 4.2.8).

It follows from the injectivity of $df(p_i)$ that the restriction of f to a sufficiently small neighborhood $V_i \subset P$ of p_i is an embedding. Its image is transverse to Q. Choosing $\varepsilon > 0$ sufficiently small and shrinking the V_i, if necessary, we may assume that the V_i are pairwise disjoint and that the tubular neighborhood $U_\varepsilon \subset M$ in (7.2.11) satisfies

$$f^{-1}(U_\varepsilon) = V_1 \cup V_2 \cup \cdots \cup V_n.$$

Since $\operatorname{supp}(\tau_Q) \subset U_\varepsilon$ we obtain $\operatorname{supp}(f^*\tau_Q) \subset f^{-1}(U_\varepsilon) = \bigcup_{i=1}^k V_i$ and hence

$$\int_P f^*\tau_Q = \sum_{i=1}^n \int_{V_i} f^*\tau_Q = \sum_{i=1}^n \int_{V_i} (\exp^{-1} \circ f)^* \tau_\varepsilon. \tag{7.2.17}$$

Here the second equation uses the exponential map $\exp : TQ_\varepsilon^\perp \to U_\varepsilon$ and the Thom form $\tau_\varepsilon = \exp^* \tau_Q \in \Omega_c^n(TQ^\perp)$ with support in $TQ_\varepsilon^\perp$.

Now choose a local trivialization

$$\psi_i : TQ^\perp|_{W_i} \to W_i \times \mathbb{R}^\ell$$

of the normal bundle $TQ^\perp$ over a contractible neighborhood $W_i \subset Q$ of $f(p_i)$ such that the open set $TQ^\perp_\varepsilon|_{W_i}$ is mapped diffeomorphically onto the product $W_i \times B_\varepsilon$. Here $B_\varepsilon \subset \mathbb{R}^\ell$ denotes the open ball of radius ε centered at zero. Let $\tau_i \in \Omega^\ell(W_i \times B_\varepsilon)$ be the Thom form defined by $\psi_i^*\tau_i = \tau_\varepsilon$. Then, by equation (7.2.17), we have

$$\int_P f^*\tau_Q = \sum_{i=1}^n \int_{V_i} (\exp^{-1} \circ f)^*\tau_\varepsilon = \sum_{i=1}^n \int_{V_i} (\psi_i \circ \exp^{-1} \circ f)^*\tau_i. \qquad (7.2.18)$$

Consider the composition

$$f_i := \mathrm{pr}_2 \circ \psi_i \circ \exp^{-1} \circ f|_{V_i} : V_i \to B_\varepsilon.$$

If $\varepsilon > 0$ is chosen sufficiently small, then this is a diffeomorphism; it is orientation preserving if $\nu(p_i; Q, f) = 1$, and is orientation reversing if $\nu(p_i; Q, f) = -1$. Since W_i is contractible, there exists a homotopy $h_t : V_i \to W_i$ such that

$$h_0 \equiv f(p_i), \qquad h_1 = \mathrm{pr}_1 \circ \psi_i \circ \exp^{-1} \circ f|_{V_i} : V_i \to W_i.$$

Thus

$$h_1 \times f_i = \psi_i \circ \exp^{-1} \circ f|_{V_i} : V_i \to W_i \times B_\varepsilon.$$

Moreover, the pullback of the Thom form $\tau_i \in \Omega^\ell(W_i \times B_\varepsilon)$ under the homotopy $h_t \times f_i$ has compact support in $[0,1] \times V_i$.

With this notation in place it follows from Corollary 5.3.9 and Stokes' Theorem 5.2.12 that

$$\begin{aligned}\int_{V_i} (\psi_i \circ \exp^{-1} \circ f)^*\tau_i = \int_{V_i} (h_1 \times f_i)^*\tau_i &= \int_{V_i} (h_0 \times f_i)^*\tau_i \\ &= \nu(p_i; Q, f) \int_{\{f(p_i)\} \times B_\varepsilon} \tau_i \\ &= \nu(p_i; Q, f).\end{aligned}$$

Here the third equality follows from Exercise 5.2.11 and the last equality follows from the fact that the integral of τ_i over each slice $\{q\} \times B_\varepsilon$ is equal to one. Combining this with (7.2.18) we find

$$\int_P f^*\tau_Q = \sum_{i=1}^n \int_{V_i} (\psi_i \circ \exp^{-1} \circ f)^*\tau_i = \sum_{i=1}^n \nu(p_i; Q, f) = Q \cdot f.$$

This proves Theorem 7.2.18. □

Proof of Theorem 6.4.8. By Lemma 7.2.17, the closed ℓ-form $\tau_Q \in \Omega_c^\ell(M)$, constructed in (7.2.13) via the Thom class on the normal bundle $TQ^\perp$, is Poincaré dual to Q as in §6.4.3. Thus Theorem 7.2.18 yields

$$Q \cdot f = \int_P f^*\tau_Q = \int_M \tau_Q \wedge \tau_f = (-1)^{\ell(m-\ell)} \int_Q \tau_f.$$

Here the second equality follows from the definition of the cohomology class $[\tau_f] \in H_c^{m-\ell}(M)$, Poincaré dual to the map f, via equation (6.4.6) in §6.4.3 with $\omega = \tau_Q$. The last equality holds by Lemma 7.2.17 with $\omega = \tau_f$. This proves Theorem 6.4.8. □

Let P and Q be compact oriented submanifolds of M without boundary and suppose that

$$\dim(P) + \dim(Q) = \dim(M).$$

Then Theorem 6.4.8 asserts that

$$P \cdot Q = \int_M \tau_P \wedge \tau_Q.$$

By Lemma 7.2.17 we may choose τ_P and τ_Q with support in arbitrarily small neighborhoods of P and Q, respectively, arising from Thom forms on the normal bundles as in (7.2.13). If P is transverse to Q, then the exterior product $\tau_P \wedge \tau_Q$ is supported near the intersection points of P and Q, and the contribution to the integral is precisely the intersection number near each intersection point. This is the geometric content of Theorem 6.4.8.

Example 7.2.19. Consider the manifold $M = \mathbb{R}^2$ and the submanifolds

$$P = \mathbb{R} \times \{0\}, \qquad Q = \{0\} \times \mathbb{R},$$

Thus P and Q are the x-axis and the y-axis, respectively, in the Euclidean plane with their standard orientations. We choose Thom forms

$$\tau_P = \rho(y)\, dy, \qquad \tau_Q = -\rho(x)\, dx,$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is a smooth compactly supported function with integral one. Then the exterior product

$$\tau_P \wedge \tau_Q = \rho(x)\rho(y) dx \wedge dy$$

is a compactly supported 2-form on $\mathbb{R}^2$ with integral one. This is also the intersection index of P and Q at the unique intersection point.

7.3 The Euler Class

In §7.3.1 we introduce the Euler number of an oriented rank-m vector bundle over a compact oriented m-manifold without boundary as the self-intersection number of the zero section. In analogy to the Poincaré–Hopf Theorem this number can also be defined as the algebraic count of the zeros of a section with only isolated zeros, and it agrees with the integral of the pullback of a Thorm form under a section. More generally, the Euler class is the pullback of the Thom class under a section, whether or not the rank agrees with the dimension of the base. §7.3.2 establishes the basic properties of the Euler class and shows that it is Poincaré dual to the zero set of a transverse section. The Euler class is used in §7.3.3 to establish the product structure on the de Rham cohomology of complex projective space.

7.3.1 The Euler Number

Let $\pi : E \to M$ be a vector bundle. To define the Euler number of E under suitable hypotheses, we will specialize Theorem 7.2.18 to the case where M is replaced by E, the submanifold Q is replaced by the zero section $Z = \{0_p \,|\, p \in M\} \subset E$, and the map $f : P \to M$ is replaced by a section $s : M \to E$. In this case the normal bundle of Z is the vector bundle E itself, and the dimension condition $\dim(P) + \dim(Q) = \dim(M)$ in intersection theory translates into the condition $\operatorname{rank}(E) = \dim(M) = m$.

Definition 7.3.1 (Vertical Derivative). *Let $s : M \to E$ be a section of a vector bundle. A point $p \in M$ is called a* **zero of** s *if $s(p) = 0_p \in E_p$ is the zero element of the fiber $E_p = \pi^{-1}(p)$. The* **vertical derivative of** s **at a zero** $p \in M$ *is the liner map $Ds(p) : T_pM \to E_p$ defined as follows. Let $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ be a local trivialization such that $p \in U_\alpha$ and consider the vector space isomorphism $\psi_\alpha(p) := \mathrm{pr}_2 \circ \psi_\alpha|_{E_p} : E_p \to V$ and the section in local coordinates $s_\alpha := \mathrm{pr}_2 \circ \psi_\alpha \circ s|_{U_\alpha} : U_\alpha \to V$. Then the vertical derivative $Ds(p) : T_pM \to E_p$ is defined by*

$$Ds(p)v := \psi_\alpha(p)^{-1} ds_\alpha(p)v \tag{7.3.1}$$

for $v \in T_pM$. Thus we have a commutative diagram

$$\begin{tikzcd}
 & V & \\
T_pM \arrow[ur, "ds_\alpha(p)"] \arrow[rr, "Ds(p)"] & & E_p \arrow[ul, "\psi_\alpha(p)"']
\end{tikzcd}.$$

The reader may verify that the linear map (7.3.1) *is independent of the choice of α with $p \in U_\alpha$ (provided that $s(p) = 0_p$).*

Exercise 7.3.2. Show that there is a natural splitting

$$T_{0_p}E \cong T_pM \oplus E_p, \qquad p \in M, \tag{7.3.2}$$

of the tangent bundle of E along the zero section. The inclusion of T_pM into $T_{0_p}E$ is given by the derivative of the zero section. If $s : M \to E$ is a section and $p \in M$ is a zero of s, show that $Ds(p) : T_pM \to E_p$ is the composition of the usual derivative $ds(p) : T_pM \to T_{0_p}E$ with the projection $T_{0_p}E \to E_p$ onto the vertical subspace in the splitting (7.3.2).

Exercise 7.3.3. Show that a section $s : M \to E$ is **transverse to the zero section** if and only if the vertical derivative $Ds(p) : T_pM \to E_p$ is surjective for every $p \in M$ with $s(p) = 0_p$. We write $s \pitchfork 0$ to mean that s is transverse to the zero section.

Exercise 7.3.4. Let E be a real rank-n vector bundle over a smooth m-manifold M and let $s : M \to E$ be a smooth section of E. Assume s is transverse to the zero section. Then the **zero set**

$$s^{-1}(0) := \{p \in M \,|\, s(p) = 0_p\}$$

of s is a smooth submanifold of M of dimension $m - n$ and

$$T_ps^{-1}(0) = \ker\, Ds(p)$$

for every $p \in M$ with $s(p) = 0_p$. **Hint:** Use Lemma 4.1.2.

Exercise 7.3.5 (Transversality). Let $\pi : E \to M$ be a vector bundle of finite type over a manifold with boundary. Prove that there exists a smooth section $s : M \to E$ such that s and $s|_{\partial M}$ are transverse to the zero section.

Hint 1: Show that there exist finitely many sections

$$s_1, \dots, s_\ell : M \to E$$

such that the vectors $s_1(p), \dots, s_\ell(p)$ span the fiber E_p for every $p \in M$ (see Exercise 7.1.16 and Step 1 in the proof of Lemma 4.1.6).

Hint 2: Define the map $\mathscr{S} : \mathbb{R}^\ell \times M \to E$ by

$$\mathscr{S}(\lambda, p) := \sum_{i=1}^{\ell} \lambda_i s_i(p) \in E_p$$

for $\lambda = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{R}^\ell$ and $p \in M$. This is a section of the pullback bundle $\mathbb{R}^\ell \times E$ over $\mathbb{R}^\ell \times M$. Prove that $\mathscr{S}$ and $\mathscr{S}|_{\mathbb{R}^\ell \times \partial M}$ are transverse to the zero section.

Hint 3: Use Exercise 7.3.4 to show that the set

$$\mathscr{Z} := \left\{(\lambda, p) \in \mathbb{R}^\ell \times M \,\middle|\, \mathscr{S}(\lambda, p) = 0_p\right\}$$

is a smooth submanifold of $\mathbb{R}^\ell \times M$ with boundary $\partial\mathscr{Z} = \mathscr{Z} \cap (\mathbb{R}^\ell \times \partial M)$.
Hint 4: Let $\lambda \in \mathbb{R}^\ell$ be a common regular value of the projections

$$\mathscr{Z} \to \mathbb{R}^\ell : (\lambda, p) \mapsto \lambda, \qquad \partial\mathscr{Z} \to \mathbb{R}^\ell : (\lambda, p) \mapsto \lambda.$$

Define the section $s : M \to E$ by $s(p) := \mathscr{S}(\lambda, p)$ for $p \in M$. Prove that both s and $s|_{\partial M}$ are transverse to the zero section.

Corollary 7.3.6 (**Euler Number**). *Let E be an oriented vector bundle of rank m over a compact oriented m-manifold M without boundary and let $\tau \in \Omega_c^m(E)$ be a Thom form. Let $s : M \to E$ be a smooth section that is transverse to the zero section and define the* **index of a zero** *p of s by*

$$\iota(p, s) := \begin{cases} +1, & \text{if } Ds(p) : T_pM \to E_p \text{ is orientation preserving,} \\ -1, & \text{if } Ds(p) : T_pM \to E_p \text{ is orientation reversing.} \end{cases} \qquad (7.3.3)$$

Then

$$\int_M s^*\tau = Z \cdot s = \sum_{s(p)=0_p} \iota(p, s). \qquad (7.3.4)$$

This integral is independent of s and is called the **Euler number of** *E.*

Proof. The second equality in (7.3.4) follows from Theorem 7.2.18. The third equality in (7.3.4) follows from the definition of the intersection number in Theorem 4.2.8 and the fact that the intersection index of the zero section Z with $s(M)$ at a zero p of s is $\iota(p, s)$. That the Euler number is independent of s follows from the homotopy invariance of the intersection number or, alternatively, from Cartan's formula and the fact that the Thom form τ is closed. This proves Corollary 7.3.6. □

Exercise 7.3.7. Let E be as in Corollary 7.3.6. Define the index $\iota(p, s) \in \mathbb{Z}$ of an **isolated zero of a section** $s : M \to E$. Prove that equation (7.3.4) in Corollary 7.3.6 continues to hold for sections with only isolated zeros. **Hint:** See the proof of the Poincaré–Hopf Theorem.

By Corollary 7.3.6 the Euler number is the self-intersection number of the zero section in E. One can show as in Chapter 2 that the right hand side in (7.3.4) is independent of the choice of the section s, assuming it is transverse to the zero section, and use this to define the Euler number of E in the case $\operatorname{rank}(E) = \dim(M)$. Thus the definition of the Euler number extends to the case where E is an orientable manifold (and M is not).

Example 7.3.8 (Euler characteristic). Consider the tangent bundle of a compact oriented m-manifold M without boundary. A section of $E = TM$ is a vector field $X \in \mathrm{Vect}(M)$ and it is transverse to the zero section if and only if its zeros are all nondegenerate. Hence, by Corollary 7.3.6 the **Euler number of the tangent bundle** is given by

$$\int_M X^*\tau = \sum_{X(p)=0} \iota(p, X)$$

for every vector field X with only nondegenerate zeros and every Thom form $\tau \in \Omega^m_c(TM)$. This gives another proof of the part of the Poincaré–Hopf Theorem 2.3.1 which asserts that the sum of the indices of the zeros of a vector field (with only nondegenerate zeros) is a topological invariant. By Theorem 6.4.9 this invariant is given by

$$\int_M X^*\tau = \chi(M) = \sum_{i=0}^{m} (-1)^i \dim(H^i(M)).$$

In other words, *the Euler number of the tangent bundle of M is the Euler characteristic of M.*

Example 7.3.9 (Self-Intersection Number). Let M be an oriented Riemannian $2n$-manifold without boundary and let $Q \subset M$ be a compact oriented n-dimensional submanifold without boundary. Then by Theorem 4.3.7 and Corollary 7.3.6, the Euler number of the normal bundle $TQ^\perp$ is the self-intersection number $Q \cdot Q$ (see Remark 4.3.17 and Corollary 7.3.13).

Exercise 7.3.10 (Complex Line Bundles over $\mathbb{C}\mathrm{P}^1$). Think of $\mathbb{C}\mathrm{P}^1$ as the space of all 1-dimensional complex linear subspaces $\ell \subset \mathbb{C}^2$. Let $d \in \mathbb{Z}$ and consider the complex line bundle $H^d \to \mathbb{C}\mathrm{P}^1$ defined by

$$H^d := \frac{(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}}{\mathbb{C}^*}, \qquad [z_0 : z_1; \zeta] \equiv [\lambda z_0 : \lambda z_1; \lambda^d \zeta].$$

Here $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ denotes the multiplicative group of nonzero complex numbers. Think of H^d as an oriented real rank-2 vector bundle over $\mathbb{C}\mathrm{P}^1$. Prove that the Euler number of H^d is d. (**Hint:** Find a section of H^d that is transverse to the zero section and use (7.3.4).) Show that $H^{-1} \to \mathbb{C}\mathrm{P}^1$ is naturally isomorphic to the tautological bundle over $\mathbb{C}\mathrm{P}^1$ whose fiber over ℓ is the line ℓ itself. Show that $H \to \mathbb{C}\mathrm{P}^1$ is the bundle whose fiber over ℓ is the dual space $\mathrm{Hom}^{\mathbb{C}}(\ell, \mathbb{C})$. Show that the bundle H^d is isomorphic to H^{-d} by an isomorphism that is orientation reversing on each fiber.

7.3.2 The Euler Class

Let us now drop the condition that the rank of the bundle is equal to the dimension of the base. Instead of a characteristic number we will then obtain a characteristic de Rham cohomology class.

Definition 7.3.11 (Euler Class). *Let $\pi : E \to M$ be an oriented rank-n vector bundle of finite type over a smooth manifold M (possibly with boundary). The* **Euler class of** *E is the de Rham cohomology class*

$$e(E) := [s^*\tau] = s^*\tau(E) \in H^n(M)$$

where $\tau \in \Omega^n_{\mathrm{vc}}(E)$ is a Thom form on E and $s : M \to E$ is a smooth section.

Since any two sections of E are smoothly homotopic, it follows from Theorem 6.1.1 and Corollary 7.2.10 that the cohomology class of $s^*\tau$ is independent of the choices of s and τ. Thus the Euler class is well defined.

Remark 7.3.12 (Euler Class and Euler Number). Let $\pi : E \to M$ be an oriented rank-n vector bundle over a compact oriented n-manifold M without boundary. Then the integral of (a representative of) the cohomology class $e(E)$ over M is the Euler number by Corollary 7.3.6. It is denoted by

$$\int_M e(E) := \int_M s^*\tau,$$

where $\tau \in \Omega^m_c(E)$ is a Thom form and $s : M \to E$ is a smooth section.

Corollary 7.3.13 (Euler Class and Self-Intersection). *Let M be an oriented Riemannian $2n$-manifold without boundary, let $Q \subset M$ be an oriented n-dimensional submanifold without boundary, and denote by $TQ^\perp$ the normal bundle of Q. Then the Euler number of the normal bundle $TQ^\perp$ is the self-intersection number of Q, i.e.*

$$\int_Q e(TQ^\perp) = Q \cdot Q. \tag{7.3.5}$$

Proof. Let $\tau \in \Omega^n_c(TQ^\perp)$ be a Thom form and let $Y : Q \to TQ^\perp$ be a normal vector field with only nondegenerate zeros (see Exercise 7.3.5). Then

$$\int_Q e(TQ^\perp) = \int_Q Y^*\tau = \sum_{Y(p)=0_p} \iota(p, Y) = Q \cdot Q.$$

Here the second equality follows from the definition of the Euler class, the third equality follows from Corollary 7.3.6, and the last equality follows from Theorem 4.3.7. This proves Corollary 7.3.13. □

Remark 7.3.14 (Euler Class for Odd Rank). Let $\pi : M \to E$ be as in Definition 7.3.11. If $n = \mathrm{rank}(E)$ is odd, then

$$e(E) = 0,$$

To see this, choose a Thom form $\tau \in \Omega^n_{\mathrm{vc}}(E)$, let $\iota : M \to E$ be the zero section, and denote by $\psi : E \to E$ the involution given by

$$\psi(e) := -e$$

for $e \in E$. Then $\widetilde{\tau} := -\psi^*\tau \in \Omega^n_{\mathrm{vc}}(E)$ is another Thom form because n is odd. Hence there exists a $\beta \in \Omega^{n-1}_{\mathrm{vc}}(E)$ such that $d\beta = \tau - \widetilde{\tau} = \tau + \psi^*\tau$. This implies $d\iota^*\beta = \iota^*\tau + \iota^*\psi^*\tau = 2\iota^*\tau$ and hence $e(E) = [\iota^*\tau] = 0$.

The next theorem relates the Euler class to Poincaré duality. When E is an oriented rank-n bundle over a compact oriented m-manifold M without boundary, it shows that the Euler class $e(E) \in H^n(M)$ is Poincaré dual to the zero set $Q = s^{-1}(0)$ of a transverse section as explained in §6.4.3.

Theorem 7.3.15 (Euler Class and Integration). *Let $\pi : E \to M$ be an oriented rank-n vector bundle of finite type over an oriented m-manifold M without boundary and let $s : M \to E$ be a smooth section that is transverse to the zero section such that $s^{-1}(0)$ is a compact subset of M. Let $\tau \in \Omega^n_{\mathrm{vc}}(E)$ be a Thom form and let $\omega \in \Omega^{m-n}_c(M)$ be closed. Then*

$$\int_M \omega \wedge s^*\tau = \int_{s^{-1}(0)} \omega. \tag{7.3.6}$$

(See below for our choice of orientation of $s^{-1}(0)$.)

Proof. Choose a Riemannian metric on M. Orient the zero set

$$Q := s^{-1}(0) = \{q \in M \,|\, s(q) = 0_q\}$$

so that for each $q \in Q$ orientations match in the direct sum decomposition

$$T_qM = T_qQ \oplus T_qQ^\perp.$$

Here $T_qQ^\perp$ is oriented such that the isomorphism

$$Ds(q) : T_qQ^\perp \to E_q$$

is orientation preserving. Choose $\varepsilon > 0$ such that the map $\exp : TQ^\perp_\varepsilon \to U_\varepsilon$ in (7.2.11) is a diffeomorphism (Theorem 4.3.8).

Since the zero set of s is contained in U_ε, There exists a neighborhood $U \subset E$ of the zero section such that $s^{-1}(U) \subset U_\varepsilon$. For example, the set $U := E \setminus s(M \setminus U_\varepsilon)$ is an open neighborhood of the zero section with this property. Assume first that our Thom form τ is supported in U and so

$$\operatorname{supp}(s^*\tau) \subset s^{-1}(U) \subset U_\varepsilon.$$

This implies that the pullback of the differential form $s^*\tau \in \Omega^n(M)$ under the exponential map $\exp : TQ^\perp_\varepsilon \to U_\varepsilon$ defines a closed n-form

$$\tau_\varepsilon := \begin{cases} \exp^* s^*\tau & \text{in } TQ^\perp_\varepsilon, \\ 0 & \text{in } TQ^\perp \setminus TQ^\perp_\varepsilon \end{cases} \in \Omega^n_c(TQ^\perp).$$

We prove that

$$\pi_*\tau_\varepsilon = 1. \tag{7.3.7}$$

To see this, observe that the map $s \circ \exp : TQ^\perp_\varepsilon \to E$ sends $(q,0)$ to 0_q and agrees on the zero section up to first order with Ds. Hence we can homotop the map $s \circ \exp$ to the vector bundle isomorphism $Ds : TQ^\perp \to E|_Q$. An explicit homotopy $F : [0,1] \times TQ^\perp_\varepsilon \to E$ is given by

$$F(t,q,v) := f_t(q,v) := \begin{cases} t^{-1}s(\exp_q(tv)) \in E_{\exp_q(tv)}, & \text{if } t > 0, \\ Ds(q)v, & \text{if } t = 0, \end{cases}$$

for $q \in Q = s^{-1}(0)$ and $v \in T_qM$ such that $v \perp T_qQ$ and $|v| < \varepsilon$. That F is smooth can be seen by choosing local trivializations on E. Hence F is a smooth homotopy connecting the maps

$$f_0 = Ds, \qquad f_1 = s \circ \exp.$$

Moreover, F extends smoothly to the closure of $[0,1] \times TQ^\perp_\varepsilon$ and the image of the set $[0,1] \times \partial TQ^\perp_\varepsilon$ under F does not intersect the zero section of E. Shrinking U, if necessary, we may assume that

$$f_t(\partial TQ^\perp_\varepsilon) \subset M \setminus U, \qquad U \cap E|_Q \subset f_t(TQ^\perp_\varepsilon) \qquad \text{for } 0 \le t \le 1.$$

Choose the Thom form $\tau \in \Omega^n_c(E)$ with support in U. Then it follows from our choice of U that the forms $f_t^*\tau$ have uniform compact support in $TQ^\perp_\varepsilon$. Hence, for each $q \in Q$, we have

$$\int_{T_qQ^\perp_\varepsilon} \tau_\varepsilon = \int_{T_qQ^\perp_\varepsilon} (s \circ \exp)^*\tau = \int_{T_qQ^\perp_\varepsilon} f_1^*\tau = \int_{T_qQ^\perp_\varepsilon} f_0^*\tau = 1.$$

Here the last equality uses the fact that $f_0 = Ds : TQ^\perp \to E|_Q$ is an orientation preserving vector bundle isomorphism. Thus $\pi_*\tau_\varepsilon = 1$ as claimed.

Thus we have proved that $\tau_\varepsilon = (s \circ \exp)^*\tau$ is a Thom form on $TQ^\perp$ with support in $TQ^\perp_\varepsilon$. Hence τ_ε satisfies the conditions in (7.2.12) and the closed n-form $\tau_Q := s^*\tau \in \Omega^n(M)$ with support in U_ε satisfies condition (7.2.13), i.e. $\tau_Q|_{U_\varepsilon} = (\exp^{-1})^*\tau_\varepsilon$. Hence it follows from Lemma 7.2.17 that τ_Q satisfies equation (7.3.6) for every closed form $\omega \in \Omega^{m-n}(M)$. If ω has compact support, then the left hand side of (7.3.6) is independent of the choice of the Thom form τ by Corollary 7.2.10. This proves Theorem 7.3.15. □

Example 7.3.16. The hypothesis that ω has compact support cannot be removed in Theorem 7.3.15. Consider the trivial bundle $E = M \times \mathbb{R}^m$ over the oriented m-manifold $M := \{x \in \mathbb{R}^m \,|\, |x| > 1\}$, let $s : M \to E$ be the section $s(x) := (x, x)$, choose a Thom form $\tau \in \Omega^m(E)$ with support contained in $\{(x,\xi) \in E \,|\, |\xi| \geq 1\}$, and let $\omega = 1 \in \Omega^0(M)$. Then the left hand side of (7.3.6) vanishes while the right hand side is one.

Exercise 7.3.17. Deduce Corollary 7.3.6 from Theorem 7.3.15 as the special case where M is compact, $\mathrm{rank}(E) = \dim(M)$ so that $Q = s^{-1}(0)$ is a zero-dimensional manifold, and $\omega = 1 \in \Omega^0(M)$ is the constant function one.

Theorem 7.3.18 (Properties of the Euler class). *The Euler Class satisfies the following conditions.*

(Zero) *Let $\pi : E \to M$ be an oriented vector bundle of finite type over a smooth manifold M. If E admits a nowhere vanishing section, then the Euler class of E vanishes.*

(Functoriality) *Let $\pi : E \to M$ be an oriented vector bundle of finite type over a smooth manifold M and let $f : M' \to M$ be a smooth map defined on another smooth manifold M'. Then the pullback bundle $f^*E \to M'$ has finite type and its Euler class is the pullback of the Euler class of E, i.e.*

$$e(f^*E) = f^*e(E).$$

(Sum) *The Euler class of the Whitney sum of two oriented vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ of finite type over a smooth manifold M is the cup product of the Euler classes, i.e.*

$$e(E_1 \oplus E_2) = e(E_1) \cup e(E_2).$$

Proof. If $s : M \to E$ is a nowhere vanishing section, then the complement of the image of s is a neighborhood of the zero section. Hence, by Corollary 7.2.10 there exists a Thom form $\tau \in \Omega^n_{\mathrm{vc}}(E)$ with support in $E \setminus s(M)$. For this Thom form we have $s^*\tau = 0$ and this proves the *(Zero)* property.

To prove *(Functoriality)* recall that

$$f^*E = \left\{(p', e) \in M' \times E \,|\, f(p') = \pi(e)\right\}.$$

If E admits a trivialization over an open set $U \subset M$, then the pullback bundle admits a trivialization over the open set $f^{-1}(U) \subset M'$. Thus f^*E has finite type. Define the map $\widetilde{f} : f^*E \to E$ as the projection of the set $f^*E \subset M' \times E$ onto the second factor, i.e.

$$\widetilde{f}(p', e) := e$$

for $p' \in M'$ and $e \in E_{f(p')}$. Then the restriction of $\widetilde{f}$ to each fiber is an orientation preserving vector space isomorphism. Now let $n := \mathrm{rank}(E)$ and let $\tau \in \Omega^n_{\mathrm{vc}}(E)$ be a Thom form. Then $\widetilde{f}^*\tau \in \Omega^n_{\mathrm{vc}}(f^*E)$ is a Thom form on the pullback bundle by Exercise 7.2.13. Now let $s : M \to E$ be a section of E. Then there exists a section $f^*s : M' \to f^*E$ defined by

$$(f^*s)(p') := (p', s(f(p'))) \qquad \text{for } p' \in M'.$$

This section satisfies $\widetilde{f} \circ (f^*s) = s \circ f : M \to f^*M$ and hence

$$(f^*s)^*\widetilde{f}^*\tau = (\widetilde{f} \circ (f^*s))^*\tau = (s \circ f)^*\tau = f^*(s^*\tau).$$

This proves *(Functoriality)* of the Euler class.

To prove the *(Sum)* property abbreviate

$$E := E_1 \oplus E_2$$

and observe that there are two obvious projections

$$\mathrm{pr}_i : E \to E_i, \qquad i = 1, 2.$$

Let $n_i := \mathrm{rank}(E_i)$ and let $\tau_i \in \Omega^{n_i}_{\mathrm{vc}}(E_i)$ be a Thom form on E_i. Then

$$\tau := \mathrm{pr}_1^*\tau_1 \wedge \mathrm{pr}_2^*\tau_2 \in \Omega^{n_1+n_2}_{\mathrm{vc}}(E)$$

is a Thom form on E, by Fubini's theorem. A section $s : M \to E$ can be expressed as a direct sum $s = s_1 \oplus s_2$ of two sections $s_i : M \to E_i$. Then we have $\mathrm{pr}_i \circ s = s_i$ and hence

$$s^*\tau = s^*\left(\mathrm{pr}_1^*\tau_1 \wedge \mathrm{pr}_2^*\tau_2\right) = s_1^*\tau_1 \wedge s_2^*\tau_2.$$

This proves Theorem 7.3.18. □

7.3.3 The Product Structure on $H^*(\mathbb{C}\mathrm{P}^n)$

We examine the ring structure on the de Rham cohomology of $\mathbb{C}\mathrm{P}^n$, where multiplication is the cup product with unit

$$1 \in H^0(M).$$

We already know from Example 6.4.16 that the odd-dimensional de Rham cohomology vanishes and that

$$H^{2k}(\mathbb{C}\mathrm{P}^n) \cong \mathbb{R}, \qquad k = 0, 1, \dots, n.$$

Throughout we identify $\mathbb{C}\mathrm{P}^k$ with a submanifold of $\mathbb{C}\mathrm{P}^n$ when $k \le n$; thus

$$\mathbb{C}\mathrm{P}^k = \left\{ [z_0 : z_1 : \cdots : z_k : 0 : \cdots : 0] \in \mathbb{C}\mathrm{P}^n \,\middle|\, |z_0|^2 + \cdots + |z_k|^2 > 0 \right\}.$$

In particular $\mathbb{C}\mathrm{P}^0$ is the single point $[1 : 0 : \cdots : 0]$. Let

$$h \in H^2(\mathbb{C}\mathrm{P}^n)$$

be the class dual to the submanifold $\mathbb{C}\mathrm{P}^{n-1}$ as defined in §6.4.3. Thus

$$\int_{\mathbb{C}\mathrm{P}^n} a \cup h = \int_{\mathbb{C}\mathrm{P}^{n-1}} a \tag{7.3.8}$$

for every $a \in H^{2n-2}(\mathbb{C}\mathrm{P}^n)$.

Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ denote the multiplicative group of nonzero complex numbers and consider the **complex line bundle**

$$\pi : H \to \mathbb{C}\mathrm{P}^n$$

defined as the quotient

$$H := \frac{(\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}}{\mathbb{C}^*} \to \mathbb{C}\mathrm{P}^n,$$

where the equivalence relation is given by

$$[z_0 : z_1 : \cdots : z_n; \zeta] \equiv [\lambda z_0 : \lambda z_1 : \cdots : \lambda z_n; \lambda \zeta]$$

for $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, $\zeta \in \mathbb{C}$, and $\lambda \in \mathbb{C}^*$. The fibers of this bundle are one-dimensional complex vector spaces; hence the term *complex line bundle*. One can also think of H as an oriented real rank-2 bundle over $\mathbb{C}\mathrm{P}^n$.

Theorem 7.3.19. *For $k = 0, 1, \dots, n$ define the de Rham cohomology class $h^k \in H^{2k}(\mathbb{C}\mathrm{P}^n)$ as the k-fold cup product of h with itself, i.e.*

$$h^k := \underbrace{h \cup \cdots \cup h}_{k \text{ times}} \in H^{2k}(\mathbb{C}\mathrm{P}^n).$$

In particular, $h^0 = 1 \in H^0(\mathbb{C}\mathrm{P}^n)$ is the empty product and $h^1 = h$. These classes have the following properties.

(i) *h is the Euler class of the oriented real rank-2 bundle $H \to \mathbb{C}\mathrm{P}^n$.*

(ii) *The cohomology class h^k dual to the submanifold $\mathbb{C}\mathrm{P}^{n-k}$; thus, for every cohomology class $a \in H^{2n-2k}(\mathbb{C}\mathrm{P}^n)$, we have*

$$\int_{\mathbb{C}\mathrm{P}^n} a \cup h^k = \int_{\mathbb{C}\mathrm{P}^{n-k}} a. \tag{7.3.9}$$

(iii) *For $k = 0, \dots, n$ we have*

$$\int_{\mathbb{C}\mathrm{P}^k} h^k = 1. \tag{7.3.10}$$

(iv) *For every compact oriented $2k$-manifold P without boundary and every smooth map $f : P \to \mathbb{C}\mathrm{P}^n$ we have*

$$\int_P f^* h^k = f \cdot \mathbb{C}\mathrm{P}^{n-k}. \tag{7.3.11}$$

Proof. Geometrically one can think of $\mathbb{C}\mathrm{P}^n$ is as the set of complex one-dimensional subspaces of $\mathbb{C}^{n+1}$, i.e.

$$\mathbb{C}\mathrm{P}^n = \left\{\ell \subset \mathbb{C}^{n+1} \,\middle|\, \ell \text{ is a 1-dimensional complex subspace}\right\}.$$

The **tautological complex line bundle** over $\mathbb{C}\mathrm{P}^n$ is the bundle whose fiber over ℓ is the line ℓ itself. In this formulation H is the dual of the tautological bundle so that the fiber of H over $\ell \in \mathbb{C}\mathrm{P}^n$ is the dual space

$$H_\ell = \ell^* = \mathrm{Hom}^{\mathbb{C}}(\ell, \mathbb{C}).$$

Thus H can be identified with the set of all pairs (ℓ, ϕ) where $\ell \subset \mathbb{C}^{n+1}$ is a 1-dimensional complex subspace and $\phi : \ell \to \mathbb{C}$ is a complex linear map. (**Exercise:** Verify this.) In this second formulation every complex linear map $\Phi : \mathbb{C}^{n+1} \to \mathbb{C}$ defines a section $s : \mathbb{C}\mathrm{P}^n \to H$ which assigns to every $\ell \in \mathbb{C}\mathrm{P}^n$ the restriction $s(\ell) := \Phi|_\ell$. An example, in our previous formulation, is the projection onto the last coordinate:

$$s\big([z_0 : z_1 : \cdots : z_n]\big) := [z_0 : z_1 : \cdots : z_n; z_n].$$

This section is transverse to the zero section and its zero set is the projectve subspace $s^{-1}(0) = \mathbb{C}\mathrm{P}^{n-1} \subset \mathbb{C}\mathrm{P}^n$ with its complex orientation. By Theorem 7.3.15 the Euler class $e(H) \in H^2(\mathbb{C}\mathrm{P}^n)$ is dual to the zero set of any transverse section of H. Hence it follows from from our definitions that $h := e(H)$. This proves (i).

By Theorem 7.3.18 the restriction of h to each projective subspace $\mathbb{C}\mathrm{P}^{i+1}$ is the Euler class of the restriction of the bundle H. Hence

$$\int_{\mathbb{C}P^{i+1}} a \cup h = \int_{\mathbb{C}\mathrm{P}^i} a$$

for every $a \in H^{2i}(\mathbb{C}\mathrm{P}^n)$ by Theorem 7.3.15. By induction, we obtain

$$\int_{\mathbb{C}\mathrm{P}^{i+k}} a \cup h^k = \int_{\mathbb{C}\mathrm{P}^i} a$$

for all $i, k \geq 0$ with $i + k \leq n$ and every $a \in H^{2i}(\mathbb{C}\mathrm{P}^n)$. With $i = n - k$ this proves (ii) and, with $i = 0$ and $a = 1 \in H^0(\mathbb{C}\mathrm{P}^n)$, this proves (iii). Now let P be a compact oriented $2k$-manifold without boundary and let $f : P \to \mathbb{C}\mathrm{P}^n$ be a smooth map. By (ii) the cohomology class h^k is dual to the submanifold $Q := \mathbb{C}\mathrm{P}^{n-k}$ as in §6.4.3. Thus, by Theorem 6.4.8, we have

$$f \cdot \mathbb{C}\mathrm{P}^{n-k} = (-1)^{2k(2n-2k)} \int_P f^*h^k = \int_P f^*h^k.$$

This proves (iv) and Theorem 7.3.19. □

Corollary 7.3.20. *Let $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ be a smooth map. Then there exists an integer $d \in \mathbb{Z}$ such that*

$$L(f) = 1 + d + d^2 + \cdots + d^n. \tag{7.3.12}$$

Proof. Since $H^2(\mathbb{C}\mathrm{P}^n) = \mathbb{R}h$, there is a real number d such that $f^*h = dh$. To prove that d is an integer, we compute

$$d = d\int_{\mathbb{C}\mathrm{P}^1} h = \int_{\mathbb{C}\mathrm{P}^1} f^*h = (f|_{\mathbb{C}\mathrm{P}^1}) \cdot \mathbb{C}\mathrm{P}^{n-1} \in \mathbb{Z}.$$

Here the first equality uses (7.3.10) and the last equality uses (7.3.11). For $i = 0, 1, \ldots, n$ we have $H^{2i}(\mathbb{C}\mathrm{P}^n) = \mathbb{R}h^i$ by part (iii) of Theorem 7.3.19 and $f^*h^i = d^i h^i$, and hence $\mathrm{trace}(f^* : H^{2i}(\mathbb{C}\mathrm{P}^n) \to H^{2i}(\mathbb{C}\mathrm{P}^n)) = d^i$. Moreover, $H^k(\mathbb{C}\mathrm{P}^n) = 0$ in odd degrees, and so (7.3.12) follows from (6.4.10) in Theorem 6.4.9. This proves Corollary 7.3.20. □

Remark 7.3.21. Equation (7.3.9) can be viewed as a special instance of the general fact, not proved in this book, that the the cup product of two closed forms dual to transverse submanifolds $P, Q \subset M$ is dual to the intersection $P \cap Q$ (with the appropriate careful choice of orientations). Theorem 6.4.8 can also be interpreted as an example of this principle.

Remark 7.3.22. By equation (7.3.11), the class $h^k \in H^{2k}(\mathbb{C}\mathrm{P}^n)$ is **integral** in the sense that the integral of h^k over every compact oriented $2k$-dimensional submanifold $Q \subset \mathbb{C}\mathrm{P}^n$ without boundary is an integer. By equation (7.3.10), the class h^k generates the additive subgroup of all integral classes in $H^{2k}(\mathbb{C}\mathrm{P}^n)$ (also called the **integral lattice**) in the sense that every integral cohomology class in $H^{2k}(\mathbb{C}\mathrm{P}^n)$ is an integer multiple of h^k. Here we use the fact that $H^{2k}(\mathbb{C}\mathrm{P}^n)$ is a one-dimensional real vector space (see Example 6.4.16).

Remark 7.3.23. The definition of the *integral lattice* in Remark 7.3.22 is rather primitive but suffices for our purposes. The correct definition involves a cohomology theory over the integers such as, for example, the singular cohomology. De Rham's theorem asserts that the de Rham cohomology group $H^*_{\mathrm{dR}}(M)$ is naturally isomorphic to the singular cohomology $H^*_{\mathrm{sing}}(M;\mathbb{R})$ with real coefficients. Moreover, there is a natural homomorphism $H^*_{\mathrm{sing}}(M;\mathbb{Z}) \to H^*_{\mathrm{sing}}(M;\mathbb{R})$. The correct definition of the integral lattice $\Lambda \subset H^*_{\mathrm{dR}}(M)$ is as the subgroup (in fact the subring) of all those de Rham cohomology classes whose images under de Rham's isomorphism in $H^*_{\mathrm{sing}}(M;\mathbb{R})$ have integral lifts, i.e. belong to the image of the homomorphism $H^*_{\mathrm{sing}}(M;\mathbb{Z}) \to H^*_{\mathrm{sing}}(M;\mathbb{R})$. The relation between these two definitions of the integral lattice is not at all obvious. It is related to the question of which integral singular homology classes can be represented by submanifolds. However, in the case of $\mathbb{C}\mathrm{P}^n$ these subtleties do not play a role.

Remark 7.3.24. By Theorem 7.3.19 the cohomology class $h \in H^2(\mathbb{C}\mathrm{P}^n)$ is a multiplicative generator of $H^*(\mathbb{C}\mathrm{P}^n)$, i.e. every element $a \in H^*(\mathbb{C}\mathrm{P}^n)$ can be expressed as a sum $a = c_0 + c_1 h + c_2 h^2 + \cdots + c_n h^n$ with real coefficients c_i. Think of the c_i as the coefficients of a polynomial

$$p(u) = c_0 + c_1 u + c_2 u^2 + \cdots + c_n u^n$$

in one variable, so that $a = p(h)$. Thus we have a ring isomorphism

$$\frac{\mathbb{R}[u]}{\langle u^{n+1} = 0 \rangle} \longrightarrow H^*(\mathbb{C}\mathrm{P}^n) : p \mapsto p(h).$$

The integral lattice in $H^*(\mathbb{C}\mathrm{P}^n)$, as defined in Remark 7.3.22, is the image of the subring of polynomials with integer coefficients under this isomorphism.

We shall return to the Euler class of a real rank-2 bundle in §8.3.3 with an alternative definition and several examples.

Chapter 8

Connections and Curvature

This chapter begins in §8.1 by introducing the concepts of connection and parallel transport. §8.2 introduces the curvature of a connection, followed by a discussion of gauge transformations and flat connections. With the basic notions in place we turn to Chern–Weil theory in §8.3. As a first application we give another definition of the Euler class of an oriented real rank-2 bundle and discuss several examples. Our main application is the introduction of the Chern classes of complex vector bundles in §8.4. We list their axioms and prove their existence via Chern–Weil theory. Some applications of the Chern classes to geometric questions are discussed in §8.5. The chapter closes with a brief outlook to some deeper results in differential topology.

8.1 Connections

We introduce bundle valued differential forms and connections in §8.1.1, discuss parallel transport in §8.1.2, and examine structure groups in §8.1.3.

8.1.1 Vector Valued Differential Forms and Connections

Let $\pi : E \to M$ be a real rank-n vector bundle over a smooth m-manifold M. Fix an integer $k \geq 0$. A **differential k-form on M with values in E** is a collection of alternating k-forms

$$\omega_p : \underbrace{T_pM \times T_pM \times \cdots \times T_pM}_{k \text{ times}} \to E_p,$$

one for each $p \in M$, such that the map $M \to E : p \mapsto \omega_p(X_1(p), \dots, X_k(p))$ is a smooth section of E for every k vector fields $X_1, \dots, X_k \in \mathrm{Vect}(M)$.

The space of k-forms on M with values in E will be denoted by $\Omega^k(M,E)$. In particular, $\Omega^0(M,E)$ is the space of smooth sections of E. A k-form on M with values in E can also be defined as a smooth section of the vector bundle $\Lambda^k T^*M \otimes E \to M$. Thus

$$\Omega^k(M) = \Omega^0(M, \Lambda^k T^*M \otimes E).$$

Remark 8.1.1. The space $\Omega^k(M,E)$ of E-valued k-forms on M is a real vector space. Moreover, we can multiply an E-valued k-form on M by a smooth real valued function or by a real valued differential form on M using the pointwise exterior product. This gives a bilinear map

$$\Omega^\ell(M) \times \Omega^k(M,E) \to \Omega^{k+\ell}(M,E) : (\tau,\omega) \mapsto \tau \wedge \omega,$$

defined by the same formula as in the standard case where both forms are real valued. (See Definition 5.1.7.)

Remark 8.1.2. Let $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ be a family of local trivializations of E with transitions maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$. Then every global k-form $\omega \in \Omega^k(M, \mathbb{E})$ determines a family of local vector valued k-forms

$$\omega_\alpha := \mathrm{pr}_2 \circ \psi_\alpha \circ \omega|_{U_\alpha} \in \Omega^k(U_\alpha, V). \tag{8.1.1}$$

These local k-forms are related by

$$\omega_\beta = g_{\beta\alpha}\omega_\alpha. \tag{8.1.2}$$

Conversely, every collection of local k-forms $\omega_\alpha \in \Omega^k(U_\alpha, V)$ that satisfy (8.1.2) determine a global k-form $\omega \in \Omega^k(M,E)$ via (8.1.1).

Definition 8.1.3 (Connections). *Let $\pi : E \to M$ be a real vector bundle over a smooth manifold. A* **connection on** E *is a linear map*

$$\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$$

that satisfies the Leibniz rule

$$\nabla(fs) = f\nabla s + (df) \cdot s \tag{8.1.3}$$

for every $f \in \Omega^0(M)$ and every $s \in \Omega^0(M,E)$. For $p \in M$ and $v \in T_pM$ we write $\nabla_v s(p) := (\nabla s)_p(v) \in E_p$ and call this the **covariant derivative of** s **at** p **in the direction** v.

The archetypal example of a connection is the usual differential

$$d : \Omega^0(M) \to \Omega^1(M)$$

on the space of smooth real valued functions on M, thought of as sections of the trivial bundle $E = M \times \mathbb{R}$. This is a first order linear operator and the same works for vector valued functions. The next proposition shows that every connection is in a local trivialization given by a zeroth order perturbation of the operator d.

Proposition 8.1.4 (Connections). *Let $\pi : E \to M$ be a vector bundle over a smooth manifold with local trivializations*

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$$

and transitions maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(V).$$

(i) *E admits a connection.*

(ii) *For every connection ∇ on E there are 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$, called* **connection potentials***, such that*

$$(\nabla s)_\alpha = ds_\alpha + A_\alpha s_\alpha \tag{8.1.4}$$

for every $s \in \Omega^0(M, E)$, where $(\nabla s)_\alpha$ and s_α are defined by (8.1.1). The connection potentials satisfy the condition

$$A_\alpha = g_{\beta\alpha}^{-1} dg_{\beta\alpha} + g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} \tag{8.1.5}$$

for all α, β. Conversely, every collection of 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ satisfying (8.1.5) determine a connection ∇ on E via (8.1.4).

(iii) *If $\nabla, \nabla' : \Omega^0(M, E) \to \Omega^1(M, E)$ are connections on E, then there exists a 1-form $A \in \Omega^1(M, \mathrm{End}(E))$ such that*

$$\nabla' - \nabla = A.$$

Conversely, if ∇ is a connection on E, then so is $\nabla + A$ for every endomorphism valued 1-form $A \in \Omega^1(M, \mathrm{End}(E))$.

Proof. The proof has six steps.

Step 1. *For every section $s \in \Omega^0(M, E)$ and every connection ∇ on E we have $\mathrm{supp}(\nabla s) \subset \mathrm{supp}(s)$.*

Let $p_0 \in M \setminus \mathrm{supp}(s)$ and choose a smooth function $f : M \to [0,1]$ such that $f = 1$ on the support of s and $f = 0$ near p_0. Then $fs = s$ and hence

$$\nabla s = \nabla(fs) = f\nabla s + (df)s.$$

The right hand side vanishes near p_0 and hence ∇s vanishes at p_0. This proves Step 1.

Step 2. *For every connection ∇ on E and every α there is a 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ satisfying* (8.1.4).

Fix a compact subset $K \subset U_\alpha$. We first define the restriction of A_α to K. For this we choose a basis $e_1, \dots, e_n$ of V and a smooth cutoff function $\rho : M \to [0,1]$ with support in U_α such that $\rho \equiv 1$ in a neighborhood of K. For $i = 1, \dots, n$ let $s_i : M \to E$ be the smooth section defined by

$$s_i(p) := \begin{cases} \rho(p)\psi_\alpha(p)^{-1}e_i, & \text{for } p \in U_\alpha, \\ 0, & \text{for } p \in M \setminus U_\alpha. \end{cases}$$

For $p \in K$ define the linear map $(A_\alpha)_p : T_pM \to \mathrm{End}(V)$ by

$$(A_\alpha)_p(v)\sum_{i=1}^n \lambda_i e_i := \psi_\alpha(p)\sum_{i=1}^n \lambda_i \nabla_v s_i(p)$$

for $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $v \in T_pM$. By Step 1, the linear map $(A_\alpha)_p$ is independent of the choice of ρ and hence is defined for each $p \in U_\alpha$.

If $s \in \Omega^0(M, E)$ is supported in U_α, we take $K = \mathrm{supp}(s)$ and choose s_i as above. Then there exist smooth functions $f_i : M \to \mathbb{R}$, supported in K, such that

$$s = \sum_i f_i s_i, \qquad s_\alpha = \sum_i f_i e_i.$$

Hence, for $p \in K = \mathrm{supp}(s) \subset U_\alpha$, we have

$$\begin{aligned} (\nabla s)_\alpha(p; v) &= \psi_\alpha(p)\nabla_v s(p) = \psi_\alpha(p)\sum_i \nabla_v(f_i s_i)(p) \\ &= \psi_\alpha(p)\sum_i \Big(f_i(p)\nabla_v s_i(p) + (df_i(p)v)s_i(p)\Big) \\ &= (A_\alpha)_p(v)\sum_i f_i(p)e_i + \sum_i (df_i(p)v)e_i \\ &= (A_\alpha)_p(v)s_\alpha(p) + ds_\alpha(p)v. \end{aligned}$$

By Step 1, this continues to hold when s is not supported in U_α.

Step 3. *The* 1*-forms* $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ *in Step 2 satisfy* (8.1.5).

By definition we have $(\nabla s)_\beta = g_{\beta\alpha}(\nabla s)_\alpha$ and hence

$$ds_\beta + A_\beta s_\beta = g_{\beta\alpha}\left(ds_\alpha + A_\alpha s_\alpha\right)$$

on $U_\alpha \cap U_\beta$. Differentiating the identity $s_\beta = g_{\beta\alpha}s_\alpha$ we obtain

$$ds_\beta = g_{\beta\alpha}ds_\alpha + \left(dg_{\beta\alpha}\right)s_\alpha$$

and hence

$$\begin{aligned} A_\beta g_{\beta\alpha}s_\alpha &= A_\beta s_\beta \\ &= g_{\beta\alpha}A_\alpha s_\alpha + g_{\beta\alpha}ds_\alpha - ds_\beta \\ &= \bigl(g_{\beta\alpha}A_\alpha - dg_{\beta\alpha}\bigr)s_\alpha \end{aligned}$$

for every (compactly supported) smooth function $s_\alpha : U_\alpha \cap U_\beta \to V$. Thus $A_\beta g_{\beta\alpha} = g_{\beta\alpha}A_\alpha - dg_{\beta\alpha}$ on $U_\alpha \cap U_\beta$ and this proves Step 3.

Step 4. *Every collection of* 1*-forms* $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ *satisfying* (8.1.5) *determine a connection* ∇ *on* E *via* (8.1.4).

Reversing the argument in the proof of Step 3 we find that, for every smooth section $s \in \Omega^0(M, E)$, the local E-valued 1-form

$$T_pM \to E_p : v \mapsto \psi_\alpha(p)^{-1}\bigl(ds_\alpha(p)v + (A_\alpha)_p(v)s_\alpha(p)\bigr)$$

agrees on $U_\alpha \cap U_\beta$ with the corresponding 1-form with α replaced by β. Hence these 1-forms define a global smooth 1-form $\nabla s \in \Omega^1(M, E)$. This proves Step 4. In particular, we have now established assertion (ii).

Step 5. *We prove (iii).*

The difference of two connections ∇ and ∇' is linear over the functions, i.e. $(\nabla' - \nabla)(fs) = f(\nabla' - \nabla)s$ for all $f \in \Omega^0(M)$ and all $s \in \Omega^0(M, E)$. We leave it to the reader to verify that such an operator $\nabla' - \nabla$ is given by multiplication with an endomorphism valued 1-form. (**Hint:** See Step 2.)

Step 6. *We prove (i).*

Choose a partition of unity $\rho_\alpha : M \to [0, 1]$ subordinate to the cover $\{U_\alpha\}_\alpha$ and define $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ by

$$A_\alpha := \sum_\gamma \rho_\gamma g_{\gamma\alpha}^{-1}dg_{\gamma\alpha}. \tag{8.1.6}$$

These 1-forms satisfy (8.1.5) and hence determine a connection on E, by Step 4. This proves Proposition 8.1.4. □

Example 8.1.5. The Levi-Civita connection of a Riemannian metric is an example of a connection on the tangent bundle $E = TM$ (see [35]).

Exercise 8.1.6. Let $s : M \to E$ be a smooth section and $p \in M$ be a zero of s so that $s(p) = 0_p \in E_p$. Then the vertical derivative of s at p is the map

$$T_pM \to E_p : v \mapsto Ds(p)v = \nabla_v s(p)$$

for every connection ∇ on E. (See Definition 7.3.1.)

Just as the usual differential $d : \Omega^0(M) \to \Omega^1(M)$ extends to a family of linear operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$, so does a connection ∇ on a vector bundle E induce linear operators d^∇ on differential forms with values in E.

Proposition 8.1.7. *Let $\pi : E \to M$ be a vector bundle over a smooth manifold and $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ be a connection. Then there is a unique collection of operators*

$$d^\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$$

such that $d^\nabla = \nabla$ for $k = 0$ and

$$d^\nabla(\tau \wedge \omega) = (d\tau) \wedge \omega + (-1)^{\deg(\tau)} \tau \wedge d^\nabla \omega \tag{8.1.7}$$

for every $\tau \in \Omega^*(M)$ and every $\omega \in \Omega^*(M, E)$. In the local trivializations the operator d^∇ is given by

$$\left(d^\nabla \omega\right)_\alpha = d\omega_\alpha + A_\alpha \wedge \omega_\alpha \tag{8.1.8}$$

for $\omega \in \Omega^k(M, E)$ and $\omega_\alpha := \mathrm{pr}_2 \circ \pi_\alpha \circ \omega|_{U_\alpha} \in \Omega^k(U_\alpha, V)$.

Proof. Define $d^\nabla \omega$ by (8.1.8) and use equation (8.1.5) to show that $d^\nabla s$ is well defined. That this operator satisfies (8.1.7) is obvious from the definition. That equation (8.1.7) determines the operator d^∇ uniquely, follows from the fact that every k-form on M with values in E can be expressed as a finite sum of products of the form $\tau_i s_i$ with $\tau_i \in \Omega^k(M)$ and $s_i \in \Omega^0(M, E)$. This proves Proposition 8.1.7. □

Exercise 8.1.8. Show that

$$(d^\nabla \omega)(X, Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) + \omega([X, Y]) \tag{8.1.9}$$

for $\omega \in \Omega^1(M, E)$ and $X, Y \in \mathrm{Vect}(M)$ and

$$\begin{aligned}(d^\nabla \omega)(X, Y, Z) &= \nabla_X(\omega(Y, Z)) + \nabla_Y(\omega(Z, X)) + \nabla_Z(\omega(X, Y)) \\ &\quad - \omega(X, [Y, Z]) - \omega(Y, [Z, X]) - \omega(Z, [X, Y])\end{aligned} \tag{8.1.10}$$

for $\omega \in \Omega^2(M, E)$ and $X, Y, Z \in \mathrm{Vect}(M)$. **Hint:** Use (5.3.6) and (5.3.7).

8.1.2 Parallel Transport

Let ∇ be a connection on a vector bundle $\pi : E \to M$ over a smooth manifold. For every smooth path $\gamma : I \to M$ on an interval $I \subset \mathbb{R}$ the connection determines a collection of vector space isomorphisms

$$\Phi_\gamma^\nabla(t_1, t_0) : E_{\gamma(t_0)} \to E_{\gamma(t_1)}$$

between the fibers of E along γ satisfying

$$\Phi_\gamma^\nabla(t_2, t_1) \circ \Phi_\gamma^\nabla(t_1, t_0) = \Phi_\gamma^\nabla(t_2, t_0), \qquad \Phi_\gamma^\nabla(t, t) = \mathrm{id} \tag{8.1.11}$$

for $t, t_0, t_1, t_2 \in I$. These isomorphisms are called **parallel transport of ∇ along γ** and are defined as follows.

A **section of E along γ** is a smooth map $s : I \to E$ such that $\pi \circ s = \gamma$ or, equivalently, $s(t) \in E_{\gamma(t)}$ for every $t \in I$. Thus a section of E along γ is a section of the pullback bundle $\gamma^*E \to I$ and the space of sections of E along γ will be denoted by

$$\Omega^0(I, \gamma^*E) := \{s : I \to E \,|\, \pi \circ s = \gamma\}.$$

The connection determines a linear operator

$$\nabla : \Omega^0(I, \gamma^*E) \to \Omega^1(I, \gamma^*E),$$

which is called the **covariant derivative**, as follows. In the local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ a section $s \in \Omega^0(I, \gamma^*E)$ is represented by a collection of smooth curves $s_\alpha : I_\alpha \to V$ via

$$s_\alpha(t) =: \psi_\alpha(\gamma(t))s(t) \in V, \qquad t \in I_\alpha := \gamma^{-1}(U_\alpha). \tag{8.1.12}$$

These curves satisfy

$$s_\beta(t) = g_{\beta\alpha}(\gamma(t))s_\alpha(t), \qquad t \in I_\alpha \cap I_\beta \tag{8.1.13}$$

for all α, β. Conversely, any collection of smooth curves $s_\alpha : I_\alpha \to E$ satisfying (8.1.13) determines a smooth section of E along γ via (8.1.12). The covariant derivative $\nabla s(t) \in E_{\gamma(t)}$ is defined by

$$(\nabla s)_\alpha(t) = \dot{s}_\alpha(t) + A_\alpha(\dot{\gamma}(t))s_\alpha(t), \qquad t \in I_\alpha. \tag{8.1.14}$$

By (8.1.5) we have $(\nabla s)_\beta = g_{\beta\alpha}(\gamma)(\nabla s)_\alpha$ on $I_\alpha \cap I_\beta$ and hence the vector

$$\nabla s(t) := \psi_\alpha(\gamma(t))^{-1}(\nabla s)_\alpha(t) \in E_{\gamma(t)}, \qquad t \in I_\alpha, \tag{8.1.15}$$

is independent of the choice of α with $\gamma(t) \in U_\alpha$.

Let us fix a smooth curve $\gamma : I \to M$ and an *initial time* $t_0 \in I$. Then it follows from the theory of linear time dependent ordinary differential equations that, for every $e_0 \in E_{\gamma(t_0)}$, there is a unique section $s \in \Omega^0(I, \gamma^*E)$ along γ satisfying the initial value problem

$$\nabla s = 0, \qquad s(t_0) = e_0. \tag{8.1.16}$$

This section is called the **horizontal lift of γ through e_0**.

Definition 8.1.9 (Parallel Transport). *The* **parallel transport of ∇ along γ from t_0 to $t \in I$** *is the linear map*

$$\Phi_\gamma^\nabla(t, t_0) : E_{\gamma(t_0)} \to E_{\gamma(t)}$$

defined by

$$\Phi_\gamma^\nabla(t, t_0)e_0 := s(t) \tag{8.1.17}$$

for $e_0 \in E_{\gamma(t_0)}$, where $s \in \Omega^0(I, \gamma^*E)$ is the unique horizontal lift of γ through e_0.

Exercise 8.1.10. Prove that parallel transport satisfies (8.1.11).

Exercise 8.1.11 (Reparametrization). If $\phi : I' \to I$ is any smooth map between intervals and $\gamma : I \to M$ is a smooth curve, then

$$\Phi_{\gamma\circ\phi}^\nabla(t_1, t_0) = \Phi_\gamma^\nabla(\phi(t_1), \phi(t_0)) : E_{\gamma(\phi(t_0))} \to E_{\gamma(\phi(t_1))}$$

for all $t_0, t_1 \in I'$.

8.1.3 Structure Groups

Let $\mathrm{G} \subset \mathrm{GL}(V)$ be a Lie subgroup with Lie algebra

$$\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G} \subset \mathrm{End}(V)$$

Let $\pi : E \to M$ be a vector bundle with structure group G, local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$, and transition maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{G}$. The bundle of **endomorphisms of E** is defined by

$$\mathrm{End}(E) := \left\{ (p, \xi) \;\middle|\; \begin{array}{l} p \in M,\ \xi : E_p \to E_p \text{ is a linear map,} \\ p \in U_\alpha \implies \psi_\alpha(p) \circ \xi \circ \psi_\alpha(p)^{-1} \in \mathfrak{g} \end{array} \right\}. \tag{8.1.18}$$

Thus $\mathrm{End}(E)$ is a vector bundle whose fibers are isomorphic to the Lie algebra $\mathfrak{g}$. The space of sections of $\mathrm{End}(E)$ carries a Lie algebra structure, understood pointwise. Differential forms with values in $\mathrm{End}(E)$ are in local trivializations represented by differential forms on U_α with values in $\mathfrak{g}$.

Proposition 8.1.12. *Let* $\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$ *be a connection on* E *with connection potentials* $A_\alpha \in \Omega^0(U_\alpha, \mathrm{End}(V))$.

(i) *The connection potentials* $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ *take values in* $\mathfrak{g}$ *if and only if parallel transport preserves the structure group, i.e. for every smooth path* $\gamma : I \to M$ *and all* $t_0, t_1 \in I$ *with* $\gamma(t_0) \in U_\alpha$ *and* $\gamma(t_1) \in U_\beta$ *we have*

$$\psi_\beta(\gamma(t_1)) \circ \Phi_\gamma^\nabla(t_1, t_0) \circ \psi_\alpha(\gamma(t_0))^{-1} \in \mathrm{G}. \tag{8.1.19}$$

∇ *is called a* G**-connection on** E *if it satisfies these equivalent conditions.*

(ii) *If* ∇ *is a* G*-connection and* $A \in \Omega^1(M, \mathrm{End}(E))$*, then* $\nabla + A$ *is a* G *connection. If* $\nabla, \nabla' : \Omega^0(M,E) \to \Omega^1(M,E)$ *are* G*-connections, then*

$$\nabla' - \nabla \in \Omega^1(M, \mathrm{End}(E)).$$

(iii) *Every* G*-bundle admits a* G*-connection.*

Proof. It suffices to prove (i) for curves $\gamma : I \to U_\alpha$. If $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$, then

$$\xi(t) := A_\alpha(\dot\gamma(t)) \in \mathfrak{g}$$

for every $t \in I$. Thus $\xi : I \to \mathfrak{g}$ is a smooth curve in the Lie algebra of G and hence the differential equation

$$\dot g(t) + \xi(t) g(t) = 0, \qquad g(t_0) = \mathbb{1},$$

has a unique solution $g : I \to \mathrm{G} \subset \mathrm{GL}(V)$. Now parallel transport along γ from t_0 to t is given by

$$\Phi_\gamma(t, t_0) = \psi_\alpha(\gamma(t))^{-1} \circ g(t) \circ \psi_\alpha(\gamma(t_0)) : E_{\gamma(t_0)} \to E_{\gamma(t)}$$

and hence satisfies (8.1.19). Reversing this argument we see that (8.1.19) for every smooth path $\gamma : I \to U_\alpha$ implies $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$. This proves (i). Assertion (ii) follows follows immediately from (i) and Proposition 8.1.4. Assertion (iii) follows from the explicit formula (8.1.6) in the proof of Proposition 8.1.4. This proves Proposition 8.1.12. □

Example 8.1.13 (**Oriented Vector Bundle).** Let V be an oriented vector space and $\mathrm{G} = \mathrm{GL}^+(V)$ be the group of orientation preserving automorphisms of V. Vector bundles with structure group $\mathrm{GL}^+(V)$ are oriented vector bundles (see Definition 7.1.22).

Example 8.1.14 (Riemannian Vector Bundle). Let V be a finite-dimensional oriented real Hilbert space and $\mathrm{G} = \mathrm{SO}(V)$ be the group of orientation preserving orthogonal transformations of V. If $\pi : E \to M$ is a vector bundle with structure group $\mathrm{SO}(V)$, then the local trivializations induce orientations as well as inner products

$$E_p \times E_p \to \mathbb{R} : (e_1, e_2) \mapsto \langle e_1, e_2 \rangle_p$$

on the fibers. The inner products fit together smoothly in the sense that the map $M \to \mathbb{R} : p \mapsto \langle s_1(p), s_2(p) \rangle_p$ is smooth for every pair of smooth sections $s_1, s_2 \in \Omega^0(M, E)$. Such a family of inner products is called a **Riemannian structure** on E and a vector bundle E with a Riemannian structure is called a **Riemannian vector bundle**.

A connection ∇ on a Riemannian vector bundle $\pi : E \to M$ is called a **Riemannian connection** iff it satisfies the Leibniz rule

$$d\langle s_1, s_2 \rangle = \langle \nabla s_1, s_2 \rangle + \langle s_1, \nabla s_2 \rangle \tag{8.1.20}$$

for all $s_1, s_2 \in \Omega^0(M, E)$. **Exercise:** Prove that every oriented Riemannian vector bundle admits a system of local trivializations whose transition maps take values in $\mathrm{SO}(V)$. Prove that Riemannian connections are the $\mathrm{SO}(V)$-connections in Proposition 8.1.12. In other words, a connection is Riemannian if and only if parallel transport preserves the inner product. Prove that $\mathrm{End}(E)$ is the bundle of skew-symmetric endomorphisms of E.

Example 8.1.15 (Complex Vector Bundle). Let V be a complex vector space and $\mathrm{G} = \mathrm{GL}_{\mathbb{C}}(V)$ be the group of complex linear automorphisms of V. If $\pi : E \to M$ is a vector bundle with structure group $\mathrm{GL}_{\mathbb{C}}(V)$, then the local trivializations induce complex structures on the fibers of the vector bundle that fit together smoothly, i.e. a vector bundle automorphism

$$J : E \to E, \qquad J^2 = -1\!\!1.$$

The pair (E, J) is called a **complex vector bundle**.

A connection ∇ on a complex vector bundle $\pi : E \to M$ is called a **complex connection** iff it is complex linear, i.e.

$$\nabla(Js) = J\nabla s \tag{8.1.21}$$

for all $s \in \Omega^0(M, E)$. **Exercise:** Prove that every complex vector bundle admits a system of local trivializations whose transition maps take values in $\mathrm{GL}_{\mathbb{C}}(V)$. Prove that complex connections are the $\mathrm{GL}_{\mathbb{C}}(V)$-connections in Proposition 8.1.12. In other words, a connection is complex linear if and only if parallel transport is complex linear. Prove that $\mathrm{End}(E)$ is the bundle of complex linear endomorphisms of E.

Example 8.1.16 (Hermitian Vector Bundle). A **Hermitian vector space** is a complex vector space V equipped with a bilinear form

$$V \times V \to \mathbb{C} : (u, v) \mapsto \langle u, v\rangle_c$$

whose real part is an inner product and that is complex anti-linear in the first variable and complex linear in the second variable. Thus, for $u, v \in V$ and $\lambda \in \mathbb{C}$, we have

$$\langle \lambda u, v\rangle_c = \bar{\lambda}\langle u, v\rangle_c, \qquad \langle u, \lambda v\rangle_c = \lambda\langle u, v\rangle_c.$$

Such a bilinear form is called a **Hermitian form** on V. Note that the complex structure is skew-symmetric with respect to the inner product

$$\langle \cdot, \cdot\rangle := \mathrm{Re}\,\langle \cdot, \cdot\rangle_c,$$

and that any such inner product uniquely determines a Hermitian form. The group of **unitary automorphisms** of a Hermitian vector space V is

$$\mathrm{U}(V) := \{g \in \mathrm{GL}_{\mathbb{C}}(V) \,|\, \langle gu, gv\rangle_c = \langle u, v\rangle_c \,\forall\, u, v \in V\}.$$

For $V = \mathbb{C}^n$ we use the standard notation $\mathrm{U}(n) := \mathrm{U}(\mathbb{C}^n)$.

If $\pi : E \to M$ is a vector bundle with structure group $\mathrm{U}(V)$, then the local trivializations induce Hermitian structures on the fibers of the vector bundle that fit together smoothly. Thus E is both a complex and a Riemannian vector bundle and the complex structure is skew-symmetric with respect to the Riemannian structure:

$$\langle e_1, Je_2\rangle + \langle Je_1, e_2\rangle = 0, \qquad e_1, e_2 \in E_p.$$

The Hermitian form on the fibers of E is then given by

$$\langle e_1, e_2\rangle_c = \langle e_1, e_2\rangle + \mathbf{i}\langle Je_1, e_2\rangle, \qquad e_1, e_2 \in E_p.$$

A complex vector bundle with such a structure is called a **Hermitian vector bundle**. Every Hermitian vector bundle admits a system of local trivializations whose transition maps take values in $\mathrm{U}(V)$. Thus vector bundles with structure group $\mathrm{U}(V)$ are Hermitian vector bundles.

A connection ∇ on a Hermitian vector bundle $\pi : E \to M$ is called a **Hermitian connection** iff it is complex linear and Riemannian, i.e. iff it satisfies (8.1.20) and (8.1.21). Thus the Hermitian connections are the $\mathrm{U}(V)$-connections in Proposition 8.1.12. In other words, a connection is Hermitian if and only if parallel transport preserves the Hermitian structure. Moreover, End(E) is the bundle of skew-Hermitian endomorphisms of E.

Exercise 8.1.17. Every complex vector bundle E admits a Hermitian structure. Any two Hermitian structures on E are related by a complex linear automorphism of E. **Hint:** Let V be a complex vector space and $\mathscr{H}(V)$ be the space of Hermitian forms on V compatible with the given complex structure. Show that $\mathscr{H}(V)$ is a convex subset of a (real) vector space and that $\mathrm{GL}_{\mathbb{C}}(V)$ acts tansitively on $\mathscr{H}(V)$. Describe Hermitian structures in local trivializations.

Remark 8.1.18 (Pullback Connection). We close this section with a brief discussion of the pullback of a vector bundle and a connection under a smooth map. Let

$$\pi : E \to M$$

be a vector bundle with structure group $\mathrm{G} \subset \mathrm{GL}(V)$, local trivializations $\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V$, and transition maps $g_{\beta\alpha} : U_\alpha \times U_\beta \to \mathrm{G}$, let ∇ be a G-connection on E with connection potentials $A_\alpha^\nabla \in \Omega^1(U_\alpha, \mathfrak{g})$, and let

$$f : M' \to M$$

be a smooth map (between manifolds). Then the pullback bundle

$$f^*E = \big\{(p;, e) \in M' \times E \,|\, f(p') = \pi(e)\big\}$$

is a G-bundle over M' and that the G connection ∇ on E induces a G-connection $f^*\nabla$ on f^*E. To see this, note that the local trivializations of E induce local trivializations of the pullback bundle over $f^{-1}(U_\alpha)$ given by

$$f^*\psi_\alpha : f^*E|_{f^{-1}(U_\alpha)} \to f^{-1}(U_\alpha) \times V, \qquad (f^*\psi_\alpha)(p', e) := (p', \mathrm{pr}_2 \circ \psi_\alpha(e)).$$

Thus

$$(f^*\psi_\alpha)(p') = \psi_\alpha(f(p')) : (f^*E)_{p'} = E_{f(p')} \to V$$

for $p' \in f^{-1}(U_\alpha)$ and the resulting transition maps are given by

$$f^*g_{\beta\alpha} = g_{\beta\alpha} \circ f : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to \mathrm{G}.$$

The connection potentials of the **pullback connection** $f^*\nabla$ are, by definition, the 1-forms

$$A_\alpha^{f^*\nabla} := f^*A_\alpha^\nabla \in \Omega^1(f^{-1}(U_\alpha), \mathfrak{g}).$$

Thus f^*E is a G-bundle and $f^*\nabla$ is a G-connection on f^*E.
Exercise: Show that the 1-forms $A_\alpha^{f^*\nabla}$ satisfy equation (8.1.5) with $g_{\beta\alpha}$ replaced by $f^*g_{\beta\alpha}$ and hence define a G-connection on f^*E.
Exercise: Show that the covariant derivative of a section along a curve is an example of a pullback connection.

8.2 Curvature

We introduce the curvature of a connection and prove the Bianchi identity in §8.2.1, discuss gauge transformations in §8.2.2, and examine flat connections in §8.2.3.

8.2.1 Curvature and the Bianchi Identity

In contrast to the exterior differential on differential forms, the operator d^∇ does not, in general, define a cochain complex. The failure of $d^\nabla \circ d^\nabla$ to vanish gives rise to the definition of the curvature of a connection.

Proposition 8.2.1. *Let $\pi : E \to M$ be a vector bundle over a smooth manifold and $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ be a connection.*

(i) *There is a unique endomorphism valued 2-form $F^\nabla \in \Omega^2(M, \mathrm{End}(E))$, called the* **curvature of the connection** *∇, such that*

$$d^\nabla d^\nabla s = F^\nabla s \tag{8.2.1}$$

for every $s \in \Omega^0(M, E)$. In local trivializations the curvature is given by

$$(F^\nabla s)_\alpha = F_\alpha s_\alpha, \qquad F_\alpha := dA_\alpha + A_\alpha \wedge A_\alpha \in \Omega^2(U_\alpha, \mathrm{End}(V)). \tag{8.2.2}$$

Moreover, on $U_\alpha \cap U_\beta$ we have

$$g_{\beta\alpha} F_\alpha = F_\beta g_{\beta\alpha}. \tag{8.2.3}$$

(ii) *For every $\omega \in \Omega^k(M, E)$ we have*

$$d^\nabla d^\nabla \omega = F^\nabla \wedge \omega. \tag{8.2.4}$$

(iii) *For $X, Y \in \mathrm{Vect}(M)$ and $s \in \Omega^0(M, E)$ we have*

$$F^\nabla(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s + \nabla_{[X,Y]} s. \tag{8.2.5}$$

(iv) *If ∇ is a* G*-connection, then $F^\nabla \in \Omega^2(M, \mathrm{End}(E))$. (See* (8.1.18)*.)*

Proof. We prove (i). Define $F_\alpha \in \Omega^2(U_\alpha, \mathrm{End}(V))$ by (8.2.2). Then, for every $s \in \Omega^0(M, E)$, we have

$$\begin{aligned}(d^\nabla d^\nabla s)_\alpha &= d\big(ds_\alpha + A_\alpha s_\alpha\big) + A_\alpha \wedge \big(ds_\alpha + A_\alpha s_\alpha\big) \\ &= d(A_\alpha s_\alpha) + A_\alpha \wedge ds_\alpha + (A_\alpha \wedge A_\alpha) s_\alpha \\ &= \big(dA_\alpha + A_\alpha \wedge A_\alpha\big) s_\alpha \\ &= F_\alpha s_\alpha.\end{aligned} \tag{8.2.6}$$

Hence on $U_\alpha \cap U_\beta$:

$$g_{\beta\alpha}F_\alpha s_\alpha = g_{\beta\alpha}\left(d^\nabla d^\nabla s\right)_\alpha = \left(d^\nabla d^\nabla s\right)_\beta = F_\beta s_\beta = F_\beta g_{\beta\alpha} s_\alpha.$$

This shows that the F_α satisfy equation (8.2.3) and therefore determine a global endomorphism valued 2-form $F^\nabla \in \Omega^2(M, \mathrm{End}(E))$ via

$$(F^\nabla s)_\alpha := F_\alpha s_\alpha$$

for $s \in \Omega^0(M, .E)$. By (8.2.6) this global 2-form satisfies (8.2.1) and it is uniquely determined by this condition. Thus we have proved (i).

We prove (ii). Given $\tau \in \Omega^\ell(M)$ and $s \in \Omega^0(M, E)$, we have

$$\begin{aligned} d^\nabla d^\nabla(\tau s) &= d^\nabla\left((d\tau)s + (-1)^\ell \tau \wedge d^\nabla s\right) \\ &= \tau \wedge d^\nabla d^\nabla s = \tau \wedge F^\nabla s = F^\nabla \wedge (\tau s). \end{aligned}$$

Since every k-form $\omega \in \Omega^k(M, E)$ can be expressed as a finite sum of k-forms of the form τs we deduce that F^∇ satisfies (8.2.4) for all k. This proves (ii).

We prove (iii). Let $X, Y \in \mathrm{Vect}(M)$ and $s \in \Omega^0(M, E)$. It follows from equation 8.1.9 in Exercise 8.1.8 that

$$\begin{aligned} F^\nabla(X, Y)s &= \nabla_X\left(d^\nabla s(Y)\right) - \nabla_Y\left(d^\nabla s(X)\right) + d^\nabla s([X, Y]) \\ &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s + \nabla_{[X,Y]} s. \end{aligned}$$

This proves (iii).

We prove (iv). If ∇ is a G-connection, then

$$(F_\alpha)_p(u, v) = (dA_\alpha)_p(u, v) + [A_\alpha(u), A_\alpha(v)] \in \mathfrak{g}$$

for all $p \in U_\alpha$ and $u, v \in T_pM$. This proves (iv) and Proposition 8.2.1. □

Remark 8.2.2. A connection on a vector bundle $\pi : E \to M$ induces a connection on the endomorphism bundle $\mathrm{End}(E) \to M$. The corresponding operator

$$d^\nabla : \Omega^k(M, \mathrm{End}(E)) \to \Omega^{k+1}(M, \mathrm{End}(E))$$

is uniquely determined by the Leibniz rule

$$d^\nabla(\Phi s) = (d^\nabla \Phi)s + (-1)^{\deg(\Phi)}\Phi \wedge d^\nabla s$$

for $\Phi \in \Omega^k(M, \mathrm{End}(E))$ and $s \in \Omega^0(M, E)$. **Exercise:** If the operator d^∇ on $\Omega^*(M, \mathrm{End}(E))$ is defined by this formula, prove that

$$d^\nabla(\Phi \wedge \Psi) = (d^\nabla \Phi) \wedge \Psi + (-1)^{\deg(\Phi)}\Phi \wedge d^\nabla \Psi$$

for $\Phi, \Psi \in \Omega^*(M, \mathrm{End}(E))$. Deduce that the operator d^∇ on $\Omega^*(M, \mathrm{End}(E))$ arises from a connection on $\mathrm{End}(E)$.

Proposition 8.2.3 (**Bianchi Identity**). *Every connection ∇ on a vector bundle $\pi : E \to M$ satisfies the* **Bianchi identity**

$$d^\nabla F^\nabla = 0. \tag{8.2.7}$$

Proof 1. By definition of the operator

$$d^\nabla : \Omega^2(M, \mathrm{End}(E)) \to \Omega^3(M, \mathrm{End}(E))$$

we have

$$(d^\nabla F^\nabla)s = d^\nabla(F^\nabla s) - F^\nabla \wedge d^\nabla s = d^\nabla(d^\nabla d^\nabla s) - (d^\nabla d^\nabla)d^\nabla s = 0$$

for $s \in \Omega^0(M, E)$. □

Proof 2. In the local trivializations we have

$$\begin{aligned}\big((d^\nabla F^\nabla)s\big)_\alpha &= \big(d^\nabla F^\nabla s - F^\nabla \wedge d^\nabla s\big)_\alpha \\ &= d(F_\alpha s_\alpha) + A_\alpha \wedge F_\alpha s_\alpha - F_\alpha \wedge (ds_\alpha + A_\alpha s_\alpha) \\ &= \big(dF_\alpha + A_\alpha \wedge F_\alpha - F_\alpha \wedge A_\alpha\big)s_\alpha \\ &= \big(d(A_\alpha \wedge A_\alpha) + A_\alpha \wedge dA_\alpha - (dA_\alpha) \wedge A_\alpha\big)s_\alpha \\ &= 0\end{aligned}$$

for $s \in \Omega^0(M, E)$. □

Proof 3. It follows from (8.1.10) that

$$\begin{aligned}&\big(d^\nabla F^\nabla(X, Y, Z)\big)s \\ &= d^\nabla(F^\nabla s)(X, Y, Z) - (F^\nabla \wedge d^\nabla s)(X, Y, Z) \\ &= \nabla_X\left(F^\nabla(Y, Z)s\right) + \nabla_Y\left(F^\nabla(Z, X)s\right) + \nabla_Z\left(F^\nabla(X, Y)s\right) \\ &\quad - F^\nabla(X, [Y, Z])s - F^\nabla(Y, [Z, X])s - F^\nabla(Z, [X, Y])s \\ &\quad - F^\nabla(Y, Z)\nabla_X s - F^\nabla(Z, X)\nabla_Y s - F^\nabla(X, Y)\nabla_Z s \\ &= 0.\end{aligned}$$

for $X, Y, Z \in \mathrm{Vect}(M)$ and $s \in \Omega^0(M, E)$. Here the last equation follows from (8.2.5) by direct calculation. □

Example 8.2.4. If ∇ is the Levi-Civita connection on the tangent bundle of a Riemannian manifold, then (8.2.5) shows that $F^\nabla \in \Omega^2(M, \mathrm{End}(TM))$ is the Riemann curvature tensor and (8.2.7) is the second Bianchi identity.

8.2.2 Gauge Transformations

Let $\pi : E \to V$ be a vector bundle with structure group $\mathrm{G} \subset \mathrm{GL}(V)$, local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$, and transition maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{G}.$$

A **gauge transformation** of E is a vector bundle automorphism $u : E \to E$ such that the vector space isomorphism

$$u_\alpha(p) := \psi_\alpha(p) \circ u(p) \circ \psi_\alpha(p)^{-1} : V \to V \tag{8.2.8}$$

is an element of G for every α and every $p \in U_\alpha$. The group

$$\mathscr{G}(E) := \left\{ u : E \to E \,|\, \psi_\alpha(p) \circ u(p) \circ \psi_\alpha(p)^{-1} \in \mathrm{G}\; \forall\, \alpha\; \forall\, p \in U_\alpha \right\},$$

of gauge transformations is called the **gauge group of** E.

In the local trivializations a gauge transformation is represented by the maps $u_\alpha : U_\alpha \to \mathrm{G}$ in (8.2.8). For all α and β these maps satisfy

$$g_{\beta\alpha} u_\alpha = u_\beta g_{\beta\alpha} \tag{8.2.9}$$

on $U_\alpha \cap U_\beta$. Conversely, every collection of smooth maps $u_\alpha : U_\alpha \to \mathrm{G}$ satisfying (8.2.9) determines a gauge transformation $u \in \mathscr{G}(E)$ via (8.2.8). The gauge group can be thought of as an infinite-dimensional analogue of a Lie group with Lie algebra

$$\mathrm{Lie}(\mathscr{G}(E)) = \Omega^0(M, \mathrm{End}(E)).$$

If $\xi : M \to \mathrm{End}(E)$ is a section, then the pointwise exponential map gives rise to a gauge transformation $u = \exp(\xi)$. Hencer the gauge group $\mathscr{G}(E)$ is infinite-dimensional (unless G is a discrete group or M is a finite set).

Let us denote the space of G-connections on E by

$$\mathscr{A}(E) := \left\{ \nabla : \Omega^0(M, E) \to \Omega^1(M, E) \,|\, \nabla \text{ is a G-connection} \right\}.$$

By Proposition 8.1.12 this space is nonempty and the difference of two G-connections is a 1-form on M with values in $\mathrm{End}(E)$. Thus $\mathscr{A}(E)$ is an affine space with corresponding vector space $\Omega^1(M, \mathrm{End}(E))$. The gauge group $\mathscr{G}(E)$ acts on the space of k-forms with values in E in the obvious manner by composition and it acts on the space of G-connections (contravariantly) by conjugation. We denote this action by

$$u^*\nabla = u^{-1} \circ \nabla \circ u : \Omega^0(M, E) \to \Omega^1(M, E)$$

for $\nabla \in \mathscr{A}(E)$ and $u \in \mathscr{G}(E)$. The connection potentials of $u^*\nabla$ are

$$A_\alpha^{u^*\nabla} = u_\alpha^{-1} du_\alpha + u_\alpha^{-1} A_\alpha^\nabla u_\alpha \in \Omega^1(U_\alpha, \mathfrak{g}). \tag{8.2.10}$$

Lemma 8.2.5. *The curvature of the connection $u^*\nabla$ is given by*

$$F^{u^*\nabla} = u^{-1} \circ F^{\nabla} \circ u \in \Omega^2(M, \mathrm{End}(E)) \tag{8.2.11}$$

and in the local trivialisations by

$$F_\alpha^{u^*\nabla} = u_\alpha^{-1} F_\alpha^{\nabla} u_\alpha \in \Omega^2(U_\alpha, \mathfrak{g}).$$

The parallel transport of the connection $u^*\nabla$ is given by

$$\Phi_\gamma^{u^*\nabla}(t_1, t_0) = u(\gamma(t_1))^{-1} \circ \Phi_\gamma(t_1, t_0) \circ u(\gamma(t_0)) : E_{\gamma(t_0)} \to E_{\gamma(t_1)} \tag{8.2.12}$$

for every smooth path $\gamma : I \to M$ and all $t_0, t_1 \in I$.

Proof. Equation (8.2.11) follows directly from the definitions. To prove equation (8.2.12) we choose a smooth curve $\gamma : I \to U_\alpha$ and a smooth vector field $s(t) \in E_{\gamma(t)}$ along γ and abbreviate

$$\widetilde{s} := u^{-1}s, \qquad \widetilde{\nabla} := u^*\nabla, \qquad \widetilde{A}_\alpha := u_\alpha^{-1} du_\alpha + u_\alpha^{-1} A_\alpha u_\alpha.$$

In the local trivialization over U_α we have

$$s_\alpha(t) = \psi_\alpha(\gamma(t))^{-1} s(t)$$

and

$$\widetilde{s}_\alpha(t) = \psi_\alpha(\gamma(t))^{-1} u(\gamma(t)) s(t)$$

and hence

$$s_\alpha(t) = u_\alpha(\gamma(t)) \widetilde{s}_\alpha(t).$$

Differentiating this equation we obtain

$$\begin{aligned}
(\nabla s)_\alpha &= \dot{s}_\alpha + A_\alpha(\dot{\gamma}) s_\alpha \\
&= u_\alpha(\gamma) \frac{d}{dt} \widetilde{s}_\alpha + \big(du_\alpha(\gamma)\dot{\gamma}\big) \widetilde{s}_\alpha + A_\alpha(\dot{\gamma}) u_\alpha(\gamma) \widetilde{s}_\alpha \\
&= u_\alpha(\gamma) \left(\frac{d}{dt} \widetilde{s}_\alpha + \widetilde{A}_\alpha(\dot{\gamma}) \widetilde{s}_\alpha \right) \\
&= \big(u \widetilde{\nabla} \widetilde{s}\big)_\alpha.
\end{aligned}$$

Thus we have proved that

$$(u^*\nabla)(u^{-1}s) = u^{-1}(\nabla s). \tag{8.2.13}$$

In particular, $\nabla s \equiv 0$ if and only if $(u^*\nabla)(u^{-1}s) \equiv 0$. This proves (8.2.12) and Lemma 8.2.5. □

8.2.3 Flat Connections

A connection $\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$ on a vector bundle $\pi : E \to M$ is called a **flat connection** if its curvature vanishes. By Proposition 8.2.1 a flat connection gives rise to a cochain complex

$$\Omega^0(M,E) \xrightarrow{d^\nabla} \Omega^1(M,E) \xrightarrow{d^\nabla} \Omega^2(M,E) \xrightarrow{d^\nabla} \cdots \xrightarrow{d^\nabla} \Omega^m(M,E). \tag{8.2.14}$$

The cohomology of this complex will be denoted by

$$H^k(M,\nabla) := \frac{\ker\, d^\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)}{\mathrm{im}\, d^\nabla : \Omega^{k-1}(M,E) \to \Omega^k(M,E)}.$$

The de Rham cohomology of M is the cohomology associated to the trivial connection $\nabla = d$ on the vector bundle $E = M \times \mathbb{R}$. The cohomology of the cochain complex (8.2.14) for a general flat connection ∇ on E is also called **de Rham cohomology with twisted coefficients in** E. We shall see that a vector bundle need not admit a flat connection.

To understand flat connections geometrically, we observe that every connection ∇ on a vector bundle $\pi : E \to M$ determines a **horizontal subbundle** $H \subset TE$ of the tangent bundle of E. It is defined by

$$H_e := \left\{ \left.\frac{d}{dt}\right|_{t=0} s(t) \,\middle|\, s : \mathbb{R} \to E,\ s(0) = e,\ \nabla s \equiv 0 \right\} \tag{8.2.15}$$

for $e \in E$. Note that the map $s : \mathbb{R} \to E$ in this definition is a section of E along the curve $\gamma := \pi \circ s : \mathbb{R} \to M$. The image of H_e under the derivative of a local trivialization $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ with

$$p := \pi(e) \in U_\alpha$$

is the subspace

$$d\psi_\alpha(e)H_e = \left\{ (\hat{p},\hat{v}) \in T_pM \times V \,\middle|\, \hat{v} + (A_\alpha)_p(\hat{p})v = 0 \right\}.$$

Here $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(V))$ is the connection potential of ∇.

Theorem 8.2.6. *Let ∇ be a connection on a vector bundle $\pi : E \to M$. The following are equivalent.*

(i) *The curvature of ∇ vanishes.*

(ii) *The horizontal subbundle $H \subset TE$ is involutive.*

(iii) *The parallel transport isomorphism $\Phi^\nabla_\gamma(1,0) : E_{\gamma(0)} \to E_{\gamma(1)}$ depends only on the homotopy class of $\gamma : [0,1] \to M$ with fixed endpoints.*

Proof. We prove that (i) implies (iii). Let $p_0, p_1 \in M$ and

$$[0,1] \times [0,1] \to M : (\lambda, t) \mapsto \gamma(\lambda, t) = \gamma_\lambda(t)$$

be a smooth homotopy with fixed endpoints

$$\gamma_\lambda(0) = p_0, \qquad \gamma_\lambda(1) = p_1, \qquad 0 \le \lambda \le 1.$$

Fix an element $e_0 \in E_{p_0}$ and, for $0 \le \lambda \le 1$, denote by $s_\lambda : [0,1] \to E$ the horizontal lift of γ_λ through e_0. Then it follows from the theory of ordinary differential equations that the map

$$[0,1] \times [0,1] \to E : (\lambda, t) \mapsto s(\lambda, t) := s_\lambda(t)$$

is smooth. Let $\nabla_\lambda s$ be the covariant dervative of the vector field $\lambda \mapsto s(\lambda, t)$ along the curve $\lambda \mapsto \gamma(\lambda, t)$ with t fixed and similarly with λ and t interchanged. Then

$$F^\nabla(\partial_\lambda \gamma, \partial_t \gamma)s = \nabla_\lambda \nabla_t s - \nabla_t \nabla_\lambda s \tag{8.2.16}$$

This is the analogue of equation (8.2.5) for sections along 2-parameter curves. The proof is left as an exercise for the reader. Since $\nabla_t s \equiv 0$, by defintion, and $F^\nabla \equiv 0$, by (i), we obtain

$$\nabla_t \nabla_\lambda s \equiv 0.$$

For $t = 1$ this implies that the curve $[0,1] \to E_{p_1} : \lambda \mapsto s_\lambda(1)$ is constant. Thus we have proved that (i) implies (iii).

We prove that (iii) implies (ii). Choose a Riemannian metric on M and fix an element $e_0 \in E$. Let $U_0 \subset M$ be a geodesic ball centered at the point $p_0 := \pi(e_0)$, whose radius is smaller than the injectivity radius r_0 of M at p_0. Then there is a unique smooth map $\xi : U_0 \to T_{p_0}M$ such that

$$\exp_{p_0}(\xi(p)) = p, \qquad |\xi(p)| < r_0$$

We define a smooth section $s : U_0 \to E$ over U_0 by

$$s(p) := \Phi_{\gamma_p}(1,0)e_0 \in E_p, \qquad \gamma_p(t) := \exp_{p_0}(t\xi(p))$$

If $\gamma : [0,1] \to U_0$ is any smooth curve connecting p_0 to p, then γ is homotopic to γ_p with fixed endpoints and hence $s(\gamma(1)) = \Phi_\gamma(1,0)e_0$. The same argument for the restriction of γ to the interval $[0,t]$ shows that

$$s(\gamma(t)) = \Phi_\gamma(t,0)e_0, \qquad 0 \le t \le 1.$$

Differentiating this equation at $t = 1$ we obtain

$$ds(p)\dot\gamma(1) = \left.\frac{d}{dt}\right|_{t=1} s(\gamma(t)) \in H_{s(p)}.$$

This holds for every smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = p_0$ and $\gamma(1) = p$. Since $\dot\gamma(1)$ can be chosen arbitrarily we obtain $\operatorname{im} ds(p) \subset H_{s(p)}$. Since $\dim(H_{s(p)}) = \dim(M) = \dim(T_pM)$ for every $p \in M$ we have

$$s(p_0) = e_0, \qquad \operatorname{im} ds(p) = H_{s(p)} \quad \forall\, p \in U_0.$$

Thus we have found a submanifold of E through e_0 that is tangent to H. Hence H is integrable and, by the Theorem of Frobenius, it is therefore involutive. Thus we have proved that (iii) implies (ii).

We prove that (ii) implies (i). A vector field $X \in \operatorname{Vect}(M)$ has a unique **horizontal lift** $X^\# \in \operatorname{Vect}(E)$ such that

$$d\pi \circ X^\# = X \circ \pi, \qquad X^\#(e) \in H_e \quad \forall\, e \in E.$$

We show that the Lie bracket of two such lifts is given by

$$[X^\#, Y^\#](e) = [X,Y]^\#(e) + F^\nabla\big(X(\pi(e)), Y(\pi(e))\big). \tag{8.2.17}$$

This equation is meaningful because $F^\nabla(X(\pi(e)), Y(\pi(e))) \in E_e \subset T_eE$. To prove (8.2.17) we observe that the restriction of $X^\#$ to $\pi^{-1}(U_\alpha)$ is the pullback under ψ_α of the vector field $X^\#_\alpha \in \operatorname{Vect}(U_\alpha \times V)$ given by

$$X^\#_\alpha(p,v) = (X(p), -(A_\alpha \circ X)(p)v)$$

for $p \in U_\alpha$ and $v \in V$. Hence $\operatorname{pr}_1 \circ [X^\#_\alpha, Y^\#_\alpha] = [X,Y]$ and

$$\begin{aligned}
\operatorname{pr}_2[X^\#_\alpha, Y^\#_\alpha](p,v) &= (A_\alpha \circ X)(p)(A_\alpha \circ Y)(p)v \\
&\quad - \mathcal{L}_Y(A_\alpha \circ X)(p)v \\
&\quad - (A_\alpha \circ Y)(p)(A_\alpha \circ X)(p)v \\
&\quad + \mathcal{L}_X(A_\alpha \circ Y)(p)v \\
&= [A_\alpha(X(p)), A_\alpha(Y(p))]v \\
&\quad + dA_\alpha(X(p), Y(p))v - A_\alpha([X,Y](p))v \\
&= F_\alpha(X(p), Y(p))v - A_\alpha([X,Y](p))v.
\end{aligned}$$

Here the second equality follows from equation (8.1.9) for the trivial connection on the bundle $U_\alpha \times \operatorname{End}(V)$, and the last equality follows from (8.2.2). This proves (8.2.17). It follows from (8.2.17) that the connection ∇ is flat whenever the horizontal subbundle $H \subset TE$ is involutive. Thus we have proved that (ii) implies (i). This proves Theorem 8.2.6. □

Fix a vector space V and a Lie subgroup $\mathrm{G} \subset \mathrm{GL}(V)$. Every flat G-connection ∇ on a vector bundle $\pi : E \to M$ with structure group G gives rise to a group homomorphism $\rho^\nabla : \pi_1(M, p_0) \to \mathrm{G}$, defined by

$$\rho^\nabla(\gamma) := \psi_\alpha(p_0) \circ \Phi_\gamma(1,0) \circ \psi_\alpha(p_0)^{-1} \in \mathrm{G} \subset \mathrm{GL}(V) \qquad (8.2.18)$$

for every smooth loop $\gamma : [0,1] \to M$ with endpoints $\gamma(0) = \gamma(1) = p_0$. Here the map $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ is a local trivialization with $p_0 \in U_\alpha$. By Proposition 8.1.12, the right hand side of (8.2.18) is an element of the structure group G and, by Theorem 8.2.6, it depends only on the homotopy class of γ with fixed endpoints. The notation ρ^∇ is slightly misleading as the homomorphism depends on a choice of the local trivialization ψ_α. However, different choices of the local trivialization result in conjugate homomorphisms. Moreover, different choices of the base point result in conjugate representations, by equation (8.1.11). And Lemma 8.2.5 shows that the gauge group $\mathscr{G}(E)$ acts on the space $\mathscr{A}^{\mathrm{flat}}(E)$ of flat G-connections on E and that the representations ρ^∇ and $\rho^{u^*\nabla}$ are conjugate for every $\nabla \in \mathscr{A}^{\mathrm{flat}}(E)$ and every $u \in \mathscr{G}(E)$. Thus the correspondence $\nabla \mapsto \rho^\nabla$ defines a map

$$\mathscr{M}^{\mathrm{flat}}(E) := \frac{\mathscr{A}^{\mathrm{flat}}(E)}{\mathscr{G}(E)} \to \frac{\mathrm{Hom}(\pi_1(M), \mathrm{G})}{\text{conjugacy}}. \qquad (8.2.19)$$

This map need not be bijective as different representations $\rho : \pi_1(M) \to \mathrm{G}$ may arise from flat connections on non-isomorphic G-bundles. However it extends to a bijective correspondence in the following sense.

Exercise 8.2.7. Prove the following assertions.

(I) For every homomorphism $\rho : \pi_1(M) \to \mathrm{G}$ there exists a flat G-connection ∇ on some G-bundle $E \to M$ such that ρ^∇ is conjugate to ρ.

(II) If (E, ∇) and (E', ∇') are flat G-bundles with fibers isomorphic to V such that ρ^∇ and $\rho^{\nabla'}$ are conjugate, then (E, ∇) and (E', ∇') are isomorphic. In particular, the map (8.2.19) is injective.

Hint: Use parallel transport to prove part (II). To prove part (I), choose a universal cover $\widetilde{M} \to M$ and define E as the quotient

$$E = \frac{\widetilde{M} \times V}{\pi_1(M, p_0)}.$$

Here the fundamental group acts on V throught ρ. Sections of E are ρ-equivariant maps $s : \widetilde{M} \to V$. Since the additive group $\mathbb{R}$ is isomorphic to the group $\mathrm{GL}^+(\mathbb{R})$ via the exponential map, this gives another proof of Exercise 6.5.20.

8.3 Chern–Weil Theory

This section discusses invariant polynomials on Lie algebras (§8.3.1), characteristic classes (§8.3.2), and the Euler class of a rank-2 bundle (§8.3.3).

8.3.1 Invariant Polynomials

We assume throughout that V is a real vector space and $\mathrm{G} \subset \mathrm{GL}(V)$ is a Lie subgroup with Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) \subset \mathrm{End}(V)$. An **invariant polynomial of degree** d on $\mathfrak{g}$ is a degree-d polynomial $p : \mathfrak{g} \to \mathbb{R}$ such that

$$p(g\xi g^{-1}) = p(\xi) \tag{8.3.1}$$

for every $\xi \in \mathfrak{g}$ and every $g \in \mathrm{G}$. The polynomial condition can be expressed as follows. Choose a basis $e_1, \dots, e_N$ of $\mathfrak{g}$ and write the elements of $\mathfrak{g}$ as

$$\xi = \sum_{i=1}^{N} \xi^i e_i, \qquad \xi^i \in \mathbb{R}.$$

Then a polynomial of degree d on $\mathfrak{g}$ is a map of the form

$$p(\xi) = \sum_{|\nu|=d} a_\nu \xi^\nu, \qquad \xi^\nu := (\xi^1)^{\nu_1}(\xi^2)^{\nu_2}\cdots(\xi^N)^{\nu_N}, \tag{8.3.2}$$

where the sum runs over all multi-indices $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{N}_0^N$ satisfying

$$|\nu| := \nu_1 + \nu_2 + \cdots + \nu_N = d.$$

Definition 8.3.1. *Let $\pi : E \to M$ be a vector bundle with the structure group* G *and local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V$ and let ∇ be a* G*-connection on E. Then an invariant polynomial $p : \mathfrak{g} = \mathrm{Lie}(\mathrm{G}) \to \mathbb{R}$ of degree d determines a differential form $p(F^\nabla) \in \Omega^{2d}(M)$ as follows.*

Let $F_\alpha \in \Omega^2(U_\alpha, \mathfrak{g})$ be given by (8.2.2) *and write*

$$F_\alpha =: \sum_{i=1}^{N} \omega_\alpha^i e_i, \qquad \omega_\alpha^i \in \Omega^2(U_\alpha).$$

If p has the form (8.3.2)*, we define*

$$p(F^\nabla)|_{U_\alpha} := \sum_{|\nu|=d} a_\nu \omega_\alpha^\nu, \qquad \omega_\alpha^\nu := (\omega_\alpha^1)^{\nu_1} \wedge (\omega_\alpha^2)^{\nu_2} \wedge \cdots \wedge (\omega_\alpha^N)^{\nu_N}.$$

It follows from (8.2.3) *and the invariance of p that these definitions agree on the intersection $U_\alpha \cap U_\beta$ for all α and β. The reader may verify that the differential form $p(F^\nabla) \in \Omega^{2d}(M)$ is independent of the choice of the basis of $\mathfrak{g}$ used to define it.*

8.3.2 Characteristic Classes

Let $\pi : E \to M$ be a vector bundle with structure group G. and let $p : \mathfrak{g} \to \mathbb{R}$ be an invariant polynomial of degree d on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$.

Theorem 8.3.2 (Chern–Weil). (i) *The differential form* $p(F^\nabla) \in \Omega^{2d}(M)$ *is closed for every* G*-conection* ∇ *on* E*.*
(ii) *The de Rham cohomology class of* $p(F^\nabla) \in \Omega^{2d}(M)$ *is independent of the choice of the* G*-conection* ∇*.*
(iii) *If* $f : M' \to M$ *is a smooth map, then* $p(F^{f^*\nabla}) = f^*p(F^\nabla)$*.*

By Theorem 8.3.2 every invariant polynmial $p : \mathfrak{g} \to \mathbb{R}$ of degree d on the Lie algebra of the structure group G determines a **characteristic de Rham cohomology class**

$$p(E) := [p(F^\nabla)] \in H^{2d}(M)$$

for every vector bundle $\pi : E \to M$ with structure group G. Namely, by Proposition 8.1.12, there is a G-connection ∇ on E and, by Theorem 8.3.2, the differential form $p(F^\nabla) \in \Omega^{2d}(M)$ associated to such a connection is closed and its cohomology class is independent of ∇. It follows also from Theorem 8.3.2 that the characteristic classes of G-bundles over different manifolds are related under pullback by smooth maps $f : M' \to M$ via

$$p(f^*E) = f^*p(E).$$

Since $p(F^\nabla) = 0$ for every flat G-connection ∇, a G-bundle with a nontrivial characteristic class does not admit a flat G-connection.

Proof of Theorem 8.3.2. We prove (i). The Lie bracket on $\mathfrak{g}$ determines structure constants $c_{ij}^k \in \mathbb{R}$ such that

$$[e_i, e_j] = \sum_{k=1}^N c_{ij}^k e_k, \qquad i, j = 1, \ldots, N.$$

By invariance of the polynomial we haver $p(\exp(t\eta)\xi\exp(-\eta)) = p(\xi)$ for all $\xi, \eta \in \mathfrak{g}$ and all $t \in \mathbb{R}$. Differentiate this identity at $t = 0$ to obtain

$$dp(\xi)[\eta, \xi] = \left.\frac{d}{dt}\right|_{t=0} p(\exp(t\eta)\xi\exp(-\eta)) = 0.$$

For $k = 1, \ldots, N$ define the polynomial $p_k : \mathfrak{g} \to \mathbb{R}$ of degree $d - 1$ by

$$p_k(\xi) := dp(\xi)e_k.$$

Then, for $i = 1, \dots, N$, we have

$$0 = dp(\xi)[e_i, \xi] = \sum_{j=1}^{N} \xi^j dp(\xi)[e_i, e_j] = \sum_{j,k=1}^{N} c_{ij}^k \xi^j p_k(\xi).$$

Replacing ξ by the 2-form

$$\omega_\alpha = \sum_{i=1}^{N} \omega_\alpha^i e_i = F_\alpha^\nabla \in \Omega^2(U_\alpha, \mathfrak{g})$$

of Definition 8.3.1, we obtain

$$\sum_{j,k=1}^{m} c_{ij}^k p_k(\omega_\alpha) \wedge \omega_\alpha^i, \qquad i = 1, \dots, N. \tag{8.3.3}$$

Now write the connection potentials $A_\alpha^\nabla \in \Omega^1(U_\alpha, \mathfrak{g})$ in the form

$$A_\alpha^\nabla = \sum_{i=1}^{N} a_\alpha^i e_i, \qquad a_\alpha^i \in \Omega^1(U_\alpha).$$

Then the Bianchi identity takes the form

$$\begin{aligned} 0 = (d^\nabla F^\nabla)_\alpha &= dF_\alpha^\nabla + [A_\alpha^\nabla \wedge F_\alpha^\nabla] \\ &= \sum_{k=1}^{N} (d\omega_\alpha^k) e_k + \sum_{i,j=1}^{N} a_\alpha^i \wedge \omega_\alpha^j [e_i, e_j] \\ &= \sum_{k=1}^{N} \left(d\omega_\alpha^k + \sum_{i,j=1}^{N} c_{ij}^k a_\alpha^i \wedge \omega_\alpha^j \right) e_k. \end{aligned}$$

Hence

$$d\omega_\alpha^k + \sum_{i,j=1}^{N} c_{ij}^k a_\alpha^i \wedge \omega_\alpha^j = 0, \qquad k = 1, \dots, N. \tag{8.3.4}$$

Combining equations (8.3.3) and (8.3.4), we obtain

$$d(p(\omega_\alpha)) = \sum_{k=1}^{N} p_k(\omega_\alpha) \wedge d\omega_k^\alpha = - \sum_{i,j,k=1}^{N} c_{ij}^k p_k(\omega_\alpha) \wedge a_\alpha^i \wedge \omega_\alpha^j = 0.$$

Here the first equality is left as an exercise for the reader, the second equality follows from (8.3.4), and the last equality follows from (8.3.3). Thus we have proved part (i).

We prove part (ii). Let ∇^0 and ∇^1 be two G-connections on E with connection potentials $A_\alpha^0 \in \Omega^1(U_\alpha, \mathfrak{g})$ and $A_\alpha^1 \in \Omega^1(U_\alpha, \mathfrak{g})$, respectively. Then Proposition 8.1.12 shows that, for $t \in \mathbb{R}$, the operator

$$\nabla^t := (1-t)\nabla^0 + t\nabla^1 : \Omega^0(M,E) \to \Omega^1(M,E)$$

is a G-connection on E with connection potentials

$$A_\alpha^t := tA_\alpha^1 + (1-t)A_\alpha^0 \in \Omega^1(U_\alpha, \mathfrak{g}).$$

Define a connection $\widetilde{\nabla}$ on the vector bundle $\widetilde{E} := E \times \mathbb{R}$ over $\widetilde{M} := M \times \mathbb{R}$ as follows. The local trivializations are given by

$$\widetilde{\psi}_\alpha : \pi^{-1}(U_\alpha) \times \mathbb{R} \to (U_\alpha \times \mathbb{R}) \times V, \qquad \widetilde{\psi}(e,t) := ((p,t), \mathrm{pr}_2 \circ \psi_\alpha(e)).$$

The connection potentials of $\widetilde{\nabla}$ in these trivializations are the 1-forms

$$\widetilde{A}_\alpha \in \Omega^1(U_\alpha \times \mathbb{R}, \mathfrak{g}), \qquad (\widetilde{A}_\alpha)_{(p,t)}(\hat{p}, \hat{t}) := (A_\alpha^t)_p(\hat{p})$$

for $p \in U_\alpha$, $\hat{p} \in T_pM$, and $t, \hat{t} \in \mathbb{R}$. Then

$$F_\alpha^{\widetilde{\nabla}} = F_\alpha^{\nabla^t} - \partial_t A_\alpha^t \wedge dt \in \Omega^2(U_\alpha \times \mathbb{R}, \mathfrak{g})$$

and hence

$$p(F^{\widetilde{\nabla}}) = \omega(t) + \tau(t) \wedge dt \in \Omega^{2d}(M \times \mathbb{R}),$$

where

$$\omega(t) := p(F^{\nabla^t}) \in \Omega^{2d}(M), \qquad t \in \mathbb{R},$$

and

$$\mathbb{R} \to \Omega^{2d-1}(M) : t \mapsto \tau(t)$$

is a smooth family of $(2d-1)$-forms on M. By part (i) the $2d$-form $p(F^{\widetilde{\nabla}})$ on the manifold $\widetilde{M} = M \times \mathbb{R}$ is closed. Thus, by equation (6.3.2) in the proof of Theorem 6.3.8, we have

$$0 = d^{M\times\mathbb{R}} p(F^{\widetilde{\nabla}}) = d^M\omega(t) + \bigl(d^M\beta(t) + \partial_t\omega(t)\bigr) \wedge dt.$$

This implies $\partial_t\omega(t) = -d^M\beta(t)$ for every t and hence

$$p(F^{\nabla^1}) - p(F^{\nabla^0}) = \omega(1) - \omega(0) = \int_0^1 \partial_t\omega(t)\,dt = -d^M \int_0^1 \beta(t)\,dt.$$

Thus $p(F^{\nabla^1}) - p(F^{\nabla^0})$ is exact and this proves part (ii).

We prove part (iii). By Remark 8.1.18 the curvature of the pullback connection $f^*\nabla$ is in the local trivializations $f^*\psi_\alpha$ given by the 2-forms

$$F_\alpha^{f^*\nabla} = f^*F_\alpha^\nabla \in \Omega^1(f^{-1}(U_\alpha), \mathfrak{g}).$$

Hence it follows directly from the definitions that $p(F^{f^*\nabla}) = f^*p(F^\nabla)$. This proves part (iii) and Theorem 8.3.2. □

8.3.3 The Euler Class of an Oriented Rank-2 Bundle

Let $\pi : E \to M$ be an oriented Riemannian real rank-2 bundle over a smooth manifold. By Example 8.1.14 E is a vector bundle with structure group

$$\mathrm{SO}(2) = \left\{ g = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \,\middle|\, a, c \in \mathbb{R},\, a^2 + c^2 = 1 \right\}.$$

Its Lie algebra consists of all skew-symmetric real 2×2-matrices:

$$\mathfrak{so}(2) = \left\{ \xi = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{R} \right\}.$$

The linear map $e : \mathfrak{so}(2) \to \mathbb{R}$ defined by

$$e(\xi) := \frac{-\lambda}{2\pi}$$

is invariant under conjugation. (Note, however, that $e(g^{-1}\xi g) = -e(\xi)$ whenever $g \in \mathrm{O}(n)$ has determinant -1. Thus we must assume that E is oriented.) Hence there is a characteristic class

$$e(E) := [e(F^\nabla)] \in H^2(M), \tag{8.3.5}$$

where ∇ is a Riemannian connection on E. If we change the Riemannian structure on E, then there is an orientation preserving automorphism of E intertwining the two inner products. (Prove this!) Thus the characteristic class $e(E)$ is independent of the choice of the Riemannian metric. We prove that (8.3.5) is the **Euler class of** E whenever M is a compact oriented manifold without boundary (see Definition 7.3.11).

Theorem 8.3.3. *If E is an oriented real rank-2 bundle over a compact oriented manifold M without boundary, then* (8.3.5) *is the Euler class of E.*

Proof. Choose a smooth section $s : M \to E$ that is transverse to the zero section and denote

$$Q := s^{-1}(0).$$

Choose a Riemannian metric on M and let

$$\exp : TQ^\perp_\varepsilon \to U_\varepsilon$$

be the tubular neighborhood diffeomorphism in (7.2.11). Multiplying s by a suitable positive function on M we may assume that

$$p \in M \setminus U_{\varepsilon/3} \qquad \Longrightarrow \qquad |s(p)| = 1.$$

We claim that there exists a Riemannian connection ∇ on E such that

$$\nabla s = 0 \qquad \text{on} \qquad M \setminus U_{\varepsilon/2}. \tag{8.3.6}$$

To see this, choose on open cover $\{U_\alpha\}$ of M such that E admits a trivialization over each set U_α and one of the sets is $U_{\alpha_0} = M \setminus \overline{U}_{\varepsilon/3}$. In particular, we can use s to trivialize E over U_{α_0}. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to this open cover suuch that $\rho_{\alpha_0} = 1$ on $M \setminus U_{\varepsilon/2}$. Then (8.1.6) defines a Riemannian connection that satisfies (8.3.6). it follows from (8.3.6) that $F^\nabla s = d^\nabla \nabla s = 0$ on $M \setminus U_{\varepsilon/2}$. Since F^∇ is a 2-form with values in the skew-symmetric endomorphisms of E we deduce that

$$F^\nabla = 0 \qquad \text{on} \qquad M \setminus U_{\varepsilon/2}. \tag{8.3.7}$$

By (8.3.7) the 2-form the 2-form

$$\tau_\varepsilon := \exp^* e(F^\nabla) \in \Omega^2_c(TQ^\perp_\varepsilon)$$

has compact support. We prove below that τ_ε is a Thom form on $TQ^\perp$. It then follows from Lemma 7.2.17 with $\tau_Q = e(F^\nabla)$ that

$$\int_M \omega \wedge e(F^\nabla) = \int_Q \omega = \int_M \omega \wedge s^*\tau$$

for every closed form $\omega \in \Omega^{m-2}(M)$ and every Thom form $\tau \in \Omega^2_c(E)$, where the last equality follows from Theorem 7.3.15. By Poincaré duality in Theorem 6.4.1 this implies that $e(F^\nabla) - s^*\tau$ is exact, which proves the assertion. Thus it remains to prove that τ_ε is indeed a Thom form on $TQ^\perp$.

To see this, fix a point $q_0 \in Q$ and choose a positive orthonormal basis

$$u, v \in T_{q_0}Q^\perp, \qquad |u| = |v| = 1, \qquad \langle u, v\rangle = 0.$$

We define a smooth map $\gamma : \mathbb{D} \to U_\varepsilon$ on the closed unit disk $\mathbb{D} \subset \mathbb{R}^2$ by

$$\gamma(z) := \exp_{q_0}(\varepsilon(xu + yv)).$$

for $z = (x, y) \in \mathbb{D}$. (The exponential map extends to the closure of $TQ^\perp_\varepsilon$.) This is an orientation preserving embedding of $\mathbb{D}$ into a fiber of the normal bundle $\overline{TQ}^\perp_\varepsilon$ followed by the exponential map. Moreover, we have

$$\int_{\mathbb{D}} \gamma^* e(F^\nabla) = \int_{\mathbb{D}} e(F^{\gamma^*\nabla}) = 1.$$

Here the first equality follows from part (iii) of Theorem 8.3.2 and the second equality follows from Lemma 8.3.4 below by choosing a positive orthonormal trivialization of the pullback bundle $\gamma^*E \to \mathbb{D}$ (for example via radial parallel transport). Hence $\pi_*\tau_\varepsilon = 1$ and this proves Theorem 8.3.3. □

Lemma 8.3.4. *Let* $\mathbb{D} \subset \mathbb{R}^2$ *be the closed unit disk with coordinates* $z = (x, y)$ *and let* $s : \mathbb{D} \to \mathbb{R}^2$ *and* $\xi, \eta : \mathbb{D} \to \mathfrak{so}(2)$ *be smooth functions. Suppose that*

$$\left\{\begin{array}{ll} s(z) = 0, & \text{for } z = 0, \\ s(z) \neq 0, & \text{for } z \neq 0, \\ |s(z)| = 1, & \text{for } |z| \geq 1/2, \end{array}\right. \qquad \det(ds(0)) > 0,$$

and that the Riemannian connection

$$\nabla := d + A, \qquad A := \xi dx + \eta dy \in \Omega^1(\mathbb{D}, \mathfrak{so}(2))$$

satisfies $\nabla s = 0$ *for* $|z| \geq 1/2$*. Then*

$$\int_{\mathbb{D}} e(F^\nabla) = 1.$$

Proof. Identify $\mathbb{R}^2$ with $\mathbb{C}$ via $z = x + \mathbf{i}y$ and think of s as a vector field on $\mathbb{D}$. For $0 \leq r < 1$ define the curve $\gamma_r : S^1 \to S^1$ by

$$\gamma_r(e^{\mathbf{i}\theta}) := s(re^{\mathbf{i}\theta}).$$

Then the index formula for vector fields in Lemma 2.3.3 shows that

$$1 = \deg(\gamma_r) = \frac{1}{2\pi\mathbf{i}} \int_0^{2\pi} \gamma_r(\theta)^{-1}\dot{\gamma}_r(\theta)\, d\theta, \qquad 1/2 \leq r \leq 1. \tag{8.3.8}$$

To see this, choose a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\gamma_r(\theta) = e^{\mathbf{i}\phi(\theta)}$$

for all θ. Then

$$\phi(\theta + 2\pi) = \phi(\theta) + 2\pi \deg(\gamma_r)$$

and this proves (8.3.8).

At this point it is convenient to identify $\mathfrak{so}(2)$ with the imaginary axis via the isomorphism

$$\iota : \mathfrak{so}(2) \to \mathbf{i}\mathbb{R}, \qquad \iota\left(\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}\right) := \mathbf{i}\lambda.$$

Thus $\xi \in \mathfrak{so}(2)$ acts on $\mathbb{R}^2 \cong \mathbb{C}$ by multiplication with $\iota(\xi)$ and

$$e(F^\nabla) = \frac{\mathbf{i}}{2\pi}\iota(F^\nabla) = \frac{\mathbf{i}}{2\pi} d\iota(A), \qquad \iota(A) = \iota(\xi)\, dx + \iota(\eta)\, dy.$$

The condition $\nabla s = 0$ for $|z| = 1$ takes the form

$$\partial_x s(e^{\mathbf{i}\theta}) + \iota(\xi(e^{\mathbf{i}\theta}))s(e^{\mathbf{i}\theta}) = 0, \qquad \partial_y s(e^{\mathbf{i}\theta}) + \iota(\eta(e^{\mathbf{i}\theta}))s(e^{\mathbf{i}\theta}) = 0$$

and this gives

$$\dot\gamma_1(\theta) = \bigl(\sin(\theta)\iota(\xi(e^{\mathbf{i}\theta})) - \cos(\theta)\iota(\eta(e^{\mathbf{i}\theta}))\bigr)\gamma_1(\theta).$$

Hence

$$\begin{aligned}
\int_{\mathbb{D}} e(F^\nabla) &= \frac{\mathbf{i}}{2\pi}\int_{\mathbb{D}} d\iota(A) \\
&= \frac{\mathbf{i}}{2\pi}\int_{S^1} \iota(A) \\
&= \frac{\mathbf{i}}{2\pi}\int_{S^1} \bigl(\iota(\xi)\,dx + \iota(\eta)\,dy\bigr) \\
&= \frac{\mathbf{i}}{2\pi}\int_0^{2\pi} \bigl(\cos(\theta)\iota(\eta(e^{\mathbf{i}\theta})) - \sin(\theta)\iota(\xi(e^{\mathbf{i}\theta}))\bigr)\,d\theta \\
&= -\frac{\mathbf{i}}{2\pi}\int_0^{2\pi} \gamma_1(\theta)^{-1}\dot\gamma_1(\theta)\,d\theta \\
&= 1.
\end{aligned}$$

The last equation follows from (8.3.8) and this proves Lemma 8.3.4. □

Corollary 8.3.5. *An oriented Riemannian rank-2 vector bundle E over M admits a flat Riemannian connection if and only if its Euler class $e(E)$ vanishes in the de Rham cohomology group $H^2(M)$.*

Proof. If E admits a flat Riemannian connection ∇, then $e(F^\nabla) = 0$ and so its Euler class vanishes by Theorem 8.3.3. Conversely, assume $e(E) = 0$ and let ∇ be any Riemannian connection on E. Then $e(F^\nabla)$ is exact. Hence there is a 1-form $\alpha \in \Omega^1(M)$ such that $e(F^\nabla) = d\alpha$. Since the linear map $e : \mathfrak{so}(2) \to \mathbb{R}$ is a vector space isomorphism, there exists a unique 1-form $A \in \Omega^1(M, \mathrm{End}(E))$ such that $e(A) = \alpha$. Hence $\nabla - A$ is a flat Riemannian connection. This proves Corollary 8.3.5. □

Exercise 8.3.6. Let $\pi : E \to M$ be an oriented real rank-2 bundle over a connected simply connected manifold M with vanishing Euler class $e(E) = 0$ in de Rham cohomology. Prove that E admits a global trivialization. **Hint:** Use the existence of a flat Riemannian connection on E in Corollary 8.3.5.

Example 8.3.7. Consider the vector bundle

$$E := \frac{S^2 \times \mathbb{R}^2}{\sim} \to \mathbb{R}\mathrm{P}^2$$

where the equivalence relation on $S^2 \times \mathbb{R}^2$ is given by $(x, \zeta) \sim (-x, -\zeta)$ for $x \in S^2$ and $\zeta \in \mathbb{R}^2$. By the Borsuk–Ulam Theorem 1.5.1 this vector bundle does not admit a nonzero section and hence has no global trivialization. It is oriented as a vector bundle (although the base manifold $\mathbb{R}\mathrm{P}^2$ is not orientable) and its Euler class vanishes in the de Rham cohomology group $H^2(\mathbb{R}\mathrm{P}^2) = 0$. **Exercise:** Find a flat Riemannian connection on E.

Example 8.3.7 shows that the assertion of Exercise 8.3.6 does not extend to non simply connected manifolds. The problem is that the Euler class in Chern–Weil theory is only defined with real coefficients. The definition of the Euler class can be refined with integer coefficients. This requires a cohomology theory over the integers which we do not develop here. The Euler class of an oriented rank-2 bundle is then an integral cohomology class. In particular, $H^2(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and the Euler class of the bundle in Example 8.3.7 is the unique nontrivial element of $H^2(\mathbb{R}\mathrm{P}^2; \mathbb{Z})$. More generally, oriented rank-2 bundles are classified by their Euler classes in integral cohomology: two oriented rank-2 bundles over M are isomorphic if and only if they have the same Euler class in $H^2(M; \mathbb{Z})$.

Example 8.3.8 (Complex Line Bundles over the Torus). A complex line bundle over the torus

$$\mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m$$

can be described by a **cocycle**

$$\mathbb{Z}^m \to C^\infty(\mathbb{R}^m, S^1) : k \mapsto \phi_k$$

which satisfies

$$\phi_{k+\ell}(x) = \phi_\ell(x + k)\phi_k(x)$$

for $x \in \mathbb{R}^m$ and $k, \ell \in \mathbb{Z}^m$. The associated complex line bundle is

$$E_\phi := \frac{\mathbb{R}^m \times \mathbb{C}}{\mathbb{Z}^m}, \qquad [x, \zeta] \equiv [x + k, \phi_k(x) z] \quad \forall\, k \in \mathbb{Z}^m.$$

A section of E_ϕ is a smooth map $s : \mathbb{R}^m \to \mathbb{C}$ such that

$$s(x + k) = \phi_k(x) s(x)$$

for $x \in \mathbb{R}^m$ and $k \in \mathbb{Z}^m$.

A Hermitian connection on E_ϕ has the form

$$\nabla = d + A, \qquad A = \sum_{i=1}^{n} A_i(x) dx^i,$$

where the functions $A_i : \mathbb{R}^m \to \mathbf{i}\mathbb{R}$ satisfy the condition

$$A_i(x+k) - A_i(x) = -\phi_k(x)^{-1} \frac{\partial \phi_k}{\partial x^i}(x).$$

for all $x \in \mathbb{R}^m$ and all $k \in \mathbb{Z}^m$. This can be used to compute the Euler class of the bundle.

For example, any integer matrix $B \in \mathbb{Z}^{m\times m}$ determines a cocycle

$$\phi_k^B(x) = \exp(2\pi \mathbf{i} k^T B x). \tag{8.3.9}$$

A Hermitian connection on E_{ϕ^B} is then given by

$$\nabla^B = d + A, \qquad A := -2\pi\mathbf{i} \sum_{i,j=1}^{m} x^i B_{ij} dx^j. \tag{8.3.10}$$

Its curvature is the imaginary valued 2-form

$$F^{\nabla^B} = dA = -2\pi\mathbf{i} \sum_{i<j} (B_{ij} - B_{ji})\, dx^i \wedge dx^j.$$

Hence the bundle E^{ϕ^B} has the Euler class

$$e(E_{\phi^B}) = \sum_{i<j}^{m} C_{ij} [dx^i \wedge dx^j] \in H^2(\mathbb{T}^m), \qquad C := B - B^T.$$

This bundle admits a trivialization whenever B is symmetric and it admits a square root whenever B is skew-symmetric. (Prove this.) Another cocycle with the same Euler class is given by

$$\phi_k(x) = \varepsilon(k) \exp(\pi\mathbf{i} k^T C x), \qquad \varepsilon(k+\ell) = \varepsilon(k)\varepsilon(\ell)\exp(\pi\mathbf{i} k^T C \ell),$$

with $\varepsilon(k) = \pm 1$. If $C = B - B^T$, then the numbers

$$\varepsilon(k) = \exp(\pi\mathbf{i} k^T B k)$$

satisfy this condition.

Two cocycles ϕ and ψ are called **equivalent** iff there exists a smooth map

$$u : \mathbb{R}^m \to S^1$$

that satisfies the condition

$$\psi_k(x) = u(x+k)^{-1}\phi_k(x)u(x)$$

for all $x \in \mathbb{R}^m$ and $k \in \mathbb{Z}^m$. We claim that every cocycle ϕ is equivalent to one of the form (8.3.9). To see this, we use the fact that every 2-dimensional de Rham cohomology class on $\mathbb{T}^m$ with integer periods can be represented by a 2-form with constant integer coefficients (see Example 6.4.12). This implies that there is a skew-symmetric integer matrix

$$C = -C^T \in \mathbb{Z}^{m\times m}$$

such that the Euler class of E_ϕ is

$$e(E_\phi) = \sum_{i<j} C_{ij}[dx^i \wedge dx^j].$$

Now the argument in the Proof of Corollary 8.3.5 shows that there is Hermitian connection ∇ on E_ϕ with constant curvature

$$F^\nabla = -2\pi\mathbf{i}\sum_{i<j} C_{ij}dx^i \wedge dx^j.$$

Choose an integer matrix $B \in \mathbb{Z}^{m\times m}$ such that

$$C = B - B^T$$

and consider the connection ∇^B in (8.3.10). It has the same curvature as ∇ and hence there exists a smooth function $\xi : \mathbb{R}^m \to \mathbf{i}\mathbb{R}$ such that

$$\nabla = \nabla^B + d\xi.$$

This implies that the gauge transformation

$$u := \exp(\xi) : \mathbb{R}^m \to S^1$$

transforms ϕ^B into ϕ.

Exercise: Fill in the details. Prove that the complex line bundles E_ϕ and E_ψ associated to equivalent cocycles are isomorphic.

8.4 Chern Classes

Our main application of Chern–Weil theory is the definition of the Chern classes in de Rham cohomology. We begin with the axioms for the Chern classes and their construction in §8.4.1, prove their existence and uniqueness in §8.4.2, and discuss several examples in §8.4.3.

8.4.1 Axioms and Construction

We have already used the fact that a complex Hermitian line bundle can be regarded as an oriented real rank-2 bundle. Conversely, an oriented real Riemannian rank-2 bundle has a unique complex structure compatible with the inner product and the orientation, and can therefore be considered as a **complex Hermitian line bundle**. In this setting a Hermitian connection is the same as a Riemannian connection. In the complex notation the curvature F^∇ of a Hermitian connection is an imaginary valued 2-form on M, the Bianchi identity asserts that it is closed, and the real valued closed 2-form

$$e(F^\nabla) = \frac{\mathbf{i}}{2\pi} F^\nabla \in \Omega^2(M)$$

is a representative of the Euler class. (See Lemma 8.3.4.) This is also the first Chern class of E, when regarded as a complex complex line bundle.

More generally, the Chern classes of complex vector bundles are characteristic classes in the even-dimensional cohomology of the base manifold. They are uniquely characterized by certain axioms which we now formulate in our de Rham cohomology setting. We will see that, in order to compute the Chern classes of specific vector bundles, it suffices in many cases to know that they exist and which axioms they satisfy, without knowing how they are constructed.

Just as in the case of the Euler class, the definition of the Chern classes can be extended to cohomology theories with integer coefficients, and then they may carry more information about the vector bundle, that cohomology classes with real coefficients cannot detect. However, this goes beyond the scope of the present book.

Theorem 8.4.1 (**Chern Class).** *There exists a unique functor, called the* **Chern class***, that assigns to every complex rank-n bundle $\pi : E \to M$ over a compact manifold a de Rham cohomology class*

$$c(E) = c_0(E) + c_1(E) + \cdots + c_n(E) \in H^*(M)$$

with $c_i(E) \in H^{2i}(M)$ and $c_0(E) = 1$ and satisfies the following axioms.

(Naturality) *Isomorphic vector bundles over M have the same Chern class.*

(Zero) *The Chern class of the trivial bundle $E = M \times \mathbb{C}^n$ is $c(E) = 1$.*

(Functoriality) *The Chern class of the pullback of a complex vector bundle $\pi : E \to M$ under a smooth map is the pullback of the Chern class of E, i.e.*

$$c(f^*E) = f^*c(E).$$

(Sum) *The Chern class of the Whitney sum $E_1 \oplus E_2$ of two complex vector bundles over M is the cup product of the Chern classes:*

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2).$$

(Euler Class) *The top Chern class of a complex rank-n bundle $\pi : E \to M$ over a compact oriented manifold M without boundary is the Euler class*

$$c_n(E) = e(E).$$

The proof of Theoirem 8.4.1 is deferred to §8.4.2.

Remark 8.4.2. It follows from the *(Euler Class*) axiom that the anti-tautological line bundle $H \to \mathbb{C}\mathrm{P}^n$ with fiber $H_\ell = \ell^*$ over $\ell \in \mathbb{C}\mathrm{P}^n$ has first Chern class

$$c_1(H) = h \in H^2(\mathbb{C}\mathrm{P}^n) \tag{8.4.1}$$

where h is the positive integral generator of $H^2(CP^n)$ whose integral over the submanifold $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^n$ with its complex orientation is equal to one. (See Theorem 7.3.19.) In fact, the proof of Theorem 8.4.1 shows that the *(Euler Class*) axiom can be replaced by the (*Normalization*) axiom (8.4.1).

We now give a construction of the Chern classes via Chern–Weil theory which works equally well for arbitrary base manifolds M, compact or not. Every complex vector bundle E admits a Hermitian structure and that any two Hermitian structures on E are related by a complex automorphism of E (see Example 8.1.16 and Exercise 8.1.17). A Hermitian vector bundle of complex rank n is a vector bundle with the structure group

$$\mathrm{G} = \mathrm{U}(n) = \left\{ g \in \mathbb{C}^{n\times n} \,|\, g^*g = \mathbb{1} \right\}.$$

Here $g^* := \bar{g}^T$ denotes the conjugate transpose of $g \in \mathbb{C}^{n\times n}$. The Lie algebra of $\mathrm{U}(n)$ is the real vector space of skew-Hermitian complex $n \times n$-matrices

$$\mathfrak{g} = \mathfrak{u}(n) = \left\{ \xi \in \mathbb{C}^{n\times n} \,|\, \xi^* + \xi = \mathbb{1} \right\}.$$

The eigenvalues of a matrix $\xi \in \mathfrak{u}(n)$ are purely imaginary and those of the matrix $\mathbf{i}\xi/2\pi$ are real. The kth **Chern polynomial** $c_k : \mathfrak{u}(n) \to \mathbb{R}$ is defined as the kth symmetric function in the eigenvalues of $\mathbf{i}\xi/2\pi$. Thus

$$c_k(\xi) := \sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k},$$

where the real numbers $x_1, \dots, x_n$ are the eigenvalues of the matrux $\mathbf{i}\xi/2\pi$ with repetitions according to multiplicity. In particular, we have

$$\begin{aligned} c_0(\xi) &= 1, \\ c_1(\xi) &= \sum_i x_i = \mathrm{trace}\left(\frac{\mathbf{i}\xi}{2\pi}\right), \\ c_2(\xi) &= \sum_{i<j} x_i x_j, \\ c_n(\xi) &= x_1x_2\cdots x_n = \det\left(\frac{\mathbf{i}\xi}{2\pi}\right). \end{aligned}$$

Thus $c_k : \mathfrak{u}(n) \to \mathbb{R}$ is an invariant polynomial of degree k and we define the **kth Chern class** of a rank-n Hermitian vector bundle $\pi : E \to M$ by

$$c_k(E) := [c_k(F^\nabla)] \in H^{2k}(M), \tag{8.4.2}$$

where ∇ is a Hermitian connection on E. By Theorem 8.3.2 this cohomology class is independent of the choice of the Hermitian connection ∇.

8.4.2 Proof of Existence and Uniqueness

We will prove that the classes (8.4.2) satisfy the axioms of Theorem 8.4.1. Here is a technical lemma which will be needed later in the proof. We remark that Lemma 8.4.3 is the only place where the compactness assumption on the base enters the proof of Theorem 8.4.1

Lemma 8.4.3. *Every complex vector bundle over a compact manifold M admits an embedding into the trivial bundle $M \times \mathbb{C}^N$ for some $N \in \mathbb{N}$.*

Proof. Let $\pi : E \to M$ be a complex rank-n bundle over a compact manifold. Choose a system of local trivializations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$, $i = 1, \dots, \ell$, such that the U_i cover M, and a partition of unity $\rho_i : M \to [0,1]$ subordinate to this cover. Define the map $\iota : E \to M \times \mathbb{C}^{\ell n}$ by

$$\iota(e) := \big(\pi(e), \rho_1(\pi(e))\mathrm{pr}_2(\psi_1(e)), \dots, \rho_\ell(\pi(e))\mathrm{pr}_2(\psi_\ell(e))\big).$$

This map is a proper injective immersion and restricts to a linear embedding into $\{p\} \times \mathbb{C}^{\ell n}$ on each fiber E_p. This proves Lemma 8.4.3. □

Proof of Theorem 8.4.1. The cohomology classes (8.4.2) are invariants of complex vector bundles, because every complex vector bundle admits a Hermitian structure and any two Hermitian structures on a complex vector bundle are isomorphic (Exercise 8.1.17). These classes satisfy the *(Naturality)* and *(Zero)* axioms by definitions and they satisfy the *(Functoriality)* axiom by Theorem 8.3.2. To prove the *(Sum)* axiom we observe that the Chern polynomials are the coefficients of the characteristic polynomial

$$p_t(\xi) := \det\left(\mathbb{1} + t\frac{\mathbf{i}\xi}{2\pi}\right) = \sum_{k=0}^{n} c_k(\xi)t^k.$$

In particular, for $t = 1$, we have

$$c(\xi) = \sum_{k=0}^{n} c_k(\xi) = \prod_{i=1}^{n}(1 + x_i) = \det\left(\mathbb{1} + \frac{\mathbf{i}\xi}{2\pi}\right)$$

and hence $c(\xi \oplus \eta) = c(\xi)c(\eta)$ for the direct sum of two skew-Hermitian matrices. This implies

$$c(F^{\nabla_1 \oplus \nabla_2}) = c(F^{\nabla_1} \oplus F^{\nabla_2}) = c(F^{\nabla_1}) \wedge c(F^{\nabla_2})$$

for the direct sum of two Hermitian connections on two Hermitian vector bundles over M and this proves the *(Sum)* axiom.

It remains to prove the *(Euler Class)* axiom. By Theorem 8.3.3 the first Chern class of a complex line bundle is equal to the Euler class in $H^2(M)$. With this understood, it follows from the (*Sum*) axiom for the Euler class (Theorem 7.3.18) and for the Chern class (already established) that the *(Euler Class*) axiom holds for Whitney sums of complex line bundles.

An example is the partial flag manifold

$$\mathcal{F}(n, N) := \left\{ (\Lambda_i)_{i=0}^{n} \;\middle|\; \begin{array}{l} \Lambda_i \text{ is a complex subspace of } \mathbb{C}^N, \\ \dim_{\mathbb{C}}(\Lambda_i) = i,\ \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n \end{array} \right\}.$$

There is a complex rank-n bundle

$$E(n, N) \to \mathcal{F}(n, N)$$

whose fiber over the flag $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n$ is the subspace Λ_n. It is a direct sum of the complex line bundles $L_i \to \mathcal{F}(n, N)$, $i = 1, \dots, n$, whose fiber over the same flag is the intersection $\Lambda_i \cap \Lambda_{i-1}^{\perp}$. Hence it follows from what we have already proved that the top Chern class of the bundle $E(n, N) \to \mathcal{F}(n, N)$ agrees with its Euler class, i.e. $c_n(E(n, N)) = e(E(n, N)) \in H^{2n}(\mathcal{F}(n, N))$.

Now consider the Grassmannian

$$\mathrm{G}_n(\mathbb{C}^N) := \bigl\{\Lambda \subset \mathbb{C}^N \,|\, \Lambda \text{ is an } n\text{-dimensional complex subspace}\bigr\}$$

of complex n-planes in $\mathbb{C}^N$. It carries a tautological bundle

$$E_n(\mathbb{C}^N) \to \mathrm{G}_n(\mathbb{C}^N)$$

whose fiber over an n-dimensional complex subspace $\Lambda \subset \mathbb{C}^N$ is the subspace itself. There is an obvious map

$$\pi : \mathcal{F}(n, N) \to \mathrm{G}_n(\mathbb{C}^N)$$

which sends a partial flag $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n$ in $\mathbb{C}^N$ with $\dim_{\mathbb{C}}(\Lambda_i) = i$ to the subspace Λ_n. We have

$$\pi^* E_n(\mathbb{C}^n) = E(n, N) \to \mathcal{F}(n, N)$$

and hence, by *(Functoriality)*,

$$\pi^* c_n(E_n(\mathbb{C}^N)) = c_n(E(n, N)) = e(E(n, N)) = \pi^* e(E_n(\mathbb{C}^N)).$$

At this point we use (without proof) the fact that the map

$$\pi^* : H^*(\mathrm{G}_n(\mathbb{C}^N)) \to H^*(\mathcal{F}(n, N)) \tag{8.4.3}$$

is injective. This implies

$$c_n(E_n(\mathbb{C}^N)) = e(E_n(\mathbb{C}^N)) \in H^{2n}(\mathrm{G}_n(\mathbb{C}^N)) \tag{8.4.4}$$

for every pair of integers $N \geq n \geq 0$.

By Lemma 8.4.3, a complex line bundle $\pi : E \to M$ over a compact manifold can be embedded into the trivial bundle $M \times \mathbb{C}^N$ for a suitable integer $N \in \mathbb{N}$. Such an embedding can be expressed as a smooth map

$$f : M \to \mathrm{G}_n(\mathbb{C}^N)$$

into the Grassmannian of complex n-planes in $\mathbb{C}^N$ such that E is isomorphic to the pullback of the tautological bundle $E_n(\mathbb{C}^N) \to \mathrm{G}_n(\mathbb{C}^N)$. Hence it follows from (8.4.4) and *(Functoriality)* that

$$c_n(E) = f^* c_n(E_n(\mathbb{C}^N)) = f^* e(E_n(\mathbb{C}^N)) = e(E).$$

This proves the existence of Chern classes satisfying the five axioms.

To prove uniqueness, we first observe that the Chern classes of complex line bundles over compact oriented manifolds without boundary are determined by the *(Euler Class)* axiom. Second, the Chern classes of the bundle $E(n, N)$ are determined by those of line bundles via the *(Naturality)* and *(Sum)* axioms, as it is isomorphic to a direct sum of complex line bundles. Third, the Chern classes of the tautological bundle

$$E_n(\mathbb{C}^N) \to \mathrm{G}_n(\mathbb{C}^N)$$

are determined by those of $E(n, N)$ via *(Functoriality)*, because the homomorphism (8.4.3) is injective. Fourth, the Chern classes of any complex vector bundle E over a compact manifold M are determined by those of $E_n(\mathbb{C}^N)$ via *(Naturality)* and *(Functoriality)*, as there is a map

$$f : M \to \mathrm{G}_n(\mathbb{C}^N)$$

for some N such that E is isomorphic to the pullback bundle $f^*E_n(\mathbb{C}^N)$:

$$E \cong f^*E_n(\mathbb{C}^N).$$

This proves Theorem 8.4.1. □

We remark that the map

$$\pi : \mathcal{F}(n, N) \to \mathrm{G}_n(\mathbb{C}^N)$$

is a fibration with fibers diffeomorphic to the flag manifold $\mathcal{F}(n, n)$. One can use the spectral sequence of this fibration to prove that the map (8.4.3) is injective. This can be viewed as an extension of the Künneth formula, but it goes beyond the scope of the present book. For details see Bott and Tu [3].

We also remark that Theorem 8.4.1 continues to hold for noncompact base manifolds M. The only place where we have used compactness of M is in Lemma 8.4.3, which in turn was used for proving uniqueness. If we replace the Grassmannian with the classifying space of the unitary group $\mathrm{U}(n)$ (which can be represented as the direct limit of the Grassmanians $\mathrm{G}_n(\mathbb{C}^N)$ as N tends to ∞), then complex rank-n bundles over noncompact manifolds M can be represented as pullbacks of the tautological bundle under maps to this classifying space or, equivalently, be embedded into the product of M with an infinite-dimensional complex vector space. This can be used to extend Theorem 8.4.1 to complex vector bundles over noncompact base manifolds or, in fact, over arbitrary topological spaces.

8.4.3 Examples and Exercises

Exercise 8.4.4 (**Euler Number**). Let $\pi : E \to M$ be a complex rank-n bundle over a compact oriented $2n$-manifold without boundary. Show that the top Chern number

$$\int_M c_n(E) = \int_M \det\left(\frac{\mathbf{i}}{2\pi} F^\nabla\right) = \sum_{s(p)=0_p} \iota(p,s)$$

is the Euler number of E, without using the *(Euler Class)* axiom. **Hint:** Assume s is transverse to the zero section and let p_i be the zeros of s. Show that s can be chosen with norm one outside of a disjoint collection of neighborhoods U_i of the p_i and that the connection can be chosen such that $\nabla s = 0$ on the complement of the U_i. Show that

$$\det\left(\frac{\mathbf{i}}{2\pi} F^\nabla\right) = 0 \qquad \text{on} \quad M \setminus \bigcup_i U_i.$$

Now use the argument in the proof Lemma 8.3.4 to show that for each i

$$\int_{U_i} \det\left(\frac{\mathbf{i}}{2\pi} F^\nabla\right) = \iota(p_i, s).$$

Exercise 8.4.5 (**First Pontryagin Class**). Let $\pi : E \to M$ be a real vector bundle and consider the tensor product $E \otimes_\mathbb{R} \mathbb{C}$. This is a complex vector bundle and **Pontryagin classes** of E are defined as the even Chern classes of $E \otimes_\mathbb{R} \mathbb{C}$:

$$p_i(E) := (-1)^i c_{2i}(E \otimes_\mathbb{R} \mathbb{C}) \in H^{4i}(X).$$

Show that the odd Chern classes of $E \otimes_\mathbb{R} \mathbb{C}$ vanish. Show that

$$p_1(E) = c_1(E)^2 - 2c_2(E)$$

whenever E is itself a complex vector bundle. If E is a Hermitian vector bundle and ∇ is a Hermitian connection on E, show that the first Pontryagin class can be represented by the real valued closed 4-form $\frac{1}{4\pi}\mathrm{trace}(F^\nabla \wedge F^\nabla)$:

$$p_1(E) = \frac{1}{4\pi}\left[\mathrm{trace}\left(F^\nabla \wedge F^\nabla\right)\right] \in H^4(M). \tag{8.4.5}$$

Hint: The endomorphism valued 4-form $F^\nabla \wedge F^\nabla \in \Omega^4(M, \mathrm{End}(E))$ is defined like the exterior product of scalar 2-forms, with the product of real numbers replaced by the composition of endomorphisms. Express the 4-form (8.4.5) in the form $p_1(F^\nabla)$ for a suitable invariant degree-2 polynomial $p_1 : \mathfrak{u}(2) \to \mathbb{R}$.

Example 8.4.6 (Tensor Products of Complex Line Bundles). Let

$$\pi_1 : E_1 \to M, \qquad \pi_2 : E_2 \to M$$

be complex line bundles and consider the tensor product

$$E := E_1 \otimes E_2 := \left\{ (p, e_1 \otimes e_2) \,\middle|\, \begin{array}{l} p \in M,\, e_1 \in E_1,\, e_2 \in E_2, \\ \pi_1(e_1) = \pi_2(e_2) = p \end{array} \right\}.$$

This is again a complex line bundle over M and its first Chern class is the sum of the first Chern classes of E_1 and E_2:

$$c_1(E_1 \otimes E_2) = c_1(E_1) + c_1(E_2). \tag{8.4.6}$$

(Here we use the formula (8.4.2) as the definition of the first Chern class in the case of a noncompact base manifold.) To see this, we choose Hermitian structures on E_1 and E_2 and Hermitian local trivializations over an open cover $\{U_\alpha\}_\alpha$ of M with transition maps $g_{i,\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{U}(1) = S^1$. These give rise, in an obvious manner, to a Hermitian structure on the tensor product $E = E_1 \otimes E_2$ and to local trivializations of E with transition maps

$$g_{\beta\alpha} = g_{1,\beta\alpha} \cdot g_{2,\beta\alpha} : U_\alpha \cap U_\beta \to S^1.$$

For $i = 1, 2$ choose a Hermitian connection ∇_i on E_i with connection potentials

$$A_{i,\alpha} \in \Omega^1(U_\alpha, \mathbf{i}\mathbb{R}).$$

They determine a connection ∇ on E via the Leibniz rule

$$\nabla(s_1 \otimes s_2) := (\nabla_1 s_1) \otimes s_2 + s_1 \otimes (\nabla_2 s_2)$$

for $s_1 \in \Omega^0(M, E_1)$ and $s_2 \in \Omega^0(M, E_2)$. The connection potentials of ∇ are

$$A_\alpha = A_{1,\alpha} + A_{2,\alpha} \in \Omega^1(U_\alpha, \mathbf{i}\mathbb{R}).$$

Hence the curvature of F^∇ is given by

$$F^\nabla = F^{\nabla_1} + F^{\nabla_2} \in \Omega^2(M, \mathbf{i}\mathbb{R}).$$

In fact, the restriction of F^∇ to U_α is just the differential of A_α. Since $c_1(E)$ is the cohomology class of the real valued closed 2-form $\frac{\mathbf{i}}{2\pi} F^\nabla \in \Omega^2(M)$, this implies equation (8.4.6).

Example 8.4.7 (The Inverse of a Complex Line Bundle). Let $\mathbb{E} \to M$ be a complex line bundle with transition maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}.$$

Then there is a complex line bundle

$$E^{-1} \to M,$$

unique up to isomorphism, with transition maps

$$g_{\beta\alpha}^{-1} : U_\alpha \cap U_\beta \to \mathbb{C}^*.$$

Its tensor product with E is isomorphic to the trivial bundle. Hence

$$c_1(E^{-1}) = -c_1(E)$$

by equation (8.4.6).

Example 8.4.8 (Complex Line Bundles over $\mathbb{C}\mathrm{P}^n$). For $d \in \mathbb{Z}$ consider the complex line bundle

$$H^d := \frac{S^{2n+1} \times \mathbb{C}}{S^1} \to \mathbb{C}\mathrm{P}^n$$

where the circle S^1 acts on $S^{2n+1} \times \mathbb{C}$ by

$$\lambda \cdot (z_0, \dots, z_n; \zeta) := (\lambda z_0, \lambda z_1, \cdots : \lambda z_n; \lambda^d \zeta)$$

for $(z_0, \dots, z_n) \in S^{2n+1} \subset \mathbb{C}^{n+1}$, $\zeta \in \mathbb{C}$, and $\lambda \in S^1$. The equivalence classes in H^d are denoted by

$$[z_0 : z_1 : \cdots : z_n; \zeta] \equiv [\lambda z_0 : \lambda z_1 : \cdots : \lambda z_n; \lambda^d \zeta].$$

For $d = 0$ this is the trivial bundle, for $d > 0$ it is the d-fold tensor product of the line bundle $H \to \mathbb{C}\mathrm{P}^n$ in Theorem 7.3.19, and we have

$$H^{-d} \cong (H^d)^{-1}.$$

Hence, by Theorem 7.3.19, equation (8.4.6), and Example 8.4.7, we have

$$c_1(H^d) = dh$$

for every $d \in \mathbb{Z}$. Here $h \in H^2(\mathbb{C}\mathrm{P}^n)$ is the positive integral generator with integral one over the submanifold $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^n$.

8.5 Chern Classes in Geometry

This section discusses several topics in geometry where the Chern classes play an important role. In §8.5.1 we introduce the notions of a complex manifold and a holomorphic line bundle and compute the Chern class of $\mathbb{CP}^n$. §8.5.2 is devoted to the adjunction formula and §8.5.3 examines the relation between the first Chern class and the self-intersection numbers of a compact oriented 2-dimensional submanifold of a complex surface. In §8.5.4 we discuss the Hirzebruch Signature Theorem without proof and in §8.5.5 we explain how this result can be used to compute the cohomology of a complex hypersurface of $\mathbb{CP}^3$. The sections closes with a discussion of almost complex structures on smooth 4-manifolds in §8.5.6.

8.5.1 Complex Manifolds and Holomorphic Line Bundles

We begin with the definition of a complex manifold.

Definition 8.5.1 (Complex Manifold). *A* **complex n-manifold** *is a real $2n$-dimensional manifold X equipped with an atlas*

$$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{C}^n$$

such that the transition maps

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are holomorphic maps between open subsets of $\mathbb{C}^n$. This means that the real derivative of $\phi_\beta \circ \phi_\alpha^{-1}$ at every point is given by multiplication with a complex $n \times n$-matrix. A complex 1-manifold is called a **complex curve** *and a complex 2-manifold is called a* **complex surface***. Thus a complex curve has real dimension two and a complex surface has real dimension four.*

Complex manifolds are oriented and their tangent bundles inherit complex structures from the coordinate charts. Thus the tangent bundle TX of a complex manifold has Chern classes. If X is a complex n-manifold with an atlas as above, a smooth function

$$f : U \to \mathbb{C}$$

on an open subset $U \subset X$ is called **holomorphic** iff the function

$$f \circ \phi_\alpha^{-1} : \phi_\alpha(U \cap U_\alpha) \to \mathbb{C}$$

is holomorphic for each α. Equivalently, the derivative $df(p) : T_pX \to \mathbb{C}$ is complex linear for every $p \in U$.

Example 8.5.2 (The Chern Class of $\mathbb{C}\mathrm{P}^n$). The complex projective space $\mathbb{C}\mathrm{P}^n$ is a complex manifold and hence its tangent bundle has Chern classes. In the geometric description of $\mathbb{C}\mathrm{P}^n$ as the space of complex lines in $\mathbb{C}^{n+1}$ the tangent space of $\mathbb{C}\mathrm{P}^n$ at a point $\ell \in \mathbb{C}\mathrm{P}^n$ is given by

$$T_\ell \mathbb{C}\mathrm{P}^n = \mathrm{Hom}^{\mathbb{C}}(\ell, \ell^\perp).$$

Geometrically, every line in $\mathbb{C}^{n+1}$ sufficiently close to ℓ is the graph of a complex linear map from ℓ to $\ell^\perp$. Moreover, each complex linear map from ℓ to itself is given by multiplication with a complex number. Thus $\mathrm{Hom}^{\mathbb{C}}(\ell,\ell) = \mathbb{C}$ and hence $T_\ell\mathbb{C}\mathrm{P}^n \oplus \mathbb{C} \cong \mathrm{Hom}^{\mathbb{C}}(\ell, \ell^\perp \oplus \ell) = \mathrm{Hom}^{\mathbb{C}}(\ell, \mathbb{C}^{n+1})$. Thus the direct sum of $T\mathbb{C}\mathrm{P}^n$ with the trivial bundle $H^0 = \mathbb{C}\mathrm{P}^n \times \mathbb{C}$ is the $(n+1)$-fold direct sum of the bundle $H \to \mathbb{C}\mathrm{P}^n$ in Theorem 7.3.19 with itself, i.e.

$$T\mathbb{C}\mathrm{P}^n \oplus H^0 = \underbrace{H \oplus H \oplus \cdots \oplus H}_{n+1 \text{ times}}.$$

Since $c(H) = 1 + h$ it follows from the *(Zero)* and *(Sum)* axioms that

$$c(T\mathbb{C}\mathrm{P}^n) = (1+h)^{n+1},$$

where $h \in H^2(\mathbb{C}\mathrm{P}^n)$ is the positive integral generator as in Theorem 7.3.19.

Definition 8.5.3 (Holomorphic Line Bundle). *A* **holomorphic line bundle** *over a complex manifold X is a complex line bundle*

$$\pi : E \to X$$

equipped with local trivializations such that the transition maps

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

are holomorphic. A **holomorphic section** *of such a holomorphic line bundle E is a section $s : X \to E$ that, in the local trivializations, is represented by holomorphic functions $s_\alpha : U_\alpha \to \mathbb{C}$. The notion makes sense because the s_α are related by $s_\beta = g_{\beta\alpha} s_\alpha$ on $U_\alpha \cap U_\beta$ and the $g_{\beta\alpha}$ are holomorphic.*

If we choose a Hermitian structure on a holomorphic line bundle and Hermitian trivializations, the transition maps will no longer be holomorphic, by the maximum principle, unless they are locally constant. It is therefore often more convenient to use the original holomorphic trivializations.

Example 8.5.4 (Holomorphic Line Bundles over $\mathbb{C}\mathrm{P}^n$). The line bundle $H^d \to \mathbb{C}\mathrm{P}^n$ in Example 8.4.8 admits the structure of a holomorphic line bundle. More precisely, the standard atlas $\phi_i : U_i \to \mathbb{C}^n$ defined by

$$U_i := \{[z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \mid z_i \neq 0\}$$

and

$$\phi_i([z_0 : \cdots : z_n]) := \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right)$$

has holomorphic transition maps. A trivialization of H^d over U_i is the map $\psi_i : H^d|_{U_i} \to U_i \times \mathbb{C}$ defined by

$$\psi_i([z_0 : \cdots : z_n; \zeta]) := \left([z_0 : \cdots : z_n], \frac{\zeta}{z_i^d}\right).$$

The transition maps $g_{ji} : U_i \cap U_j \to \mathbb{C}^*$ are then given by

$$g_{ji}([z_0 : \cdots : z_n]) = \left(\frac{z_i}{z_j}\right)^d$$

and they are evidently holomorphic. For $d \geq 0$ every homogeneous complex polynomial $p : \mathbb{C}^{n+1} \to \mathbb{C}$ of degree d determines a holomorphic section

$$s([z_0 : \cdots : z_n]) = [z_0 : \cdots : z_n; p(z_0, \dots, z_n)]$$

of H^d. It turns out that these are all the holomorphic sections of H^d and that the only holomorphic section of H^d for $d < 0$ is the zero section. However the proof of these facts would take us too far afield into the realm of algebraic geometry. An excellent reference is the book [11] by Griffiths and Harris.

8.5.2 The Adjunction Formula

Let X be a compact connected complex surface and let $C \subset X$ be a **smooth complex curve**. Thus C is a compact submanifold without boundary whose tangent space T_xC at each point $x \in C$ is a one-dimensional complex subspace of T_xX. In particular, C is a compact oriented 2-manifold without boundary. The **adjunction formula** asserts

$$\langle c_1(TX), C\rangle = \chi(C) + C \cdot C, \tag{8.5.1}$$

where $C \cdot C$ denotes the self-intersection number of C, $\chi(C)$ denotes the Euler characteristic of C, and $\langle c_1(TX), C\rangle$ denotes the integral of (a representative of) the first Chern class $c_1(TX) \in H^2(X)$ over C.

To prove the adjunction formula we choose a Riemannian metric on X such that the complex structure on each tangent space T_xX is a skew symmetric automorphism. Thus both the tangent bundle of C and the normal bundle $TC^\perp$ are complex vector bundles over C and the restriction of TX to C is the direct sum

$$TX|_C = TC \oplus TC^\perp.$$

By the *(Euler Class)* axiom for the Chern classes and Example 7.3.8 we have

$$\langle c_1(TC), C\rangle = \langle e(TC), C\rangle = \chi(C).$$

Using the *(Euler Class)* axiom again we obtain

$$\langle c_1(TC^\perp), C\rangle = \langle e(TC^\perp), C\rangle = C \cdot C.$$

Here the last equality follows from Corollary 7.3.13. Now the *(Sum)* axiom for the Chern classes asserts that

$$\langle c_1(TX), C\rangle = \langle c_1(TC), C\rangle + \langle c_1(TC^\perp), C\rangle$$

and this proves (8.5.1).

Now suppose that $\pi : E \to X$ is a holomorphic line bundle over a compact connected complex surface without boundary and $s : X \to E$ is a holomorphic section that is transverse to the zero section. Then it follows directly from the definitions that its zero set $C := s^{-1}(0)$ is a compact complex curve without boundary. Let us also assume that C is connected and denote by g the genus of C, understood as a compact connected oriented 2-manifold without boundary. By Example 6.4.15 we have

$$\chi(C) = 2 - 2g$$

and hence the adjunction formula (8.5.1) takes the form

$$\begin{aligned} 2 - 2g &= \langle c_1(TX), C\rangle - C \cdot C \\ &= \langle c_1(TX) - c_1(E), C\rangle \\ &= \int_X \big(c_1(TX) - c_1(E)\big) \cup c_1(E) \end{aligned} \tag{8.5.2}$$

Here the second equality follows from the fact that the vertical derivative Ds along $C = s^{-1}(0)$ furnishes an isomorphism form the normal bundle $TC^\perp$ to the restriction $E|_C$. The last equality follows from the fact that the Euler class $c_1(E) = e(E)$ is dual to C, by Theorem 7.3.15.

Example 8.5.5 (Degree-d Curves in $\mathbb{C}\mathrm{P}^2$). As a specific example we take $X = \mathbb{C}\mathrm{P}^2$ and $E = H^d$. Suppose that $p : \mathbb{C}^3 \to \mathbb{C}$ is a homogeneous complex degree-d polynomial and that the resulting holomorphic section $s : \mathbb{C}\mathrm{P}^2 \to H^d$ is transverse to the zero section (see Example 8.5.4). Then the zero set of s is a smooth degree-d curve

$$C_d = \left\{[z_0 : z_1 : z_2] \in \mathbb{C}\mathrm{P}^2 \,|\, p(z_0, z_1, z_2) = 0\right\}.$$

By Example 8.4.8 we have $c_1(H^d) = dh$ and by Example 8.5.2 we have $c_1(T\mathbb{C}\mathrm{P}^2) = 3h$. Thus equation (8.5.2) asserts that the genus $g = g(C_d)$ of the complex curve C_d satisfies the equation

$$2 - 2g = (3 - d)d \int_{\mathbb{C}\mathrm{P}^2} h \cup h = 3d - d^2.$$

Here the second equality follows from (7.3.10). Thus we have proved that

$$g(C_d) = \frac{(d-1)(d-2)}{2}. \tag{8.5.3}$$

This is the original version of the adjunction formula. One can verify it geometrically by deforming a degree-d curve to a union of d *generic* lines in $\mathbb{C}\mathrm{P}^2$. Any two of these lines intersect in exactly one point and *"generic"* means here that these points are pairwise distinct. Thus we end up with a total of $d(d-1)/2$ intersection points. Performing a *connected sum* operation at each of the intersection points one can verify the formula (8.5.3).

A compact connected oriented 2-dimensional submanifold $\Sigma \subset \mathbb{C}\mathrm{P}^2$ without boundary is said to **represent the cohomology class** dh iff

$$dh = [\tau_\Sigma]$$

is Poincaré dual to Σ as in §6.4.3. Thus our complex degree-d curve C_d is such a representative of the class dh. A remarkable fact is that every representative of the class dh has at least the genus of C_d, i.e.

$$g(\Sigma) \geq \frac{(d-1)(d-2)}{2}. \tag{8.5.4}$$

This is the socalled **Thom Conjecture** which was open for many years and was finally settled in the nineties by Kronheimer and Mrowka [20], using Donaldson theory. They later extended their result to much greater generality and proved, with the help of Seiberg–Witten theory, that every 2-dimensional symplectic submanifold with nonnegative self-intersection number in a symplectic 4-manifold minimizes the genus in its cohomology class. For an exposition see their book [21]. The case of negative self-intersection number was later settled by Ozsvath and Szabo [32].

8.5.3 Chern Class and Self-Intersection

Let X be a complex surface and let

$$\Sigma \subset X$$

be a compact oriented 2-dimensional submanifold without boundary. Then the integral of the first Chern class of TX over Σ agrees modulo two with the self-intersection number:

$$\langle c_1(TX), \Sigma\rangle \equiv \Sigma \cdot \Sigma \mod 2. \tag{8.5.5}$$

To see this, choose any complex structure on each of the real rank-2 bundles $T\Sigma$ and $T\Sigma^\perp$. Then the same argument as in the proof of the adjunction formula (8.5.1) shows that the integral of the first Chern class of this new complex structure on $TX|_\Sigma$ over Σ is the sum

$$\chi(\Sigma) + \Sigma \cdot \Sigma.$$

Since the Euler characteristic $\chi(\Sigma)$ is even and the integrals of the first Chern classes of $TX|_\Sigma$ with both complex structures agree modulo two, by Exercise 8.5.6 below, this proves (8.5.5).

Exercise 8.5.6 (Complex Rank-2 Bundles over Real 2-Manifolds). Let Σ be compact connected oriented 2-manifold without boundary.

(i) There are precisely two oriented real rank 4-bundles over Σ, one trivial and one nontrivial.

(ii) Every oriented real rank 4-bundle admits a complex structure compatible with the orientation.

(iii) A complex rank-2-bundle $\pi : E \to \Sigma$ admits a real trivialization if and only if its first Chern number $\langle c_1(E), \Sigma\rangle = \int_\Sigma c_1(E)$ is even.

Hint 1: An elegant proof of these facts can be given by means of the Stiefel–Whitney classes (see Milnor–Stasheff [28]).

Hint 2: Consider the trivial bundle $\Sigma \times \mathbb{R}^4$ and identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ via $x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ where $\mathbf{i}^2 + \mathbf{j}^2 + \mathbf{k}^2 = -1$ and $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. Show that every complex structure on $\mathbb{H}$ that is compatible with the inner product and orientation has the form

$$J_\lambda = \lambda_1\mathbf{i} + \lambda_2\mathbf{j} + \lambda_3\mathbf{k}, \qquad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1.$$

Thus a complex structure on $E = \Sigma \times \mathbb{H}$ that is compatible with the metric and orientation has the form $z \mapsto J_{\lambda(z)}$ where $\lambda : \Sigma \to S^2$ is a smooth map. Prove that the first Chern number of (E, J_λ) is given by

$$\int_\Sigma c_1(E, J_\lambda) = 2 \deg(\lambda : \Sigma \to S^2).$$

Use the ideas in the next hint.

Hint 3: Here is a sketch of a proof that the first Chern numbers of any two complex structures on an oriented real rank 4-bundle $\pi : E \to \Sigma$ agree modulo two. By transversality every real vector bundle whose rank is bigger than the dimension of the base has a nonvanishing section (see Chapter 4). Hence E has two linearly independent sections s_1 and s_2. Denote by $\Lambda \subset E$ the subbundle spanned by s_1 and s_2. Given a complex structure J on E denote by $E_1 \subset E$ the complex subbundle spanned by s_1 and Js_1. Thus E_1 has a global trivialization and so the first Chern number of the complex line bundle E/E_1 agrees with the first Chern number of (E, J). Show that this number agrees modulo two with the Euler number of the oriented real rank-2 bundle E/Λ. To see this, think of s_2 as a section of E/E_1 and of Js_1 as a section of E/Λ. Both sections have the same zeros: the points $z \in \Sigma$ where Λ_z is a complex subspace of E_z. Prove that the transversality conditions for both sections are equivalent. Compare the indices of the zeros.

Hint 4: Choose an closed disk $D \subset \Sigma$ and show via parallel transport that the restrictions of E to both D and $\overline{\Sigma \setminus D}$ admit global trivializations. This requires the existence of a pair-of-pants decomposition of Σ (see Hirsch [13]). Assuming this we obtain two trivializations over the boundary

$$\Gamma := \partial D \cong S^1.$$

These differ by a loop in the structure group. In the complex case this construction gives rise to a loop

$$g : S^1 \to \mathrm{U}(2) \subset \mathrm{SO}(4).$$

In the real case we get a loop in SO(4). Prove that, in the complex case with the appropriate choice of orientations, the first Chern number of E is given by

$$\int_\Sigma c_1(E) = \deg(\det \circ g : S^1 \to S^1).$$

Prove that a loop $g : S^1 \to \mathrm{U}(2)$ is contractible in SO(4) if and only if the degree of the composition $\det \circ g : S^1 \to S^1$ has even degree.

8.5.4 The Hirzebruch Signature Theorem

Let X be a compact connected oriented smooth 4-manifold without boundary. Then Poincaré duality (Theorem 6.4.1) asserts that the Poincaré pairing

$$H^2(X) \times H^2(X) \to \mathbb{R} : ([\omega],[\tau]) \mapsto \int_X \omega \wedge \tau, \tag{8.5.6}$$

is nondegenerate. The pairing (8.5.6) is a symmetric bilinear form, also called the **intersection form of** X and denoted by

$$Q_X : H^2(X) \times H^2(X) \to \mathbb{R}.$$

Thus the second Betti number $b_2(X) = dim H^2(X)$ is a sum

$$b_2(X) = b^+(X) + b^-(X)$$

where $b^+(X)$ is the maximal dimension of a subspace of $H^2(X)$ on which the intersection form Q_X is positive definite and $b^-(X)$ is the maximal dimension of a subspace of $H^2(X)$ on which Q_X is negative definite. Equivalently, $b^+(X)$ is the number of positive entries and $b^-(X)$ is the number of negative entries in any diagonalization of Q_X. The **signature** of X is defined by

$$\sigma(X) := b^+(X) - b^-(X).$$

The **Hirzebruch Signature Theorem** asserts that, if X is a complex surface, then

$$\int_X c_1(TX) \cup c_1(TX) = 2\chi(X) + 3\sigma(X). \tag{8.5.7}$$

Equivalently, the signature is one third of the integral of the cohomology class

$$c_1(TX)^2 - 2c_2(TX) \in H^4(X)$$

over X. The class $c_1^2 - 2c_2$ is the first Pontryagin class and is also defined for arbitrary real vector bundles $E \to X$ (see Exercise 8.4.5). Thus equation (8.5.7) can be expressed in the form $\sigma(X) = \frac{1}{3}p_1(TX)$. (Here we use the same notation $p_1(TX)$ for a 4-dimensional de Rham cohomology class and for its integral over X.) In this form the Hirzebruch Signature Theorem remains valid for all compact connected oriented smooth 4-manifold without boundary. It is a deep theorem in differential topology and its proof goes beyond the scope of this book.

As an explicit example consider the complex projective plane

$$X = \mathbb{C}\mathrm{P}^2, \qquad c_1(X) = 3h, \qquad \chi(X) = 3, \qquad \sigma(X) = 1,$$

Another example is

$$X = S^2 \times S^2, \qquad c_1(X) = 2a + 2b, \qquad \chi(X) = 4, \qquad \sigma(X) = 0.$$

Here we choose as a basis of $H^2(S^2 \times S^2)$ the cohomology classes a and b of two volume forms with integral one on the two factors, pulled back to the product. The intersection form is in this basis given by

$$Q_X \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A third example is the 4-torus $X = \mathbb{T}^4 = \mathbb{C}^2/\mathbb{Z}^4$ with its standard complex structure. In this case both Chern classes are zero and $\chi(\mathbb{T}^4) = \sigma(\mathbb{T}^4) = 0$. **Exercise:** Verify the last equality by choosing a suitable basis of $H^2(\mathbb{T}^4)$. Verify the Hirzebruch signature formula in all three cases.

8.5.5 Hypersurfaces of $\mathbb{C}\mathrm{P}^3$

An interesting class of complex 4-manifolds is given by complex hypersurfaces of $\mathbb{C}\mathrm{P}^3$. Their cohomology groups can be computed with the help of the the Chern classes and the Hirzebruch Signature Theorem by a beautiful argument due to Milnor. Here is how this works.

Consider the holomorphic line bundle

$$H^d \to \mathbb{C}\mathrm{P}^3$$

in Example 8.5.4 and let $p : \mathbb{C}^4 \to \mathbb{C}$ be a homogeneous complex degree-d polynomial. Assume that the resulting holomorphic section $s : \mathbb{C}\mathrm{P}^3 \to H^d$ is transverse to the zero section. Then its zero set

$$X_d := \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \,|\, p(z_0, z_1, z_2, z_3) = 0 \right\}.$$

is a complex submanifold of $\mathbb{C}\mathrm{P}^3$ and hence is a complex surface. In this case the **Lefschetz Hyperplane Theorem** asserts that X_d is connected and simply connected. (More generally, the Lefschetz Hyperplane Theorem, asserts that the zero set of a transverse holomorphic section of a *"sufficiently nice"* holomorphic line bundle inherits the homotopy and cohomology groups of the ambient manifold below the middle dimension; *"nice"* means that the line bundle has lots of holomorphic sections or, in technical terms, is *"ample"*. The line bundle $H^d \to \mathbb{C}\mathrm{P}^n$ satisfies this condition for $d > 0$.)

We will prove that

$$\begin{aligned}\chi(X_d) &= d^3 - 4d^2 + 6d, \\ \sigma(X_d) &= \frac{4d - d^3}{3}, \\ b^+(X_d) &= \frac{d^3 - 6d^2 + 11d - 3}{3}, \\ b^-(X_d) &= \frac{2d^3 - 6d^2 + 7d - 3}{3}.\end{aligned} \tag{8.5.8}$$

To see this, we first observe that, by Poincaré duality and the the Lefschetz Hyperplane Theorem, we have

$$b_0(X_d) = b_4(X_d) = 1, \qquad b_1(X_d) = b_3(X_d) = 0.$$

Hence

$$\chi(X_d) = 2 + b^+ + b^-$$

and so the last two equations in (8.5.8) follow from the first two. Next we choose a Riemannian metric on $\mathbb{C}\mathrm{P}^3$ with respect to which the standard complex structure is skew-symmetric (for example the Fubini–Study metric [11]). This gives a splitting

$$T\mathbb{C}\mathrm{P}^3|_{X_d} = TX_d \oplus TX_d^\perp$$

into complex subbundles. The vertical derivative of s along X again provides us with an isomorphism $Ds : TX_d^\perp \to E|_{X_d}$. Thus, by the *(Sum)* axiom for the Chern classes and Example 8.5.2, we have

$$(1 + h)^4 = c(T\mathbb{C}\mathrm{P}^3) = c(TX_d)c(TX_d^\perp) = c(TX_d)(1 + dh).$$

Here we think of the cohomology classes on $\mathbb{C}\mathrm{P}^3$ as their restrictions to X_d. Abbreviating

$$c_1 := c_1(TX_d), \qquad c_2 := c_2(TX_d),$$

we obtain

$$1 + 4h + 6h^2 = (1 + c_1 + c_2)(1 + dh) = 1 + (c_1 + dh) + (c_2 + dhc_1)$$

and hence

$$c_1 = (4 - d)h, \qquad c_2 = 6h^2 - dhc_1 = (d^2 - 4d + 6)h^2.$$

Since X_d is the zero set of a smooth section of H^d it is dual to the Euler class $e(H^d) = c_1(H^d) = dh$ (see Example 8.4.8), by Theorem 7.3.15. Hence

$$\int_{X_d} h \cup h = d \int_{\mathbb{C}\mathrm{P}^3} h \cup h \cup h = d.$$

Here the second equality follows from (7.3.10). Combining the last three equations we find

$$\chi(X_d) = \int_{X_d} c_2(TX) = (d^2 - 4d + 6) \int_{X_d} h \cup h = d^3 - 4d^2 + 6d$$

and

$$\int_{X_d} c_1(TX_d) \cup c_1(X_d) = (d-4)^2 \int_{X_d} h \cup h = d(d-4)^2.$$

Hence the Hirzebruch signature formula gives

$$\sigma(X_d) = \frac{d(d-4)^2 - 2d^3 + 8d^2 - 12d}{3} = \frac{4d - d^3}{3}$$

and this proves (8.5.8).

The first two examples are $X_1 \cong \mathbb{C}\mathrm{P}^2$ and $X_2 \cong S^2 \times S^2$. The reader may verify that the numbers in equation (8.5.8) match in these cases. The cubic surfaces in $\mathbb{C}\mathrm{P}^3$ are all diffeomorphic to $\mathbb{C}\mathrm{P}^2$ with six points *blown up*. This blowup construction is an operation in algebraic geometry, where one removes a point in the manifold and replaces it by the set of all complex lines through the origin in the tangent space at that point. Such a blowup admits in a canonical way the structure of a complex manifold [11]. An alternative description of X_3 is as a connected sum

$$X_3 = \mathbb{C}\mathrm{P}^2 \# 6\overline{\mathbb{C}\mathrm{P}}^2.$$

Here $\overline{\mathbb{C}\mathrm{P}}^2$ refers to the complex projective plane with the orientation reversed, which is not a complex manifold. (Its signature is minus one and the number $2\chi(\overline{\mathbb{C}\mathrm{P}}^2) + 3\sigma(\overline{\mathbb{C}\mathrm{P}}^2) = 3$ is not the integral of the square of any 2-dimensional cohomology class.) The symbol # refers to the connected sum operation where one cuts out balls from the two manifolds and glues the complements together along their boundaries, which are diffeomorphic to the 3-sphere. The resulting manifold is oriented and the numbers $b^\pm$ are additive under this operation Thus

$$\chi(X_3) = 9, \qquad \sigma(X_3) = -5, \qquad b^+(X_3) = 1, \qquad b^-(X_3) = 6$$

and this coincides with (8.5.8) for $d = 3$.

Particularly interesting examples are the quartic surfaces in $\mathbb{C}\mathrm{P}^3$. They are **K3-surfaces**. These can be uniquely characterized (up to diffeomorphism) as compact connected simply connected complex surfaces without boundary whose first Chern classes vanish. These manifolds do not all admit complex embeddings into $\mathbb{C}\mathrm{P}^3$ but the surfaces of type X_4 are examples. They have characteristic numbers

$$\chi(X_4) = 24, \qquad \sigma(X_4) = -16, \qquad b^+(X_4) = 3, \qquad b^-(X_4) = 19,$$

which one can read off equation (8.5.8). One can also deduce these numbers from the Hirzebruch signature formula, which in this case takes the form $0 = 2\chi + 3\sigma = 4 + 5b^+ - b^-$. That the number b^+ must be equal to 3 follows from the existence of a Ricci-flat Kähler metric, a deep theorem of Yau, and this implies that the complex exterior power $\Lambda^{2,0}T^*X$ has a global nonvanishing holomorphic section. Therefore the dimension p_g of the space of holomorphic sections of this bundle is equal to one, and it then follows from Hodge theory that $b^+ = 1 + 2p_g = 3$. The details of this lie again much beyond what is covered in the present book.

The distinction between the cases

$$d < 4, \qquad d = 4, \qquad d > 4$$

for hypersurfaces of $\mathbb{C}\mathrm{P}^3$ is analogous to the distinction of complex curves in terms of the genus. For curves in $\mathbb{C}\mathrm{P}^2$ these are the cases $d < 3$ (genus zero/positive curvature), $d = 3$ (genus one/zero curvature), and $d > 3$ (higher genus/negative curvature). In the present situation the case $d < 4$ gives examples of Fano surfaces analogous to the 2-sphere, the $K3$-surfaces with $d = 4$ correspond to the 2-torus allthough they do not admit flat metrics, and for $d > 4$ the manifold X_d is an example of a **surface of general type** in analogy with higher genus curves.

Exercise 8.5.7. Show that the polynomial $p(z_0, \dots, z_n) = z_0^d + \cdots + z_n^d$ on $\mathbb{C}^{n+1}$ gives rise to a holomorphic section $s : \mathbb{C}\mathrm{P}^n \to H^d$ that is transverse to the zero section. Hence its zero set X_d is a smooth complex hypersurface of $\mathbb{C}\mathrm{P}^n$. Prove that its first Chern class is zero whenever $d = n + 1$. Kähler manifolds with this property are called **Calabi–Yau manifolds**. The $K3$-surfaces are examples. The quintic hypersurfaces of $\mathbb{C}\mathrm{P}^4$ are examples of Calabi–Yau 3-folds and they play an important role in geometry and physics.

Exercise 8.5.8. Compute the Betti numbers of a degree-d hypersurface in $\mathbb{C}\mathrm{P}^4$. **Hint:** The Lefschetz Hyperplane Theorem asserts in this case that $b_0(X_d) = b_2(X_d) = 1$ and $b_1(X_d) = 0$.

8.5.6 Almost Complex Structures on Four-Manifolds

Let X be a $2n$-manifold. An **almost complex structure** on X is an automorphism $J : TX \to TX$ of the tangent bundle such that $J^2 = -\mathbb{1}$. The tangent bundle of any complex manifold has such a structure, as the multiplication by $\mathbf{i} = \sqrt{-1}$ in the coordinate charts carries over to the tangent bundle. However, not every almost complex structure arises from a complex structure (except in real dimension two).

Let us now assume that X is a compact connected oriented smooth 4-manifold without boundary. Let J be an almost complex structure on X and denote its first Chern class in de Rham cohomology by

$$c := c_1(TX, J) \in H^2(X).$$

This is an integral class in that the number $c\cdot\Sigma = \langle c, \Sigma\rangle = \int_\Sigma c$ is an integer for every compact oriented 2-dimensional submanifold $\Sigma \subset X$. Moreover, equation (8.5.5) carries over to the almost complex setting so that

$$c \cdot \Sigma \equiv \Sigma \cdot \Sigma \mod 2 \tag{8.5.9}$$

for every Σ as above. The Hirzebruch signature formula also continues to hold in the almost complex setting and hence

$$c^2 = 2\chi(X) + 3\sigma(X). \tag{8.5.10}$$

Here we abbreviate $c^2 := \langle c^2, X\rangle = \int_X c^2 \in \mathbb{Z}$. It turns out that, conversely, for every integral de Rham cohomology class $c \in H^2(X)$ that satisfies (8.5.9) and (8.5.10) there is an almost complex structure J on X with $c_1(TX, J) = c$. We will not prove this here. However, this can be used to examine which 4-manifolds admit almost complex structures and to understand their first Chern classes.

Exercise 8.5.9. Consider the 4-manifold $X = \mathbb{C}\mathrm{P}^2 \# k\overline{\mathbb{C}\mathrm{P}}^2$ (the projective plane with k points blown up). This manifold admits a complex structure by a direct construction in algebraic geometry [11]. Verify that it admits an almost complex structure by finding all integral classes $c \in H^2(X)$ that satisfy (8.5.9) and (8.5.10). Start with $k = 0, 1, 2$. (For $k \geq 2$ there are infinitely many such classes and, by work of T.C. Wall, they can all be represented by complex structures for $2 \leq k \leq 8$.)

Exercise 8.5.10. The k-fold connected sum $X = k\mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \cdots \# \mathbb{C}\mathrm{P}^2$ admits an almost complex structure if and only if k is odd.

Exercise 8.5.11. Which integral class $c \in H^2(\mathbb{T}^4)$ is the first Chern class of an almost complex structure on $\mathbb{T}^4$?

8.6 Low-Dimensional Manifolds

The examples in the previous section show that there is a rich world of manifolds out there whose study is the subject of differential topology and other related areas of mathematics, including complex, symplectic, and algebraic topology. The present notes only scratch the surface of some of these areas. One fundamental question in differential topology is how to tell if two manifolds of the same dimension m are diffeomorphic, or perhaps not diffeomorphic as the case may be. In this closing section we discuss some classical and some more recent answers to this question.

The easiest case is of course $m = 1$. We prove in Appendix A.4 that every compact connected smooth 1-manifold *without boundary* is diffeomorphic to the circle and in the case of *nonempty boundary* is diffeomorphic to the closed unit interval. We have seen that this observation plays a central role in the definitions of degree and intersection number, and in fact throughout differential topology.

The next case is $m = 2$, where this question is also completely understood, although the proof is considerably harder. Two compact connected oriented smooth 2-manifolds without boundary are diffeomorphic if and only if they have the same genus. As pointed out in Example 6.4.15, a beautiful proof of this theorem, based on Morse theory, is contained in the book of Hirsch [13]. The result generalizes to all compact 2-manifolds with or without boundary, and orientable or not. Both in the orientable and in the nonorientable case the diffeomorphism type of a compact connected 2-manifold is determined by the Euler characteristic and the number of boundary components. The proof is also contained in [13]. This does not mean, however, that the study of 2-manifolds has now been settled. For example the study of real 2-manifolds equipped with complex structures (called **Riemann surfaces**) is a rich field of research with connections to many areas of mathematics such as algebraic geometry, number theory, and dynamical systems. A classical result is the **uniformization theorem**, which asserts that every connected simply connected Riemann surface is holomorphically diffeomorphic to either the complex plane, or the open unit disk in the complex plane, or the 2-sphere with its standard complex structure. In particular, it is not necessary to assume that the Riemann surface is paracompact; paracompactness is a consequence of uniformization. This is a partial answer to a complex analogue of the aforementioned question. We remark that interesting objects associated to oriented 2-manifolds are the mapping class group (diffeomorphisms up to isotopy) and Teichmüller space (complex structures up to diffeomorphisms isotopic to the identity).

The compact connected manifolds without boundary in dimensions one and two are not simply connected except for the 2-sphere. Let us now turn to the higher-dimensional case and focus on simply connected manifolds. In dimension three a central question, which was open for about a century, is the following.

Three-Dimensional Poincaré Conjecture. *Every compact connected simply connected* 3*-manifold* M *without boundary is diffeomorphic to* S^3*.*

This conjecture has recently (in the early years of the 21st century) been confirmed by Grigory Perelman. His proof is a modification of an earlier program by Richard Hamilton to use the socalled **Ricci flow** on the space of all Riemannian metrics on M. The idea is, roughly speaking, to start with an arbitrary Riemannian metric and use it as an initial condition for the Ricci flow. It is then a hard problem in geometry and nonlinear parabolic partial differential equations to understand the behavior of the metric under this flow. The upshot is that, through lot of hard analysis and deep geometric insight, Perelman succeded in proving that the flow does converge to a round metric (with constant sectional curvature). Then a standard result in differential geometry provides a diffeomorphism to the 3-sphere. The proof of the Poincaré conjecture is one of the deepest theorems in differential topology and is an example of the power of analytical tools to settle questions in geometry and topology. There are now many expositions of Perelman's proof of the three-dimensional Poincaré conjecture, beyond Perelman's original papers, too numerous to discuss here. An example is the detailed book by Morgan and Tian [29].

The higher-dimensional analogue of the the Poincaré conjecture asserts that every compact connected simply connected smooth m-manifold M without boundary whose integral cohomology is isomorphic to that of the m-sphere, i.e.

$$H^k(M;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } k = 0 \text{ and } k = m, \\ 0, & \text{for } 1 \le k \le m-1, \end{cases}$$

is homeomorphic (or diffeomorphic) to the m-sphere. Here one distingushes between the **topological Poincaré conjecture** (which asserts the existence of a homeomorphism) and the **smooth Poincaré conjecture** (which asserts the existence of a diffeomorphism). The smooth Poincaré conjecture is still open in dimension four. However, the topological Poinaré conjecture in dimension four was settled by Michael Freedman in the 1980s.

Remarkably, the higher-dimensional Poincaré conjecture is much easier to understand than in dimensions three and four. The topological Poincaré conjecture in all dimensions $m \geq 6$ was settled long ago by Stephen Smale with the methods of Morse theory, and the proof was then extended to dimension $m = 5$ by Christopher Zeeman. A beautiful exposition of this subject is Milnor's book [27]. In many dimensions there are socalled **exotic spheres** that are homeomorphic but not diffeomorphic to the m-sphere. Examples are Milnor's famous exotic 7-spheres [24]. Later work by Kervaire and Milnor [19] showed that there are precisely 27 diffeomorphism classes of exotic spheres in dimension seven.

Let us now turn to compact connected simply connected smooth 4-manifolds X without boundary and with $H^2(X) \neq 0$. The intersection form

$$Q_X : H^2(X) \times H^2(X) \to \mathbb{R}$$

is then a diffeomorphism invariant and so are the numbers $b^+(X)$ and $b^-(X)$ (see §8.5.4). They are determined by the Euler characteristic and signature of X. In fact, more is true. The intersection form can be defined on integral cohomology and Poincaré duality over the integers asserts that it remains nondegenerate over the integers (which can be proved with the same methods as Theorem 6.4.1 once an integral cohomology theory has been set up). This means that the intersection form is represented by a symmetric integer matrix with determinant ±1 in any integral basis of $H^2(X;\mathbb{Z})$.

This leads to the issue of understanding quadratic forms over the integers. One must distinguish between the even and odd case, where **even** means that the number $Q(a,a)$ is even for every integer vector a and **odd** means that $Q(a,a)$ is odd for some integer vector a. Thus an oriented smooth 4-manifold X is called **even** iff the self-intersection number of every compact oriented 2-dimensional submanifold $\Sigma \subset X$ without boundary is even and it is called **odd** is the self-intersection number is odd for some Σ. This property (being even or odd) is called the **parity of** X. For example, it follows from (8.5.5) that a hypersurface $X_d \subset \mathbb{C}\mathrm{P}^3$ of degree d is odd if and only if d is odd. (Exercise: Prove this using the fact that $c_1(X_d) = (4-d)h$. Find a surface with odd self-intersection number when d is odd.)

Examples of even quadratic forms are

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad E_8 := \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Both matrices are symmetric and have determinant ±1. The second matrix is the Cartan matrix associated to the Dynkin diagram E_8 and is positive definite. A quadratic form (over the integers) is called **indefinite** iff both b^+ and b^- are nonzero. The classification theorem for nondegenerate quadratic forms over the integers asserts that every indefinite nondegenerate quadratic form is diagonalizable over the integers in the odd case (with entries ±1 on the diagonal) and in the even case is isomorphic to a direct sum of copies of H and $\pm E_8$. It follows, for example, that the self-intersection form of a $K3$-surface is isomorphic to $3H - 2E_8$. However, there are many positive (or negative) definite exotic quadratic forms. A deep theorem of Donaldson, that he proved in the early 1980s, asserts that the intersection form of a smooth 4-manifold is diagonalizable, whenever it is positive or negative definite. Thus the exotic forms do not appear as intersection forms of smooth 4-manifolds.

Donaldson's Diagonalizability Theorem. *If X is a compact connected oriented smooth 4-manifold without boundary with definite intersection form, then its intersection form is diagonalizable over the integers.*

Combining this with the aforementioned known facts about quadratic forms over the integers, we see that two compact connected simply connected oriented smooth 4-manifolds without boundary have isomorphic intersection forms over the integers if and only if they have the same Euler characteristic, signature, and parity. Now a deep theorem of Michael Freedman asserts that two compact connected simply connected oriented smooth 4-manifolds without boundary are homeomorphic if and only if they have isomorphic intersection forms over the integers. In the light of Donaldson's theorem Freedman's result can be rephrased as follows.

Freedman's Theorem. *Two compact connected simply connected oriented smooth 4-manifold without boundary are homeomorphic if and only if they have the same Euler characteristic, signature, and parity.*

A corollary is the Topological Poincaré Conjecture in Dimension Four. A natural question is if Freedman's theorem can be strengthened to provide a diffeomorphism. The answer is negative. In the early 1980s, around the same time when Freedman proved his theorem, Donaldson discovered remarkable invariants of compact oriented smooth 4-manifolds without boundary by studying the anti-self-dual Yang–Mills equations with structure group SU(2). He proved that the resulting invariants are nontrivial for Kähler surfaces whereas they are trivial for every connected sum $X_1 \# X_2$ with $b^+(X_i) > 0$. Thus two such manifolds cannot be diffeomorphic.

Donaldson's Theorem. *Let X be a compact connected simply connected Kähler surface without boundary and assume $b^+(X) \geq 2$. Then X is not diffeomorphic to any connected sum $k\mathbb{C}\mathrm{P}^2 \# \ell\overline{\mathbb{C}\mathrm{P}}^2$.*

The only candidate for such a connected sum would be with $k = b^+(X)$ and $\ell = b^-(X)$. Since $k \geq 2$, this manifold has trivial Donaldson invariants and so cannot be diffeomorphic to X. To make the statement interesting we also have to assume that X is odd. Then the two manifolds are homeomorphic, by Freedman's theorem. An infinite sequence of examples is provided by hypersurfaces $X_d \subset \mathbb{C}\mathrm{P}^3$ of odd degree $d \geq 5$ (see §8.5.5). These are connected simply connected Kähler surfaces, satisfy $b^+(X_d) \geq 2$, and they are odd. Hence Donaldson's theorem applies, and Freedman's theorem furnishes a homeomorphism to a connected sum of $\mathbb{C}\mathrm{P}^2$'s and $\overline{\mathbb{C}\mathrm{P}}^2$'s.

A beautiful introduction to Donaldson theory can be found in the book by Donaldson and Kronheimer [8]. The book includes a proof of Donaldson's Diagonalizability Theorem, which is also based on the study of anti-self-dual SU(2)-instantons. When Seiberg–Witten theory was discovered in 1994, Taubes proved that all symplectic 4-manifolds have nontrivial Seiberg–Witten invariants. Since the Seiberg–Witten invariants of connected sums have the same vanishing properties as Donaldson invariants, this gave rise to an extension of Donaldson's theorem with *"Kähler surface"* replaced by *"symplectic* 4*-manifold"*. Both Donaldson and Seiberg–Witten theory are important topics in the study of 3- and 4-manifolds with a wealth of results in various directions, the Kronheimer–Mrowka proof of the Thom conjecture being just one example (§8.5.2). In a nutshell one can think of these as intersection theories in suitable infinite-dimensional settings. This shows again the power of analytical methods in topology and geometry.

Appendix A

Notes

The appendix explains some foundational material that is used throughout this book. §A.1 examines paracompact topological spaces, §A.2 shows how to construct partitions of unity, §A.3 uses partitions of unity to embed second countable Hausdorff manifolds into Euclidean space, and §A.4 establishes the classification of one-manifolds following Milnor [26]. §A.5 discusses Riemannian metrics and the Levi-Civita connection and §A.6 reviews some background material about geodesics and the exponential map.

A.1 Paracompactness

The main result of this section asserts that every second countable locally compact Hausdorff space is paracompact. Here are the relevant definitions.

Definition A.1.1. *Let M be a topological space.*

- *M is called* **locally compact** *iff for every open set $U \subset M$ and every element $p \in U$ there exists a compact set K such that $p \in \mathrm{int}(K) \subset K \subset U$.*

- *M is called* **σ-compact** *iff there exists a sequence of compact sets $K_i \subset M$ such that $K_i \subset \mathrm{int}(K_{i+1})$ for all $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} K_i = M$,*

- *M is called* **second countable** *iff its topology has a countable base, i.e. there exists a a sequence of open sets $V_i \subset M$ such that for every open set $U \subset M$ and every $p \in U$ there exists an $i \in \mathbb{N}$ such that $p \in V_i \subset U$.*

- *M is called* **paracompact** *iff every open cover $\{U_\alpha\}_{\alpha \in A}$ of M admits a* **locally finite** *refinement, i.e. there exists an open cover $\{V_\beta\}_{\beta \in B}$ such that every set V_β is contained in one of the sets U_α and every element $p \in M$ has an open neighborhood W such that $\#\{\beta \in B \,|\, W \cap V_\beta \neq \emptyset\} < \infty$.*

We will use the basic facts that every compact subset of a Hausdorff topological space is closed and that every closed subset of a compact set is compact. We will also use the axiom of choice whenever convenient.

Lemma A.1.2. *Let M be a locally compact Hausdorff space, let $U \subset M$ be an open set, and let $K \subset U$ be a compact set. Then there exists an open set $V \subset M$ such that $\overline{V}$ is compact and*

$$K \subset V \subset \overline{V} \subset U.$$

Proof. Since $K \subset U$ and M is locally compact, every element $p \in K$ has a compact neighborhhod $B_p \subset U$. Since M is a Hausdorff space, the set B_p is closed. Hence $V_p := \mathrm{int}(B_p)$ is an open neighborhood of p such that

$$p \in V_p \subset \overline{V}_p \subset B_p \subset U.$$

Since $\{V_p\}_{p\in K}$ is an open cover of K and K is compact, there exist finitely elements $p_1, \dots, p_\ell \in K$ such that

$$K \subset V_{p_1} \cup \cdots \cup V_{p_\ell} =: V.$$

This set V is open. Its closure $\overline{V} = \overline{V}_{p_1} \cup \cdots \cup \overline{V}_{p_\ell}$ is a closed subset of the compact set $B := B_{p_1} \cup \cdots \cup B_{p_\ell}$ and hence is itself compact. Moreover, $\overline{V} \subset B \subset U$ and this proves Lemma A.1.2. □

Lemma A.1.3. *Let M be a second countable locally compact Hausdorff space. Then M is σ-compact.*

Proof. Let $\{V_i\}_{i\in\mathbb{N}}$ be a countable base for the topology of M and define

$$I := \{i \in \mathbb{N} \,|\, \overline{V}_i \text{ is compact}\}.$$

Then, by Lemma A.1.2, the collection $\{V_i\}_{i\in I}$ is still a countable base for the topology of M. For $k \in \mathbb{N}$ define the sets

$$U_k := \bigcup_{i\in I_k} V_i, \qquad B_k := \bigcup_{i\in I_k} \overline{V}_i, \qquad I_k := \{i \in I \,|\, i \le k\}.$$

Then, for every $k \in \mathbb{N}$, the set B_k is compact and hence is contained in the set U_ℓ for some $\ell \in \mathbb{N}$. For $k \in \mathbb{N}$ let $\nu(k) > k$ be the smallest integer bigger than k such that $B_k \subset U_{\nu(k)}$. Define the sequence $k_1 < k_2 < k_3 < \cdots$ inductively by $k_{i+1} := \nu(k_i)$ for $i \in \mathbb{N}$. Then the set $K_i := B_{k_i}$ is compact and, for every $i \in \mathbb{N}$, we have

$$K_i \subset U_{\nu(k_i)} \subset \mathrm{int}(B_{\nu(k_i)}) = \mathrm{int}(B_{k_{i+1}}) = \mathrm{int}(K_{i+1}).$$

Moreover $\bigcup_{i\in\mathbb{N}} K_i = \bigcup_{i\in I} \overline{V}_i = M$. This proves Lemma A.1.3. □

Theorem A.1.4. *Let M be a second countable locally compact Hausdorff space. Then M is paracompact.*

Proof. By Lemma A.1.3 there exists a sequence of compact sets $K_i \subset M$ that satisfies the conditions

$$K_i \subset \mathrm{int}(K_{i+1}) \qquad \text{for all } i \in \mathbb{N}, \qquad \bigcup_{i\in\mathbb{N}} K_i = M.$$

Let $K_i := \emptyset$ for $i \le 0$ and, for $i \in \mathbb{N}$, define

$$B_i := \overline{K_i \setminus K_{i-1}}, \qquad W_i := \mathrm{int}(K_{i+1}) \setminus K_{i-2}.$$

Then $\bigcup_{i\in\mathbb{N}} B_i = M$ and, for each $i \in \mathbb{N}$, the set B_i is compact, the set W_i is open, and $B_i \cap K_{i-2} \subset B_i \cap \mathrm{int}(K_{i-1}) = \emptyset$, and hence

$$B_i \subset W_i, \qquad W_i \cap W_{i+3} = \emptyset.$$

Now let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Then, for each $i \in \mathbb{N}$, the collection

$$\{W_i \cap U_\alpha\}_{\alpha\in A}$$

is an open cover of B_i and so has a finite subcover

$$B_i \subset \bigcup_{j=1}^{m_i} \left(W_i \cap U_{\alpha_{ij}}\right), \qquad \alpha_{i1}, \dots \alpha_{im_i} \in A.$$

It follows that the collection

$$\left\{W_i \cap U_{\alpha_{ij}} \,|\, i \in \mathbb{N},\, j = 1, \dots, m_i\right\}$$

is a locally finite refinement of the open cover $\{U_\alpha\}_{\alpha\in A}$ of M. Namely, each $p_0 \in M$ belongs to one of the sets W_{i_0}, and this set intersects only those sets $W_i \cap U_{\alpha_{ij}}$ with $i_0 - 2 \le i \le i_0 + 2$. This proves Theorem A.1.4. □

We remark that every locally compact Hausdorff space M is regular (i.e. for every closed subset $A \subset M$ and every point $p \in M \setminus A$ there exist disjoint open sets $U, V \subset M$ such that $A \subset U$ and $p \in V$). Hence it follows from Urysohn's Metrization Theorem [31, Theorem 34.1] that every second countable locally compact Hausdorff space is metrizable. Using this fact one can deduce Theorem A.1.4 from a general theorem which asserts that every metric space is paracompact [31, Theorem 41.4].

A.2 Partitions of Unity

This section establishes the existence of a partition of unity subordinate to any given open cover on any paracompact Hausdorff manifold.

Definition A.2.1. *Let M be a smooth manifold with boundary. A* **partition of unity** *on M is a collection of smooth functions $\rho_\alpha : M \to [0,1]$, indexed by the elements of a set A, such that each point $p \in M$ has an open neighborhood $V \subset M$ on which only finitely many ρ_α do not vanish, i.e.*

$$\#\left\{\alpha \in A \,\middle|\, \rho_\alpha|_V \not\equiv 0\right\} < \infty, \tag{A.2.1}$$

and, for every $p \in M$, we have

$$\sum_{\alpha \in A} \rho_\alpha(p) = 1. \tag{A.2.2}$$

If $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M, then a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ (indexed by the same set A) is called **subordinate to the cover** *if each ρ_α is supported in U_α, i.e.* $\mathrm{supp}(\rho_\alpha) := \overline{\{p \in M \,|\, \rho_\alpha(p) \neq 0\}} \subset U_\alpha$ *for all $\alpha \in A$.*

Theorem A.2.2 (Partitions of unity). *Let M be a smooth manifold with boundary whose topology is paracompact and Hausdorff. Then, for every open cover of M, there exists a partition of unity subordinate to that cover.*

Lemma A.2.3. *Let M be a smooth m-manifold with boundary whose topology is Hausdorff. Then, for every open set $V \subset M$ and every compact set $K \subset V$, there exists a smooth function $\kappa : M \to \mathbb{R}$ with compact support such that $\kappa \geq 0$, $\mathrm{supp}(\kappa) \subset V$, and $\kappa(p) > 0$ for all $p \in K$.*

Proof. Assume first that $K = \{p_0\}$ is a single point. Choose a compact neighborhood $C \subset V$ of p_0. Since M is Hausdorff C is closed and hence the open set $U := \mathrm{int}(C)$ satisfies $p_0 \in U \subset \overline{U} \subset C \subset V$. Shrinking U, if necessary, we may assume that there exists a coordinate chart $\phi : U \to \Omega$ with values in an open set $\Omega \subset \mathbb{H}^m$. Choose a smooth function $\kappa_0 : \Omega \to \mathbb{R}$ with compact support such that $\kappa \geq 0$ and $\kappa_0(\phi(p_0)) > 0$, and define $\kappa : M \to \mathbb{R}$ by $\kappa|_U := \kappa_0 \circ \phi$ and $\kappa|_{M \setminus U} := 0$. Then $\kappa \geq 0$, κ is snooth, $\mathrm{supp}(\kappa) \subset V$, and $\kappa(p_0) > 0$. This proves the lemma in the case where K is a point.

Now let K be any compact subset of V. Then, by the first part of the proof, there exists a collection of smooth functions $\kappa_p : M \to \mathbb{R}$, one for every $p \in K$, such that $\kappa_p \geq 0$, $\mathrm{supp}(\kappa_p) \subset V$, and $\kappa_p(p) > 0$. Since K is compact there are finitely many points $p_1, \dots, p_k \in K$ such that the sets $\left\{p \in M \,|\, \kappa_{p_j}(p) > 0\right\}$ cover K. Hence the function $\kappa := \sum_j \kappa_{p_j}$ is nonnegative, supported in V, and positive on K. This proves Lemma A.2.3. $\square$

Lemma A.2.4. *Let M be a topological space and let $\{V_i\}_{i\in I}$ is a locally finite collection of open sets in M. Then $\overline{\bigcup_{i\in I} V_i} = \bigcup_{i\in I} \overline{V}_i$.*

Proof. The set $\bigcup_{i\in I} \overline{V}_i$ is obviously contained in the closure of $\bigcup_{i\in I} V_i$. To prove the converse choose a point $p_0 \in M \setminus \bigcup_{i\in I} \overline{V}_i$. Since the collection $\{V_i\}_{i\in I}$ is locally finite, there exists an open neighborhood U of p_0 such that the set $I_0 := \{i \in I \,|\, V_i \cap U \neq \emptyset\}$ is finite. Hence $U_0 := U \setminus \bigcup_{i\in I_0} \overline{V}_i$ is an open neighborhood of p_0 and $U_0 \cap V_i = \emptyset$ for every $i \in I$. Thus $p_0 \notin \overline{\bigcup_{i\in I} V_i}$. This proves Lemma A.2.4. □

Proof of Theorem A.2.2. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. We prove in four steps that there is a partition of unity subordinate to this cover. The proofs of steps one and two are taken from [31, Lemma 41.6].

Step 1. *There exists a locally finite open cover $\{V_i\}_{i\in I}$ of M such that, for every $i \in I$, the closure $\overline{V}_i$ is compact and contained in one of the sets U_α.*

Denote by $\mathcal{V} \subset 2^M$ the set of all open sets $V \subset M$ such that $\overline{V}$ is compact and $\overline{V} \subset U_\alpha$ for some $\alpha \in A$. Since M is a locally compact Hausdorff space, this collection $\mathcal{V}$ is an open cover of M. (Namely, if $p \in M$, then there exists an index $\alpha \in A$ such that $p \in U_\alpha$; since M is locally compact, there exists a compact neighborhood $K \subset U_\alpha$ of p; since M is Hausdorff, the set K is closed and thus $V := \mathrm{int}(K)$ is an open neighborhood of p with $\overline{V} \subset K \subset U_\alpha$.) Since M is paracompact, the open cover $\mathcal{V}$ has a locally finite refinement $\{V_i\}_{i\in I}$. This cover satisfies the requirements of Step 1.

Step 2. *There exists a collection of compact sets $K_i \subset V_i$, one for each $i \in I$, such that $M = \bigcup_{i\in I} K_i$.*

Denote by $\mathcal{W} \subset 2^M$ the set of all open sets $W \subset M$ such that $\overline{W} \subset V_i$ for some $i \in I$. Since M is a locally compact Hausdorff space, the collection $\mathcal{W}$ is an open cover of M. Since M is paracompact, this open cover $\mathcal{W}$ has a locally finite refinement $\{W_j\}_{j\in J}$. Hence, by the axiom of choice, there exists a map $J \to I : j \mapsto i_j$ such that

$$\overline{W}_j \subset V_{i_j} \qquad \text{for all } j \in J.$$

Since the collection $\{W_j\}_{j\in J}$ is locally finite, so is the collection $\{W_j\}_{j\in J,\, i_j=i}$ for every $i \in I$. Hence it follows from Lemma A.2.4 that

$$K_i := \overline{\bigcup_{i_j=i} W_j} = \bigcup_{i_j=i} \overline{W}_j \subset V_i$$

for every $i \in I$. Since $\overline{V}_i$ is compact so is K_i. This proves Step 2.

Step 3. *There exists a partition of unity subordinate to the cover* $\{V_i\}_{i\in I}$.

Choose a collection of compact sets $K_i \subset V_i$ for $i \in I$ as in Step 2. Then, by Lemma A.2.3 and the axiom of choice, there is a collection of smooth functions $\kappa_i : M \to \mathbb{R}$ with compact support such that

$$\mathrm{supp}(\kappa_i) \subset V_i, \qquad \kappa_i \geq 0, \qquad \kappa_i|_{K_i} > 0 \qquad \text{for all } i \in I.$$

Since the cover $\{V_i\}_{i\in I}$ is locally finite, the sum $\kappa := \sum_{i\in I} \kappa_i : M \to \mathbb{R}$ is **locally finite** (i.e. each point in M has a neighborhood in which only finitely many terms do not vanish) and thus defines a smooth function on M. This function is everywhere positive, because each summand is nonnegative and, for each $p \in M$, there exists an index $i \in I$ such that $p \in K_i$ and hence $\kappa_i(p) > 0$. Thus the funtions $\chi_i := \kappa_i/\kappa$ define a partition of unity satisfying $\mathrm{supp}(\chi_i) \subset V_i$ for every $i \in I$ as required.

Step 4. *There exists a partition of unity subordinate to the cover* $\{U_\alpha\}_{\alpha\in A}$.

Let $\{\chi_i\}_{i\in I}$ be the partition of unity constructed in Step 3. By the axiom of choice there exists a map $I \to A : i \mapsto \alpha_i$ such that $V_i \subset U_{\alpha_i}$ for each $i \in I$. For $\alpha \in A$ define $\rho_\alpha : M \to [0,1]$ by

$$\rho_\alpha := \sum_{\alpha_i = \alpha} \chi_i.$$

Here the sum runs over all indices $i \in I$ with $\alpha_i = \alpha$. This sum is locally finite and hence is a smooth function on M. Moreover, each point in M has an open neighborhood in which only finitely many of the ρ_α do not vanish. Hence the sum of the ρ_α is a well defined function on M and

$$\sum_{\alpha\in A} \rho_\alpha = \sum_{\alpha\in A} \sum_{\alpha_i=\alpha} \chi_i = \sum_{i\in I} \chi_i \equiv 1.$$

This shows that the functions ρ_α form a partition of unity. To prove the inclusion $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$ we consider the open sets $W_i := \{p \in M \,|\, \chi_i(p) > 0\}$ for $i \in I$. Since $W_i \subset V_i$, this collection is locally finite. Hence, by Lemma A.2.4, we have

$$\mathrm{supp}(\rho_\alpha) = \overline{\bigcup_{\alpha_i=\alpha} W_i} = \bigcup_{\alpha_i=\alpha} \overline{W}_i = \bigcup_{\alpha_i=\alpha} \mathrm{supp}(\chi_i) \subset \bigcup_{\alpha_i=\alpha} V_i \subset U_\alpha.$$

This proves Theorem A.2.2. □

A.3 Embedding a Manifold into Euclidean Space

Recall the notation $\mathbb{H}^m := \{x = (x_1, \dots, x_m) \in \mathbb{R}^m \,|\, x_m \geq 0\}$ for the m-dimensional upper half space in (1.2.7).

Theorem A.3.1. *Let M be a smooth m-manifold with boundary whose topology is second countable and Hausdorff. Then the following holds.*

(i) *There exists an embedding $f : M \to \mathbb{R}^{2m+1}$ with a closed image.*

(ii) *There exists an embedding $f : M \to \mathbb{H}^{2m+2}$ with a closed image such that $f(\partial M) = f(M) \cap \partial\mathbb{H}^{2m+2}$ and f is transverse to $\partial\mathbb{H}^{2m+2}$.*

Proof. The proof has six steps.

Step 1. *Let $U \subset M$ be an open set and let $K \subset U$ be a compact set. Then there exists a $k \in \mathbb{N}$, a smooth map $f : M \to \mathbb{R}^k$, and an open set $V \subset M$ such that $K \subset V \subset U$ and the map $f|_V : V \to \mathbb{R}^k$ is an injective immersion.*

Choose a smooth atlas $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ on M (consisting of coordinate charts $\phi_\alpha : U_\alpha \to \mathbb{H}^m$ as in Definition 1.2.13) such that only finitely many U_α intersect K and, for all $\alpha \in A$, we have

$$U_\alpha \cap K \neq \emptyset \qquad \Longrightarrow \qquad U_\alpha \subset U.$$

Choose $\alpha_1, \dots, \alpha_\ell \in A$ such that $\{\alpha_1, \dots, \alpha_\ell\} = \left\{\alpha \in A \,\middle|\, U_\alpha \cap K \neq \emptyset\right\}$. By Theorem A.1.4 and Theorem A.2.2 choose a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ subordinate to the open cover $\{U_\alpha\}_{\alpha \in A}$ of M. For $i = 1, \dots, \ell$ abbreviate

$$\phi_i := \phi_{\alpha_i}, \quad \rho_i := \rho_{\alpha_i}, \quad V_i := \left\{p \in U_{\alpha_i} \,\middle|\, \rho_i(p) > 0\right\}, \quad V := V_1 \cup \dots \cup V_\ell.$$

Then $K \subset V \subset U$. Let $k := \ell(m+1)$ and define $f : M \to \mathbb{R}^k$ by

$$f(p) := \begin{pmatrix} \rho_1(p) \\ \rho_1(p)\phi_1(p) \\ \vdots \\ \rho_\ell(p) \\ \rho_\ell(p)\phi_\ell(p) \end{pmatrix} \qquad \text{for } p \in M.$$

Then the restriction $f|_V : V \to \mathbb{R}^k$ is injective. Namely, if $p_0, p_1 \in V$ satisfy $f(p_0) = f(p_1)$, then $I := \left\{i \,\middle|\, \rho_i(p_0) > 0\right\} = \left\{i \,\middle|\, \rho_i(p_1) > 0\right\} \neq \emptyset$ and, for $i \in I$, we have $\rho_i(p_0) = \rho_i(p_1)$, hence $\phi_i(p_0) = \phi_i(p_1)$, and so $p_0 = p_1$. Moreover, for every $p \in V$ the derivative $df(p) : T_pM \to \mathbb{R}^k$ is injective. This proves Step 1.

Step 2. *Let* $f : M \to \mathbb{R}^k$ *be an injective immersion and let* $\mathcal{A} \subset \mathbb{R}^{(2m+1)\times k}$ *be a nonempty open set. Then there exists a matrix* $A \in \mathcal{A}$ *such that the map* $Af : M \to \mathbb{R}^{2m+1}$ *is an injective immersion.*

Define $\mathrm{int}(M) := M \setminus \partial M$ and for $i = 0, 1, 2, 3, 4$ define the manifold W_i and the map $F_i : \mathcal{A} \times W_i \to \mathbb{R}^{2m+1}$ by

$$\begin{array}{ll} W_0 := \left\{(p,q) \in \mathrm{int}(M) \times \mathrm{int}(M) \,\middle|\, p \neq q\right\}, & F_0(A,p,q) := A(f(p) - f(q)), \\ W_1 := \mathrm{int}(M) \times \partial M, & F_1(A,p,q) := A(f(p) - f(q)), \\ W_2 := \left\{(p,q) \in \partial M \times \partial M \,\middle|\, p \neq q\right\}, & F_2(A,p,q) := A(f(p) - f(q)), \\ W_3 := \left\{(p,v) \in TM \,\middle|\, p \in \mathrm{int}(M),\, v \neq 0\right\}, & F_3(A,p,v) := Adf(p)v, \\ W_4 := \left\{(p,v) \in TM \,\middle|\, p \in \partial M,\, v \neq 0\right\}, & F_4(A,p,v) := Adf(p)v. \end{array}$$

Then for each i the map $F_i : \mathcal{A} \times W_i \to \mathbb{R}^{2m+1}$ is smooth. Moreover, the zero vector in $\mathbb{R}^{2m+1}$ is a regular value of F_0, F_1, F_2 because f is injective and of F_3, F_4 because f is an immersion. Hence, by Lemma 1.2.7 the sets

$$\begin{array}{ll} \mathcal{M}_i := F_i^{-1}(0) = \left\{(A,p,q) \in \mathcal{A} \times W_i \,\middle|\, Af(p) = Af(q)\right\}, & i = 0, 1, 2, \\ \mathcal{M}_i := F_i^{-1}(0) = \left\{(A,p,v) \in \mathcal{A} \times W_i \,\middle|\, Adf(p)v = 0\right\}, & i = 3, 4, \end{array}$$

are smooth manifolds and

$$\dim(\mathcal{M}_i) = \dim(\mathcal{A}) + \dim(W_i) - 2m - 1 < \dim(\mathcal{A}).$$

Since M is a second countable Hausdorff manifold, so is each $\mathcal{M}_i$. Hence Sard's Theorem 1.2.1 asserts that the canonical projections

$$\mathcal{M}_i \to \mathcal{A} : (A,p,q) \mapsto A =: \pi_i(A,p,q), \qquad i = 0, 1, 2, 3, 4,$$

have a common regular value $A \in \mathcal{A}$. Since $\dim(\mathcal{M}_i) < \dim(\mathcal{A})$ for each i, this implies

$$A \in \mathcal{A} \setminus \bigcup_{i=0}^{4} \pi_i(\mathcal{M}_i).$$

Hence $Af : M \to \mathbb{R}^{2m+1}$ is an injective immersion and this proves Step 2.

If M is compact and has an empty boundary, the result follows from Steps 1 and 2 with $K = U = M$. In the noncompact case the proof requires two more steps to construct an embedding into $\mathbb{R}^{4m+4}$, a further step to reduce the dimension to $2m + 1$, and a final step to obtain an embedding into $\mathbb{H}^{2m+2}$ that is transverse to the boundary and maps ∂M to $\partial\mathbb{H}^{2m+2}$.

Step 3. *Assume M is not compact. Then there exists a sequence of open sets $U_i \subset M$, a sequence of smooth functions $\rho_i : M \to [0,1]$ with compact support, and a sequence of compact sets $K_i \subset U_i$ such that*

$$\mathrm{supp}(\rho_i) \subset U_i, \qquad K_i = \rho_i^{-1}(1) \subset U_i, \qquad U_i \cap U_j = \emptyset$$

for all $i, j \in \mathbb{N}$ with $|i-j| \ge 2$ and $M = \bigcup_{i=1}^\infty K_i$.

Every manifold is locally compact. Since M is also second countable and Hausdorff, Lemma A.1.3 asserts that there exists a sequence of compact sets $C_i \subset M$ such that $C_i \subset \mathrm{int}(C_{i+1})$ for all $i \in \mathbb{N}$ and $M = \bigcup_{i\in\mathbb{N}} C_i$. As in the proof of Theorem A.1.4 let $C_0 := \emptyset$ and define

$$B_i := \overline{C_i \setminus C_{i-1}} \tag{A.3.1}$$

for $i \in \mathbb{N}$. Then $M = \bigcup_{i\in\mathbb{N}} B_i$. We prove that

$$B_i = C_i \setminus \mathrm{int}(C_{i-1}) \tag{A.3.2}$$

for all $i \in \mathbb{N}$. To see this, note first that every compact subset of M is closed because M is Hausdorff. Hence the right hand side in (A.3.2) is a closed set containing $C_i \setminus C_{i-1}$ and so $B_i \subset C_i \setminus \mathrm{int}(C_{i-1})$. To prove the converse inclusion, observe that $C_i \setminus B_i \subset C_{i-1}$, hence $\mathrm{int}(C_i) \setminus B_i$ is an open subset of C_{i-1}, hence $\mathrm{int}(C_i) \setminus B_i \subset \mathrm{int}(C_{i-1})$, and hence $\mathrm{int}(C_i) \setminus \mathrm{int}(C_{i-1}) \subset B_i$. Since $C_i \setminus \mathrm{int}(C_i) \subset C_i \setminus C_{i-1} \subset B_i$ by (A.3.1), this proves (A.3.2).

It follows from (A.3.2) that, for all $i \in \mathbb{N}$, we have

$$B_i \subset W_i := \mathrm{int}(C_{i+1}) \setminus C_{i-2}, \qquad W_i \cap B_{i+2} = \emptyset. \tag{A.3.3}$$

Since B_i is compact, the set W_i is open, and M is a locally compact Hausdorff space, it follows from Lemma A.1.2 by induction that there exists a sequence of open sets $U_i \subset M$ with compact closure such that, for all $i \in \mathbb{N}$,

$$B_i \subset U_i \subset \overline{U}_i \subset W_i, \qquad \overline{U}_i \cap \overline{U}_{i+2} = \emptyset. \tag{A.3.4}$$

This sequence satisfies $U_i \cap U_j = \emptyset$ whenever $|i-j| \ge 2$, because for $j \ge i+3$ we have $U_i \subset W_i \subset C_{i+1}$ and $U_j \subset W_j \subset M \setminus C_{j-2} \subset M \setminus C_{i+1}$.

Now M is paracompact by Theorem A.1.4. Hence Theorem A.2.2 asserts that, for each $i \in \mathbb{N}$, there exists of a partition of unity subordinate to the open cover $M = U_i \cup (M \setminus B_i)$, and hence a smooth function $\rho_i : M \to [0,1]$ such that $\mathrm{supp}(\rho_i) \subset U_i$ and $\rho_i|_{B_i} \equiv 1$. Thus $K_i := \rho_i^{-1}(1)$ is a sequence of compact sets such that $B_i \subset K_i \subset U_i$ for all i and hence $M = \bigcup_{i\in\mathbb{N}} K_i$. Tthis proves Step 3.

Step 4. *Assume M is not compact. Then there exists an embedding*

$$f : M \to \mathbb{R}^{4m+4}$$

with a closed image and a pair of orthonormal vectors $x, y \in \mathbb{R}^{4m+4}$ such that, for every $\varepsilon > 0$, there exists a compact set $K \subset M$ satisfying

$$\sup_{p\in M\setminus K} \inf_{s,t\in\mathbb{R}} \left| \frac{f(p)}{|f(p)|} - sx - ty \right| < \varepsilon. \tag{A.3.5}$$

Assume M is not compact and let K_i, U_i, ρ_i be as in Step 3. Then, by Steps 1 and 2, there exists a sequence of smooth maps $g_i : M \to \mathbb{R}^{2m+1}$ such that $g_i|_{M\setminus U_i} \equiv 0$, the restriction $g_i|_{K_i} : K_i \to \mathbb{R}^{2m+1}$ is injective, and the derivative $dg_i(p) : T_pM \to \mathbb{R}^{2m+1}$ is injective for all $p \in K_i$ and all $i \in \mathbb{N}$. Let $\xi \in \mathbb{R}^{2m+1}$ be a unit vector and define the maps $f_i : M \to \mathbb{R}^{2m+1}$ by

$$f_i(p) := \rho_i(p) \left(i\xi + \frac{g_i(p)}{\sqrt{1 + |g_i(p)|^2}} \right) \tag{A.3.6}$$

for $p \in M$ and $i \in \mathbb{N}$. Then the restriction $f_i|_{K_i} : K_i \to \mathbb{R}^{2m+1}$ is injective, the derivative $df_i(p) : T_pM \to \mathbb{R}^{2m+1}$ is injective for all $p \in K_i$, and

$$\mathrm{supp}(f_i) \subset U_i, \qquad f_i(K_i) \subset B_1(i\xi), \qquad f_i(M) \subset B_{i+1}(0).$$

Define the maps $f^{\mathrm{odd}}, f^{\mathrm{ev}} : M \to \mathbb{R}^{2m+1}$ and $\rho^{\mathrm{odd}}, \rho^{\mathrm{ev}} : M \to \mathbb{R}$ by

$$\rho^{\mathrm{odd}}(p) := \begin{cases} \rho_{2i-1}(p), & \text{if } i \in \mathbb{N} \text{ and } p \in U_{2i-1}, \\ 0, & \text{if } p \in M \setminus \bigcup_{i\in\mathbb{N}} U_{2i-1}, \end{cases}$$

$$f^{\mathrm{odd}}(p) := \begin{cases} f_{2i-1}(p), & \text{if } i \in \mathbb{N} \text{ and } p \in U_{2i-1}, \\ 0, & \text{if } p \in M \setminus \bigcup_{i\in\mathbb{N}} U_{2i-1}, \end{cases}$$

$$\rho^{\mathrm{ev}}(p) := \begin{cases} \rho_{2i}(p), & \text{if } i \in \mathbb{N} \text{ and } p \in U_{2i}, \\ 0, & \text{if } p \in M \setminus \bigcup_{i\in\mathbb{N}} U_{2i}, \end{cases}$$

$$f^{\mathrm{ev}}(p) := \begin{cases} f_{2i}(p), & \text{if } i \in \mathbb{N} \text{ and } p \in U_{2i}, \\ 0, & \text{if } p \in M \setminus \bigcup_{i\in\mathbb{N}} U_{2i}, \end{cases}$$

and define the map $f : M \to \mathbb{R}^{4m+4}$ by

$$f(p) := \left(\rho^{\mathrm{odd}}(p), f^{\mathrm{odd}}(p), \rho^{\mathrm{ev}}(p), f^{\mathrm{ev}}(p) \right)$$

for $p \in M$.

We prove that f is injective. To see this, note that

$$\begin{aligned} p \in K_{2i-1} \qquad &\Longrightarrow \qquad \begin{cases} 2i-2 < \left|f^{\mathrm{odd}}(p)\right| < 2i, \\ |f^{\mathrm{ev}}(p)| < 2i+1, \end{cases} \\ p \in K_{2i} \qquad &\Longrightarrow \qquad \begin{cases} 2i-1 < |f^{\mathrm{ev}}(p)| < 2i+1, \\ \left|f^{\mathrm{odd}}(p)\right| < 2i+2, \end{cases} \end{aligned} \tag{A.3.7}$$

Now let $p_0, p_1 \in M$ such that $f(p_0) = f(p_1)$. Assume first that $p_0 \in K_{2i-1}$. Then $\rho^{\mathrm{odd}}(p_1) = \rho^{\mathrm{odd}}(p_0) = 1$ and hence $p_1 \in \bigcup_{j\in\mathbb{N}} K_{2j-1}$. By (A.3.7), we also have $2i-2 < |f^{\mathrm{odd}}(p_1)| = |f^{\mathrm{odd}}(p_0)| < 2i$ and hence $p_1 \in K_{2i-1}$. This implies $f_{2i-1}(p_1) = f^{\mathrm{odd}}(p_1) = f^{\mathrm{odd}}(p_0) = f_{2i-1}(p_0)$ and so $p_0 = p_1$. Now assume $p_0 \in K_{2i}$. Then $\rho^{\mathrm{ev}}(p_1) = \rho^{\mathrm{ev}}(p_0) = 1$ and hence $p_1 \in \bigcup_{j\in\mathbb{N}} K_{2j}$. By (A.3.7), we also have $2i-1 < |f^{\mathrm{ev}}(p_1)| = |f^{\mathrm{ev}}(p_0)| < 2i+1$, so $p_1 \in K_{2i}$, which implies $f_{2i}(p_1) = f^{\mathrm{ev}}(p_1) = f^{\mathrm{ev}}(p_0) = f_{2i}(p_0)$, and so again $p_0 = p_1$. This shows that f is injective. That f is an immersion follows from the fact that the derivative $df_i(p)$ is injective for all $p \in K_i$ and all $i \in \mathbb{N}$.

We prove that f is proper and has a closed image. Let $(p_\nu)_{\nu\in\mathbb{N}}$ be a sequence in M such that the sequence $(f(p_\nu))_{\nu\in\mathbb{N}}$ in $\mathbb{R}^{4m+4}$ is bounded. Choose $i \in \mathbb{N}$ such that $|f^{\mathrm{odd}}(p_\nu)| < 2i$ and $|f^{\mathrm{ev}}(p_\nu)| < 2i+1$ for all $\nu \in \mathbb{N}$. Then $p_\nu \in \bigcup_{j=1}^{2i} K_j$ for all $\nu \in \mathbb{N}$ by (A.3.7). Hence $(p_\nu)_{\nu\in\mathbb{N}}$ has a convergent subsequence. Thus $f : M \to \mathbb{R}^{4m+4}$ is an embedding with a closed image.

Define $x := (0,\xi,0,0), y := (0,0,0,\xi) \in \mathbb{R} \times \mathbb{R}^{2m+1} \times \mathbb{R} \times \mathbb{R}^{2m+1}$. Let $(p_\nu)_{\nu\in\mathbb{N}}$ be a sequence in M that does not have a convergent subsequence and choose $i_\nu \in \mathbb{N}$ such that $p_\nu \in K_{2i_\nu-1} \cup K_{2i_\nu}$ for all $\nu \in \mathbb{N}$. Then i_ν tends to infinity. If $p_\nu \in K_{2i_\nu-1}$ for all ν, then $\limsup_{\nu\to\infty}|f^{\mathrm{odd}}(p_\nu)|^{-1}|f^{\mathrm{ev}}(p_\nu)| \le 1$ by (A.3.7). Passing to a subsequence, still denoted by $(p_\nu)_{\nu\in\mathbb{N}}$, we assume that the limit $\lambda := \lim_{\nu\to\infty}|f^{\mathrm{odd}}(f_\nu)|^{-1}|f^{\mathrm{ev}}(p_\nu)|$ exists. Then $0 \le \lambda \le 1$ and, by (A.3.6), we have

$$\begin{aligned} &\lim_{\nu\to\infty} \frac{f^{\mathrm{odd}}(p_\nu)}{|f^{\mathrm{odd}}(p_\nu)|} = \xi, \quad \lim_{\nu\to\infty} \frac{f^{\mathrm{ev}}(p_\nu)}{|f^{\mathrm{odd}}(p_\nu)|} = \lambda\xi, \quad \lim_{\nu\to\infty} \frac{|f^{\mathrm{odd}}(p_\nu)|}{|f(p_\nu)|} = \frac{1}{\sqrt{1+\lambda^2}}, \\ &\lim_{\nu\to\infty} \frac{f(p_\nu)}{|f(p_\nu)|} = \left(0, \frac{\xi}{\sqrt{1+\lambda^2}}, 0, \frac{\lambda\xi}{\sqrt{1+\lambda^2}}\right) = \frac{1}{\sqrt{1+\lambda^2}}x + \frac{\lambda}{\sqrt{1+\lambda^2}}y. \end{aligned}$$

Similarly, if $p_\nu \in K_{2i_\nu}$ for all ν, there exists a subsequence such that the limit $\lambda := \lim_{\nu\to\infty}|f^{\mathrm{ev}}(p_\nu)|^{-1}|f^{\mathrm{odd}}(p_\nu)|$ exists and, by (A.3.6), this implies

$$\lim_{\nu\to\infty} \frac{f(p_\nu)}{|f(p_\nu)|} = \left(0, \frac{\lambda\xi}{\sqrt{1+\lambda^2}}, 0, \frac{\xi}{\sqrt{1+\lambda^2}}\right) = \frac{\lambda}{\sqrt{1+\lambda^2}}x + \frac{1}{\sqrt{1+\lambda^2}}y.$$

This shows that the vectors x and y satisfy the requirements of Step 4.

Step 5. *We prove part (i).*

Assume M is not compact an $m \geq 1$. Choose an embeding $f : M \to \mathbb{R}^{4m+4}$ and vectors $x, y \in \mathbb{R}^{4m+4}$ as in Step 4 and define

$$\mathcal{A} := \left\{ A \in \mathbb{R}^{(2m+1)\times(4m+4)} \,\middle|\, Ax \text{ and } Ay \text{ are linearly independent} \right\}.$$

Since $m \geq 1$, this is a nonempty open subset of $\mathbb{R}^{(2m+1)\times(4m+4)}$. We prove that the map $Af : M \to \mathbb{R}^{2m+1}$ is proper and has a closed image for every $A \in \mathcal{A}$. To see this, fix a matrix $A \in \mathcal{A}$. Let $(p_\nu)_{\nu\in\mathbb{N}}$ be a sequence in M that does not have a convergent subsequence. Then by Step 4 there exists a subsequence, still denoted by $(p_\nu)_{\nu\in\mathbb{N}}$, and real numbers $s, t \in \mathbb{R}$ such that

$$s^2 + t^2 = 1, \qquad \lim_{\nu\to\infty} \frac{f(p_\nu)}{|f(p_\nu)|} = sx + ty, \qquad \lim_{\nu\to\infty} |f(p_\nu)| = \infty.$$

This implies

$$\lim_{\nu\to\infty} \frac{Af(p_\nu)}{|f(p_\nu)|} = sAx + tAy \neq 0$$

and hence $\lim_{\nu\to\infty} |Af(p_\nu)| = \infty$. Thus the preimage of every compact subset of $\mathbb{R}^{2m+1}$ under the map $Af : M \to \mathbb{R}^{2m+1}$ is a compact subset of M, and hence Af is proper and has a closed image.

By Step 2 there exists a matrix $A \in \mathcal{A}$ such that $Af : M \to \mathbb{R}^{2m+1}$ is an injective immersion and so is an embedding. This proves Step 5.

Step 6. *We prove part (ii).*

Let $f : M \to \mathbb{R}^{2m+1}$ be as in part (i). Let $\{U_\alpha, \phi_\alpha\}_\alpha$ be an atlas on M, so each $\phi_\alpha : U_\alpha \to \Omega_\alpha$ is a coordinate chart with values in an open set $\Omega_\alpha \subset \mathbb{H}^m$. By Theorem A.2.2 choose a partition of unity $\{\rho_\alpha\}_\alpha$ subordinate to the opern cover $\{U_\alpha\}_\alpha$. Then the function $g := \sum_\alpha \rho_\alpha \phi_{\alpha,m} : M \to \mathbb{R}$ is positive on $M \setminus \partial M$ and $g(p) = 0$, $dg(p)v < 0$ for every $p \in \partial M$ and every outward pointing tangent vector $v \in T_pM$. Hence the map $F := (f, g) : M \to \mathbb{H}^{2m+2}$ satisfies the requirements of part (ii). This proves Theorem A.3.1. □

The **Whitney Embedding Theorem** asserts that every second countable Hausdorff m-manifold M admits an embedding $f : M \to \mathbb{R}^{2m}$. The proof is based on the *Whitney Trick* and goes beyond the scope of this book. The next exercise shows that Whitney's theorem is sharp.

Exercise A.3.2. The manifold $\mathbb{R}\mathrm{P}^2$ cannot be embedded into $\mathbb{R}^3$. The same is true for the **Klein bottle** $K := \mathbb{R}^2/\equiv$ where the equivalence relation is given by $[x, y] \equiv [x + k, (-1)^k y + \ell]$ for $x, y \in \mathbb{R}$ and $k, \ell \in \mathbb{Z}$.

Corollary A.3.3. *Let* $n := 2m+2 \geq 4$ *and let* M *be a compact* m*-manifold with boundary. Then there exists an embedding* $f : M \to \mathbb{H}^n$ *and a compact* $(m-1)$*-manifold* $Q \subset \mathbb{R}^{n-1}$ *without boundary (see Figure A.1) such that*

$$\begin{aligned} f(\partial M) &= f(M) \cap \partial \mathbb{H}^n = Q \times \{0\}, \\ f(M) &\cap \big(\mathbb{R}^{n-1} \times [0,1]\big) = Q \times [0,1]. \end{aligned} \tag{A.3.8}$$

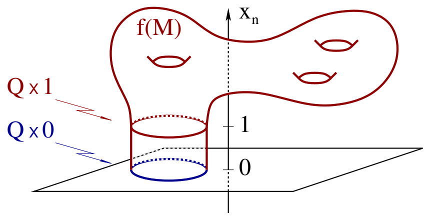

Figure A.1: A product embedding near the boundary

Proof. By part (i) of Theorem A.3.1 choose an embedding $f : M \to \mathbb{R}^{n-1}$ and as in Step 6 of the proof choose a smooth function $g : M \to \mathbb{R}$ such that g is positive on $M \setminus \partial M$ and $g(q) = 0$ and $dg(q)v < 0$ for every $q \in \partial M$ and every outward pointing tangent vector $v \in T_qM$. Choose $\varepsilon > 0$ so small that the interval $[0, 3\varepsilon]$ contains no critical value of g and define

$$U_\varepsilon := \{p \in M \,|\, g(p) < 3\varepsilon\}.$$

Choose any Riemannian metric on M (see §A.5) and let $\nabla g \in \mathrm{Vect}(M)$ be the gradient vector field of g with respect to this metric. For $q \in \partial M$ let $[0, 3\varepsilon) \to M : t \mapsto \phi_t(q)$ be the solution of the differential equation

$$\partial_t \phi_t(q) = \frac{\nabla g(\phi_t(q))}{|\nabla g(\phi_t(q))|^2}, \qquad \phi_0(q) = q.$$

Then $g(\phi_t(q)) = t$ for $q \in \partial M$ and $0 \leq t < 3\varepsilon$ and the map

$$\partial M \times [0, 3\varepsilon) \to U_\varepsilon : (q, t) \mapsto \phi_t(q) \tag{A.3.9}$$

is a diffeomorphism. In particular, the map (A.3.9) is surjective, because M is compact and the interval $[0, 3\varepsilon]$ does not contain any critical value of g. Now choose a smooth cutoff function $\beta : [0, 3\varepsilon] \to [0, 3\varepsilon]$ such that $\beta(t) = 0$ for $0 \leq t \leq \varepsilon$ and $\beta(t) = t$ for $2\varepsilon \leq t \leq 3\varepsilon$. Define $F : M \to \mathbb{R}^n$ by

$$F(p) := \left\{ \begin{array}{ll} (f(p), g(p)/\varepsilon), & \text{for } p \in M \setminus U_\varepsilon, \\ (f(\phi_{\beta(t)}(q)), t/\varepsilon), & \text{for } p = \phi_t(q),\ q \in \partial M,\ 0 \leq t < 3\varepsilon. \end{array} \right.$$

Then F is an embedding that satisfies (A.3.8) with $Q := f(\partial M)$. This proves Corollary A.3.3. □

A.4 Classifying Smooth One-Manifolds

The purpose of this section is to prove the following result. We follow almost verbatim the beautiful exposition by Milnor in [26, Appendix]

Theorem A.4.1. *Every nonempty connected smooth* 1*-dimensional manifold is diffeomorphic either to the circle* $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ *or to some interval of real numbers containing more than one point.*

An interval of real numbers is a connected subset of $\mathbb{R}$. An interval $I \subset \mathbb{R}$ which contains more than one point may be finite or infinite; closed, open, or half–open. Every such interval is diffeomorphic to either $[0,1]$, $(0,1)$, or $[0,1)$. (For example, use an affine transformation of $\mathbb{R}$, followed by the diffeomorphism $f(x) = x/\sqrt{1-x^2}$ when the source interval is bounded and the target interval is unbounded.) Thus there are precisely four distinct diffeomorphism classes of nonempty connected smooth 1-manifolds.

Throughout the remainder of this section we will assume (without loss of generality by Theorem A.3.1) that $M \subset \mathbb{R}^n$ is a nonempty connected smooth 1-dimensional submanifold with boundary of a Euclidean space $\mathbb{R}^n$. The proof of Theorem A.4.1 will make use of the concept of *arc-length.*

Definition A.4.2. *Let* $I \subset \mathbb{R}$ *be an interval containing more than one point. A map* $f : I \to M$ *is called a* **parametrization by arc-length** *iff* f *maps* I *diffeomorphically onto an open subset of* M *and* f *satisfies the condition*

$$|f'(t)| = 1 \qquad \text{for all } t \in I.$$

Here $f'(t) := df(t)1 \in \mathbb{R}^n$ *denotes the derivative of* f*, understood as a smooth map from* I *to* $\mathbb{R}^n$*, at* $t \in I$ *and*

$$|x| := \sqrt{x_1^2 + \cdots + x_n^2}$$

denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$*.*

Remark A.4.3. **(i)** If $f : I \to M$ maps I diffeomorphically onto an open subset of M, then $t_0 \in I$ is a boundary point of I if and only if $f(t_0)$ is a boundary point of M; in particular, if I has a boundary point, so does M.

(ii) Let $g : J \to M$ be a diffeomorphism from an interval J containing more than one point onto an open subset of M. Let $t_0 \in J$ and define

$$\sigma(t) := \int_{t_0}^{t} |g'(s)|\, ds \qquad \text{for } t \in J.$$

Then $I := \sigma(J) \subset \mathbb{R}$ is an interval, the map $\sigma : J \to I$ is a diffeomorphism, and the map $f := g \circ \sigma^{-1} : I \to M$ is a parametrization by arc-length.

Lemma A.4.4. *Let $f : I \to M$ and $g : J \to M$ be parametrizations by arc-length such that $f(I) \cap g(J) \neq \emptyset$. Then the intersection $f(I) \cap g(J)$ has at most two connected components. If $f(I) \cap g(J)$ has only one connected component, then f can be extended to a parametrization by arc-length of the union $f(I) \cup g(J)$. If $f(I) \cap g(J)$ has two connected components, then M is diffeomorphic to S^1.*

Proof. Define $U := f^{-1}(g(J)) \subset I$ and $V := g^{-1}(f(I)) \subset J$. Then U is a nonempty relatively open subset of I, V is a nonempty relatively open subset of J, and the transition map

$$\rho := g^{-1} \circ f : U \to V$$

is smooth and has the derivative ±1 at every point. Its graph

$$\Gamma := \left\{(s,t) \in I \times J \,\middle|\, f(s) = g(t)\right\}$$

is a closed subset of $I \times J$ consisting of line segments of slope ±1. Hence each of these line segments cannot end in the interior of the rectangle $I \times J$ but must extend to the boundary. Since Γ is the graph of a bijective map ρ from $U \subset I$ to $V \subset J$, there can be at most one line segment ending on each of the four edges of the rectangle $I \times J$. Hence there are at most two such line segments, i.e. Γ has either one or two connected components. Moreover, if Γ has two connected components, then they must both have the same slope. (See Figure A.2.)

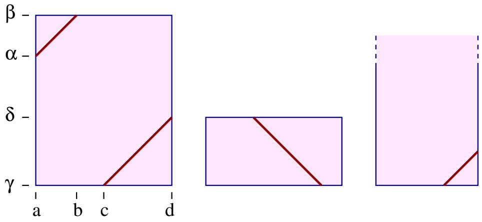

Figure A.2: Three of the possibilities for Γ

If Γ has only one connected component, then ρ extends uniquely to an affine map, still denoted by $\rho : \mathbb{R} \to \mathbb{R}$, of the form $\rho(t) = a+t$ or $\rho(t) = a-t$. Hence the map $F : I \cup \rho^{-1}(J) \to M$, defined by

$$F(t) := \left\{ \begin{array}{rl} f(t), & \text{if } t \in I, \\ g(\rho(t)), & \text{if } \rho(t) \in J, \end{array} \right. \qquad \text{for } t \in I \cup \rho^{-1}(J), \tag{A.4.1}$$

is the required parametrization by arc-length of the union $f(I) \cup g(J)$.

Now assume that Γ has two connected components and that they both have slope $+1$ (otherwise compose β with the map $t \mapsto -t$). Then they must be arranged as in the rectangle on the left in Figure A.2. Thus

$$U = (a,b) \cup (c,d) \subset I = (a,d), \qquad V = (\gamma,\delta) \cup (\alpha,\beta) \subset J = (\gamma,\beta),$$

where $a < b \le c < d$, $\gamma < \delta \le \alpha < \beta$, and $b-a = \beta-\alpha$, $d-c = \delta-\gamma$. Here the intervals I and J are open, because none of the points $g(\alpha), g(\delta), f(b), f(c)$ is a boundary point of M. After translating the interval J, if necessary, we may assume that $\gamma = c$ and $\delta = d$, so that

$$a < b \le c < d \le \alpha < \beta$$

and $f(t) = g(t)$ for $c < t < d$ and $f(t) = g(\alpha - a + t)$ for $a < t < b$. This implies that the map $h : S^1 \to M$, defined by the formula

$$h(\cos(\theta), \sin(\theta)) := \begin{cases} f(t), & \text{if } a < t := (\alpha - a)\theta/2\pi < d, \\ g(t), & \text{if } c < t := (\alpha - a)\theta/2\pi < \beta, \end{cases}$$

is well-defined, injective, and smooth. The image $h(S^1) = f(I) \cup g(J)$ of this map is open and compact, and hence must be all of M, because M is connected. This proves Lemma A.4.4. □

Proof of Theorem A.4.1. Let $\mathcal{F}$ be the set of all pairs (I, f) consiting of an interval $I \subset \mathbb{R}$ that contains more than one point and a parametrization by arc-length $f : I \to M$ of an open subset $f(I) \subset M$. This set is nonempty, because M is nonempty, and it is partially ordered by the relation

$$(I, f) \preccurlyeq (J, g) \qquad \overset{\text{def}}{\Longleftrightarrow} \qquad I \subset J \text{ and } g|_I = f.$$

Every chain $\{(I_\alpha, f_\alpha)\}_{\alpha \in A}$ in $\mathcal{F}$ has a supremum given by $I := \bigcup_\alpha I_\alpha$ and $f(t) := f_\alpha(t)$ for $t \in I_\alpha$. Hence it follows from Zorn's Lemma that $\mathcal{F}$ has a maximal element. Let $(I, f) \in \mathcal{F}$ be such a maximal element and assume that M is not diffeomorphic to S^1.

We prove that $f(I)$ is a closed subset of M. Let $p \in M \cap \overline{f(I)}$. Then there exists a parametrization by arc-length $g : J \to M$ of an open neighborhood $g(J) \subset M$ of p. Since p belongs to the closure of $f(I)$ with respect to the relative topology of M, it follows that $f(I) \cap g(J) \neq \emptyset$. Since M is not diffeomorphic to S^1, it follows from Lemma A.4.4 that $f(I) \cap g(J)$ is connected. Hence it follows again from Lemma A.4.4 that f extends to a parametrization by arc-length of the union $f(I) \cup g(J)$. Since (I, f) is maximal, this implies $p \in g(J) \subset f(I)$. Thus we have proved that $f(I)$ is a closed subset of M and, since M is connected, it follows that $f(I) = M$. Thus $f : I \to M$ is a diffeomorphism, and this proves Theorem A.4.1. □

A.5 Riemannian Metrics

This section establishes the existence of Riemannian metrics on paracompact Hausdorff manifolds and discusses the Levi-Civita connection.

Definition A.5.1 (Riemannian Metric). *Let M be a smooth m-manifold (possibly with boundary) and let $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ be an atlas on M. A* **Riemannian metric** *on M is a collection of inner products*

$$T_pM \times T_pM \to \mathbb{R} : (v, w) \mapsto g_p(v, w), \tag{A.5.1}$$

one for every $p \in M$, such that the map $g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m \times m}$, defined by

$$g_{\alpha,ij}(\phi_\alpha(p)) := g_p\left(\frac{\partial}{\partial x_i}(p), \frac{\partial}{\partial x_j}(p)\right) \tag{A.5.2}$$

for $p \in U_\alpha$ and $i, j = 1, \dots, m$, is smooth for every $\alpha \in A$. We will also denote the inner product by $\langle v, w \rangle_p := g_p(v, w)$ and drop the subscript p if the base point is understood from the context. A smooth manifold equipped with a Riemannian metric is called a **Riemannian manifold**.

For different coordinate charts the maps g_α and g_β are related by

$$g_\alpha(x) = d\phi_{\beta\alpha}(x)^T g_\beta(\phi_{\beta\alpha}(x)) d\phi_{\beta\alpha}(x) \tag{A.5.3}$$

for $x \in \phi_\alpha(U_\alpha \cap U_\beta)$, where $\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \ \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ denotes the transition map (see Definition 1.1.2). Conversely, every collection of smooth maps $g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m \times m}$ with values in the space of positive definite matrices that satisfies (A.5.3) for all $\alpha, \beta \in A$ determines a Riemannian metric on M via (A.5.2).

Let (M, g) be a Riemannian manifold. The norm of a tangent vector $v \in T_pM$ determined by this metric is given by $|v| := |v|_p := \sqrt{\langle v, v \rangle_p}$ and the length of a smooth curve $\gamma : [0, 1] \to M$ is defined by

$$L(\gamma) := \int_0^1 |\dot{\gamma}(t)| \, dt. \tag{A.5.4}$$

Now assume that M is connected. Then the set

$$\Omega_{p,q} := \{\gamma : [0, 1] \to M \,|\, \gamma \text{ is smooth, } \gamma(0) = p, \, \gamma(1) = q\}.$$

of smooth curves joining p to q is nonempty, and the formula

$$d(p, q) := \inf_{\gamma \in \Omega_{p,q}} L(\gamma)$$

for $p, q \in M$ defines a distance function on M that induces the manifold topology (see [35, Lemma 4.7.1]).

Lemma A.5.2. *Let M be a smooth m-manifold whose topology is Hausdorff. Then the following are equivalent.*

(i) *M admits a Riemannian metric.*

(ii) *The topology on M is metrizable.*

(iii) *M is paracompact.*

Proof. That (i) implies (ii) was proved above under the assumption that M is connected. If M is disconnected, define $d'(p,q) := d(p,q)/(1+d(p,q))$ whenever $\Omega_{p,q} \neq \emptyset$, and $d'(p,q) := 1$ whenever $\Omega_{p,q} = \emptyset$. Then d' is a distance function that induces the manifold topology of M. That (ii) implies (iii) follows from a general theorem which asserts that every metric space is paracompact (see [31, Thm 41.4]). To prove that (iii) implies (i), choose an atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ on M. Since M is paracompact, Theorem A.2.2 asserts that there exists a partition of unity $\{\rho_\alpha\}_{\alpha\in A}$, subordinate to the cover $\{U_\alpha\}_{\alpha\in A}$. For every such partition of unity the formula

$$\langle v, w\rangle_p := \sum_{p\in U_\alpha} \rho_\alpha(p) \left\langle d\phi_\alpha(p)v, d\phi_\alpha(p)w\right\rangle_{\mathbb{R}^m} \qquad \text{for } v, w \in T_pM$$

defines a Riemannian metric on M. This proves Lemma A.5.2. □

The next lemma uses the concept of a connection

$$\nabla : \Omega^0(M, TM) \to \Omega^1(M, TM)$$

for the tangent bundle $E = TM$ of a Riemannian manifold (M, g) as introduced in §8.1.1. The connection ∇ is called **torsion-free** if

$$[X, Y] = \nabla_Y X - \nabla_X Y \tag{A.5.5}$$

for all $X, Y \in \mathrm{Vect}(M) = \Omega^0(M, TM)^1$ and it is called **Riemannian** iff it satisfies the Leibniz rule

$$\mathcal{L}_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{A.5.6}$$

for all $X, Y, Z \in \mathrm{Vect}(M)$ (see Example 8.1.14).

Lemma A.5.3. *Every Riemannian manifold admits a unique torsion-free Riemannian connection, called the* **Levi-Civita connection**.

Proof. See [35, Lemma 5.2.7]. □

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[1] Our sign convention for the Lie bracket is explained in [35, §2.4.3]

To describe the Levi-Civita connection in local coordinates, let (M, g) be a Riemannian m-manifold, fix a coordinate chart $\phi : U \to \Omega$ on an open set $U \subset M$ with values in an open set $\Omega \subset \mathbb{H}^m$, denote by $g_{ij} : \Omega \to \mathbb{R}$ the associated metric tensor, and let $g^{ij} : \Omega \to \mathbb{R}$ be the inverse tensor so that

$$\sum_{j=1}^{m} g_{ij} g^{jk} = \delta_i^k$$

for $i, k = 1, \ldots, m$. In these coordinates a smooth vector field $X \in \mathrm{Vect}(M)$ is represented by a smooth map $\xi = (\xi^1, \ldots, \xi^m) : \Omega \to \mathbb{R}^m$ defined by

$$\xi(\phi(p)) := d\phi(p)X(p)$$

for $p \in U$. In the notation (1.1.6) this equation can be written as

$$X|_U = \sum_{i=1}^{m} (\xi^i \circ \phi) \frac{\partial}{\partial x^i}$$

Let $Y \in \mathrm{Vect}(M)$ be another smooth vector field represented by the function $\eta : \Omega \to \mathbb{R}^m$ so that $\eta(\phi(p)) := d\phi(p)Y(p)$ for $p \in U$.

Lemma A.5.4 (Christoffel Symbols). *Let $Z := \nabla_X Y$ be the covariant derivative of the vector field Y in the direction of the vector field X and denote by $\zeta = (\zeta^1, \ldots, \zeta^m) : \Omega \to \mathbb{R}^m$ the local coordinates of the vector field Z so that $\zeta(\phi(p)) = d\phi(p)(\nabla_X Y)(p)$ for $p \in U$. Then*

$$\zeta^k = \sum_{i=1}^{m} \frac{\partial \eta^k}{\partial x^i} \xi^i + \sum_{i,j=1}^{m} \Gamma_{ij}^k \xi^i \eta^j, \tag{A.5.7}$$

for $k = 1, \ldots, m$, where the $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ are the **Christoffel symbols**

$$\Gamma_{ij}^k := \sum_{\ell=1}^{m} g^{k\ell} \frac{1}{2} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right) \tag{A.5.8}$$

for $i, j, k = 1, \ldots, m$.

Proof. In local coordinates every connection ∇ on TM is given by an equation of the form (A.5.7) for suitable functions $\Gamma_{ij}^k : \Omega \to \mathbb{R}$. The torsion-free and Riemannian conditions on ∇ then take the form

$$\Gamma_{ij}^k = \Gamma_{ji}^k, \qquad \frac{\partial g_{ij}}{\partial x^\ell} = \sum_{k=1}^{m} \left(g_{ik} \Gamma_{j\ell}^k + g_{jk} \Gamma_{i\ell}^k \right). \tag{A.5.9}$$

These equations taken together are equivalent to (A.5.8). For more details see [35, Lemma 3.6.5]. □

One useful application of the Levi–Civita connection is that it assigns a *covariant derivative* to each vecttor field along a curve and hence gives rise to *parallel transport.* Assume that M is a Riemannian m-manifold and denote by ∇ its Levi-Civita connection. Let $\gamma : I \to M$ be a smooth curve.

Definition A.5.5. *A* **vector field along** γ *is a collection of tangent vectors*

$$X(t) \in T_{\gamma(t)}M,$$

one for each $t \in I$*, such that the map* $I \to TM : t \mapsto (\gamma(t), X(t))$ *is smooth. The space of vector fields along* γ *will be denoted by* $\mathrm{Vect}(\gamma)$*.*

The **covariant derivative of a vector field** $X \in \mathrm{Vect}(\gamma)$ *along* γ *is the unique vector field* $\nabla X \in \mathrm{Vect}(\gamma)$ *that satisfies the condition*

$$d\phi(\gamma(t))\nabla X(t) = \zeta(t), \qquad \zeta^k(t) := \dot{\xi}^k(t) + \sum_{i,j} \Gamma^k_{ij}(x(t))\xi^i(t)\dot{x}^j(t),$$

for each coordinate chart $\phi : U \to \Omega \subset \mathbb{R}^m$ *and each* $t \in I_\phi := \gamma^{-1}(U) \subset I$*, where the* $\Gamma^k_{ij} : \Omega \to \mathbb{R}$ *are the Christoffel symbols and the curves* $x : I_\phi \to \Omega$ *and* $\xi : I_\phi \to \mathbb{R}^m$ *are defined by*

$$x(t) := \phi(\gamma(t)), \qquad \xi(t) := d\phi(\gamma(t))X(t) \qquad \text{for } t \in I_\phi.$$

A vector field $X \in \mathrm{Vect}(\gamma)$ *is called* **parallel** *iff* $\nabla X = 0$*.*

Theorem A.5.6. *For every* $t_0 \in I$ *and every* $v_0 \in T_{\gamma(t_0)}M$ *there exists a unique parallel vector field* X *along* γ *such that* $X(t_0) = v_0$*.*

Proof. See [35, Theorem 3.3.4]. □

Definition A.5.7 (Parallel transport). *For* $t_0 \in I$ *define the map*

$$\Phi_\gamma(t, t_0) : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$$

by $\Phi_\gamma(t, t_0)v_0 := X(t)$*, where* $X \in \mathrm{Vect}(\gamma)$ *is the unique parallel vector field along* γ *satisfying* $X(t_0) = v_0$*. The collection of maps* $\Phi(t, t_0)$ *for* $t, t_0 \in I$ *is called* **parallel transport along** γ*.*

Since the Levi-Civita connection is Riemannian, parallel transport preserves the inner product. In particular, Theorem A.5.6 provides an easy and natural method of extending a frame of a tangent space $T_{p_0}M$ over any curve passing through p_0. Similarly, a frame of the normal to a smooth submanifold $P \subset M$ at a point $p_0 \in P$ extends by parallel transport over any curve in P passing through p_0.

A.6 The Exponential Map

Let (M, g) be a Riemannian m-manifold without boundary and denote by ∇ the Levi-Civita connection. Via pullback the Levi-Civita connection induces a covariant derivative operator on the space $\mathrm{Vect}(\gamma) := \Omega^0(I, \gamma^*TM)$ of smooth vector fields along any smooth curve $\gamma : I \to M$, and this pullback connection will be denoted by the same symbol $\nabla : \mathrm{Vect}(\gamma) \to \mathrm{Vect}(\gamma)$ (see §8.1.2 and Remark 8.1.18).

Definition A.6.1. *Let $I \subset \mathbb{R}$ be an interval. A smooth curve $\gamma : I \to M$ is called a* **geodesic** *iff it satisfies the equation $\nabla\dot{\gamma} = 0$, i.e. the covariant derivative of its derivative vanishes everywhere.*

Geodesics are solutions of a second order differential equation. Namely, if $\phi : U \to \Omega$ is a local coordinate chart and the $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ are the Christoffel symbols as in Lemma A.5.4, then a smooth curve $\gamma : I \to U$ is a geodesic if and only if the curve $c = (c^1, \ldots, c^m) := \phi \circ \gamma : I \to \Omega$ satisfies the second order differential equation

$$\ddot{c}^k + \sum_{i,j=1}^m \Gamma_{ij}^k(c)\dot{c}^i\dot{c}^j = 0, \qquad k = 1, \ldots, m. \tag{A.6.1}$$

As an aside, the reader may verify that (A.6.1) is the Euler–Lagrange equation associated to the energy functional

$$E(c) := \frac{1}{2}\int_I \sum_{i,j=1}^m g_{ij}(c(t))\dot{c}^i(t)\dot{c}^j(t)\,dt$$

on the space of smooth curves $c : I \to \Omega$. In the intrinsic formulation, a smooth curve $\gamma : I \to M$ is a geodesic if and only if the map $(\gamma, \dot{\gamma}) : I \to TM$ is an integral curve of a suitable vector field on TM, called the *geodesic spray* (see [35, Lemma 4.3.3]). This implies that, for every $p \in M$ and every $v \in T_pM$, there exists a unique geodesic $\gamma : I_{p,v} \to M$ on a maximal open existence interval $I_{p,v} \subset \mathbb{R}$ containing the origin such that

$$\gamma(0) = p, \qquad \dot{\gamma}(0) = v \tag{A.6.2}$$

(see [35, Lemma 4.3.4]). These geodesics give rise to an exponential map

$$\exp_p : V_p \to M, \qquad V_p := \{v \in T_pM \,|\, 1 \in I_{p,v}\}, \tag{A.6.3}$$

defined by $\exp_p(v) := \gamma(1)$, where $\gamma : I_{p,v} \to M$ is the unique geodesic satisfying (A.6.2). The exponential map is smooth because it is obtained from the integral curves of a smooth vector field on the tangent bundle. Moreover it has the following properties.

Lemma A.6.2. **(i)** *The set*

$$V := \bigcup_{p\in M} \{p\} \times V_p \subset TM$$

is open and the map

$$V \to M : (p, v) \mapsto \exp_p(v)$$

is smooth.

(ii) *Let* $p \in M$ *and* $v \in T_pM$*. Then the unique geodesic* $\gamma : I_{p,v} \to M$ *that satisfies* (A.6.2) *is given by* $I_{p,v} = \{t \in \mathbb{R} \,|\, tv \in V_p\}$ *and*

$$\gamma(t) = \exp_p(tv)$$

for $t \in I_{p,v}$*.*

(iii) *The derivative of the exponential map* (A.6.3) *at the origin is the identity, i.e.* $d\exp_p(0) = \mathrm{id}_{T_pM}$ *for all* $p \in M$*.*

Proof. See [35, Lemma 4.3.6 & Corollary 4.3.7]. □

Exercise A.6.3. Assume $\dim(M) = 1$. Prove that a curve $\gamma : I \to M$ is a geodesic if and only if the function $I \to \mathbb{R} : t \mapsto |\dot\gamma(t)|$ is constant.

It follows from part (iii) of Lemma A.6.2 and the Inverse Function Theorem 1.1.19 that, for $r > 0$ sufficiently small, the exponential map restricts to a diffeomorphism from the ball

$$B_r(p) := \{v \in T_pM \,|\, |v| < r\} \tag{A.6.4}$$

of radius r in the tangent space onto its image

$$U_r(p) = \left\{\exp_p(v) \,|\, v \in T_pM,\ |v| < r\right\}. \tag{A.6.5}$$

The supremum of the numbers $r > 0$ for which this holds is called the **injectivity radius of** (M, g) **at** p and will be denoted by

$$\mathrm{inj}(p; M) := \sup\left\{ r > 0 \;\middle|\; \begin{array}{l} \exp_p : B_r(p) \to U_r(p) \\ \text{is a diffeomorphism} \end{array} \right\}. \tag{A.6.6}$$

In [35, §4.5] it is shown that geodesics minimize the distance on small time intervals and that the set $U_r(p)$ is the ball of radius r in the metric space (M, d) whenever $0 < r < \mathrm{inj}(p; M)$. Here is a precise formulation of the result.

Theorem A.6.4 (Existence of Minimal Geodesics). *Let (M, g) be a Riemannian m-manifold, fix a point $p \in M$, and let $r > 0$ be smaller than the injectivity radius of M at p. Let $v \in T_pM$ such that $|v| < r$. Then*

$$d(p, q) = |v|, \qquad q := \exp_p(v),$$

and a curve $\gamma \in \Omega_{p,q}$ has minimal length $L(\gamma) = |v|$ if and only if there is a smooth map $\beta : [0, 1] \to [0, 1]$ satisfying

$$\beta(0) = 0, \qquad \beta(1) = 1, \qquad \dot\beta \geq 0$$

such that $\gamma(t) = \exp_p(\beta(t)v)$ for $0 \leq t \leq 1$.

Proof. See [35, Theorem 4.5.4]. ☐

A key ingredient in the proof of Theorem A.6.4 is the Gauß Lemma which is also used in the proof of the Tubular Neighborhood Theorem 4.3.8.

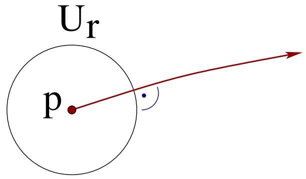

Figure A.3: The Gauß Lemma.

Lemma A.6.5 (Gauß Lemma). *Let M, p, r be as in Theorem A.6.4, let $I \subset \mathbb{R}$ be an open interval, and let $w : I \to V_p$ be a smooth curve whose norm $|w(t)| =: r$ is constant. Define*

$$\alpha(s, t) := \exp_p(sw(t))$$

for $(s, t) \in \mathbb{R} \times I$ with $sw(t) \in V_p$. Then

$$\left\langle \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle \equiv 0.$$

Thus the geodesics through the point p are orthogonal to the boundaries of the balls $U_r(p)$ in (A.6.5) *(see Figure A.3).*

Proof. See [35, Lemma 4.5.5]. ☐

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