Introduction to Lattices and Order

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LSSU Math 400

Subsection 1

Lattices as Ordered Sets

Upper and Lower Bounds

- Let P be an ordered set and let $S \subseteq P$.
  An element $x \in P$ is an **upper bound** of S if $s \leq x$ for all $s \in S$.
  An element $x \in P$ is a **lower bound** of S if $x \leq s$ for all $s \in S$.
- The set of all upper bounds of S is denoted by S^u (read "S **upper**"):
  $$S^u = \{x \in P : (\forall s \in S)\ s \leq x\}.$$
- The set of all lower bounds is denoted S^ℓ ("S **lower**"):
  $$S^\ell = \{x \in P : (\forall s \in S)\ s \geq x\}.$$
- Since $\leq$ is transitive,
  - S^u is always an up-set;
  - S^ℓ is always a down-set.

Least Upper and Greatest Lower Bounds

- If S^u has a least element x, then x is the **least upper bound** of S. Equivalently, x is the **least upper bound** of S if
  (i) x is an upper bound of S;
  (ii) $x \leq y$, for all upper bounds y of S.
- If S^ℓ has a greatest element x, then x is called the **greatest lower bound** of S.
- Since least elements and greatest elements are unique, least upper bounds and greatest lower bounds are unique when they exist.
- The least upper bound of S is also called the **supremum** of S and is denoted by $\sup S$.
- The greatest lower bound of S is also called the **infimum** of S and is denoted by $\inf S$.

Top and Bottom

- We discuss P itself with respect to suprema and infima:
  - If P has a top element, then $P^u = \{\top\}$; thus, $\sup P = \top$.
  - When P has no top element, we have $P^u = \varnothing$.
    Hence, $\sup P$ does not exist.
  - If P has a bottom element, then $\inf P = \bot$.
- We turn to $S = \varnothing$ with respect to suprema and infima:
  - Every element $x \in P$ satisfies (vacuously) $s \leq x$, for all $s \in S$. Thus, $\varnothing^u = P$ and, hence, $\sup \varnothing$ exists if and only if P has a bottom element, and in that case $\sup \varnothing = \bot$.
  - If P has a top element, then $\inf \varnothing = \top$.

Joins and Meets

- We write:
  - $x \vee y$ (read as "x **join** y") in place of $\sup\{x, y\}$ when it exists;
  - $x \wedge y$ (read as "x **meet** y") in place of $\inf\{x, y\}$ when it exists.
- Similarly we write:
  - $\bigvee S$ (the "**join of** S") instead of $\sup S$ and
  - $\bigwedge S$ (the "**meet of** S") instead of $\inf S$

  when these exist.
- It is sometimes necessary to indicate that the join or meet is being found in a particular ordered set P, in which case we write

$$\bigvee_P S \quad \text{or} \quad \bigwedge_P S.$$

- If S is of the form $S = \{A_i\}_{i \in I}$, where I is some indexing set, we write $\bigvee_{i \in I} A_i$ for $\bigvee\{A_i : i \in I\}$ and $\bigwedge_{i \in I} A_i$ for $\bigwedge\{A_i : i \in I\}$.

Lattices and Complete Lattices

Definitions

Let P be a non-empty ordered set.

(i) If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a **lattice**.

(ii) If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a **complete lattice**.

(1) Let P be any ordered set. Suppose $x, y \in P$ and $x \leq y$. Then

$$\begin{array}{c|c} \{x,y\}^u = \uparrow y & \{x,y\}^\ell = \downarrow x \\ x \vee y = y & x \wedge y = x \end{array}$$

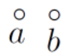
In particular, since $\leq$ is reflexive, we have $x \vee x = x$ and $x \wedge x = x$.

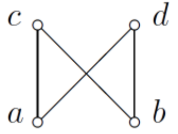
Remarks on Lattices and Complete Lattices

(2) In an ordered set P, the least upper bound $x \vee y$ of $\{x, y\}$ may fail to exist for two different reasons:

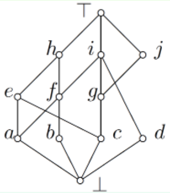
(a) because x and y have no common upper bound;

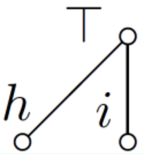
(b) because they have no least upper bound.

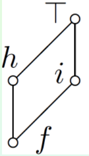




(3) Consider the ordered set drawn below.







Since $\{b, c\}^u = \{\top, h, i\}$ has distinct minimal elements, h and i, it cannot have a least element. Hence $b \vee c$ does not exist.

Since $\{a, b\}^u = \{\top, h, i, f\}$ has a least element, f, $a \vee b = f$.

Further Remarks on Lattices and Complete Lattices

(4) Let P be a lattice. Then, for all $a, b, c, d \in P$,

(i) $a \leq b$ implies $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$;

(ii) $a \leq b$ and $c \leq d$ imply $a \vee c \leq b \vee d$ and $a \wedge c \leq b \wedge d$.

(i) Using the definitions of join and meet, we get:

$$\left.\begin{array}{r} a \leq b \leq b \vee c \\ c \leq b \vee c \end{array}\right\} \Rightarrow a \vee c \leq b \vee c;$$

$$\left.\begin{array}{l} a \wedge c \leq a \leq b \\ a \wedge c \leq c \end{array}\right\} \Rightarrow a \wedge c \leq b \wedge c.$$

(ii) Using Part (i), we get

$$a \vee c \leq b \vee c \leq b \vee d$$
$$a \wedge c \leq b \wedge c \leq b \wedge d.$$

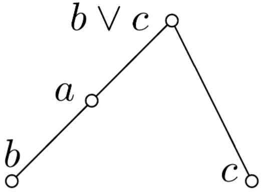
Further Remarks on Lattices and Complete Lattices

(5) Let P be a lattice. Let $a, b, c \in P$ and assume that $b \leq a \leq b \vee c$. Since $c \leq b \vee c$, we have $(b \vee c) \vee c = b \vee c$, by (1). Thus, by (4)(i),

$$b \vee c \leq a \vee c \leq (b \vee c) \vee c = b \vee c,$$

whence $a \vee c = b \vee c$.

Thus, when calculating joins and meets on a diagram, once we know the join of b and c, the join of c with the intermediate element a is forced.



Example I: Some Linear Orders

- Let P be a non-empty ordered set.
  If $x \leq y$, then $x \vee y = y$ and $x \wedge y = x$.
  Hence, to show that P is a lattice, it suffices to prove that $x \vee y$ and $x \wedge y$ exist in P for all noncomparable pairs $x, y \in P$.
- In particular, every chain is a lattice in which

$$x \vee y = \max\{x, y\} \quad \text{and} \quad x \wedge y = \min\{x, y\}.$$

  - Thus, each of $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ and $\mathbb{N}$ is a lattice under its usual order.
    None of them is complete; every one lacks a top element, and a complete lattice must have top and bottom elements.
  - If $x < y$ in $\mathbb{R}$, then the closed interval $[x, y]$ is a complete lattice (by the completeness axiom for $\mathbb{R}$).
  - Failure of completeness in $\mathbb{Q}$ is more fundamental than in $\mathbb{R}$.
    In $\mathbb{Q}$, it is not only the lack of top and bottom elements which causes problems; for example, the set $\{s \in \mathbb{Q} : s^2 < 2\}$ has upper bounds but no least upper bound in $\mathbb{Q}$.

Example II: Powersets

- For any set X, the ordered set $\langle \mathcal{P}(X); \subseteq \rangle$ is a complete lattice in which

$$\bigvee \{A_i : i \in I\} = \bigcup \{A_i : i \in I\} \quad \text{and} \quad \bigwedge \{A_i : i \in I\} = \bigcap \{A_i : i \in I\}.$$

- We indicate the index set by subscripting, e.g., instead of $\bigcup \{A_i : i \in I\}$ we shall write $\bigcup_{i \in I} A_i$ or simply $\bigcup A_i$.
- We verify the assertion about meets (a dual proof works for joins);

  Let $\{A_i\}_{i \in I}$ be a family of elements of $\mathcal{P}(X)$. Since $\bigcap_{i \in I} A_i \subseteq A_j$, for all $j \in I$, it follows that $\bigcap_{i \in I} A_i$ is a lower bound for $\{A_i\}_{i \in I}$.
  Also, if $B \in \mathcal{P}(X)$ is a lower bound of $\{A_i\}_{i \in I}$, then $B \subseteq A_i$, for all $i \in I$ and, hence, $B \subseteq \bigcap_{i \in I} A_i$. Thus, $\bigcap_{i \in I} A_i$ is indeed the greatest lower bound of $\{A_i\}_{i \in I}$ in $\mathcal{P}(X)$.

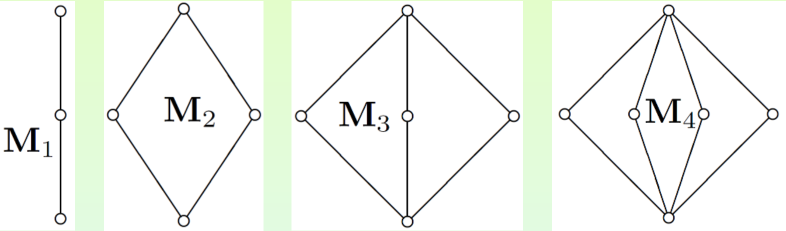
Example III: Lattices of Sets

- Let $\varnothing \neq \mathcal{L} \subseteq \mathcal{P}(X)$. Then $\mathcal{L}$ is called
  - a **lattice of sets** if it is closed under finite unions and intersections;
  - a **complete lattice of sets** if it is closed under arbitrary unions and intersections.
- If $\mathcal{L}$ is a lattice of sets, then $\langle \mathcal{L}; \subseteq \rangle$ is a lattice in which $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.
- Similarly, if $\mathcal{L}$ is a complete lattice of sets, then $\langle \mathcal{L}; \subseteq \rangle$ is a complete lattice with join given by set union and meet given by set intersection.
- Let P be an ordered set and consider the ordered set $\mathcal{O}(P)$ of all down-sets of P.

  If $\{A_i\}_{i \in I} \subseteq \mathcal{O}(P)$, then $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ both belong to $\mathcal{O}(P)$. Hence $\mathcal{O}(P)$ is a complete lattice of sets, called the **down-set lattice** of P.

Example IV: The Ordered Sets M_n

- The ordered set $\boldsymbol{M}_n$ (for $n \geq 1$) is easily seen to be a lattice:



Let $x, y \in \boldsymbol{M}_n$, with $x \parallel y$. Then x and y are in the central antichain of $\boldsymbol{M}_n$ and, hence, $x \vee y = \top$ and $x \wedge y = \bot$.

Example V: The Ordered Set $\langle \mathbb{N}_0; \preccurlyeq \rangle$

- Consider the ordered set $\langle \mathbb{N}_0; \preccurlyeq \rangle$ of non-negative integers ordered by division.
- Recall that k is the **greatest common divisor** (or **highest common factor**) of m and n if
  (a) k divides both m and n (that is, $k \preccurlyeq m$ and $k \preccurlyeq n$);
  (b) if j divides both m and n, then j divides k (that is, $j \preccurlyeq k$, for all lower bounds j of $\{m, n\}$).

  Thus, the greatest common divisor of m and n is precisely the meet of m and n in $\langle \mathbb{N}_0; \preccurlyeq \rangle$.
- Dually, the join of m and n in $\langle \mathbb{N}_0; \preccurlyeq \rangle$ is given by their **least common multiple**.
- These statements remain valid when m or n equals 0.
- Thus, $\langle \mathbb{N}_0; \preccurlyeq \rangle$ is a lattice in which

$$m \vee n = \mathrm{lcm}\{m, n\} \quad \text{and} \quad m \wedge n = \gcd\{m, n\}.$$

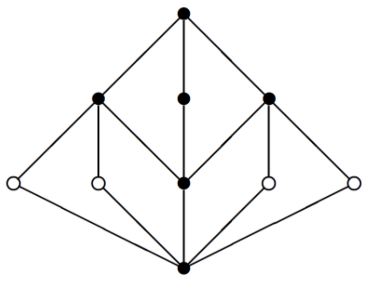
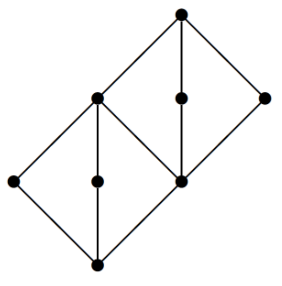
- $\langle \mathbb{N}_0; \preccurlyeq \rangle$ is actually a complete lattice.

Lattices of Subgroups

- Assume that G is a group and $\langle \text{Sub}G; \subseteq \rangle$ is its ordered set of subgroups.
- Let $H, K \in \text{Sub}G$.
  - It is always the case that $H \cap K \in \text{Sub}G$, whence $H \wedge K$ exists and equals $H \cap K$.
  - $H \cup K$ is is not a subgroup in general. Nevertheless, $H \vee K$ does exist in Sub$G$, as (rather tautologically) the subgroup $\langle H \cup K \rangle$ generated by $H \cup K$. Unfortunately, there is no convenient general formula for $H \vee K$.
- Normal subgroups are more amenable.
  - Meet is again given by $\cap$;
  - Join in $\mathcal{N}$-Sub$G$ has a particularly compact description:
    If H, K are normal subgroups of G, then $HK := \{hk : h \in H, k \in K\}$ is also a normal subgroup of G.
    It follows easily that the join in $\mathcal{N}$-Sub$G$ is given by $H \vee K = HK$.

Examples of Lattices of Subgroups

- The lattices Sub$G$ and $\mathcal{N}$-Sub$G$ for the group, D_4, of symmetries of a square and for the group $\mathbb{Z}_2 \times \mathbb{Z}_4$.

The elements of $\mathcal{N}$-Sub$G$ are shaded.

Subsection 2

Lattices as Algebraic Structures

Lattices as Algebraic Structures

- Given a lattice L, we may define binary operations **join** and **meet** on the non-empty set L by

$$a \vee b := \sup\{a, b\} \quad \text{and} \quad a \wedge b := \inf\{a, b\}, \quad a, b \in L.$$

- The operations $\vee : L^2 \to L$ and $\wedge : L^2 \to L$ are order-preserving.

The Connecting Lemma

Let L be a lattice and let $a, b \in L$. Then the following are equivalent:

(i) $a \leq b$;

(ii) $a \vee b = b$;

(iii) $a \wedge b = a$.

- We have shown that (i) implies both (ii) and (iii).
  Assume (ii). Then b is an upper bound for $\{a, b\}$, whence $b \geq a$. Thus (i) holds. Similarly, (iii) implies (i).

Properties of $\vee$ and $\wedge$

Theorem

Let L be a lattice. Then $\vee$ and $\wedge$ satisfy, for all $a, b, c \in L$,

(L1) $(a \vee b) \vee c = a \vee (b \vee c)$ (**associative laws**)

$(\text{L1})^{\partial}$ $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

(L2) $a \vee b = b \vee a$ (**commutative laws**)

$(\text{L2})^{\partial}$ $a \wedge b = b \wedge a$

(L3) $a \vee a = a$ (**idempotency laws**)

$(\text{L3})^{\partial}$ $a \wedge a = a$

(L4) $a \vee (a \wedge b) = a$ (**absorption laws**)

$(\text{L4})^{\partial}$ $a \wedge (a \vee b) = a$.

- By the **Duality Principle for lattices** it is enough to consider (L1)-(L4).

Proof of the Properties

- We have already proven (L3).
- (L2) is immediate because, for any set S, $\sup S$ is independent of the order in which the elements of S are listed.
- (L4) follows easily from the Connecting Lemma: Since $a \wedge b \leq a$, we get $a \vee (a \wedge b) = a$.
- We prove (L1).

  It is enough, by (L2), to show that $(a \vee b) \vee c = \sup \{a, b, c\}$. This is the case if $\{a \vee b, c\}^u = \{a, b, c\}^u$. But

$$\begin{array}{rcl} d \in \{a, b, c\}^u & \iff & d \in \{a, b\}^u \text{ and } d \geq c \\ & \iff & d \geq a \vee b \text{ and } d \geq c \\ & \iff & d \in \{a \vee b, c\}^u. \end{array}$$

From Algebraic Structures to Ordered Structures

Theorem

Let $\langle L; \vee, \wedge \rangle$ be a non-empty set equipped with two binary operations which satisfy (L1)-(L4) and $(\text{L1})^\partial$-$(\text{L4})^\partial$.

(i) For all $a, b \in L$, we have $a \vee b = b$ if and only if $a \wedge b = a$.

(ii) Define $\leq$ on L by $a \leq b$ if $a \vee b = b$. Then $\leq$ is an order relation.

(iii) With $\leq$ as in (ii), $\langle L; \leq \rangle$ is a lattice in which the original operations agree with the induced operations, that is, for all $a, b \in L$,

$$a \vee b = \sup\{a, b\} \quad \text{and} \quad a \wedge b = \inf\{a, b\}.$$

- Assume $a \vee b = b$. Then $a = a \wedge (a \vee b)$ (by $(\text{L4})^\partial$) $= a \wedge b$ (by assumption).

  Conversely, assume $a \wedge b = a$. Then $b = b \vee (b \wedge a)$ (by (L4)) $= b \vee (a \wedge b)$ (by $(\text{L2})^\partial$) $= b \vee a$ (by assumption) $= a \vee b$ (by (L2)).

From Algebraic Structures to Ordered Structures (Cont'd)

- Now define $\leq$ as in (ii). Then $\leq$ is
  - reflexive by (L3): $a \vee a \stackrel{(L3)}{=} a \Rightarrow a \leq a$;
  - antisymmetric by (L2): $a \leq b \;\&\; b \leq a \Rightarrow a \vee b = b \;\&\; b \vee a = a \stackrel{(L2)}{\Rightarrow} a = b$;
  - transitive by (L1): $a \leq b \;\&\; b \leq c \Rightarrow a \vee b = b \;\&\; b \vee c = c \Rightarrow a \vee c = a \vee (b \vee c) \stackrel{(L1)}{=} (a \vee b) \vee c = b \vee c = c \Rightarrow a \leq c$;
- To show that $\sup\{a, b\} = a \vee b$ in the ordered set $\langle L; \leq \rangle$, we must check:
  - $a \vee b \in \{a, b\}^u$: $a \vee (a \vee b) = (a \vee a) \vee b = a \vee b \Rightarrow a \leq a \vee b$ and $b \vee (a \vee b) = b \vee (b \vee a) = (b \vee b) \vee a = b \vee a = a \vee b \Rightarrow b \leq a \vee b$;
  - $d \in \{a, b\}^u$ implies $d \geq a \vee b$: $(a \vee b) \vee d = (a \vee b) \vee (d \vee d) = ((a \vee b) \vee d) \vee d = (a \vee (b \vee d)) \vee d = (a \vee (d \vee b)) \vee d = ((a \vee d) \vee b) \vee d = (a \vee d) \vee (b \vee d) = d \vee d = d \Rightarrow a \vee b \leq d$;

The characterization of inf is obtained by duality.

Stocktaking: Algebra and Order

- We have shown that lattices can be completely characterized in terms of the join and meet operations.
- We may henceforth say "let L be a lattice", replacing L by $\langle L; \leq \rangle$ or by $\langle L; \vee, \wedge \rangle$ if we want to emphasize that we are thinking of it as a special kind of ordered set or as an algebraic structure.
- In a lattice L, associativity of $\vee$ and $\wedge$ allows us to write iterated joins and meets unambiguously without brackets.
- An easy induction shows that these correspond to sups and infs in the expected way:

$$\bigvee\{a_1, \ldots, a_n\} = a_1 \vee \cdots \vee a_n \quad \text{and} \quad \bigwedge\{a_1, \ldots, a_n\} = a_1 \wedge \cdots \wedge a_n,$$

  for $a_1, \ldots, a_n \in L, n \geq 1$;
- Consequently, $\bigvee F$ and $\bigwedge F$ exist for any finite, non-empty subset F of a lattice.

Bounded Lattices

- Let L be a lattice.
- It may happen that $\langle L; \leq \rangle$ has top and bottom elements $\top$ and $\bot$;
- When thinking of L as $\langle L; \vee, \wedge \rangle$, we say:
  - L has a **one** if there exists $1 \in L$, such that $a = a \wedge 1$, for all $a \in L$;
  - L has a **zero** if there exists $0 \in L$, such that $a = a \vee 0$, for all $a \in L$.
- The lattice $\langle L; \vee, \wedge \rangle$ has a:
  - one if and only if $\langle L; \leq \rangle$ has a top element $\top$ and, in that case, $1 = \top$;
  - zero if and only if $\langle L; \leq \rangle$ has a bottom element $\bot$ and, in that case, $0 = \bot$.
- A lattice $\langle L; \vee, \wedge \rangle$ possessing 0 and 1 is called **bounded**.
- A finite lattice is automatically bounded, with $1 = \bigvee L$ and $0 = \bigwedge L$.

  Example: $\langle \mathbb{N}_0; \mathrm{lcm}, \mathrm{gcd} \rangle$ is bounded, with $1 = 0$ and $0 = 1$.

Subsection 3

Sublattices, Products and Homomorphisms

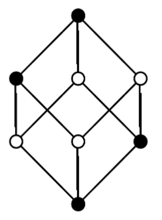
Sublattices

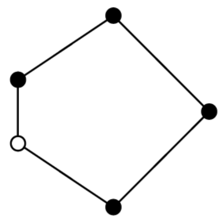
Definition (Sublattice)

Let L be a lattice and $\varnothing \neq M \subseteq L$. Then M is a **sublattice** of L if

$$a, b \in M \quad \text{implies} \quad a \vee b \in M \text{ and } a \wedge b \in M.$$

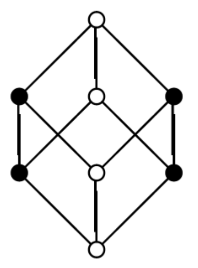
- We denote the collection of all sublattices of L by $\mathrm{Sub}L$ and let $\mathrm{Sub}_0 L = \mathrm{Sub}L \cup \{\varnothing\}$; both are ordered by inclusion.
- Examples:
  (1) Any one-element subset of a lattice is a sublattice. More generally, any non-empty chain in a lattice is a sublattice. (To test that a non-empty subset M is a sublattice, it suffices to consider non-comparable elements a, b.)
  (2) In the diagrams the shaded elements form sublattices:

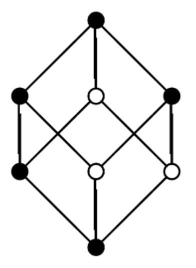




More Examples of Sublattices

(3) In the diagrams below the shaded elements do not form sublattices:





(3) A subset M of a lattice $\langle L; \leq\rangle$ may be a lattice in its own right without being a sublattice of L, e.g., the right picture above.

Products

- Let L and K be lattices.
- Define $\vee$ and $\wedge$ coordinatewise on $L \times K$, as follows:

$$\begin{array}{rcl}(\ell_1, k_1) \vee (\ell_2, k_2) & = & (\ell_1 \vee \ell_2, k_1 \vee k_2), \\ (\ell_1, k_1) \wedge (\ell_2, k_2) & = & (\ell_1 \wedge \ell_2, k_1 \wedge k_2).\end{array}$$

- It is routine to check that $L \times K$ satisfies the identities (L1)-(L4)$^\partial$ and therefore is a lattice.
- Also

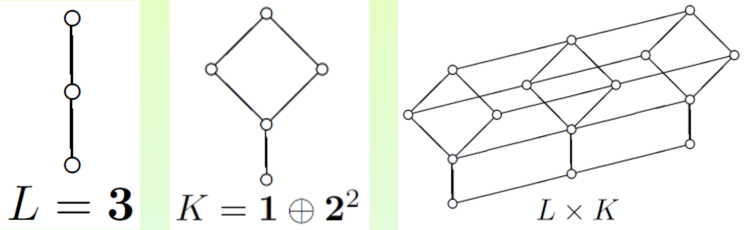
$$\begin{array}{rcl}(\ell_1, k_1) \vee (\ell_2, k_2) = (\ell_2, k_2) & \iff & \ell_1 \vee \ell_2 = \ell_2 \text{ and } k_1 \vee k_2 = k_2 \\ & \iff & \ell_1 \leq \ell_2 \text{ and } k_1 \leq k_2 \\ & \iff & (\ell_1, k_1) \leq (\ell_2, k_2),\end{array}$$

  with respect to the order on $L \times K$.

  Hence the lattice formed by taking the ordered set product of lattices L and K is the same as that obtained by defining $\vee$ and $\wedge$ coordinatewise on $L \times K$.

An Example

- The product of the lattices $L = \mathbf{3}$ and $K = \mathbf{1} \oplus \mathbf{2}^2$:



  Notice how (isomorphic copies) of L and K sit inside $L \times K$ as the sublattices $L \times \{0\}$ and $\{0\} \times K$.
- The product of lattices L and K always contains sublattices isomorphic to L and K.
- Iterated products and powers are defined in the obvious way.
- It is also possible to define the product of an infinite family of lattices.

Homomorphisms

Definition

Let L and K be lattices. A map $f : L \to K$ is said to be a **homomorphism** (or, for emphasis, **lattice homomorphism**) if f is **join-preserving** and **meet-preserving**, i.e., for all $a, b \in L$,

$$f(a \vee b) = f(a) \vee f(b) \quad \text{and} \quad f(a \wedge b) = f(a) \wedge f(b).$$

A bijective homomorphism is a (**lattice**) **isomorphism**.
If $f : L \to K$ is a one-to-one homomorphism, then the sublattice $f(L)$ of K is isomorphic to L and we refer to f as an **embedding** (of L into K).

Remarks on Lattice Homomorphisms

(1) The inverse of an isomorphism is a homomorphism and hence is also an isomorphism:
Let $f : L \to K$ be an isomorphism, $a', b' \in K$, such that $a' = f(a), b' = f(b)$. Then, for the join (and dually for the meet)

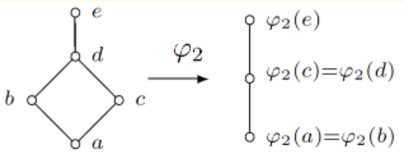
$$\begin{array}{rcl} f^{-1}(a' \vee b') & = & f^{-1}(f(a) \vee f(b)) \\ & = & f^{-1}(f(a \vee b)) \\ & = & a \vee b \\ & = & f^{-1}(f(a)) \vee f^{-1}(f(b)) \\ & = & f^{-1}(a') \vee f^{-1}(b'); \end{array}$$

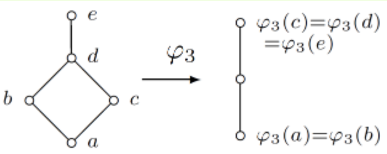
(2) We write $L \rightarrowtail K$ to indicate that the lattice K has a sublattice isomorphic to the lattice L.
We will see, next, that $M \rightarrowtail L$ implies $M \hookrightarrow L$.

(3) For bounded lattices L and K it is often appropriate to consider homomorphisms $f : L \to K$, such that $f(0) = 0$ and $f(1) = 1$. Such maps are called $\{0, 1\}$-**homomorphisms**.

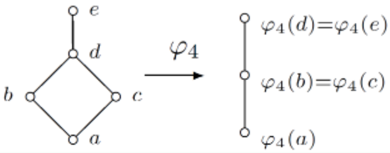
Examples of Mappings

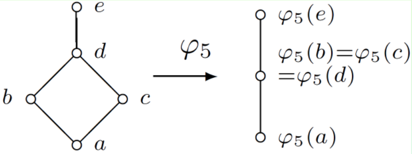
- The maps φ_2 and φ_3 are homomorphisms:





- The maps φ_4 and φ_5 are order preserving but not homomorphisms:





- In general an order-preserving map may not be a homomorphism.

Order and Lattice Isomorphisms

Proposition

Let L and K be lattices and $f : L \to K$ a map.

(i) The following are equivalent:
  (a) f is order-preserving;
  (b) $(\forall a, b \in L) f(a \vee b) \geq f(a) \vee f(b)$;
  (c) $(\forall a, b \in L) f(a \wedge b) \leq f(a) \wedge f(b)$.

  In particular, if f is a homomorphism, then f is order-preserving.

(ii) f is a lattice isomorphism if and only if it is an order-isomorphism.

(i) Since $a \leq a \vee b, b \leq a \vee b, a \wedge b \leq a$ and $a \wedge b \leq b$, we get

$$\left.\begin{array}{l} f(a) \leq f(a \vee b) \\ f(b) \leq f(a \vee b) \end{array}\right\} \Rightarrow f(a) \vee f(b) \leq f(a \vee b);$$

$$\left.\begin{array}{l} f(a \wedge b) \leq f(a) \\ f(a \wedge b) \leq f(b) \end{array}\right\} \Rightarrow f(a \wedge b) \leq f(a) \wedge f(b).$$

Order and Lattice Isomorphisms (Cont'd)

(ii) Assume that f is a lattice isomorphism. Then, by the Connecting Lemma, $a \le b$ iff $a \vee b = b$ iff $f(a \vee b) = f(b)$ iff $f(a) \vee f(b) = f(b)$ iff $f(a) \le f(b)$, whence, f is an order-embedding, and so is an order-isomorphism.

- Conversely, assume that f is an order-isomorphism. Then f is bijective. By (i) and duality, to show that f is a lattice isomorphism it suffices to show that

$$f(a) \vee f(b) \ge f(a \vee b), \quad \text{for all } a, b \in L.$$

Since f is surjective, there exists $c \in L$, such that $f(a) \vee f(b) = f(c)$. Then $f(a) \le f(c)$ and $f(b) \le f(c)$. Since f is an order-embedding, it follows that $a \le c$ and $b \le c$, whence $a \vee b \le c$. Because f is order-preserving, $f(a \vee b) \le f(c) = f(a) \vee f(b)$, as required.

Subsection 4

Ideals and Filters

Ideals

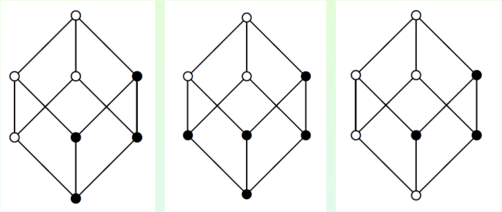
Definition

Let L be a lattice. A non-empty subset J of L is called an **ideal** if

(i) $a, b \in J$ implies $a \vee b \in J$,

(ii) $a \in L$, $b \in J$ and $a \leq b$ imply $a \in J$.

- More compactly, an ideal is a non-empty down-set closed under join.



An ideal and two non-ideals.

- Every ideal J of a lattice L is a sublattice, since $a \wedge b \leq a$ for any $a, b \in L$.

Filters

Definition

Let L be a lattice. A non-empty subset G of L is called a **filter** if

(i) $a, b \in G$ implies $a \wedge b \in G$,

(ii) $a \in L$, $b \in G$ and $a \geq b$ imply $a \in G$.

- The set of all ideals of L is denoted by $\mathcal{I}(L)$.
- The set of all filters of L is denoted by $\mathcal{F}(L)$.
- An ideal or filter is called **proper** if it does not coincide with L.
  - An ideal J of a lattice with 1 is proper if and only if $1 \notin J$;
  - Dually, a filter G of a lattice with 0 is proper if and only if $0 \notin G$.
- For each $a \in L$, the set $\downarrow a$ is an ideal, known as the **principal ideal** generated by a.
- Dually, $\uparrow a$ is the **principal filter** generated by a.

Examples

(1) In a finite lattice, every ideal or filter is principal:
- The ideal J equals $\downarrow \bigvee J$.
- The filter G equals $\uparrow \bigwedge G$.

(2) Let L and K be bounded lattices and $f : L \to K$ a $\{0,1\}$-homomorphism. Then $f^{-1}(0)$ is an ideal and $f^{-1}(1)$ is a filter in L.

(3) The following are ideals in $\mathcal{P}(X)$:
  (a) all subsets not containing a fixed element of X;
  (b) all finite subsets (this ideal is non-principal if X is infinite).

(4) Let $(X;\mathcal{T})$ be a topological space and let $x \in X$. Then the set $\{V \subseteq X : (\exists U \in \mathcal{T}) x \in U \subseteq V\}$ is a filter in $\mathcal{P}(X)$. It is called the **filter of neighborhoods** of x.

Subsection 5

Complete Lattices and $\bigcap$-Structures

Complete Lattices: Basic Properties

- Recall that a **complete lattice** is defined to be a non-empty, ordered set P, such that the join (supremum), $\bigvee S$, and the meet (infimum), $\bigwedge S$, exist for every subset S of P.
- The following are immediate consequences of the definitions of least upper bound and greatest lower bound:

Lemma

Let P be an ordered set, let $S, T \subseteq P$ and assume that $\bigvee S, \bigvee T, \bigwedge S$ and $\bigwedge T$ exist in P.

(i) $s \leq \bigvee S$ and $s \geq \bigwedge S$, for all $s \in S$.

(ii) Let $x \in P$; then $x \leq \bigwedge S$ if and only if $x \leq s$, for all $s \in S$.

(iii) Let $x \in P$; then $x \geq \bigvee S$ if and only if $x \geq s$, for all $s \in S$.

(iv) $\bigvee S \leq \bigwedge T$ if and only if $s \leq t$, for all $s \in S$ and all $t \in T$.

(v) If $S \subseteq T$, then $\bigvee S \leq \bigvee T$ and $\bigwedge S \geq \bigwedge T$.

Proof of the Basic Properties

(i) $\bigvee S$ is an upper bound of S and $s \in S$. Hence, $s \leq \bigvee S$.
$\bigwedge S$ is a lower bound of S and $s \in S$. Hence, $\bigwedge S \leq s$.

(ii) Suppose $x \leq \bigwedge S$. Since $\bigwedge S \leq s$, for all $s \in S$, we get, by transitivity, $x \leq s$, for all $s \in S$.
Suppose $x \leq s$, for all $s \in S$. This means that x is a lower bound of S. Since $\bigwedge S$ is a greatest lower bound of S, $x \leq \bigwedge S$.

(iii) Dual to Part (ii).

(iv) Suppose $\bigvee S \leq \bigwedge T$. Let $s \in S$ and $t \in T$. Then $s \leq \bigvee S \leq \bigwedge T \leq t$.
Assume, conversely, that, for all $s \in S$ and all $t \in T$, $s \leq t$. By Part (ii), $s \leq \bigwedge T$. By Part (iii), $\bigvee S \leq \bigwedge T$.

(v) Suppose $S \subseteq T$.
- $\bigvee T$ is an upper bound of T. Since $S \subseteq T$, $\bigvee T$ is an upper bound of S. $\bigvee S$ is the least upper bound of S. Hence, $\bigvee S \leq \bigvee T$.
- $\bigwedge T$ is a lower bound of T. Since $S \subseteq T$, $\bigvee T$ is also a lower bound of S. $\bigwedge S$ is the greatest lower bound of S. Hence, $\bigwedge T \leq \bigwedge S$.

Join and Meet and Set Unions

Lemma

Let P be a lattice, let $S, T \subseteq P$ and assume that $\bigvee S, \bigvee T, \bigwedge S$ and $\bigwedge T$ exist in P. Then

$$\bigvee(S \cup T) = (\bigvee S) \vee (\bigvee T) \quad \text{and} \quad \bigwedge(S \cup T) = (\bigwedge S) \wedge (\bigwedge T).$$

- $\bigvee(S \cup T)$ is an upper bound of $S \cup T$. Thus, $\bigvee(S \cup T)$ is an upper bound of S and of T. Since $\bigvee S$ is the least upper bound of S, $\bigvee S \leq \bigvee(S \cup T)$. Since $\bigvee T$ is the least upper bound of T, $\bigvee T \leq \bigvee(S \cup T)$. Since $(\bigvee S) \vee (\bigvee T)$ is the least upper bound of $\{\bigvee S, \bigvee T\}$, $(\bigvee S) \vee (\bigvee T) \leq \bigvee(S \cup T)$.
  $(\bigvee S) \vee (\bigvee T)$ is an upper bound of $\{\bigvee S, \bigvee T\}$. By transitivity, $(\bigvee S) \vee (\bigvee T)$ is an upper bound of $S \cup T$. Since $\bigvee(S \cup T)$ is the least upper bound of $S \cup T$, $\bigvee(S \cup T) \leq (\bigvee S) \vee (\bigvee T)$.
  By antisymmetry, $\bigvee(S \cup T) = (\bigvee S) \vee (\bigvee T)$.
  The second equality can be shown similarly.

On Finite Joins and Meets

- Using the preceding lemma, we get, using induction,

Lemma

Let P be a lattice. Then $\bigvee F$ and $\bigwedge F$ exist for every finite, non-empty subset F of P.

- Let $F = \{x_1, x_2, \ldots, x_n\}$, $n \geq 1$. Then:
  - $\bigvee\{x_1\} = x_1$;
  - $\bigvee\{x_1, x_2\} = x_1 \vee x_2$;
  - $\bigvee\{x_1, x_2, \ldots, x_n\} = \bigvee\{x_1, x_2, \ldots, x_{n-1}\} \vee x_n$.

  Similarly, we may show that the finite meet $\bigwedge F$ also exists.

Corollary

Every finite lattice is complete.

Joins and Meets and Order-Preserving Maps

Definition

Let P and Q be ordered sets and $\varphi : P \to Q$ a map. Then we say that

- φ **preserves existing joins** if whenever $\bigvee S$ exists in P then $\bigvee \varphi(S)$ exists in Q and $\varphi(\bigvee S) = \bigvee \varphi(S)$;
- φ **preserves existing meets** if whenever $\bigwedge S$ exists in P then $\bigwedge \varphi(S)$ exists in Q and $\varphi(\bigwedge S) = \bigwedge \varphi(S)$

Lemma

Let P and Q be ordered sets and $\varphi : P \to Q$ be an order-preserving map.

(i) Assume that $S \subseteq P$ is such that $\bigvee S$ exists in P and $\bigvee \varphi(S)$ exists in Q. Then $\varphi(\bigvee S) \geq \bigvee \varphi(S)$. Dually, $\varphi(\bigwedge S) \leq \bigwedge \varphi(S)$ if both meets exist.

(ii) Assume now that $\varphi : P \to Q$ is an order-isomorphism. Then φ preserves all existing joins and meets.

Proof of the Lemma

(i) $\bigvee S$ is an upper bound of S: $S \leq \bigvee S$. φ is order preserving: $\varphi(S) \leq \varphi(\bigvee S)$. $\bigvee \varphi(S)$ is the least upper bound of $\varphi(S)$. Hence, $\bigvee \varphi(S) \leq \varphi(\bigvee S)$.

$\bigwedge S$ is a lower bound of S: $\bigwedge S \leq S$. φ is order-preserving: $\varphi(\bigwedge S) \leq \varphi(S)$. $\bigwedge \varphi(S)$ is the greatest lower bound of $\varphi(S)$. Hence, $\varphi(\bigwedge S) \leq \bigwedge \varphi(S)$.

(ii) Assume φ is an order isomorphism. In particular, it is surjective. Thus, there exists $x \in P$, such that $\bigvee \varphi(S) = \varphi(x)$. Thus, for all $s \in S$, $\varphi(s) \leq \varphi(x)$. Since φ is order reflecting, $S \leq x$. Since $\bigvee S$ is the least upper bound of S, $\bigvee S \leq x$. Since φ is order preserving, $\varphi(\bigvee S) \leq \varphi(x)$. Thus, $\varphi(\bigvee S) \leq \bigvee \varphi(S)$. Equality follows by Part (i) and antisymmetry.

Preservation of meets can be shown similarly.

Subsets of Complete Lattices

- The next lemma is useful for showing that certain subsets of complete lattices are themselves complete lattices.

Lemma

Let Q be a subset, with the induced order, of some ordered set P and let $S \subseteq Q$. If $\bigvee_P S$ exists and belongs to Q, then $\bigvee_Q S$ exists and equals $\bigvee_P S$ (and dually for $\bigwedge_Q S$).

- For any $x \in S$, we have $x \le \bigvee_P S$. since $\bigvee_P S \in Q$, by hypothesis, it acts as an upper bound for S in Q. Further, if y is any upper bound for S in Q, it is also an upper bound for S in P and so $y \ge \bigvee_P S$.

Corollary

Let $\mathcal{L}$ be a family of subsets of a set X and let $\{A_i\}_{i\in I}$ be a subset of $\mathcal{L}$.

(i) If $\bigcup_{i\in I} A_i \in \mathcal{L}$, then $\bigvee_{\mathcal{L}}\{A_i : i \in I\}$ exists and equals $\bigcup_{i\in I} A_i$.

(ii) If $\bigcap_{i\in I} A_i \in \mathcal{L}$, then $\bigwedge_{\mathcal{L}}\{A_i : i \in I\}$ exists and equals $\bigcap_{i\in I} A_i$.

Consequently, any (complete) lattice of sets is a (complete) lattice with joins and meets given by union and intersection.

Synthesizing Joins Using Meets

- To show that an ordered set is a complete lattice requires only half as much work as the definition would have us believe.

Lemma

Let P be an ordered set such that $\bigwedge S$ exists in P, for every non-empty subset S of P. Then $\bigvee S$ exists in P, for every subset S of P which has an upper bound in P; indeed, $\bigvee S = \bigwedge S^u$.

- Let $S \subseteq P$ and assume that S has an upper bound in P. Thus, $S^u \neq \varnothing$. Hence, by assumption, $a = \bigwedge S^u$ exists in P. We claim that $\bigvee S = a$.

  For all $s \in S$ and all $u \in S^u$, $s \leq u$. Consequently, for all $s \in S$, $s \leq \bigwedge S^u = a$. Thus, a is an upper bound of S.

  Suppose b is also an upper bound of S. By definition, $b \in S^u$. Hence, $a = \bigwedge S^u \leq b$. Therefore, a is the least upper bound of S, i.e., $a = \bigvee S$.

Complete Lattices in Terms of Arbitrary Meets

Theorem

Let P be a non-empty ordered set. Then the following are equivalent:

(i) P is a complete lattice;

(ii) $\bigwedge S$ exists in P, for every subset S of P;

(iii) P has a top element, $\top$, and $\bigwedge S$ exists in P for every non-empty subset S of P.

- It is trivial that (i) implies (ii).

  (ii) implies (iii) since the meet of the empty subset of P exists only if P has a top element.

  It follows easily from the previous lemma that (iii) implies (i).

Complete Lattices of Sets

Corollary

Let X be a set and let $\mathcal{L}$ be a family of subsets of X, ordered by inclusion, such that:

(a) $\bigcap_{i\in I} A_i \in \mathcal{L}$, for every non-empty family $\{A_i\}_{i\in I} \subseteq \mathcal{L}$, and
(b) $X \in \mathcal{L}$.

Then $\mathcal{L}$ is a complete lattice in which

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i, \quad \bigvee_{i\in I} A_i = \bigcap \{B \in \mathcal{L} : \bigcup_{i\in I} A_i \subseteq B\}.$$

- To show that $\langle \mathcal{L}; \subseteq \rangle$ is a complete lattice, it suffices to show that $\mathcal{L}$ has a top element and that the meet of every nonempty subset of $\mathcal{L}$ exists in $\mathcal{L}$. By (b), $\mathcal{L}$ has a top element, namely X. Let $\{A_i\}_{i\in I}$ be a non-empty subset of $\mathcal{L}$. Then (a) gives $\bigcap_{i\in I} A_i \in \mathcal{L}$. Therefore $\bigwedge_{i\in I} A_i$ exists and is given by $\bigcap_{i\in I} A_i$. Thus, $\langle \mathcal{L}; \subseteq \rangle$ is a complete lattice. Since X is an upper bound of $\{A_i\}_{i\in I}$ in $\mathcal{L}$, $\bigvee_{i\in I} A_i = \bigwedge \{A_i : i \in I\}^u = \bigcap \{B \in \mathcal{L} : (\forall i \in I) A_i \subseteq B\} = \bigcap \{B \in \mathcal{L} : \bigcup_{i\in I} A_i \subseteq B\}$.

Intersection Structures

Definitions

If $\mathcal{L}$ is a non-empty family of subsets of X which satisfies

$$\bigcap_{i \in I} A_i \in \mathcal{L}, \text{ for every non-empty family } \{A_i\}_{i \in I} \subseteq \mathcal{L},$$

then $\mathcal{L}$ is called an **intersection structure** (or **$\bigcap$-structure**) on X. If $\mathcal{L}$ also satisfies $X \in \mathcal{L}$, we refer to it as a **topped intersection structure** on X. An alternative term is **closure system**.

- In a complete lattice $\mathcal{L}$ of this type:
  - the meet is just set intersection, but
  - in general the join is not set union.

Algebraic $\bigcap$-Intersection Structures

- Each of the following is a topped $\bigcap$-structure and so forms a complete lattice under inclusion:
  - the subgroups, $\text{Sub}G$, of a group G;
  - the normal subgroups, $\mathcal{N}\text{-Sub}G$, of a group G;
  - the equivalence relations on a set X;
  - the subspaces, $\text{Sub}V$ of a vector space V;
  - the convex subsets of a real vector space;
  - the subrings of a ring;
  - the ideals of a ring;
  - $\text{Sub}_0 L$, the sublattices of a lattice L, with the empty set adjoined (note that $\text{Sub}L$ is not closed under intersections, except when $|L| = 1$);
  - the ideals of a lattice L with 0 (or, if L has no zero element, the ideals of L with the empty set added), and dually for filters.

  These families all belong to a class of $\bigcap$-structures, called **algebraic $\bigcap$-structures** because of their provenance.

Topological $\bigcap$-Intersection Structures

- The closed subsets of a topological space are closed under finite unions and finite intersections and hence form a lattice of sets in which $A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

  In fact, the closed sets form a topped $\bigcap$-structure and, consequently, the lattice of closed sets is complete.
  - Meet is given by intersection;
  - The join of a family of closed sets is not their union but is obtained by forming the closure of their union.
- Since the open subsets of a topological space are closed under arbitrary union and include the empty set, they form a complete lattice under inclusion.

  By the dual version of the preceding corollary, join and meet are given by
  $$\bigvee_{i \in I} A_i = \bigcup_{i \in I} A_i \quad \text{and} \quad \bigwedge_{i \in I} A_i = \text{Int}(\bigcap_{i \in I} A_i),$$
  where $\text{Int}(A)$ denotes the interior of A.

The Knaster-Tarski Fixpoint Theorem

- Given an ordered set P and a map $F : P \to P$, an element $x \in P$ is called a **fixpoint** of F if $F(x) = x$.

The Knaster-Tarski Fixpoint Theorem

Let L be a complete lattice and $F : L \to L$ an order-preserving map. Then

$$\alpha := \bigvee \{x \in L : x \leq F(x)\}$$

is a fixpoint of F. Further, α is the greatest fixpoint of F.
Dually, F has a least fixpoint, given by $\bigwedge \{x \in L : F(x) \leq x\}$.

- Let $H = \{x \in L : x \leq F(x)\}$. For all $x \in H$, $x \leq \alpha$, so $x \leq F(x) \leq F(\alpha)$. Thus, $F(\alpha) \in H^u$, whence $\alpha \leq F(\alpha)$. Since F is order-preserving, $F(\alpha) \leq F(F(\alpha))$. This says $F(\alpha) \in H$, so $F(\alpha) \leq \alpha$.
  If β is any fixpoint of F, then $\beta \in H$, so $\beta \leq \alpha$.

Subsection 6

Chain Conditions and Completeness

Finiteness Conditions

- We know that every finite lattice is complete.
- There are various finiteness conditions, of which "P is finite" is the strongest, which will guarantee that a lattice P is complete.

Definition

Let P be an ordered set.

(i) If $C = \{c_0, c_1, \ldots, c_n\}$ is a finite chain in P with $|C| = n + 1$, then we say that the **length** of C is n.

(ii) P is said to have **length** n, written $\ell(P) = n$, if the length of the longest chain in P is n.

(iii) P is of **finite length** if it has length n for some $n \in \mathbb{N}_0$.

(iv) P has **no infinite chains** if every chain in P is finite.

(v) P satisfies the **ascending chain condition**, (**ACC**), if given any sequence $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$ of elements of P, there exists $k \in \mathbb{N}$, such that $x_k = x_{k+1} = \cdots$.

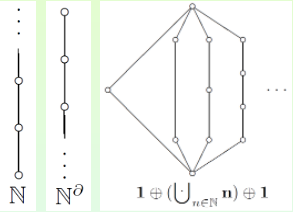
The dual of the ACC is the **descending chain condition**, (**DCC**).

Examples

(1) The lattices M_n are of length 2. A lattice of finite length has no infinite chains and so satisfies both (ACC) and (DCC).

(2) The lattice $\langle \mathbb{N}_0; \leqslant \rangle$ satisfies (DCC) but not (ACC).

(3)



The chain $\mathbb{N}$ satisfies (DCC) but not (ACC). Dually, $\mathbb{N}^\partial$ satisfies (ACC) but not (DCC). The lattice $\mathbf{1} \oplus (\dot\bigcup_{n\in\mathbb{N}} \boldsymbol{n}) \oplus \mathbf{1}$ is the simplest example of a lattice which has no infinite chains but is not of finite length.

(4) It can be shown that a vector space V is finite dimensional if and only if Sub$V$ is of finite length, in which case $\dim V = \ell(\text{Sub}V)$.

ACC and Maximal Elements

Lemma

An ordered set P satisfies (ACC) if and only if every non-empty subset A of P has a maximal element.

Informal Proof: We shall prove the contrapositive in both directions, i.e., we prove that P has an infinite ascending chain if and only if there is a non-empty subset A of P which has no maximal element.

- Assume that $x_1 < x_2 < \cdots < x_n < \cdots$ is an infinite ascending chain in P. Then, clearly, $A = \{x_n : n \in \mathbb{N}\}$ has no maximal element.
- Conversely, assume that A is a non-empty subset of P which has no maximal element. Let $x_1 \in A$. Since x_1 is not maximal in A, there exists $x_2 \in A$, with $x_1 < x_2$. Similarly, there exists $x_3 \in A$, with $x_2 < x_3$. Continuing in this way (the Axiom of Choice is needed) we obtain an infinite ascending chain in P.

ACC, DCC and Infinite Chains

Theorem

An ordered set P has no infinite chains if and only if it satisfies both (ACC) and (DCC).

- If P has no infinite chains, then it satisfies both (ACC) and (DCC).
  Suppose that P satisfies both (ACC) and (DCC) and contains an infinite chain C. Note that if A is a non-empty subset of C, then A has a maximal element m, by the preceding lemma. If $a \in A$, then, since C is a chain, we have $a \le m$ or $m \le a$.
  - But $m \le a$ implies $m = a$, by the maximality of m.
  - Hence, $a \le m$, for all $a \in A$. So every non-empty subset of C has a greatest element.

  Let x_1 be the greatest element of C; let x_2 be the greatest element of $C \setminus \{x_1\}$; in general let x_{n+1} be the greatest element of $C \setminus \{x_1, x_2, \ldots, x_n\}$. Then $x_1 \succ x_2 \succ \cdots \succ x_n \succ \cdots$ is an infinite, descending, covering chain in P, contradicting the (DCC).

Chain Conditions and Completeness

- Lattices with no infinite chains are complete:

Theorem

Let P be a lattice.

(i) If P satisfies (ACC), then for every non-empty subset A of P, there exists a finite subset F of A, such that $\bigvee A = \bigvee F$ (which exists in P).

(ii) If P has a bottom element and satisfies (ACC), then P is complete.

(iii) If P has no infinite chains, then P is complete.

- Assume that P satisfies (ACC) and let A be a non-empty subset of P. Then, $B := \{\bigvee F : F \text{ is a finite non-empty subset of } A\}$ is a well-defined subset of P. Since B is non-empty, B has a maximal element $m = \bigvee F$, for some finite subset F of A. Let $a \in A$. Then $\bigvee(F \cup \{a\}) \in B$ and $m = \bigvee F \leq \bigvee(F \cup \{a\})$. Since m is maximal in B, $m = \bigvee F = \bigvee(F \cup \{a\})$. As $m = \bigvee(F \cup \{a\})$, we have $a \leq m$, whence m is an upper bound of A.

Chain Conditions and Completeness (Cont'd)

- Let $x \in P$ be an upper bound of A. Then x is an upper bound of F, since $F \subseteq A$. Hence $m = \bigvee F \leq x$. Thus, m is the least upper bound of A, i.e., $\bigvee A = m = \bigvee F$.

  (ii) follows from (i) and a preceding result.

  A lattice with no infinite chains has a bottom element and satisfies (ACC), whence (iii) follows from (ii).

Subsection 7

Join-Irreducible Elements

Join- and Meet-Irreducible Elements

Definition

Let L be a lattice. An element $x \in L$ is **join-irreducible** if:

(i) $x \neq 0$ (in case L has a zero);

(ii) $x = a \vee b$ implies $x = a$ or $x = b$, for all $a, b \in L$.

Condition (ii) is equivalent to the more pictorial:

(ii)′ $a < x$ and $b < x$ imply $a \vee b < x$, for all $a, b \in L$.

Definition

Let L be a lattice. An element $x \in L$ is **meet-irreducible** if:

(i) $x \neq 1$ (in case L has a one);

(ii) $x = a \wedge b$ implies $x = a$ or $x = b$, for all $a, b \in L$.

Condition (ii) is equivalent to the more pictorial:

(ii)′ $x < a$ and $x < b$ imply $x < a \wedge b$, for all $a, b \in L$.

Join-Dense and Meet-Dense Subsets

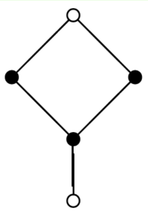
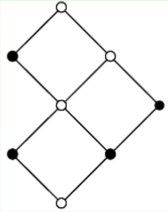
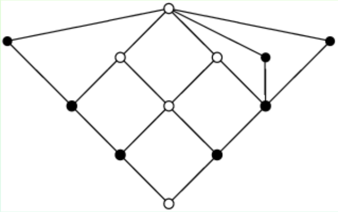
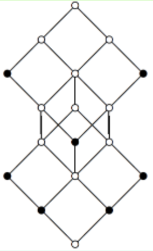
- We denote:
  - the set of join-irreducible elements of L by $\mathcal{J}(L)$;
  - the set of meet-irreducible elements by $\mathcal{M}(L)$.

  Each of these sets inherits L's order relation, and will be regarded as an ordered set.
- Let P be an ordered set and let $Q \subseteq P$.
  - Q is called **join-dense in** P if for every element $a \in P$, there is a subset A of Q such that $a = \bigvee_P A$;
  - Q is called **meet-dense in** P if for every element $a \in P$, there exists a subset A of Q such that $a = \bigwedge_P A$.

Examples I

(1) In a chain, every non-zero element is join-irreducible. Thus, if L is an n-element chain, then $\mathcal{J}(L)$ is an $(n-1)$-element chain.

(2) In a finite lattice L, an element is join-irreducible if and only if it has exactly one lower cover. This makes $\mathcal{J}(L)$ extremely easy to identify from a diagram of L.

Examples II

(3) Consider the lattice $\langle \mathbb{N}_0; \text{lcm}, \text{gcd} \rangle$. A non-zero element $m \in \mathbb{N}_0$ is join-irreducible if and only if m is of the form p^r, where p is a prime and $r \in \mathbb{N}$.

(4) In a lattice $\mathcal{P}(X)$ the join-irreducible elements are exactly the singleton sets, $\{x\}$, for $x \in X$.

(5) It is easily seen that the lattice of open subsets of $\mathbb{R}$ (that is, subsets which are unions of open intervals) has no join-irreducible elements.

Some Remarks

- We have excluded 0 from being regarded as join-irreducible.
  - Note that we can never write 0 as a non-empty join, $\bigvee_P A$, unless $0 \in A$.
  - To compensate for this restriction, we have not excluded $A = \varnothing$ in the definition of join-density, noting that $\bigvee_P \varnothing = 0$ in a lattice P with zero.

  Insisting that 0 is not join-irreducible is the lattice-theoretic equivalent of declaring that 1 is not a prime number.
- Our examples have shown that join-irreducible elements do not necessarily exist in infinite lattices.

  On the other hand, it is easy to see that in a finite lattice every element is a join of join-irreducible elements.

DCC and Join-Irreducibles

Proposition

Let L be a lattice satisfying (DCC).

(i) Suppose $a, b \in L$ and $a \nleq b$. Then, there exists $x \in \mathcal{J}(L)$, such that $x \leq a$ and $x \nleq b$.

(ii) $a = \bigvee\{x \in \mathcal{J}(L) : x \leq a\}$, for all $a \in L$.

These conclusions hold in particular if L is finite.

(i) Let $a \nleq b$ and let $S := \{x \in L : x \leq a \text{ and } x \nleq b\}$. The set S is non-empty since it contains a. Hence, since L satisfies (DCC), there exists a minimal element x of S. We claim that x is join-irreducible. Suppose that $x = c \vee d$, with $c < x$ and $d < x$. By the minimality of x, neither c nor d lies in S. We have $c < x \leq a$, so $c \leq a$, and, similarly, $d \leq a$. Therefore $c, d \notin S$ implies $c \leq b$ and $d \leq b$. But then $x = c \vee d \leq b$, a contradiction. Thus $x \in \mathcal{J}(L) \cap S$, proving (i).

DCC and Join-Irreducibles (Cont'd)

(ii) Let $a \in L$ and let $T := \{x \in \mathcal{J}(L) : x \leq a\}$. Clearly a is an upper bound of T. Let c be an upper bound of T. We claim that $a \leq c$. Suppose that $a \nleq c$; then $a \nleq a \wedge c$. By (i), there exists $x \in \mathcal{J}(L)$, with $x \leq a$ and $x \nleq a \wedge c$. Hence $x \in T$ and, consequently, $x \leq c$, since c is an upper bound of T. Thus x is a lower bound of $\{a, c\}$ and consequently $x \leq a \wedge c$, a contradiction. Hence $a \leq c$, as claimed. This proves that $a = \bigvee T$ in L, whence (ii) holds.

Chain Conditions and Join Density

- Part (iii) below is an analogue of (the existence portion of) the Fundamental Theorem of Arithmetic.

Theorem

Let L be a lattice.

(i) If L satisfies (DCC), then $\mathcal{J}(L)$ and, more generally, any subset Q which contains $\mathcal{J}(L)$ is join-dense in L.

(ii) If L satisfies (ACC) and Q is join-dense in L, then, for each $a \in L$, there exists a finite subset F of Q, such that $a = \bigvee F$.

(iii) If L has no infinite chains, then, for each $a \in L$, there exists a finite subset F of $\mathcal{J}(L)$, such that $a = \bigvee F$.

(iv) If L has no infinite chains, then Q is join-dense in L if and only if $\mathcal{J}(L) \subseteq Q$.

Chain Conditions and Join Density (Cont'd)

(i) This an immediate consequence of Part (ii) of the previous proposition.

(ii) This follows immediately from a previous result.

(iii) No infinite chains implies both (ACC) and (DCC), so (iii) is a consequence of (i) and (ii).

(iv) One direction follows from (i).

In the other direction, assume that Q is join-dense in L and let $x \in \mathcal{J}(L)$. By (ii), there is a finite subset F of Q such that $x = \bigvee F$. Since x is join-irreducible we have $x \in F$ and, hence, $x \in Q$. Thus, $\mathcal{J}(L) \subseteq Q$.