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55

Francesco Oliveri

Lie Symmetries of Differential Equations

Theory, Applications, and Symbolic Computation

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To Mimma, Maria Lucia and Elena Sofia. Although they likely will not read this book, they should know that, without doing Mathematics, I would be "a lost soul swimming in a fish bowl".

Preface

The main aim of this book is to give significant applications to concrete examples of interest in Applied Mathematics and Mathematical Physics of both classical Lie theory and some of its recent developments, including many original contributions contained in several scientific papers. At the same time, in order to have an almost self-contained monograph, I cannot exempt myself from giving a short account of the relevant theoretical arguments justifying the proposed methods. In fact, this is *not* a book where the reader can find (only) some useful recipes to deal with differential equations; rather, the considered applications are intended to show the extraordinary power and the algorithmic nature of the general methods of Lie group analysis of differential equations.

Moreover, two computer algebra packages (freely available), developed to manage quickly and efficiently the cumbersome computations involved in Lie group analysis of differential equations, are described in detail.

The idea of this book originated from the lecture notes prepared in 2009 on the occasion of an advanced course delivered at the Summer School of Mathematical Physics in Ravello (Italy). Those lecture notes along the years considerably grew and now contain some of the topics covered in a semester course on *Lie Symmetries of Differential Equations*, which is part of the higher programme in Mathematics at the University of Messina.

Chapters 1–7 review well known classical results; their content is suitable for an introductory course in Lie group analysis of differential equations. On the contrary, the remaining chapters mainly contain original results appeared in the last thirty years in papers published in peer–reviewed journals, and most of the topics could be of interest especially to postdocs and researchers.

Chapter 1 briefly introduces the evolution of the concept of symmetry and its modern use in Mathematical Physics (Galilean group and dimensional analysis). Chapters 2 and 3 review the main theoretical arguments about Lie groups of transformations and Lie algebras, whereas Chap. 4 focuses on the prolongation formulae for managing Lie groups of transformations leaving differential equations invariant.

Applications of Lie symmetries to ordinary differential equations are discussed in Chap. 5: it is shown the use of symmetries for lowering the order of ordinary differential equations (possibly reducing them to quadrature), determining integrating factors or first integrals. Also, second order ordinary differential equations admitting at least two symmetries spanning a two–dimensional Lie algebra are classified and solved. Chapter 6 deals with partial differential equations, discussing the determination of invariant solutions, and generation of new solutions starting from a given solution by using symmetries. Moreover, the problem of the classification of inequivalent invariant solutions using optimal systems of Lie subalgebras is analyzed.

Chapter 7 is concerned with differential equations admitting a variational formulation. The invariance of the Lagrangian action under a Lie group of transformations, and Noether theorem for constructing first integrals (in the context of classical mechanics), or conservation laws in the case of a general Lagrangian function of order $r \geq 1$ in n independent variables, are discussed.

Chapter 8 investigates the so called inverse problem of Lie groups analysis of differential equations. After reviewing the basics of the differential geometry of differential equations, the notions of *strongly Lie remarkable equations* and *weakly Lie remarkable equations* are introduced, and necessary and sufficient conditions are established. Various examples of differential equations—both partial and ordinary—uniquely determined by Lie algebras of symmetries are characterized.

Chapter 9 is devoted to the use of Lie symmetries admitted by partial differential equations for constructing invertible point mappings suitable to transform a given equation into an equivalent, and often more convenient, form. Various problems are addressed: the transformation to autonomous form of general partial differential equations, or to linear form when the equations are of first order, the transformation of nonhomogeneous and/or nonautonomous quasilinear first order systems of partial differential equations to homogeneous and autonomous form, the transformation of first order nonlinear differential equations polynomial in the derivatives into autonomous first order equations polynomially homogeneous in the derivatives. Several examples of physical interest are used to illustrate how the procedures work.

Chapter 10 deals with conditional symmetries and equivalence transformations. Conditional symmetries generalize the nonclassical method first introduced by Bluman and Cole in 1969. On the contrary, equivalence transformations, extensively studied by Ovsiannikov (but also considered by Lie himself), constitute a nice framework for investigating the invariance of classes of differential equations containing unspecified functions.

Chapter 11 presents an approach to approximate symmetries of differential equations containing small terms that is consistent with the basic principles of perturbative analysis. Moreover, it permits to preserve, and adapt to the approximate context, many relevant properties of exact Lie symmetries of differential equations.

Finally, Chap. 12 describes two computer programs. The first one, written in the Computer Algebra System REDUCE, is able to perform almost automatically much of the computations described in this book (in particular, it is able to determine point, contact, variational and conditional symmetries, both exact and approximate).

The second one, SymboLie, written in Wolfram Mathematica™, is devoted to find families of optimal Lie subalgebras of a finite–dimensional Lie algebra.

It is really a pleasure to warmly thank some of the friends and colleagues with whom I have significantly worked and/or discussed over the years: Luca Amata, Ljudmila Bordag, Giampaolo Cicogna, Carmela Currò, Rosa Di Salvo, Andrea Donato, Giuseppe Gaeta, Matteo Gorgone, Nail Ibragimov, Joseph Krasil'schchik, Lucia Margheriti, Giovanni Manno, Maxim Pavlov, Giuseppe Saccomandi, Mariano Torrisi, Rita Tracinà, Antonino Valenti, Raffaele Vitolo are just few of them!

I am particularly indebted to Matteo Gorgone who carefully read the material in this book; his comments and suggestions were precious for helping me to fix various typos, and improve the clearness and the global coherence of the notations.

The author is grateful to the unknown reviewers for their time and effort in reading a preliminary version of this book; their valuable comments and suggestions were useful to eliminate inaccuracies so improving the quality of the book. Special thanks go to Paolo Maria Mariano for his support and encouragement to publish this book.

This book wishes to honor the memory of Andrea Donato (1947–2000), who, besides having been my mentor and a good friend, initiated me in 1985 to the study of Lie symmetries of differential equations.

Messina, Italy — Francesco Oliveri
August 2025

Competing Interests The author has no conflicts of interest to declare that are relevant to the content of this book.

This book is entirely my own work.

Results arising from books or papers by other authors are always clearly attributed.

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Chapter 1
Symmetries



In the beginning was the symmetry.
Werner Heisenberg

Tell me why should symmetry be of importance?
Mao Zedong

1.1 What Is Meant by Symmetry?

Symmetries, and their application to differential equations, are the leading actor of this book. We start by recalling that the word *symmetry* comes from ancient Greek: σύν (together) and μέτρον (measure). Some widespread meanings of the word symmetry are illustrated in the following quotation extracted from the 1952 book "Symmetry" by Weyl [71]:

> The word symmetry is used in our everyday language in two meanings. In the one sense symmetric means something like well–proportioned, well–balanced, and symmetry denotes that sort of concordance of several parts by which they integrate into a whole. Beauty is bound up with symmetry. […]
> In this sense the idea is by no means restricted to spatial objects; the synonym "harmony" points more toward its acoustical and musical than its geometric applications. *Ebenmass* is a good German equivalent for the Greek symmetry; for like it carries also the connotation of "middle measure," the mean toward which the virtuous should strive in their actions according to Aristotle's *Nichomachean Ethics*, and which Galen in *De temperamentis* describes as the state of mind which is equally removed from both extremes: σύμμετρον ὅπερ ἐκατέρου τῶν ἄκρων ἀπέχει. The image of the balance provides a natural link to the second sense in which the word symmetry is used in modern times: *bilateral symmetry*, the symmetry of left and right, which is so conspicuous in the structure of the higher animals, especially the human body. Now this bilateral symmetry is a strictly

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Fig. 1.1 Évariste Galois (Bourg La Reine, October 25, 1811—Paris, May 31, 1832). *Source* https://commons.wikimedia.org/wiki/File:Evariste_galois.jpg

geometric and, in contrast to the vague notion of symmetry discussed before, an absolutely precise concept. A body, a spatial configuration, is symmetric with respect to a given plane E if it is carried into itself by reflection in E. Take any line l perpendicular to E and any point p on l: there exists one and only one point p' on l which has the same distance from E but lies on the other side. The point p' coincides with p only if p is on E.

In the nineteenth century, symmetry appears unexpectedly in the work of Évariste Galois (Fig. 1.1), who exploited the permutations of the roots of an algebraic equation to prove that there is no general formula for solving by radicals a polynomial equation of degree greater than four. The central idea of Galois theory is based on the permutations of the roots such that any algebraic equation satisfied by the roots is still satisfied after the roots have been permuted. Thus, it emerged the idea of a symmetry as a transformation, and the algebraic structure of a *group* of symmetries as a set of transformations, containing the identity transformation, the inverse transformation of each element, and such that the composition of two symmetries yields a symmetry too. Roughly speaking, a symmetry of a mathematical object is a transformation that does not transform the object. In some sense, symmetries behave like a skilled pickpocket: they are actions that leave no trace.

Symmetries are of course strictly related to geometrical objects, and they cannot be considered a geometrical accident, or a lucky and rare phenomenon. The link between symmetry and geometry assumed an essential and effective role after the famous lecture by Felix Klein (Fig. 1.2) in 1872 (now known as *Erlangen Program* [34]), where it was clearly stated that every kind of Geometry (Affine, Projective, Conformal,



Fig. 1.2 Felix Christian Klein (Düsseldorf, April 25, 1849—Göttingen June 22, 1925). *Source* https://commons.wikimedia.org/wiki/File:Felix_Christian_Klein.jpg

Differential, Euclidean or non-Euclidean, Topology, …) can be defined in terms of symmetries: *the geometrical properties are simply invariant properties with respect to a group of transformations*.

1.2 Symmetries in Mathematics and Physics

Symmetry (together with principle of parsimony, *Ockham razor*) through the history of science revealed often successful in expanding the sphere of our knowledge of the world.

The request of symmetry in the formulation of the *laws of nature* accounts for their features that are independent of some unessential circumstances. For instance, the reproducibility of experiments in different places at different times relies on the invariance of the laws of nature under space translation and rotation (homogeneity and isotropy of space, respectively: *there is neither a privileged position nor a privileged direction!*), and time translation (homogeneity of time: *there is no privileged instant!*). Without such regularities, our understanding of physical events would probably be very limited. An important implication of symmetry in Mathematics and Physics is the existence of conservation laws. This connection has been established in 1918, when Noether [36, 52] (Fig. 1.3) showed the intimate relation between continuous symmetries and conservation laws.



Fig. 1.3 Emmy Noether (Erlangen, March 23, 1882—Bryn Mawr, April 14, 1935). *Source* https://commons.wikimedia.org/wiki/File:EmmyNoether.png

The concept of symmetry fascinated through the centuries many artists and scientists, from the ancient Greeks to Kepler, in his efforts for determining the orbits of the planets, to Newton, who embodied in the laws of mechanics as a symmetry principle what is now known as Galilean invariance, i.e., the equivalence of motion in different inertial frames.

In the nineteenth century, a great advance arose when Marius Sophus Lie (Fig. 1.4) began to investigate the continuous groups of transformations leaving differential equations invariant, creating what is now called the *symmetry analysis of differential equations*. The original Lie's aim was that of setting a general theory for the integration of ordinary differential equations similar to that developed by Évariste Galois and Niels Abel (see [73] for a good historical account) for algebraic equations. More information about Sophus Lie's life and the mathematical relevance of his contributions can be found in the book by Stubhaug [63] and in some papers (see, for instance, Fritzsche [22]).

Thus, symmetry analysis of differential equations was developed and applied by Sophus Lie during the period 1872–1899 [37–39]. This theory allows one to derive solutions of differential equations in a completely algorithmic way. Despite this feature, the Lie approach to differential equations was not exploited for half a century, and only the abstract theory of Lie groups grew. As far as we know, the

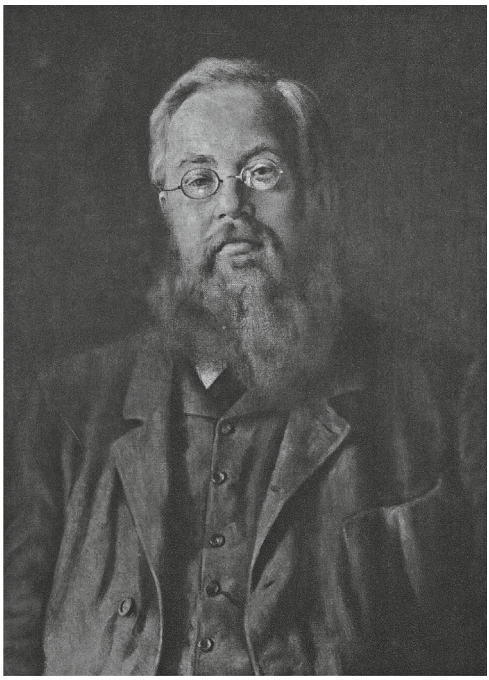


Fig. 1.4 Marius Sophus Lie (Nordfjordeid, December 17, 1842—Christiania, February 18, 1899). Lithograph after painting by Erik Werenskiold. Courtesy of Municipal Archives of Trondheim, Norway. *Source* https://commons.wikimedia.org/wiki/File:Sophus_Lie_(1842_-_1899).jpg

term Lie group (groupe de M. Lie) appears for the first time in a paper by Arthur Tresse [67] in 1893. Moreover, the term Lie algebra was coined by Hermann Weyl in 1928; before Weyl, Lie algebras were referred to as infinitesimal groups. It was in 1940, with the work of Birkhoff [4] and Sedov [61] on dimensional analysis, that the theory gave relevant results in concrete applied problems. Further, it was the Russian school, under the guidance mainly of Ovsiannikov [55, 56], that since 1960 began to use systematically the methods of symmetry analysis of differential equations for the explicit construction of solutions of any sort of problems, even complicated, of Mathematical Physics. The English translations by Bluman [55] and Ames [56] of the books by Ovsiannikov gave a great diffusion of Lie methods, and, as a consequence, many significant results have been achieved by a high number of researchers either from a theoretical or an applied point of view.

Lie theory is powerful, versatile, and fundamental for the development of systematic procedures leading to the integration by quadrature (or, at least, to lowering the order) of ordinary differential equations, to the determination of invariant solutions of initial and boundary value problems, to the transformation of differential equations in equivalent forms more easy to handle (e.g., autonomous or linear), to the determination of conservation laws, to the implementation of numerical schemes, …

Sophus Lie pointed out the main problem he addressed with his theory in a foreword to his lecture *Differentialgleichungen* given in Leipzig (1891):

> The older examinations on ordinary differential equations as found in standard books are not systematic. The writers developed special integration theories for homogeneous differential equations, for linear differential equations, and other special integrable forms of differential equations. However, the mathematicians did not realize that these special theories are all contained in the term infinitesimal transformations, which is closely connected with the term of a one parametric group.

The application of Lie theory to differential equations is completely algorithmic but usually involves a lot of cumbersome and tedious calculations. For instance, it is not uncommon to have to handle hundreds of differential and/or algebraic equations for obtaining the symmetries admitted by a system of differential equations. Today, nobody can disregard the facilities offered by Computer Algebra Systems, both commercial (like Maple™ [41], or Wolfram Mathematica™ [72]) and open source (like Maxima,[1] REDUCE [30], or SageMath [66]); use of specifically designed packages greatly helps the analysis of algebraic/differential manipulations that can be often done almost automatically without requiring much user work [31–33, 53, 59, 60, 62]. In this book, the program RELIE [53] (written in REDUCE) and the program SYMBOLIE (written in Wolfram Mathematica™) will be illustrated.

Classical Lie theory is a source for various generalizations. Among these generalizations we may quote the potential symmetries [5, 7], the nonclassical symmetries first introduced by Bluman and Cole [6], and now part of the more general method of conditional symmetries, the equivalence transformations for classes of differential equations [56], the approximate symmetries [2, 19, 23] for differential equations involving small parameters, the generalized symmetries [54], which in turn extend contact symmetries introduced by Lie himself, λ-symmetries [47–51], μ-symmetries [11, 24], …

The key idea of Lie theory of symmetry analysis of differential equations relies on the invariance of the equations under a transformation of independent and dependent variables. These transformations form a local group of point transformations establishing a diffeomorphism on the space of independent and dependent variables, mapping solutions of the equations to other solutions of the same equations. Any transformation of the independent and dependent variables in turn induces a transformation of the derivatives. Typical examples of symmetries are translations, rotations, and dilations. These transformations do not exhaust the possible symmetries useful to analyze differential equations. An important role is often played by discrete symmetries such as reflections. They are symmetries under the transformations of a discrete group, i.e., a topological group with a discrete topology whose elements form a finite or a countable set.

In Lie theory of differential equations, we may consider a *direct* problem and an *inverse* one. The direct problem consists in finding the symmetries admitted by a differential equation (or by a class of differential equations). On the contrary, the inverse problem aims to find the most general form of a differential equation (or a class of differential equations) admitting a given set of Lie symmetries.

[1] https://maxima.sourceforge.io.

The main aim of this book is to give significant applications to concrete examples of interest in Applied Mathematics and Mathematical Physics of both classical Lie theory and some of its recent developments, including many original contributions. At the same time, according to the maxim "*The best practice is a good theory*" (attributed to Ernst Mach), we present the relevant theoretical arguments justifying the proposed methods. This reflects on the structure of this book. In order to have an almost self-contained monograph, we cover in Chaps. 2–7 many classical arguments of modern group analysis; on the contrary, Chaps. 8–12 mainly contain original results.

We close with a quotation from the Preface of a book [57] published in Italian by Ernesto Pascal in 1903 (a few years later than Lie's death), where the fundamental role of Lie theory in the analysis of differential equations is clearly stressed:

> Questa teoria, pur di data così recente, si è andata meravigliosamente e rapidamente sviluppando e ha preso presto un cotal posto fra le sue consorelle, che oggidì un qualunque giovine che percorra gli studii delle matematiche superiori, non può più non conoscerla, almeno nei suoi fondamenti e principii.[2]

1.3 Symmetries in Continuum Mechanics and Thermodynamics

Continuous media are described (in local form) by balance laws of mass, momentum, angular momentum, energy, and entropy [29]:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,\\
&\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} - \mathsf{T}) = \rho \mathbf{b}, \qquad \mathsf{T} = \mathsf{T}^T,\\
&\frac{\partial (\rho(\varepsilon + v^2/2))}{\partial t} + \nabla \cdot \big((\rho\varepsilon + \rho v^2/2)\mathbf{v} - \mathsf{T} \cdot \mathbf{v} + \mathbf{q}\big) = h + \rho \mathbf{b} \cdot \mathbf{v},\\
&\frac{\partial (\rho s)}{\partial t} + \nabla \cdot (\rho s \mathbf{v} + \mathbf{J}_s) = \sigma_s,
\end{aligned}
$$

where ρ is the mass density, $\mathbf{v}$ is the velocity, ε is the specific internal energy, T is the Cauchy stress tensor, $\mathbf{q}$ is the heat flux, s is the specific entropy, $\mathbf{J}_s$ is the entropy flux, $\mathbf{b}$ is the specific body forces, h is the internal energy production, and σ_s is the entropy production.

The balance laws contain more unknowns than equations. So, we need to supplement them by the constitutive equations to describe the behavior of the material at hand. Larger sets of balance laws (e.g., Grad's moments of Boltzmann

[2] This theory, although so recent, has been beautifully and quickly developing and took such an important place that nowadays any young pupil who is addressing higher mathematical studies can no longer not know it, at least in its foundations and principles.

equation [28], or Extended Thermodynamics [46], continua with microstructures [10], continua with internal variables [42], …) can be also considered.

Although there are many degrees of freedom, three fundamental constraints do exist:

- Determinism: Any material behavior at the present depends only on its past history: The initial state of a mechanical system (the totality of positions and velocities of its points at some moment of time) uniquely determines all of its motion.
- Material symmetry: The structural symmetries of a material must reflect on constitutive equations [29, 45].
- Entropy principle: In every admissible process, the entropy production σ_s cannot be negative [68].

A series of experimental facts (at least approximately) is at the basis of classical mechanics [1]:

- the space is three-dimensional and Euclidean, and time is one-dimensional;
- Galilei principle of relativity: there exist coordinate systems (called inertial) possessing the following two properties:

  1. all the laws of nature at all instants of time are the same in all inertial coordinate systems;
  2. all coordinate systems in uniform rectilinear motion with respect to an inertial one are themselves inertial.

Therefore, the universe is modeled as a four-dimensional affine space A^4 (the set of *world points* or events). The parallel displacements of the universe A^4 give the vector space $\mathbb{R}^4$. There is a linear mapping $t: A^4 \times A^4 \to \mathbb{R}$. The time interval between the events $\mathbf{a}, \mathbf{b} \in A^4$ is the number $t(\mathbf{b} - \mathbf{a})$.

If $t(\mathbf{b} - \mathbf{a}) = 0$, then the two events are called simultaneous. The distance $d: A^4 \times A^4 \to \mathbb{R}^+$ between *simultaneous events*

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\| = \sqrt{(\mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b})}, \qquad \mathbf{a}, \mathbf{b} \in A^4$$

is given by a scalar product on the space $\mathbb{R}^4$. This distance makes every space of simultaneous events into a three-dimensional Euclidean space E^3.

A space A^4, equipped with a Galilean space-time structure, is called a *Galilean space*.

The Galilean group consists of all transformations preserving the structure of Galilean space (intervals of time and distances between simultaneous events). There are three relevant subgroups:

- Uniform motion of frames with velocity $\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)$ (Galilei's relativity principle):

$$(t, x_1, x_2, x_3) \mapsto (t, x_1 + \alpha_1 t, x_2 + \alpha_2 t, x_3 + \alpha_3 t),$$

  whereupon $v_k \mapsto v_k + \alpha_k$, $k = 1, 2, 3$.

- Translation of time and space coordinates:

$$(t, x_1, x_2, x_3) \mapsto (t + \alpha_4, x_1 + \alpha_5, x_2 + \alpha_6, x_3 + \alpha_7).$$

- Rotation of the coordinate axes:

$$(t, \mathbf{x}) \mapsto (t, \mathsf{Q}\mathbf{x}),$$

where Q is an arbitrary orthogonal matrix (depending on three parameters).

Every Galilean transformation of the space $\mathbb{R} \times \mathbb{R}^3$ can be written in a unique way as the composition of a rotation, a translation, and a uniform motion. This group involves ten arbitrary parameters. Additional transformations, corresponding to the invariance of physical laws under an arbitrary change of the fundamental units of measurable quantities, fall in the domain of dimensional analysis.

As far as the constitutive equations are concerned, they are often derived from a macroscopic point of view in accordance with experiments; anyway, transformation groups are involved. A constitutive equation (for stress tensor, heat flux, entropy flux, …) is just a mathematical relation between forces and motion, in the sense that if forces are applied to a body of a solid, liquid, or gaseous material they will cause it to undergo a motion in which the motion itself will differ according to the nature of the body. In classical continuum mechanics, the forces of interest include pure mechanical stresses, as well as thermal or electromagnetic effects. The general principles commonly accepted for assigning constitutive relations have a long history (see the review [21]). Hooke in 1678 observed that a spring scale at the bottom of a deep mine or at the top of a mountain gives the same response to a given force. Later, in 1829, Poisson and Cauchy independently realized that a spring scale may serve for measuring the change of gravity acceleration and also the centrifugal force of a rotating system. By attaching a spring with a terminal mass to the center of a horizontal, uniformly rotating table, the force exerted by the spring in response to a given elongation is independent of the observer. So, Poisson and Cauchy were forced to impose the theoretical requirement that only frame-independent terms should enter a constitutive equation of a material, laying the bases for what is now called *frame-indifference principle*.

In the formulation of the frame-indifference principle by Truesdell and Noll [69], constitutive equations must be *objective* with respect to a change of observer; the latter is defined by the transformations (forming a group):

$$t^\star = t + t_0, \qquad \mathbf{x}^\star = \mathsf{Q}(t)\mathbf{x} + \mathbf{c}(t), \tag{1.1}$$

where t_0 is constant, whereas $\mathbf{c}(t)$ is an arbitrary vector, and $\mathsf{Q}(t)$ an arbitrary orthogonal matrix, both time dependent. Scalar functions f are objective if $f^\star = f$, vector functions $\mathbf{u}$ are objective if $\mathbf{u}^\star = \mathsf{Q}\mathbf{u}$, tensorial functions T are objective if $\mathsf{T}^\star = \mathsf{Q}\mathsf{T}\mathsf{Q}^T$, …

Finally, entropy principle imposes additional constraints to the constitutive equations. In fact, the constitutive functions characterizing the properties of a continuous material must be assigned in such a way the second law of thermodynamics is satisfied by arbitrary thermodynamic processes. Thus, second law is regarded as a restriction on the constitutive equations and not on the thermodynamic processes. As a consequence of such a statement, it is not necessary to verify a posteriori if a solution of the balance equations satisfies second law of thermodynamics, because such a requirement is ensured a priori by the form of the constitutive equations. There are two classical methods for the exploitation of entropy principle: Coleman–Noll procedure [15–17] and Liu procedure [40]. In both methods, the constraints imposed on the entropy inequality by the balance laws (eliminating time derivatives in the Coleman–Noll approach, or introducing Lagrange multipliers in the Liu approach) are used. This means that the entropy inequality is evaluated on the balance equations. Nevertheless, this is not always sufficient in the presence of nonlocal constitutive equations, i.e., constitutive equations depending on some derivatives of the unknown fields.

For this reason, an extension of these two classical techniques has been recently proposed (see [12–14, 25], and references therein). They require to use as constraints in the entropy inequality both the balance equations and some of their differential consequences. The extended procedures revealed able to dismantle some physical properties which are not obtained by applying the classical methods. A notable example is given in [27], where the extended Liu procedure has been successfully applied for explicitly characterizing the constitutive equations of a viscous Korteweg fluid [35], i.e., a fluid where the stress tensor depends on first- and second-order gradients of mass density (see also [26] concerned with mixtures of viscous Korteweg fluids).

1.4 Dimensional Analysis

Important Lie groups (scaling groups) are commonly used also by physicists and engineers who have no idea of the theory but know the importance of describing physical relations in dimensionless form. The circumstance that physical laws need to have the same form whatever the scales used to measure lengths, masses, time intervals, …has been clearly stated in 1873 by Maxwell [43]:

> *The formula at which we arrive must be such that a person of any nation, by substituting for the different symbols the numerical value of the quantities as measured by his own national units, would arrive at a true result.*

From a mathematical point of view, this means that a relation involving some physical quantities must be independent of the scales we adopt for the fundamental quantities [4]. The implications of this requirement are mathematically stated by the Buckingham π-theorem. This was first proved by the French mathematician J. Bertrand in 1878 in considering problems arising in electrodynamics and heat conduction phenomena [3]. The technique of using the "method of dimensions"

Table 1.1 Base units of International System (SI)

Symbol	Name	Quantity
s	Second	Time
m	Meter	Length
kg	Kilogram	Mass
A	Ampere	Electric current
K	Kelvin	Thermodynamic temperature
mol	Mole	Amount of substance
cd	Candela	Luminous intensity

became widely known after the works of Lord Rayleigh. A general formulation of the theorem for the case of arbitrarily many quantities was given first by Vaschy [70] in 1892, then in 1911 by Federman [20] and Riabouchinsky [58], and again in 1914 by Buckingham [8] (see also [9]). The notation used by Buckingham for the dimensionless variables (π_i) is at the origin of the name given to the theorem.

The modern form of the metric system, coordinated by the *Bureau International des Poids et Mesures*,[3] is the International System of Units (SI). In the SI, there are 7 base units (see Table 1.1) and 22 coherent derived units, represented as products of powers of the base units, and possessing special names and symbols.

Consider a physical relation expressing a measurable quantity u in terms of n measurable quantities $(u_1, u_2, \ldots, u_n)$,

$$u = F(u_1, u_2, \ldots, u_n), \tag{1.2}$$

where the function F is not specified. Suppose that the quantities $(u_1, \ldots, u_n, u)$ are measured in terms of m fundamental dimensions $(L_1, \ldots, L_m)$.

Each quantity has a dimension expressed as a monomial of fundamental dimensions; the dimensions of the quantities u and u_j are expressed as

$$[u] = L_1^{a_1} \cdots L_m^{a_m}, \qquad [u_j] = L_1^{a_{1,j}} \cdots L_m^{a_{m,j}}, \qquad j = 1, \ldots, n,$$

where a_j and $a_{j,k}$ are real numbers. A dimensionless quantity u is such that $[u] = 1$. Equivalently, we can associate to the measurable quantities u and u_j their dimension column vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}, \qquad \mathbf{a}_j = \begin{pmatrix} a_{1,j} \\ \vdots \\ a_{m,j} \end{pmatrix}.$$

[3] International Bureau of Weights and Measures.

A dimensionless quantity is associated to the null m-dimensional vector. Define the $m \times n$ dimension matrix of the quantities $u_1, \ldots, u_n$

$$\mathsf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix},$$

and let $r = \mathrm{rank}(\mathsf{A})$.

Given a set of fundamental dimensions, a measurable quantity has a value that depends on the chosen system of units. A change in the system of units corresponds to a positive rescaling of the fundamental dimensions; this in turn induces a positive rescaling of each measurable quantity. Thus, relation (1.2) with quantities measured with the system of units $L_1, \ldots, L_m$ must hold also when we adopt a new system of units, i.e.:

$$u' = F(u'_1, \ldots, u'_n),$$

where $u', u'_1, \ldots, u'_n$ are measured in the system of units $L'_1, \ldots, L'_m$. With such an assumption, we can prove the Buckingham π-theorem.

Theorem 1.1 (Buckingham π-theorem) *The physical relation*

$$u = F(u_1, \ldots, u_n)$$

involving m fundamental dimensions $L_1, \ldots, L_m$ can be written in the dimensionless form:

$$\pi = G(\pi_1, \ldots, \pi_r), \tag{1.3}$$

where r is the rank of the $m \times n$ dimension matrix A of the quantities $u_1, \ldots, u_n$; moreover,

$$\pi = u \prod_{j=1}^{n} u_j^{y_j}, \qquad \pi_i = \prod_{j=1}^{n} u_j^{x_{j,i}}, \quad i = 1, \ldots, r, \tag{1.4}$$

with suitable real values y_j and $x_{j,i}$.

Proof Let

$$[u] = L_1^{a_1} \cdots L_m^{a_m}, \qquad [u_j] = L_1^{a_{1,j}} \cdots L_m^{a_{m,j}}, \qquad j = 1, \ldots, n,$$

and rescale the unit of the fundamental dimension L_m, say

$$L'_m = \lambda L_m,$$

λ being a positive real number. Thence,

$$u' = \lambda^{a_m} u, \qquad u'_j = \lambda^{a_{m,j}} u_j, \quad j = 1, \ldots, n,$$

and

$$\lambda^{a_m} u = F(\lambda^{a_{m,1}} u_1, \ldots, \lambda^{a_{m,n}} u_n).$$

The case $a_{m,1} = \cdots = a_{m,n} = 0$ and $a_m \neq 0$ must be excluded as it implies $u \equiv 0$. The case $a_{m,1} = \cdots = a_{m,n} = a_m = 0$ is trivial as it means that the dimension L_m is not really involved. Therefore, without loss of generality, we can assume, possibly by a permutation of the quantities $u_1, \ldots, u_n$, $a_{m,n} \neq 0$. Introducing the quantities

$$\begin{aligned} v &= u u_n^{-a_m/a_{m,n}}, \\ v_j &= u_j u_n^{-a_{m,j}/a_{m,n}}, \qquad j = 1, \ldots, n-1, \\ v_n &= u_n, \end{aligned}$$

it is

$$v' = v, \qquad v'_i = v_i \ (i = 1, \ldots, n-1), \qquad v'_n = \lambda^{a_{m,n}} v_n,$$

whereupon

$$v = \widehat{F}(v_1, \ldots, v_{n-1}, \lambda^{a_{m,n}} v_n)$$

for all λ, implying that $\widehat{F}$ cannot depend on v_n.

Thus, we are led to a relation, expressing a measurable quantity in terms of $n-1$ quantities, involving $m-1$-dimensional units. Repeating the procedure for the remaining dimensions, we arrive to the dimensionless relation (1.3) along with (1.4).

Moreover, from

$$\begin{aligned} 1 = [\pi] &= L_1^{a_1 + \sum_{j=1}^n a_{1,j} y_j} \cdots L_m^{a_m + \sum_{j=1}^n a_{m,j} y_j}, \\ 1 = [\pi_i] &= L_1^{\sum_{j=1}^n a_{1,j} x_{j,i}} \cdots L_m^{\sum_{j=1}^n a_{m,j} x_{j,i}}, \qquad i = 1, \ldots, r, \end{aligned}$$

it follows that $\mathbf{y} = (y_1, \ldots, y_n)^T$ (^{T} denotes the transposition) is a solution of the linear system $\mathsf{A}\mathbf{y} = -\mathbf{a}$, and that $\mathbf{x}_i = (x_{1,i}, \ldots, x_{n,i})^T$, $i = 1, \ldots, r$, are r linearly independent solutions of the linear system $\mathsf{A}\mathbf{x} = \mathbf{0}$. □

1.4.1 Application of Dimensional Analysis to First Atomic Explosion

One of the most impressive use of the Buckingham π-theorem is concerned with the computation of the energy relaxed in the first atomic explosion, the Trinity test, conducted by the United States Army on July 16, 1945 in a desert area of New Mexico (named by Spanish conquistadors *Jornada del Muerto*[4]), as part of the Manhattan Project, coordinated by J. Robert Oppenheimer. The test consisted in an implosion of a plutonium bomb similar to the *Fat Man* bomb later detonated over Nagasaki, Japan, on August 9, 1945.

Some years later, a series of pictures of the explosion, along with a size scale, and time stamps, were released and published in a popular magazine. Based on these photographs, the British physicist Geoffrey Ingram Taylor in 1950 [64, 65] estimated the power released by the explosion (which was still a secret at that time). In fact, using the radius R of the *ball of fire* which spread out from the point where explosion occurred at different times, dimensional analysis yielded an estimate of the energy E generated in the explosion. Therefore, the energy relaxed was computed from purely kinematical data.

The pictures (in Fig. 1.5, a photograph taken 25 ms after the explosion is displayed) showed that the ball of fire assumes at first the form of a rough sphere that reached the ground in <1 ms. After the impact, the shock wave produced continued to expand as a nearly perfect luminous hemisphere.

First, we have to identify the relevant variables in our problem: the radius R of the fireball, the elapsed time t since the explosion, the energy E relaxed, the initial density ρ and the initial pressure p of the air. Of course, there can be lots of additional variables, but as long as they do not have a great relevance to the problem, we can neglect them.

Assume the relation

$$R = F(t, E, \rho, p), \tag{1.5}$$

and use the dimensions L_1 (for length), L_2 (for mass), and L_3 (for time). The dimension matrix A of (t, E, ρ, p) and the dimension vector $\mathbf{a}$ of R are

$$\mathsf{A} = \begin{pmatrix} 0 & 2 & -3 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & -2 \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The rank of the matrix A is 3. So, there is one linearly independent solution of

$$\mathsf{A}\mathbf{x} = \mathbf{0},$$

[4] Route of the dead man.

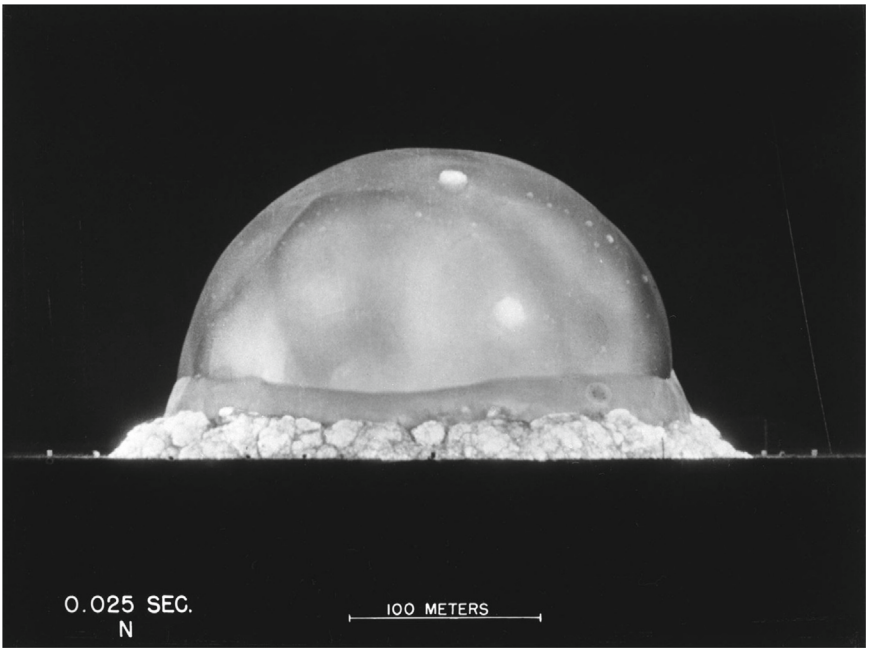


Fig. 1.5 Photograph of the ball of fire (25 ms after the explosion) produced by the Trinity test. Courtesy of US Government Defense Threat Reduction Agency. *Source* https://commons.wikimedia.org/wiki/File:Trinity_Test_Fireball_25ms.jpg

say

$$\mathbf{x} = \left(\frac{6}{5}, -\frac{2}{5}, -\frac{3}{5}, 1\right)^T .$$

As a solution of

$$\mathsf{A}\mathbf{y} = -\mathbf{a}$$

we choose

$$\mathbf{y} = \left(-\frac{2}{5}, -\frac{1}{5}, \frac{1}{5}, 0\right)^T .$$

Thus, introducing the dimensionless variables

$$\pi = R\left(\frac{Et^2}{\rho}\right)^{-1/5}, \qquad \pi_1 = p\left(\frac{t^6}{E^2\rho^3}\right)^{1/5},$$

relation (1.5) becomes

$$R = \left(\frac{Et^2}{\rho}\right)^{1/5} G\left(\pi_1\right).$$

Assuming the function G to be continuous at $\pi_1 = 0$, whence $G(\pi_1) \approx G(0) \neq 0$ for $\pi_1 \ll 1$ (this is reasonable due to the time duration of the phenomenon), we obtain

$$R = \left(\frac{Et^2}{\rho}\right)^{1/5} G(0).$$

A plot of $\log(R)$ versus $\log(t)$ on the data deduced from the photographs of Trinity test allowed Taylor to take $G(0) \approx 1$. At $t = 0.006\,\mathrm{s}$ the radius of the shock wave was approximately 80 m; furthermore, taking as the density of the air $\rho = 1.2\,\mathrm{kg\,m^{-3}}$, the estimate of the energy relaxed by the explosion

$$E = 1.092 \cdot 10^{14}\ \text{joule},$$

corresponding to about 25 kt. Notice that the actual value of the energy relaxed was still classified in 1950. Different values of the relaxed energy can be found in the literature (see, for instance, the 2011 paper by Deakin [18], or the recent paper by Mone and Seshadri [44] who revisited Taylor's analysis of the Trinity test), but, undoubtedly, the result achieved by Taylor is impressive!

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Chapter 2
Lie Groups of Transformations



2.1 Introduction

In view of its application to differential equations, Lie group theory—with special emphasis on Lie groups of transformations—is briefly reviewed. The interested reader may refer to a wide set of available monographs [1–23] for a more detailed exposition and for learning more about the (many) topics not covered here.

Lie groups are special groups that, in addition to group properties, have the structure of a differentiable manifold. In fact, a Lie group G is a C^∞ manifold endowed with a group structure in which the group operation and the inversion of an element are analytic functions. This means that the group elements can be continuously varied; for any two points a and b in the manifold, there is a multiplication $\omega(a, b)$,[1] and this group operation is consistent with the continuous structure of the manifold. This implies that if two more points a' and b' (close to a and b, respectively) are considered, then the product $\omega(a', b')$ is close to the product $\omega(a, b)$.

Despite appearances, the coexistence of the manifold structure with the group properties is a severe restriction, and very few structures (manifold + group) are Lie groups.

Groups arise as an algebraic abstraction of the notion of symmetry, whereas manifolds generalize the familiar concepts of curves and surfaces in the three-dimensional space. Roughly speaking, a manifold is a space that locally looks like a Euclidean space, but whose global character might be quite different. The union of such two different mathematical concepts combines, and significantly extends, both the algebraic methods of group theory and the multi-variable calculus.

Remarkably, the resulting theory can be applied to a wide range of physical and mathematical problems.

[1] Usually, the composition of two elements of a group is denoted as $a \cdot b$; the reason of the notation $\omega(a, b)$ will soon become clear.

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2.2 Groups, Lie Groups

Definition 2.1 (*Group*) A *group* G is a nonempty set of elements endowed with a law of composition (binary operation)

$$\omega\colon G \times G \to G$$

satisfying the axioms

1. the associative property holds:

$$\forall a, b, c \in G,\ \omega(a, \omega(b, c)) = \omega(\omega(a, b), c);$$

2. there exists the identity element:

$$\exists\, e \in G \text{ such that } \forall a \in G,\ \omega(e, a) = a = \omega(a, e);$$

3. every element has an inverse:

$$\forall a \in G,\ \exists a^{-1} \in G \text{ such that } \omega(a^{-1}, a) = e = \omega(a, a^{-1}).$$

Remark 2.1 It is a simple matter to prove that the identity element e and the inverse a^{-1} of the element $a \in G$ are unique.

Definition 2.2 (*Abelian group*) A group G is *Abelian* if

$$\forall\, a, b \in G,\ \omega(a, b) = \omega(b, a),$$

i.e., the group operation is commutative.

Definition 2.3 (*Subgroup*) A *subgroup* H of G is a subset of G which is closed with respect to the law of composition of G and the inversion.

Definition 2.4 (*Lie group*) A Lie group G is a group that carries the structure of an n-dimensional differentiable manifold in such a way that both the group operation and the operation of inversion of an element are smooth maps between manifolds.

A Lie group is a special instance of a topological group.

Definition 2.5 (*Topological group*) A set G endowed with a group operation and a topology is said to be a topological group if the mappings which define its group structure are continuous, i.e., if the group composition and the inversion of an element are continuous.

Example 2.1 The additive group $\mathbb{R}$ is a Lie group because $\mathbb{R}$ is a differentiable manifold and the sum (group operation) is analytic.

Example 2.2 The additive group $\mathbb{Z}$ is *not* a Lie group because its elements cannot be continuously varied. Nevertheless, $\mathbb{Z}$ can be endowed with the discrete topology, where every subset is considered both open and closed. So, any open neighborhood of an element $x \in \mathbb{Z}$ will contain only x. $\mathbb{Z}$ is a manifold where each point has an open neighborhood homeomorphic to $\mathbb{R}^0$ (a single point), and the group operation is smooth and compatible with this zero-dimensional structure.

Example 2.3 The general linear group $\mathrm{GL}(m;\mathbb{R})$ defined by

$$\mathrm{GL}(m;\mathbb{R}) = \{\mathsf{A}\colon\ \det(\mathsf{A}) \neq 0\},$$

where A is a real $m \times m$ matrix, is a Lie group. The manifold structure can be identified with the open subset of the linear space of all $m \times m$ nonsingular matrices. The group composition (ordinary matrix multiplication) and the inversion of a nonsingular matrix are analytic functions.

Definition 2.6 (*Lie subgroup*) A subset H of a Lie group G is a Lie subgroup if H is a subgroup of the abstract group G and a submanifold of the analytic manifold G.

Example 2.4 The special linear group $\mathrm{SL}(m;\mathbb{R})$ defined by

$$\mathrm{SL}(m;\mathbb{R}) = \{\mathsf{A} \in \mathrm{GL}(m;\mathbb{R})\colon\ \det(\mathsf{A}) = 1\}$$

is a Lie subgroup of the Lie group $\mathrm{GL}(m;\mathbb{R})$.

Example 2.5 The orthogonal group $\mathrm{O}(m;\mathbb{R})$ defined by

$$\mathrm{O}(m;\mathbb{R}) = \{\mathsf{A} \in \mathrm{GL}(m;\mathbb{R})\colon\ \mathsf{A}^T\mathsf{A} = \mathsf{I}_m\}$$

(the superscript ^{T} denotes the transposition operator, and I_m is the $m \times m$ identity matrix) is a Lie subgroup of the Lie group $\mathrm{GL}(m;\mathbb{R})$.

Example 2.6 The special orthogonal group $\mathrm{SO}(m;\mathbb{R})$ defined by

$$\mathrm{SO}(m;\mathbb{R}) = \{\mathsf{A} \in \mathrm{GL}(m;\mathbb{R})\colon\ \mathsf{A}^T\mathsf{A} = \mathsf{I}_m,\ \det(\mathsf{A}) = 1\}$$

is a Lie subgroup of the Lie group $\mathrm{GL}(m;\mathbb{R})$ (and also a Lie subgroup of the orthogonal group $\mathrm{O}(m;\mathbb{R})$).

Example 2.7 (*Symplectic matrix*) Let $m = 2n$; a matrix A of order m such that

$$\mathsf{A}^T\mathsf{J}\mathsf{A} = \mathsf{J},$$

where

$$\mathsf{J} = \begin{pmatrix} \mathsf{0}_n & -\mathsf{I}_n \\ \mathsf{I}_n & \mathsf{0}_n \end{pmatrix},$$

$\mathsf{0}_n$ and I_n being the null and identity matrix of order n, respectively, is called symplectic. It is straightforward to verify that

1. J and I_{2n} are symplectic;
2. the product of two symplectic matrices is symplectic;
3. symplectic matrices are nonsingular;
4. the transpose of a symplectic matrix is symplectic;
5. the inverse of a symplectic matrix is symplectic, and has the form

$$\mathsf{A}^{-1} = -\mathsf{J}\mathsf{A}^T\mathsf{J}.$$

Thus, the set of symplectic matrices of order $2n$ is a multiplicative Lie group, denoted as $\mathrm{SP}(n;\mathbb{R})$. It is a Lie subgroup of $\mathrm{GL}(m;\mathbb{R})$, and, because of the constraints of the definition, is a manifold with dimension $(2n^2+n)$. For $n=1$, it is $\mathrm{SP}(1;\mathbb{R}) = \mathrm{SL}(2;\mathbb{R})$.

Definition 2.7 (*Compact Lie group*) A Lie group G is compact if its manifold is a closed and bounded submanifold of $\mathbb{R}^n$, for some n.

Definition 2.8 (*Connected Lie group*) A Lie group G is connected if its manifold is connected.

Remark 2.2 A topological space is connected if it is not the disjoint union of two open sets. Since a manifold locally looks like a Euclidean space, any connected manifold is *pathwise connected*, i.e., for any couple of points there is a smooth curve joining them. For instance, $\mathrm{GL}(m;\mathbb{R})$ (and $\mathrm{O}(m;\mathbb{R})$ too) is not connected, as it is the disjoint union of the two connected Lie groups

$$\mathrm{GL}^+(m;\mathbb{R}) = \{\mathsf{A}\colon\ \det(\mathsf{A}) > 0\},$$
$$\mathrm{GL}^-(m;\mathbb{R}) = \{\mathsf{A}\colon\ \det(\mathsf{A}) < 0\}.$$

In the remainder of this book, we mainly consider connected Lie groups.

In Mathematical Physics, groups typically occur as transformations. Therefore, let us introduce Lie groups of transformations, starting with one-parameter ones.

2.3 One-Parameter Lie Groups of Transformations

Definition 2.9 (*Groups of transformations*) Let $M \subseteq \mathbb{R}^n$ be a manifold, and $S \subseteq \mathbb{R}$. The set of transformations

$$\mathbf{Z}\colon M \times S \to M,$$
$$\mathbf{z} \mapsto \mathbf{z}' = \mathbf{Z}(\mathbf{z};a),$$

depending on the real parameter a, constitutes a *real one-parameter group of transformations* on M if:

1. for each value of the parameter $a \in S$, the transformations are one-to-one onto M;
2. S, with the law of composition $\omega\colon S \times S \to S$, is a group with identity e;
3. $\mathbf{z}' = \mathbf{z}$ when $a = e$, i.e., $\mathbf{Z}(\mathbf{z}; e) = \mathbf{z}$;
4. if $\mathbf{z}' = \mathbf{Z}(\mathbf{z}; a)$, $\mathbf{z}'' = \mathbf{Z}(\mathbf{z}'; b)$, then $\mathbf{z}'' = \mathbf{Z}(\mathbf{z}; \omega(a, b))$, $\forall\, a, b \in S$.

Remark 2.3 $\mathbf{Z}(\mathbf{z}; a)$ is the *group action* on $\mathbf{z}$.

Definition 2.10 (*Lie group of transformations*) A group of transformations defines a *real one-parameter Lie group of transformations* if, besides satisfying the axioms of Definition 2.9, possesses the following properties:

1. a is a continuous parameter, i.e., S is an open interval in $\mathbb{R}$ (possibly the whole set of real numbers);
2. $\mathbf{Z}$ is C^∞ with respect to $\mathbf{z}$ in M, and an analytic function of a in S;
3. $\forall\, a, b \in S$, $\omega(a, b)$ and a^{-1} are analytic functions.

Example 2.8 (*Translations*) Let $M = \mathbb{R}^2$, and let $\mathbf{z}_0 \in \mathbb{R}^2$ be a fixed point of the plane. The set of transformations

$$\mathbf{z}' = \mathbf{z} + a\mathbf{z}_0$$

($a \in \mathbb{R}$ and $\omega(a, b) = a + b$) defines a one-parameter Lie group of transformations, called *group of translations*.

Example 2.9 (*Scaling*) Let $M = \mathbb{R}^2$. The transformation

$$\mathbf{z}' = a\mathbf{z}$$

($a \in \mathbb{R}^+$ and $\omega(a, b) = a \cdot b$) defines a one-parameter Lie group of transformations, called *group of scaling*.

Example 2.10 (*Rotations in the plane*) Let $M = \mathbb{R}^2$, and let A be the 2×2 matrix

$$\mathsf{A} = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}.$$

The transformation

$$\mathbf{z}' = \mathsf{A}\mathbf{z}$$

($a \in \mathbb{R}$ and $\omega(a, b) = a + b$) characterizes the clockwise rotation by the angle a in the plane.

Definition 2.11 (*Lie group homomorphism*) A Lie group homomorphism is a smooth map

$$\phi\colon G \to G'$$

between two Lie groups G (with operation ω) and G' (with operation ω') respecting the group operations:

$$\phi(\omega(a,b)) = \omega'(\phi(a),\phi(b)), \qquad \forall\, a,b \in G.$$

If ϕ has a smooth inverse, an isomorphism between G and G' is determined.

Example 2.11 The Lie group $\mathbb{R}^+$ consisting of all positive real numbers, the ordinary multiplication being the group operation, is isomorphic to the additive Lie group $\mathbb{R}$. The exponential function

$$\phi\colon \mathbb{R} \to \mathbb{R}^+, \qquad \phi(x) = \exp(x)$$

yields the isomorphism.

Lemma 2.1 *Let S be an open interval of $\mathbb{R}$, and let $\omega\colon S \times S \to S$ an operation rendering S a group, e being the identity element and a^{-1} the inverse of a. If $\omega(a,b)$ and the inversion are analytic, then there exists an analytic function $\phi\colon S \to \mathbb{R}$ such that*

$$\phi(\omega(a,b)) = \phi(a) + \phi(b),$$

i.e., ϕ is a group homomorphism.

Proof Define the function

$$\Omega(a) = \left.\frac{\partial\omega(p,q)}{\partial q}\right|_{(p,q)=(a^{-1},a)},$$

and the Cauchy initial value problem

$$\frac{d\phi}{da} = \Omega(a), \quad \phi(e) = 0. \tag{2.1}$$

Integration of (2.1) yields

$$\phi(a) = \int_e^a \Omega(x)dx.$$

Due to the hypotheses, the function ϕ is analytic. It represents the group homomorphism we are looking for. What is needed is the proof of the equality

$$\phi(\omega(a,b)) = \phi(a) + \phi(b),$$

that is,

$$\int_e^{\omega(a,b)} \Omega(x)dx = \int_e^a \Omega(x)dx + \int_e^b \Omega(x)dx. \tag{2.2}$$

By differentiating both sides of last condition with respect to b, we get

$$\Omega(\omega(a,b))\frac{\partial\omega(a,b)}{\partial b} = \Omega(b), \tag{2.3}$$

and integrating between e and b,

$$\int_e^{\omega(a,b)} \Omega(x)dx = \int_e^b \Omega(x)dx + \varphi(a).$$

In particular, choosing $b = e$, we obtain

$$\varphi(a) = \int_e^a \Omega(x)dx,$$

implying that (2.2) and (2.3) are equivalent.

Now, since ω is a group operation, we have

$$\omega(c, \omega(a,b)) = \omega(\omega(c,a), b);$$

differentiating the latter with respect to b, we get

$$\frac{\partial\omega(c,\omega(a,b))}{\partial\omega(a,b)}\frac{\partial\omega(a,b)}{\partial b} = \frac{\partial\omega(\omega(c,a),b)}{\partial b}.$$

In particular, choosing $c = (\omega(a,b))^{-1} = \omega(b^{-1}, a^{-1})$, one obtains

$$\begin{aligned}\Omega(\omega(a,b))\frac{\partial\omega(a,b)}{\partial b} &= \frac{\partial\omega(\omega(\omega(a,b)^{-1},a),b)}{\partial b}\\ &= \frac{\partial\omega(\omega(\omega(b^{-1},a^{-1}),a),b)}{\partial\omega(a,b)}\\ &= \frac{\partial\omega(\omega(b^{-1},\omega(a^{-1},a)),b)}{\partial b}\\ &= \frac{\partial\omega(\omega(b^{-1},e),b)}{\partial b} = \frac{\partial\omega(b^{-1},b)}{\partial b} = \Omega(b)\end{aligned}$$

that is the relation (2.3), equivalent to (2.2). □

Remark 2.4 Lemma 2.1, in the case of one-parameter real Lie groups of transformations, allows us to assume, without loss of generality, the group operation as the ordinary sum.

Example 2.12 Consider the scaling group

$$z' = (1+a)z, \qquad z \in \mathbb{R}, \tag{2.4}$$

with $a \in S =]-1, \infty[$, where the group operation is such that

$$\omega(a,b) = a + b + ab, \qquad a^{-1} = -\frac{a}{1+a}, \qquad e = 0.$$

It is

$$\Omega(a) = \left.\frac{\partial \omega(a,b)}{\partial b}\right|_{(a^{-1},a)} = \frac{1}{1+a},$$

and

$$\phi\colon S \to \mathbb{R}, \qquad \phi(a) = \log(1+a).$$

The function ϕ is such that

$$\begin{aligned}\phi(\omega(a,b)) &= \phi(a+b+ab) = \log((1+a)(1+b)) \\ &= \log(1+a) + \log(1+b) = \phi(a) + \phi(b).\end{aligned}$$

Therefore, setting $\alpha = \log(1+a)$, the scaling group (2.4) can be written as

$$z' = \exp(\alpha) z, \ \ \alpha \in \mathbb{R},$$

where the group operation is the ordinary sum.

Example 2.13 ([17]) Consider $S =]-1, 1[$ with the group operation

$$\omega(a,b) = \frac{2ab - a - b}{ab - 1};$$

with identity $e = 0$; the inverse

$$a^{-1} = \frac{a}{2a-1}$$

exists for $a \in]-\frac{1}{2}, \frac{1}{2}[$. This is a classical example of a local Lie group. It is

$$\Omega(a) = \left.\frac{\partial \omega(a,b)}{\partial b}\right|_{(a^{-1},a)} = \frac{1}{(a-1)^2},$$

whereupon

$$\phi\colon S \to \mathbb{R}, \qquad \phi(a) = \int_0^a \frac{1}{(x-1)^2} dx = \frac{a}{1-a}.$$

The function ϕ is the required group homomorphism because

$$\phi(\omega(a,b)) = \phi\left(\frac{2ab-a-b}{ab-1}\right) = \frac{a}{1-a} + \frac{b}{1-b} = \phi(a) + \phi(b).$$

2.4 Infinitesimal Transformations

Consider a one-parameter (a) Lie group of transformations

$$\mathbf{z}' = \mathbf{Z}(\mathbf{z}; a) \tag{2.5}$$

that, because of Lemma 2.1, we may assume to involve as a law of composition the sum with identity $e = 0$. By expanding in powers of a around $a = 0$, we get

$$\mathbf{z}' = \mathbf{z} + a\left.\frac{\partial \mathbf{Z}(\mathbf{z}; a)}{\partial a}\right|_{a=0} + \frac{a^2}{2}\left.\frac{\partial^2 \mathbf{Z}(\mathbf{z}; a)}{\partial a^2}\right|_{a=0} + \cdots = \mathbf{z} + a\left.\frac{\partial \mathbf{Z}(\mathbf{z}; a)}{\partial a}\right|_{a=0} + O(a^2).$$

By setting

$$\boldsymbol{\zeta}(\mathbf{z}) = \left.\frac{\partial \mathbf{Z}(\mathbf{z}; a)}{\partial a}\right|_{a=0},$$

the transformation

$$\mathbf{z}' = \mathbf{z} + a\boldsymbol{\zeta}(\mathbf{z}) + O(a^2)$$

is called the *infinitesimal transformation* of the Lie group of transformations (2.5); the components of $\boldsymbol{\zeta}(\mathbf{z})$ are called the *infinitesimals* (or infinitesimal generators) of the group (2.5).

Lie first fundamental theorem shows that the infinitesimal transformation contains the essential information for characterizing a one-parameter Lie group of transformations.

Theorem 2.1 (Lie first fundamental theorem) *The Lie group of transformations* (2.5) *is equivalent to the solution of the initial value problem for the system of first-order differential equations (called Lie equations)*

$$\frac{d\mathbf{z}'}{da} = \boldsymbol{\zeta}(\mathbf{z}'), \qquad \mathbf{z}'(0) = \mathbf{z}. \tag{2.6}$$

Proof If $\mathbf{z}' = \mathbf{Z}(\mathbf{z}; a)$, taking into account that

$$\mathbf{Z}(\mathbf{z}; a + \varepsilon) = \mathbf{Z}(\mathbf{z}'; \varepsilon),$$

and expanding both sides in powers of ε around $\varepsilon = 0$, we get

$$\mathbf{Z}(\mathbf{z}; a+\varepsilon) = \mathbf{Z}(\mathbf{z}; a) + \varepsilon \frac{\partial \mathbf{Z}(\mathbf{z}; a)}{\partial a} + O(\varepsilon^2) = \mathbf{z}' + \varepsilon \frac{d\mathbf{z}'}{da} + O(\varepsilon^2),$$
$$\mathbf{Z}(\mathbf{z}'; \varepsilon) = \mathbf{Z}(\mathbf{z}'; 0) + \varepsilon \left.\frac{\partial \mathbf{Z}(\mathbf{z}'; \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} + O(\varepsilon^2) = \mathbf{z}' + \varepsilon \boldsymbol{\zeta}(\mathbf{z}') + O(\varepsilon^2),$$

where $\mathbf{z}'$ is given by (2.5).

Comparison of these expressions implies that $\mathbf{z}' = \mathbf{Z}(\mathbf{z}; a)$ satisfies the initial value problem (2.6).

Conversely, since the infinitesimals $\boldsymbol{\zeta}(\mathbf{z}')$ and their first-order partial derivatives are continuous, the Cauchy theorem for the initial value problem (2.6) implies that the solution of (2.6) exists and is unique. This solution has to be (2.5). □

Theorem 2.1 means that the infinitesimal transformation is de facto equivalent to the Lie group of transformations, and so justifies the term *infinitesimal generator of the group* given to $\boldsymbol{\zeta}(\mathbf{z})$.

Example 2.14 Consider in $\mathbb{R}^2$ the infinitesimal generator $\boldsymbol{\zeta} = (z_1, z_2)$. The Lie equations are

$$\frac{dz_1'}{da} = z_1', \quad z_1'(0) = z_1,$$

$$\frac{dz_2'}{da} = z_2', \quad z_2'(0) = z_2,$$

whose integration yields

$$z_1' = \exp(a) z_1, \qquad z_2' = \exp(a) z_2,$$

corresponding to the group of uniform scalings in the plane.

Example 2.15 Take in $\mathbb{R}^2$ the infinitesimal generator $\boldsymbol{\zeta} = (z_2, -z_1)$. The Lie equations are

$$\frac{dz_1'}{da} = z_2', \qquad z_1'(0) = z_1,$$

$$\frac{dz_2'}{da} = -z_1', \quad z_2'(0) = z_2,$$

whose integration yields

$$z_1' = z_1 \cos a + z_2 \sin a, \qquad z_2' = -z_1 \sin a + z_2 \cos a,$$

corresponding to the group of counterclockwise rotations in the plane.

Example 2.16 Consider in $\mathbb{R}^3$ the infinitesimal generator $\boldsymbol{\zeta} = (z_2, 0, 1)$. The Lie equations are

$$\frac{dz_1'}{da} = z_2', \quad z_1'(0) = z_1,$$

$$\frac{dz_2'}{da} = 0, \quad z_2'(0) = z_2,$$

$$\frac{dz_3'}{da} = 1, \quad z_3'(0) = z_3,$$

whose integration gives

$$z_1' = z_1 + az_2, \qquad z_2' = z_2, \qquad z_3' = z_3 + a. \tag{2.7}$$

By interpreting z_1 as a space coordinate, z_2 as a time coordinate, and z_3 as a velocity, the one-parameter Lie group (2.7) characterizes the Galilean frame invariance in one dimension (a is the velocity of one inertial frame with respect to the other one).

Example 2.17 Consider in $\mathbb{R}^2$ the infinitesimal generator $\boldsymbol{\zeta} = (z_1^2, z_1 z_2)$. The integration of Lie equations,

$$\frac{dz_1'}{da} = z_1'^2, \quad z_1'(0) = z_1,$$

$$\frac{dz_2'}{da} = z_1' z_2', \quad z_2'(0) = z_2,$$

yields

$$z_1' = \frac{z_1}{1 - az_1}, \qquad z_2' = \frac{z_2}{1 - az_1}. \tag{2.8}$$

Notice that transformations (2.8) become singular when $1 - az_1 = 0$. Therefore, they only make sense if defined in a neighborhood of $a = 0$. In such a case we have a *local Lie group of transformations*.

2.5 Infinitesimal Generator and Invariance

The *infinitesimal generator* $\boldsymbol{\zeta}(\mathbf{z})$ (which is a vector field) of the one-parameter Lie group of transformations (2.5) allows us to introduce the first-order differential operator (the *symbol*, according to Lie's terminology)

$$\Xi = \boldsymbol{\zeta}(\mathbf{z}) \cdot \nabla = \zeta_1(\mathbf{z}) \frac{\partial}{\partial z_1} + \cdots + \zeta_n(\mathbf{z}) \frac{\partial}{\partial z_n}, \tag{2.9}$$

which is called *infinitesimal operator* of the Lie group (with a slight abuse of language, we can refer to it using again the term *infinitesimal generator*).

For any differentiable function $F(\mathbf{z})$ it is

$$\Xi(F) = \boldsymbol{\zeta}(\mathbf{z}) \cdot \nabla F = \zeta_1(\mathbf{z}) \frac{\partial F}{\partial z_1} + \cdots + \zeta_n(\mathbf{z}) \frac{\partial F}{\partial z_n},$$

and, in particular,

$$\Xi(\mathbf{z}) = \boldsymbol{\zeta}(\mathbf{z}).$$

A one-parameter Lie group of transformations, which by Theorem 2.1 is equivalent to its infinitesimal transformation, is also equivalent to the infinitesimal operator Ξ in (2.9).

The following theorem shows that the use of the infinitesimal generator leads to an algorithm for finding the formal solution of the initial value problem (2.6).

Theorem 2.2 *The one-parameter Lie group of transformations* (2.5) *is equivalent to*

$$\mathbf{z}' = \exp(a\,\Xi)(\mathbf{z}) = \mathbf{z} + a\,\Xi(\mathbf{z}) + \frac{a^2}{2}\,\Xi^2(\mathbf{z}) + \cdots = \sum_{k=0}^{\infty} \frac{a^k}{k!}\,\Xi^k(\mathbf{z}), \tag{2.10}$$

where the operator Ξ *is defined by* (2.9)*, and* $\Xi^k(\mathbf{z}) = \Xi\left(\Xi^{k-1}(\mathbf{z})\right)$*, in particular,* $\Xi^0(\mathbf{z}) = \mathbf{z}$*.*

Proof Let us have

$$\Xi = \zeta_1(\mathbf{z}) \frac{\partial}{\partial z_1} + \cdots + \zeta_n(\mathbf{z}) \frac{\partial}{\partial z_n},$$

and

$$\Xi' = \zeta_1(\mathbf{z}') \frac{\partial}{\partial z_1'} + \cdots + \zeta_n(\mathbf{z}') \frac{\partial}{\partial z_n'},$$

where $\mathbf{z}' = \mathbf{Z}(\mathbf{z}; a)$, whose expansion in Taylor series around $a = 0$ yields

$$\mathbf{z}' = \sum_{k=0}^{\infty} \frac{a^k}{k!} \left.\frac{\partial^k \mathbf{Z}(\mathbf{z}; a)}{\partial a^k}\right|_{a=0} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \left.\frac{d^k \mathbf{z}'}{da^k}\right|_{a=0}.$$

For any differentiable function $F(\mathbf{z})$ it is

$$\frac{dF(\mathbf{z}')}{da} = \sum_{i=1}^{n} \frac{\partial F(\mathbf{z}')}{\partial z_i'} \frac{dz_i'}{da} = \sum_{i=1}^{n} \zeta_i(\mathbf{z}') \frac{\partial F(\mathbf{z}')}{\partial z_i'} = \Xi'\left(F(\mathbf{z}')\right);$$

in particular, it follows

$$\begin{aligned}
&\frac{d\mathbf{z}'}{da} = \Xi'(\mathbf{z}'),\\
&\frac{d^2\mathbf{z}'}{da^2} = \frac{d}{da}\left(\frac{d\mathbf{z}'}{da}\right) = \Xi'\left(\Xi'(\mathbf{z}')\right) = \Xi'^2\left(\mathbf{z}'\right),\\
&\cdots,
\end{aligned}$$

whereupon

$$\left.\frac{d^k\mathbf{z}'}{da^k}\right|_{a=0} = \Xi^k\left(\mathbf{z}\right), \qquad k \in \mathbb{N}$$

holds true, so that (2.10) follows. □

Henceforth, the Taylor series expansion around $a = 0$ of a function $\mathbf{Z}(\mathbf{z}; a)$, which defines a Lie group of transformations (2.5), is determined by the coefficient of its $O(a)$ term, i.e., by the infinitesimals $\boldsymbol{\zeta}(\mathbf{z})$.

Thus, one can find explicitly a one-parameter Lie group of transformations from its infinitesimal transformation, by expressing the group in terms of a power series (2.10), called *Lie series*.

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.1 *If $F(\mathbf{z})$ is infinitely differentiable, then*

$$F(\mathbf{z}') = F\left(\exp(a\,\Xi)(\mathbf{z})\right) = \exp(a\,\Xi)\left(F(\mathbf{z})\right).$$

Proof

$$\begin{aligned}
F(\mathbf{z}') &= F(\mathbf{z}) + a\left.\frac{dF(\mathbf{z}')}{da}\right|_{a=0} + \frac{a^2}{2!}\left.\frac{d^2F(\mathbf{z}')}{da^2}\right|_{a=0} + \cdots\\
&= F(\mathbf{z}) + a\,\Xi(F(\mathbf{z})) + \frac{a^2}{2!}\Xi^2(F(\mathbf{z})) + \cdots\\
&= \exp(a\,\Xi)\left(F(\mathbf{z})\right).
\end{aligned}$$

□

Now, we can introduce the concept of invariance of a function with respect to a Lie group of transformations, and prove the related invariance criterion.

Definition 2.12 (*Invariant function*) An infinitely differentiable function $F(\mathbf{z})$ is said an *invariant function* of the Lie group of transformations (2.5) if and only if the condition

$$F(\mathbf{z}') \equiv F(\mathbf{z})$$

holds true.

If $F(\mathbf{z})$ is an invariant function of (2.5), then it is simply called an *invariant* of (2.5).

The invariance of a function is characterized in a very simple way by means of the infinitesimal operator of the group, as shown in the following theorem.

Theorem 2.3 *An infinitely differentiable function $F(\mathbf{z})$ is invariant under (2.5) if and only if*

$$\Xi\left(F(\mathbf{z})\right) = 0.$$

Proof From Corollary 2.1, it follows:

$$F(\mathbf{z}') = F\left(\exp(a\,\Xi)(\mathbf{z})\right) = \exp(a\,\Xi)\left(F(\mathbf{z})\right) = \sum_{k=0}^{\infty} \frac{a^k}{k!}\,\Xi^k\left(F(\mathbf{z})\right)$$

$$= F(\mathbf{z}) + a\,\Xi\left(F(\mathbf{z})\right) + \frac{a^2}{2}\,\Xi^2\left(F(\mathbf{z})\right) + \cdots \tag{2.11}$$

Suppose $F(\mathbf{z}') = F(\mathbf{z})$. Then, (2.11) implies $\Xi\left(F(\mathbf{z})\right) = 0$. Conversely, if $\Xi\left(F(\mathbf{z})\right) = 0$, then $\Xi^k\left(F(\mathbf{z})\right) = 0$ for all positive integers k, whence (2.11) implies that $F(\mathbf{z}') = F(\mathbf{z})$. □

Remark 2.5 From a geometrical point of view, the condition $\Xi\left(F(\mathbf{z})\right) = \boldsymbol{\zeta}(\mathbf{z}) \cdot \nabla F = 0$ means that the vector field $\boldsymbol{\zeta}(\mathbf{z})$ must be tangent to the surface $F(\mathbf{z}) = 0$.

It can be defined also the invariance of a surface in $\mathbb{R}^n$ with respect to a Lie group of transformations.

Definition 2.13 (*Invariant surface*) A surface $F(\mathbf{z}) = 0$, where $F(\mathbf{z})$ is infinitely differentiable, is said an *invariant surface* with respect to the one-parameter Lie group (2.5) if $F(\mathbf{z}') = 0$ when $F(\mathbf{z}) = 0$.

As a consequence of Theorem 2.3, the following result immediately follows.

Theorem 2.4 *A surface $F(\mathbf{z}) = 0$ is invariant with respect to the group (2.5) if and only if*

$$\Xi\left(F(\mathbf{z})\right) = 0 \text{ when } F(\mathbf{z}) = 0,$$

i.e.,

$$\Xi\left(F(\mathbf{z})\right) = \Lambda(\mathbf{z})F(\mathbf{z})$$

for some smooth function (multiplier) $\Lambda(\mathbf{z})$ nowhere vanishing.

Theorems 2.3 and 2.4 can be easily extended to functions with values in $\mathbb{R}^m$ and to general manifolds, respectively.

Example 2.18 Given the infinitesimal operator of the rotation in the plane,

$$\Xi = z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2},$$

the functions $F(z_1, z_2)$ that are left invariant satisfy the condition

$$z_2 \frac{\partial F}{\partial z_1} - z_1 \frac{\partial F}{\partial z_2} = 0,$$

whereupon $F(z_1, z_2) = F(z_1^2 + z_2^2)$, as one expects.

Definition 2.14 (*Invariant family of surfaces*) The family of surfaces

$$F(\mathbf{z}) = c, \qquad \mathbf{z} \in \mathbb{R}^n,$$

where the constant c labels the different surfaces of the family, is an invariant family of surfaces for a Lie group of transformations if

$$F(\mathbf{z}') = c' \text{ when } F(\mathbf{z}) = c.$$

From this definition, it follows that

$$c' = C(c; a).$$

Theorem 2.5 ([4]) *A family of surfaces, $F(\mathbf{z}) = c$, is an invariant family of surfaces for the Lie group of transformations* (2.5) *if and only if*

$$\Xi(F) = \chi(F),$$

for some infinitely differentiable function $\chi(F)$.

Proof Let $F(\mathbf{z}) = c$ be an invariant family of surfaces for the Lie group (2.5). Then, expanding in power series around $a = 0$

$$F(\mathbf{z}') = C(c; a),$$

we get

$$F(\mathbf{z}) + a\,\Xi(F(\mathbf{z})) + O(a^2) = c + a\chi(c) + O(a^2),$$

where $\chi(c) = \left.\dfrac{\partial C}{\partial c}\right|_{a=0}$. So,

$$\Xi(F(\mathbf{z})) = \chi(F(\mathbf{z})).$$

Conversely, suppose $\Xi(F) = \chi(F)$ for some differentiable function $\chi(F)$. Then, $\Xi^2(F(\mathbf{z})) = \chi'(F)\chi(F)$, and, in general, $\Xi^k(F(\mathbf{z})) = \chi_k(F), k > 0$, for some functions $\chi_k(F)$. Consequently, if $F(\mathbf{z}) = c$, then

$$\begin{aligned} F(\mathbf{z}') &= F(\mathbf{z}) + a\,\Xi(F(\mathbf{z})) + \frac{a^2}{2}\Xi^2(F(\mathbf{z})) + \cdots \\ &= F(\mathbf{z}) + \sum_{k=1}^{\infty} \frac{a^k}{k!}\chi_k(F(\mathbf{z})) = c + \sum_{k=1}^{\infty} \frac{a^k}{k!}\chi_k(c) = C(c;a) = c'. \end{aligned}$$

□

Two different types of invariant families of surfaces may be considered. The trivial type is when each surface in the family is itself invariant. This type is characterized by $\chi(F) = 0$. The nontrivial type is when no surface in the family is itself invariant, i.e., each surface is mapped to a different one. In such a case, as the representation of a family of surfaces is not unique, and so $G(F(\mathbf{z})) = G(c)$ for any invertible function G characterizes the same family, then the invariance of the family is guaranteed by the condition:

$$\Xi(G(F(\mathbf{z}))) = G'(F)\Xi(F) = G'(F)\chi(F).$$

The choice $G'(F) = 1/\chi(F)$ allows us to restate Theorem 2.5 for families where no surface is itself invariant as follows.

Theorem 2.6 ([4]) *A family of surfaces, $F(\mathbf{z}) = c$, is an invariant family of surfaces for the Lie group of transformations $\mathbf{z}' = \mathbf{Z}(\mathbf{z}; a)$ if and only if*

$$\Xi(F) = 1.$$

Example 2.19 Consider in $\mathbb{R}^3$ the family of spherical surfaces

$$z_1^2 + z_2^2 + z_3^2 = c, \qquad c > 0. \tag{2.12}$$

The infinitesimal operators

$$\Xi_1 = z_2\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_2}, \quad \Xi_2 = z_3\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_3}, \quad \Xi_3 = z_3\frac{\partial}{\partial z_2} - z_2\frac{\partial}{\partial z_3},$$

corresponding to the rotations in the planes z_1z_2, z_1z_3, and z_2z_3, respectively, leave each surface in the family invariant; in fact, it is

$$\Xi_1\left(z_1^2 + z_2^2 + z_3^2\right) = \Xi_2\left(z_1^2 + z_2^2 + z_3^2\right) = \Xi_3\left(z_1^2 + z_2^2 + z_3^2\right) = 0.$$

On the contrary, the infinitesimal operator

$$\Xi = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3},$$

corresponding to a uniform scaling of z_1, z_2, and z_3, maps a spherical surface into a different one. In fact,

$$\Xi\left(z_1^2+z_2^2+z_3^2\right)\Big|_{z_1^2+z_2^2+z_3^2=c}=2c,$$

and the associated finite transformation

$$z_1'=\exp(a)z_1,\quad z_2'=\exp(a)z_2,\quad z_3'=\exp(a)z_3,$$

maps the surface (2.12) into

$$z_1'^2+z_2'^2+z_3'^2=\exp(2a)c=c'.$$

According to Theorem 2.6, we can write the surface in (2.12) as

$$\frac{1}{2c}(z_1^2+z_2^2+z_3^2)=\frac{1}{2},$$

whence

$$\Xi\left(\frac{1}{2c}(z_1^2+z_2^2+z_3^2)\right)\Big|_{(z_1^2+z_2^2+z_3^2)/(2c)=1/2}=1.$$

2.6 Canonical Variables

Given in $\mathbb{R}^n$ the one-parameter Lie group of transformations (2.5), let us make the change of variables defined by the one-to-one and (at least) C^1 transformation

$$\mathbf{y}=\mathbf{Y}(\mathbf{z}),\qquad \mathbf{y}=(y_1,\ldots,y_n). \tag{2.13}$$

If

$$\Xi=\sum_{i=1}^{n}\zeta_i(\mathbf{z})\frac{\partial}{\partial z_i}$$

is the infinitesimal operator in terms of the coordinates $\mathbf{z}$, the corresponding operator in terms of $\mathbf{y}$ has a (possibly) different representation, say

$$\widetilde{\Xi}=\sum_{i=1}^{n}\widetilde{\zeta}_i(\mathbf{y})\frac{\partial}{\partial y_i}.$$

We should have the same group action, even if the infinitesimals have different expressions, whence the following theorem is stated.

Theorem 2.7 *Given*

$$\Xi = \sum_{i=1}^{n} \zeta_i(\mathbf{z}) \frac{\partial}{\partial z_i},$$

and the one-to-one and (at least) C^1 transformation (2.13), it is

$$\widetilde{\Xi} = \sum_{i=1}^{n} \widetilde{\zeta}_i(\mathbf{y}) \frac{\partial}{\partial y_i},$$

where

$$\widetilde{\zeta}_i(\mathbf{y}) = \Xi(y_i), \qquad i = 1, \ldots, n.$$

Proof Let $F(\mathbf{z})$ be a smooth function, and $\mathbf{y} = \mathbf{Y}(\mathbf{z})$ (together with its inverse $\mathbf{z} = \mathbf{Y}^{-1}(\mathbf{y})$) a smooth transformation; we have

$$F(\mathbf{z}) = F(\mathbf{Y}^{-1}(\mathbf{y})) = \widetilde{F}(\mathbf{y}).$$

Using the chain rule, it is

$$\begin{aligned}
\Xi(F(\mathbf{z})) &= \sum_{i=1}^{n} \zeta_i \frac{\partial}{\partial z_i} \left(F(\mathbf{Y}^{-1}(\mathbf{y}))\right) = \sum_{i=1}^{n} \zeta_i \frac{\partial}{\partial z_i} \left(\widetilde{F}(\mathbf{y})\right) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \zeta_i \frac{\partial \widetilde{F}(\mathbf{y})}{\partial y_j} \frac{\partial y_j}{\partial z_i} \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \zeta_i \frac{\partial y_j}{\partial z_i} \right) \frac{\partial \widetilde{F}(\mathbf{y})}{\partial y_j} \\
&= \sum_{j=1}^{n} \Xi(y_j) \frac{\partial \widetilde{F}(\mathbf{y})}{\partial y_j} = \sum_{j=1}^{n} \widetilde{\zeta}_j \frac{\partial \widetilde{F}(\mathbf{y})}{\partial y_j} = \widetilde{\Xi}(\widetilde{F}(\mathbf{y})),
\end{aligned}$$

where

$$\widetilde{\zeta}_j = \Xi\left(y_j\right) = \sum_{i=1}^{n} \zeta_i \frac{\partial y_j}{\partial z_i}, \qquad j = 1, \ldots, n.$$

□

Definition 2.15 (*Canonical variables*) A change of coordinates (2.13) defines a set of *canonical coordinates* (also called *rectifying coordinates*) for the one-parameter Lie group of transformations (2.5) if, in terms of such coordinates, the group (2.5) becomes

$$\begin{aligned}
y_i' &= y_i, \qquad i = 1, \ldots, n-1, \\
y_n' &= y_n + a,
\end{aligned} \tag{2.14}$$

i.e., a translation of only one component, say the nth one.

Theorem 2.8 *For any Lie group of transformations* (2.5) *there exists a set of canonical coordinates* $\mathbf{y} = (y_1(\mathbf{z}), \ldots, y_n(\mathbf{z}))$ *such that* (2.5) *is equivalent to* (2.14).

Proof From Theorem 2.3, we have

$$y_i' = y_i \quad \text{if and only if} \quad \Xi(y_i) = 0, \quad i = 1, \ldots, n-1,$$

and, from Theorem 2.7,

$$y_n' = y_n + a \quad \text{if and only if} \quad \Xi(y_n) = 1.$$

Thus, the first-order linear partial differential equations

$$\zeta_1(\mathbf{z})\frac{\partial y_i}{\partial z_1} + \cdots + \zeta_n(\mathbf{z})\frac{\partial y_i}{\partial z_n} = 0, \qquad i = 1, \ldots, n-1, \tag{2.15a}$$

$$\zeta_1(\mathbf{z})\frac{\partial y_n}{\partial z_1} + \cdots + \zeta_n(\mathbf{z})\frac{\partial y_n}{\partial z_n} = 1 \tag{2.15b}$$

characterize the canonical variables. In fact, Eq. (2.15a) has $n-1$ functionally independent solutions: these solutions are the $n-1$ essential constants appearing in the general solution of the system of n first-order ordinary differential equations

$$\frac{d\mathbf{z}}{da} = \boldsymbol{\zeta}(\mathbf{z}) \tag{2.16}$$

resulting from the characteristic equations of the system (2.15a) (because of the autonomous form of the system (2.16), the nth constant is *unessential*). The nth canonical variable is a particular solution of the nonhomogeneous first-order linear partial differential equation (2.15b). □

Remark 2.6 In terms of the canonical coordinates, the infinitesimal operator of the Lie group assumes its simplest form:

$$\widetilde{\Xi} = \frac{\partial}{\partial y_n}.$$

Remark 2.7 Hereafter, to light the notation, we will remove the "tilde" in the notation used for the infinitesimal operator when a different set of variables (say, canonical) is used.

Example 2.20 Take the Lie group in $\mathbb{R}^2$ generated by

$$\Xi = z_1^2\frac{\partial}{\partial z_1} + z_1 z_2\frac{\partial}{\partial z_2}.$$

The canonical variables (y_1, y_2) are such that

$$\Xi(y_1) = z_1^2 \frac{\partial y_1}{\partial z_1} + z_1 z_2 \frac{\partial y_1}{\partial z_2} = 0,$$
$$\Xi(y_2) = z_1^2 \frac{\partial y_2}{\partial z_1} + z_1 z_2 \frac{\partial y_2}{\partial z_2} = 1.$$

A possible choice is

$$y_1 = \frac{z_1}{z_2}, \qquad y_2 = -\frac{1}{z_2},$$

so that the infinitesimal operator becomes

$$\Xi = \frac{\partial}{\partial y_2}.$$

Example 2.21 Consider the Lie group in $\mathbb{R}^3$ generated by

$$\Xi = \alpha z_1 \frac{\partial}{\partial z_1} + \beta z_2 \frac{\partial}{\partial z_2} + \gamma z_3 \frac{\partial}{\partial z_3}, \qquad \alpha, \beta, \gamma \text{ constants}, \gamma \neq 0.$$

The canonical variables (y_1, y_2, y_3) are such that

$$\Xi(y_1) = \alpha z_1 \frac{\partial y_1}{\partial z_1} + \beta z_2 \frac{\partial y_1}{\partial z_2} + \gamma z_3 \frac{\partial y_1}{\partial z_3} = 0,$$
$$\Xi(y_2) = \alpha z_1 \frac{\partial y_2}{\partial z_1} + \beta z_2 \frac{\partial y_2}{\partial z_2} + \gamma z_3 \frac{\partial y_2}{\partial z_3} = 0,$$
$$\Xi(y_3) = \alpha z_1 \frac{\partial y_3}{\partial z_1} + \beta z_2 \frac{\partial y_3}{\partial z_2} + \gamma z_3 \frac{\partial y_3}{\partial z_3} = 1.$$

A possible choice is

$$y_1 = \frac{z_1}{z_3^{\alpha/\gamma}}, \qquad y_2 = \frac{z_2}{z_3^{\beta/\gamma}}, \qquad y_3 = \frac{1}{\gamma} \ln z_3;$$

so, the infinitesimal operator becomes

$$\Xi = \frac{\partial}{\partial y_3}.$$

2.7 Multi-Parameter Lie Groups of Transformations

A Lie group of transformations may depend on many parameters as well,

$$\begin{aligned} &\mathbf{Z}\colon M \times S \to M, \\ &\mathbf{z} \mapsto \mathbf{z}' = \mathbf{Z}(\mathbf{z}; \mathbf{a}), \end{aligned} \tag{2.17}$$

where M is an open connected domain of $\mathbb{R}^n$, $\mathbf{a} = (a_1, \ldots, a_r) \in S$, and S is an open connected subset of $\mathbb{R}^r$. Moreover, $\mathbf{Z}(\mathbf{z}; \mathbf{a})$ is C^∞ with respect to $\mathbf{z}$, and analytic with respect to $\mathbf{a}$.

Let the law of composition of parameters rendering S a group be denoted by

$$\boldsymbol{\omega}(\mathbf{a}, \mathbf{b}) = (\omega_1(\mathbf{a}, \mathbf{b}), \ldots, \omega_r(\mathbf{a}, \mathbf{b}))$$

with identity $\mathbf{e} = (e_1, \ldots, e_r)$; also in this case $\boldsymbol{\omega}\colon S \times S \to S$, as well as the inversion of any element $\mathbf{a} \in S$, should be assumed analytic in their domain of definition.

The relations (2.17), for arbitrary choices of the components of $\mathbf{a}$, characterize r-parameter, or $(r - \nu)$-parameter transformations, the second situation being associated to the existence of $(r - \nu)$ functions $A_1(\mathbf{a}), \ldots, A_{r-\nu}(\mathbf{a})$ such that the function $\mathbf{Z}$ depends on $\mathbf{a}$ through $A_1(\mathbf{a}), \ldots, A_{r-\nu}(\mathbf{a})$. If $\nu = 0$, then we say that the parameters $(a_1, \ldots, a_r)$ are *essential*.

Let us suppose the existence of functions $A_1(\mathbf{a}), \ldots, A_{r-\nu}(\mathbf{a})$ (with $0 < \nu < r$), consider a function $\varphi(\mathbf{a})$, and construct the Jacobian matrix

$$\mathsf{J} = \begin{pmatrix} \dfrac{\partial \varphi}{\partial a_1} & \cdots\cdots & \dfrac{\partial \varphi}{\partial a_{r-\nu+1}} \\ \dfrac{\partial A_1}{\partial a_1} & \cdots\cdots & \dfrac{\partial A_1}{\partial a_{r-\nu+1}} \\ \vdots & \vdots \quad \vdots & \vdots \\ \dfrac{\partial A_{r-\nu}}{\partial a_1} & \cdots\cdots & \dfrac{\partial A_{r-\nu}}{\partial a_{r-\nu+1}} \end{pmatrix}$$

with respect to $r - \nu + 1$ arbitrarily chosen components of $\mathbf{a}$ (possibly relabeling them to choose the first $r - \nu + 1$ components). Then, the condition $\det(\mathsf{J}) = 0$ yields a linear partial differential equation for φ,

$$\sum_{k=1}^{r-\nu+1} \alpha_k(\mathbf{a}) \frac{\partial \varphi}{\partial a_k} = 0,$$

which is a special case of the general equation:

$$\sum_{k=1}^{r} \alpha_k(\mathbf{a}) \frac{\partial \varphi}{\partial a_k} = 0. \tag{2.18}$$

This equation is obviously satisfied by $\varphi = A_1, \ldots, \varphi = A_{r-\nu}$; since any function of $A_1, \ldots, A_{r-\nu}$ is a solution, then $\mathbf{Z}$, assumed to be function of $A_1, \ldots, A_{r-\nu}$, is a solution. Choosing different groups of $r - \nu + 1$ parameters, we have that if the parameters $a_1, \ldots, a_r$ are not all essential, then the function $\mathbf{Z}$ must satisfy to some equations like (2.18). Vice versa, if the function $\mathbf{Z}$ satisfies some equation like (2.18), then $\mathbf{Z}$ depends on the parameters $a_1, \ldots, a_r$ through some functions $A_1(\mathbf{a}), \ldots, A_{r-\nu}(\mathbf{a})$. Thus, the following theorem remains proved.

Theorem 2.9 *In the family of transformations* $\mathbf{z}' = \mathbf{Z}(\mathbf{z}; \mathbf{a})$*, the parameters* $a_1, \ldots, a_r$ *are essential if and only if the functions* $\mathbf{Z}$ *do not satisfy any equation like*

$$\sum_{k=1}^{r} \alpha_k(\mathbf{a}) \frac{\partial \mathbf{Z}}{\partial a_k} = 0.$$

Example 2.22 Let $z \in \mathbb{R}$; the family of transformations

$$z' = Z(z; \mathbf{a}) = \frac{a_1 z + a_2}{a_3 z + a_4}, \qquad a_1 a_4 - a_2 a_3 \neq 0;$$

characterizes a multi-parameter Lie group; moreover, the group operation is

$$\boldsymbol{\omega}(\mathbf{a}, \mathbf{b}) = (a_1 b_1 + a_3 b_2, a_2 b_1 + a_4 b_2, a_1 b_3 + a_3 b_4, a_2 b_3 + a_4 b_4)$$

with identity element $\mathbf{e} = (1, 0, 0, 1)$, and inverse

$$\left(\frac{a_4}{a_1 a_4 - a_2 a_3}, \frac{-a_2}{a_1 a_4 - a_2 a_3}, \frac{-a_3}{a_1 a_4 - a_2 a_3}, \frac{a_1}{a_1 a_4 - a_2 a_3} \right).$$

As it is identically

$$\alpha_1(\mathbf{a}) \frac{\partial Z}{\partial a_1} + \frac{a_2}{a_1} \alpha_1(\mathbf{a}) \frac{\partial Z}{\partial a_2} + \frac{a_3}{a_1} \alpha_1(\mathbf{a}) \frac{\partial Z}{\partial a_3} + \frac{a_4}{a_1} \alpha_1(\mathbf{a}) \frac{\partial Z}{\partial a_4} = 0,$$

with $\alpha_1(\mathbf{a})$ arbitrary, the parameters are not all essential. On the contrary, for the family of transformations

$$z' = Z(z; \mathbf{a}) = \frac{z + a_1}{a_2 z + a_3}, \qquad \mathbf{a} = (a_1, a_2, a_3),$$

an equation like

$$\alpha_1(\mathbf{a}) \frac{\partial Z}{\partial a_1} + \alpha_2(\mathbf{a}) \frac{\partial Z}{\partial a_2} + \alpha_3(\mathbf{a}) \frac{\partial Z}{\partial a_3} = 0$$

cannot be satisfied whatever the functions $\alpha_1(\mathbf{a}), \alpha_2(\mathbf{a}), \alpha_3(\mathbf{a})$ (not identically vanishing) are chosen. Thus, the three parameters are essential.

When dealing with r-parameter Lie groups of transformations, we will implicitly assume the r parameters to be essential.

Lemma 2.2 *Let*

$$\mathbf{z}' = \mathbf{Z}(\mathbf{z}; \mathbf{a})$$

be an r-parameter Lie group of transformations. Then the identity

$$\mathbf{Z}(\mathbf{Z}(\mathbf{z}; \mathbf{a}); \boldsymbol{\omega}(\mathbf{a}^{-1}, \mathbf{a} + \boldsymbol{\varepsilon})) = \mathbf{Z}(\mathbf{z}; \mathbf{a} + \boldsymbol{\varepsilon}),$$

where $\boldsymbol{\varepsilon}, \mathbf{a} + \boldsymbol{\varepsilon} \in S$, holds true.

Proof

$$\begin{aligned}\mathbf{Z}(\mathbf{Z}(\mathbf{z}; \mathbf{a}); \boldsymbol{\omega}(\mathbf{a}^{-1}, \mathbf{a} + \boldsymbol{\varepsilon})) &= \mathbf{Z}(\mathbf{z}; \boldsymbol{\omega}(\mathbf{a}, \boldsymbol{\omega}(\mathbf{a}^{-1}, \mathbf{a} + \boldsymbol{\varepsilon}))) \\ &= \mathbf{Z}(\mathbf{z}; \boldsymbol{\omega}(\boldsymbol{\omega}(\mathbf{a}, \mathbf{a}^{-1}), \mathbf{a} + \boldsymbol{\varepsilon})) \\ &= \mathbf{Z}(\mathbf{z}; \boldsymbol{\omega}(\mathbf{e}, \mathbf{a} + \boldsymbol{\varepsilon})) = \mathbf{Z}(\mathbf{z}; \mathbf{a} + \boldsymbol{\varepsilon}).\end{aligned}$$

□

Lemma 2.3 *Let*

$$\begin{aligned}&\boldsymbol{\omega}\colon S \times S \to S, \\ &(\mathbf{a}, \mathbf{b}) \mapsto \boldsymbol{\omega}(\mathbf{a}, \mathbf{b})\end{aligned}$$

be a binary analytic operation rendering an open connected domain $S \subseteq \mathbb{R}^r$ a group. For $\mathbf{a} \in S$ and $\boldsymbol{\varepsilon}$ small enough (in norm) such that $\mathbf{a} + \boldsymbol{\varepsilon} \in S$, it is

$$\boldsymbol{\omega}(\mathbf{a}^{-1}, \mathbf{a} + \boldsymbol{\varepsilon}) = \mathbf{e} + \Omega(\mathbf{a})\boldsymbol{\varepsilon} + O\left(\|\boldsymbol{\varepsilon}\|^2\right),$$

where $\Omega(\mathbf{a})$ is the $r \times r$ matrix with entries

$$\Omega^{\alpha}_{\beta}(\mathbf{a}) = \left.\frac{\partial \omega_{\alpha}(\mathbf{p}, \mathbf{q})}{\partial q_{\beta}}\right|_{(\mathbf{p},\mathbf{q})=(\mathbf{a}^{-1},\mathbf{a})}.$$

Proof Straightforward, by expanding in power series,

$$\begin{aligned}&\boldsymbol{\omega}(\mathbf{a}^{-1}, \mathbf{a} + \boldsymbol{\varepsilon}) \\ &\quad = \boldsymbol{\omega}(\mathbf{a}^{-1}, \mathbf{a}) + \nabla_{\mathbf{q}}\boldsymbol{\omega}(\mathbf{p}, \mathbf{q})\big|_{(\mathbf{p},\mathbf{q})=(\mathbf{a}^{-1},\mathbf{a})}\,\boldsymbol{\varepsilon} + O\left(\|\boldsymbol{\varepsilon}\|^2\right) \\ &\quad = \mathbf{e} + \Omega(\mathbf{a})\boldsymbol{\varepsilon} + O\left(\|\boldsymbol{\varepsilon}\|^2\right),\end{aligned}$$

where $\nabla_{\mathbf{q}} \equiv \left(\dfrac{\partial}{\partial q_1}, \ldots, \dfrac{\partial}{\partial q_r}\right)$. □

Definition 2.16 (*Generators of an r-parameter Lie group*) Let

$$\mathbf{z}' = \mathbf{Z}(\mathbf{z}; \mathbf{a})$$

be an r-parameter Lie group of transformations. The vectors

$$\boldsymbol{\zeta}_\alpha(\mathbf{z}) = \left.\frac{\partial \mathbf{Z}(\mathbf{z}; \mathbf{a})}{\partial a_\alpha}\right|_{\mathbf{a}=\mathbf{e}}, \qquad \alpha = 1, \ldots, r$$

are the generators of the r-parameter Lie group. These generators may be collected in the $r \times n$ *infinitesimal matrix*

$$\chi(\mathbf{z}) = \begin{pmatrix} \zeta_{11} & \cdots & \zeta_{1n} \\ \vdots & \vdots & \vdots \\ \zeta_{r1} & \ldots & \zeta_{rn} \end{pmatrix}.$$

We associate to each generator the differential operator

$$\Xi_\alpha = \boldsymbol{\zeta}_\alpha(\mathbf{z}) \cdot \boldsymbol{\nabla} = \sum_{i=1}^{n} \zeta_{\alpha i}(\mathbf{z}) \frac{\partial}{\partial z_i}, \qquad \alpha = 1, \ldots, r.$$

Finally, let $\boldsymbol{\nabla}_{\mathbf{a}} \mathbf{Z}(\mathbf{z}; \mathbf{a}) = \boldsymbol{\nabla}_{\mathbf{a}} \mathbf{z}'$ be the $n \times r$ matrix with entries $\dfrac{\partial z_i'}{\partial a_\alpha}$.

The following theorem is the Lie first fundamental theorem for r-parameter Lie groups.

Theorem 2.10 (Lie first fundamental theorem) *The r-parameter Lie group of transformations*

$$\mathbf{z}' = \mathbf{Z}(\mathbf{z}; \mathbf{a}) \tag{2.19}$$

is the solution of the initial value problem for the system of $n \cdot r$ first-order partial differential equations (in some neighborhood of $\mathbf{a} = \mathbf{e}$):

$$\frac{\partial z_i'}{\partial a_\alpha} = \sum_{\gamma=1}^{r} \zeta_{\gamma i}(\mathbf{z}') \Omega_\alpha^\gamma(\mathbf{a}), \qquad i = 1, \ldots, n, \ \alpha = 1, \ldots, r,$$
$$z_i'(\mathbf{e}) = z_i, \qquad i = 1, \ldots, n,$$

or, in compact form,

$$\boldsymbol{\nabla}_{\mathbf{a}} \mathbf{z}' = \chi^T(\mathbf{z}') \Omega(\mathbf{a}),$$
$$\mathbf{z}'(\mathbf{e}) = \mathbf{z}.$$

Proof Due to Lemma 2.2, it is

$$\mathbf{Z}\left(\mathbf{Z}(\mathbf{z};\mathbf{a});\boldsymbol{\omega}(\mathbf{a}^{-1},\mathbf{a}+\boldsymbol{\varepsilon})\right)=\mathbf{Z}(\mathbf{z};\mathbf{a}+\boldsymbol{\varepsilon}).$$

Let us expand both sides in Taylor series around $\boldsymbol{\varepsilon}=\mathbf{0}$, and take into account Lemma 2.3:

$$\begin{aligned}
\mathbf{Z}\left(\mathbf{Z}(\mathbf{z};\mathbf{a});\boldsymbol{\omega}(\mathbf{a}^{-1},\mathbf{a}+\boldsymbol{\varepsilon})\right)&=\mathbf{Z}\left(\mathbf{Z}(\mathbf{z};\mathbf{a});\mathbf{e}+\Omega(\mathbf{a})\boldsymbol{\varepsilon}+O\left(\|\boldsymbol{\varepsilon}\|^2\right)\right)\\
&=\mathbf{Z}\left(\mathbf{z}';\mathbf{e}+\Omega(\mathbf{a})\boldsymbol{\varepsilon}+O\left(\|\boldsymbol{\varepsilon}\|^2\right)\right)\\
&=\mathbf{Z}(\mathbf{z}';\mathbf{e})+\nabla_{\mathbf{a}}\mathbf{Z}(\mathbf{z}';\mathbf{a})\big|_{\mathbf{a}=\mathbf{e}}\,\Omega(\mathbf{a})\boldsymbol{\varepsilon}+O\left(\|\boldsymbol{\varepsilon}\|^2\right)\\
&=\mathbf{z}'+\chi^T(\mathbf{z}')\Omega(\mathbf{a})\boldsymbol{\varepsilon}+O\left(\|\boldsymbol{\varepsilon}\|^2\right),\\
\mathbf{Z}(\mathbf{z};\mathbf{a}+\boldsymbol{\varepsilon})&=\mathbf{Z}(\mathbf{z};\mathbf{a})+\nabla_{\mathbf{a}}\mathbf{Z}(\mathbf{z};\mathbf{a})\boldsymbol{\varepsilon}+O\left(\|\boldsymbol{\varepsilon}\|^2\right)\\
&=\mathbf{z}'+(\nabla_{\mathbf{a}}\mathbf{z}')\boldsymbol{\varepsilon}+O\left(\|\boldsymbol{\varepsilon}\|^2\right);
\end{aligned}$$

so, (2.19) satisfies the differential equations:

$$\nabla_{\mathbf{a}}\mathbf{z}'=\chi^T(\mathbf{z}')\Omega(\mathbf{a}). \tag{2.20}$$

This is a set of $n\cdot r$ partial differential equations to be integrated with the initial condition $\mathbf{z}'(\mathbf{e})=\mathbf{z}$. □

This means that also for r-parameter Lie groups of transformations the infinitesimal version contains all the information of the group.

Remark 2.8 Evaluating (2.20) for $\mathbf{a}=\mathbf{e}$, it follows that $\Omega(\mathbf{e})$ is the $r\times r$ identity matrix.

Example 2.23 The two-parameter family of transformations

$$\begin{aligned}
z_1'&=\exp(a_1)z_1+a_2,\\
z_2'&=\exp(2a_1)z_2,
\end{aligned} \tag{2.21}$$

with the law of composition

$$(\omega_1(\mathbf{a},\mathbf{b}),\omega_2(\mathbf{a},\mathbf{b}))=(a_1+b_1,\exp(b_1)a_2+b_2),$$

defines a two-parameter Lie group of transformations with $z_1'=z_1$, $z_2'=z_2$ when $\mathbf{a}=(0,0)$.

Since

$$\left.\frac{\partial z_1'}{\partial a_1}\right|_{\mathbf{a}=(0,0)}=z_1,\quad \left.\frac{\partial z_2'}{\partial a_1}\right|_{\mathbf{a}=(0,0)}=2z_2,$$
$$\left.\frac{\partial z_1'}{\partial a_2}\right|_{\mathbf{a}=(0,0)}=1,\quad \left.\frac{\partial z_2'}{\partial a_2}\right|_{\mathbf{a}=(0,0)}=0,$$

the infinitesimal matrix is given by

$$\chi(\mathbf{z}) = \begin{pmatrix} z_1 & 2z_2 \\ 1 & 0 \end{pmatrix},$$

and the corresponding infinitesimal operators of (2.21) are

$$\Xi_1 = z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2},$$
$$\Xi_2 = \frac{\partial}{\partial z_1}.$$

Moreover, it is

$$\Omega(\mathbf{a}) = \begin{pmatrix} 1 & 0 \\ -a_2 & 1 \end{pmatrix}.$$

According to Theorem 2.10, the group (2.21) can be recovered by solving the initial value problem for the system of partial differential equations

$$\frac{\partial z_1'}{\partial a_1} = z_1' - a_2, \quad \frac{\partial z_2'}{\partial a_1} = 2z_2',$$
$$\frac{\partial z_1'}{\partial a_2} = 1, \qquad \frac{\partial z_2'}{\partial a_2} = 0,$$

with the initial condition

$$z_1'(0,0) = z_1, \qquad z_2'(0,0) = z_2.$$

Remark 2.9 The r-parameter Lie group of transformations (2.17) can be recovered also integrating the (ordinary) Lie equations corresponding to each of the infinitesimal operators $\Xi_1, \ldots, \Xi_r$, and then making their composition

$$\mathbf{z}' = \prod_{\alpha=1}^{r} \exp(a_\alpha \Xi_\alpha)\, \mathbf{z}.$$

As far as Example 2.23 is concerned, we have

$$\exp(a_1 \Xi_1)\mathbf{z} = (\exp(a_1) z_1, \exp(2a_1) z_2),$$
$$\exp(a_2 \Xi_2)\mathbf{z} = (z_1 + a_2, z_2),$$

and we have the two-parameter Lie group

$$\mathbf{z}' = \exp(a_1 \Xi_1) \exp(a_2 \Xi_2)\mathbf{z} = (\exp(a_1) z_1 + a_2, \exp(2a_1) z_2). \tag{2.22}$$

Changing the order of the one-parameter Lie groups in the composition, we have the same Lie group, though its form *may* appear different. In fact,

$$\mathbf{z}' = \exp(a_2\,\Xi_2)\exp(a_1\,\Xi_1)\mathbf{z} = (\exp(a_1)(z_1 + a_2), \exp(2a_1)z_2)$$

is not essentially different from (2.22).

Remark 2.10 The one-parameter (a) Lie group of transformations

$$\mathbf{z}' = \exp(a\,\Xi)\mathbf{z} = \exp\left(a\sum_{\alpha=1}^{r}\sigma_\alpha\,\Xi_\alpha\right)\mathbf{z},$$

obtained by exponentiating the infinitesimal generator

$$\Xi = \sum_{\alpha=1}^{r}\sigma_\alpha\,\Xi_\alpha = \sum_{j=1}^{N}\zeta_j(\mathbf{z})\frac{\partial}{\partial z_j},$$

where $\sigma_1, \ldots, \sigma_r$ are fixed real constants, and

$$\zeta_j(\mathbf{z}) = \sum_{\alpha=1}^{r}\sigma_\alpha\zeta_{\alpha j}(\mathbf{z}), \qquad j = 1, \ldots, N$$

defines a one-parameter (a) subgroup of the r-parameter Lie group of transformations.

For the infinitesimal operators of an r-parameter Lie group, a new operation can be defined (*Lie bracket*) in such a way they gain the structure of a Lie algebra (see also Chap. 3).

2.8 Lie Brackets

Definition 2.17 (*Lie bracket*) Let G be an r-parameter Lie group of transformations along with the associated infinitesimal operators Ξ_μ $(\mu = 1, \ldots, r)$. The *Lie bracket* (or commutator) of two operators Ξ_μ and Ξ_ν is

$$\begin{aligned}\left[\Xi_\mu, \Xi_\nu\right] &= \Xi_\mu\,\Xi_\nu - \Xi_\nu\,\Xi_\mu \\ &= \sum_{i=1}^{n}\left(\zeta_{\mu i}(\mathbf{z})\frac{\partial}{\partial z_i}\left(\sum_{j=1}^{n}\zeta_{\nu j}(\mathbf{z})\frac{\partial}{\partial z_j}\right) - \zeta_{\nu i}(\mathbf{z})\frac{\partial}{\partial z_i}\left(\sum_{j=1}^{n}\zeta_{\mu j}(\mathbf{z})\frac{\partial}{\partial z_j}\right)\right).\end{aligned}$$

Proposition 2.1 *The Lie bracket of two first-order differential operators is still a first-order differential operator:*

$$[\Xi_\mu, \Xi_\nu] = \Xi_\mu \, \Xi_\nu - \Xi_\nu \, \Xi_\mu = \sum_{i=1}^{n} \widehat{\zeta}_i(\mathbf{z}) \frac{\partial}{\partial z_i},$$

where

$$\widehat{\zeta}_i(\mathbf{z}) = \sum_{j=1}^{n} \left(\zeta_{\mu j}(\mathbf{z}) \frac{\partial \zeta_{\nu i}(\mathbf{z})}{\partial z_j} - \zeta_{\nu j}(\mathbf{z}) \frac{\partial \zeta_{\mu i}(\mathbf{z})}{\partial z_j} \right).$$

Proof Straightforward. □

Proposition 2.2 *The Lie bracket is skew-symmetric, say*

$$[\Xi_\mu, \Xi_\nu] = -[\Xi_\nu, \Xi_\mu],$$

and bilinear, say

$$[\Xi_\lambda, \alpha \Xi_\mu + \beta \Xi_\nu] = \alpha [\Xi_\lambda, \Xi_\mu] + \beta [\Xi_\lambda, \Xi_\nu],$$

where $\alpha, \beta \in \mathbb{R}$.

Proof Both properties follow from the definition. □

Theorem 2.11 (Lie second fundamental theorem) *The commutator of any two infinitesimal operators of an r-parameter Lie group of transformations is an infinitesimal operator of the r-parameter Lie group, say*

$$[\Xi_\mu, \Xi_\nu] = \sum_{\lambda=1}^{r} C_{\mu\nu}^{\lambda} \Xi_\lambda, \qquad \mu, \nu = 1, \ldots, r, \tag{2.23}$$

where the constant coefficients $C_{\mu\nu}^{\lambda}$ *are called* structure constants.

Proof The proof follows immediately from the integrability conditions

$$\frac{\partial^2 z_i'}{\partial a_\mu \partial a_\nu} = \frac{\partial^2 z_i'}{\partial a_\nu \partial a_\mu}, \qquad i = 1, \ldots n, \quad \mu, \nu = 1, \ldots, r.$$

In fact, using (2.20) and the Einstein convention of sums over repeated indices, it is

$$\begin{aligned}\frac{\partial^2 z_i'}{\partial a_\mu a_\nu} &= \frac{\partial}{\partial a_\mu}\left(\zeta_{\rho i}(\mathbf{z}')\Omega_\nu^\rho(\mathbf{a})\right) = \frac{\partial \zeta_{\rho i}(\mathbf{z}')}{\partial z_j'}\frac{\partial z_j'}{\partial a_\mu}\Omega_\nu^\rho + \zeta_{\rho i}(\mathbf{z}')\frac{\partial \Omega_\nu^\rho}{\partial a_\mu} \\ &= \frac{\partial \zeta_{\rho i}(\mathbf{z}')}{\partial z_j'}\zeta_{\lambda j}(\mathbf{z}')\Omega_\mu^\lambda\Omega_\nu^\rho + \zeta_{\rho i}(\mathbf{z}')\frac{\partial \Omega_\nu^\rho}{\partial a_\mu}, \\ \frac{\partial^2 z_i'}{\partial a_\nu a_\mu} &= \frac{\partial}{\partial a_\nu}\left(\zeta_{\lambda i}(\mathbf{z}')\Omega_\mu^\lambda(\mathbf{a})\right) = \frac{\partial \zeta_{\lambda i}(\mathbf{z}')}{\partial z_j'}\frac{\partial z_j'}{\partial a_\nu}\Omega_\mu^\lambda + \zeta_{\lambda i}(\mathbf{z}')\frac{\partial \Omega_\mu^\lambda}{\partial a_\nu} \\ &= \frac{\partial \zeta_{\lambda i}(\mathbf{z}')}{\partial z_j'}\zeta_{\rho j}(\mathbf{z}')\Omega_\nu^\rho\Omega_\mu^\lambda + \zeta_{\lambda i}(\mathbf{z}')\frac{\partial \Omega_\mu^\lambda}{\partial a_\nu},\end{aligned}$$

whereupon

$$\left(\zeta_{\lambda j}(\mathbf{z}')\frac{\partial \zeta_{\rho i}(\mathbf{z}')}{\partial z_j'} - \zeta_{\rho j}(\mathbf{z}')\frac{\partial \zeta_{\lambda i}(\mathbf{z}')}{\partial z_j'}\right)\Omega_\mu^\lambda\Omega_\nu^\rho = \zeta_{\lambda i}(\mathbf{z}')\left(\frac{\partial \Omega_\mu^\lambda}{\partial a_\nu} - \frac{\partial \Omega_\nu^\lambda}{\partial a_\mu}\right). \tag{2.24}$$

Finally, evaluating (2.24) for $\mathbf{a} = \mathbf{e}$, we have

$$\left(\zeta_{\mu j}(\mathbf{z})\frac{\partial \zeta_{\nu i}(\mathbf{z})}{\partial z_j} - \zeta_{\nu j}(\mathbf{z})\frac{\partial \zeta_{\mu i}(\mathbf{z})}{\partial z_j}\right) = C_{\mu\nu}^\lambda \zeta_{\lambda i}(\mathbf{z}),$$

where

$$C_{\mu\nu}^\lambda = \left(\frac{\partial \Omega_\mu^\lambda}{\partial a_\nu} - \frac{\partial \Omega_\nu^\lambda}{\partial a_\mu}\right)\Bigg|_{\mathbf{a}=\mathbf{e}}$$

are suitable constants. □

Theorem 2.12 (Jacobi identity) *For any three operators of an r-parameter Lie group, the Jacobi identity is satisfied, i.e.:*

$$\left[\Xi_\lambda, \left[\Xi_\mu, \Xi_\nu\right]\right] + \left[\Xi_\mu, \left[\Xi_\nu, \Xi_\lambda\right]\right] + \left[\Xi_\nu, \left[\Xi_\lambda, \Xi_\mu\right]\right] = 0.$$

Proof

$$\begin{aligned}\left[\Xi_\lambda, \left[\Xi_\mu, \Xi_\nu\right]\right] &= \left[\Xi_\lambda, \Xi_\mu\Xi_\nu - \Xi_\nu\Xi_\mu\right] \\ &= \Xi_\lambda\Xi_\mu\Xi_\nu - \Xi_\lambda\Xi_\nu\Xi_\mu - \Xi_\mu\Xi_\nu\Xi_\lambda + \Xi_\nu\Xi_\mu\Xi_\lambda \\ &= \Xi_\lambda\Xi_\mu\Xi_\nu - \Xi_\lambda\Xi_\nu\Xi_\mu - \Xi_\mu\Xi_\nu\Xi_\lambda + \Xi_\nu\Xi_\mu\Xi_\lambda \\ &\quad + \Xi_\mu\Xi_\lambda\Xi_\nu - \Xi_\mu\Xi_\lambda\Xi_\nu + \Xi_\nu\Xi_\lambda\Xi_\mu - \Xi_\nu\Xi_\lambda\Xi_\mu \\ &= -\Xi_\mu(\Xi_\nu\Xi_\lambda - \Xi_\lambda\Xi_\nu) + (\Xi_\nu\Xi_\lambda - \Xi_\lambda\Xi_\nu)\Xi_\mu \\ &\quad - \Xi_\nu(\Xi_\lambda\Xi_\mu - \Xi_\mu\Xi_\lambda) + (\Xi_\lambda\Xi_\mu - \Xi_\mu\Xi_\lambda)\Xi_\nu \\ &= -\left[\Xi_\mu, \left[\Xi_\nu, \Xi_\lambda\right]\right] - \left[\Xi_\nu, \left[\Xi_\lambda, \Xi_\mu\right]\right].\end{aligned}$$

□

Theorem 2.13 (Lie third fundamental theorem) *The structure constants defined by the commutation relations* (2.23) *satisfy the relations*

$$C^{\rho}_{\mu\nu} = -C^{\rho}_{\nu\mu},$$
$$\sum_{\theta=1}^{r} \left(C^{\theta}_{\nu\lambda} C^{\rho}_{\mu\theta} + C^{\theta}_{\mu\nu} C^{\rho}_{\lambda\theta} + C^{\theta}_{\lambda\mu} C^{\rho}_{\nu\theta} \right) = 0.$$

Proof These relations follow immediately from the antisymmetry of Lie brackets, and Jacobi identity. □

Some properties of Lie brackets are very important and will be used in other parts of this book.

Theorem 2.14 *The Lie bracket of two operators is invariant in form with respect to any invertible change of variables.*

Proof Let $\mathbf{y} = \mathbf{Y}(\mathbf{z})$ be a change of variables. It is

$$\Xi_{\mu} = \sum_{i=1}^{n} \zeta_{\mu i} \frac{\partial}{\partial z_i}, \qquad \widetilde{\Xi}_{\mu} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \zeta_{\mu i} \frac{\partial y_j}{\partial z_i} \right) \frac{\partial}{\partial y_j},$$
$$\Xi_{\nu} = \sum_{i=1}^{n} \zeta_{\nu i} \frac{\partial}{\partial z_i}, \qquad \widetilde{\Xi}_{\nu} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \zeta_{\nu i} \frac{\partial y_j}{\partial z_i} \right) \frac{\partial}{\partial y_j}.$$

Hence, using the convention of sums over repeated indices,

$$\begin{aligned}
\left[\widetilde{\Xi}_{\mu}, \widetilde{\Xi}_{\nu} \right] &= \left(\widetilde{\zeta}_{\mu j} \frac{\partial \widetilde{\zeta}_{\nu k}}{\partial y_j} - \widetilde{\zeta}_{\nu j} \frac{\partial \widetilde{\zeta}_{\mu k}}{\partial y_j} \right) \frac{\partial}{\partial y_k} \\
&= \left(\zeta_{\mu i} \frac{\partial y_j}{\partial z_i} \frac{\partial}{\partial y_j} \left(\zeta_{\nu \ell} \frac{\partial y_k}{\partial z_\ell} \right) - \zeta_{\nu i} \frac{\partial y_j}{\partial z_i} \frac{\partial}{\partial y_j} \left(\zeta_{\mu \ell} \frac{\partial y_k}{\partial z_\ell} \right) \right) \frac{\partial}{\partial y_k} \\
&= \left(\zeta_{\mu i} \frac{\partial y_j}{\partial z_i} \frac{\partial \zeta_{\nu \ell}}{\partial z_m} \frac{\partial z_m}{\partial y_j} \frac{\partial y_k}{\partial z_\ell} - \zeta_{\nu i} \frac{\partial y_j}{\partial z_i} \frac{\partial \zeta_{\mu \ell}}{\partial z_m} \frac{\partial z_m}{\partial y_j} \frac{\partial y_k}{\partial z_\ell} \right) \frac{\partial}{\partial y_k} \\
&= \left(\left(\zeta_{\mu i} \frac{\partial \zeta_{\nu \ell}}{\partial z_m} - \zeta_{\nu i} \frac{\partial \zeta_{\mu \ell}}{\partial z_m} \right) \frac{\partial z_m}{\partial y_j} \frac{\partial y_j}{\partial z_i} \frac{\partial y_k}{\partial z_\ell} \right) \frac{\partial}{\partial y_k} \\
&= \left(\left(\zeta_{\mu i} \frac{\partial \zeta_{\nu \ell}}{\partial z_m} - \zeta_{\nu i} \frac{\partial \zeta_{\mu \ell}}{\partial z_m} \right) \delta_{mi} \frac{\partial y_k}{\partial z_\ell} \right) \frac{\partial}{\partial y_k} \\
&= \left(\left(\zeta_{\mu i} \frac{\partial \zeta_{\nu \ell}}{\partial z_i} - \zeta_{\nu i} \frac{\partial \zeta_{\mu \ell}}{\partial z_i} \right) \frac{\partial y_k}{\partial z_\ell} \right) \frac{\partial}{\partial y_k} \\
&= \widetilde{\left[\Xi_{\mu}, \Xi_{\nu} \right]}.
\end{aligned}$$

□

This theorem means that the structure constants do not change under a coordinate transformation.

It is convenient to display the commutators of a Lie algebra through its *commutator table*, whose (μ, ν)th entry is $[\Xi_\mu, \Xi_\nu]$: the commutator table is a skew-symmetric matrix. Moreover, the structure constants are easily deduced from the commutator table.

Example 2.24 (*Group of rigid motions in* $\mathbb{R}^2$) The group of rigid motions in $\mathbb{R}^2$ preserves distances between any two points in $\mathbb{R}^2$ and it is the three-parameter Lie group of transformations of rotations and translations in $\mathbb{R}^2$ given by

$$\begin{aligned} z_1' &= z_1 \cos a_1 + z_2 \sin a_1 + a_2, \\ z_2' &= -z_1 \sin a_1 + z_2 \cos a_1 + a_3. \end{aligned}$$

The corresponding infinitesimal operators are

$$\Xi_1 = z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2}, \qquad \Xi_2 = \frac{\partial}{\partial z_1}, \qquad \Xi_3 = \frac{\partial}{\partial z_2}, \tag{2.25}$$

and the commutator table of the associated Lie algebra is

	Ξ_1	Ξ_2	Ξ_3
Ξ_1	0	Ξ_3	$-\Xi_2$
Ξ_2	$-\Xi_3$	0	0
Ξ_3	Ξ_2	0	0

Example 2.25 (*Similitude group in* $\mathbb{R}^2$) The similitude group in $\mathbb{R}^2$ consists of uniform scalings and rigid motions in $\mathbb{R}^2$. It is the four-parameter Lie group of transformations in $\mathbb{R}^2$ given by

$$\begin{aligned} z_1' &= \exp(a_4)(z_1 \cos a_1 + z_2 \sin a_1) + a_2, \\ z_2' &= \exp(a_4)(-z_1 \sin a_1 + z_2 \cos a_1) + a_3. \end{aligned}$$

The infinitesimal operators are Ξ_1, Ξ_2, Ξ_3 of (2.25) and

$$\Xi_4 = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}.$$

The corresponding commutator table is

	Ξ_1	Ξ_2	Ξ_3	Ξ_4
Ξ_1	0	Ξ_3	$-\Xi_2$	0
Ξ_2	$-\Xi_3$	0	0	Ξ_2
Ξ_3	Ξ_2	0	0	Ξ_3
Ξ_4	0	$-\Xi_2$	$-\Xi_3$	0

Finally, the following result can be proved.

Theorem 2.15 *If a regularly assigned manifold $F(\mathbf{z}) = 0$ is invariant with respect to the operators Ξ_μ and Ξ_ν, then it is invariant with respect to their commutator $[\Xi_\mu, \Xi_\nu]$.*

Proof Since the manifold $F(\mathbf{z}) = 0$ is invariant with respect to the Lie groups generated by Ξ_μ and Ξ_ν, it is

$$\begin{aligned}\Xi_\mu(F(\mathbf{z})) &= \Lambda_\mu(\mathbf{z})F(\mathbf{z}),\\ \Xi_\nu(F(\mathbf{z})) &= \Lambda_\nu(\mathbf{z})F(\mathbf{z}),\end{aligned}$$

where $\Lambda_\mu(\mathbf{z})$ and $\Lambda_\nu(\mathbf{z})$ are suitable Lagrange multipliers. Then,

$$\begin{aligned}\left[\Xi_\mu, \Xi_\nu\right](F(\mathbf{z})) &= \Xi_\mu\Xi_\nu(F(\mathbf{z})) - \Xi_\nu\Xi_\mu(F(\mathbf{z}))\\ &= \Xi_\mu(\Lambda_\nu(\mathbf{z})F(\mathbf{z})) - \Xi_\nu(\Lambda_\mu(\mathbf{z})F(\mathbf{z}))\\ &= \Xi_\mu(\Lambda_\nu(\mathbf{z}))F(\mathbf{z}) + \Lambda_\nu(\mathbf{z})\Xi_\mu(F(\mathbf{z}))\\ &\quad - \Xi_\nu(\Lambda_\mu(\mathbf{z}))F(\mathbf{z}) - \Lambda_\mu(\mathbf{z})\Xi_\nu(F(\mathbf{z}))\\ &= \left(\Xi_\mu(\Lambda_\nu(\mathbf{z})) - \Xi_\nu(\Lambda_\mu(\mathbf{z}))\right)F(\mathbf{z}) = \Lambda(\mathbf{z})F(\mathbf{z}).\end{aligned}$$

So, the manifold $F(\mathbf{z}) = 0$ is invariant with respect to $[\Xi_\mu, \Xi_\nu]$. □

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Chapter 3
Lie Algebras



3.1 Lie Algebras

Lie groups are intimately connected to Lie algebras [8, 10–13, 20–22], yielding a linearized framework to study the local properties of Lie groups. In this chapter, we introduce the first basic notions about Lie algebras. As it will become soon clear, the algebraic structure of the Lie symmetries of differential equations plays a crucial role both for obtaining exact solutions, deriving first integrals or conservation laws, classifying differential equations, transforming to suitable equivalent form, as well as implementing numerical schemes [9, 17].

Definition 3.1 (*Field*) A field $\mathbb{K}$ is a nonempty set endowed with two operations (addition and multiplication), $(\mathbb{K}, +, \cdot)$, satisfying the axioms:

1. the two operations are associative and commutative;
2. there exist two distinct elements (0 and 1) such that

$$a + 0 = a, \qquad a \cdot 1 = a, \quad \forall\, a \in \mathbb{K};$$

3. there exists the additive inverse,

$$\forall a \in \mathbb{K},\ \exists\, -a \in \mathbb{K} \text{ such that } a + (-a) = 0;$$

4. there exists the multiplicative inverse,

$$\forall a \in \mathbb{K},\ a \neq 0,\ \exists a^{-1} \in \mathbb{K} \text{ such that } a \cdot a^{-1} = 1;$$

5. multiplication distributes over addition,

$$\forall\, a, b, c \in \mathbb{K},\ a \cdot (b + c) = a \cdot b + a \cdot c.$$

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Thus, a field $\mathbb{K}$ is an Abelian group with respect to the addition with identity 0, $\mathbb{K}\backslash\{0\}$ is an Abelian group with respect to the multiplication with identity 1, and the distributive property of multiplication over addition holds.

Definition 3.2 (*Characteristic of a field*) The characteristic of a field indicates whether the sum of the multiplicative identity added to itself a finite number of times produces the additive identity. This characteristic is 0 or a prime number.

Definition 3.3 (*Vector space*) A vector space V over a field $\mathbb{K}$ is a nonempty set of elements (vectors) endowed with two operations, vector addition between vectors producing a vector, and scalar multiplication between an element of $\mathbb{K}$ (scalar) and a vector producing a vector, satisfying the axioms:

1. closure properties,
$$\forall\, \mathbf{u}, \mathbf{v} \in V,\ \mathbf{u} + \mathbf{v} \in V;$$
$$\forall\, a \in \mathbb{K}, \forall\, \mathbf{u} \in V,\ a\mathbf{u} \in V;$$
2. vector addition is associative and commutative;
3. there exists a zero vector, $\mathbf{0}$, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$, for any vector $\mathbf{u} \in V$;
4. for any vector $\mathbf{u} \in V$, there exists an additive inverse $(-\mathbf{u}) \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
5. for any vector $\mathbf{u} \in V$, $1\mathbf{u} = \mathbf{u}$, where 1 is the multiplicative identity of the field $\mathbb{K}$;
6. for any scalars $a, b \in \mathbb{K}$, and any vector $\mathbf{u} \in V$, $a(b\mathbf{u}) = (a \cdot b)\mathbf{u}$;
7. for any scalars $a, b \in \mathbb{K}$, and any vector $\mathbf{u} \in V$, $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$;
8. for any scalar $a \in \mathbb{K}$, and any vectors $\mathbf{u}, \mathbf{v} \in V$, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.

Definition 3.4 (*Basis and dimension of a vector space*) Let V be a vector space over the field $\mathbb{K}$. A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is linearly independent if

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = \mathbf{0}, \qquad a_i \in \mathbb{K},\ i = 1, \ldots, n,$$

holds only when $a_1 = a_2 = \cdots = a_n = 0$. A basis of V is a set of linearly independent vectors that span the entire vector space, i.e., any vector in V can be written as a unique linear combination of the vectors of the basis. The cardinality of the basis is the dimension of the vector space.

Definition 3.5 (*Lie algebra*) Let $\mathbb{K}$ be a field. A *Lie algebra* $\mathbb{L}$ over $\mathbb{K}$ is a vector space over $\mathbb{K}$ equipped with a bilinear map (*Lie bracket*),

$$[\cdot, \cdot]\colon \mathbb{L} \times \mathbb{L} \to \mathbb{L},$$
$$(x, y) \mapsto [x, y],$$

satisfying the following properties:

$$[x, x] = 0, \ \forall x \in \mathbb{L}, \tag{3.1a}$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \ \forall x, y, z \in \mathbb{L}, \tag{3.1b}$$

the relation (3.1b) being the so-called Jacobi identity.

Due to the bilinearity of the Lie bracket and to (3.1a), it is

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x],$$

whereupon

$$[x, y] = -[y, x], \quad \forall x, y \in \mathbb{L}, \tag{3.2}$$

so that the Lie bracket is skew-symmetric. Moreover, if $\mathbb{K}$ does not have characteristic 2, Eq. (3.2) implies (3.1a); to prove this, it is sufficient to choose $y = x$ in (3.2).

The following properties are easily proved:

1. $[x, 0] = 0 = [0, x]$, for all $x \in \mathbb{L}$;
2. if $[x, y] \neq 0$, then $x, y \in \mathbb{L}$ are linearly independent over $\mathbb{K}$.

The *dimension* of a Lie algebra $\mathbb{L}$ is its dimension as a vector space; if it has finite dimension r, then the notation $\mathbb{L}_r$ will be used.

Definition 3.6 (*Abelian Lie algebra*) A Lie algebra $\mathbb{L}$ is *Abelian* if

$$[x, y] = 0, \quad \forall x, y \in \mathbb{L}.$$

Definition 3.7 (*Lie subalgebra*) Given a Lie algebra $\mathbb{L}$, a *Lie subalgebra* of $\mathbb{L}$ is a vector space $\mathbb{L}' \subseteq \mathbb{L}$ such that $[x, y] \in \mathbb{L}'$ for all $x, y \in \mathbb{L}'$.

Example 3.1 Let $\mathbb{R}^3$ be the Euclidean three-dimensional vector space. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, we can define the vector product

$$\mathbf{x} \wedge \mathbf{y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

We have a three-dimensional Lie algebra:

1. $\mathbf{x} \wedge \mathbf{y} = -\mathbf{y} \wedge \mathbf{x}$;
2. $(\alpha \mathbf{x} + \beta \mathbf{y}) \wedge \mathbf{z} = \alpha(\mathbf{x} \wedge \mathbf{z}) + \beta(\mathbf{y} \wedge \mathbf{z})$;
3. $\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z}) + \mathbf{z} \wedge (\mathbf{x} \wedge \mathbf{y}) + \mathbf{y} \wedge (\mathbf{z} \wedge \mathbf{x}) = \mathbf{0}$.

Example 3.2 If V is a vector space over the field $\mathbb{K}$, the set of all linear maps from V to V is again a vector space over $\mathbb{K}$, denoted by $\mathrm{GL}(V)$.

The vector space $\mathrm{GL}(V)$ becomes a Lie algebra, denoted by $\mathfrak{gl}(V)$ and known as *general linear algebra*, with the Lie bracket

$$[x, y] := x \circ y - y \circ x, \ \forall x, y \in \mathfrak{gl}(V),$$

where $\circ$ denotes the composition of maps.

There is a matrix version for the general linear algebra $\mathfrak{gl}(V)$: the vector space $\mathrm{GL}(n; \mathbb{K})$ of all $n \times n$ matrices over $\mathbb{K}$ with the Lie bracket

$$[x, y] := xy - yx,$$

where xy is the usual matrix product of x and y.

Example 3.3 Let $\mathfrak{sl}(n; \mathbb{K})$ be the subset of $\mathfrak{gl}(n; \mathbb{K})$ of the traceless matrices. For arbitrary traceless matrices x and y, xy is traceless, so $\mathfrak{sl}(n; \mathbb{K})$ is a subalgebra of $\mathfrak{gl}(n; \mathbb{K})$.

Example 3.4 The Lie algebra $\mathfrak{t}(m; \mathbb{K})$ of diagonal matrices in $\mathfrak{gl}(n; \mathbb{K})$ equipped with the Lie bracket of $\mathfrak{gl}(n; \mathbb{K})$ is an Abelian subalgebra of $\mathfrak{gl}(n; \mathbb{K})$.

Example 3.5 Let $\mathfrak{b}(n; \mathbb{K})$ be the subspace of upper-triangular matrices in $\mathfrak{gl}(n; \mathbb{K})$ equipped with the Lie bracket of $\mathfrak{gl}(n; \mathbb{K})$, it becomes a Lie algebra.

Example 3.6 (*Hamiltonian matrix*) Let $m = 2n$; a matrix A of order $2n$ is said Hamiltonian if

$$\mathsf{A}^T\mathsf{J} + \mathsf{J}\mathsf{A} = \mathsf{0}_{2n},$$

where

$$\mathsf{J} = \begin{pmatrix} \mathsf{0}_n & -\mathsf{I}_n \\ \mathsf{I}_n & \mathsf{0}_n \end{pmatrix},$$

$\mathsf{0}_n$ and I_n being the null and identity matrix of order n, respectively.

It is immediate to prove that the following statements are equivalent:

1. A is a Hamiltonian matrix;
2. $\mathsf{A} = \mathsf{J}\mathsf{S}$, S being a symmetric matrix;
3. $\mathsf{J}\mathsf{A}$ is symmetric.

Moreover, J is trivially Hamiltonian as

$$\mathsf{J}^T\mathsf{J} + \mathsf{J}\mathsf{J} = \mathsf{I}_{2n} - \mathsf{I}_{2n} = \mathsf{0}_{2n}.$$

The set of Hamiltonian matrices is a vector subspace of the vector space $\mathrm{GL}(2n; \mathbb{R})$. Moreover, it is also closed with respect to the Lie bracket: if A_1, A_2 are Hamiltonian matrices, then $\mathsf{A}_1 = \mathsf{J}\mathsf{S}_1$, $\mathsf{A}_2 = \mathsf{J}\mathsf{S}_2$ (S_1, S_2 symmetric matrices), whence

$$[\mathsf{A}_1, \mathsf{A}_2] = \mathsf{A}_1\mathsf{A}_2 - \mathsf{A}_2\mathsf{A}_1 = \mathsf{J}\mathsf{S}_1\mathsf{J}\mathsf{S}_2 - \mathsf{J}\mathsf{S}_2\mathsf{J}\mathsf{S}_1 = \mathsf{J}(\mathsf{S}_1\mathsf{J}\mathsf{S}_2 - \mathsf{S}_2\mathsf{J}\mathsf{S}_1),$$

where

$$(\mathsf{S}_1\mathsf{J}\mathsf{S}_2 - \mathsf{S}_2\mathsf{J}\mathsf{S}_1)^T = (\mathsf{S}_1\mathsf{J}\mathsf{S}_2 - \mathsf{S}_2\mathsf{J}\mathsf{S}_1).$$

So, the set of Hamiltonian matrices of order $2n$ is a Lie algebra, usually denoted as $\mathfrak{sp}(n;\mathbb{R})$, and is a subalgebra of $\mathfrak{gl}(m;\mathbb{R})$. Because of the constraints imposed by the definition of Hamiltonian matrices, the dimension of this Lie algebra is $(2n^2+n)$. For $n=1$, Hamiltonian matrices are traceless, so that $\mathfrak{sp}(1;\mathbb{R}) = \mathfrak{sl}(2;\mathbb{R})$.

The Lie algebra $\mathfrak{sp}(n;\mathbb{R})$ is the one corresponding to the Lie group $\mathrm{SP}(n;\mathbb{R})$ of symplectic matrices defined in Example 2.7. It is simple to prove that, if A is a Hamiltonian matrix of order $2n$, the matrix $\exp(\mathsf{A}t)$ is symplectic. Conversely, if $\mathsf{A}(t)$ is a time-dependent symplectic matrix of order $2n$, then the matrix $\left.\dfrac{d\mathsf{A}(t)}{dt}\right|_{t=0}$ is Hamiltonian. For this reason, Hamiltonian matrices are also called infinitesimal symplectic matrices.

Example 3.7 (*Fermionic operators*) Let $\mathbb{H}$ be a Hilbert space and consider n linear operators $(\mathsf{b}_1,\ldots,\mathsf{b}_n)$ acting on $\mathbb{H}$, and satisfying the canonical anticommutation relations (CAR)

$$\{\mathsf{a}_j, \mathsf{a}_k^\dagger\} = \delta_{jk}\mathsf{I}, \qquad \{\mathsf{a}_j, \mathsf{a}_k\} = \{\mathsf{a}_j^\dagger, \mathsf{a}_k^\dagger\} = 0, \tag{3.3}$$

for all $j,k=1,\ldots,n$, where $\{\mathsf{u},\mathsf{v}\} = \mathsf{u}\mathsf{v}+\mathsf{v}\mathsf{u}$ is the anticommutator, δ_{jk} is the Kronecker delta, I is the identity operator in $\mathbb{H}$, and $\mathsf{a}_j^\dagger$ is the adjoint (transpose and conjugate complex) of a_j. In quantum field theory, $\mathsf{b}_j^\dagger$ and b_j are known as fermionic creation and annihilation operators, respectively. The set

$$\left\{\mathsf{a}_j^\dagger\mathsf{a}_k : \, j,k=1,\ldots,n\right\}$$

is a Lie algebra. In fact, using (3.3),

$$\begin{aligned}
[\mathsf{a}_j^\dagger\mathsf{a}_k, \mathsf{a}_\ell^\dagger\mathsf{a}_m] &= \mathsf{a}_j^\dagger\mathsf{a}_k\mathsf{a}_\ell^\dagger\mathsf{a}_m - \mathsf{a}_\ell^\dagger\mathsf{a}_m\mathsf{a}_j^\dagger\mathsf{a}_k \\
&= \mathsf{a}_j^\dagger\left(\delta_{k\ell}\mathsf{I} - \mathsf{a}_\ell^\dagger\mathsf{a}_k\right)\mathsf{a}_m - \mathsf{a}_\ell^\dagger\left(\delta_{jm}\mathsf{I} - \mathsf{a}_j^\dagger\mathsf{a}_m\right)\mathsf{a}_k \\
&= \delta_{k\ell}\mathsf{a}_j^\dagger\mathsf{a}_m - \delta_{jm}\mathsf{a}_\ell^\dagger\mathsf{a}_k.
\end{aligned}$$

Example 3.8 (*Bosonic ladder operators*) Let $\mathbb{H}$ be a Hilbert space and consider n linear operators $(\mathsf{b}_1,\ldots,\mathsf{b}_n)$ acting on $\mathbb{H}$, and satisfying the canonical commutation relations (CCR)

$$[\mathsf{b}_j, \mathsf{b}_k^\dagger] = \delta_{jk}\mathsf{I}, \qquad [\mathsf{b}_j, \mathsf{b}_k] = [\mathsf{b}_j^\dagger, \mathsf{b}_k^\dagger] = 0, \tag{3.4}$$

for all $j,k=1,\ldots,n$, where $[\mathsf{u},\mathsf{v}] = \mathsf{u}\mathsf{v}-\mathsf{v}\mathsf{u}$ is the commutator, δ_{jk} the Kronecker delta, I the identity operator in $\mathbb{H}$, and $\mathsf{b}_j^\dagger$ is the adjoint (transpose and conjugate

complex) of b_j. In quantum field theory, $\mathsf{b}_j^\dagger$ and b_j are known as bosonic creation and annihilation operators, respectively.

The operators

$$\left\{\mathsf{b}_j^\dagger \mathsf{b}_k : \ j, k = 1, \ldots, n\right\}$$

form a Lie algebra. In fact, using (3.4),

$$\begin{aligned}[\mathsf{b}_j^\dagger \mathsf{b}_k, \mathsf{b}_\ell^\dagger \mathsf{b}_m] &= \mathsf{b}_j^\dagger \mathsf{b}_k \mathsf{b}_\ell^\dagger \mathsf{b}_m - \mathsf{b}_\ell^\dagger \mathsf{b}_m \mathsf{b}_j^\dagger \mathsf{b}_k \\ &= \mathsf{b}_j^\dagger \left(\delta_{k\ell}\mathsf{I} + \mathsf{b}_\ell^\dagger \mathsf{b}_k\right) \mathsf{b}_m - \mathsf{b}_\ell^\dagger \left(\delta_{jm}\mathsf{I} + \mathsf{b}_j^\dagger \mathsf{b}_m\right) \mathsf{b}_k \\ &= \delta_{k\ell} \mathsf{b}_j^\dagger \mathsf{b}_m - \delta_{jm} \mathsf{b}_\ell^\dagger \mathsf{b}_k.\end{aligned}$$

Remark 3.1 Ladder operators (either fermionic or bosonic) play a central role in quantum field theory. Bosons and fermions are the two classes of ladders which are likely the most known since the beginning of quantum mechanics, the reason being that they describe interactions between particles (the bosons), and the particles themselves (the fermions). We could think of photons and electrons, just to give two concrete examples of bosons and fermions, respectively. Besides quantum mechanics [15], ladder operators, and the *number representation*, in the last 15 years revealed useful also in the modelization of classical macroscopic systems [4–6]. Ladder operators generalizing fermions and bosons are the quons. They satisfy the q-mutation relations

$$[\mathsf{c}_j, \mathsf{c}_k^\dagger]_q = \delta_{jk}\mathsf{I}, \qquad [\mathsf{c}_j, \mathsf{c}_k]_q = [\mathsf{c}_j^\dagger, \mathsf{c}_k^\dagger]_q = 0,$$

where $-1 \leq q \leq 1$, $[\mathsf{u}, \mathsf{v}]_q = \mathsf{uv} - q\mathsf{vu}$ is the q-mutator. It is easy to ascertain that the operators

$$\left\{\mathsf{c}_j^\dagger \mathsf{c}_k : \ j, k = 1, \ldots, n\right\}$$

built of quonic operators do not form a Lie algebra unless $q = -1$ (fermions) or $q = 1$ (bosons).

Definition 3.8 (*Ideal of a Lie algebra*) An *ideal* of a Lie algebra $\mathbb{L}$ is a subspace $\mathbb{J}$ of $\mathbb{L}$ such that

$$[x, y] \in \mathbb{J}, \ \forall x \in \mathbb{J}, \ \forall y \in \mathbb{L}.$$

As an example, $\mathfrak{sl}(n; \mathbb{K})$ is an ideal of $\mathfrak{gl}(n; \mathbb{K})$.

Ideals in a Lie algebra theory play a role similar to that of normal subgroups in group theory, and two-sided ideals in ring theory. In fact, because the Lie bracket is skew-symmetric, there is no need to distinguish between left and right ideals as in the case of non-commutative rings.

The Lie algebra $\mathbb{L}$ and $\{0\}$ are ideals of $\mathbb{L}$, called the *trivial ideals* of $\mathbb{L}$. Any ideal different from $\mathbb{L}$ is called a *proper ideal*.

Remark 3.2 From the definition, it follows that an ideal is also a subalgebra, but the converse is not necessarily true. For example, $\mathfrak{b}(2;\mathbb{R})$ is a subalgebra of $\mathfrak{gl}(2;\mathbb{R})$ but not an ideal. In fact, taking

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{gl}(2;\mathbb{R}), \qquad \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \in \mathfrak{b}(2;\mathbb{R}),$$

it is

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\right] = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \notin \mathfrak{b}(2;\mathbb{R}).$$

It is possible to introduce some operations to construct new ideals from two given ideals $\mathbb{I}$ and $\mathbb{J}$ of a Lie algebra $\mathbb{L}$. First, we observe that $\mathbb{I} \cap \mathbb{J}$ is an ideal of $\mathbb{L}$. In fact, $\mathbb{I} \cap \mathbb{J}$ is a vector subspace of $\mathbb{L}$, and for $x \in \mathbb{L}$ and $y \in \mathbb{I} \cap \mathbb{J}$, we have $[x, y] \in \mathbb{I}$ since $\mathbb{I}$ is an ideal, and similarly $[x, y] \in \mathbb{J}$. Hence, $[x, y] \in \mathbb{I} \cap \mathbb{J}$.

The union of ideals, however, is not generally an ideal. Anyway, the following result holds true.

Proposition 3.1 *Let $\mathbb{L}$ be a Lie algebra, and let $\mathbb{I}$ and $\mathbb{J}$ be ideals of $\mathbb{L}$. Then*

$$\mathbb{I} + \mathbb{J} = \{x + y \colon x \in \mathbb{I}, y \in \mathbb{J}\}$$

is an ideal.

Proof As $\mathbb{I} + \mathbb{J}$ is a vector subspace of $\mathbb{L}$, let us show that it is an ideal. Let $x \in \mathbb{L}$ and $z \in \mathbb{I} + \mathbb{J}$. Then there exist $y_1 \in \mathbb{I}$ and $y_2 \in \mathbb{J}$ such that $z = y_1 + y_2$. Thus:

$$[x, z] = [x, y_1 + y_2] = [x, y_1] + [x, y_2] \in \mathbb{I} + \mathbb{J}.$$

□

The ideal $\mathbb{I} + \mathbb{J}$ is called the *sum* of the two ideals $\mathbb{I}$ and $\mathbb{J}$.

3.2 Center and Derived Algebras

We can define another operation between ideals: given two ideals $\mathbb{I}$ and $\mathbb{J}$ of a Lie algebra $\mathbb{L}$ over the field $\mathbb{K}$, we denote by $[\mathbb{I}, \mathbb{J}]$ the set

$$[\mathbb{I}, \mathbb{J}] = \left\{ \sum_{i=1}^{n} \alpha_i [x_i, y_i] \colon \alpha_i \in \mathbb{K}, x_i \in \mathbb{I}, y_i \in \mathbb{J}, n \in \mathbb{N} \right\},$$

which is called the *product* of the ideals $\mathbb{I}$ and $\mathbb{J}$.

Proposition 3.2 *Let $\mathbb{L}$ be a Lie algebra over $\mathbb{K}$, and let $\mathbb{I}$ and $\mathbb{J}$ be ideals of $\mathbb{L}$. Then $[\mathbb{I}, \mathbb{J}]$ is an ideal.*

Proof Clearly, $[\mathbb{I}, \mathbb{J}]$ is a vector subspace of $\mathbb{L}$. Let $x \in \mathbb{L}$ and $z \in [\mathbb{I}, \mathbb{J}]$. Then there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$, $x_1, \ldots, x_n \in \mathbb{I}$, and $y_1, \ldots, y_n \in \mathbb{J}$ such that

$$z = \sum_{i=1}^{n} \alpha_i [x_i, y_i].$$

It is

$$\begin{aligned}
[x, z] &= \left[x, \sum_{i=1}^{n} \alpha_i [x_i, y_i] \right] = \sum_{i=1}^{n} \alpha_i [x, [x_i, y_i]] \\
&= \sum_{i=1}^{n} \alpha_i \left(-[x_i, [y_i, x]] - [y_i, [x, x_i]] \right) \\
&= \sum_{i=1}^{n} \alpha_i \left(-[x_i, [y_i, x]] + [[x, x_i], y_i] \right) \in [\mathbb{I}, \mathbb{J}],
\end{aligned}$$

because $x_i \in \mathbb{I}$, $[y_i, x] \in \mathbb{J}$, $[x, x_i] \in \mathbb{I}$, and $y_i \in \mathbb{J}$ for each $i = 1, \ldots, n$. □

From this proposition, it follows that $[\mathbb{L}, \mathbb{L}]$ is an ideal of $\mathbb{L}$; so, we can give the following definition.

Definition 3.9 (*Derived algebra*) Let $\mathbb{L}$ be a Lie algebra. The ideal $\mathbb{L}^{(1)} = [\mathbb{L}, \mathbb{L}]$ is called the *derived algebra* of $\mathbb{L}$.

Given a Lie algebra $\mathbb{L}$ and a subset $S \subseteq \mathbb{L}$, we can define the set of elements of $\mathbb{L}$ that commute with all elements of S:

$$\mathbb{Z}_{\mathbb{L}}(S) = \{x \in \mathbb{L} \colon [x, s] = 0 \text{ for every } s \in S\}.$$

The set $\mathbb{Z}_{\mathbb{L}}(S)$ is called the *centralizer* of S in $\mathbb{L}$. It is easy to verify that $\mathbb{Z}_{\mathbb{L}}(S)$ is a subalgebra of $\mathbb{L}$. Given $x, y \in \mathbb{Z}_{\mathbb{L}}(S)$ and $s \in S$, the Jacobi identity yields

$$[[x, y], s] = -[[y, s], x] - [[s, x], y] = [0, x] - [0, y] = 0,$$

whence $[x, y] \in \mathbb{Z}_{\mathbb{L}}(S)$.

Definition 3.10 (*Center of a Lie algebra*) Let $\mathbb{L}$ be a Lie algebra. The set of elements of $\mathbb{L}$ that commute with all elements of $\mathbb{L}$, say

$$\mathbb{Z}(\mathbb{L}) = \{x \in \mathbb{L} \colon [x, y] = 0 \ \forall y \in \mathbb{L}\},$$

is called the *center* of $\mathbb{L}$.

Proposition 3.3 *Let $\mathbb{L}$ be a Lie algebra. $\mathbb{L}$ is Abelian if and only if $\mathbb{Z}(\mathbb{L}) = \mathbb{L}$.*

Proof If $\mathbb{L}$ is an Abelian Lie algebra, for any $x \in \mathbb{L}$, we have $[x, y] = 0$ for every $y \in \mathbb{L}$, so $x \in \mathbb{Z}(\mathbb{L})$ and thus $\mathbb{L} = \mathbb{Z}(\mathbb{L})$. Conversely, if $\mathbb{L} = \mathbb{Z}(\mathbb{L})$, then $[x, y] = 0$ for all $x, y \in \mathbb{L}$, i.e., $\mathbb{L}$ is Abelian. □

Note that the center of $\mathbb{L}$ is the centralizer of $\mathbb{L}$ itself, i.e., $\mathbb{Z}(\mathbb{L}) = \mathbb{Z}_{\mathbb{L}}(\mathbb{L})$. The following property holds.

Proposition 3.4 *Let $\mathbb{L}$ be a Lie algebra over a field $\mathbb{K}$. Then $\mathbb{Z}(\mathbb{L})$ is an ideal.*

Proof $\mathbb{Z}(\mathbb{L})$ is a vector subspace of $\mathbb{L}$: taking $\alpha, \beta \in \mathbb{K}$, $x, y \in \mathbb{Z}(\mathbb{L})$, and $z \in \mathbb{L}$, it is

$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z] = 0,$$

so that $\alpha x + \beta y \in \mathbb{Z}(\mathbb{L})$. Moreover, let $x \in \mathbb{L}$ and $y \in \mathbb{Z}(\mathbb{L})$, for any $z \in \mathbb{L}$; using the Jacobi identity, we have

$$[[x, y], z] = -[[y, z], x] - [[z, x], y] = -[0, x] - 0 = 0,$$

i.e., $[x, y] \in \mathbb{Z}(\mathbb{L})$. □

Given a vector space V and a subspace $W \subseteq V$, it is known that we can construct the quotient vector space V/W. Let us now define the quotient Lie algebra.

Let $\mathbb{L}$ be a Lie algebra over $\mathbb{K}$, and let $\mathbb{I}$ be an ideal of $\mathbb{L}$. We know that $\mathbb{I}$ is a particular subspace of $\mathbb{L}$; so, we can consider the quotient space $\mathbb{L}/\mathbb{I}$.

Theorem 3.1 (Quotient Lie algebra) *Let $\mathbb{L}$ be a Lie algebra over $\mathbb{K}$, and let $\mathbb{I}$ be an ideal of $\mathbb{L}$. On $\mathbb{L}/\mathbb{I}$, let us define the bracket*

$$\begin{aligned} &[\cdot, \cdot] : \mathbb{L}/\mathbb{I} \times \mathbb{L}/\mathbb{I} \to \mathbb{L}/\mathbb{I} \\ &(x + \mathbb{I}, y + \mathbb{I}) \mapsto [x + \mathbb{I}, y + \mathbb{I}] = [x, y] + \mathbb{I}. \end{aligned}$$

The vector space $\mathbb{L}/\mathbb{I}$ is a Lie algebra called quotient Lie algebra *(or* factor Lie algebra*) over $\mathbb{K}$.*

Proof The Lie bracket is well defined. For $x, y \in \mathbb{L}$, it is $[x + \mathbb{I}, y + \mathbb{I}] = [x, y] + \mathbb{I}$. Let $x', y' \in \mathbb{L}$ such that $x + \mathbb{I} = x' + \mathbb{I}$ and $y + \mathbb{I} = y' + \mathbb{I}$. We need to show that

$$[x + \mathbb{I}, y + \mathbb{I}] = [x' + \mathbb{I}, y' + \mathbb{I}] \iff [x, y] - [x', y'] \in \mathbb{I}.$$

Since x is equivalent to x', there exists $z_1 \in \mathbb{I}$ such that $x = x' + z_1$. Similarly, there exists $z_2 \in \mathbb{I}$ such that $y = y' + z_2$. Then

$$[x, y] - [x', y'] = [x' + z_1, y' + z_2] - [x', y'] = [x', z_2] + [z_1, y'] + [z_1, z_2] \in \mathbb{I}.$$

The Lie bracket is bilinear. We prove linearity in the first argument. Let $\alpha, \beta \in \mathbb{K}$ and $x, y, z \in \mathbb{L}$. We have

$$\begin{aligned}
[\alpha(x+\mathbb{I})+\beta(y+\mathbb{I}), z+\mathbb{I}] &= [(\alpha x+\mathbb{I})+(\beta y+\mathbb{I}), z+\mathbb{I}] \\
&= [(\alpha x+\beta y)+\mathbb{I}, z+\mathbb{I}] \\
&= [\alpha x+\beta y, z]+\mathbb{I} \\
&= (\alpha[x,z]+\beta[y,z])+\mathbb{I} \\
&= \alpha([x,z]+\mathbb{I})+\beta([y,z]+\mathbb{I}) \\
&= \alpha[x+\mathbb{I}, z+\mathbb{I}]+\beta[y+\mathbb{I}, z+\mathbb{I}].
\end{aligned}$$

Moreover, $[\cdot,\cdot]$ is skew-symmetric and satisfies Jacobi identity. Let $x, y \in \mathbb{L}$,

$$[x+\mathbb{I}, y+\mathbb{I}] = [x,y]+\mathbb{I} = -[y,x]+\mathbb{I} = -[y+\mathbb{I}, x+\mathbb{I}].$$

Finally,

$$\begin{aligned}
&[x+\mathbb{I}, [y+\mathbb{I}, z+\mathbb{I}]]+[y+\mathbb{I}, [z+\mathbb{I}, x+\mathbb{I}]]+[z+\mathbb{I}, [x+\mathbb{I}, y+\mathbb{I}]] \\
&\quad = [x+\mathbb{I}, [y,z]+\mathbb{I}]+[y+\mathbb{I}, [z,x]+\mathbb{I}]+[z+\mathbb{I}, [x,y]+\mathbb{I}] \\
&\quad = ([x,[y,z]]+\mathbb{I})+([y,[z,x]]+\mathbb{I})+([z,[x,y]]+\mathbb{I}) \\
&\quad = ([x,[y,z]]+[y,[z,x]]+[z,[x,y]])+\mathbb{I} \\
&\quad = 0+\mathbb{I} = \mathbb{I}.
\end{aligned}$$

□

3.3 Solvable, Nilpotent, Simple, and Semisimple Lie Algebras

The ideals constitute an important ingredient for classifying Lie algebras. Previously, we introduced a special ideal of a Lie algebra $\mathbb{L}$ known as the derived algebra: $\mathbb{L}^{(1)} = [\mathbb{L}, \mathbb{L}]$. The following lemma holds.

Lemma 3.1 *Let $\mathbb{L}$ be a Lie algebra and $\mathbb{I} \subseteq \mathbb{L}$ an ideal. Then the quotient $\mathbb{L}/\mathbb{I}$ is Abelian if and only if $\mathbb{I}$ contains the derived algebra $\mathbb{L}^{(1)}$.*

Proof The quotient $\mathbb{L}/\mathbb{I}$ is Abelian if and only if for every $x, y \in \mathbb{L}$ we have

$$[x+\mathbb{I}, y+\mathbb{I}] = [x,y]+\mathbb{I} = \mathbb{I},$$

i.e., for every $x, y \in \mathbb{L}$, we have $[x, y] \in \mathbb{I}$. This holds if and only if $\mathbb{L}^{(1)} \subseteq \mathbb{I}$, because $\mathbb{I}$ is a vector subspace of $\mathbb{L}$, and $\mathbb{L}^{(1)}$ is generated by the set $\{[x, y]\colon x, y \in \mathbb{L}\}$. □

This lemma says that the derived algebra $\mathbb{L}^{(1)}$ is the smallest ideal of $\mathbb{L}$ such that the quotient $\mathbb{L}/\mathbb{L}^{(1)}$ is Abelian. By a similar reasoning, we can say that the derived algebra of $\mathbb{L}^{(1)}$, denoted by $\mathbb{L}^{(2)}$, is the smallest ideal of $\mathbb{L}^{(1)}$ such that the quotient $\mathbb{L}^{(1)}/\mathbb{L}^{(2)}$ is Abelian, and so on. Thus, we can define the *derived series* of $\mathbb{L}$ as follows.

Definition 3.11 (*Derived series of a Lie algebra*) Let $\mathbb{L}$ be a Lie algebra. Consider the ideals

$$\mathbb{L}^{(0)} = \mathbb{L}, \quad \mathbb{L}^{(1)} = [\mathbb{L}, \mathbb{L}], \quad \ldots, \quad \mathbb{L}^{(k)} = [\mathbb{L}^{(k-1)}, \mathbb{L}^{(k-1)}], \quad \ldots.$$

The sequence $\mathbb{L} \supseteq \mathbb{L}^{(1)} \supseteq \mathbb{L}^{(2)} \supseteq \cdots$ is called the *derived series* of $\mathbb{L}$.

This notion is useful to define *solvable* algebras.

Definition 3.12 (*Solvable Lie algebra*) A Lie algebra $\mathbb{L}$ is said to be *solvable* if there exists a finite positive integer m such that $\mathbb{L}^{(m)} = \{0\}$.

It is easily recognized that an Abelian Lie algebra is solvable. The reason of the attribute "solvable" (coined by Sophus Lie himself) to these algebras relies on their remarkable property of allowing the multiple order reduction of an ordinary differential equation.

If $\mathbb{L}$ is solvable, then the derived series of $\mathbb{L}$ gives an *approximation* of $\mathbb{L}$ via a finite sequence of ideals whose quotients are Abelian. The converse also holds.

Lemma 3.2 *Let $\mathbb{L}$ be a Lie algebra, and let $\mathbb{I}_0, \mathbb{I}_1, \ldots, \mathbb{I}_m$, with $m \in \mathbb{N}$, be ideals of $\mathbb{L}$ such that*

$$\mathbb{L} = \mathbb{I}_0 \supseteq \mathbb{I}_1 \supseteq \cdots \supseteq \mathbb{I}_{m-1} \supseteq \mathbb{I}_m = \{0\},$$

and the quotients $\mathbb{I}_{k-1}/\mathbb{I}_k$ are Abelian for every $1 \leq k < m$. Then $\mathbb{L}$ is solvable.

Proof We will show that $\mathbb{L}^{(m)} = \{0\}$. By hypothesis, the quotient $\mathbb{L}/\mathbb{I}_1$ is Abelian, so using Lemma 3.1, we have $\mathbb{L}^{(1)} \subseteq \mathbb{I}_1$. Suppose this inclusion holds for $k-1$, i.e., $\mathbb{L}^{(k-1)} \subseteq \mathbb{I}_{k-1}$, and prove it for k. By assumption, the quotient $\mathbb{I}_{k-1}/\mathbb{I}_k$ is Abelian, so by Lemma 3.1, we have $[\mathbb{I}_{k-1}, \mathbb{I}_{k-1}] \subseteq \mathbb{I}_k$. By the inductive hypothesis, we have $\mathbb{L}^{(k-1)} \subseteq \mathbb{I}_{k-1}$, so that

$$\mathbb{L}^{(k)} = [\mathbb{L}^{(k-1)}, \mathbb{L}^{(k-1)}] \subseteq [\mathbb{I}_{k-1}, \mathbb{I}_{k-1}] \subseteq \mathbb{I}_k,$$

and the claim remains proved. □

Theorem 3.2 *For any r-dimensional solvable Lie algebra $\mathbb{L}_r$, there exists a chain*

$$\mathbb{L}_r \supset \mathbb{L}_{r-1} \supset \ldots \supset \mathbb{L}_1$$

of subalgebras of dimensions $r, r-1, \ldots, 1$, respectively, such that $\mathbb{L}_s$ is an ideal of $\mathbb{L}_{s+1}$ for $s = 1, \ldots, r-1$.

Proof Let $\mathbb{L}_r^{(m)} = \{0\}$, and let m $(0 < m < \infty)$ be the smallest positive integer such that $\mathbb{L}_r^{(m)} = \{0\}$. It follows that $\mathbb{L}_r^{(m-1)}$ is Abelian or one-dimensional.

Let us show the existence of a chain of ideals of dimensions from 1 to r. If $\mathbb{L}_r^{(m-1)}$ is k-dimensional $(1 < k < r)$, we can easily construct (in many ways) a complete chain of ideals $\mathbb{L}_j$ of dimension j $(1 \leq j < k)$. So, we have the chain

$$\mathbb{L}_r \supset \mathbb{L}_r^{(1)} \ldots \supset \mathbb{L}_r^{(m-1)} \supset \mathbb{L}_{k-1} \supset \ldots \mathbb{L}_1.$$

If this is not the required complete chain of ideals (in the sense that the length of the subchain $\mathbb{L}_r \supset \mathbb{L}_r^{(1)} \ldots \supset \mathbb{L}_r^{(m-1)}$ is less than $r - k + 1$), then we may fill the gaps constructing the intermediate ideals. This is done by taking into account that adding an element of $\mathbb{L}_r^{(j)}$ to $\mathbb{L}_r^{(j-1)}$ produces an ideal one dimension higher than $\dim(\mathbb{L}_r^{(j-1)})$. We have to continue this procedure until all the gaps in the chain of ideals are filled. □

Two-dimensional Lie algebras are solvable, as stated by the following theorem.

Theorem 3.3 *Every two-dimensional Lie algebra is solvable.*

Proof Given the basis $\{x_1, x_2\}$ of a two-dimensional Lie algebra, two cases may arise:

1. $[x_1, x_2] = 0$, whence the Lie algebra is Abelian and trivially solvable;
2. $[x_1, x_2] \neq 0$: in this case the Lie bracket gives an element of the vector space spanned by x_1 and x_2, say

$$[x_1, x_2] = ax_1 + bx_2,$$

for some constants a and b not both vanishing; if $ab = 0$ all is ok; if $ab \neq 0$, we choose as basis

$$\{\widetilde{x}_1, \widetilde{x}_2\} = \left\{ax_1 + bx_2, \frac{1}{a}x_2\right\},$$

so that

$$[\widetilde{x}_1, \widetilde{x}_2] = \widetilde{x}_1.$$

□

Another important class of Lie algebras is that of *nilpotent* Lie algebras.

Definition 3.13 (*Central descending series*) Let $\mathbb{L}$ be a Lie algebra. Consider the ideals

$$\mathbb{L}^0 = \mathbb{L}, \quad \mathbb{L}^1 = \mathbb{L}^{(1)}, \quad \mathbb{L}^2 = [\mathbb{L}, \mathbb{L}^1], \quad \ldots, \quad \mathbb{L}^k = [\mathbb{L}, \mathbb{L}^{k-1}], \quad \ldots.$$

The sequence $\mathbb{L} \supseteq \mathbb{L}^1 \supseteq \mathbb{L}^2 \supseteq \cdots$ is called the *central descending series*.

Definition 3.14 (*Nilpotent Lie algebra*) A Lie algebra $\mathbb{L}$ is said to be *nilpotent* if there exists a finite positive integer m such that $\mathbb{L}^m = \{0\}$.

Remark 3.3 Every nilpotent Lie algebra is solvable. The converse is not generally true. For example, the Lie algebra $\mathfrak{b}(n; \mathbb{K})$ of upper-triangular $n \times n$ matrices is solvable, but it is not nilpotent for $n \geq 2$.

Definition 3.15 (*Simple Lie algebra*) A Lie algebra $\mathbb{L}$ is said to be *simple* if $\mathbb{L}$ has only trivial ideals.

Of course, simple Lie algebras cannot be Abelian.

Example 3.9 Let $\mathbb{L}$ be the three-dimensional Lie algebra with basis $\{x_1, x_2, x_3\}$ with Lie bracket

$$[x_1, x_2] = x_3, \qquad [x_2, x_3] = x_1, \qquad [x_3, x_1] = x_2.$$

It is easy to verify that this Lie algebra has no proper ideals. This algebra can be realized taking the Lie algebra associated to the group SO(3) of the rotations in $\mathbb{R}^3$.

Definition 3.16 (*Semisimple Lie algebra*) A Lie algebra is said to be *semisimple* if it is the direct sum of two simple Lie algebras.

A semisimple Lie algebra has neither solvable nor Abelian ideals different from $\{0\}$.

3.4 Structure Constants

If $\mathbb{L}_r$ is an r-dimensional Lie algebra with basis $\{x_1, \ldots, x_r\}$, any element x of the Lie algebra Ξ can be represented as

$$x = \sum_{i=1}^{r} f^i x_i,$$

where the $f^i \in \mathbb{K}$, $i = 1, \ldots, r$, are constants.

The Lie bracket of any two elements in the basis of $\mathbb{L}_r$ is a linear combination of the basis elements, say

$$[x_i, x_j] = \sum_{k=1}^{r} c_{ij}^k x_k, \qquad i, j = 1, \ldots, r, \tag{3.5}$$

where the constants c_{ij}^k are called *structure constants*, and Eqs. (3.5) are known as *commutation relations*.

As the Lie bracket is skew-symmetric and satisfies Jacobi identity, we have some constraints on the structure constants, namely:

$$c_{ij}^k = -c_{ji}^k,$$
$$\sum_{m=1}^{r} \left(c_{ij}^m c_{mk}^\ell + c_{jk}^m c_{mi}^\ell + c_{ki}^m c_{mj}^\ell\right) = 0,$$

$\forall\, i, j, k, \ell \in \{1, \ldots, r\}$.

For an Abelian Lie algebra all structure constants are zero. Moreover, the property of a Lie algebra to be solvable has a straightforward consequence on their structure constants.

Theorem 3.4 *An r-dimensional Lie algebra spanned by $\{x_1, \ldots, x_r\}$ is solvable if it is possible to choose the basis in such a way*

$$[x_i, x_j] = \sum_{k=1}^{j-1} c_{ij}^k x_k, \qquad 1 \leq i < j \leq r.$$

Proof The proof follows by choosing a basis such that the chain of subalgebras of dimensions $r, r-1, \ldots, 1$, respectively, with $\mathbb{L}_s$ ideal of $\mathbb{L}_{s+1}$ for $s = 1, \ldots, r-1$ have bases

$$\{x_1, \ldots, x_r\} \supset \{x_1, \ldots, x_{r-1}\} \supset \ldots \supset \{x_1, , x_2\} \supset \{x_1\}.$$

□

It is convenient to display the commutators of a Lie algebra through its *commutator table* (see Examples 2.24 and 2.25), whose (i, j) entry is $[x_i, x_j]$: the commutator table is a skew-symmetric matrix. Of course, the structure constants are easily deduced from the commutator table.

Structure constants enter the Cartan criterion for semisimple Lie algebras: an r-dimensional Lie algebra is semisimple if and only if the matrix whose (i, ℓ) entry is $\sum_{j,k=1}^{r} c_{ij}^k c_{\ell k}^j$ is nonsingular.

3.5 Homomorphisms of Lie Algebras

Definition 3.17 (*Lie algebra homomorphism*) If $\mathbb{L}$ and $\mathbb{L}'$ are Lie algebras over the field $\mathbb{K}$, a map

$$\varphi : \mathbb{L} \to \mathbb{L}'$$

is a *Lie algebra homomorphism* if it is linear and

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]' \qquad \forall\, x, y \in \mathbb{L},$$

where $[\cdot, \cdot]$ and $[\cdot, \cdot]'$ are the Lie brackets in $\mathbb{L}$ and $\mathbb{L}'$, respectively; φ is an *isomorphism* if it is also bijective ($\mathbb{L} \cong \mathbb{L}'$). An isomorphism from $\mathbb{L}$ into $\mathbb{L}$ is called *automorphism*. The set of all automorphisms of $\mathbb{L}$ is a group denoted by $\mathrm{Aut}(\mathbb{L})$.

In the finite-dimensional case, isomorphic Lie algebras have the same dimension.

If $\mathbb{J}$ is an ideal of $\mathbb{L}$, there is a *natural homomorphism* φ of the Lie algebra $\mathbb{L}$ on the factor algebra $\mathbb{L}/\mathbb{J}$, defined by $\varphi(u) = U$, if $u \in U$. The ideal $\mathbb{J}$ is the kernel of this homomorphism.

Theorem 3.5 (Isomorphism theorems)

1. *Let $\varphi : \mathbb{L} \to \mathbb{L}'$ be a homomorphism of Lie algebras. Then $Ker\,\varphi$ is an ideal of $\mathbb{L}$ and $Im\,\varphi$ is a subalgebra of $\mathbb{L}'$, and*

$$\mathbb{L}/Ker\,\varphi \cong Im\,\varphi.$$

2. *If $\mathbb{I}$ and $\mathbb{J}$ are ideals of a Lie algebra $\mathbb{L}$, then $(\mathbb{I} + \mathbb{J})/\mathbb{J} \cong \mathbb{I}/(\mathbb{I} \cap \mathbb{J})$.*
3. *If $\mathbb{I}$ and $\mathbb{J}$ are ideals of a Lie algebra $\mathbb{L}$ such that $\mathbb{I} \subseteq \mathbb{J}$, then $\mathbb{J}/\mathbb{I}$ is an ideal of $\mathbb{L}/\mathbb{I}$ and $(\mathbb{L}/\mathbb{I})/(\mathbb{J}/\mathbb{I}) \cong \mathbb{L}/\mathbb{J}$.*

If $\mathbb{L}$ is a Lie algebra and V is a vector space, a *linear representation* of $\mathbb{L}$ in V is a homomorphism

$$\rho : \mathbb{L} \to \mathrm{GL}(V).$$

Every Lie algebra defines, by its own structure, one linear representation, called the *adjoint representation*

$$\begin{aligned} &\mathrm{ad} : \mathbb{L} \to \mathfrak{gl}(\mathbb{L}), \\ &x \mapsto \mathrm{ad}_x(y) = [x, y] \qquad \text{for } x, y \in \mathbb{L}. \end{aligned}$$

It follows from the bilinearity of the Lie bracket that the map ad_x is linear for each $x \in \mathbb{L}$ and the map $x \mapsto \mathrm{ad}_x$ is itself linear. Therefore, to show that ad is a homomorphism, it needs to check that

$$\mathrm{ad}_{[x,y]} = \mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x \quad \forall\, x, y \in \mathbb{L},$$

but this is equivalent to Jacobi identity.

The kernel of ad is the center of $\mathbb{L}$.

Definition 3.18 (*Derivation of a Lie algebra*) A *derivation* of $\mathbb{L}$ is a linear map

$$\gamma : \mathbb{L} \to \mathbb{L}$$

such that

$$\gamma([x, y]) = [\gamma(x), y] + [x, \gamma(y)], \ \forall x, y \in \mathbb{L}.$$

The derivations of $\mathbb{L}$ form a Lie algebra with the bracket

$$[\gamma_1, \gamma_2] = \gamma_1 \circ \gamma_2 - \gamma_2 \circ \gamma_1.$$

This Lie algebra, denoted by $\mathrm{Der}(\mathbb{L})$, is called *algebra of derivations of the Lie algebra* $\mathbb{L}$.

The elements of $\mathrm{ad}(\mathbb{L})$ are special derivations called *inner derivations*. In other words, an inner derivation is a map

$$\begin{aligned} &\gamma : \mathbb{L} \to \mathbb{L}, \\ &y \mapsto \gamma(y) = [x, y] \quad \text{for some } x \in \mathbb{L}. \end{aligned}$$

Each derivation $\gamma \in \mathbb{L}$ determines an automorphism, denoted by $\exp(\gamma)$ and defined by

$$\exp(\gamma)(y) = y + \gamma(y) + \frac{\gamma^2(y)}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{\gamma^j(y)}{j!}. \tag{3.6}$$

Definition 3.19 (*Inner automorphisms*) Using Formula (3.6) with an inner derivation $\gamma = \mathrm{ad}_x$, we get

$$\exp(\mathrm{ad}_x(y)) = y + [x, y] + \frac{1}{2}[x, [x, y]] + \cdots \tag{3.7}$$

The *inner automorphisms* of $\mathbb{L}$ consist of the smallest subgroup of $\mathrm{Aut}(\mathbb{L})$, containing all automorphisms of form (3.7), where x runs through all elements of $\mathbb{L}$. This group is denoted by $\mathrm{Int}(\mathbb{L})$.

3.6 Lie Subgroups and Lie Subalgebras

Let us consider a group G, and use the notation $\omega(a, b) = a \cdot b$ $(a, b \in G)$ for the group operation. An automorphism of the group G is a bijective map

$$\phi : G \to G$$

such that

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b).$$

In the group of all automorphisms of the group G, there are the automorphisms

$$\phi_a : G \to G, \qquad \forall a \in G$$
$$\phi_a(b) = a \cdot b \cdot a^{-1},$$

called *inner automorphisms* of the group G. As $a \cdot b \cdot a^{-1} = b$ is equivalent to saying $a \cdot b = b \cdot a$, the existence and number of inner automorphisms different from the identity is a sort of measure of the failure of the commutative law in the group.

Definition 3.20 (*Conjugate subgroups*) A subgroup $H_1 \subseteq G$ is *conjugate* to a subgroup $H_2 \subseteq G$ if there exists $a \in G$ such that $H_2 = aH_1a^{-1}$, i.e., the subgroups H_1 and H_2 are connected by inner automorphisms of the group.

This relation of conjugacy between subgroups is an equivalence relation, and the corresponding equivalence classes are said *conjugacy classes*.

Considering an r-parameter Lie group of point transformations, we can associate to it the Lie algebras of infinitesimal generators. Classifying the subgroups of a Lie group up to conjugacy can be reduced to the corresponding problem of classification of Lie subalgebras. In the remainder of this chapter, let us refer to real Lie algebras.

The determination of similar Lie subalgebras, and then associating to them similar Lie subgroups, is more effective algorithmically because the group of inner automorphisms of a Lie algebra $\mathbb{L}$, $\mathrm{Int}(\mathbb{L})$, is always a group of linear transformations on the main space, whereas the group $\mathrm{Int}(G)$ is necessarily not.

Let $\mathbb{L}_r$ be a real r-dimensional Lie algebra with basis $\{x_1, \ldots, x_r\}$; after fixing the basis, $\mathbb{L}_r$ is uniquely characterized by its structure constants. By taking two elements $x, y \in \mathbb{L}_r$, they can be decomposed according to the basis in the form:

$$x = \sum_{i=1}^{r} f^i x_i, \qquad y = \sum_{i=1}^{r} g^i x_i,$$

where f^i and g^i are suitable elements belonging to $\mathbb{R}$, whence it follows

$$[x, y] = \sum_{i,j=1}^{r} f^i g^j \left[x_i, x_j\right] = \sum_{i,j,k=1}^{r} f^i g^j c_{ij}^k x_k. \tag{3.8}$$

Relation (3.8) implies that, for the *coordinates* $\mathbf{f} \equiv (f^1, \ldots, f^r)$ and $\mathbf{g} \equiv (g^1, \ldots, g^r)$ of the elements x and y with respect to the basis $\{x_1, \ldots, x_r\}$, we may introduce a Lie bracket in $\mathbb{R}^r$ such that

$$\left[\mathbf{f}, \mathbf{g}\right]^k = \sum_{i,j=1}^{r} f^i g^j c_{ij}^k, \qquad k = 1, \ldots, r.$$

We note that any $x \in \mathbb{L}$ spans a one-dimensional subalgebra with elements ax ($a \in \mathbb{R}$), i.e., one-dimensional subalgebras always exist in any Lie algebra $\mathbb{L}$. On the contrary, as far as the two-dimensional subalgebras of a real Lie algebra $\mathbb{L}$ are concerned, their existence is not always guaranteed (for instance, the three-dimensional Lie algebra in Example 3.9 does not possess two-dimensional subalgebras).

Let $\{x_1, \ldots, x_r\}$ be a basis of the real Lie algebra $\mathbb{L}_r$, and c_{ij}^k its structure constants in this basis. A generic Lie subalgebra of dimension $1 < d < r$ is spanned by the elements

$$y_i = \sum_{k=1}^{r} h_i^k x_k, \qquad i = 1, \ldots, d, \tag{3.9}$$

where the $d \times r$ matrix $\left\| h_i^\alpha \right\|$ has rank d, provided that

$$\left[y_i, y_j \right] = \sum_{k=1}^{d} \lambda_{ij}^k x_k, \qquad i, j = 1, \ldots, d, \tag{3.10}$$

λ_{ij}^k being suitable constants.

By using (3.9), relations (3.10) yield

$$\sum_{\ell,m=1}^{r} h_i^\ell h_j^m c_{\ell m}^n = \sum_{k=1}^{d} \lambda_{ij}^k h_k^n, \qquad i, j = 1, \ldots, d, n = 1, \ldots, r, \tag{3.11}$$

clarifying the fact that the determination of a Lie subalgebra requires the solution of a system of quadratic equations. In fact, what is needed is to find the entries h_i^k of a matrix with rank d such that relations (3.11) hold true for some constants λ_{ij}^k. Of course, this problem can be addressed using a computer and suitable algorithms based, for instance, on Gröbner bases; nevertheless, this can be computationally hard for high values of r and d. Luigi Bianchi was aware of this circumstance when in his book of 1918 [7] he wrote:

> Il problema è così ricondotto, come si era enunciato, ad un problema algebrico. In pratica però questo metodo, teoricamente soddisfacente, offrirebbe presto difficoltà insormontabili, e val meglio sostituirlo con un altro che utilizza ulteriori proprietà del gruppo aggiunto.[1]

Let $\mathbb{L}_r$ be a real Lie algebra; as already observed, the vector space $\mathbb{R}^r$ gains the structure of a Lie algebra sharing the same structure constants as $\mathbb{L}_r$. Due to this, we may consider a different perspective to the computation of the inner automorphisms of a finite-dimensional Lie algebra that reveals useful for classifying Lie subalgebras.

[1] The problem is thus brought back, as it was stated, to an algebraic problem. In practice, this method, theoretically satisfactory, would soon give rise to insurmountable difficulties, and it is better to replace it with another one that uses further properties of the adjoint group.

Definition 3.21 Let $\mathbb{L}_r$ an r-dimensional Lie algebra, and introduce the Lie algebra $\mathbb{L}_r^A$ spanned by the following first-order differential operators:

$$E_j = \sum_{i,k=1}^{r} c_{ij}^k f^i \frac{\partial}{\partial f^k}, \qquad j = 1, \ldots, r.$$

The algebra $\mathbb{L}_r^A$ generates through the integration of the Lie equations,

$$\frac{d\widetilde{f}^k}{dt} = \sum_{i=1}^{r} c_{ij}^k \widetilde{f}^i, \qquad j, k = 1, \ldots, r,$$
$$\widetilde{f}^k(0) = f^k,$$

the group of inner automorphisms of the Lie algebra $\mathbb{L}_r$.

Definition 3.22 (*Similar subalgebras*) Two Lie subalgebras $\mathbb{L}'$ and $\mathbb{L}''$ (with the same dimension) of a Lie algebra $\mathbb{L}$ are similar if there exists an inner automorphism $\phi \in \mathrm{Int}(\mathbb{L})$ such that $\phi(\mathbb{L}') = \mathbb{L}''$. The similarity is an equivalence relation, indeed $\mathrm{id}_{\mathbb{L}} \in \mathrm{Int}(\mathbb{L})$, and if $\phi, \psi \in \mathrm{Int}(\mathbb{L})$ then $\phi^{-1}, \phi \circ \psi \in \mathrm{Int}(\mathbb{L})$.

Any subalgebra of a Lie algebra $\mathbb{L}_r$ is completely defined by its generators, which are linear combinations of the generators of the Lie algebra $\mathbb{L}_r$. Hence, the subalgebra is completely defined by the coefficients of these linear combinations. Taking into account Definition 3.21, actions of automorphisms can be transferred to these coefficients.

Definition 3.23 (*Optimal subalgebras* [16, 18, 19]) For a given finite-dimensional Lie algebra $\mathbb{L}_r$, with inner automorphisms $\mathcal{A} \equiv \mathrm{Int}(\mathbb{L}_r)$, the optimal system of d-dimensional Lie subalgebras $\Theta_{\mathcal{A}}^d(\mathbb{L}_r)$ is the set of representatives of the classes of similar subalgebras [18, 19]. Then, the complete optimal system of Lie subalgebras is

$$\Theta_{\mathcal{A}}(\mathbb{L}_r) \equiv \{\Theta_{\mathcal{A}}^1(\mathbb{L}_r), \Theta_{\mathcal{A}}^2(\mathbb{L}_r), \ldots, \Theta_{\mathcal{A}}'^{r-1}(\mathbb{L}_r)\}.$$

We observe that an optimal system of Lie subalgebras is not unique, and different algorithms for its construction lead to different systems $\Theta_{\mathcal{A}}(\mathbb{L}_r)$. All these systems are equivalent in the following sense: if $\Theta_{\mathcal{A}}(\mathbb{L}_r)$ and $\Theta'_{\mathcal{A}}(\mathbb{L}_r)$ are two optimal systems, for any $\theta \in \Theta_{\mathcal{A}}(\mathbb{L}_r)$ there is an automorphism $\phi \in \mathcal{A}$ such that $\phi(\theta) \in \Theta'_{\mathcal{A}}(\mathbb{L}_r)$, and vice versa.

Thus, the optimal system of subalgebras of a Lie algebra $\mathbb{L}_r$ with inner automorphisms $\mathcal{A} = \mathrm{Int}(\mathbb{L}_r)$ is a set of subalgebras $\Theta_{\mathcal{A}}(\mathbb{L}_r)$ such that

1. there are no two elements of this set which can be transformed into each other by inner automorphisms of the Lie algebra $\mathbb{L}_r$;
2. any subalgebra of the Lie algebra $\mathbb{L}_r$ can be transformed into one of subalgebras of the set $\Theta_{\mathcal{A}}(\mathbb{L}_r)$.

The union of the elements of the optimal system of given dimensionality s is called *optimal system of order s* and denoted by the symbol $\Theta^s_{\mathcal{A}}$; since the dimension of an algebra is invariant under automorphisms, the solution of the classification problem for a finite-dimensional Lie algebra $\mathbb{L}_r$ must be tables of optimal systems for every $s = 1, \ldots, r-1$.

Besides the inner automorphisms, one has also to take into account a uniform scaling of all generators: in fact, any subalgebra is transformed into a similar subalgebra under this operation. The problem of constructing an optimal system of subalgebras reduces to obtaining the maximum possible number of zero coordinates in the basis of the subalgebras.

The determination of optimal systems is relatively easy for Lie algebras of small dimensionality, but also in this circumstance the solution is accomplished by a method requiring choices from a few possibilities at certain stages of the work, and the *simplicity* of the results obtained is not always clear.

Example 3.10 ([14]) Consider the three-dimensional solvable Lie algebra with basis $\{x_1, x_2, x_3\}$ whose nonzero Lie brackets are

$$[x_1, x_3] = x_1, \qquad [x_2, x_3] = x_2.$$

The inner automorphisms are given by

$$A_1 = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -t_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \exp(t_3) & 0 & 0 \\ 0 & \exp(t_3) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where t_1, t_2, and t_3 are the parameters of the automorphisms A_1, A_2, and A_3, respectively.

The two-dimensional subalgebras can be of two types: subalgebras including the generator x_3, and those without x_3. There is only one subalgebra without the generator x_3:

$$\{x_1, x_2\}.$$

For the second type of subalgebras, after scaling and taking into account linear combinations, a basis of them can be chosen as follows:

$$\{q_1 x_1 + q_2 x_2 + x_3,\ p_1 x_1 + p_2 x_2\},$$

with p_1 and p_2 such that $p_1^2 + p_2^2 \neq 0$.

By action of the automorphisms A_1 and A_2, the coefficients q_1 and q_2 are transformed to zero, whereas the coefficients p_1 and p_2 remain unchanged. Hence, all classes of these subalgebras are

$$\{x_2, x_3\}, \qquad \{x_1 + p_2 x_2, x_3\},$$

with an arbitrary p_2. Thus, the list of all dissimilar two-dimensional subalgebras consists of the subalgebras

$$\{x_1, x_2\}, \quad \{x_2, x_3\}, \quad \{x_1 + p_2 x_2, x_3\}.$$

Similarly one can classify all one-dimensional dissimilar subalgebras. They are

$$\{x_1 + p_2 x_2\}, \qquad \{x_2\}, \qquad \{x_3\}.$$

As we will show in Chap. 6, the characterization of systems of optimal Lie subalgebras is very important in the applications of Lie group theory to differential equations. For instance, the optimal systems of subalgebras of the algebra of point symmetries of a partial differential equation serve to characterize inequivalent group invariant solutions. Moreover, in Chap. 12, we will illustrate the package SYMBOLIE [1–3], able to compute optimal systems of families of Lie subalgebras of a finite-dimensional Lie algebra.

3.7 Lie Algebras of Operators

The basis of a generic Lie algebra has been heretofore denoted as $\{x_1, \dots, x_r\}$, and the structure constants as c_{ij}^k. In the following, as we are mainly interested to finite-dimensional Lie algebras spanned by infinitesimal generators of symmetries of differential equations, we will denote the basis as $\{\Xi_1, \dots, \Xi_r\}$ (this is the notation used also in SYMBOLIE [3]), and the corresponding structure constants as C_{ij}^k. Moreover, even if real ($\mathbb{K} = \mathbb{R}$) and complex ($\mathbb{K} = \mathbb{C}$) Lie algebras are of special relevance in the application of Lie groups of transformations to differential equations, we shall restrict ourselves to consider real Lie algebras.

As proved in Chap. 2, an r-parameter Lie group of transformations in $\mathbb{R}^n$ is characterized by its infinitesimal operators

$$\Xi_\mu = \sum_{i=1}^{n} \zeta_{\mu i}(\mathbf{z}) \frac{\partial}{\partial z_i}, \qquad \mu = 1, \dots, r.$$

We also defined the Lie bracket (or commutator) of two operators Ξ_μ and Ξ_ν,

$$[\Xi_\mu, \Xi_\nu] = \Xi_\mu \Xi_\nu - \Xi_\nu \Xi_\mu,$$

still producing a first-order differential operator, say

$$[\Xi_\mu, \Xi_\nu] = \sum_{j=1}^{n} \widehat{\zeta}_j(\mathbf{z}) \frac{\partial}{\partial z_j},$$

where

$$\widehat{\zeta}_j(\mathbf{z}) = \sum_{i=1}^{n} \left(\zeta_{\mu i}(\mathbf{z}) \frac{\partial \zeta_{\nu j}(\mathbf{z})}{\partial z_i} - \zeta_{\nu i}(\mathbf{z}) \frac{\partial \zeta_{\mu j}(\mathbf{z})}{\partial z_i} \right).$$

As a consequence of this definition, the operation of commutation is bilinear, skew-symmetric, and satisfies the Jacobi identity.

Moreover, we also proved that the infinitesimal generators of a multi-parameter Lie group of transformations are closed under the operation of commutation, and that the Lie bracket is form invariant with respect to an invertible change of coordinates. This means that the structure constants do not change by a transformation of coordinates. On the contrary, the structure constants change if a different basis of the Lie algebra is used.

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Chapter 4
Lie Groups of Transformations of Differential Equations



4.1 Introduction

The study of differential equations within the framework of Lie symmetry analysis requires some further elements. In particular, we need to recognize the different role of independent and dependent variables, and devise a method to extend the group action to derivatives. In fact, the continuous transformations of independent and dependent variables induce naturally the transformations of derivatives. This means that the infinitesimal operator of the Lie group of transformations need to be suitably *prolonged* to the so-called *jet space*. Then, we can establish the Lie infinitesimal criterion for computing the Lie symmetries admitted by differential equations. The interested reader may refer to a wide set of available monographs [3, 4, 6, 7, 10, 19–25, 27, 29–32, 36, 37] for more details and different viewpoints.

In considering Lie groups of transformations associated to a given differential equation $\mathcal{E}$ involving $n \geq 1$ independent variables

$$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{D}_n \subseteq \mathbb{R}^n,$$

and $m \geq 1$ dependent variables

$$\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{D}_m \subseteq \mathbb{R}^m,$$

where $\mathbb{D}_n$ and $\mathbb{D}_m$ are open domains, let us split the variable $\mathbf{z}$ in

$$\mathbf{z}' = \mathbf{Z}(\mathbf{z}; a)$$

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as $\mathbf{z} = (\mathbf{x}, \mathbf{u})$, so that the one-parameter Lie group of transformations is rewritten as

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}, \mathbf{u}; a), \quad \mathbf{u}' = \mathbf{U}(\mathbf{x}, \mathbf{u}; a), \tag{4.1}$$

acting on the space $\mathbb{D} \equiv \mathbb{D}_n \times \mathbb{D}_m \subseteq \mathbb{R}^{n+m}$ of the variables $(\mathbf{x}, \mathbf{u})$.

As a differential equation involves, in addition to $\mathbf{x}$ and $\mathbf{u}$, the derivatives up to some finite order of the latter with respect to the former, we have to determine the transformation of such derivatives. This is accomplished by *prolonging* the action of the group. In fact, the transformations (4.1) determine suitable transformations for such derivatives.

Let $\mathbf{u}^{(1)}$ denote the vector whose $n \cdot m$ components are all first-order (partial) derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$,

$$\mathbf{u}^{(1)} \equiv \left(\frac{\partial u_1}{\partial x_1}, \ldots, \frac{\partial u_1}{\partial x_n}, \ldots, \frac{\partial u_m}{\partial x_1}, \ldots, \frac{\partial u_m}{\partial x_n} \right),$$

and, in general, let $\mathbf{u}^{(k)}$ denote the vector whose components are the $m \cdot \binom{n+k-1}{k}$ kth-order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$. An rth-order differential equation can be written in the general form:

$$\mathbf{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}\right) = \mathbf{0}, \tag{4.2}$$

where $\mathbf{\Delta} \equiv \left(\Delta_1, \ldots, \Delta_q\right)$ are suitable smooth functions depending at most on the indicated arguments.

Let us introduce the *jet space*[1] $J^r(\mathbb{R}^{n+m}, n)$, whose coordinates, denoted for simplicity by $\mathbf{z}$, are the independent variables $\mathbf{x}$, the dependent variables $\mathbf{u}$, and the derivatives of the latter variables with respect to the former ones up to the order r. Its dimension is easily computed:

$$\dim(J^r(\mathbb{R}^{n+m}, n)) = n + m \sum_{i=0}^{r} \binom{n+i-1}{i} = n + m\binom{n+r}{r} = N.$$

Each Δ_j $(j = 1, \ldots, q)$ is a function from (an open subset of) $J^r(\mathbb{R}^{n+m}, n)$ to $\mathbb{R}$. Setting

$$q' = \operatorname{rank} \left\| \frac{\partial(\Delta_1, \ldots, \Delta_q)}{\partial(z_1, \ldots, z_N)} \right\|,$$

Equation (4.2) from a geometrical point of view can be thought of as a $(N - q')$-dimensional manifold $\mathcal{E}$ in the N-dimensional jet space $J^r(\mathbb{R}^{n+m}, n)$.

[1] Some additional details about the geometry of jet spaces will be given in Chap. 8.

The transformations of the derivatives, induced by the transformations of independent and dependent variables, lead to natural extensions (*prolongations*) of the one-parameter Lie group of transformations (4.1). While the one-parameter Lie group of transformations (4.1) acts on the space $(\mathbf{x}, \mathbf{u})$, the prolonged group acts on the jet space $J^r(\mathbb{R}^{n+m}, n)$.

With such a viewpoint, we may investigate the invariance of a differential equation with respect to a one-parameter Lie group of transformations.

Let

$$\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x}) \equiv (\Theta_1(\mathbf{x}), \ldots, \Theta_m(\mathbf{x}))$$

be a solution of $\mathcal{E}$.

Definition 4.1 A Lie group of transformations of the form (4.1) admitted by $\mathcal{E}$ has the two equivalent properties:

1. a transformation of the group maps any solution of (4.2) into a (possibly different) solution of (4.2);
2. a transformation of the group maps the manifold $\mathcal{E}$ into itself, i.e., (4.2) has the same form (up to a multiplicative function that is nowhere vanishing) in terms of both the variables $(\mathbf{x}, \mathbf{u})$ and the transformed variables $(\mathbf{x}', \mathbf{u}')$.

The first definition of the group of transformations admitted by a differential equation is apparently simple, but has only a theoretical value. In fact, it requires that we know at least a solution of the differential equation. On the contrary, the second definition does not require the knowledge of a solution, and has a geometrical interpretation: the group of transformations leaves a differential equation invariant if the manifold in the jet space defined by the equation is invariant. This second viewpoint allows us to check the invariance by using the infinitesimal criterion on the differential equation, provided that we *prolong* the action of the group to the jet space. Roughly speaking, we need to find the prolonged vector fields that are tangent to the manifold characterized by the equation (also called the *frame*, or the *skeleton* of the equation).

4.2 Prolongations of a One-Parameter Lie Group of Transformations

Hereafter, we derive the *prolonged* transformations of a Lie group of point transformations for managing the derivatives appearing in a differential equation.

Two main cases may be distinguished (although they are not really different), according to the number of independent variables: $n = 1$ and $n > 1$. In the first case, we have ordinary differential equations, in the second one partial differential equations.

4.2.1 One Independent Variable

Suppose we have an rth-order ordinary differential equation involving m unknowns $\mathbf{u}(x) \equiv (u_1(x), \ldots, u_m(x))$, and we are interested to find the one-parameter Lie group of transformations of the form

$$x' = X(x, \mathbf{u}; a), \qquad \mathbf{u}' = \mathbf{U}(x, \mathbf{u}; a) \tag{4.3}$$

leaving it invariant. The *natural* prolongation of (4.3) from the space $(x, \mathbf{u})$ to the prolonged space $(x, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$, where

$$\mathbf{u}^{(k)} = \frac{d^k \mathbf{u}}{dx^k} = \left(\frac{d^k u_1}{dx^k}, \ldots, \frac{d^k u_m}{dx^k} \right) \equiv \left(u_1^{(k)}, \ldots, u_m^{(k)} \right)$$

$(k = 1, \ldots, r)$, is obtained by requiring that the contact conditions, linking the differentials dx, $d\mathbf{u}$, $d\mathbf{u}^{(k)}$ $(k = 1, \ldots, r-1)$,

$$\begin{aligned} d\mathbf{u} &= \mathbf{u}^{(1)} dx, \\ d\mathbf{u}^{(k-1)} &= \mathbf{u}^{(k)} dx, \qquad 1 < k \le r, \end{aligned}$$

are preserved, so that

$$d\mathbf{u}' = \mathbf{u}^{(1)\prime} dx', \tag{4.4a}$$

$$d\mathbf{u}^{(k-1)\prime} = \mathbf{u}^{(k)\prime} dx', \qquad 1 < k \le r. \tag{4.4b}$$

By differentiating (4.3), we have

$$\begin{aligned} dx' &= \frac{\partial X}{\partial x} dx + \frac{\partial X}{\partial u_1} du_1 + \cdots + \frac{\partial X}{\partial u_m} du_m, \\ d\mathbf{u}' &= \frac{\partial \mathbf{U}}{\partial x} dx + \frac{\partial \mathbf{U}}{\partial u_1} du_1 + \cdots + \frac{\partial \mathbf{U}}{\partial u_m} du_m; \end{aligned}$$

so, from the first-order contact condition (4.4a), we get

$$\mathbf{u}^{(1)\prime} = \mathbf{U}^{(1)}(x, \mathbf{u}, \mathbf{u}^{(1)}; a) = \frac{\dfrac{\partial \mathbf{U}}{\partial x} + \displaystyle\sum_{\alpha=1}^{m} \frac{\partial \mathbf{U}}{\partial u_\alpha} u_\alpha^{(1)}}{\dfrac{\partial X}{\partial x} + \displaystyle\sum_{\alpha=1}^{m} \frac{\partial X}{\partial u_\alpha} u_\alpha^{(1)}}.$$

By induction, higher order prolongations are immediately determined. Assuming

$$\mathbf{u}^{(k-1)\prime} = \mathbf{U}^{(k-1)}(x, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k-1)}; a), \quad k > 1;$$

the contact condition (4.4b) yields

$$\begin{aligned}\mathbf{u}^{(k)\prime} &= \mathbf{U}^{(k)}(x, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}; a) \\ &= \frac{\dfrac{\partial \mathbf{U}^{(k-1)}}{\partial x} + \displaystyle\sum_{\alpha=1}^{m} \left(\frac{\partial \mathbf{U}^{(k-1)}}{\partial u_\alpha} u_\alpha^{(1)} + \cdots + \frac{\partial \mathbf{U}^{(k-1)}}{\partial u_\alpha^{(k-1)}} u_\alpha^{(k)} \right)}{\dfrac{\partial X}{\partial x} + \displaystyle\sum_{\alpha=1}^{m} \frac{\partial X}{\partial u_\alpha} u_\alpha^{(1)}}.\end{aligned}$$

Therefore, the following theorem can be stated.

Theorem 4.1 *The one-parameter Lie group of transformations* (4.3)*, acting on the* $(x, \mathbf{u})$ *space, prolongs naturally to the one-parameter Lie group*

$$\begin{aligned} x' &= X(x, \mathbf{u}; a), \\ \mathbf{u}' &= \mathbf{U}(x, \mathbf{u}; a), \\ \mathbf{u}^{(1)\prime} &= \mathbf{U}^{(1)}(x, \mathbf{u}, \mathbf{u}^{(1)}; a), \\ &\cdots, \\ \mathbf{u}^{(r)\prime} &= \mathbf{U}^{(r)}(x, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}; a), \end{aligned}$$

where

$$\mathbf{U}^{(k)} = \frac{\dfrac{\partial \mathbf{U}^{(k-1)}}{\partial x} + \displaystyle\sum_{\alpha=1}^{m} \left(\frac{\partial \mathbf{U}^{(k-1)}}{\partial u_\alpha} u_\alpha^{(1)} + \cdots + \frac{\partial \mathbf{U}^{(k-1)}}{\partial u_\alpha^{(k-1)}} u_\alpha^{(k)} \right)}{\dfrac{\partial X}{\partial x} + \displaystyle\sum_{\alpha=1}^{m} \frac{\partial X}{\partial u_\alpha} u_\alpha^{(1)}}, \quad k = 1, \ldots, r,$$

and

$$\mathbf{U}^{(0)} = \mathbf{U}(x, \mathbf{u}; a),$$

acting on the $(x, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$ *space.*

Proof Straightforward. □

Example 4.1 (*Rotation group*) Let $m = 1$, and consider the rotation group

$$\begin{aligned} x' &= X(x, u; a) = x \cos a + u \sin a, \\ u' &= U(x, u; a) = -x \sin a + u \cos a. \end{aligned}$$

We immediately derive

$$
\begin{aligned}
u^{(1)\prime} &= \frac{-\sin a + u^{(1)} \cos a}{\cos a + u^{(1)} \sin a}, \\
u^{(2)\prime} &= \frac{u^{(2)}}{(\cos a + u^{(1)} \sin a)^3}, \\
u^{(3)\prime} &= \frac{(u^{(1)} \sin a + \cos a) u^{(3)} - 3(u^{(2)})^2 \sin a}{(\cos a + u^{(1)} \sin a)^5}, \\
&\cdots
\end{aligned}
$$

Remark 4.1 By defining the total derivative operator with respect to x,

$$
\mathcal{D}_x = \frac{\partial}{\partial x} + \sum_{\alpha=1}^{m} \left(u_\alpha^{(1)} \frac{\partial}{\partial u_\alpha} + \cdots + u_\alpha^{(\ell+1)} \frac{\partial}{\partial u_\alpha^{(\ell)}} + \cdots \right),
$$

for a differentiable function $F(x, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$, we have

$$
\mathcal{D}_x(F) = \frac{\partial F}{\partial x} + \sum_{\alpha=1}^{m} \left(u_\alpha^{(1)} \frac{\partial F}{\partial u_\alpha} + \cdots + u_\alpha^{(r+1)} \frac{\partial F}{\partial u_\alpha^{(r)}} \right).
$$

Therefore, the rth prolongation may be written as

$$
\begin{aligned}
&x' = X(x, \mathbf{u}; a), \\
&\mathbf{u}' = \mathbf{U}(x, \mathbf{u}; a), \\
&\mathbf{u}^{(1)\prime} = \mathbf{U}^{(1)}(x, \mathbf{u}, \mathbf{u}^{(1)}; a) = \frac{\mathcal{D}_x(\mathbf{U}(x, \mathbf{u}; a))}{\mathcal{D}_x(X(x, \mathbf{u}; a))}, \\
&\cdots \\
&\mathbf{u}^{(r)\prime} = \mathbf{U}^{(r)}(x, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}; a) = \frac{\mathcal{D}_x(\mathbf{U}^{(r-1)}(x, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r-1)}; a))}{\mathcal{D}_x(X(x, \mathbf{u}; a))}.
\end{aligned}
$$

Things go similarly when we consider infinitesimal transformations.

Theorem 4.2 (Prolonged infinitesimal transformations) *The infinitesimal transformation*

$$
\begin{aligned}
x' &= x + a\xi(x, \mathbf{u}) + O(a^2), \\
\mathbf{u}' &= \mathbf{u} + a\boldsymbol{\eta}(x, \mathbf{u}) + O(a^2)
\end{aligned}
$$

has the rth prolongation given by

$$
\begin{aligned}
&x' = x + a\xi(x, \mathbf{u}) + O(a^2),\\
&\mathbf{u}' = \mathbf{u} + a\boldsymbol{\eta}(x, \mathbf{u}) + O(a^2),\\
&\mathbf{u}^{(1)\prime} = \mathbf{u}^{(1)} + a\boldsymbol{\eta}^{(1)}(x, \mathbf{u}, \mathbf{u}^{(1)}) + O(a^2),\\
&\dots\\
&\mathbf{u}^{(r)\prime} = \mathbf{u}^{(r)} + a\boldsymbol{\eta}^{(r)}(x, \mathbf{u}, \mathbf{u}^{(1)}, \dots, \mathbf{u}^{(r)}) + O(a^2),
\end{aligned}
$$

where

$$
\boldsymbol{\eta}^{(k)}(x, \mathbf{u}, \mathbf{u}^{(1)}, \dots, \mathbf{u}^{(k)}) \equiv (\eta_1^{(k)}, \dots, \eta_m^{(k)}), \qquad k = 1, \dots, r
$$

is given by

$$
\boldsymbol{\eta}^{(k)}(x, \mathbf{u}, \mathbf{u}^{(1)}, \dots, \mathbf{u}^{(k)}) = \left.\frac{\partial \mathbf{U}^{(k)}(x, \mathbf{u}, \mathbf{u}^{(1)}, \dots, \mathbf{u}^{(k)}; a)}{\partial a}\right|_{a=0}.
$$

Formulae for the prolonged infinitesimals $\boldsymbol{\eta}^{(k)}$ $(k = 1, \dots, r)$ *are easily obtained by recursion:*

$$
\boldsymbol{\eta}^{(k)} = \mathcal{D}_x(\boldsymbol{\eta}^{(k-1)}) - \mathbf{u}^{(k)}\mathcal{D}_x(\xi), \tag{4.5}
$$

with $\boldsymbol{\eta}^{(0)} = \boldsymbol{\eta}(x, \mathbf{u})$.

Proof The proof of formulae (4.5) follows immediately by taking into account that the operators of derivative with respect to a and total derivative $\mathcal{D}_x$ are commuting, so that

$$
\begin{aligned}
\boldsymbol{\eta}^{(1)} &= \left.\left(\frac{\partial}{\partial a}\left(\frac{\mathcal{D}_x(\mathbf{U}(x, \mathbf{u}; a))}{\mathcal{D}_x(X(x, \mathbf{u}; a))}\right)\right)\right|_{a=0}\\
&= \left.\frac{\mathcal{D}_x(X)\dfrac{\partial}{\partial a}\left(\mathcal{D}_x(\mathbf{U})\right) - \mathcal{D}_x(\mathbf{U})\dfrac{\partial}{\partial a}\left(\mathcal{D}_x(X)\right)}{(\mathcal{D}_x(X))^2}\right|_{a=0}\\
&= \left.\frac{\mathcal{D}_x(X)\mathcal{D}_x\left(\dfrac{\partial \mathbf{U}}{\partial a}\right) - \mathcal{D}_x(\mathbf{U})\mathcal{D}_x\left(\dfrac{\partial X}{\partial a}\right)}{(\mathcal{D}_x(X))^2}\right|_{a=0}\\
&= \mathcal{D}_x(\boldsymbol{\eta}) - \mathbf{u}^{(1)}\mathcal{D}_x(\xi),
\end{aligned}
$$

and, for $k > 1$,

$$\begin{aligned}\boldsymbol{\eta}^{(k)} &= \left(\frac{\partial}{\partial a}\left(\frac{\mathcal{D}_x(\mathbf{U}^{(k-1)}(x,\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(k-1)};a))}{\mathcal{D}_x(X)}\right)\right)\Bigg|_{a=0}\\ &= \left.\frac{\mathcal{D}_x(X)\dfrac{\partial}{\partial a}\left(\mathcal{D}_x(\mathbf{U}^{(k-1)})\right)-\mathcal{D}_x(\mathbf{U}^{(k-1)})\dfrac{\partial}{\partial a}\left(\mathcal{D}_x(X)\right)}{(\mathcal{D}_x(X))^2}\right|_{a=0}\\ &= \left.\frac{\mathcal{D}_x(X)\mathcal{D}_x\left(\dfrac{\partial\mathbf{U}^{(k-1)}}{\partial a}\right)-\mathcal{D}_x(\mathbf{U}^{(k-1)})\mathcal{D}_x\left(\dfrac{\partial X}{\partial a}\right)}{(\mathcal{D}_x(X))^2}\right|_{a=0}\\ &= \mathcal{D}_x(\boldsymbol{\eta}^{(k-1)})-\mathbf{u}^{(k)}\mathcal{D}_x(\xi).\end{aligned}$$

□

Theorem 4.3 (Prolonged infinitesimal operator) *The infinitesimal operator of a Lie group of transformations,*

$$\Xi = \xi(x,\mathbf{u})\frac{\partial}{\partial x}+\sum_{\alpha=1}^{m}\eta_\alpha(x,\mathbf{u})\frac{\partial}{\partial u_\alpha},$$

has the rth prolongation recursively given by

$$\begin{aligned}&\Xi^{(0)} = \Xi,\\ &\Xi^{(k)} = \Xi^{(k-1)}+\sum_{\alpha=1}^{m}\eta_\alpha^{(k)}(x,\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(k)})\frac{\partial}{\partial u_\alpha^{(k)}},\qquad k=1,\ldots,r.\end{aligned}$$

Proof Immediate. □

Moreover, we can state the following theorem.

Theorem 4.4 *The prolonged infinitesimals $\boldsymbol{\eta}^{(k)}$, satisfying the recursion relation*

$$\boldsymbol{\eta}^{(k)} = \mathcal{D}_x(\boldsymbol{\eta}^{(k-1)})-\mathbf{u}^{(k)}\mathcal{D}_x(\xi) \tag{4.6}$$

($k\geq 1$), where $\boldsymbol{\eta}^{(0)}=\boldsymbol{\eta}(x,\mathbf{u})$, are given by the closed formula

$$\boldsymbol{\eta}^{(k)} = \mathcal{D}_x^k(\boldsymbol{\eta}(x,\mathbf{u}))-\sum_{i=1}^{k}\binom{k}{i}\mathbf{u}^{(k-i+1)}\mathcal{D}_x^i(\xi), \tag{4.7}$$

where $\mathcal{D}_x^k=\mathcal{D}_x(\mathcal{D}_x^{k-1})$, and $\mathcal{D}_x^0$ is the identity operator.

Proof First of all, we observe that formula (4.7) is correct for $k = 1$. For $k > 1$, let us prove that relation (4.7), assumed true for $k - 1$, satisfies the recursive formula (4.6). In fact, it is

$$\begin{aligned}
\boldsymbol{\eta}^{(k)} &= \mathcal{D}_x(\boldsymbol{\eta}^{(k-1)}) - \mathbf{u}^{(k)}\mathcal{D}_x(\xi) \\
&= \mathcal{D}_x\left(\mathcal{D}_x^{k-1}(\boldsymbol{\eta}) - \sum_{i=1}^{k-1}\binom{k-1}{i}\mathbf{u}^{(k-i)}\mathcal{D}_x^i(\xi)\right) - \mathbf{u}^{(k)}\mathcal{D}_x(\xi) \\
&= \mathcal{D}_x^k(\boldsymbol{\eta}) - \sum_{i=1}^{k-1}\binom{k-1}{i}\left(\mathbf{u}^{(k-i+1)}\mathcal{D}_x^i(\xi) + \mathbf{u}^{(k-i)}\mathcal{D}_x^{i+1}(\xi)\right) - \mathbf{u}^{(k)}\mathcal{D}_x(\xi) \\
&= \mathcal{D}_x^k(\boldsymbol{\eta}) - k\mathbf{u}^{(k)}\mathcal{D}_x(\xi) - \sum_{i=2}^{k-1}\binom{k-1}{i}\mathbf{u}^{(k-i+1)}\mathcal{D}_x^i(\xi) \\
&\quad - \sum_{i=1}^{k-2}\binom{k-1}{i}\mathbf{u}^{(k-i)}\mathcal{D}_x^{i+1}(\xi) - \mathbf{u}^{(1)}\mathcal{D}_x^k(\xi) \\
&= \mathcal{D}_x^k(\boldsymbol{\eta}) - k\mathbf{u}^{(k)}\mathcal{D}_x(\xi) - \mathbf{u}^{(1)}\mathcal{D}_x^k(\xi) \\
&\quad - \sum_{i=2}^{k-1}\left(\binom{k-1}{i} + \binom{k-1}{i-1}\right)\mathbf{u}^{(k-i+1)}\mathcal{D}_x^i(\xi) \\
&= \mathcal{D}_x^k(\boldsymbol{\eta}) - k\mathbf{u}^{(k)}\mathcal{D}_x(\xi) - \sum_{i=2}^{k-1}\binom{k}{i}\mathbf{u}^{(k-i+1)}\mathcal{D}_x^i(\xi) - \mathbf{u}^{(1)}\mathcal{D}_x^k(\xi) \\
&= \mathcal{D}_x^k(\boldsymbol{\eta}) - \sum_{i=1}^{k}\binom{k}{i}\mathbf{u}^{(k-i+1)}\mathcal{D}_x^i(\xi).
\end{aligned}$$

□

Remark 4.2 The prolonged infinitesimals $\boldsymbol{\eta}^{(k)}$ have the following properties:

1. the components of $\boldsymbol{\eta}^{(k)}$ are polynomial in the components of $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}$ with coefficients that are linear homogeneous in $\xi(x, \mathbf{u})$, the components of $\boldsymbol{\eta}(x, \mathbf{u})$, and their partial derivatives up to the kth order;
2. the components of $\boldsymbol{\eta}^{(k)}$ are linear in the components of $\mathbf{u}^{(k)}$ for $k \geq 2$.

Example 4.2 (*Rotation group*) From

$$\begin{aligned}
x' &= X(x, u; a) = x\cos a + u\sin a, \\
u' &= U(x, u; a) = -x\sin a + u\cos a,
\end{aligned}$$

we deduce

$$\xi = u, \qquad \eta = -x, \qquad \Xi = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u},$$

and

$$\begin{aligned}
&\eta^{(1)} = -(1 + (u^{(1)})^2),\\
&\eta^{(2)} = -3u^{(1)}u^{(2)},\\
&\eta^{(3)} = -(3(u^{(2)})^2 + 4u^{(1)}u^{(3)}),\\
&\dots
\end{aligned}$$

For instance, the third-order prolonged infinitesimal operator reads

$$\begin{aligned}
\Xi^{(3)} = {} & u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u} - (1 + (u^{(1)})^2)\frac{\partial}{\partial u^{(1)}}\\
& - 3u^{(1)}u^{(2)}\frac{\partial}{\partial u^{(2)}} - (3(u^{(2)})^2 + 4u^{(1)}u^{(3)})\frac{\partial}{\partial u^{(3)}}.
\end{aligned}$$

Example 4.3 (*Scaling group*) The transformation

$$\begin{aligned}
&x' = X(x, u; a) = \exp(a)x,\\
&u' = U(x, u; a) = \exp(\alpha a)u, \qquad \alpha \text{ constant},
\end{aligned}$$

yields

$$\xi = x, \qquad \eta = \alpha u, \qquad \Xi = x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u},$$

and

$$\eta^{(k)} = (\alpha - k)u^{(k)}, \qquad k \geq 1.$$

For instance, the third-order prolonged infinitesimal operator is

$$\Xi^{(3)} = x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u} + (\alpha - 1)u^{(1)}\frac{\partial}{\partial u^{(1)}} + (\alpha - 2)u^{(2)}\frac{\partial}{\partial u^{(2)}} + (\alpha - 3)u^{(3)}\frac{\partial}{\partial u^{(3)}}.$$

4.2.2 More Than One Independent Variable

Now, consider the case with $n > 1$ independent variables, and derive the *natural* prolongation of

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}, \mathbf{u}; a), \qquad \mathbf{u}' = \mathbf{U}(\mathbf{x}, \mathbf{u}; a) \tag{4.8}$$

from the $(\mathbf{x}, \mathbf{u})$ space to the prolonged space $(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$, where a generic element of the set of all partial kth-order partial derivatives $\mathbf{u}^{(k)}$ is denoted as[2]

$$u_{\alpha,i_1\ldots i_k} = \frac{\partial^k u_\alpha}{\partial x_{i_1} \ldots \partial x_{i_k}}.$$

This is accomplished by requiring once again that the contact conditions, relating differentials dx_j, du_α, $du_{\alpha,i}$, …,

$$du_\alpha = \sum_{j=1}^{n} u_{\alpha,j} dx_j,$$

$$du_{\alpha,i_1\ldots i_{k-1}} = \sum_{j=1}^{n} u_{\alpha,i_1\ldots i_{k-1}j} dx_j, \quad k > 1$$

are preserved, so that

$$du'_\alpha = \sum_{j=1}^{n} u'_{\alpha,j} dx'_j, \tag{4.9a}$$

$$du'_{\alpha,i_1\ldots i_{k-1}} = \sum_{j=1}^{n} u'_{\alpha,i_1\ldots i_{k-1}j} dx'_j, \qquad k > 1. \tag{4.9b}$$

By differentiating the generic components of (4.8), we have

$$dx'_i = \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j} dx_j + \sum_{\beta=1}^{m} \frac{\partial X_i}{\partial u_\beta} du_\beta,$$

$$du'_\alpha = \sum_{j=1}^{n} \frac{\partial U_\alpha}{\partial x_j} dx_j + \sum_{\beta=1}^{m} \frac{\partial U_\alpha}{\partial u_\beta} du_\beta;$$

thus, using the first-order contact condition (4.9a), we get

$$\sum_{i=1}^{n} u'_{\alpha,i} \mathcal{D}_j(X_i) = \mathcal{D}_j(U_\alpha),$$

[2] Note the difference between the notation for the derivatives of $u_\alpha(x)$, $u_\alpha^{(k)} = \dfrac{d^k u_\alpha}{dx^k}$, and the partial derivatives of $u_\alpha(x_1, \ldots, x_n)$, $u_{\alpha,i_1\ldots i_k} = \dfrac{\partial^k u_\alpha}{\partial x_{i_1} \ldots \partial x_{i_k}}$. If we have only one dependent variable u, the derivatives will be denoted as $u^{(k)} = \dfrac{d^k u}{dx^k}$, and $u_{,i_1\ldots i_k} = \dfrac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}}$, respectively.

$\mathcal{D}_j$ being the total derivative with respect to x_j, i.e.:

$$\mathcal{D}_j = \frac{\partial}{\partial x_j} + \sum_{\alpha=1}^{m} \left(u_{\alpha,j} \frac{\partial}{\partial u_\alpha} + \cdots + u_{\alpha,i_1 \ldots i_k j} \frac{\partial}{\partial u_{\alpha,i_1 \ldots i_k}} + \cdots \right).$$

By introducing a matrix A with entries $A_{ij} = \mathcal{D}_j(X_i)$, and the row vectors

$$\begin{aligned} \mathbf{B}_\alpha^{(1)} &= (\mathcal{D}_1(U_\alpha), \ldots, \mathcal{D}_n(U_\alpha)), \\ \mathbf{u}_\alpha^{(1)} &= \left(u_{\alpha,1}, \ldots, u_{\alpha,n}\right), \\ \mathbf{u}_\alpha^{(1)\prime} &= \left(u'_{\alpha,1}, \ldots, u'_{\alpha,n}\right), \end{aligned}$$

we have

$$\mathbf{u}_\alpha^{(1)\prime} = \mathbf{B}_\alpha^{(1)} \mathsf{A}^{-1} = \mathbf{U}_\alpha^{(1)}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}; a), \tag{4.10}$$

yielding the desired transformation of first-order derivatives.

Higher order prolongations are inductively determined following the same ideas. Assuming

$$\mathbf{u}^{(k-1)\prime} = \mathbf{U}^{(k-1)}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k-1)}; a), \qquad k > 1,$$

and using the higher order contact condition (4.9b), we get

$$\sum_{j=1}^{n} u'_{\alpha,i_1 \ldots i_{k-1} j} \mathcal{D}_\ell \left(X_j\right) = \mathcal{D}_\ell \left(U^{(k-1)}_{\alpha,i_1 \ldots i_{k-1}}\right).$$

Therefore, by introducing the row vectors

$$\begin{aligned} \mathbf{B}^{(k)}_{\alpha,i_1 \ldots i_{k-1}} &= \left(\mathcal{D}_1 \left(U^{(k-1)}_{\alpha,i_1 \ldots i_{k-1}}\right), \ldots, \mathcal{D}_n \left(U^{(k-1)}_{\alpha,i_1 \ldots i_{k-1}}\right)\right), \\ \mathbf{u}^{(k)}_{\alpha,i_1 \ldots i_{k-1}} &= \left(u_{\alpha,i_1 \ldots i_{k-1} 1}, \ldots, u_{\alpha,i_1 \ldots i_{k-1} n}\right), \\ \mathbf{u}^{(k)\prime}_{\alpha,i_1 \ldots i_{k-1}} &= \left(u'_{\alpha,i_1 \ldots i_{k-1} 1}, \ldots, u'_{\alpha,i_1 \ldots i_{k-1} n}\right), \end{aligned}$$

we have

$$\mathbf{u}^{(k)\prime}_{\alpha,i_1 \ldots i_{k-1}} = \mathbf{B}^{(k)}_{\alpha,i_1 \ldots i_{k-1}} \mathsf{A}^{-1} = \mathbf{U}^{(k)}_{\alpha,i_1 \ldots i_k}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}; a). \tag{4.11}$$

Collecting such relations for all α ($1 \le \alpha \le m$) and all $i_1, \ldots, i_k$ ($1 \le i_1 \le \cdots \le i_k \le n$), we may write

$$\mathbf{u}^{(k)\prime} = \mathbf{U}^{(k)}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}; a).$$

Thus, the following theorem remains proved.

Theorem 4.5 *The one-parameter Lie group of transformations* (4.8) *acting on the* $(\mathbf{x}, \mathbf{u})$ *space prolongs naturally to the one-parameter Lie group acting on the* $(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$ *space:*

$$\begin{aligned}
&\mathbf{x}' = \mathbf{X}(\mathbf{x}, \mathbf{u}; a),\\
&\mathbf{u}' = \mathbf{U}(\mathbf{x}, \mathbf{u}; a),\\
&\mathbf{u}^{(1)\prime} = \mathbf{U}^{(1)}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}; a),\\
&\ldots\\
&\mathbf{u}^{(r)\prime} = \mathbf{U}^{(r)}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}; a).
\end{aligned}$$

The prolongations of infinitesimal transformations are easily computed as shown by next theorem.

Theorem 4.6 *The infinitesimal transformations*

$$\begin{aligned}
&\mathbf{x}' = \mathbf{x} + a\boldsymbol{\xi}(\mathbf{x}, \mathbf{u}) + O(a^2),\\
&\mathbf{u}' = \mathbf{u} + a\boldsymbol{\eta}(\mathbf{x}, \mathbf{u}) + O(a^2)
\end{aligned}$$

have the r*th-order prolongation given by*

$$\begin{aligned}
&\mathbf{x}' = \mathbf{x} + a\boldsymbol{\xi}(\mathbf{x}, \mathbf{u}) + O(a^2),\\
&\mathbf{u}' = \mathbf{u} + a\boldsymbol{\eta}(\mathbf{x}, \mathbf{u}) + O(a^2),\\
&\mathbf{u}^{(1)\prime} = \mathbf{u}^{(1)} + a\boldsymbol{\eta}^{(1)}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}) + O(a^2),\\
&\ldots\\
&\mathbf{u}^{(r)\prime} = \mathbf{u}^{(r)} + a\boldsymbol{\eta}^{(r)}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}) + O(a^2),
\end{aligned}$$

where the component of the extended infinitesimal $\boldsymbol{\eta}^{(1)}$ *corresponding to* $u_{\alpha,i}$ *is*

$$\eta_{[\alpha,i]} = \mathcal{D}_i\left(\eta_\alpha\right) - \sum_{j=1}^{n} u_{\alpha,j}\mathcal{D}_i\left(\xi_j\right),$$

and, in general, the component of the extended infinitesimal $\boldsymbol{\eta}^{(k)}$ $(k > 1)$ *corresponding to* $u_{\alpha,i_1\ldots i_k}$ *is*

$$\eta_{[\alpha,i_1\ldots i_k]} = \mathcal{D}_{i_k}\left(\eta_{[\alpha,i_1\ldots,i_{k-1}]}\right) - \sum_{j=1}^{n} u_{\alpha,i_1\ldots i_{k-1}j}\mathcal{D}_{i_k}\left(\xi_j\right).$$

Proof Recalling that $A_{ij} = \mathcal{D}_j(X_i)$, we have

$$\begin{aligned}
&A_{ij}\big|_{a=0} = \delta_{ij},\\
&\left.\frac{\partial A_{ij}}{\partial a}\right|_{a=0} = \left.\frac{\partial}{\partial a}\left(\mathcal{D}_j(X_i)\right)\right|_{a=0} = \mathcal{D}_j\left.\left(\frac{\partial X_i}{\partial a}\right)\right|_{a=0} = \mathcal{D}_j\left(\xi_i\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbf{B}_{\alpha}^{(1)}\big|_{a=0} = \left(u_{\alpha,1},\ldots,u_{\alpha,n}\right),\\
&\mathbf{B}_{\alpha,i_1\ldots i_{k-1}}^{(k)}\Big|_{a=0} = \left(u_{\alpha,i_1\ldots i_{k-1}1},\ldots,u_{\alpha,i_1\ldots i_{k-1}n}\right),\\
&\frac{\partial \mathbf{B}_{\alpha}^{(1)}}{\partial a}\Big|_{a=0} = \frac{\partial}{\partial a}\left(\mathcal{D}_1(U_\alpha),\ldots,\mathcal{D}_n(U_\alpha)\right)\Big|_{a=0}\\
&\qquad = \left(\mathcal{D}_1\left(\frac{\partial U_\alpha}{\partial a}\right),\ldots,\mathcal{D}_n\left(\frac{\partial U_\alpha}{\partial a}\right)\right)\Big|_{a=0}\\
&\qquad = \left(\mathcal{D}_1\left(\eta_\alpha\right),\ldots,\mathcal{D}_n\left(\eta_\alpha\right)\right),\\
&\frac{\partial \mathbf{B}_{\alpha,i_1\ldots i_{k-1}j}^{(k)}}{\partial a}\Big|_{a=0} = \frac{\partial}{\partial a}\left(\mathcal{D}_1\left(U_{\alpha,i_1\ldots i_{k-1}}^{(k-1)}\right),\ldots,\mathcal{D}_n\left(U_{\alpha,i_1\ldots i_{k-1}}^{(k-1)}\right)\right)\Big|_{a=0}\\
&\qquad = \left(\mathcal{D}_1\left(\frac{\partial U_{\alpha,i_1\ldots i_{k-1}}^{(k-1)}}{\partial a}\right),\ldots,\mathcal{D}_n\left(\frac{\partial U_{\alpha,i_1\ldots i_{k-1}}^{(k-1)}}{\partial a}\right)\right)\Big|_{a=0}\\
&\qquad = \left(\mathcal{D}_1\left(\eta_{\alpha,i_1\ldots i_{k-1}}\right),\ldots,\mathcal{D}_n\left(\eta_{\alpha,i_1\ldots i_{k-1}}\right)\right).
\end{aligned}$$

Therefore, taking into account (4.10), from

$$\frac{\partial \mathbf{u}_{\alpha}^{(1)\prime}}{\partial a}\Big|_{a=0} = \left(\frac{\partial \mathbf{B}_{\alpha}^{(1)}}{\partial a}\mathsf{A}^{-1} - \mathbf{B}_{\alpha}^{(1)}\mathsf{A}^{-1}\frac{\partial \mathsf{A}}{\partial a}\mathsf{A}^{-1}\right)\Big|_{a=0},$$

we deduce

$$\eta_{[\alpha,i]} = \mathcal{D}_i\left(\eta_\alpha\right) - \sum_{j=1}^{n} u_{\alpha,j}\mathcal{D}_i\left(\xi_j\right),$$

while, taking into account (4.11), from

$$\frac{\partial \mathbf{u}_{\alpha,i_1\ldots i_{k-1}}^{(k)\prime}}{\partial a}\Big|_{a=0} = \left(\frac{\partial \mathbf{B}_{\alpha,i_1\ldots i_{k-1}}^{(k)}}{\partial a}\mathsf{A}^{-1} - \mathbf{B}_{\alpha,i_1\ldots i_{k-1}}^{(k)}\mathsf{A}^{-1}\frac{\partial \mathsf{A}}{\partial a}\mathsf{A}^{-1}\right)\Big|_{a=0},$$

we deduce

$$\eta_{[\alpha,i_1\ldots i_k]} = \mathcal{D}_{i_k}\left(\eta_{[\alpha,i_1\ldots i_{k-1}]}\right) - \sum_{j=1}^{n} u_{\alpha,i_1\ldots i_{k-1}j}\mathcal{D}_{i_k}\left(\xi_j\right).$$

□

Theorem 4.7 *The infinitesimal operator*

$$\Xi = \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\alpha}$$

has the rth-order prolongation defined recursively by

$$\Xi^{(0)} = \Xi,$$
$$\Xi^{(k)} = \Xi^{(k-1)} + \sum_{\alpha=1}^{m} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \eta_{[\alpha, i_1 \ldots i_k]} \frac{\partial}{\partial u_{\alpha, i_1 \ldots i_k}},$$

$k = 1, \ldots, r$.

Also in this case, Remark 4.2 holds true.

4.3 Lie Groups Admitted by Differential Equations

The infinitesimals of the Lie groups of transformations leaving a given differential equation invariant can be found by means of a straightforward algorithm that will be described below. Remarkably, the search of such one-parameter Lie groups of transformations leads usually to multi-parameter Lie groups of transformations.

Let

$$\mathbf{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}\right) = \mathbf{0} \tag{4.12}$$

($\mathbf{\Delta} \equiv (\Delta_1, \ldots, \Delta_q)$) be an rth-order system of q differential equations with n independent variables $\mathbf{x}$ and m dependent variables $\mathbf{u}$.

Equation (4.12) can be considered as characterizing a manifold in the *jet space* $J^r(\mathbb{R}^{n+m}, n)$ with coordinates $(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$.

The second point of view in Definition 4.1 can be formulated as follows.

Definition 4.2 The one-parameter Lie group of transformations (4.1) leaves the (4.12) *invariant* (equivalently, we say that it is *admitted* by (4.12)) if and only if its rth-order prolongation leaves the manifold defined by (4.12) invariant.

Theorem 2.4 allows us to prove the following result, which leads directly to the algorithm for the computation of the symmetries admitted by a given differential equation.

Theorem 4.8 (Infinitesimal criterion) *Let*

$$\Xi = \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\alpha}$$

be the infinitesimal operator corresponding to (4.1)*, and* $\Xi^{(r)}$ *the* r*th-order prolonged infinitesimal operator. The Lie group* (4.1) *is admitted by* (4.12) *if and only if*

$$\Xi^{(r)} \left(\boldsymbol{\Delta} \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)} \right) \right) \Big|_{\boldsymbol{\Delta}=\mathbf{0}} = \mathbf{0}, \tag{4.13}$$

i.e., the expression

$$\Xi^{(r)} \left(\boldsymbol{\Delta} \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)} \right) \right)$$

is vanishing when evaluated on $\Delta \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)} \right) = \mathbf{0}$*. Equivalently, it has to be*

$$\begin{aligned} &\Xi^{(r)} \left(\Delta_j \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)} \right) \right) = \\ &\quad = \sum_{k=1}^{q} \Lambda_{jk} \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)} \right) \Delta_k \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)} \right) \end{aligned}$$

$(j = 1, \ldots, q)$ *for some Lagrange multipliers* Λ_{jk}*.*

If (4.12) can be solved with respect to q derivatives, we do not need to introduce the Lagrange multipliers, and we eliminate in the invariance condition (4.13) these derivatives.

If the differential equation is polynomial in the derivatives, the invariance condition (4.13) is polynomial in the components of $(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$, with coefficients expressed as linear combinations of the unknowns ξ_i, η_α and their partial derivatives. Because (4.12) has been used in the invariance condition (4.13), the derivatives of $\mathbf{u}$ therein appearing are independent and can be arbitrarily varied. By equating to zero the coefficients of these derivatives in this polynomial, one obtains an overdetermined system of linear and homogeneous differential equations for the infinitesimals, called *determining equations*. The integration of the latter leads to the infinitesimals of the group. The infinitesimals involve arbitrary constants (and in some cases arbitrary functions), and thus we have de facto multi-parameter Lie groups.

Example 4.4 Consider the simplest second-order ordinary differential equation

$$u^{(2)} = 0 \qquad \left(u^{(2)} = \frac{d^2 u}{dx^2} \right), \tag{4.14}$$

and look for its Lie symmetries. We need the second-order prolongation of the infinitesimal operator, say

$$\Xi^{(2)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \eta^{(1)} \frac{\partial}{\partial u^{(1)}} + \eta^{(2)} \frac{\partial}{\partial u^{(2)}},$$

where $\xi \equiv \xi(x,u)$, $\eta \equiv \eta(x,u)$, and

$$\eta^{(1)} = \frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial x}\right) u^{(1)} - \frac{\partial \xi}{\partial u}(u^{(1)})^2,$$
$$\eta^{(2)} = \frac{\partial^2 \eta}{\partial x^2} + \left(2\frac{\partial^2 \eta}{\partial x \partial u} - \frac{\partial^2 \xi}{\partial x^2}\right) u^{(1)} + \left(\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi}{\partial x \partial u}\right)(u^{(1)})^2 - \frac{\partial^2 \xi}{\partial u^2}(u^{(1)})^3 + \left(\frac{\partial \eta}{\partial u} - 2\frac{\partial \xi}{\partial x}\right) u^{(2)} - 3\frac{\partial \xi}{\partial u} u^{(1)} u^{(2)}.$$

The invariance condition of (4.14), $\Xi^{(2)}(u^{(2)})\big|_{u^{(2)}=0} = 0$, gives

$$\frac{\partial^2 \eta}{\partial x^2} + \left(2\frac{\partial^2 \eta}{\partial x \partial u} - \frac{\partial^2 \xi}{\partial x^2}\right) u^{(1)} + \left(\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi}{\partial x \partial u}\right)(u^{(1)})^2 - \frac{\partial^2 \xi}{\partial u^2}(u^{(1)})^3 = 0.$$

This is a polynomial in $u^{(1)}$ yielding the *determining equations*

$$\begin{aligned} &2\frac{\partial^2 \eta}{\partial x \partial u} - \frac{\partial^2 \xi}{\partial x^2} = 0, \quad \frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi}{\partial x \partial u} = 0, \\ &\frac{\partial^2 \eta}{\partial x^2} = 0, \qquad\qquad \frac{\partial^2 \xi}{\partial u^2} = 0, \end{aligned} \tag{4.15}$$

for the unknowns $\xi(x,u)$ and $\eta(x,u)$. The integration of these determining equations gives the solution

$$\begin{aligned} \xi(x,u) &= a_1 + a_3 x + a_4 u + a_7 x^2 + a_8 x u, \\ \eta(x,u) &= a_2 + a_5 x + a_6 u + a_7 x u + a_8 u^2, \end{aligned} \tag{4.16}$$

$a_1, \ldots, a_8$ being arbitrary constants. As (4.15) is a linear and homogeneous system of partial differential equations, if we have two solutions also their linear combination is a solution. Therefore, (4.16) may be seen as linear combinations (through the constants $a_1, \ldots, a_8$) of fundamental solutions, giving eight different Lie symmetries (obtained by setting one constant equal to 1 and the remaining ones to 0), whose corresponding infinitesimal operators are

$$\begin{aligned} &\Xi_1 = \frac{\partial}{\partial x}, && \Xi_2 = \frac{\partial}{\partial u}, \\ &\Xi_3 = x\frac{\partial}{\partial x}, && \Xi_4 = u\frac{\partial}{\partial x}, \\ &\Xi_5 = x\frac{\partial}{\partial u}, && \Xi_6 = u\frac{\partial}{\partial u}, \\ &\Xi_7 = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, && \Xi_8 = xu\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u}. \end{aligned} \tag{4.17}$$

Table 4.1 Table of Lie brackets (commutator table) of symmetries of $u_{,2} = 0$

$[\,,\,]$	Ξ_1	Ξ_2	Ξ_3	Ξ_4	Ξ_5	Ξ_6	Ξ_7	Ξ_8
Ξ_1	0	0	Ξ_1	0	Ξ_2	0	$2\Xi_3 + \Xi_6$	Ξ_4
Ξ_2	0	0	0	Ξ_1	0	Ξ_2	Ξ_5	$\Xi_3 + 2\Xi_6$
Ξ_3	$-\Xi_1$	0	0	$-\Xi_4$	Ξ_5	0	Ξ_7	0
Ξ_4	0	$-\Xi_1$	Ξ_4	0	$\Xi_6 - \Xi_3$	$-\Xi_4$	Ξ_8	0
Ξ_5	$-\Xi_2$	0	$-\Xi_5$	$\Xi_3 - \Xi_6$	0	Ξ_5	0	Ξ_7
Ξ_6	0	$-\Xi_2$	0	Ξ_4	$-\Xi_5$	0	0	Ξ_8
Ξ_7	$-2\Xi_3 - \Xi_6$	$-\Xi_5$	$-\Xi_7$	$-\Xi_8$	0	0	0	0
Ξ_8	$-\Xi_4$	$-\Xi_3 - 2\Xi_6$	0	0	$-\Xi_7$	$-\Xi_8$	0	0

The infinitesimal operators (4.17) are the elements of a basis of an eight-dimensional vector space which is also a Lie algebra, as it can be verified by computing the Lie brackets of all possible couples of operators (see Table 4.1).

Example 4.5 (*Symmetries of linear heat equation*) Let us consider the linear second-order partial differential equation

$$u_{,1} - u_{,22} = 0 \tag{4.18}$$

($u_{,1} = \partial u/\partial x_1, u_{,22} = \partial^2 u/\partial x_2^2$; interpreting x_1 as the time, x_2 as a space coordinate, and $u(x_1, x_2)$ as a temperature, (4.18) is the Fourier equation for heat conduction), and let us determine its Lie symmetries. We need the second-order prolongation of the infinitesimal operator, say

$$\begin{aligned}\Xi^{(2)} = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \eta \frac{\partial}{\partial u} + \eta_{[,1]} \frac{\partial}{\partial u_{,1}} + \eta_{[,2]} \frac{\partial}{\partial u_{,2}} \\ + \eta_{[,11]} \frac{\partial}{\partial u_{,11}} + \eta_{[,12]} \frac{\partial}{\partial u_{,12}} + \eta_{[,22]} \frac{\partial}{\partial u_{,22}},\end{aligned}$$

where ξ_1, ξ_2, and η depend on (x_1, x_2, u).

The invariance condition,

$$\Xi^{(2)}(u_{,1} - u_{,22})\big|_{u_{,1}-u_{,22}=0} = 0,$$

yields

$$\begin{aligned}
&\frac{\partial \eta}{\partial x_1} - \frac{\partial^2 \eta}{\partial x_2^2} - \left(2\frac{\partial^2 \eta}{\partial x_2 \partial u} + \frac{\partial \xi_2}{\partial x_1} - \frac{\partial^2 \xi_2}{\partial x_2^2}\right) u_{,2} \\
&\quad + \left(2\frac{\partial^2 \xi_2}{\partial x_2 \partial u} - \frac{\partial^2 \eta}{\partial u^2}\right)(u_{,2})^2 + \frac{\partial^2 \xi_2}{\partial u^2}(u_{,2})^3 + 2\frac{\partial \xi_1}{\partial x_2} u_{,12} \\
&\quad + 2\frac{\partial \xi_1}{\partial u} u_{,2} u_{,12} + \left(2\frac{\partial \xi_2}{\partial x_2} - \frac{\partial \xi_1}{\partial x_1} + \frac{\partial^2 \xi_1}{\partial x_2^2}\right) u_{,22} \\
&\quad + 2\left(\frac{\partial^2 \xi_1}{\partial x_2 \partial u} + \frac{\partial \xi_2}{\partial u}\right) u_{,2} u_{,22} + \frac{\partial^2 \xi_1}{\partial u^2}(u_{,2})^2 u_{,22} = 0,
\end{aligned}$$

where Eq. (4.18) has been used to eliminate $u_{,1}$. It is recovered a polynomial in the derivatives $u_{,2}$, $u_{,12}$, and $u_{,22}$ whose coefficients, involving the infinitesimals and their partial derivatives, must be vanishing. Hence, the following set of determining equations (which are linear and homogeneous partial differential equations) is found:

$$\begin{aligned}
&\frac{\partial \eta}{\partial x_1} - \frac{\partial^2 \eta}{\partial x_2^2} = 0, \qquad 2\frac{\partial^2 \eta}{\partial x_2 \partial u} + \frac{\partial \xi_2}{\partial x_1} - \frac{\partial^2 \xi_2}{\partial x_2^2} = 0, \\
&2\frac{\partial^2 \xi_2}{\partial x_2 \partial u} - \frac{\partial^2 \eta}{\partial u^2} = 0, \quad 2\frac{\partial \xi_2}{\partial x_2} - \frac{\partial \xi_1}{\partial x_1} + \frac{\partial^2 \xi_1}{\partial x_2^2} = 0, \\
&\frac{\partial^2 \xi_1}{\partial x_2 \partial u} + \frac{\partial \xi_2}{\partial u} = 0, \quad \frac{\partial^2 \xi_2}{\partial u^2} = 0, \qquad \frac{\partial \xi_1}{\partial x_2} = 0, \\
&\frac{\partial \xi_1}{\partial u} = 0, \qquad \frac{\partial^2 \xi_1}{\partial u^2} = 0.
\end{aligned} \tag{4.19}$$

Equations (4.19) can be easily integrated, and we obtain the solution

$$\begin{aligned}
\xi_1 &= a_1 + 2a_3 x_1 + a_4 x_1^2, \\
\xi_2 &= a_2 + a_3 x_2 + a_4 x_1 x_2 + a_5 x_1, \\
\eta &= -\frac{a_4}{4}(x_2^2 + 2x_1)u - \frac{a_5}{2} x_2 u + a_6 u + f(x_1, x_2),
\end{aligned}$$

where $a_1, \ldots, a_6$ are arbitrary constants, and $f(x_1, x_2)$ a solution to Eq. (4.18), i.e.:

$$f_{,1} - f_{,22} = 0.$$

In this case, the corresponding infinitesimal operators live in an infinite-dimensional vector space (because of the presence of the function f) that it is also a Lie algebra, which is spanned by

$$
\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2},\\
&\Xi_3 = 2x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2},\\
&\Xi_4 = x_1^2\frac{\partial}{\partial x_1} + x_1x_2\frac{\partial}{\partial x_2} - \frac{x_2^2 + 2x_1}{4}u\frac{\partial}{\partial u},\\
&\Xi_5 = x_1\frac{\partial}{\partial x_2} - \frac{x_2u}{2}\frac{\partial}{\partial u},\\
&\Xi_6 = u\frac{\partial}{\partial u}, \qquad \Xi_f = f(x_1, x_2)\frac{\partial}{\partial u}.
\end{aligned}
\tag{4.20}
$$

The nonzero Lie brackets are

$$
\begin{aligned}
&[\Xi_1, \Xi_3] = 2\Xi_1, \quad [\Xi_1, \Xi_4] = \Xi_3 - \frac{1}{2}\Xi_6,\\
&[\Xi_1, \Xi_5] = \Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_2,\\
&[\Xi_2, \Xi_4] = \Xi_5, \quad [\Xi_2, \Xi_6] = -\frac{1}{2}\Xi_6,\\
&[\Xi_3, \Xi_4] = 2\Xi_4, \quad [\Xi_3, \Xi_5] = \Xi_5,
\end{aligned}
$$

and

$$
\begin{aligned}
&\big[\Xi_1, \Xi_f\big] = \Xi_{f,1}, \quad \big[\Xi_2, \Xi_f\big] = \Xi_{f,2},\\
&\big[\Xi_3, \Xi_f\big] = \Xi_g, \quad \big[\Xi_4, \Xi_f\big] = \Xi_h,\\
&\big[\Xi_5, \Xi_f\big] = \Xi_m, \quad \big[\Xi_6, \Xi_f\big] = \Xi_{-f},
\end{aligned}
\tag{4.21}
$$

where

$$
\begin{aligned}
&g(x_1, x_2) = 2x_1f_{,1} + x_2f_{,2},\\
&h(x_1, x_2) = x_1^2f_{,1} + x_1x_2f_{,2} + \frac{x_2^2 + 2x_1}{4}f,\\
&m(x_1, x_2) = x_1f_{,2} + \frac{x}{2}f.
\end{aligned}
\tag{4.22}
$$

The Lie brackets in (4.21) yield infinitesimal operators still belonging to the Lie algebra, since $-f$, $f_{,1}$, $f_{,2}$, and the functions $g(x_1, x_2)$, $h(x_1, x_2)$, $m(x_1, x_2)$, defined by (4.22), satisfy the Fourier equation.

The infinitesimal operators $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5, \Xi_6\}$ span a six-dimensional Lie subalgebra of the infinite-dimensional Lie algebra of symmetries of linear heat equation.

Remark 4.3 Examples 4.4 and 4.5, although concerned with very simple differential equations, clearly show that looking for their Lie symmetries requires lengthy and tedious calculations. As the procedure for finding the Lie symmetries admitted by a differential equation is completely algorithmic, it is convenient to use a Computer Algebra System with a dedicated package. Many specialized programs are available in the literature (see [3, 9, 11, 16–18, 33–36]). In Chap. 12, the REDUCE [15] package

RELIE [28], able to do many computations in Lie symmetry analysis without much effort, is described in detail.

In a session of the Computer Algebra System REDUCE , the code

```
in "relie.red" $
jetorder:=2 $
xvar:={x} $
uvar:={u} $
diffeqs:={u_xx} $
leadders:={u_xx} $
relie(4) $
;end;
```

computes the Lie symmetries of the ordinary differential equation $u^{(2)} = 0$, and the code

```
in "relie.red" $
jetorder:=2 $
xvar:={x1,x2} $
uvar:={u} $
diffeqs:={u_x1-u_x2x2} $
leadders:={u_x1} $
relie(4) $
;end;
```

computes the Lie symmetries of Fourier equation $u_{,1} - u_{,22} = 0$ (see Example 12.3 for more details).

Both computations require a few seconds!

4.4 Lie Brackets and Prolongations

An important property of Lie brackets and prolonged vector fields needs to be explicited.

Theorem 4.9 *The Lie bracket of the kth ($k \geq 1$) prolongations of two generators Ξ_1 and Ξ_2 is the same as the prolongation of their Lie bracket,*

$$\left[\Xi_1^{(k)}, \Xi_2^{(k)}\right] = [\Xi_1, \Xi_2]^{(k)},$$

i.e., the operations of prolongation and taking the Lie bracket commute.

Proof To prove the claim, due to the recursive definition of the higher order prolongations, it is sufficient to limit ourselves to first-order prolongations. Let us consider

the infinitesimal operators Ξ_1 and Ξ_2 involving the variables $\mathbf{x}$ and $\mathbf{u}$. Since the operation of Lie bracket is invariant in form with respect to a change of variables, let us introduce the canonical variables associated to Ξ_1 (still denoting them with $\mathbf{x}$ and $\mathbf{u}$ to simplify the notation), whereupon

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha \frac{\partial}{\partial u_\alpha}.$$

Thus,

$$[\Xi_1, \Xi_2] = \sum_{i=1}^{n} \frac{\partial \xi_i}{\partial x_1} \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \frac{\partial \eta_\alpha}{\partial x_1} \frac{\partial}{\partial u_\alpha},$$

whereas the first-order prolonged operators become

$$\begin{aligned} \Xi_1^{(1)} &= \Xi_1, \\ \Xi_2^{(1)} &= \Xi_2 + \sum_{\alpha=1}^{m} \sum_{k=1}^{n} \eta_{[\alpha,k]} \frac{\partial}{\partial u_{\alpha,k}}; \end{aligned}$$

moreover, it is

$$\eta_{[\alpha,k]} = \mathcal{D}_k(\eta_\alpha) - \mathcal{D}_k(\xi_j) u_{\alpha,j}.$$

So,

$$\begin{aligned} \left[\Xi_1^{(1)}, \Xi_2^{(1)}\right] &= [\Xi_1, \Xi_2] + \left[\Xi_1, \sum_{\alpha=1}^{m} \sum_{k=1}^{n} \eta_{[\alpha,k]} \frac{\partial}{\partial u_{\alpha,k}}\right] = \\ &= \sum_{i=1}^{n} \frac{\partial \xi_i}{\partial x_1} \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \frac{\partial \eta_\alpha}{\partial x_1} \frac{\partial}{\partial u_\alpha} + \sum_{\alpha=1}^{m} \sum_{k=1}^{n} \frac{\partial \eta_{[\alpha,k]}}{\partial x_1} \frac{\partial}{\partial u_{\alpha,k}}. \end{aligned}$$

On the other hand, the prolongation of the commutator $[\Xi_1, \Xi_2]$ is

$$[\Xi_1, \Xi_2]^{(1)} = [\Xi_1, \Xi_2] + \sum_{\alpha=1}^{m} \sum_{k=1}^{n} \widehat{\eta}_{[\alpha,k]} \frac{\partial}{\partial u_{\alpha,k}},$$

where

$$\begin{aligned} \widehat{\eta}_{[\alpha,k]} &= \mathcal{D}_k\left(\frac{\partial \eta_\alpha}{\partial x_1}\right) - \mathcal{D}_k\left(\frac{\partial \xi_j}{\partial x_1}\right) u_{\alpha,j} = \\ &= \frac{\partial}{\partial x_1}\left(\mathcal{D}_k(\eta_\alpha) - \mathcal{D}_k(\xi_j) u_{\alpha,j}\right) = \frac{\partial \eta_{[\alpha,k]}}{\partial x_1}. \end{aligned}$$

Hence,

$$[\Xi_1, \Xi_2]^{(1)} = \left[\Xi_1^{(1)}, \Xi_2^{(1)}\right].$$

□

Remark 4.4 Theorem 4.9 allows us to say that if a differential equation admits the Lie symmetries generated by the operators Ξ_1 and Ξ_2, then the Lie symmetry generated by $[\Xi_1, \Xi_2]$ is admitted as well. In other words, the Lie symmetries of a differential equation are the elements of a Lie algebra.

4.5 Beyond Lie Point Symmetries

The Lie groups heretofore considered are *point transformations* since they establish a mapping between *points* $(\mathbf{x}, \mathbf{u})$ and $(\mathbf{x}', \mathbf{u}')$. These transformations naturally prolong to the rth-order jet space $J^r(\mathbb{R}^{n+m}, n)$. Nothing prevents us to consider more general transformations that are not prolongations of point transformations, where a *point* $(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}) \in J^r(\mathbb{R}^{n+m}, n)$ is moved to another point in $J^r(\mathbb{R}^{n+m}, n)$ by a one-parameter group of transformations which is not the rth prolongation of a point transformation. Below we analyze these transformations for ordinary and partial differential equations.

4.5.1 Contact Symmetries of Ordinary Differential Equations

Let us consider the second-order ordinary differential equation

$$u_{,2} = F(u, u_{,1}), \tag{4.23}$$

that, for arbitrary function F, admits only the symmetry generated by

$$\Xi = \frac{\partial}{\partial x}.$$

We will see in Chap. 5 that the admitted Lie symmetry allows us to set $u_{,1} = v(u)$, whence $u_{,2} = \dfrac{dv}{du} u_{,1} = v v_{,1}$; as a consequence, Eq. (4.23) reduces to the first-order ordinary differential equation:

$$v_{,1} = \frac{F(u, v)}{v}.$$

The latter cannot admit any symmetry coming from symmetries of (4.23) (we say that there are no *inherited* symmetries) because the second-order ordinary differential equation, for an arbitrary function $F(u, u_{,1})$, admits only the translation with respect to x. Nevertheless, every first-order ordinary differential equation admits Lie point

symmetries, say those generated by

$$\Xi = \xi(u, v)\frac{\partial}{\partial u} + \eta(u, v)\frac{\partial}{\partial v},$$

for some functions ξ and η (for instance, the symmetry with $\eta = \xi F$, with ξ arbitrary function); this symmetry, in terms of the original variables, reads

$$\Xi = \xi(u, u^{(1)})\frac{\partial}{\partial u} + \eta(u, u^{(1)})\frac{\partial}{\partial u^{(1)}}$$

that cannot be the first prolongation of a Lie point symmetry because ξ depends on $u_{,1}$.

These considerations suggest to introduce more general transformations whose infinitesimal generators depend also on derivatives.

Thus, let us consider one-parameter groups of transformations of the form:

$$\begin{aligned} x' &= X(x, u, u^{(1)}, \ldots, u^{(r)}; a), \\ u' &= U(x, u, u^{(1)}, \ldots, u^{(r)}; a), \\ u^{(1)'} &= U^{(1)}(x, u, u^{(1)}, \ldots, u^{(r)}; a), \\ &\ldots \\ u^{(r)\prime} &= U^{(r)}(x, u, u^{(1)}, \ldots, u^{(r)}; a); \end{aligned} \tag{4.24}$$

these are called *contact transformations of order* r. Of course, we have to preserve contact conditions, say

$$\begin{aligned} du' &= U^{(1)}dx', \\ du^{(k)\prime} &= U^{(k+1)}dx', \qquad k = 1, \ldots, r-1, \end{aligned}$$

and this will imply that $U^{(1)}$ must depend on $u^{(r+1)}$, ..., $U^{(r)}$ must depend on $u^{(2r)}$, contrarily to the hypotheses. To avoid this contradiction, let us switch to the infinitesimal version of (4.24):

$$\begin{aligned} x' &= x + a\xi(x, u, u^{(1)}, \ldots, u^{(r)}) + O(a^2), \\ u' &= u + a\eta(x, u, u^{(1)}, \ldots, u^{(r)}) + O(a^2), \\ u^{(1)\prime} &= u + a\eta^{(1)}(x, u, u^{(1)}, \ldots, u^{(r)}) + O(a^2), \\ &\ldots \\ u^{(r)\prime} &= u + a\eta^{(r)}(x, u, u^{(1)}, \ldots, u^{(r)}) + O(a^2), \end{aligned}$$

where, in order the contact conditions be preserved, it is

$$\eta^{(k)} = \mathcal{D}_x(\eta^{(k-1)}) - u^{(k)}\mathcal{D}_x(\xi), \qquad k = 1, \ldots, r,$$

with $\eta^{(0)} = \eta$.

In order $\eta^{(k)}$ $(k = 1, \ldots, r)$ to be independent of derivatives of u of order higher than r, it is easily recognized that we have to impose the conditions

$$\frac{\partial \eta^{(k-1)}}{\partial u^{(r)}} - u^{(k)}\frac{\partial \xi}{\partial u^{(r)}} = 0, \qquad k = 1, \ldots, r. \tag{4.25}$$

Taking into account the recursive relation for $\eta^{(r-1)}$ and that

$$\frac{\partial}{\partial u^{(r)}}\mathcal{D}_x = \mathcal{D}_x\frac{\partial}{\partial u^{(r)}} + \frac{\partial}{\partial u^{(r-1)}},$$

condition (4.25), for $k = r$, gives

$$\begin{aligned}
0 &= \frac{\partial}{\partial u^{(r)}}\left(\mathcal{D}_x(\eta^{(r-2)}) - u^{(r-1)}\mathcal{D}_x(\xi)\right) - u^{(r)}\frac{\partial \xi}{\partial u^{(r)}} \\
&= \mathcal{D}_x\left(\frac{\partial \eta^{(r-2)}}{\partial u^{(r)}}\right) + \frac{\partial \eta^{(r-2)}}{\partial u^{(r-1)}} - u^{(r-1)}\mathcal{D}_x\left(\frac{\partial \xi}{\partial u^{(r)}}\right) \\
&\quad - u^{(r-1)}\frac{\partial \xi}{\partial u^{(r-1)}} - u^{(r)}\frac{\partial \xi}{\partial u^{(r)}} \\
&= \mathcal{D}_x\left(\frac{\partial \eta^{(r-2)}}{\partial u^{(r)}} - u^{(r-1)}\frac{\partial \xi}{\partial u^{(r)}}\right) + \frac{\partial \eta^{(r-2)}}{\partial u^{(r-1)}} - u^{(r-1)}\frac{\partial \xi}{\partial u^{(r-1)}} \\
&= \frac{\partial \eta^{(r-2)}}{\partial u^{(r-1)}} - u^{(r-1)}\frac{\partial \xi}{\partial u^{(r-1)}}.
\end{aligned}$$

Thus, it is

$$\begin{aligned}
\frac{\partial \eta^{(r-2)}}{\partial u^{(r-1)}} &= u^{(r-1)}\frac{\partial \xi}{\partial u^{(r-1)}}, \\
\frac{\partial \eta^{(r-2)}}{\partial u^{(r)}} &= u^{(r-1)}\frac{\partial \xi}{\partial u^{(r)}}.
\end{aligned}$$

Cross differentiation of the latter conditions yields

$$\frac{\partial \xi}{\partial u^{(r)}} = 0,$$

whence

$$\frac{\partial \eta^{(k)}}{\partial u^{(r)}} = 0, \qquad k = 0, \ldots, r-1.$$

Therefore, a contact transformation of order r must have all the infinitesimals ξ, η, $\eta^{(1)}$, …, $\eta^{(r-1)}$ independent of $u^{(r)}$, and so we have a contact transformation of order $r-1$ prolonged to the rth-order jet space.

We can repeat the above reasoning, and prove that the maximum order of contact transformation becomes $r-2$, $r-3$, …The process breaks down when $r=1$, and so we deduce that the generators of a contact transformation are prolongation of vector fields having the form

$$\Xi = \xi(x,u,u^{(1)})\frac{\partial}{\partial x} + \eta(x,u,u^{(1)})\frac{\partial}{\partial x} + \eta^{(1)}(x,u,u^{(1)})\frac{\partial}{\partial u^{(1)}},$$

subject to the condition

$$\frac{\partial \eta}{\partial u^{(1)}} = u^{(1)}\frac{\partial \xi}{\partial u^{(1)}}.$$

Introducing the function $\Omega(x,u,u^{(1)})$, the latter condition is satisfied by

$$\xi = \frac{\partial \Omega}{\partial u^{(1)}}, \qquad \eta = u^{(1)}\frac{\partial \Omega}{\partial u^{(1)}} - \Omega.$$

Because of its definition, it is also

$$\eta^{(1)} = -\frac{\partial \Omega}{\partial x} - u^{(1)}\frac{\partial \Omega}{\partial u^{(1)}},$$

whereupon the contact transformation, with generating function $\Omega(x,u,u_{,1})$, is represented by the vector field

$$\Xi = \frac{\partial \Omega}{\partial u^{(1)}}\frac{\partial}{\partial x} + \left(u^{(1)}\frac{\partial \Omega}{\partial u^{(1)}} - \Omega\right)\frac{\partial}{\partial u} + \left(-\frac{\partial \Omega}{\partial x} - u^{(1)}\frac{\partial \Omega}{\partial u^{(1)}}\right)\frac{\partial}{\partial u^{(1)}}.$$

To have a proper contact symmetry we need a generating function Ω which is not linear in $u^{(1)}$; if this is not the case, we have just the first prolongation of a point symmetry.

We cannot have proper contact symmetries if the differential equation involves more than one dependent variable. In fact, considering a contact transformation generated by the operator

$$\Xi = \xi(x,\mathbf{u},\mathbf{u}^{(1)})\frac{\partial}{\partial x} + \sum_{\alpha=1}^{m}\left(\eta_\alpha(x,\mathbf{u},\mathbf{u}^{(1)})\frac{\partial}{\partial u_\alpha} + \eta_\alpha^{(1)}(x,\mathbf{u},\mathbf{u}^{(1)})\frac{\partial}{\partial u_\alpha^{(1)}}\right),$$

we have to satisfy the conditions

$$\frac{\partial \eta_\alpha}{\partial u^{(1)}_\beta} = u^{(1)}_\alpha \frac{\partial \xi}{\partial u^{(1)}_\beta},$$
$$\frac{\partial \eta_\alpha}{\partial u^{(1)}_\gamma} = u^{(1)}_\alpha \frac{\partial \xi}{\partial u^{(1)}_\gamma}.$$

Cross differentiation yields

$$\delta_{\alpha\gamma} \frac{\partial \xi}{\partial u^{(1)}_\beta} = \delta_{\alpha\beta} \frac{\partial \xi}{\partial u^{(1)}_\gamma}.$$

If there are more than one dependent variable, we can choose $\alpha = \beta$ and $\alpha \neq \gamma$, and deduce

$$\frac{\partial \xi}{\partial u^{(1)}_\gamma} = 0,$$

whereupon it is also

$$\frac{\partial \eta_\alpha}{\partial u^{(1)}_\gamma} = 0.$$

This means that contact symmetries of ordinary differential equations involving more than one dependent variable are prolongations of Lie point symmetries.

4.5.2 Contact Symmetries of Partial Differential Equations

Using similar arguments, we can define contact symmetries also for scalar partial differential equations starting from

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}, u, \mathbf{u}^{(1)}; a),$$
$$u' = U(\mathbf{x}, u, \mathbf{u}^{(1)}; a),$$
$$\mathbf{u}^{(1)\prime} = \mathbf{U}_1(\mathbf{x}, u, \mathbf{u}^{(1)}; a),$$

to which corresponds the infinitesimal operator

$$\Xi = \sum_{i=1}^{n} \xi_i(\mathbf{x}, u, \mathbf{u}^{(1)}) \frac{\partial}{\partial x_i} + \eta(\mathbf{x}, u, \mathbf{u}^{(1)}) \frac{\partial}{\partial u} + \sum_{i=1}^{n} \eta_{[,i]}(\mathbf{x}, u, \mathbf{u}^{(1)}) \frac{\partial}{\partial u_{,i}},$$

where

$$\frac{\partial \eta}{\partial u_{,i}} = \sum_{k=1}^{n} u_{,k} \frac{\partial \xi_k}{\partial u_{,i}}, \qquad i = 1, \ldots, n.$$

Also in this case, there exists a generating function $\Omega(\mathbf{x}, u, \mathbf{u}^{(1)})$ such that

$$\xi_i = \frac{\partial \Omega}{\partial u_{,i}}, \qquad \eta = \sum_{i=1}^{n} u_{,i} \frac{\partial \Omega}{\partial u_{,i}} - \Omega, \qquad \eta_{[,i]} = -\left(u_{,i} \frac{\partial \Omega}{\partial u} + \frac{\partial \Omega}{\partial x_i} \right).$$

4.6 Concluding Remarks

The symmetries of differential equations heretofore presented are transformations in the space of independent and dependent variables (*point symmetries*), or transformations in the space of independent and dependent variables, and first-order derivatives (*contact* symmetries [13, 26]), transforming solutions into solutions. Both kinds of symmetries have a geometrical meaning.

If this geometrical interpretation is relaxed, a further enlargement is represented by the so-called *generalized symmetries*; roughly speaking, they are transformations in the space of the independent variables, dependent variables, and *all* (partial) derivatives [1, 29]. The generalized symmetries are introduced as formal vector fields whose components depend also on higher order derivatives. They prove useful in many contexts (see, for instance, [2, 3, 5, 8, 12, 14]). Anyway, the generalized symmetries are beyond the scope of this book and will not be discussed.

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Chapter 5
Lie Symmetries of Ordinary Differential Equations



5.1 Introduction

In this chapter, we use Lie symmetries to analyze ordinary differential equations. Sophus Lie proved that if a given ordinary differential equation admits a one-parameter Lie group of point transformations, then the order of the equation can be algorithmically lowered by one. Hence, the solution of the reduced equation and a quadrature give the solution of the original ordinary differential equation. For a first-order ordinary differential equation, the Lie method yields its quadrature, or, equivalently, enables us to find a first integral or an integrating factor. Furthermore, an ordinary differential equation of order n admitting an algebra of Lie symmetries possessing an r-dimensional ($r \leq n$) solvable Lie subalgebra, can be reduced to an ordinary differential equation of order $(n - r)$: in this case, the solution of the original ordinary differential equation is found by solving the reduced equation plus r quadratures.

The reader may refer to [7–10, 25, 26] for further elements and examples.

5.2 Basic Examples

Example 5.1 The first-order ordinary differential equation

$$u^{(1)} = f(x) \qquad \left(u^{(1)} = \frac{du}{dx}\right) \tag{5.1}$$

is easily solved by quadrature,

$$u = \int f(x)dx + c, \qquad c \text{ constant.}$$

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It is evident that Eq. (5.1) is invariant with respect to the Lie group (translation of the dependent variable)

$$x' = x, \qquad u' = u + a \tag{5.2}$$

because

$$u^{(1)\prime} = \frac{du'}{dx'} = \frac{du}{dx} = u^{(1)}, \qquad f(x') = f(x).$$

Conversely, the general equation

$$u^{(1)} = f(x, u)$$

is invariant with respect to (5.2) if the function f does not depend on u. This means that reduction by quadrature of (5.1) is possible if and only if Eq. (5.1) is invariant with respect to (5.2).

Example 5.2 The first-order homogeneous ordinary differential equation

$$u^{(1)} = f\left(\frac{u}{x}\right) \tag{5.3}$$

is invariant with respect to the scaling group

$$x' = \exp(a)x, \qquad u' = \exp(a)u; \tag{5.4}$$

in fact,

$$u^{(1)\prime} = \frac{du'}{dx'} = \frac{du}{dx} = u^{(1)}, \qquad f\left(\frac{u'}{x'}\right) = f\left(\frac{u}{x}\right).$$

By introducing the canonical variables

$$r = \frac{u}{x}, \qquad s = \log u,$$

the Lie group (5.4) assumes the form

$$r' = r, \qquad s' = s + a.$$

From

$$ds = \frac{1}{u}du, \qquad dr = -\frac{u}{x^2}dx + \frac{1}{x}du,$$

we obtain

$$u^{(1)} = \frac{r\dfrac{ds}{dr}}{\dfrac{ds}{dr} - \dfrac{1}{r}},$$

and Eq. (5.3) is mapped to

$$\frac{ds}{dr} = g(r), \qquad g(r) = \frac{f(r)}{rf(r) - r^2};$$

the latter is solved by quadrature,

$$s = \int g(r)dr + c;$$

in terms of the original variables, we get the solution

$$u = c' \exp\left(\int^{u/x} g(r)dr\right), \qquad c' \text{ constant.}$$

Example 5.3 It is easy to check that the first-order ordinary differential equation

$$u^{(1)} = -u^3 + \frac{1}{x\sqrt{x}} \tag{5.5}$$

is invariant with respect to the scaling group

$$x' = \exp(-2a)x, \qquad u' = \exp(a)u; \tag{5.6}$$

in fact, we have

$$u^{(1)'} = \frac{du'}{dx'} = \exp(3a)\frac{du}{dx} = \exp(3a)u^{(1)},$$
$$-(u')^3 + \frac{1}{x'\sqrt{x'}} = \exp(3a)\left(-u^3 + \frac{1}{x\sqrt{x}}\right).$$

By introducing the canonical variables

$$r = xu^2, \qquad s = \log u,$$

of the Lie group (5.6), we obtain

$$u^{(1)} = \frac{u^3\dfrac{ds}{dr}}{1 - 2r\dfrac{ds}{dr}},$$

and Eq. (5.5) is transformed to

$$\frac{ds}{dr} = \frac{-1 + r^{-3/2}}{1 + 2r^{-1/2} - 2r},$$

that can be solved by quadrature,

$$s = \int \frac{-1 + r^{-3/2}}{1 + 2r^{-1/2} - 2r} dr + c, \qquad c \text{ constant.}$$

The above examples show how to use a Lie symmetry admitted by a first-order ordinary differential equation. Introducing (if needed) the canonical variables, the equation is solved by quadrature.

5.3 First-Order Ordinary Differential Equations

Assume a general first-order ordinary differential equation

$$u^{(1)} = f(x, u) \tag{5.7}$$

to be left invariant by the Lie group of point transformations generated by

$$\Xi = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}. \tag{5.8}$$

This means that, by considering the first-order prolongation of the infinitesimal operator Ξ in (5.8),

$$\Xi^{(1)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \left(\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial x}\right)u^{(1)} - \frac{\partial \xi}{\partial u}(u^{(1)})^2\right)\frac{\partial}{\partial u^{(1)}},$$

the invariance condition

$$\Xi^{(1)}\left(u^{(1)} - f(x, u)\right)\Big|_{u^{(1)}=f(x,u)} = 0 \tag{5.9}$$

holds. The condition (5.9) implies that the infinitesimals $\xi(x, u)$ and $\eta(x, u)$ satisfy the linear partial differential equation

$$\frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial x}\right)f - \frac{\partial \xi}{\partial u}f^2 - \xi\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial u} = 0. \tag{5.10}$$

If we are able to find a solution to (5.10) for the infinitesimals $\xi(x, u)$ and $\eta(x, u)$, by introducing the canonical variables $r(x, u)$ and $s(x, u)$ such that

$$\Xi(r) = 0, \qquad \Xi(s) = 1,$$

the Lie group generated by (5.8) can be written as

$$r' = r, \qquad s' = s + a. \tag{5.11}$$

Furthermore, in terms of the canonical coordinates $r(x, u)$ and $s(x, u)$, it is

$$\frac{ds}{dr} = \frac{\dfrac{\partial s}{\partial x} + \dfrac{\partial s}{\partial u}u^{(1)}}{\dfrac{\partial r}{\partial x} + \dfrac{\partial r}{\partial u}u^{(1)}} = \frac{\dfrac{\partial s}{\partial x} + \dfrac{\partial s}{\partial u}f(x, u)}{\dfrac{\partial r}{\partial x} + \dfrac{\partial r}{\partial u}f(x, u)}. \tag{5.12}$$

Due to

$$\Xi(r) = \xi\frac{\partial r}{\partial x} + \eta\frac{\partial r}{\partial u} = 0,$$

relation (5.12) is nonsingular if $f(x, u) \neq \eta/\xi$. Determining x and u in terms of r and s, and using in (5.12), one obtains

$$\frac{ds}{dr} = F(r, s).$$

However, as the latter equation must be invariant with respect to the Lie group (5.11), the function F cannot depend on s, i.e., we have

$$\frac{ds}{dr} = F(r),$$

that can be solved by quadrature.

Remark 5.1 The Lie symmetries admitted by a first-order ordinary differential equation cannot be found in general, as we have two unknowns, ξ and η, and only one condition, say (5.10).

By direct substitution, it can be easily seen that (5.10) always admits the solution $\eta = \xi f$ for any ξ. Nevertheless, this solution (yielding a so-called trivial symmetry) is useless to reduce the equation to quadrature, unless we know how to solve the given differential equation. In fact, the canonical variables associated to the trivial symmetry

$$\Xi = \xi(x, u)\frac{\partial}{\partial x} + \xi(x, u)f(x, u)\frac{\partial}{\partial u}$$

are found from the conditions

$$\xi(x, u)\frac{\partial r}{\partial x} + \xi(x, u)f(x, u)\frac{\partial r}{\partial u} = 0,$$
$$\xi(x, u)\frac{\partial s}{\partial x} + \xi(x, u)f(x, u)\frac{\partial s}{\partial u} = 1,$$

whose solution requires the integration of the characteristic equation

$$\frac{dx}{\xi(x,u)} = \frac{du}{\xi(x,u)f(x,u)},$$

i.e., we need the solution to Eq. (5.7) itself.

Anyway, we may consider a special ansatz for the infinitesimals η and/or ξ (for instance, polynomial in x and u of a fixed degree) and check if the invariance condition is satisfied (see Example 5.5).

5.3.1 Integrating Factors

The first-order ordinary differential equation (5.7) can be rewritten as

$$g(x,u)dx + h(x,u)du = 0, \qquad f(x,u) = -\frac{g(x,u)}{h(x,u)}. \tag{5.13}$$

The general solution is a family of curves

$$\phi(x,u) = c, \qquad c \text{ constant},$$

such that

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial u}du = 0$$

is equivalent to (5.13), and this implies

$$h(x,u)\frac{\partial\phi}{\partial x} - g(x,u)\frac{\partial\phi}{\partial u} = 0. \tag{5.14}$$

If a Lie group of the form (5.8) is admitted by the differential equation, it leaves the family of solution curves invariant, that is, transforms a solution curve to another solution curve. Therefore, if Ξ is the infinitesimal generator of the Lie group, it has to be

$$\Xi(\phi(x,u)) = \xi(x,u)\frac{\partial\phi}{\partial x} + \eta(x,u)\frac{\partial\phi}{\partial u} = 1. \tag{5.15}$$

From conditions (5.14) and (5.15), it follows

$$\frac{\partial\phi}{\partial x} = \frac{g(x,u)}{\xi(x,u)g(x,u) + \eta(x,u)h(x,u)},$$
$$\frac{\partial\phi}{\partial u} = \frac{h(x,u)}{\xi(x,u)g(x,u) + \eta(x,u)h(x,u)};$$

this is correct if

$$\xi(x,u)g(x,u)+\eta(x,u)h(x,u)\neq 0 \quad \Leftrightarrow \quad \eta(x,u)\neq \xi(x,u)f(x,u),$$

i.e., we have a nontrivial symmetry, whence we deduce that

$$\mu(x,u)=\frac{1}{\xi(x,u)g(x,u)+\eta(x,u)h(x,u)} \tag{5.16}$$

is an integrating factor. Conversely, if $\mu(x,u)$ is an integrating factor, then any $\xi(x,u)$, $\eta(x,u)$ satisfying (5.16) give the infinitesimals of the Lie group leaving Eq. (5.7) invariant.

A simple application is when in Eq. (5.13) $g(x,u)$ and $h(x,u)$ are homogeneous functions of the same order α, that is

$$g(\lambda x,\lambda u)=\lambda^{\alpha}g(x,u), \qquad h(\lambda x,\lambda u)=\lambda^{\alpha}h(x,u),$$

with $\lambda\in\mathbb{R}^{+}$. This implies

$$\begin{aligned} x\frac{\partial g}{\partial x}+u\frac{\partial g}{\partial u}&=\alpha g,\\ x\frac{\partial h}{\partial x}+u\frac{\partial h}{\partial u}&=\alpha h. \end{aligned}$$

It is easy to observe that in such a case (5.13) is invariant with respect to the Lie group of transformations

$$x'=\lambda x, \qquad u'=\lambda u,$$

whose generator is

$$\Xi=x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}.$$

Thus, the integrating factor is immediately found:

$$\mu(x,u)=\frac{1}{xg(x,u)+uh(x,u)}.$$

Example 5.4 The first-order ordinary differential equation

$$u^{(1)}=\frac{x^2u^2+7x^3u-4u^4}{xu^3-2x^2u^2+3x^4} \tag{5.17}$$

can be written equivalently as

$$(x^2u^2+7x^3u-4u^4)dx+(-xu^3+2x^2u^2-3x^4)du=0. \tag{5.18}$$

We may easily determine an integrating factor by observing that equation (5.17) (as well as (5.18)) is invariant with respect to the Lie group generated by

$$\Xi = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

Consequently, an integrating factor arises, i.e.,

$$\mu(x,u) = \frac{1}{x(x^2u^2 + 7x^3u - 4u^4) + u(-xu^3 + 2x^2u^2 - 3x^4)}.$$

Example 5.5 The ordinary differential equation [25]

$$u^{(1)} = \frac{u^3 - 3x^2u + u}{-x^3 + 3xu^2 + x}, \tag{5.19}$$

by direct inspection, does not admit simple symmetries like scalings or translations. Looking for the infinitesimals as third degree polynomials in x and u we obtain:

$$\begin{aligned}\xi &= a_1(u^3 - 3x^2u + u) + a_2(-x^3 + 3xu^2 + x),\\ \eta &= a_1(x^3 - 3xu^2 - x) + a_2(u^3 - 3x^2u + u).\end{aligned}$$

We observe that for $a_1 = 0$ we have a trivial symmetry, i.e.,

$$\frac{\eta}{\xi} = \frac{u^3 - 3x^2u + u}{-x^3 + 3xu^2 + x}.$$

On the contrary, setting $a_2 = 0$, we have a nontrivial symmetry generated by the vector field

$$\Xi = (u^3 - 3x^2u + u)\frac{\partial}{\partial x} + (x^3 - 3xu^2 - x)\frac{\partial}{\partial u}.$$

Instead of looking for the canonical variables (whose determination appears not so easy), let us integrate (5.19) by means of an integrating factor. So, we have

$$\int \frac{(u^3 - 3x^2u + u)dx + (x^3 - 3xu^2 - x)du}{(u^3 - 3x^2u + u)^2 + (-x^3 + 3xu^2 + x)^2} = c, \qquad c \text{ arbitrary constant},$$

yielding

$$\frac{1}{2}\arctan\left(\frac{u}{x+1}\right) + \frac{1}{2}\arctan\left(\frac{u}{x-1}\right) - \arctan\left(\frac{u}{x}\right) = c,$$

whence

$$\frac{(x^2+u^2)^2 - x^2 + u^2}{xu} = c_1, \qquad c_1 \text{ arbitrary constant},$$

i.e., the solution in implicit form.

5.4 Higher Order Ordinary Differential Equations

Now, let us move to the application of Lie groups of point transformations to the study of a second or higher order ordinary differential equation like

$$u^{(n)} = f(x, u, u^{(1)}, \ldots, u^{(n-1)}), \qquad n \geq 2. \tag{5.20}$$

From a geometrical point of view, it defines a surface in the jet space with coordinates $(x, u, u^{(1)}, \ldots, u^{(n)})$.

Assume that Eq. (5.20) admits a one-parameter Lie group of point transformations with infinitesimal operator

$$\Xi = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}. \tag{5.21}$$

The invariance of (5.20) under a Lie group of point transformations constructively leads to reducing (5.20) to an $(n-1)$th-order ordinary differential equation plus a quadrature.

When $n \geq 2$, the admitted infinitesimals of Lie groups of point transformations satisfy an overdetermined system of linear determining equations that have only a finite number of linearly independent solutions. In practice, we are usually able to compute explicitly (if they exist) the admitted Lie groups of point transformations of an nth-order ordinary differential equation in the case $n \geq 2$.

In the next subsections, we illustrate the procedures to reduce the order of an ordinary differential equation admitting one-parameter Lie groups of transformations by means of either canonical variables or differential invariants; moreover, also the case of multi-parameter Lie groups is considered.

5.4.1 Order Reduction through Canonical Variables

Theorem 5.1 ([7, 9]) *Suppose a nontrivial one-parameter Lie group of transformations with infinitesimal generator* (5.21) *is admitted by Eq.* (5.20)*, with* $n \geq 2$*. Let* $r(x, u)$*,* $s(x, u)$ *be the corresponding canonical coordinates satisfying* $\Xi(r) = 0$*,* $\Xi(s) = 1$*. Then, the nth-order ordinary differential equation* (5.20) *reduces to an* $(n-1)$*th-order ordinary differential equation,*

$$\frac{d^{n-1}z}{dr^{n-1}} = F\left(r, z, \frac{dz}{dr}, \ldots, \frac{d^{n-2}z}{dr^{n-2}}\right), \tag{5.22}$$

where $z = \dfrac{ds}{dr}$.

Proof In terms of the canonical coordinates $r(x,u)$ and $s(x,u)$ it is

$$\frac{ds}{dr} = \frac{\dfrac{\partial s}{\partial x} + \dfrac{\partial s}{\partial u}u^{(1)}}{\dfrac{\partial r}{\partial x} + \dfrac{\partial r}{\partial u}u^{(1)}}. \tag{5.23}$$

By differentiating (5.23), we get

$$\frac{d^2s}{dr^2} = \frac{d\left(\dfrac{ds}{dr}\right)}{dr} = u^{(2)} f_1\left(r, s, \frac{ds}{dr}\right) + g_1\left(r, s, \frac{ds}{dr}\right), \tag{5.24}$$

where

$$\begin{aligned}
f_1\left(r, s, \frac{ds}{dr}\right) &= \frac{1}{\left(\dfrac{\partial r}{\partial x} + \dfrac{\partial r}{\partial u}u^{(1)}\right)^3}\left(\frac{\partial r}{\partial x}\frac{\partial s}{\partial u} - \frac{\partial r}{\partial u}\frac{\partial s}{\partial x}\right),\\
g_1\left(r, s, \frac{ds}{dr}\right) &= \frac{1}{\left(\dfrac{\partial r}{\partial x} + \dfrac{\partial r}{\partial x}u^{(1)}\right)^3}\left((u^{(1)})^3\left(\frac{\partial r}{\partial u}\frac{\partial^2 s}{\partial u^2} - \frac{\partial s}{\partial u}\frac{\partial^2 r}{\partial u^2}\right)\right.\\
&\quad + (u^1)^2\left(\frac{\partial r}{\partial x}\frac{\partial^2 s}{\partial u^2} + 2\left(\frac{\partial r}{\partial u}\frac{\partial^2 s}{\partial x \partial u} - \frac{\partial s}{\partial u}\frac{\partial^2 r}{\partial x \partial u}\right) - \frac{\partial s}{\partial x}\frac{\partial^2 r}{\partial u^2}\right)\\
&\quad + u^{(1)}\left(\frac{\partial r}{\partial u}\frac{\partial^2 s}{\partial x^2} + 2\left(\frac{\partial r}{\partial x}\frac{\partial^2 s}{\partial x \partial u} - \frac{\partial s}{\partial x}\frac{\partial^2 r}{\partial x \partial u}\right) - \frac{\partial s}{\partial u}\frac{\partial^2 r}{\partial x^2}\right)\\
&\quad \left. + \left(\frac{\partial r}{\partial x}\frac{\partial^2 s}{\partial x^2} - \frac{\partial s}{\partial x}\frac{\partial^2 r}{\partial x^2}\right)\right).
\end{aligned}$$

Solving (5.23) with respect to $u^{(1)}$, we have

$$u^{(1)} = \frac{\dfrac{\partial s}{\partial x} - \dfrac{\partial r}{\partial x}\dfrac{ds}{dr}}{\dfrac{\partial r}{\partial u}\dfrac{ds}{dr} - \dfrac{\partial s}{\partial u}},$$

that, used in (5.24), yields

$$u^{(2)} = \frac{d^2s}{dr^2}F_1\left(r, s, \frac{ds}{dr}\right) + G_1\left(r, s, \frac{ds}{dr}\right),$$

with $F_1 = 1/f_1$ and $G_1 = -g_1/f_1$. Since r and s are canonical coordinates, it is $\dfrac{\partial r}{\partial x}\dfrac{\partial s}{\partial u} - \dfrac{\partial r}{\partial u}\dfrac{\partial s}{\partial x} \neq 0$, and hence $f_1 \neq 0$. Proceeding inductively, we have

$$\frac{d^k s}{dr^k} = u^{(k)} f_1\left(r, s, \frac{ds}{dr}\right) + g_{k-1}\left(r, s, \frac{ds}{dr}, \ldots, \frac{d^{k-1}s}{dr^{k-1}}\right)$$

for some function $g_{k-1}\left(r, s, \frac{ds}{dr}, \ldots, \frac{d^{k-1}s}{dr^{k-1}}\right)$. This leads us to write

$$u^{(k)} = \frac{d^k s}{dr^k} F_1\left(r, s, \frac{ds}{dr}\right) + G_{k-1}\left(r, s, \frac{ds}{dr}, \ldots, \frac{d^{k-1}s}{dr^{k-1}}\right),$$

where

$$G_{k-1} = -\frac{g_{k-1}}{f_1}, \qquad k \geq 2.$$

Finally, the original ordinary differential equation may be written in the normal form

$$\frac{d^n s}{dr^n} = F\left(r, s, \frac{ds}{dr}, \ldots, \frac{d^{n-1}s}{dr^{n-1}}\right) \tag{5.25}$$

for some function $F\left(r, s, \frac{ds}{dr}, \ldots, \frac{d^{n-1}s}{dr^{n-1}}\right)$. Since (5.25) is invariant with respect to the translation in s, the function F does not depend on s. This allows us to introduce $z = \frac{ds}{dr}$ and write (5.20) in the form (5.22) as an $(n-1)$th-order ordinary differential equation. □

Remark 5.2 If $z = \phi(r; c_1, \ldots, c_{n-1})$ ($c_1, \ldots, c_{n-1}$ arbitrary constants) is the general solution of Eq. (5.22), then the quadrature

$$s(x, u) = \int^{r(x,u)} \phi(t; c_1, \ldots, c_{n-1}) dt + c_n$$

gives the general solution of (5.20), where also c_n is an arbitrary constant.

Example 5.6 Consider the second-order linear homogeneous ordinary differential equation

$$u^{(2)} + p(x)u^{(1)} + q(x)u = 0. \tag{5.26}$$

It admits the Lie group of symmetries (scaling of the dependent variable)

$$x' = x, \qquad u' = \exp(a)u. \tag{5.27}$$

Introducing the canonical coordinates

$$r = x, \qquad s = \log u,$$

it is

$$z = \frac{ds}{dr} = \frac{u^{(1)}}{u},$$

whereupon

$$u^{(1)} = zu, \qquad u^{(2)} = zu^{(1)} + u\frac{dz}{dr} = z^2u + u\frac{dz}{dr}.$$

Therefore, the equation can be rewritten as

$$\frac{dz}{dr} + z^2 + p(r)z + q(r) = 0,$$

known as Riccati equation. If we know a particular solution $z_0(r)$ of Riccati equation, then, setting

$$z = z_0 + \frac{1}{w},$$

we obtain the linear equation

$$\frac{dw}{dr} - p(r)w - 1 = 0$$

whose solution is

$$w(r) = \exp\left(\int p(r)dr\right)\int\left(\exp\left(-\int p(r)dr\right)\right)dr + c, \quad c \text{ constant.}$$

5.4.2 Order Reduction through Differential Invariants

For a Lie group generated by

$$\Xi = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$$

an invariant $J(x,u)$ is found by solving the condition

$$\Xi(J) = \xi(x,u)\frac{\partial J}{\partial x} + \eta(x,u)\frac{\partial J}{\partial u} = 0.$$

Taking the kth prolongation of a Lie group, we may define the differential invariant of order k as follows.

Definition 5.1 (*Differential invariant*) A *differential invariant* of order $k \geq 1$ of the Lie group generated by

$$\Xi = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$$

is a function $J_k(x, u, u^{(1)}, \ldots, u^{(k)})$, with $\dfrac{\partial J_k}{\partial u^{(k)}} \neq 0$, such that

$$\Xi^{(k)}(J_k) = \xi \frac{\partial J_k}{\partial x} + \eta \frac{\partial J_k}{\partial u} + \eta^{(1)}(x, u, u^{(1)}) \frac{\partial J_k}{\partial u^{(1)}} + \cdots + \eta^{(k)}(x, u, u^{(1)}, \ldots, u^{(k)}) \frac{\partial J_k}{\partial u^{(k)}} = 0.$$

Remark 5.3 An invariant $J(x, u)$ can be considered as a differential invariant of order zero.

Remark 5.4 If the equation

$$F(x, u, u^{(1)}, \ldots, u^{(n)}) \equiv u^{(n)} - f(x, u, u^{(1)}, \ldots, u^{(n-1)}) = 0$$

admits the Lie point symmetry corresponding to the infinitesimal generator

$$\Xi = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u},$$

i.e.,

$$\Xi^{(n)}(F)\big|_{F=0} = 0,$$

the function F must depend on its arguments through the differential invariants

$$J(x, u), \quad J_1(x, u, u^{(1)}), \quad \ldots, \quad J_n(x, u, u^{(1)}, \ldots, u^{(n)})$$

of Ξ, where

$$\Xi(J(x, u)) = 0, \quad \Xi^{(k)}(J_k(x, u, u^{(1)}, \ldots, u^{(k)})) = 0, \qquad k = 1, \ldots, n$$

with

$$\frac{\partial J_k}{\partial u^{(k)}} \neq 0, \qquad k = 1, \ldots, n.$$

The differential invariants of order k are found by integrating the characteristic equations

$$\frac{dx}{\xi(x, u)} = \frac{du}{\eta(x, u)} = \frac{du^{(1)}}{\eta^{(1)}(x, u, u^{(1)})} = \ldots = \frac{du^{(k)}}{\eta^{(k)}(x, u, u^{(1)}, \ldots, u^{(k)})}.$$

However, we really need to integrate only the characteristic equations for finding the differential invariants of order 0 and 1.

In fact, since $J(x, u)$ and $J_1(x, u, u^{(1)})$ are invariants under the action of the nth prolonged group, it follows that $J_2 = \dfrac{dJ_1}{dJ}$ is a second-order differential invariant,

$J_3 = \dfrac{dJ_2}{dJ} = \dfrac{d^2J_1}{dJ^2}$ a third-order differential invariant, …, $J_n = \dfrac{dJ_{n-1}}{dJ} = \dfrac{d^{n-1}J_1}{dJ^{n-1}}$ an nth-order differential invariant.

In terms of the differential invariants, the nth-order ordinary differential equation has order $(n-1)$:

$$\frac{d^{n-1}v}{dt^{n-1}} = H\left(t, v, \frac{dv}{dt}, \ldots, \frac{d^{n-2}v}{dt^{n-2}}\right),$$

where we set $t = J$ and $v = J_1$, for some function H.

Example 5.7 By considering Eq. (5.26), the first prolongation of the Lie group (5.27) possesses the invariants

$$J(x,u) = x, \qquad J_1(x,u,u^{(1)}) = \frac{u^{(1)}}{u}.$$

The second-order differential invariant is

$$J_2 = \frac{dJ_1}{dJ} = \frac{u^{(2)}}{u} - \frac{(u^{(1)})^2}{u^2}.$$

Thus,

$$u^{(1)} = ut, \qquad u^{(2)} = u\frac{dv}{dt} + uv^2,$$

where we set $t = J$, $v = J_1$, and Eq. (5.26) transforms to

$$\frac{dv}{dt} + v^2 + p(t)v + q(t) = 0,$$

which is, as one would expect, the Riccati equation.

5.4.3 *Order Reduction with Multi-Parameter Lie Groups*

By considering the case of an nth-order ordinary differential equation which admits an r-parameter Lie group of point symmetries, it can be easily proved that if the corresponding Lie algebra is solvable then we may reduce the order to $n - r$. If $r = 2$, or the Lie algebra has a two-dimensional Lie subalgebra, then the order can be always reduced to $(n-2)$ because every two-dimensional Lie algebra is solvable (see Theorem 3.3).

Theorem 5.2 *If the equation*

$$u^{(n)} = f(x, u, u^{(1)}, \ldots, u^{(n-1)})$$

admits an r-dimensional ($r \leq n$) solvable Lie algebra as subalgebra of the Lie algebra of its point symmetries, then it is possible to reduce algorithmically its order to $(n - r)$.

Proof Let $\mathbb{L}_r$ be the solvable Lie algebra spanned by the operators

$$\{\Xi_1, \Xi_2, \ldots, \Xi_r\},$$

suitably listed such that we have

$$\left[\Xi_\mu, \Xi_\nu\right] = \sum_{\lambda=1}^{\nu-1} C_{\mu\nu}^{\lambda} \Xi_\lambda.$$

Let us introduce the canonical variables r_1 and s_1 satisfying $\Xi_1(r_1) = 0$, $\Xi_1(s_1) = 1$. The equation now becomes

$$\frac{d^{n-1} v_1}{dr_1^{n-1}} = F_1\left(r_1, v_1, \frac{dv_1}{dr_1}, \ldots, \frac{dv_1^{n-2}}{dr_1^{n-2}}\right),$$

where $v_1 = \dfrac{ds_1}{dr_1}$.

To use Ξ_2 we need to write it in terms of the new independent and dependent variables, r_1 and v_1, respectively; however, since v_1 is a derivative, we need the first prolongation $\Xi_2^{(1)}$. We have:

$$\Xi_2^{(1)} = \Xi_2^{(1)}(r_1)\frac{\partial}{\partial r_1} + \Xi_2^{(1)}(v_1)\frac{\partial}{\partial v_1}.$$

Due to the relations

$$\begin{aligned}
\Xi_1^{(1)}\Xi_2^{(1)}(r_1) &= \Xi_2^{(1)}\Xi_1^{(1)}(r_1) + C_{12}^1\Xi_1^{(1)}(r_1) = 0,\\
\Xi_1^{(1)}\Xi_2^{(1)}(v_1) &= \Xi_2^{(1)}\Xi_1^{(1)}(v_1) + C_{12}^1\Xi_1^{(1)}(v_1) = 0,
\end{aligned}$$

we deduce that $\Xi_2^{(1)}(r_1)$ is a function only of the invariant r_1 of Ξ_1, and that $\Xi_2^{(1)}(v_1)$ is a function at most of r_1 and v_1 (which is a differential invariant of order 1 for Ξ_1).[1]

By introducing the canonical variables r_2 and s_2 corresponding to $\Xi_2^{(1)}$, the equation reduces to

$$\frac{d^{n-2} v_2}{dr_2^{n-2}} = F_2\left(r_2, v_2, \frac{dv_2}{dr_2}, \ldots, \frac{dv_2^{n-3}}{dr_2^{n-3}}\right),$$

[1] If we compute $\Xi_2^{(1)}(s_1)$, this is $s_1 + J(r_1)$, but we do not need it because s_1 does not appear in the reduced equation.

where $v_2 = \dfrac{ds_2}{dr_2}$. To use Ξ_3 we need its second-order prolongation. By virtue of the relations

$$\left[\Xi_1^{(2)}, \Xi_3^{(2)}\right] = C_{13}^1 \Xi_1^{(1)} + C_{13}^2 \Xi_2^{(2)},$$
$$\left[\Xi_2^{(2)}, \Xi_3^{(2)}\right] = C_{23}^1 \Xi_1^{(1)} + C_{23}^2 \Xi_2^{(2)},$$

we have that $\Xi_3^{(2)}(r_2)$ is at most a function of r_2, and $\Xi_3^{(2)}(v_2)$ is at most a function of r_2 and v_2.

Continuing the process, we arrive to an $(n-r)$th-order differential equation

$$\frac{d^{n-r} v_r}{dr_r^{n-r}} = F_r\left(r_r, v_r, \ldots, \frac{d^{n-r-1} v_r}{dr_r^{n-r-1}}\right).$$

□

Example 5.8 (*Blasius equation*) The Blasius equation

$$u^{(3)} + \frac{1}{2} u u^{(2)} = 0 \tag{5.28}$$

is a third-order ordinary differential equation arising in some problems in fluid dynamics [6]. Looking for its Lie symmetries, it is easy to ascertain that it admits the operators

$$\Xi_1 = \frac{\partial}{\partial x}, \qquad \Xi_2 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},$$

satisfying the following commuting relation:

$$[\Xi_1, \Xi_2] = \Xi_1.$$

Let us prove that the Blasius equation can be reduced to a first-order differential equation by using in sequence Ξ_1 and Ξ_2.

The canonical variables associated to Ξ_1 are $r_1 = u$ and $s_1 = x$, whence

$$\frac{ds_1}{dr_1} = \frac{dx}{du} = \frac{1}{u^{(1)}},$$
$$\frac{d^2 s_1}{dr_1^2} = -\frac{u^{(2)}}{(u^{(1)})^2},$$
$$\frac{d^3 s_1}{dr_1^2} = -\frac{u^{(3)}}{(u^{(1)})^4} + 3\frac{(u^{(2)})^2}{(u^{(1)})^5}.$$

Solving the previous equations with respect to $u^{(1)}$, $u^{(2)}$ end $u^{(3)}$, and substituting in (5.28), we get

$$\frac{ds_1}{dr_1}\frac{d^3s_1}{dr_1^3} - 3\left(\frac{d^2s_1}{dr_1^2}\right)^2 + \frac{1}{2}r_1\left(\frac{ds_1}{dr_1}\right)^2\frac{d^2s_1}{dr_1^2} = 0.$$

Setting $v_1 = \dfrac{ds_1}{dr_1}$, we get the second-order equation

$$v_1\frac{d^2v_1}{dr_1^2} - 3\left(\frac{dv_1}{dr_1}\right)^2 + \frac{1}{2}r_1v_1^2\frac{dv_1}{dr_1} = 0. \tag{5.29}$$

The first-order prolongation of Ξ_2, in the variables r_1 and v_1, has the form

$$\Xi_2 = -r_1\frac{\partial}{\partial r_1} + 2v_1\frac{\partial}{\partial v_1},$$

and its canonical variables are

$$r_2 = v_1r_1^2, \qquad s_2 = \frac{1}{2}\ln v_1.$$

Computing

$$\frac{ds_2}{dr_2} = \frac{1}{2}\frac{\dfrac{dv_1}{dr_1}}{v_1\left(r_1^2\dfrac{dv_1}{dr_1} + 2r_1v_1\right)},$$

$$\frac{d^2s_2}{dr_2^2} = -\frac{6r_1v\left(\dfrac{dv_1}{dr_1}\right)^2 + r_1^2\left(\dfrac{dv_1}{dr_1}\right)^3 + 2v_1^2\left(\dfrac{dv_1}{dr_1} - r_1\dfrac{d^2v_1}{dr_1^2}\right)}{2r_1^3v_1^2\left(2v_1 + r_1\dfrac{dv_1}{dr_1}\right)^3},$$

solving with respect to $\dfrac{dv_1}{dr_1}$ and $\dfrac{d^2v_1}{dr_1^2}$, and substituting in (5.29), we obtain:

$$4r_2\frac{d^2s_2}{dr_2^2} - 4r_2^2(r_2+6)\left(\frac{ds_2}{dr_2}\right)^3 + 4r_2(r_2+2)\left(\frac{ds_2}{dr_2}\right)^2 - (r_2+2)\frac{ds_2}{dr_2} = 0.$$

Finally, setting $v_2 = \dfrac{ds_2}{dr_2}$, we obtain the first-order differential equation

$$4r_2\frac{dv_2}{dr_2} - 4r_2^2(r_2+6)v_2^3 + 4r_2(r_2+2)v_2^2 - (r_2+2)v_2 = 0.$$

Remark 5.5 It can be easily verified that in the procedure we cannot interchange the order we apply the generators: in fact, using first the operator Ξ_2, the first prolongation of Ξ_1 cannot be expressed in terms of the variables introduced by Ξ_2. This means that Ξ_1 is not inherited by the reduced equation obtained by using the canonical variables of Ξ_2.

Example 5.9 (*Chazy equation*) The Chazy equation [11]

$$u^{(3)} = 2uu^{(2)} - 3(u^{(1)})^2 \tag{5.30}$$

has been derived in 1909 as an example of a third-order differential equation with a movable singularity that is a natural boundary for its solutions. It is left invariant by the Lie point symmetries generated by

$$\Xi_1 = \frac{\partial}{\partial x}, \qquad \Xi_2 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \qquad \Xi_3 = x^2\frac{\partial}{\partial x} - (2xu+6)\frac{\partial}{\partial u},$$

satisfying the following commutation relations:

$$[\Xi_1, \Xi_2] = \Xi_1, \qquad [\Xi_1, \Xi_3] = 2\Xi_2, \qquad [\Xi_2, \Xi_3] = \Xi_3.$$

Thus, the Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3\}$ is not solvable.

Nevertheless, the two-dimensional Lie subalgebra spanned by $\{\Xi_1, \Xi_2\}$, as well as the two-dimensional subalgebra spanned by $\{\Xi_2, \Xi_3\}$, are solvable. Therefore, Eq. (5.30) can be reduced in two different ways to a first-order differential equation. Let us consider the two-dimensional Lie subalgebra spanned by $\{\Xi_2, \Xi_3\}$, and use the method of differential invariants. Because $[\Xi_2, \Xi_3] = \Xi_3$, we have to use first the symmetry generated by Ξ_3 and then the symmetry generated by Ξ_2. In fact, if t and v are such that

$$\Xi_3\left(t(x,u)\right) = 0, \qquad \Xi_3^{(1)}\left(v(x,t,u^{(1)})\right) = 0,$$

the first prolongation of Ξ_2 writes in terms of t and v, and so the corresponding symmetry is inherited by the second-order reduced equation. The zeroth-order invariant (t) and the first-order differential invariant (v) of the vector field Ξ_3 are such that

$$\begin{aligned} &x^2\frac{\partial t}{\partial x} - (2xu+6)\frac{\partial t}{\partial u} = 0 \\ &x^2\frac{\partial v}{\partial x} - (2xu+6)\frac{\partial v}{\partial u} - (2u+4xu^{(1)})\frac{\partial v}{\partial u^{(1)}} = 0, \end{aligned}$$

whose integration yields

$$t = x(xu+6), \qquad v = x^2(x^2u^{(1)} + 2xu + 6).$$

Assuming t as independent variable, and v as dependent one, second- and third-order differential invariants are

$$v^{(1)} = \frac{dv}{dt} = \frac{x^4u^{(2)} + 6x^3u^{(1)} + 6x^2u + 12x}{x^2u^{(1)} + 2xu + 6},$$
$$v^{(2)} = \frac{dv^{(1)}}{dt} = \Big(72(1+xu) + 12x^2u^2 + x^2(108 + 24xu)u^{(1)} + x^3(48 + 12xu - x^3u^{(2)})u^{(2)} + x^4(6 + 2xu + x^2u^{(1)})u^{(3)}\Big)\Big(x^2u^{(1)} + 2xu + 6\Big)^{-3}.$$

In terms of the differential invariants t, v, $v^{(1)}$ and $v^{(2)}$, the Chazy equation reduces to the second-order differential equation

$$vv^{(2)} + (v^{(1)})^2 - 2tv^{(1)} + 3v = 0$$

for the unknown $v(t)$. Moreover, the first-order prolongation of Ξ_2 becomes

$$\Xi_2^{(1)} = t\frac{\partial}{\partial t} + 2v\frac{\partial}{\partial v} + v^{(1)}\frac{\partial}{\partial v^{(1)}},$$

yielding the invariants

$$z = \frac{v}{t^2}, \qquad w = \frac{v^{(1)}}{t};$$

so, the second-order differential invariant writes

$$w^{(1)} = \frac{dw}{dz} = \frac{t(tv^{(2)} - v^{(1)})}{tv^{(1)} - 2v}.$$

Finally, we obtain the first-order differential equation

$$z(2z - w)w^{(1)} - w^2 - (z-2)w - 3z = 0$$

for the unknown $w(z)$.

5.4.4 Joint Differential Invariants

An ordinary differential equation of order n, say

$$u^{(n)} = f(x, u, u^{(1)}, \dots, u^{(n-1)}),$$

admitting a Lie symmetry, can be written in terms of the corresponding differential invariants; if more than one Lie symmetry is admitted, then it may be written (up

to a multiplicative function nowhere vanishing depending on the coordinates of the jet space) as a function of the common differential invariants of the symmetries. Denoting by

$$\Xi_\alpha = \xi_\alpha(x,u)\frac{\partial}{\partial x} + \eta_\alpha(x,u)\frac{\partial}{\partial u}, \qquad \alpha = 1,\dots r$$

the αth vector field, the fundamental differential invariants of the vector fields $\{\Xi_1,\dots,\Xi_r\}$ are solutions of the system

$$\xi_\alpha \frac{\partial J}{\partial x} + \eta_\alpha \frac{\partial J}{\partial u} + \eta_\alpha^{(1)} \frac{\partial J}{\partial u^{(1)}} + \cdots + \eta_\alpha^{(r)} \frac{\partial J}{\partial u^{(r)}} = 0, \qquad \alpha = 1,\dots r.$$

If the matrix

$$M = \begin{pmatrix} \xi_1 & \eta_1 & \eta_1^{(1)} & \cdots & \eta_1^{(r)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \cdots & \eta_2^{(r)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{r-1} & \eta_{r-1} & \eta_{r-1}^{(1)} & \cdots & \eta_{r-1}^{(r)} \\ \xi_r & \eta_r & \eta_r^{(1)} & \cdots & \eta_r^{(r)} \end{pmatrix}$$

has rank r (in this case, we say that the vector fields span an r-dimensional distribution, see Definition 8.10) then we have $r-2$ differential invariants, that are found by the method of characteristics, possibly after reducing the matrix M to row echelon form [5, 25].

Example 5.10 The third-order ordinary differential equation

$$xu^{(1)}u^{(3)} - 2x(u^{(2)})^2 - u^{(1)}u^{(2)} - (u^{(1)})^3 = 0$$

possesses the Lie symmetries spanned by

$$\Xi_1 = \frac{\partial}{\partial u}, \qquad \Xi_2 = x\frac{\partial}{\partial x}.$$

As differential invariants of both symmetries (one independent of $u^{(2)}$), satisfying the equations

$$\frac{\partial J}{\partial u} = 0, \qquad x\frac{\partial J}{\partial x} - u^{(1)}\frac{\partial J}{\partial u^{(1)}} - 2u^{(2)}\frac{\partial J}{\partial u^{(2)}} = 0,$$

let us take $t = xu^{(1)}$ and $v = x^2u^{(2)}$. We can deduce immediately the third-order differential invariant

$$v^{(1)} = \frac{dv}{dt} = \frac{2xu^{(2)} + x^2u^{(3)}}{u^{(1)} + xu^{(2)}},$$

and the equation at hand assumes the form

$$v^{(1)} = \frac{2v^2 + 3tv + t^3}{t(t+v)},$$

i.e., it can be written in terms of the differential invariants.

In [5] (see also [35]), a constructive algorithm, that reduces the determination of joint local invariants of any finite-dimensional Lie algebra of vector fields, as well as of any finite number of vector fields, to that of a commuting family of vector fields. The algorithm relies on Frobenius integrability theorem (see Theorem 8.2).

The procedure in [5] is based on the following two theorems.

Theorem 5.3 ([5]) *Let $\mathbb{L}$ be a finite-dimensional Lie algebra of vector fields defined on some open subset U of $\mathbb{R}^n$. Let $\{\Xi_1, \dots, \Xi_d\}$ be a basis of $\mathbb{L}$. Then:*

1. *The algebra of operators whose coefficient matrix is the matrix of functions obtained from the coefficients of $\Xi_1, \dots, \Xi_d$ by reducing it to reduced row echelon form is Abelian.*
2. *The local joint invariants of $\mathbb{L}$ are the same as those of the above Abelian algebra.*

Theorem 5.4 ([5]) *Let $\Xi_1, \dots, \Xi_d$ be vector fields defined on some open subset of $\mathbb{R}^n$. Then the joint invariants of $\Xi_1, \dots, \Xi_d$ are given by the following algorithm:*

Step 1 *Find the row reduced echelon form of the vector fields $\Xi_1, \dots, \Xi_d$, and let $\widetilde{\Xi}_1, \dots, \widetilde{\Xi}_r$ be the corresponding vector fields. If this is a commuting family, then stop. Otherwise go to Step 2.*

Step 2 *If some $[\widetilde{\Xi}_i, \widetilde{\Xi}_j] \neq 0$, then set $\widetilde{\Xi}_{r+1} = [\widetilde{\Xi}_i, \widetilde{\Xi}_j] \neq 0$. Go to Step 1 and substitute $\widetilde{\Xi}_1, \dots, \widetilde{\Xi}_r, \widetilde{\Xi}_{r+1}$ in place of $\Xi_1, \dots, \Xi_d$.*

This process terminates in a finite number of iterations. If $\widehat{\Xi}, \dots, \widehat{\Xi}_m$ are the commuting vector fields at the end of the above iterative process, the joint invariants of $\Xi_1, \dots, \Xi_d$ coincide with the joint invariants of $\widehat{\Xi}, \dots, \widehat{\Xi}_m$.

5.5 Second-Order Ordinary Differential Equations Reducible to Quadrature

A second-order ordinary differential equation is reducible to quadrature if it admits as subalgebra of its Lie algebra of symmetries a two-dimensional Abelian or solvable Lie algebra.

Given a second-order ordinary differential equation in normal form

$$u^{(2)} = f(x, u, u^{(1)}), \tag{5.31}$$

its admitted Lie symmetries, generated by

$$\Xi = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u},$$

are found by solving

$$\begin{aligned}&\frac{\partial^2\eta}{\partial x^2}+\left(2\frac{\partial^2\eta}{\partial x\partial u}-\frac{\partial^2\xi}{\partial x^2}\right)u^{(1)}+\left(\frac{\partial^2\eta}{\partial u^2}-2\frac{\partial^2\xi}{\partial x\partial u}\right)(u^{(1)})^2.\\&\quad-\frac{\partial^2\xi}{\partial u^2}(u^{(1)})^2+\left(\frac{\partial\eta}{\partial y}-2\frac{\partial\xi}{\partial x}-3\frac{\partial\xi}{\partial u}u^{(1)}\right)f-\xi\frac{\partial f}{\partial x}-\eta\frac{\partial f}{\partial u}.\\&\quad-\left(\frac{\partial\eta}{\partial x}+\left(\frac{\partial\eta}{\partial u}-\frac{\partial\xi}{\partial x}\right)u^{(1)}-\frac{\partial\xi}{\partial u}(u^{(1)})^2\right)\frac{\partial f}{\partial u^{(1)}}=0.\end{aligned}\tag{5.32}$$

Condition (5.32) can be viewed under two different perspectives:

1. assign f, and find ξ and η, i.e., determine the symmetries admitted by a given equation;
2. assign ξ and η, and find f, i.e., determine the most general second-order equation admitting a given symmetry.

Let us focus on the second perspective, and assume that (5.31) admits a two-dimensional Lie algebra $\mathbb{L}_2$ of symmetries generated by

$$\begin{aligned}\Xi_1&=\xi_1(x,u)\frac{\partial}{\partial x}+\eta_1(x,u)\frac{\partial}{\partial u},\\\Xi_2&=\xi_2(x,u)\frac{\partial}{\partial x}+\eta_2(x,u)\frac{\partial}{\partial u}.\end{aligned}$$

Because of Theorem 3.3, we may distinguish the case where $\mathbb{L}_2$ is Abelian from the one where $\mathbb{L}_2$ is solvable but not Abelian. In turn, in each case we may distinguish two subcases according to the rank of the distribution of $\{\Xi_1, \Xi_2\}$. Therefore, we have four cases [26]:

Case 1: $[\Xi_1, \Xi_2]=0$, $\Xi_1\vee\Xi_2\neq 0$;
Case 2: $[\Xi_1, \Xi_2]=0$, $\Xi_1\vee\Xi_2=0$;
Case 3: $[\Xi_1, \Xi_2]=\Xi_1$, $\Xi_1\vee\Xi_2\neq 0$;
Case 4: $[\Xi_1, \Xi_2]=\Xi_1$, $\Xi_1\vee\Xi_2=0$.

where $\Xi_1\vee\Xi_2=\xi_1\eta_2-\xi_2\eta_1$ (pseudo-scalar product).

Let us observe that an invertible transformation, besides preserving the Lie bracket (see Theorem 2.14), does not change the rank of a distribution. In fact, by considering an invertible change of variables,

$$t=\widehat{t}(x,u),\qquad v=\widehat{v}(x,u),\qquad \frac{\partial\widehat{t}}{\partial x}\frac{\partial\widehat{v}}{\partial u}-\frac{\partial\widehat{t}}{\partial u}\frac{\partial\widehat{v}}{\partial x}\neq 0,$$

the generators Ξ_i $(i=1,2)$ become

$$\Xi_i=\widehat{\xi}_i\frac{\partial}{\partial t}+\widehat{\eta}_i\frac{\partial}{\partial v}=\left(\xi_i\frac{\partial\widehat{t}}{\partial x}+\eta_i\frac{\partial\widehat{t}}{\partial u}\right)\frac{\partial}{\partial t}+\left(\xi_i\frac{\partial\widehat{v}}{\partial x}+\eta_i\frac{\partial\widehat{v}}{\partial u}\right)\frac{\partial}{\partial v}.$$

It is

$$\widehat{\xi}_1\widehat{\eta}_2 - \widehat{\xi}_2\widehat{\eta}_1 = (\xi_1\eta_2 - \xi_2\eta_1)\left(\frac{\partial \widehat{t}}{\partial x}\frac{\partial \widehat{v}}{\partial u} - \frac{\partial \widehat{t}}{\partial u}\frac{\partial \widehat{v}}{\partial x}\right).$$

Therefore, the four cases can be discussed assuming four simple (canonical) forms of generators, and we may always reduce the equation to quadrature by a suitable change of variables.

Case 1.

$$\Xi_1 = \frac{\partial}{\partial x}, \qquad \Xi_2 = \frac{\partial}{\partial u}. \tag{5.33}$$

Using (5.33) into (5.32), the second-order ordinary differential equation

$$u^{(2)} = f(u^{(1)}), \tag{5.34}$$

where $f(u^{(1)})$ is an arbitrary function of $u^{(1)}$, is characterized. Equation (5.34) can be easily integrated,

$$F(u^{(1)}) = x + c_1, \qquad F(u^{(1)}) = \int \frac{du^{(1)}}{f(u^{(1)})}.$$

A further integration yields

$$u = \int F^{-1}(x + c_1)dx + c_2,$$

where c_1 and c_2 are arbitrary constants.

Case 2.

$$\Xi_1 = \frac{\partial}{\partial u}, \qquad \Xi_2 = x\frac{\partial}{\partial u}. \tag{5.35}$$

Using (5.35) into (5.32), the second-order ordinary differential equation

$$u^{(2)} = f(x), \tag{5.36}$$

where $f(x)$ is an arbitrary function of x, is characterized. Equation (5.36) is immediately integrated:

$$u = \int \left(\int f(x)dx\right)dx + c_1x + c_2,$$

where c_1 and c_2 are arbitrary constants.

Case 3.

$$\Xi_1 = \frac{\partial}{\partial u}, \qquad \Xi_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}. \tag{5.37}$$

Using (5.37) into (5.32), the second-order ordinary differential equation

$$u^{(2)} = \frac{1}{x} f(u^{(1)}), \tag{5.38}$$

where $f(u^{(1)})$ is an arbitrary function of $u^{(1)}$, is characterized. Equation (5.38) can be easily integrated,

$$F(u^{(1)}) = \log x + \log c_1, \qquad F(u^{(1)}) = \int \frac{du^{(1)}}{f(u^{(1)})},$$

and a further integration yields

$$u = \int F^{-1}(\log(c_1 x))dx + c_2,$$

where c_1 and c_2 are arbitrary constants.

Case 4.

$$\Xi_1 = \frac{\partial}{\partial u}, \qquad \Xi_2 = u\frac{\partial}{\partial u}. \tag{5.39}$$

Using (5.39) into (5.32), the second-order ordinary differential equation

$$u^{(2)} = f(x)u^{(1)}, \tag{5.40}$$

where $f(x)$ is an arbitrary function of x, is characterized. Equation (5.40) is immediately integrated,

$$u = c_1 \int \exp\left(\int f(x)dx\right)dx + c_2,$$

where c_1 and c_2 are arbitrary constants.

Example 5.11 The second-order ordinary differential equation

$$x^7 u^{(2)} + x u^4 u^{(1)} - u^5 = 0 \tag{5.41}$$

admits the two-dimensional Lie algebra of symmetries spanned by the generators

$$\Xi_1 = x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \quad \Xi_2 = -x\frac{\partial}{\partial x} - \frac{5}{4}u\frac{\partial}{\partial u},$$

that are such that

$$[\Xi_1, \Xi_2] = \Xi_1, \qquad \det\begin{pmatrix} x^2 & xu \\ -x & -\frac{5}{4}u \end{pmatrix} = -\frac{1}{4}x^2 u \neq 0.$$

This means that we are in the case 3 considered before. Let us show that we may construct an invertible transformation

$$t = \widehat{t}(x, u), \qquad v = \widehat{v}(x, u)$$

(t will be the new independent variable, and v the new dependent one) such that the generators Ξ_1 and Ξ_2 assume the canonical form of case 3. The new variables t and v have to be found in such a way

$$\Xi_1(t) = 0, \quad \Xi_1(v) = 1, \quad \Xi_2(t) = t, \quad \Xi_2(v) = v,$$

i.e.,

$$\begin{aligned} x^2\frac{\partial t}{\partial x} + xu\frac{\partial t}{\partial u} &= 0, \qquad x^2\frac{\partial v}{\partial x} + xu\frac{\partial v}{\partial u} = 1, \\ x\frac{\partial t}{\partial x} + \frac{5}{4}u\frac{\partial t}{\partial u} &= -t, \qquad x\frac{\partial v}{\partial x} + \frac{5}{4}u\frac{\partial v}{\partial u} = -v. \end{aligned}$$

The integration of the latter conditions gives the invertible transformation

$$t = \left(\frac{x}{u}\right)^4, \qquad v = -\frac{1}{x}.$$

From

$$dv = \frac{1}{x^2}dx, \qquad dt = 4\frac{x^3}{u^4}dx - 4\frac{x^4}{u^5}du,$$

we obtain

$$\frac{dv}{dt} = \frac{u^5}{4x^5(u - xu^{(1)})}, \tag{5.42}$$

and then

$$\frac{d^2v}{dt^2} = \frac{u^9(-5u^2 + 10xuu^{(1)} + x^2uu^{(2)} - 5x^2(u^{(1)})^2)}{16x^9(u^3 - 3xu^2u^{(1)} + 3x^2u(u^{(1)})^2 - x^3(u^{(1)})^3)}. \tag{5.43}$$

By solving (5.42) and (5.43) with respect to $u^{(1)}$ and $u^{(2)}$, we have

$$\begin{aligned} u^{(1)} &= \frac{-u^5 + 4x^5u\dfrac{dv}{dt}}{4x^6\dfrac{dv}{dt}}, \\ u^{(2)} &= \frac{5u^9\dfrac{dv}{dt} + 4x^4u^5\dfrac{d^2v}{dt^2}}{16x^{12}\left(\dfrac{dv}{dt}\right)^3}. \end{aligned}$$

Their use in (5.41) yields

$$\frac{d^2v}{dt^2} = \frac{1}{t}\left(\left(\frac{dv}{dt}\right)^2 - \frac{5}{4}\frac{dv}{dt}\right),$$

belonging to the canonical form already considered in case 3, that can be solved by two successive quadratures.

Example 5.12 The second-order ordinary differential equation

$$u^3(u^2 + u^{(1)})(uu^{(2)} - 2(u^{(1)})^2) + k(u^{(1)})^4 = 0, \qquad k \text{ constant}, \tag{5.44}$$

admits the three-dimensional Lie algebra of symmetries spanned by the generators

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = u^2\frac{\partial}{\partial u}, \quad \Xi_3 = \left(x - \frac{1}{3u}\right)\frac{\partial}{\partial x} - \frac{2}{3}u\frac{\partial}{\partial u},$$

whose commutator table is

	Ξ_1	Ξ_2	Ξ_3
Ξ_1	0	0	Ξ_1
Ξ_2	0	0	$(\Xi_1 + 2\Xi_2)/3$
Ξ_3	$-\Xi_1$	$-(\Xi_1 + 2\Xi_2)/3$	0

It is immediate to recognize that we may consider the following two-dimensional Lie subalgebras:

$$\begin{aligned}
&\mathbb{L}_2^1 = \{\Xi_1, \Xi_2\}, \qquad [\Xi_1, \Xi_2] = 0;\\
&\mathbb{L}_2^2 = \{\Xi_1, \Xi_3\}, \qquad [\Xi_1, \Xi_3] = \Xi_1;\\
&\mathbb{L}_2^3 = \left\{\Xi_1 - \Xi_2, \frac{3}{2}\Xi_3\right\}, \qquad \left[\Xi_1 - \Xi_2, \frac{3}{2}\Xi_3\right] = \Xi_1 - \Xi_2.
\end{aligned}$$

The Lie subalgebra $\mathbb{L}_2^1$ becomes the canonical form of case 1 by introducing the new dependent variable $v = 1/u$, whereupon

$$u^{(1)} = -\frac{v^{(1)}}{v^2}, \qquad u^{(2)} = \frac{2(v^{(1)})^2 - vv^{(2)}}{v^3}.$$

Thence, Eq. (5.44) transforms into

$$v^{(2)} = \frac{k(v^{(1)})^4}{1 - v^{(1)}}.$$

The Lie subalgebra $\mathbb{L}_2^2$ becomes the canonical form of case 3 by introducing the new independent variable $t = u^{-3/2}$ and the new dependent variable $v = x - u^{-1}$, whence

$$u^{(1)} = -\frac{2u^{5/2}}{2u^{1/2} + 3v^{(1)}}, \qquad u^{(2)} = \frac{16u^{9/2} + 30u^4 v^{(1)} + 18u^{5/2} v^{(2)}}{(2u^{1/2} + 3v^{(1)})^3}.$$

Thus, Eq. (5.44) transforms into

$$v^{(2)} = -\frac{1}{t}\left(\frac{v^{(1)}}{3} - \frac{8k}{27v^{(1)}}\right),$$

belonging to canonical form of case 3.

Finally, the Lie subalgebra $\mathbb{L}_2^3$ becomes the canonical form of case 3 by introducing the new independent variable $t = (x - 1/u)^{2/3}$ and the new dependent variable $v = 1/u$. So,

$$u^{(1)} = -\frac{2v^{(1)}}{v^2(2v^{(1)} + 3t^{1/2})},$$
$$u^{(2)} = \frac{2tu^2(v^{(1)}(3 + 12tuv^{(1)} + 16t^{1/2}u(v^{(1)})^2) - 12tv^{(2)}}{t^{3/2}(2v^{(1)} + 3t^{1/2}v^3)^3},$$

and Eq. (5.44) transforms into

$$v^{(2)} = \frac{1}{t}\frac{v^{(1)}(8k(v^{(1)})^3 + 9)}{18},$$

belonging to canonical form of case 3.

Remark 5.6 It is worth of being remarked that there are second-order ordinary differential equations that do not possess any Lie symmetry and nevertheless can be solved by two successive quadratures. An example is given by the equation [2]

$$u^{(2)} = \frac{(u^{(1)})^2}{u} + pg(x)u^p u^{(1)} + g'(x)u^{p+1}, \tag{5.45}$$

where p is a nonzero constant, and $g(x)$ a nonzero arbitrary function. This equation does not admit any Lie symmetry unless

$$g(x) = a_1 \exp(a_2 x), \quad \text{or} \quad g(x) = a_1(a_2 x + a_3)^{a_4} \exp(a_5 x),$$

where a_i $(i = 1, \ldots, 5)$ are constants.

Nevertheless, Eq. (5.45) can be integrated once to give

$$\frac{u^{(1)}}{u} - g(x)u^p = c_1,$$

which is a Bernoulli equation, that can be written as a linear equation in the unknown u^{-p}, and then integrated to give

$$u = \frac{\exp(c_1 x)}{\left(c_2 - p \int (g(x) \exp(c_1 p x))\, dx\right)^{1/p}},$$

where c_1 and c_2 are the integration constants.

5.6 Linearizable Second-Order Ordinary Differential Equations

The simplest second-order ordinary differential equation

$$u^{(2)} = 0$$

admits an eight-dimensional Lie algebra of point symmetries. Lie proved that eight is the maximal dimension of the algebra of point symmetries of second-order ordinary differential equations (a proof can be found in [15]).

The question whether a second-order ordinary differential equation, say

$$v^{(2)} = F(t, v, v^{(1)}), \tag{5.46}$$

where F is a smooth function of the indicated arguments, can be mapped by an invertible point transformation like

$$u = f(t, v), \qquad x = g(t, v) \tag{5.47}$$

into the linear equation

$$u^{(2)} = 0 \tag{5.48}$$

can be posed. This question is answered by the following theorem.

Theorem 5.5 *The following statements are equivalent:*

1. *Equation* (5.46) *can be mapped by an invertible point transformation like* (5.47) *into* (5.48)*;*
2. *Equation* (5.46) *admits an eight-dimensional Lie algebra of point symmetries;*
3. *Equation* (5.46) *admits a two-dimensional Lie algebra of point symmetries spanned by the generators* Ξ_1 *and* Ξ_2 *such that* $\Xi_1 \vee \Xi_2 = 0$*;*
4. *Equation* (5.46) *is such that*

$$F = F_0(t, v) + F_1(t, v)\frac{dv}{dt} + F_2(t, v)\left(\frac{dv}{dt}\right)^2 + F_3(t, v)\left(\frac{dv}{dt}\right)^3,$$

where F_0*,* F_1*,* F_2 *and* F_3 *write in terms of the partial derivatives of* f *and* g*.*

Proof The equivalence between (1) and (2) is immediate. If such a linearizing transformation exists, since the target equation admits an eight-dimensional Lie algebra, an isomorphic Lie algebra of point symmetries has to be admitted by the source equation, and vice versa.

The equivalence between (1) and (3) is easily proved by recalling the canonical forms of second-order ordinary differential equations admitting a two-dimensional Lie algebra of point symmetries. If the source equation admits a two-dimensional Lie algebra spanned by Ξ_1 and Ξ_2 such that $\Xi_1 \vee \Xi_2 = 0$, then necessarily it can be written (possibly after a change of variables) in one of the two following canonical forms:

$$u^{(2)} = h(x),$$
$$u^{(2)} = h(x)u^{(1)},$$

where $h(x)$ is some smooth function. In the first case, the reduction to

$$u^{(2)} = 0$$

is produced by means of the replacement

$$u \to u - \int \left(\int h(x)dx \right) dx,$$

whereas in the second case by means of the replacement

$$x \to \int \exp\left(\int h(x)\, dx \right) dx.$$

Finally, the equivalence between 1) and 4) is proved by a direct approach. Since the transformation we consider is (at least locally) invertible, we may start from the linear equation and determine the most general form of a second-order ordinary differential equation equivalent to it.

In fact, from

$$\frac{d^2u}{dx^2} = 0,$$

and considering the invertible point transformation (5.47), we obtain

$$\frac{d^2v}{dt^2} = F_0(t, v) + F_1(t, v)\frac{dv}{dt} + F_2(t, v)\left(\frac{dv}{dt}\right)^2 + F_3(t, v)\left(\frac{dv}{dt}\right)^3,$$

with

$$F_0 = \frac{1}{J}\left(\frac{\partial f}{\partial t}\frac{\partial^2 g}{\partial t^2} - \frac{\partial^2 f}{\partial t^2}\frac{\partial g}{\partial t}\right),$$
$$F_1 = \frac{1}{J}\left(\frac{\partial f}{\partial v}\frac{\partial^2 g}{\partial t^2} - \frac{\partial^2 f}{\partial t^2}\frac{\partial g}{\partial v} + 2\frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial t \partial v} - 2\frac{\partial^2 f}{\partial t \partial v}\frac{\partial g}{\partial t}\right),$$
$$F_2 = \frac{1}{J}\left(\frac{\partial f}{\partial t}\frac{\partial^2 g}{\partial v^2} - \frac{\partial^2 f}{\partial v^2}\frac{\partial g}{\partial t} + 2\frac{\partial f}{\partial v}\frac{\partial^2 g f}{\partial t \partial v} - 2\frac{\partial^2 f}{\partial t \partial v}\frac{\partial g}{\partial v}\right),$$
$$F_3 = \frac{1}{J}\left(\frac{\partial f}{\partial v}\frac{\partial^2 g}{\partial v^2} - \frac{\partial^2 f}{\partial v^2}\frac{\partial g}{\partial v}\right),$$

where

$$J = \frac{\partial f}{\partial v}\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}\frac{\partial g}{\partial v} \neq 0$$

because the transformation (5.47) is invertible. □

5.7 Second-Order Ordinary Differential Equations with a Three-Dimensional Lie Algebra

The classification of second-order ordinary differential equations on the basis of their Lie point symmetries has been performed by Lie himself who proved that the maximal algebra admitted by a given second-order equation can be only of dimension 0, 1, 2, 3, or 8 [26]. Dimension 0 means that the equation does not admit a group of point transformations; it is not difficult to find examples of equations with no symmetries, like Eq. (5.45), or the first Painlevé equation $u^{(2)} = 6u^2 + x$ [14], or equations with only one symmetry, like $u^{(2)} = (x - u - 2)u^{(1)}$. We already considered the canonical forms of second-order ordinary differential equations admitting two-dimensional Lie algebras and the equations admitting eight-dimensional Lie algebras.

Let us show the second-order equations admitting at least a three-dimensional Lie algebra of point symmetries. It is known that all three-dimensional indecomposable real Lie algebras can be classified into nine types [33, 34]. Hereafter, we list simple realizations in the (x, u) plane of these Lie algebras, and the corresponding second-order ordinary differential equations $u^{(2)} = f(x, u, u^{(1)})$ admitting them. The second-order differential equations are characterized by substituting in (5.32) the infinitesimals ξ and η of the symmetries realizing each Lie algebra.

$A_{3,1}$: nonzero Lie brackets:

$$[\Xi_2, \Xi_3] = \Xi_1;$$

the vector fields

$$\Xi_1 = \frac{\partial}{\partial u}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = x\frac{\partial}{\partial u}$$

are admitted by the equation

$$u^{(2)} = c, \quad c \text{ constant.}$$

$A_{3,2}$: nonzero Lie brackets:

$$[\Xi_1, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_1 + \Xi_2;$$

the vector fields

$$\Xi_1 = \frac{\partial}{\partial u}, \quad \Xi_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \Xi_3 = x\frac{\partial}{\partial x} + (x+u)\frac{\partial}{\partial u}$$

are admitted by the equation

$$u^{(2)} = 0.$$

$A_{3,3}$: nonzero Lie brackets:

$$[\Xi_1, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_2;$$

the vector fields

$$\Xi_1 = \frac{\partial}{\partial u}, \quad \Xi_2 = x\frac{\partial}{\partial u}, \quad \Xi_3 = u\frac{\partial}{\partial u}$$

are admitted by the equation

$$u^{(2)} = cu^{(1)} \quad c \text{ constant.}$$

$A_{3,4}$: nonzero Lie brackets:

$$[\Xi_1, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_3] = -\Xi_2;$$

the vector fields

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = \frac{\partial}{\partial u}, \quad \Xi_3 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$$

are admitted by the equation

$$u^{(2)} = c(u_{,1})^{3/2}, \quad c \text{ constant,}$$

whereas the vector fields

$$\Xi_1 = \frac{\partial}{\partial u}, \quad \Xi_2 = x^2\frac{\partial}{\partial x}, \quad \Xi_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$$

are admitted by the equation

$$u^{(2)} = c\frac{(u^{(1)})^{3/2} - 2u^{(1)}}{x}, \quad c \text{ constant.}$$

$A_{3,5}$: nonzero Lie brackets:

$$[\Xi_1, \Xi_3] = \Xi_1, \quad [\Xi_2, \Xi_3] = a\Xi_2, \quad 0 < |a| < 1;$$

the vector fields

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = \frac{\partial}{\partial u}, \qquad \Xi_3 = x\frac{\partial}{\partial x} + au\frac{\partial}{\partial u}, \quad 0 < |a| < 1,$$

are admitted by the equation

$$u^{(2)} = c(u^{(1)})^{(a-2)/(a-1)}, \quad c \text{ constant.}$$

$A_{3,6}$: nonzero Lie brackets:

$$[\Xi_1, \Xi_3] = -\Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_1;$$

the vector fields

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = \frac{\partial}{\partial u}, \quad \Xi_3 = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}$$

are admitted by the equation

$$u^{(2)} = c(1 + (u^{(1)})^2)^{3/2}, \quad c \text{ constant.}$$

$A_{3,7}$: nonzero Lie brackets:

$$[\Xi_1, \Xi_3] = a\Xi_1 - \Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_1 + a\Xi_2, \quad a > 0;$$

the vector fields

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = \frac{\partial}{\partial u}, \quad \Xi_3 = (ax + u)\frac{\partial}{\partial x} + (au - x)\frac{\partial}{\partial u}, \quad a > 0,$$

are admitted by the equation

$$u^{(2)} = c(1 + (u^{(1)})^2)^{3/2} \exp(a \arctan(u^{(1)}), \quad c \text{ constant.}$$

$A_{3,8}$: nonzero Lie brackets:

$$[\Xi_1, \Xi_2] = \Xi_1, \quad [\Xi_1, \Xi_3] = -2\Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_3;$$

the vector fields

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = x\frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u}, \quad \Xi_3 = -x^2\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$$

are admitted by the equation

$$u^{(2)} = \frac{c(u^{(1)})^{3/2} + 2u^{(1)}(1+u^{(1)})}{x-u},$$

whereas the vector fields

$$\Xi_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \Xi_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad \Xi_3 = -x^2\frac{\partial}{\partial x} - u^2\frac{\partial}{\partial u}$$

are admitted by the equation

$$u^{(2)} = cu^{-3}, \quad c\text{constant}.$$

$A_{3,9}$: nonzero Lie brackets:

$$[\Xi_1, \Xi_2] = \Xi_3, \quad [\Xi_1, \Xi_3] = -\Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_1;$$

the vector fields

$$\Xi_1 = (1+x^2)\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \quad \Xi_2 = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \quad \Xi_3 = xu\frac{\partial}{\partial x} + (1+u^2)\frac{\partial}{\partial u}$$

are admitted by the equation

$$u^{(2)} = c\left(\frac{1+u^2+(1+x^2)(u^{(1)})^2 - 2xuu^{(1)}}{1+x^2+u^2}\right)^{3/2}, \quad c \text{ constant}.$$

The Lie algebras $A_{3,8}$ and $A_{3,9}$ are simple, the remaining ones are solvable; moreover, $A_{3,1}$ is also nilpotent. In Chap. 12, for such Lie algebras we compute by means of the program SYMBOLIE [4] the optimal systems of Lie subalgebras.

5.8 Lie Symmetries and First Integrals

In the previous sections, we have seen how to use a Lie symmetry admitted by differential equations to lower its order by one. Moreover, for an ordinary differential equation of order n, an r-dimensional solvable Lie subalgebra of the algebra of point symmetries, when $r \leq n$, allows for its reduction to an ordinary differential equation of order $n - r$. This approach may fail, for instance, for a second-order ordinary

differential equation admitting a three-dimensional Lie algebra isomorphic to $\mathfrak{so}(3)$ (the algebra $A_{3,9}$ above described); the latter does not possess two-dimensional Lie subalgebras, so that the differential equation cannot be reduced to quadrature with the methods previously described. Nevertheless, there is an alternative approach based on the existence of first integrals induced by Lie symmetries.

An nth-order ordinary differential equation,

$$u^{(n)} = f(x, u, u^{(1)}, \dots, u^{(n-1)}), \tag{5.49}$$

possesses the first integral

$$\phi(x, u, u^{(1)}, \dots, u^{(n-1)}) = c, \qquad c \text{ constant}$$

if

$$\mathcal{D}_x(\phi)|_{u^{(n)}=f} = 0, \qquad \frac{\partial \phi}{\partial u^{(n-1)}} \neq 0. \tag{5.50}$$

Introducing the operator

$$\widetilde{\mathcal{D}}_x = \mathcal{D}_x|_{u^{(n)}=f},$$

the condition (5.50) can be written as

$$\widetilde{\mathcal{D}}_x(\phi) = 0, \qquad \frac{\partial \phi}{\partial u^{(n-1)}} \neq 0.$$

The vector field

$$\Xi = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}$$

of a symmetry of (5.49) satisfies the invariance condition

$$\left(\eta^{(n)} - \xi\frac{\partial f}{\partial x} - \eta\frac{\partial f}{\partial u} - \sum_{k=1}^{n-1}\eta^{(k)}\frac{\partial f}{\partial u^{(k)}}\right)\Bigg|_{u^{(n)}=f} = 0. \tag{5.51}$$

In terms of the so-called characteristic $Q = \eta - \xi u^{(1)}$, it is

$$\eta^{(k)} = \mathcal{D}_x^k(Q) + \xi u^{(k+1)}, \qquad k = 1, \dots, n;$$

so, condition (5.51) reads

$$\widetilde{\mathcal{D}}_x^n(Q) - \frac{\partial f}{\partial u^{(n-1)}}\widetilde{\mathcal{D}}_x^{n-1}(Q) - \cdots - \frac{\partial f}{\partial u^{(1)}}\widetilde{\mathcal{D}}_x(Q) - \frac{\partial f}{\partial u}Q = 0, \tag{5.52}$$

where the condition $u^{(n)} - f = 0$ and its differential consequence have been used. Assume that Eq. (5.49) admits $r > n$ symmetries with characteristics

$Q_1, Q_2, \ldots, Q_r$. Taking $n+1$ linearly independent characteristics, let us construct the matrix

$$M = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_{n+1} \\ \widetilde{\mathcal{D}}_x(Q_1) & \widetilde{\mathcal{D}}_x(Q_2) & \cdots & \widetilde{\mathcal{D}}_x(Q_{n+1}) \\ \widetilde{\mathcal{D}}_x^2(Q_1) & \widetilde{\mathcal{D}}_x^2(Q_2) & \cdots & \widetilde{\mathcal{D}}_x^2(Q_{n+1}) \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{\mathcal{D}}_x^n(Q_1) & \widetilde{\mathcal{D}}_x^n(Q_2) & \cdots & \widetilde{\mathcal{D}}_x^n(Q_{n+1}) \end{pmatrix},$$

whose rank cannot exceed n since, due to (5.52), the last row is a linear combination of the first n rows. Let us consider the case where the rank of M is n; possibly reordering the characteristics of the symmetries admitted by Eq. (5.49), the determinant of the matrix

$$M' = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_n \\ \widetilde{\mathcal{D}}_x(Q_1) & \widetilde{\mathcal{D}}_x(Q_2) & \cdots & \widetilde{\mathcal{D}}_x(Q_n) \\ \widetilde{\mathcal{D}}_x^2(Q_1) & \widetilde{\mathcal{D}}_x^2(Q_2) & \cdots & \widetilde{\mathcal{D}}_x^2(Q_n) \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{\mathcal{D}}_x^{n-1}(Q_1) & \widetilde{\mathcal{D}}_x^{n-1}(Q_2) & \cdots & \widetilde{\mathcal{D}}_x^{n-1}(Q_n) \end{pmatrix},$$

is nonvanishing. It is easy to verify that

$$\widetilde{\mathcal{D}}_x(\det(M')) = \frac{\partial f}{\partial u^{(n-1)}} \det(M'). \tag{5.53}$$

Moreover, there must exist functions $\phi_1, \ldots, \phi_n$ such that

$$M' \begin{pmatrix} \phi_1 \\ \phi_2 \\ \cdots \\ \phi_n \end{pmatrix} = \begin{pmatrix} Q_{n+1} \\ \widetilde{\mathcal{D}}_x(Q_{n+1}) \\ \cdots \\ \widetilde{\mathcal{D}}_x^{n-1}(Q_{n+1}) \end{pmatrix}, \tag{5.54}$$

together with

$$\phi_1 \widetilde{\mathcal{D}}_x^n(Q_1) + \phi_2 \widetilde{\mathcal{D}}_x^n(Q_2) + \cdots + \phi_n \widetilde{\mathcal{D}}_x^n(Q_n) = \widetilde{\mathcal{D}}_x^n(Q_{n+1}). \tag{5.55}$$

Action of operator $\widetilde{\mathcal{D}}_x$ on (5.54), taking into account (5.54) itself and (5.55), yields

$$\widetilde{\mathcal{D}}_x(\phi_k) = 0, \qquad k = 1, \ldots, n,$$

so that we have the first integrals (possibly, some of them may be constant). The explicit expressions of the functions ϕ_k are obtained by solving the linear system (5.54).

In the case where the rank of the matrix M is $m \leq n$, the ratios of all nonzero determinants of $m \times m$ matrices

$$\begin{pmatrix} Q_{i_1} & Q_{i_2} & \cdots & Q_{i_m} \\ \widetilde{\mathcal{D}}_x(Q_{i_1}) & \widetilde{\mathcal{D}}_x(Q_{i_2}) & \cdots & \widetilde{\mathcal{D}}_x(Q_{i_m}) \\ \widetilde{\mathcal{D}}_x^2(Q_{i_1}) & \widetilde{\mathcal{D}}_x^2(Q_{i_2}) & \cdots & \widetilde{\mathcal{D}}_x^2(Q_{i_m}) \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{\mathcal{D}}_x^{m-1}(Q_{i_1}) & \widetilde{\mathcal{D}}_x^{m-1}(Q_{i_2}) & \cdots & \widetilde{\mathcal{D}}_x^{m-1}(Q_{i_m}) \end{pmatrix},$$

because of (5.53), yield either first integrals or constants. Notice that in this case, since for a first integral $\phi = c$ the condition $\frac{\partial \phi}{\partial u^{(n-1)}} \neq 0$ is required, we have first integrals only when $m = n - 1$ or $m = n$.

In general, if the obtained first integrals are less than n, it is possible to check the existence of a supplementary first integral by using (5.53).

Example 5.13 The second-order equation

$$u^{(2)} = \frac{(1 + (u^{(1)})^2)(2(xu^{(1)} - u) + k\sqrt{1 + (u^{(1)})^2}}{1 + x^2 + u^2}, \qquad k \text{ constant}$$

admits the symmetries

$$\begin{aligned} \Xi_1 &= u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, \\ \Xi_2 &= (1 + x^2 - u^2)\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}, \\ \Xi_3 &= 2xu\frac{\partial}{\partial x} + (1 - x^2 + u^2)\frac{\partial}{\partial u}; \end{aligned}$$

these vector fields give a realization of $\mathfrak{so}(3)$ in $\mathbb{R}^2$. This algebra does not admit two-dimensional Lie subalgebras, so that the equation at hand is not reducible to quadrature. Nevertheless, we can determine the first two integrals. We have the three characteristics

$$Q_1 = -x - uu^{(1)}, \qquad Q_2 = 2xu - (1 + x^2 - u^2)u^{(1)}, \qquad Q_3 = (1 - x^2 + u^2) - 2xuu^{(1)},$$

whence, considering the 2×2 matrices

$$M_{jk} = \begin{pmatrix} Q_j & Q_k \\ \widetilde{\mathcal{D}}_x(Q_j) & \widetilde{\mathcal{D}}_x(Q_k) \end{pmatrix}, \quad 1 \leq j < k \leq 3,$$

the conserved quantities are

$$\phi_1 = \frac{\det(M_{12})}{\det(M_{23})}, \qquad \phi_2 = \frac{\det(M_{13})}{\det(M_{23})}.$$

So, we have the first integrals

$$\frac{2xu + (1 - x^2 + u^2)u^{(1)} - kx\sqrt{1 + (u^{(1)})^2}}{4(u - xu^{(1)}) + k(-1 + x^2 + u^2)\sqrt{1 + (u^{(1)})^2}} = c_1,$$
$$\frac{(1 + x^2 - u^2) + 2xuu^{(1)} + ku\sqrt{1 + (u^{(1)})^2}}{4(xu^{(1)} - u) + k(1 - x^2 - u^2)\sqrt{1 + (u^{(1)})^2}} = c_2.$$

Some more examples of the procedure outlined in this section can be found in [25].

Remark 5.7 For scalar ordinary differential equations we can determine first integrals also by means of contact symmetries. For a contact symmetry the characteristic coincides with $-\Omega(x, u, u^{(1)})$, where Ω is the generating function of the contact transformation. For true contact transformations, the generating function Ω is not linear in $u^{(1)}$.

5.9 Hidden Symmetries

In Remark 5.6, it has been exhibited a second-order ordinary differential equation that can be reducible to quadrature even if it does not have any Lie point symmetry [2].

When a symmetry of a differential equation exists, one uses the symmetry to lower its order. This approach can be reversed, so that one may increase the order of the equation and look for the admitted symmetries of the new equation. Another situation may occur after the lowering of the order of a differential equation, where the reduced equation possesses symmetries not coming from the symmetries of the original equation.

In such cases, one may talk about *hidden symmetries* [1–3] When one of the effects of increasing the order of a differential equation $\mathcal{E}$ produces the birth of symmetries not possessed by $\mathcal{E}$ one has *hidden symmetries of first type*. On the contrary, when new symmetries appear after a reduction of order, then one has *hidden symmetries of second type*. Hidden symmetries of second type may arise also when looking for invariant solutions of partial differential equations. The reduced ordinary differential equation may possess symmetries not inherited from the original partial differential equation [21, 22].

5.10 Concluding Notes

Continuous symmetries are the main actors of this book. Nevertheless, it is worth noticing that many physically relevant differential equations may admit discrete symmetries (see, for instance [27] where nonclassical reductions of the Boussinesq equation are also considered). Moreover, in [20] a method for determining the most

general equation admitting a given discrete symmetry is given. A special mention is due to Hydon [23, 24] (see also the book [25]) who gave effective algorithms for computing discrete symmetries of both ordinary and partial differential equations.

Another interesting generalization of Lie symmetries is given by the C^∞-symmetries (or λ-symmetries), introduced in a series of paper by Muriel and Romero [12, 19, 28–32], At the basis of the λ-symmetries there is a modification of the prolongation formulae for vector fields in such a way there is a complete system of invariants that can be computed by derivation of lower order invariants (as in the case of classical symmetries). This allows for the order reduction of differential equations for which no Lie point symmetry exists.

After the introduction of λ-symmetries, several other types of symmetries (e.g., μ-symmetries [13]), defined through a deformation of prolongation formulae, have been introduced and collectively called *twisted symmetries* [16–18]; as classical symmetries, they reveal useful for the order reduction of ordinary differential equations or for the determination of special (invariant) solutions to partial differential equations.

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Chapter 6
Lie Symmetries of Partial Differential Equations



6.1 Introduction

In this chapter, we will consider two possible ways of using the Lie symmetries of partial differential equations, namely, the determination of particular exact solutions that are invariant with respect to a Lie group of point transformations, and the generation of new solutions starting from existing ones. The first strategy uses the infinitesimal version of the transformations, whereas the second one the finite expression of groups of transformations. The search for invariant solutions [7, 22–24] of partial differential equations is the widely used application of Lie group theory. In some cases, invariant solutions describe meaningful physical situations (for instance, self-similar solutions or traveling wave solutions). Nevertheless, it has to be remarked that, in general, not always the admitted symmetries of the partial differential equations leave physically relevant initial and/or boundary conditions invariant, thus preventing us from solving satisfactorily such problems.

6.2 Determination of Lie Symmetries of Partial Differential Equations

Hereafter, we show some examples of well-known partial differential equations, and compute the admitted Lie symmetries.

Example 6.1 (*Symmetries of Burgers equation*) Let us consider the second-order nonlinear partial differential equation for the unknown $u(x_1, x_2)$ (x_1 is the time and x_2 the space variable)

$$\frac{\partial u}{\partial x_1} + u\frac{\partial u}{\partial x_2} - \frac{\partial^2 u}{\partial x_2^2} = 0,$$

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written in compact form as

$$\Delta \equiv u_{,1} + uu_{,2} - u_{,22} = 0, \tag{6.1}$$

known as Burgers equation. It represents the simplest evolution equation containing nonlinear and dissipative terms. Equation (6.1) has been introduced first by Bateman [5] in 1915, and later by Burgers [8, 9], for describing a mathematical model of turbulence. It is related to linear heat equation by the well-known Hopf–Cole transform [10, 16]; in fact, setting

$$u = -2\frac{\partial \log(\phi(x_1, x_2))}{\partial x_2} = -2\frac{\phi_{,2}}{\phi}, \tag{6.2}$$

we get

$$\frac{\partial}{\partial x_2}\left(\frac{\phi_{,1} - \phi_{,22}}{\phi}\right) = 0,$$

whence, if ϕ is a solution of linear heat equation, then u given in (6.2) is a solution of (6.1). Burgers equation has many relevant properties, and some of these will be described in other parts of this book.

In order to find the Lie symmetries admitted by (6.1), let us consider the infinitesimal operator

$$\Xi = \xi_1(x_1, x_2, u)\frac{\partial}{\partial x_1} + \xi_2(x_1, x_2, u)\frac{\partial}{\partial x_2} + \eta(x_1, x_2, u)\frac{\partial}{\partial u}, \tag{6.3}$$

and its second-order prolongation $\Xi^{(2)}$. Requesting Eq. (6.1) to be invariant under the Lie group generated by (6.3), say

$$\Xi^{(2)}(\Delta)\big|_{\Delta=0} = 0,$$

we get the invariance condition

$$\begin{aligned}
&\left(\frac{\partial \eta}{\partial x_1} + u\frac{\partial \eta}{\partial x_2} - \frac{\partial^2 \eta}{\partial x_2^2}\right) + \left(u\frac{\partial \xi_1}{\partial x_1} - \frac{\partial \xi_2}{\partial x_1} + \eta + u^2\frac{\partial \xi_1}{\partial x_2} - u\frac{\partial \xi_2}{\partial x_2}\right.\\
&\quad \left. - 2\frac{\partial^2 \eta}{\partial x_2 \partial u} + \frac{\partial^2 \xi_2}{\partial x_2^2} - u\frac{\partial^2 \xi_1}{\partial x_2^2}\right) u_{,2} + \left(2\frac{\partial \xi_2}{\partial x_2} - \frac{\partial \xi_1}{\partial x_1} - u\frac{\partial \xi_1}{\partial x_2} + \frac{\partial^2 \xi_1}{\partial x_2^2}\right) u_{,22}\\
&\quad + 2\left(\frac{\partial^2 \xi_1}{\partial x_2 \partial u} + \frac{\partial \xi_2}{\partial u}\right) u_{,2}u_{,22} + \left(-\frac{\partial^2 \eta}{\partial u^2} + 2\frac{\partial^2 \xi_2}{\partial x_2 \partial u} - 2u\frac{\partial^2 \xi_1}{\partial x_2 \partial u}\right)(u_{,2})^2\\
&\quad + \left(\frac{\partial^2 \xi_2}{\partial u^2} - u\frac{\partial^2 \xi_1}{\partial u^2}\right)(u_{,2})^3 + \frac{\partial^2 \xi_1}{\partial u^2}(u_{,2})^2 u_{,22} + \frac{\partial \xi_1}{\partial x_2}u_{,12} + 2\frac{\partial \xi_1}{\partial u}u_{,2}u_{,12} = 0,
\end{aligned}$$

where the constraint $\Delta = 0$ has been taken into account to eliminate $u_{,1}$. The invariance condition is a polynomial in the derivatives $u_{,2}$, $u_{,12}$, and $u_{,22}$, whose coef-

ficients, involving the infinitesimals and their partial derivatives, must vanish. By annihilating the coefficients of this polynomial, a set of *determining equations* arise; the latter represent a set of overdetermined linear partial differential equations whose integration yields the expression of the infinitesimals ξ_1, ξ_2 and η:

$$\begin{aligned}
\xi_1 &= a_1 x_1^2 + a_2 x_1 + a_3, \\
\xi_2 &= (a_1 x_2 + a_4) x_1 + \frac{a_2}{2} x_2 + a_5, \\
\eta &= -\left(a_1 x_1 + \frac{a_2}{2}\right) u + a_1 x_2 + a_4,
\end{aligned}$$

where $a_1, \ldots, a_5$ are arbitrary constants. Due to this arbitrariness we may say that Burgers equation admits a five-parameter Lie group of point transformations spanned by the following infinitesimal operators:

$$\begin{aligned}
\Xi_1 &= \frac{\partial}{\partial x_1} && \text{(time translation)}, \\
\Xi_2 &= \frac{\partial}{\partial x_2} && \text{(space translation)}, \\
\Xi_3 &= 2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - u \frac{\partial}{\partial u} && \text{(scaling)}, \\
\Xi_4 &= x_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u} && \text{(Galilean transformation)}, \\
\Xi_5 &= x_1^2 \frac{\partial}{\partial x_1} + x_1 x_2 \frac{\partial}{\partial x_2} + (x_2 - x_1 u) \frac{\partial}{\partial u} && \text{(projective transformation)}.
\end{aligned}$$

Example 6.2 (*Symmetries of KdV equation*) Let us consider the Korteweg–deVries equation [20]

$$\frac{\partial u}{\partial x_1} + u \frac{\partial u}{\partial x_2} + \frac{\partial^3 u}{\partial x_2^3} = 0$$

that in compact form we rewrite as

$$\Delta \equiv u_{,1} + u u_{,2} + u_{,222} = 0. \tag{6.4}$$

By using the third-order prolongation of generator (6.3), we expand the invariance condition for Eq. (6.4),

$$\Xi^{(3)}(\Delta)\big|_{\Delta=0} = 0,$$

integrate the determining equations, so obtaining

$$\xi_1 = a_1 + 3a_4 x_1, \qquad \xi_2 = a_2 + a_3 x_1 + a_4 x_2 \qquad \eta = a_3 - 2a_4 u,$$

where $a_1, \ldots, a_4$ are arbitrary constants. Therefore, Eq. (6.4) admits a four-dimensional Lie algebra of point symmetries spanned by the vector fields

$$\begin{aligned}
\Xi_1 &= \frac{\partial}{\partial x_1} && \text{(time translation)},\\
\Xi_2 &= \frac{\partial}{\partial x_2} && \text{(space translation)},\\
\Xi_3 &= x_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u} && \text{(Galilean transformation)},\\
\Xi_4 &= 3x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - 2u \frac{\partial}{\partial u} && \text{(scaling)}.
\end{aligned}$$

Example 6.3 (*Symmetries of axially symmetric wave equation* [6]) Let us consider the axially symmetric classical wave equation (t is the time and r is the radial coordinate):

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} = 0.$$

By requiring the invariance with respect to a Lie group of point transformations and solving the associated determining equations, one obtains the Lie algebra of point symmetries spanned by

$$\begin{aligned}
\Xi_1 &= \frac{\partial}{\partial t} && \text{(time translation)},\\
\Xi_2 &= t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} && \text{(scaling of independent variables)},\\
\Xi_3 &= u \frac{\partial}{\partial u} && \text{(scaling of dependent variable)},\\
\Xi_4 &= (t^2 + r^2) \frac{\partial}{\partial t} + 2tr \frac{\partial}{\partial r} - tu \frac{\partial}{\partial u} && \text{(projective transformation)},\\
\Xi_v &= v(t, r) \frac{\partial}{\partial u} && \text{(linear superposition of solutions)},
\end{aligned}$$

where $v(t, r)$ is an arbitrary solution of the equation:

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\frac{\partial v}{\partial r} = 0;$$

henceforth, the axially symmetric wave equation admits an infinite-parameter Lie group with generators spanning an infinite-dimensional Lie algebra.

The occurrence of an infinite-dimensional Lie algebra is due to the linear form of the equation, and is also peculiar of nonlinear differential equations that can be mapped to linear form [7, 12, 13, 18].

Example 6.4 (*Symmetries of the equations of ideal gas dynamics*) Consider the three-dimensional Euler equations of ideal gas dynamics,

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial \rho}{\partial x_k} + \rho \sum_{k=1}^{3} \frac{\partial v_k}{\partial x_k} = 0,\\
&\rho \left(\frac{\partial v_i}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial v_i}{\partial x_k} \right) + \frac{\partial p}{\partial x_i} = 0, \qquad i = 1, 2, 3,\\
&\frac{\partial p}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial p}{\partial x_k} + \gamma p \sum_{k=1}^{3} \frac{\partial v_k}{\partial x_k} = 0,
\end{aligned} \tag{6.5}$$

where $\rho(t, \mathbf{x})$ is the mass density, $v_i(t, \mathbf{x})$ is the components of velocity, $p(t, \mathbf{x})$ is the pressure, and γ is the adiabatic exponent.

By applying Lie infinitesimal criterion, and solving the determining equations for the infinitesimals of the Lie symmetries, we observe that the value of the adiabatic index γ plays a role. In fact, if γ is unspecified, we have a 13-dimensional Lie algebra of point symmetries spanned by

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_{k+1} = \frac{\partial}{\partial x_k}, \qquad \Xi_{k+4} = t \frac{\partial}{\partial x_k} + \frac{\partial}{\partial v_k}, \qquad k = 1, 2, 3,\\
&\Xi_{5+i+j} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j} + v_j \frac{\partial}{\partial v_i} - v_i \frac{\partial}{\partial v_j}, \qquad 1 \le i < j \le 3,\\
&\Xi_{11} = t \frac{\partial}{\partial t} + 2\rho \frac{\partial}{\partial \rho} - \sum_{k=1}^{3} v_k \frac{\partial}{\partial v_k},\\
&\Xi_{12} = \sum_{k=1}^{3} \left(x_k \frac{\partial}{\partial x_k} + v_k \frac{\partial}{\partial v_k} \right) - 2\rho \frac{\partial}{\partial \rho},\\
&\Xi_{13} = \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}.
\end{aligned} \tag{6.6}$$

Thus, we have the invariance with respect to time translation, isometries (space translations and rotations), Galilean transformations, and three independent scalings. Furthermore, if $\gamma = 5/3$ (this is a physical value!) there is a 14th symmetry generated by the vector field

$$\Xi_{14} = t^2 \frac{\partial}{\partial t} + \sum_{k=1}^{3} \left(x_k t \frac{\partial}{\partial x_k} + (x_k - v_k t) \frac{\partial}{\partial v_k} \right) - 3\rho t \frac{\partial}{\partial \rho} - 5pt \frac{\partial}{\partial p}, \tag{6.7}$$

corresponding to a projective transformation. Ovsiannikov [25] proved that a projective transformation exists for the Euler equations of ideal gas dynamics in n space dimensions provided that $\gamma = \dfrac{n+2}{n}$.

Remark 6.1 The lengthy and tedious calculations required for determining the Lie symmetries of the system (6.5) can be automatically done by using ReLie (see Example 12.5).

6.3 Use of One-Parameter Lie Groups of Transformations

The knowledge of a one-parameter Lie group of transformations admitted by a given partial differential equation can be used to find exact solutions in two ways:

- determine particular solutions, called *invariant solutions*;
- obtain different solutions, once a special solution is known.

Let $\mathcal{E}$ a partial differential equation of order $r \geq 1$ for the unknown $\mathbf{u}(\mathbf{x}) \equiv (u_1(\mathbf{x}), \ldots, u_m(\mathbf{x}))$ where $\mathbf{x} \equiv (x_1, \ldots, x_n)$ are the independent variables, say

$$\boldsymbol{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}) = \mathbf{0}, \tag{6.8}$$

and let

$$\Xi = \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\alpha} \tag{6.9}$$

be the generator of a Lie point symmetry of (6.8) corresponding to the one-parameter Lie group

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}, \mathbf{u}; a), \qquad \mathbf{u}' = \mathbf{U}(\mathbf{x}, \mathbf{u}; a) \tag{6.10}$$

that is obtained, using (6.9), by integrating the Lie equations.

6.3.1 Invariant Solutions

The function $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ is said an *invariant solution* of the partial differential equation $\mathcal{E}$ if and only if $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ is an invariant manifold of (6.9), namely (see Theorem 2.4)

$$\Xi\,(\mathbf{u} - \boldsymbol{\Theta}(\mathbf{x}))|_{\mathbf{u}=\boldsymbol{\Theta}(\mathbf{x})} = \mathbf{0},$$

that is,

$$\sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \boldsymbol{\eta}(\mathbf{x}, \mathbf{u}), \tag{6.11}$$

and is a solution of (6.8).

Thus, an invariant solution of $\mathcal{E}$ is a solution of the overdetermined system made of the invariant manifold condition (6.11) and the given differential Eq. (6.8).

Equation (6.11) is solved by introducing the corresponding characteristic equations:

$$\frac{dx_1}{\xi_1(\mathbf{x},\mathbf{u})} = \cdots = \frac{dx_n}{\xi_n(\mathbf{x},\mathbf{u})} = \frac{du_1}{\eta_1(\mathbf{x},\mathbf{u})} = \cdots = \frac{du_m}{\eta_m(\mathbf{x},\mathbf{u})}.$$

If $J_1(\mathbf{x},\mathbf{u}), \ldots, J_{n-1}(\mathbf{x},\mathbf{u}), v_1(\mathbf{x},\mathbf{u}), \ldots, v_m(\mathbf{x},\mathbf{u})$ is a set of $m+n-1$ functionally independent invariants for Ξ, assuming $J_k(\mathbf{x},\mathbf{u})$, $k=1,\ldots,n-1$, as independent variables, and $v_\alpha(\mathbf{x},\mathbf{u})$, $\alpha=1,\ldots,m$, as new dependent variables, provided that the rank of the Jacobian matrix with entries $\dfrac{\partial v_\alpha}{\partial u_\beta}$ $(\alpha,\beta=1,\ldots,m)$ is m, then the solution of Eq. (6.11) can be expressed (at most in implicit form) as

$$v_\alpha(\mathbf{x},\mathbf{u}) = \psi_\alpha(J_1(\mathbf{x},\mathbf{u}),\ldots,J_{n-1}(\mathbf{x},\mathbf{u})), \quad \alpha=1,\ldots,m, \tag{6.12}$$

with each ψ_α arbitrary function of the invariants $J_1,\ldots,J_{n-1}$. Substituting (6.12) into (6.8), we get a *reduced* rth-order partial differential equation with $n-1$ independent variables. In many cases, it is $\xi_i(\mathbf{x})$, $i=1,\ldots,n$, whence the representation of the invariant solution (6.12) is not implicit. It is evident that, when $n=2$, the reduced equation is ordinary.

Remark 6.2 The variables $J_1,\ldots,J_{n-1}$ are often called *similarity variables*; the reason lies on the fact that the scaling invariance, i.e., the invariance under similarity transformations, was one of the first examples where this procedure has been used systematically.

Example 6.5 The linear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \tag{6.13}$$

admits, among the others, the scaling invariance

$$\Xi = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}.$$

The condition (6.11) reads as

$$t\frac{\partial u}{\partial t} + x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + x_3\frac{\partial u}{\partial x_3} = 0;$$

integrating the corresponding characteristic equations, we determine the four invariants

$$y_1 = \frac{x_1}{t}, \quad y_2 = \frac{x_2}{t}, \quad y_3 = \frac{x_3}{t}, \quad U = u;$$

therefore, the representation of the invariant solution is $u = U(y_1, y_2, y_3)$, that, inserted in (6.13), yields the *reduced* equation

$$\sum_{k=1}^{3}\left(\frac{1}{2}(y_k^2-1)\frac{\partial^2 U}{\partial y_k^2}+y_k\frac{\partial U}{\partial y_k}\right)+\sum_{1\le j<k\le 3} y_j y_k\frac{\partial^2 U}{\partial y_j\partial y_k}=0$$

involving three independent variables.

Example 6.6 (*Interaction of two waves*) Let us consider the first-order system [11]

$$\begin{aligned} u_{1,1}+u_{1,2}&=2ku_1u_2,\\ u_{2,1}-u_{2,2}&=-2ku_1u_2, \end{aligned} \tag{6.14}$$

where k is a real constant, describing the interaction of two waves of amplitude $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$, respectively, traveling along the x_2-axis (x_1 is the time) with velocities ± 1.

The infinitesimal generator of the Lie group of point transformations leaving system (6.14) invariant is

$$\begin{aligned} \Xi=&(f(\omega_1)-g(\omega_2))\frac{\partial}{\partial x_1}+(f(\omega_1)+g(\omega_2))\frac{\partial}{\partial x_2}\\ &-2\,g'(\omega_2)u_1\frac{\partial}{\partial u_1}-2f'(\omega_1)u_2\frac{\partial}{\partial u_2}, \end{aligned}$$

where $\omega_1 = x_1 + x_2$, $\omega_2 = -x_1 + x_2$, $f(\omega_1)$, and $g(\omega_2)$ are arbitrary functions of the indicated arguments, and the prime ($'$) denotes the derivative with respect to the argument. The corresponding invariant solutions of the system (6.14) are found by solving the characteristic equations

$$\frac{dx_1}{f(\omega_1)-g(\omega_2)}=\frac{dx_2}{f(\omega_1)+g(\omega_2)}=\frac{du_1}{-2g'(\omega_2)u_1}=\frac{du_2}{-2f'(\omega_1)u_2},$$

from which we find the invariant

$$\omega=\int\frac{d\omega_1}{f(\omega_1)}-\int\frac{d\omega_2}{g(\omega_2)},$$

and, ultimately, the invariant solution

$$\begin{aligned} u_1(x_1,x_2)&=\frac{\alpha\exp(k\alpha\omega)}{\alpha\beta+\exp(k\alpha\omega)}\frac{1}{g(\omega_2)},\\ u_2(x_1,x_2)&=\frac{\alpha^2\beta}{\alpha\beta+\exp(k\alpha\omega)}\frac{1}{f(\omega_1)}, \end{aligned} \tag{6.15}$$

with α, β arbitrary constants.

The solution (6.15) contains two arbitrary functions, exactly the number we expect to have the general solution. Indeed, the general solution of (6.14) is given by (6.15).

In [33], by means of the truncation of the Laurent series used to perform the Painlevé test [34, 35], it has been determined a Bäcklund transformation mapping (6.14) to the linear wave equation. More in detail, the Bäcklund transformation is

$$u_1 = \frac{1}{2k}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\log(\varphi),$$
$$u_2 = -\frac{1}{2k}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\log(\varphi),$$

where $\varphi(x_1, x_2)$ is solution of linear wave equation

$$\varphi_{,11} - \varphi_{,22} = 0. \tag{6.16}$$

Due to the known fact that the general integral of (6.16) is known (the sum of an arbitrary function of $(x_1 + x_2)$ and an arbitrary function of $(-x_1 + x_2)$), it is straightforward to recognize that (6.15) is the general solution of the system (6.14).

Example 6.7 In Example 6.4, it is shown that Eq. (6.5) of three-dimensional ideal gas dynamics with adiabatic index equal to $5/3$ admits the symmetry generated by (6.7). The corresponding invariant solutions can be easily found:

$$\rho(t, \mathbf{x}) = t^{-3}R(\boldsymbol{\omega}), \qquad \mathbf{v}(t, \mathbf{x}) = \frac{\mathbf{x}}{t} + t^{-1}\mathbf{V}(\boldsymbol{\omega}), \qquad p(t, \mathbf{x}) = t^{-5}P(\boldsymbol{\omega}),$$

where $\boldsymbol{\omega} = \dfrac{\mathbf{x}}{t}$, and $R(\boldsymbol{\omega})$, $\mathbf{V}(\boldsymbol{\omega})$, and $P(\boldsymbol{\omega})$ solutions of the reduced equations

$$\begin{aligned} &\nabla^\star(R\mathbf{V}) = 0, \\ &(\mathbf{V}\cdot\nabla^\star)\mathbf{V} + \nabla^\star P = 0, \\ &\mathbf{V}\cdot\nabla^\star P + \frac{5}{3}P\nabla^\star\cdot\mathbf{V} = 0, \end{aligned} \tag{6.17}$$

where $\nabla^\star = \left(\dfrac{\partial}{\partial\omega_1}, \dfrac{\partial}{\partial\omega_2}, \dfrac{\partial}{\partial\omega_3}\right)^T$. Identifying $\boldsymbol{\omega}$ with $\mathbf{x}$, Eq. (6.17) has the same form as the steady equations. Therefore, if

$$\{\rho(\mathbf{x}),\ \mathbf{v}(\mathbf{x}),\ p(\mathbf{x})\}$$

is a solution of steady ideal gas dynamics equations with adiabatic index equal to $5/3$, then

$$\left\{t^{-3}\rho\left(\frac{\mathbf{x}}{t}\right),\ \frac{\mathbf{x}}{t}+t^{-1}\mathbf{v}\left(\frac{\mathbf{x}}{t}\right),\ t^{-5}p\left(\frac{\mathbf{x}}{t}\right)\right\}$$

gives an unsteady solution.

This result will be used later in Theorem 10.6.

Since differential equations can admit more than one symmetry, we can determine many different invariant solutions.

Example 6.8 (*Invariant solutions of potential Burgers equation*) Let us consider the potential Burgers equation

$$u_{,1}+(u_{,2})^2-u_{,22}=0.$$

It admits an infinite-dimensional Lie algebra spanned by the vector fields

$$\begin{aligned}
&\Xi_1=\frac{\partial}{\partial x_1},\qquad \Xi_2=\frac{\partial}{\partial x_2},\qquad \Xi_3=\frac{\partial}{\partial u},\\
&\Xi_4=2x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2},\qquad \Xi_5=2x_1\frac{\partial}{\partial x_2}+x_2\frac{\partial}{\partial u},\\
&\Xi_6=4x_1^2\frac{\partial}{\partial x_1}+4x_1x_2\frac{\partial}{\partial x_2}+(2x_1+x_2^2)\frac{\partial}{\partial u},\\
&\Xi_f=f(x_1,x_2)\frac{\partial}{\partial u},\qquad f_{,1}-f_{,22}=0.
\end{aligned}\tag{6.18}$$

One may look for solutions that are invariant with respect to different subgroups; in Table 6.1, we show the representation as well as the reduced ordinary differential equations of the group invariant solutions of potential Burgers equation generated by the vector fields (6.18) of the finite Lie subalgebra of symmetries. Besides constructing invariant solutions with respect to generators of the basis of Lie algebra, we can also consider other subgroups, for instance, the one generated by $\Xi_1+c\,\Xi_2$, where c is a constant. In such a case, we have the traveling wave solutions

$$u(x_1,x_2)=\psi(x_2-cx_1)\equiv\psi(\omega),$$

where $\psi(\omega)$ satisfy the reduced ordinary differential equation:

$$-c\frac{d\psi}{d\omega}+\left(\frac{d\psi}{d\omega}\right)^2-\frac{d^2\psi}{d\omega^2}=0.$$

Notice that in general there is no reasonable limit to the number of the one-parameter subgroups we may consider!

Table 6.1 Invariant solutions of potential Burgers equation

	Invariants	Representation of the solution	Reduced equation
Ξ_1	$\begin{cases} x_2 \\ u \end{cases}$	$u = \psi(x_2)$	$\left(\frac{d\psi}{dx_2}\right)^2 - \frac{d^2\psi}{dx_2^2} = 0$
Ξ_2	$\begin{cases} x_1 \\ u \end{cases}$	$u = \psi(x_1)$	$\frac{d\psi}{dx_1} = 0$
Ξ_3	$\begin{cases} x_1 \\ x_2 \end{cases}$	No group invariant solutions	
Ξ_4	$\begin{cases} \omega = \frac{x_2}{\sqrt{x_1}} \\ u \end{cases}$	$u = \psi(\omega)$	$-\frac{1}{2}\omega\frac{d\psi}{d\omega} + \left(\frac{d\psi}{d\omega}\right)^2 - \frac{d^2\psi}{d\omega^2} = 0$
Ξ_5	$\begin{cases} x_1 \\ 4x_1u - x_2^2 \end{cases}$	$u = \frac{1}{4x_1}\left(\psi(x_1) + x_2^2\right)$	$x_1\frac{d\psi}{dx_1} - \psi - 2x_1 = 0$
Ξ_6	$\begin{cases} \omega = \frac{x_2}{x_1} \\ u - \frac{x_2^2}{4x_1} - \frac{1}{2}\log x_1 \end{cases}$	$u = \psi(\omega) + \frac{x_2^2}{4x_1} + \frac{1}{2}\log x_1$	$\left(\frac{d\psi}{d\omega}\right)^2 - \frac{d^2\psi}{d\omega^2} = 0$

Remark 6.3 It is worth of being observed that there are cases where a symmetry of a differential equation does not allow for the construction of invariant solutions (see the case of the symmetry Ξ_3 in Table 6.1, or examples in [24]).

In Sect. 6.6, we will show that it is also possible to achieve a multiple reduction of variables by using multi-parameter groups of transformations.

6.3.2 *New Solutions from a Known Solution*

As the action of a Lie group of transformations admitted by the partial differential equation $\mathcal{E}$ maps a solution to a one-parameter family of solution of the same equation (unless it is invariant), a solution of $\mathcal{E}$, which is not invariant with respect to the group, can generate new solutions. This is especially interesting when one can obtain nontrivial solutions from trivial ones.

Let us consider the one-parameter Lie group of transformations (6.10) leaving (6.8) invariant, and let

$$\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$$

be a solution of $\mathcal{E}$, which is not invariant with respect to the group (6.10).

The transformation (6.10) maps a point $(\mathbf{x}, \boldsymbol{\Theta}(\mathbf{x}))$ of the solution $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ into the point $(\mathbf{x}', \mathbf{u}')$ characterized by (6.10).

For a fixed value of the parameter a, it is

$$\mathbf{x} = \mathbf{X}(\mathbf{x}', \mathbf{u}'; -a),$$

whereupon

$$\mathbf{u}' = \mathbf{U}(\mathbf{X}(\mathbf{x}', \mathbf{u}'; -a), \boldsymbol{\Theta}(\mathbf{X}(\mathbf{x}', \mathbf{u}'; -a)); a). \tag{6.19}$$

Finally, by substituting $(\mathbf{x}', \mathbf{u}'; -a)$ with $(\mathbf{x}, \mathbf{u}; a)$ in (6.19), one may state the following theorem.

Theorem 6.1 *([7]) If $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ is a not an invariant solution of $\mathcal{E}$, admitting the symmetry group (6.10), then*

$$\mathbf{u} = \mathbf{U}(\mathbf{X}(\mathbf{x}, \mathbf{u}; a), \boldsymbol{\Theta}(\mathbf{X}(\mathbf{x}, \mathbf{u}; a)); -a)$$

implicitly defines a one-parameter family of solutions of $\mathcal{E}$.

Example 6.9 (*Solution of linear heat equation from a constant solution*) The linear heat equation (see Example 4.5)

$$u_{,1} - u_{,22} = 0$$

admits the symmetry generated by

$$\Xi = x_1^2 \frac{\partial}{\partial x_1} + x_1 x_2 \frac{\partial}{\partial x_2} - \left(\frac{x_1}{2} + \frac{x_2^2}{4}\right) u \frac{\partial}{\partial u}. \tag{6.20}$$

The related finite transformation (obtained by integrating Lie equations) is

$$x_1' = \frac{x_1}{1 - a x_1}, \qquad x_2' = \frac{x_2}{1 - a x_1},$$
$$u' = u\sqrt{1 - a x_1}\, \exp\left(-\frac{a x_2^2}{4(1 - a x_1)}\right).$$

One can obtain the inverse transformation by exchanging (x_1, x_2, u) and (x_1', x_2', u'), and replacing a by $-a$:

$$x_1 = \frac{x_1'}{1 + a x_1'}, \qquad x_2 = \frac{x_2'}{1 + a x_1'},$$
$$u = u'\sqrt{1 + a x_1'}\, \exp\left(\frac{a x_2'^2}{4(1 + a x_1')}\right).$$

By applying this transformation to the trivial solution $u = u_0$ of the heat equation (u_0 constant), the nontrivial solution

$$u = \frac{u_0}{\sqrt{1 + ax_1}} \exp\left(-\frac{ax_2^2}{4(1 + ax_1)}\right)$$

is immediately obtained.

6.4 Invariant Solutions and Initial/Boundary Value Problems

Lie symmetries constitute a powerful tool for constructing exact solutions to partial differential equations. Anyway, we have to take into account that the solutions one is interested to find in applied problems must satisfy special initial and/or boundary conditions. This implies that, in order the group invariant solution be acceptable, it is needed that also initial and/or boundary conditions be invariant [29].

As an example [14], suppose to have a differential equation for the unknown $u(x_1, x_2)$ admitting the symmetry generated by

$$\Xi = \xi_1(x_1, x_2, u)\frac{\partial}{\partial x_1} + \xi_2(x_1, x_2, u)\frac{\partial}{\partial x_2} + \eta(x_1, x_2, u)\frac{\partial}{\partial u};$$

an initial condition like $u(x_{10}, x_2) = f(x_2)$ is invariant if

$$\Xi\,(u - f(x_2))|_{u=f(x_2), x_1=x_{10}} = 0,$$

that is,

$$\eta(x_{10}, x_2, f(x_2)) - \xi_2(x_{10}, x_2, f(x_2))f'(x_2) = 0, \qquad \xi_1(x_{10}, x_2, f(x_2)) = 0.$$

Similar reasonings apply to the invariance of boundary conditions.

Example 6.10 (*Nonlinear heat equation*) The nonlinear heat equation [7]

$$u^{7/3}u_{,1} + \frac{4}{3}u_{,2}^2 - u_{,2}u_{,22} = 0 \tag{6.21}$$

admits the symmetry generated by the vector field

$$\Xi = 2(a_1 + a_2x_1)\frac{\partial}{\partial x_1} + (a_3x_2^2 + (a_2 + a_4)x_2 + a_5)\frac{\partial}{\partial x_2} - \frac{3}{2}(2a_3x_2 + a_4)\,u\frac{\partial}{\partial u},$$

where $a_1, \ldots, a_5$ are arbitrary constants. The initial condition

$$u(0, x_2) = (1 + x_2^2)^{-3/2} \exp(\arctan x_2), \tag{6.22}$$

is invariant if Ξ is such that

$$a_1 = 0, \qquad a_3 = a_5 = -\frac{3}{2}a_4, \qquad a_2 = -a_4.$$

Thus, Eq. (6.21) and the initial condition (6.22) are invariant with respect to the symmetry

$$\Xi = \frac{4}{3}x_1 \frac{\partial}{\partial x_1} + (1 + x_2^2)\frac{\partial}{\partial x_2} + (1 - 3x_2)u\frac{\partial}{\partial u}$$

that we may use to find a group invariant solution with the given initial condition.

Remark 6.4 Example 6.10 clearly shows that, if we want to determine an invariant solution satisfying an assigned initial boundary value problem, we have to use only symmetries leaving both the equation and the initial boundary data invariant.

6.5 Separable Solutions

There are several situations where one may be interested to obtain solutions to partial differential equations in terms of some functions each depending on one independent variable [17, 19, 27, 28]. These solutions are called *separable*. In these cases, the separable solutions are found by solving a set of ordinary differential equations. A classical example where this approach reveals useful is given by the search of the complete integral of Hamilton–Jacobi equation [4].

Example 6.11 (*Separable solutions of linear heat equation*) Let

$$u_{,1} - u_{,22} = 0$$

be the linear heat equation. Looking for a solution in separable form,

$$u(x_1, x_2) = \phi(x_1)\psi(x_2),$$

we obtain

$$\frac{d\phi}{dx_1}\psi - \phi\frac{d^2\psi}{dx_2^2} = 0,$$

that is compatible provided that

$$\frac{1}{\phi}\frac{d\phi}{dx_1} = c = \frac{1}{\psi}\frac{d^2\psi}{dx_2^2},$$

where c is a constant; to have a bounded solution, it is necessary that $c = -\omega^2$. Integrating the two ordinary differential equations

$$\frac{d\phi}{dx_1} + \omega^2\phi = 0, \qquad \frac{d^2\psi}{dx_2^2} + \omega^2\psi = 0,$$

we get the separable solution

$$u(x_1, x_2) = \exp(-\omega^2 x_1)\,(c_1\cos(\omega x_2) + c_2\sin(\omega x_2))\,.$$

In a recent paper [30] (see also the references therein), the authors discuss the relation between the existence of separable solutions of a partial differential equation and the structure of its symmetry operators, showing examples of separable solutions obtainable as group invariant solutions. Hereafter, let us limit ourselves to scalar partial differential equations in two independent variables, say

$$\Delta(x_1, x_2, u, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}) = 0, \tag{6.23}$$

and to additive or multiplicative separable solutions,

$$u(x_1, x_2) = \phi(x_1) + \psi(x_2), \quad \text{or} \quad u(x_1, x_2) = \phi(x_1)\psi(x_2).$$

It is easily recognized that for a linear homogeneous partial differential equation an additive separable solution can be transformed into a multiplicative separable solution and vice versa, namely:

$$\begin{aligned} v(x_1, x_2) &= \exp(u(x_1, x_2)) = \exp(\phi(x_1) + \psi(x_2)) \\ &= \exp(\phi(x_1))\exp(\psi(x_2)) = \widetilde{\phi}(x_1)\widetilde{\psi}(x_2). \end{aligned}$$

An additive separable solution of (6.23) must satisfy the constraint $u_{,12} = 0$ that, by means of the substitution $u \to \log(u)$, gives the constraint $uu_{,12} - u_{,1}u_{,2} = 0$ for multiplicative separable solutions.

Let us suppose that (6.23) admits the symmetry generated by

$$\Xi = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial u}.$$

The two functionally independent invariants are $(u - x_1)$ and x_2, so that the invariant solution has the representation

$$u = x_1 + \psi(x_2).$$

More generally, assume that (6.23) admits the symmetry generated by

$$\Xi = A(x_1, x_2, u)\left(\frac{\partial}{\partial x_1} + \frac{d\phi(x_1)}{dx_1}\frac{\partial}{\partial u}\right),$$

with $\phi(x_1)$ and $A(x_1, x_2, u)$ suitable functions of their arguments. The vector field Ξ possesses the two functionally independent invariants $u - \phi(x_1)$ and x_2, so that the invariant solution has the form of an additive separable solution:

$$u(x_1, x_2) = \phi(x_1) + \psi(x_2).$$

To have an invariant solution in separable multiplicative form, we need a symmetry generated by

$$\Xi = A(x_1, x_2, u)\left(\frac{\partial}{\partial x_1} + \frac{1}{\phi(x_1)}\frac{d\phi(x_1)}{dx_1}u\frac{\partial}{\partial u}\right),$$

whereupon the invariant solution reads as

$$u(x_1, x_2) = \phi(x_1)\psi(x_2).$$

Interchanging the independent variables, we have the additive separable solution

$$u(x_1, x_2) = \phi(x_2) + \psi(x_1),$$

which is invariant with respect to

$$\Xi = A(x_1, x_2, u)\left(\frac{\partial}{\partial x_2} + \frac{d\phi(x_2)}{dx_2}\frac{\partial}{\partial u}\right),$$

or the multiplicative separable solution

$$u(x_1, x_2) = \phi(x_2)\psi(x_1),$$

which is invariant with respect to

$$\Xi = A(x_1, x_2, u)\left(\frac{\partial}{\partial x_2} + \frac{1}{\phi(x_2)}\frac{d\phi(x_2)}{dx_2}u\frac{\partial}{\partial u}\right),$$

Of course, the function ψ involved in the separable solutions listed above must satisfy the reduced equation that we obtain inserting the separable solution into (6.23).

For instance, the separable solution in Example 6.11 is invariant with respect to the Lie group generated by

$$\Xi = \frac{\partial}{\partial x_1} - \omega^2 u \frac{\partial}{\partial u},$$

admitted by linear heat equation.

Example 6.12 The homogeneous Monge–Ampère equation for a surface in $\mathbb{R}^3$ with vanishing Gaussian curvature (see also Sect. 8.6)

$$u_{,11}u_{,22} - (u_{,12})^2 = 0$$

possesses separable solutions both in additive and multiplicative form [30]. It admits a 15-dimensional Lie algebra of point symmetries, and some of these symmetries determine the existence of invariant solutions in separable form (see Table 6.2).

Nevertheless, it has to be noticed that there are equations admitting separable solutions that are not invariant with respect to Lie point symmetries. A simple example is given by the equation of minimal surfaces in $\mathbb{R}^3$ [30].

Table 6.2 Group invariant solutions in separable form of homogeneous Monge–Ampère equation; a_k, $k = 1, \ldots, 6$, and b_j, $j = 1, \ldots, 4$, are arbitrary constants, and ψ_k, $k = 1, \ldots, 6$, arbitrary functions of the indicated argument

Symmetry	Separable solution
$\frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial u}$	$u(x_1, x_2) = a_1 x_1 + \psi_1(x_2)$
$\frac{\partial}{\partial x_2} + a_2 \frac{\partial}{\partial u}$	$u(x_1, x_2) = a_2 x_2 + \psi_2(x_1)$
$\frac{\partial}{\partial x_1} + a_3 u \frac{\partial}{\partial u}$	$u(x_1, x_2) = \psi_3(x_2) \exp(a_3 x_1)$
$\frac{\partial}{\partial x_2} + a_4 u \frac{\partial}{\partial u}$	$u(x_1, x_2) = \psi_4(x_1) \exp(a4 x_2)$
$x_1 \frac{\partial}{\partial x_1} + a_5 u \frac{\partial}{\partial u}$	$u(x_1, x_2) = k x_1^{a_5} \psi_5(x_2)$
$x_2 \frac{\partial}{\partial x_2} + a_6 u \frac{\partial}{\partial u}$	$u(x_1, x_2) = k x_2^{a_6} \psi_6(x_1)$
$x_1 \frac{\partial}{\partial x_1} + a_7 \frac{\partial}{\partial u}$	$u(x_1, x_2) = \log \lvert x_1 \rvert^{a_7} + b_1 x_2 + b_2$
$x_2 \frac{\partial}{\partial x_2} + a_8 \frac{\partial}{\partial u}$	$u(x_1, x_2) = \log \lvert x_2 \rvert^{a_8} + b_3 x_1 + b_4$

Example 6.13 The minimal surface equation

$$\left(1+(u_{,2})^2\right)u_{,11}-2u_{,1}u_{,2}u_{,12}+\left(1+(u_{,1})^2\right)u_{,22}=0$$

has the additive separable solution

$$u(x_1,x_2)=\phi_1(x_1)+\phi_2(x_2),$$

where

$$\frac{\phi_1''(x_1)}{1+\phi_1'^2(x_1)}=-\frac{\phi_2''(x_2)}{1+\phi_2'^2(x_2)}=k,$$

yielding the solution

$$u(x_1,x_2)=\frac{1}{k}\left(\log|\cos(kx_2-c_1)|-\log|\cos(kx_1-c_2)|\right)+c_3,$$

where c_1, c_2, and c_3 are arbitrary constants, if $k\neq 0$, and

$$u(x_1,x_2)=c_1x_1+c_2x_2+c_3$$

if $k=0$.

Rosenhaus et al. [30] prove that additively separated solutions of the minimal surface equation are related to its contact symmetries.

For further details, as well as examples, see [30], and references therein.

6.6 Use of r-Parameter Lie Groups of Partial Differential Equations

Given a differential equation $\mathcal{E}$, once we have determined the Lie group of point transformations leaving it invariant, various tasks can be performed when many group parameters are involved.

Of course, as shown before, one can choose one subgroup of the full group and try to determine the corresponding invariant solutions that are found as solutions of a reduced system involving one independent variable less. But, in some cases, it is also possible to use in a sequence more than one subgroup in order to obtain a reduced system, thus getting a stronger reduction of the number of independent variables.

When this is possible, there are essentially two ways to obtain such a multiple reduction of independent variables: repeating step by step the procedure used in the case of one-parameter Lie groups for each subgroup considered, or performing the reduction all-in-one.

Reducing the number of variables step by step means performing the following:

1. take a generator of a subgroup (say, Ξ, written in terms of the variables involved in the differential equation $\Delta = 0$) and build the associated similarity reduction;
2. write the original system of differential equations $\Delta = 0$ in terms of the invariants of the used symmetry, so obtaining the reduced equation $\widehat{\Delta} = 0$;
3. if a further reduction is wanted, set $\Delta = \widehat{\Delta}$ and go to step 1; of course, the symmetries admitted by $\Delta = 0$ must be rewritten (if possible) in terms of the invariants of Ξ.

This method works only if each considered subgroup is possessed by the reduced equation; of course, this is true for the first subgroup considered, but for the subsequent steps this is true only if the subgroup (written in terms of the invariants) is inherited by the reduced system. After a reduction with the symmetry generated by the vector field Ξ_1, a second reduction with a different vector field, say Ξ_2, is certainly possible if

$$[\Xi_1, \Xi_2] = \alpha \Xi_1, \qquad \alpha \text{ constant};$$

this is because the vector field Ξ_2 can be written in terms of the invariants of Ξ_1, and so it is inherited. In other terms, the two-dimensional Lie algebra spanned by $\{\Xi_1, \Xi_2\}$ is solvable. As in the multiple reduction of order of ordinary differential equations, the vector fields have to be used in a well-defined order, first Ξ_1 and then Ξ_2, unless the subalgebra is Abelian. A higher number of reduction steps can be done by using suitably the vector fields of a solvable Lie subalgebra (see also [15]).

The following example shows a situation where this is the case.

Example 6.14 As shown before, the solution invariant with respect to the scaling group of the linear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}, \tag{6.24}$$

is $u = U(y_1, y_2, y_3)$, where $y_i = x_i/t$ $(i = 1, 2, 3)$, and the function U satisfies the reduced equation:

$$\sum_{k=1}^{3} \left((y_k^2 - 1) \frac{\partial^2 U}{\partial y_k^2} + 2 y_k \frac{\partial U}{\partial y_k} \right) + 2 \left(y_1 y_2 \frac{\partial^2 U}{\partial y_1 \partial y_2} + y_1 y_3 \frac{\partial^2 U}{\partial y_1 \partial y_3} + y_2 y_3 \frac{\partial^2 U}{\partial y_2 \partial y_3} \right) = 0.$$

The operator

$$\Xi_2 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

of the rotation in the x_1x_2 plane is a second symmetry of (6.24). In terms of the new variables y_i, it reads

$$\Xi_2 = y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2},$$

which is a symmetry of the reduced wave equation. Its invariants are

$$z_1 = y_3, \qquad z_2 = y_1^2 + y_2^2 = (x_1^2 + x_2^2)/t^2, \qquad U,$$

so that the invariant solution $U = V(z_1, z_2)$ satisfies the reduced equation:

$$4z_2(1 - z_2)\frac{\partial^2 V}{\partial z_1^2} - 4z_1z_2\frac{\partial^2 V}{\partial z_1 \partial z_2} - (1 - z_1^2)\frac{\partial^2 V}{\partial z_1^2} + (4 - 6z_2)\frac{\partial V}{\partial z_2} - 2z_1\frac{\partial V}{\partial z_1} = 0.$$

Furthermore, Eq. (6.24) admits the symmetry generated by

$$\Xi_3 = t\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial t},$$

that, in terms of variables z_1, z_2, and V, reads

$$\Xi_3 = (1 - z_1^2)\frac{\partial}{\partial z_1} - 2z_1z_2\frac{\partial}{\partial z_2}.$$

Its invariants are

$$\sigma = \frac{z_2}{1 - z_1^2} = \frac{x_1^2 + x_2^2}{t^2 - x_3^2}, \qquad V$$

and the invariant solution $V = W(\sigma)$ satisfies the equation:

$$\sigma\frac{\partial^2 W}{\partial \sigma^2} + \frac{\partial W}{\partial \sigma} = 0.$$

This ordinary differential equation admits the generator $\Xi_4 = \dfrac{\partial}{\partial W}$ (but also the symmetry $\Xi_5 = \sigma\dfrac{\partial}{\partial \sigma}$ not inherited from the wave equation), so that it can be integrated yielding the particular solution

$$u(x_1, x_2, x_3) = a_1 + a_2 \log \sigma = a_1 + a_2 \log \frac{x_1^2 + x_2^2}{t^2 - x_3^2}$$

of the wave equation, where a_1 and a_2 are arbitrary constants.

Example 6.15 The linear wave equation in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}-\frac{\partial^2 u}{\partial t^2}=0$$

admits, among the others, the group of three-dimensional rotations generated by

$$\begin{aligned}\Xi_1&=\frac{\partial}{\partial\varphi},\\ \Xi_2&=\sin\varphi\frac{\partial}{\partial\theta}+\cos\varphi\cot\theta\frac{\partial}{\partial\varphi},\\ \Xi_3&=\cos\varphi\frac{\partial}{\partial\theta}-\sin\varphi\cot\theta\frac{\partial}{\partial\varphi}.\end{aligned}$$

If we start the reduction procedure with Ξ_1, then, in terms of the invariants

$$r,\qquad \theta,\qquad t,\qquad u,$$

the invariant solution is $u=U(t,r,\theta)$, where U satisfies the reduced equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)-\frac{\partial^2 U}{\partial t^2}=0.$$

This equation does not inherit the symmetries Ξ_2 and Ξ_3, so these generators cannot be used for a further reduction.

On the other hand, the wave equation has solutions $u=U(t,r)$, independent of both the coordinates φ and θ, that are invariant with respect to the Lie algebra spanned by $\{\Xi_1,\Xi_2\}$ (the same results are obtained considering the Lie algebra spanned by $\{\Xi_1,\Xi_3\}$).

6.7 H-Invariant Solutions and Factor Systems

Let G be the Lie group of transformations admitted by a differential equation $\mathcal{E}$

$$\mathbf{\Delta}(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(k)})=\mathbf{0}, \tag{6.25}$$

and $H\subseteq G$ be an r-parameter subgroup.

A solution $\mathbf{u}=\mathbf{\Theta}(\mathbf{x})$ of the differential equation $\mathcal{E}$ is called an *H-invariant solution* if the manifold $\mathbf{u}=\mathbf{\Theta}(\mathbf{x})$ is an invariant manifold with respect to the r-parameter group H generated by the operators

$$\Xi_k = \sum_{i=1}^{n} \xi_{ki} \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_{k\alpha} \frac{\partial}{\partial u_\alpha}, \qquad k = 1, \ldots, r;$$

such an H-invariant solution $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ has to satisfy (6.25) and the additional differential equations:

$$\sum_{i=1}^{n} \xi_{ki}(\mathbf{x}, \boldsymbol{\Theta}) \frac{\partial \Theta_\alpha}{\partial x_i} = \eta_{k\alpha}(\mathbf{x}, \boldsymbol{\Theta}), \qquad k = 1, \ldots, r, \ \alpha = 1, \ldots, m.$$

In other terms, we have to look for the joint invariants of the symmetries $\Xi_1, \ldots, \Xi_r$ (see also Sect. 5.4.4), i.e., defining the matrix

$$M = \begin{pmatrix} \xi_{11} & \cdots & \xi_{1n} & \eta_{11} & \cdots & \eta_{1m} \\ \xi_{21} & \cdots & \xi_{2n} & \eta_{21} & \cdots & \eta_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{r1} & \cdots & \xi_{rn} & \eta_{r1} & \cdots & \eta_{rm} \end{pmatrix},$$

we have to find the functionally independent invariants $J(\mathbf{x}, \mathbf{u})$ such that

$$M \nabla J = \mathbf{0},$$

where

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_m} \right)^T,$$

and ^{T} denotes the transpose operator. If the matrix M has rank $r^\star \leq r$, then there exist $m + n - r^\star$ functionally independent invariants $J_1(\mathbf{x}, \mathbf{u}), \ldots, J_{m+n-r^\star}(\mathbf{x}, \mathbf{u})$. When the rank of the Jacobian matrix with entries $\dfrac{\partial J_i}{\partial u_j}$ is m, we may choose $n - r^\star$ invariants (say, the first $n - r^\star$) as new independent variables (let us call them $z_1, \ldots, z_{n-r^\star}$), and the remaining as new dependent variables (let us call them $v_1, \ldots, v_m$), and express the H-invariant solutions in the form (that can be implicit):

$$v_\alpha(\mathbf{x}, \mathbf{u}) = \psi_\alpha(z_1(\mathbf{x}, \mathbf{u}), \ldots, z_{n-r^\star}(\mathbf{x}, \mathbf{u})), \qquad \alpha = 1, \ldots, m.$$

The invariant solutions discussed before are particular instances of H-invariant solutions when $r = 1$.

This allows the reduction of the initial differential equation $\mathcal{E}$ to a reduced differential equation $\mathcal{E}/H$ with $n - r^\star$ independent variables. The differential equation $\mathcal{E}/H$ is called a *factor system* or *submodel* [26]; the number $\sigma = n - r^\star$ is the number of the independent variables in the factor differential equation $\mathcal{E}/H$, and it is called the *rank of the invariant solution*, or the *rank of the submodel*.

It can be noted that, if $r^\star = n - 1$, the factor system is a system of ordinary differential equations, whereas if $r^\star = n$ the factor system is algebraic.

Example 6.16 By using the full rotation group for a multiple reduction of the wave equation in spherical coordinates, the invariants are u, r, and t, and they lead directly to the reduced equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) - \frac{\partial^2 u}{\partial t^2} = 0.$$

6.8 Classification of Invariant Solutions

Typically, we may determine invariant solutions of partial differential equations by means of many subgroups of the symmetry group. Because of the potentially infinite number of such subgroups, listing all possible invariant solutions is not a good strategy; what we really need is a way to identify an *optimal system* of group invariant solutions, from which every other invariant solution can be obtained. As remarked in Chap. 3, this is done by characterizing an optimal system of Lie subalgebras of the algebra of point symmetries admitted by a differential equation. Let us clarify some more aspects relative to equivalent group invariant solutions.

6.8.1 Classification of Lie Subgroups

As a starting point, let us focus on the problem of equivalence of solutions that are invariant under a one-parameter Lie group of transformations.

In Example 4.5, we computed the Lie symmetries of linear heat equation; in particular, let us consider the symmetries generated by the vector fields Ξ_1 and Ξ_4 in (4.20), say

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_4 = x_1^2\frac{\partial}{\partial x_1} + x_1 x_2\frac{\partial}{\partial x_2} - \frac{x_2^2 + 2x_1}{4}u\frac{\partial}{\partial u}.$$

The symmetry generated by Ξ_1 has the invariants x_2 and u, whence the invariant solution is $u = \psi(x_2)$, with ψ solution of the equation $\psi'' = 0$; so, we get the solution

$$u = c_1 x_2 + c_2, \qquad c_1, c_2 \text{ arbitrary constants.}$$

On the contrary, the symmetry generated by Ξ_4 has the invariants $\omega = \dfrac{x_2}{x_1}$ and $u\sqrt{x_1}\exp\left(\dfrac{x_2^2}{4x_1}\right)$, whence the invariant solution is $u = \dfrac{1}{\sqrt{x_1}}\exp\left(-\dfrac{x_2^2}{4x_1}\right)\psi(\omega)$ with $\psi(\omega)$ solution of the equation $\psi'' = 0$. Thus, we obtain the solution

$$u = \frac{1}{\sqrt{x_1}}\exp\left(-\frac{x_2^2}{4x_1}\right)\left(c_1\frac{x_2}{x_1} + c_2\right), \qquad c_1, c_2 \text{ arbitrary constants.} \tag{6.26}$$

Notice that in both cases the reduced ordinary differential equations have the same form.

Remarkably, the two group invariant solutions so obtained are proved to be equivalent, for one can be obtained by the action of a group admitted by linear heat equation on the other one.

Let G_r be an r-parameter Lie group of transformations admitted by a system of differential equations $\mathbf{\Delta} = \mathbf{0}$. Moreover, let $\mathbf{u} = \mathbf{\Theta}(\mathbf{x})$ be an invariant solution with respect to a one-parameter (a) subgroup of transformations $T^a \in G_r$; T^a maps $(\mathbf{x}, \mathbf{u})$ to $(\mathbf{x}', \mathbf{u}')$, namely, we can write

$$T^a(\mathbf{x}, \mathbf{u}) = (\mathbf{x}', \mathbf{u}').$$

If we set

$$\mathbf{F}(\mathbf{x}, \mathbf{u}) = \mathbf{u} - \mathbf{\Theta}(\mathbf{x}),$$

it is

$$\mathbf{F}(T^a(\mathbf{x}, \mathbf{u})) = \mathbf{F}(\mathbf{x}', \mathbf{u}') = \mathbf{0}, \quad \text{when} \quad \mathbf{F}(\mathbf{x}, \mathbf{u}) = \mathbf{0},$$

or, equivalently,

$$\mathbf{F}(\mathbf{x}', \mathbf{u}') = \Lambda(\mathbf{x}, \mathbf{u})\mathbf{F}(\mathbf{x}, \mathbf{u}),$$

Λ being an $m \times m$ matrix of suitable Lagrange multipliers.

Let $G \in G_r$ be another fixed transformation of the Lie group G_r, and set $\widetilde{F}(\mathbf{x}', \mathbf{u}') = G(F(\mathbf{x}, \mathbf{u}))$. We can prove that $\widetilde{F}$ is invariant with respect to the conjugate subgroup $\widetilde{T}^a = GT^aG^{-1}$ of the group G_r. In fact, one has

$$\widetilde{T}^a\left(\widetilde{\mathbf{F}}\right) = GT^aG^{-1}\left(\widetilde{\mathbf{F}}\right) = GT^a(\mathbf{F}) = G\left(\Lambda\mathbf{F}\right) = \left(G\left(\Lambda\right)\right)\left(G\left(\mathbf{F}\right)\right) = \widetilde{\Lambda}\widetilde{\mathbf{F}}.$$

Since s-parameter subgroups ($s > 1$) are completely determined by their one-parameter subgroups, this remark can be generalized.

Proposition 6.1 *Let G_r be the symmetry group of a system of differential equations $\mathbf{\Delta} = \mathbf{0}$ and let $H_s \subset G_r$ be an s-parameter subgroup. If $\mathbf{u} - \mathbf{\Theta}(\mathbf{x}) = \mathbf{0}$ is an H_s-invariant solution to $\mathbf{\Delta} = \mathbf{0}$, and $G \in G_r$ is any other group element, then $G(\mathbf{u} - \mathbf{\Theta}(\mathbf{x})) = \mathbf{0}$ is a $\widetilde{H}_s$-invariant solution, where $\widetilde{H}_s = GH_sG^{-1}$ is the conjugate subgroup to H_s under G.*

Example 6.17 In Example 6.9, we have seen that the linear heat equation admits the one-parameter Lie group T^a defined by the transformations

$$x_1 \mapsto \frac{x_1}{1-ax_1}, \qquad x_2 \mapsto \frac{x_2}{1-ax_1}, \qquad u \mapsto u\sqrt{1-ax_1}\exp\left(-\frac{ax^2}{4(1-ax_1)}\right),$$

whose infinitesimal generator is given by (6.20), to which corresponds the invariant solution (6.26).

Indeed, denoting by

$$F(x_1, x_2, u) \equiv u - \frac{1}{\sqrt{x_1}}\exp\left(-\frac{x_2^2}{4x_1}\right)\left(c_1\frac{x_2}{x_1} + c_2\right) = 0$$

the manifold corresponding to (6.26), one of course has

$$\widetilde{F} = T^a(F) = F.$$

Now, let G be another element of the full symmetry group, say the one defined by

$$x_1 \mapsto \frac{1}{1-x_1}, \qquad x_2 \mapsto \frac{x_2}{1-x_1}, \qquad u \mapsto u\sqrt{1-x_1}\exp\left(-\frac{x_2^2}{4(1-x_1)}\right),$$

and let G^{-1} its inverse, namely, the one yielding

$$x_1 \mapsto \frac{x_1-1}{x_1}, \qquad x_2 \mapsto \frac{x_2}{x_1}, \qquad u \mapsto u\sqrt{x_1}\exp\left(\frac{x_2^2}{4x_1}\right).$$

It is easily ascertained that the action of the conjugate subgroup $\widetilde{T}^a = GT^aG^{-1}$ on (x_1, x_2, u) yields

$$\widetilde{T}^a(x_1, x_2, u) = GT^aG^{-1}(x_1, x_2, u) = (x_1 + a, x_2, u),$$

whose generator is

$$\widetilde{\Xi} = \frac{\partial}{\partial x_1}.$$

Moreover, applying G to F, we obtain

$$\widetilde{F} = G(F) = \exp\left(\frac{x_2^2}{4(x_1-1)}\right)\sqrt{1-x_1}\,(u - c_1x_2 - c_2) = 0,$$

that is,

$$u = c_1x_2 + c_2;$$

therefore, the solution invariant with respect to Ξ_4 is transformed by a group action to the solution invariant with respect to Ξ_1.

The above considerations suggest to limit ourselves to characterize essentially different invariant solutions, i.e., those that are not obtainable using conjugate subgroups. Therefore, the problem of the classification of essentially different H-invariant solutions is reduced to the classification of subgroups of the group G_r up to conjugation. We recall that a subgroup $H \subseteq G_r$ is *equivalent* to a subgroup $\widetilde{H} \subseteq G_r$ if there exists $G \in G_r$ such that $\widetilde{H} = GHG^{-1}$; this is a relation of equivalence and the corresponding equivalence classes are said *conjugacy classes*. This problem, in turn, reduces to the corresponding problem of classification of Lie subalgebras (see Chap. 3) that can be approached more easily from an algorithmic point of view, as we shall see in Chap. 12.

6.8.2 *Equivalent Lie Subgroups and Corresponding Lie Subalgebras*

To illustrate the procedure for classifying equivalent Lie subgroups (done classifying the corresponding Lie subalgebras), let us consider one-parameter Lie subgroups.

Let $T^a \in G_r$ be a one-parameter Lie subgroup of transformations in $\mathbb{R}^n$ generated by the vector field

$$\Xi = \sum_{j=1}^{n} \zeta_j(\mathbf{z}) \frac{\partial}{\partial z_j},$$

and let $G \in G_r$ another subgroup mapping $\mathbf{z}$ to $\mathbf{z}'$. Let $\widetilde{\Xi}$ the generator of the group $\widetilde{T}^a = GT^aG^{-1}$.

Using Theorem 2.2, it is

$$T^a(\mathbf{z}) = \exp(a\,\Xi)(\mathbf{z}) = \left(\mathsf{I} + a\,\Xi + \frac{a^2}{2}\,\Xi^2 + \frac{a^3}{3!}\,\Xi^3 + \cdots\right)(\mathbf{z}),$$

and also

$$\begin{aligned}\widetilde{T}^a(\mathbf{z}) &= GT^aG^{-1}(\mathbf{z}) = G\exp(a\,\widetilde{\Xi})G^{-1}(\mathbf{z}) \\ &= G\left(\sum_{k=0}^{\infty} \frac{a^k}{k!}\,\Xi^k\right)G^{-1}(\mathbf{z}) = \sum_{k=0}^{\infty} \frac{a^k}{k!}\left(G\Xi G^{-1}\right)^k(\mathbf{z}) = \exp(aG\Xi G^{-1})(\mathbf{z}),\end{aligned}$$

whence

$$\widetilde{\Xi} = G\Xi G^{-1}. \tag{6.27}$$

Thus, two one-parameter Lie subgroups are equivalent if and only if the relation (6.27) for some group $G \in G_r$ holds true.

Remark 6.5 We note that T^a and $\widetilde{T}^a$, as the parameter changes, generate one-parameter subgroups of G_r. On the other side, Ξ and $\widetilde{\Xi}$ are two elements of the corresponding Lie algebra, and they generate two one-dimensional subalgebras. Moreover, if T^a and $\widetilde{T}^a$ are equivalent via G, then the corresponding Lie subalgebras generated by Ξ and $\widetilde{\Xi}$ are related by the same group G, i.e., the two subalgebras are *equivalent*.

Let us show how to construct, by means of the infinitesimal generators and without using the finite transformations of the Lie group, the maps establishing the equivalence among Lie subalgebras.

In the remainder of this chapter, the analysis will be limited to the equivalence among Lie subalgebras of a finite-dimensional Lie algebra $\mathbb{L}_r$ with basis $\{\Xi_1, \ldots, \Xi_r\}$.

For each generator Ξ_i in the basis, we can consider

$$\widetilde{X} = \exp(a\,\Xi_i) X \exp(-a\alpha\,\Xi_i), \tag{6.28}$$

for any generator X of the Lie algebra.

Differentiating the expression (6.28) with respect to the parameter a, we obtain

$$\begin{aligned}\frac{d\widetilde{X}}{da} &= \Xi_i \exp(a\,\Xi_i) X \exp(-a\,\Xi_i) - \exp(a\,\Xi_i) X \exp(-a\,\Xi_i)\,\Xi_i \\ &= \Xi_i \widetilde{X} - \widetilde{X}\,\Xi_i = \left[\Xi_i, \widetilde{X}\right].\end{aligned}$$

Thus, $\widetilde{X}$ satisfies the following initial value problem:

$$\frac{d\widetilde{X}}{da} = [\Xi_i, \widetilde{X}], \qquad \widetilde{X}\big|_{a=0} = X.$$

Again, calculating the second derivative, we obtain

$$\frac{d^2\widetilde{X}}{da^2} = \left[\Xi_i, \frac{d\widetilde{X}}{da}\right] = \left[\Xi_i, \left[\Xi_i, \widetilde{X}\right]\right],$$

and so on. Thus, in a neighborhood of $a = 0$, one has

$$\widetilde{X} = X + a[\Xi_i, X] + \frac{a^2}{2!}[\Xi_i, [\Xi_i, X]] + \cdots.$$

Considering the adjoint map (which is a linear map, see Chap. 3) $\mathrm{ad}_X : \mathbb{L}_r \to \mathbb{L}_r$,

$$\mathrm{ad}_X(Y) = [X, Y], \quad \text{for each } X, Y \in \mathbb{L}_r,$$

it is

$$\widetilde{X} = X + a\,\mathrm{ad}_{\Xi_i}(X) + \frac{a^2}{2!}\,\mathrm{ad}_{\Xi_i}(\mathrm{ad}_{\Xi_i}(X)) + \cdots = \\ = \sum_{k=0}^{+\infty} \frac{(a\,\mathrm{ad}_{\Xi_i}(X))^k}{k!} = \exp(\mathrm{ad}_{\Xi_i})(X). \tag{6.29}$$

If $X = \sum_{i=1}^{r} f^i\,\Xi_i$, then

$$\mathrm{ad}_X(\Xi_j) = \sum_{i=1}^{r} f^i[\Xi_i, \Xi_j] = \sum_{k=1}^{r}\left(\sum_{i=1}^{r} f^i C_{ij}^k\right),$$

whence the (k, j)-entry of the matrix associated to ad_X is $\sum_{i=1}^{r} f^i C_{ij}^k$, C_{ij}^k being the structure constants of the Lie algebra $\mathbb{L}_r$. Then, in the finite-dimensional case, we can construct the invertible transformations that map equivalent Lie subalgebras by computing the exponential of these matrices. We recall that, given an element $X \in \mathbb{L}_r$, $\exp(\mathrm{ad}_X)$ is a Lie algebra automorphism, known as *inner automorphism* of $\mathbb{L}_r$, and their set is denoted by $\mathrm{Int}(\mathbb{L}_r)$.

Example 6.18 Consider the non-Abelian two-dimensional Lie algebra $\mathbb{L}_2$ with a basis $\{\Xi_1, \Xi_2\}$ such that

$$[\Xi_1, \Xi_2] = \Xi_1.$$

A generic element $X \in \mathbb{L}_2$ has the form

$$X = f^1\Xi_1 + f^2\Xi_2.$$

In this case, it is simple to compute the inner automorphisms related to the generators Ξ_1 and Ξ_2. In fact, it is sufficient to write the series (6.29):

$$\begin{aligned}
\exp(t_1\,\mathrm{ad}_{\Xi_1})(X) &= X + t_1[\Xi_1, X] + \frac{t_1^2}{2!}[\Xi_1, [\Xi_1, X]] + \cdots \\
&= \sum_{i=1}^{2} f^i\left(\Xi_i + t_1[\Xi_1, \Xi_i] + \frac{t_1^2}{2!}[\Xi_1, [\Xi_1, \Xi_i]] + \cdots\right) \\
&= f^1\Xi_1 + f^2(\Xi_2 + t_1\Xi_1) = \left(f^1 + t_1 f^2\right)\Xi_1 + f^2\Xi_2, \\
\exp(t_2\,\mathrm{ad}_{\Xi_2})(X) &= \sum_{i=1}^{2} f^i\left(\Xi_i + t_2[\Xi_2, \Xi_i] + \frac{t_2^2}{2!}[\Xi_2, [\Xi_2, \Xi_i]] + \cdots\right) \\
&= \exp(-t_2) f^1\Xi_1 + f^2\Xi_2.
\end{aligned}$$

Example 6.19 Consider the three-dimensional Lie algebra $\mathbb{L}_3 = \mathfrak{sl}_2(\mathbb{R})$ with the basis $\{\Xi_1, \Xi_2, \Xi_3\}$ such that

$$[\Xi_1, \Xi_2] = \Xi_1, \quad [\Xi_1, \Xi_3] = -2\Xi_2, \quad [\Xi_2, \Xi_3] = \Xi_3.$$

Let us compute the matrix representation of adjoint maps ad_{Ξ_i}, with $i = 1, 2, 3$:

$$\mathrm{ad}_{\Xi_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{ad}_{\Xi_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{ad}_{\Xi_3} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

The inner automorphisms $\mathsf{A}_i = \exp(t_i \, \mathrm{ad}_{\Xi_i})$ are obtained by computing the respective exponential matrices:

$$\mathsf{A}_1 = \begin{pmatrix} 1 & t_1 & t_1^2 \\ 0 & 1 & 2t_1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathsf{A}_2 = \begin{pmatrix} \exp(-t_2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(t_2) \end{pmatrix}, \qquad \mathsf{A}_3 = \begin{pmatrix} 1 & 0 & 0 \\ -2t_3 & 1 & 0 \\ t_3^2 & -t_3 & 1 \end{pmatrix}.$$

Therefore, the two subalgebras H_1 and H_2 of $\mathbb{L}_3$ are *equivalent* if there exists an inner automorphism A such that $H_2 = \mathsf{A}H_1$.

Partitioning Lie subalgebras of a Lie algebra $\mathbb{L}_r$ in equivalence classes, we obtain an optimal system of Lie algebras (see Definition 3.23). From an optimal system of Lie subalgebras of the Lie algebra of symmetries of a differential equations we can derive an optimal system of group invariant solutions.

Example 6.20 Consider the Lie algebra $\mathbb{L}_4$ spanned by the vector fields

$$\Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_2}, \quad \Xi_3 = x_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u}, \quad \Xi_4 = 3x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - 2u \frac{\partial}{\partial u}$$

generating the Lie point symmetries of Korteweg–deVries equation [22],

$$u_{,1} + uu_{,2} + u_{,222} = 0.$$

In order to determine the inner automorphisms, we first compute the Lie brackets,

$$[\Xi_1, \Xi_3] = \Xi_2, \quad [\Xi_1, \Xi_4] = 3\Xi_1, \quad [\Xi_2, \Xi_4] = \Xi_2, \quad [\Xi_3, \Xi_4] = -2\Xi_3,$$

and then ad_{Ξ_i}, $i = 1, \ldots, 4$:

$$\mathrm{ad}_{\Xi_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; \mathrm{ad}_{\Xi_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; \mathrm{ad}_{\Xi_3} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; \mathrm{ad}_{\Xi_4} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

whence we derive $\mathsf{A}_i = \exp(t_i \operatorname{ad}_{\Xi_i})$, $i = 1, \ldots, 4$:

$$\mathsf{A}_1 = \begin{pmatrix} 1 & 0 & -t_1 & 0 \\ 0 & 1 & 0 & -3t_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathsf{A}_2 = \begin{pmatrix} 1 & 0 & 0 & -t_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathsf{A}_3 = \begin{pmatrix} 1 & t_3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathsf{A}_4 = \begin{pmatrix} \exp(t_4) & 0 & 0 & 0 \\ 0 & \exp(3t_4) & 0 & 0 \\ 0 & 0 & \exp(-2t_4) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let

$$X = f^1 \Xi_1 + f^2 \Xi_2 + f^3 \Xi_3 + f^4 \Xi_4$$

be an element of $\mathbb{L}_4$; actions of the automorphisms can be transferred to the coordinates (f^1, f^2, f^3, f^4), so that we can represent X as (f^1, f^2, f^3, f^4). Our aim is to simplify these coordinates so as to obtain a simpler representative (i.e., with the greatest number of zero coordinates). We divide the computation into several cases.

Case 1: $f^4 \neq 0$. For simplicity, we can assume $f^4 = 1$; applying A_3, it follows

$$X' = (f^1 + f^2 t_3, f^2, f^3 + 2f^4 t_3, 1),$$

and choosing $t_3 = -f^3/2$, we get a first simplification

$$X' = \left(f^1 - \frac{f^2 f^3}{2}, f^2, 0, 1 \right).$$

Then, acting with A_1 act on X' yields

$$X'' = \left(f^1 - \frac{f^2 f^3}{2}, f^2 - 3t_1, 0, 1 \right).$$

Choosing $t_1 = f^2/3$, we obtain

$$X'' = \left(f^1 - \frac{f^2 f^3}{2}, 0, 0, 1 \right).$$

Finally, acting with A_2 on X'', and choosing $t_2 = f^1 - f^2 f^3/2$, it turns out that the first component vanishes. Therefore, any one-dimensional subalgebra spanned by an element with $f^4 \neq 0$ is equivalent to the subalgebras spanned by Ξ_4.

Case 2: $f^4 = 0$. We can distinguish between two subcases: $f^3 \neq 0$ and $f^3 = 0$.
Subcase 2.1: $f^3 \neq 0$. Again, we can assume $f^3 = 1$ and act with A_1 on X. Hence,

$$X' = (f^1 - t_1, f^2, 1, 0);$$

choosing $t_1 = f^1$, we obtain $X' = f^2 \Xi_2 + \Xi_3$. Then, we can apply the inner automorphism A_4 to X' and, after rescaling, obtain

$$X'' = (0, \exp(5t_4) f^2, 1, 0),$$

whereupon

$$X'' = (0, \operatorname{sign}(f^2), 1, 0), \quad \text{with} \quad t_4 = -\frac{1}{5}\log(|f^2|).$$

Thus, any subalgebra generated by an element with $f^4 = 0$ and $f^3 \neq 0$ is equivalent to one of the three subalgebras: $\Xi_3 \pm \Xi_2$ or Ξ_3, depending on the value of f^2.
Subcase 2.2: $f^3 = 0$. If $f^2 \neq 0$, taking $f^2 = 1$, we can act with A_3 on $X = (f^1, 1, 0, 0)$, and annihilate the first coordinate by choosing $t_3 = -f^1$. If instead $f^2 = 0$, we have the one-dimensional subalgebra generated by Ξ_1.
This concludes the analysis, and the optimal system of one-dimensional Lie subalgebras is

$$\Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_1}, \quad \Xi_3 = x_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u},$$
$$\Xi_4 = 3x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - 2u \frac{\partial}{\partial u}, \quad \Xi_3 \pm \Xi_2 = x_1 \frac{\partial}{\partial x_2} \pm \frac{\partial}{\partial x_1} + \frac{\partial}{\partial u}.$$

The computations required to produce an optimal system of subalgebras are relatively simple for low-dimensional algebras, but become significantly more cumbersome as the dimension increases. In Chap. 12, a computer algebra package [1–3] written in Wolfram Mathematica™ [36], able to determine classes of optimal Lie subalgebras almost automatically, will be described.

6.9 Galilean First-Order Quasilinear Systems

Many mathematical models of physical interest (e.g., think of the equations of ideal gas dynamics) are mathematically modeled by first-order quasilinear systems of partial differential equations of the form:

$$\mathsf{A}_0(\mathbf{u})\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^{n} \mathsf{A}_i(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{F}(\mathbf{u}), \tag{6.30}$$

where $\mathbf{u}(t, \mathbf{x}) \in \mathbb{R}^m$ represents the unknown vector field, $\mathbf{x} \in \mathbb{R}^n$ ($n \geq 1$) denote the spatial coordinates (in a fixed frame reference) and $t \in \mathbb{R}$ the time, $\mathsf{A}_0(\mathbf{u})$ and $\mathsf{A}_i(\mathbf{u})$ ($i = 1, \ldots, n$) are $m \times m$ matrices with entries depending on $\mathbf{u}$, and $\mathbf{F}(\mathbf{u}) \in \mathbb{R}^m$ is a given vector function of the field $\mathbf{u}$. We assume that $\mathsf{A}_0(\mathbf{u})$ is nonsingular and that all derivatives we use exist and are continuous.

In the framework of Newtonian physics, a system of the form (6.30) has to be invariant with respect to the so-called Galilean group of transformations stating that the equations must have the same form in two reference frames differing only by constant relative motion. In order to study such kind of systems (hereafter referred to as Galilean systems) it is convenient to split the unknown vector field $\mathbf{u}$ in a pair

$$\mathbf{u} = (\mathbf{v}, \mathbf{w}), \qquad \mathbf{v} \in \mathbb{R}^n, \qquad \mathbf{w} \in \mathbb{R}^{m-n},$$

where the vector $\mathbf{v}$ assumes the meaning of a velocity.

Now, we require the system (6.30) to be left invariant by the following (Galilean) group of transformations:

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} + \mathbf{a}t, \quad t' = t, \\ \mathbf{v}' &= \mathbf{v} + \mathbf{a}, \quad \mathbf{w}' = \mathbf{w}, \end{aligned} \tag{6.31}$$

$\mathbf{x}'$ being the spatial coordinates in a new reference frame moving with a constant velocity $\mathbf{a} \equiv (a_1, \ldots, a_n)$ relative to the previous reference frame; the components of $\mathbf{a}$ are the parameters of the group. The request of invariance of (6.30) with respect to the transformations (6.31) implies some restrictions on the structure of the system (6.30). Such a structure, which is well known, has been considered by several authors [31, 32]. Hereafter, we consider a different derivation making use of the methods of infinitesimal Lie groups of transformations [21]. This approach is both straightforward and elegant: obviously, we recover the same results as in [31, 32].

The system (6.30) is Galilean if it is invariant with respect to the transformation (6.31); this characterizes an n parameter Lie group whose infinitesimal operators

$$\Xi_j = t\frac{\partial}{\partial x_j} + \frac{\partial}{\partial v_j}, \qquad j = 1, \ldots, n, \tag{6.32}$$

generating an n-dimensional Lie algebra.

If the system (6.30) is Galilean then, in the spirit of Lie theory, the first prolongations of the generators (6.32), acting on the system (6.30), give zero along the solutions of (6.30).

This leads to the following determining equations:

$$\begin{aligned} &\frac{\partial \mathsf{A}_0}{\partial v_j}(\mathsf{A}_0)^{-1}\mathbf{F} - \frac{\partial \mathbf{F}}{\partial v_j} = 0, \qquad\qquad j = 1, \dots, n, \\ &\frac{\partial \mathsf{A}_i}{\partial v_j} - \mathsf{A}^0 \delta_{ij} - \frac{\partial \mathsf{A}_0}{\partial v_j}(\mathsf{A}_0)^{-1}\mathsf{A}_i = 0, \quad i, j = 1, \dots, n, \end{aligned} \tag{6.33}$$

δ_{ij} being the Kronecker symbol; integration of (6.33) yields

$$\begin{aligned} &\mathbf{F}(\mathbf{u}) = \mathsf{A}_0(\mathbf{u})\mathbf{G}(\mathbf{w}), \\ &\mathsf{A}_j(\mathbf{u}) = \mathsf{A}_0(\mathbf{u})(v_j \mathsf{I}_m + \mathsf{B}_j(\mathbf{w})), \qquad j = 1, \dots, n, \end{aligned}$$

where I_m is the $m \times m$ identity matrix, $\mathbf{G}(\mathbf{w}) \in \mathbb{R}^m$, $\mathsf{B}_j(\mathbf{w})$ $m \times m$ matrices with entries depending at most on $\mathbf{w}$.

Thus, a Galilean system has the form

$$\mathsf{A}_0(\mathbf{u})\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{n} \mathsf{A}_0(\mathbf{u})(v_j \mathsf{I}_m + \mathsf{B}_j(\mathbf{w})\frac{\partial \mathbf{u}}{\partial x_j} - \mathsf{A}_0(\mathbf{u})\mathbf{G}(\mathbf{w}) = \mathbf{0},$$

or, equivalently, in components (by left multiplying by $(\mathsf{A}_0)^{-1}$)

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n}\left(v_j \frac{\partial u_i}{\partial x_j} + \sum_{k=1}^{m} \mathsf{B}_{ik}^{j}(\mathbf{w})\frac{\partial u_k}{\partial x_j}\right) - G_i(\mathbf{w}) = 0, \quad i = 1, \dots, n. \tag{6.34}$$

In each equation in (6.34) we can recognize the *material time derivative* of the field u_i,

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} v_j \frac{\partial u_i}{\partial x_j}, \quad i = 1, \dots, m.$$

The problem here considered falls into the class of inverse problems since we require a class of partial differential equations to be invariant with respect to an assigned Lie algebra of point symmetries. Chapter 8 will be devoted to a deeper analysis of the inverse Lie problem, and the class of Lie remarkable equations will be introduced.

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Chapter 7
Conservation Laws



7.1 Introduction

In Mathematical Physics, conservation laws have a deep relevance, as they express the conservation of physical quantities such as mass, momentum, angular momentum, energy, and electrical charge. They are also important from a theoretical point of view due to their use in investigating integrability, existence, uniqueness, and stability of solutions [6, 14], or in implementing efficient numerical methods of integration [8, 10].

In 1918, Emmy Noether [13] presented a procedure (Noether theorem) to find local conservation laws for systems of differential equations arising from a variational principle. Noether proved that a point symmetry of the action integral yields a local conservation law through an explicit formula that involves the infinitesimals of the point symmetry and the Lagrangian. In [11], a new English translation of the original paper by Emmy Noether (that she dedicated to the fiftieth anniversary of Felix Klein's doctorate) is contained. Also, many historical notes about the relevance of Noether's contributions are included.

We start considering a first-order Lagrangian in the context of analytical mechanics, so that the Noether theorem produces first integrals. Then, we move to the general problem within the framework of Lagrangian field theory, where we may have more than one independent variable and/or Lagrangians of higher order.

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7.2 Hamilton Principle and Euler–Lagrange Equations in Analytical Mechanics

In analytical mechanics [3, 7, 12], the equations of the motion of a holonomic system with n degrees of freedom subject to ideal constraints and conservative forces can be obtained starting from a scalar function, the Lagrangian $\mathcal{L}(t, q_k, \dot{q}_k)$ $(k = 1, \ldots, n)$, where $\dot{q}_k = \dfrac{dq_k}{dt}$, and using the Hamilton principle.

Given the Lagrangian $\mathcal{L}(t, q_k, \dot{q}_k)$, the Lagrangian action in the time interval $[t_1, t_2]$ is the integral

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(t, q_k, \dot{q}_k)\, dt, \tag{7.1}$$

depending on the curve

$$q_k = q_k(t), \qquad k = 1, \ldots, n,$$

representing the motion of the system in an n-dimensional configuration space.

Hamilton principle (or *least action principle*) states that the motion of the system ruled by the Lagrangian $\mathcal{L}$ in the time interval $[t_1, t_2]$ corresponds to the trajectory of the configuration space where $\mathcal{S}$ in (7.1) attains a stationary value.

In other words, among all possible curves of the configuration space from the point at time t_1 to the point at time t_2, the trajectory followed by the system is the one such that the variation of the Lagrangian action is vanishing, i.e., $\delta\mathcal{S} = 0$.

Let $q_k = q_k(t)$ $(k = 1, \ldots, n)$ be the equations of the curve representing the motion, and consider a small perturbation

$$q_k(t) + \delta q_k(t), \qquad k = 1, \ldots, n,$$

along with the constraints

$$\delta q_k(t_1) = \delta q_k(t_2) = 0, \qquad k = 1, \ldots, n,$$

that is, the varied curve has the ending points coinciding with those of the non-varied curve.

This perturbation of the trajectory produces a variation of the Lagrangian action,

$$\begin{aligned}\delta\mathcal{S} &= \int_{t_1}^{t_2} \mathcal{L}(t, q_k + \delta q_k, \dot{q}_k + \delta\dot{q}_k)dt - \int_{t_1}^{t_2} \mathcal{L}(t, q_k, \dot{q}_k)dt \\ &= \int_{t_1}^{t_2} \left(\mathcal{L}(t, q_k + \delta q_k, \dot{q}_k + \delta\dot{q}_k) - \mathcal{L}(t, q_k, \dot{q}_k)\right) dt = \int_{t_1}^{t_2} \delta\mathcal{L}.\end{aligned}$$

Taking the Taylor expansion of the Lagrangian of the varied motion, and neglecting higher order terms in δq_k and $\delta \dot{q}_k$, we obtain

$$\begin{aligned}
\delta \mathcal{S} &= \int_{t_1}^{t_2} \sum_{k=1}^{n} \left(\frac{\partial \mathcal{L}}{\partial q_k} \delta q_k + \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta \dot{q}_k \right) dt \\
&= \int_{t_1}^{t_2} \sum_{k=1}^{n} \left(\frac{\partial \mathcal{L}}{\partial q_k} \delta q_k - \mathcal{D}_t \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) \delta q_k + \mathcal{D}_t \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta q_k \right) \right) dt \\
&= \int_{t_1}^{t_2} \left(\sum_{k=1}^{n} \left(\frac{\partial \mathcal{L}}{\partial q_k} - \mathcal{D}_t \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) \right) \delta q_k \right) dt + \left[\sum_{k=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} \left(\sum_{k=1}^{n} \left(\frac{\partial \mathcal{L}}{\partial q_k} - \mathcal{D}_t \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) \right) \delta q_k \right) dt,
\end{aligned}$$

where we made use of the conditions that δq_k's are vanishing at $t = t_1$ and $t = t_2$, and $\mathcal{D}_t$ is the total time derivative.

It is $\delta \mathcal{S} = 0$ if and only if

$$\sum_{k=1}^{n} \left(\frac{\partial \mathcal{L}}{\partial q_k} - \mathcal{D}_t \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) \right) \delta q_k = 0.$$

Finally, the arbitrariness and independence of δq_k's lead to the Euler–Lagrange equations, say

$$\frac{\partial \mathcal{L}}{\partial q_k} - \mathcal{D}_t \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) = 0, \qquad k = 1, \ldots, n.$$

7.2.1 Equivalent Lagrangians

There are cases where different Lagrangians produce the same Euler–Lagrange equations. In some of these cases, it is possible to construct an equivalence relation among such Lagrangians. Using Hamilton principle, it is possible to characterize equivalent Lagrangians.

Definition 7.1 (*Null Lagrangian*) A Lagrangian such that the corresponding Euler–Lagrange equations are identically satisfied is called a *null Lagrangian*.

Theorem 7.1 *$\mathcal{L}$ is a null Lagrangian if it is the total time derivative of a function $\mathcal{F}(t, q_1, \ldots, q_n)$ depending on t and the Lagrangian coordinates q_k, say*

$$\mathcal{L} = \mathcal{D}_t(\mathcal{F}(t, q_1, \ldots, q_n)).$$

Proof If

$$\mathcal{L} = \mathcal{D}_t(\mathcal{F}(t, q_1, \ldots, q_n))$$

for some smooth function $\mathcal{F}(t, q_k)$, then

$$\mathcal{L} = \sum_{k=1}^{n} \frac{\partial \mathcal{F}}{\partial q_k}\dot{q}_k + \frac{\partial \mathcal{F}}{\partial t},$$

whence

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{\partial \mathcal{F}}{\partial q_k}.$$

Consequently, we have the Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_k} - \mathcal{D}_t\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial}{\partial q_k}\left(\mathcal{D}_t(\mathcal{F})\right) - \mathcal{D}_t\left(\frac{\partial \mathcal{F}}{\partial q_k}\right) \equiv 0,$$

where the Schwarz theorem on mixed derivatives has been used. Furthermore,

$$S = \int_{t_1}^{t_2} \mathcal{D}_t(\mathcal{F})dt = \mathcal{F}(t_2, q_k(t_2)) - \mathcal{F}(t_1, q_k(t_1)).$$

So, $\delta S = 0$ for arbitrary variations δq_k's vanishing at t_1 and t_2. □

Definition 7.2 (*Equivalent Lagrangians*) The Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ are *equivalent* if they differ by the total time derivative of a smooth function $\mathcal{F}(t, q_k)$, i.e.:

$$\mathcal{L}_2 = \mathcal{L}_1 + \mathcal{D}_t(\mathcal{F}(t, q_k)).$$

Remark 7.1 Of course, two equivalent Lagrangians produce the same Euler–Lagrange equations.

The converse is in general not true: two Lagrangians yielding the same Euler–Lagrange equations are not necessarily equivalent, as illustrated in the following examples.

Example 7.1 The Lagrangians

$$\mathcal{L}_1 = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2, \qquad \mathcal{L}_2 = \lambda\left(\frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2\right)$$

($\lambda \neq 0, 1$) yield the same Euler–Lagrange equation,

$$\ddot{q} + \frac{k}{m}q = 0,$$

but are not equivalent according to Definition 7.2.

Example 7.2 The Lagrangians

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{2}\dot{q}^2 \qquad \text{(free particle with unitary mass)},\\ \mathcal{L}_2 &= \sqrt{1+\dot{q}^2} \quad \text{(geodesic of the plane)}\end{aligned}$$

lead to the same solution but are not equivalent. In these two cases, Euler–Lagrange equations admit a first integral,

$$\dot{q} = c, \qquad \frac{\dot{q}}{\sqrt{1+\dot{q}^2}} = c',$$

whence we deduce

$$q = ct + q_0, \qquad q = \frac{c'}{\sqrt{1-c'^2}}t + q_0,$$

that in the (t, q)-plane represent a straight line (the geodesic).

Remark 7.2 As two Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ such that

$$\mathcal{L}_2 = \lambda\mathcal{L}_1, \qquad \lambda \text{ nonvanishing constant},$$

lead to the same Euler–Lagrange equations, we can give a weaker definition for equivalent Lagrangians [7]: $\mathcal{L}_1$ and $\mathcal{L}_2$ are equivalent if their linear combination with constant coefficients is equal to the total time derivative of a smooth function $\mathcal{F}(t, q_k)$, i.e.:

$$\lambda_1\mathcal{L}_1(t, q_k, \dot{q}_k) + \lambda_2\mathcal{L}_2(t, q_k, \dot{q}_k) = \mathcal{D}_t(\mathcal{F}(t, q_k)),$$

λ_1 and λ_2 being constants.

7.2.2 Variational Symmetries

Given the Lie group of infinitesimal transformations

$$\begin{aligned}t' &= t + a\tau(t, q_1, \ldots, q_n) + O(a^2),\\ q'_k &= q_k + a\eta_k(t, q_1, \ldots, q_n) + O(a^2), \qquad k = 1, \ldots, n,\end{aligned}$$

generated by the vector field

$$\Xi = \tau\frac{\partial}{\partial t} + \sum_{k=1}^{n}\eta_k\frac{\partial}{\partial q_k},$$

let us impose that it leaves the Lagrangian action invariant, that is,

$$\int_{t_1'}^{t_2'} \mathcal{L}(t', q_k', \dot{q}_k')dt' = \int_{t_1}^{t_2} \mathcal{L}(t, q_k, \dot{q}_k)dt.$$

Taking into account that

$$\begin{aligned} dt' &= \left(1 + a\mathcal{D}_t(\tau) + O(a^2)\right) dt, \\ \dot{q}_k' &= \dot{q}_k + a\psi_k + O(a^2), \end{aligned}$$

where

$$\psi_k = \mathcal{D}_t(\eta_k) - \mathcal{D}_t(\tau)\dot{q}_k,$$

it is

$$\begin{aligned} 0 &= \int_{t_1'}^{t_2'} \mathcal{L}(t', q_k', \dot{q}_k')dt' - \int_{t_1}^{t_2} \mathcal{L}(t, q_k, \dot{q}_k)dt = \int_{t_1'}^{t_2'} \mathcal{L}'dt' - \int_{t_1}^{t_2} \mathcal{L}dt \\ &= \int_{t_1}^{t_2} \left(\mathcal{L} + a\left(\tau \frac{\partial \mathcal{L}}{\partial t} + \sum_{k=1}^{n} \left(\eta_k \frac{\partial \mathcal{L}}{\partial q_k} + \psi_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) \right) + O(a^2) \right) \\ &\times \left(1 + a\mathcal{D}_t(\tau) + O(a^2)\right) dt - \int_{t_1}^{t_2} \mathcal{L}dt \\ &= \int_{t_1}^{t_2} \left(a\left(\mathcal{L}\mathcal{D}_t(\tau) + \tau \frac{\partial \mathcal{L}}{\partial t} + \sum_{k=1}^{n} \left(\eta_k \frac{\partial \mathcal{L}}{\partial q_k} + \psi_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) \right) + O(a^2) \right) dt \\ &= \int_{t_1}^{t_2} \left(a\left(\mathcal{L}\mathcal{D}_t(\tau) + \Xi^{(1)}(\mathcal{L}) \right) + O(a^2) \right) dt, \end{aligned}$$

$\Xi^{(1)}$ being the first-order prolongation of Ξ.

Therefore, we can state the following result.

Theorem 7.2 *The Lagrangian action is invariant if*

$$\mathcal{L}\mathcal{D}_t(\tau) + \Xi^{(1)}(\mathcal{L}) = 0.$$

Definition 7.3 (*Variational symmetry*) A symmetry of the Lagrangian action is called a *variational symmetry*.

Since an equivalent Lagrangian can be obtained adding the total time derivative of an arbitrary smooth function $\mathcal{F}(t, q_1, \ldots, q_n)$, we can give a more general definition, that of a *divergence symmetry*.

Definition 7.4 (*Divergence symmetry*) A *divergence symmetry* is a symmetry preserving the Lagrangian action up to a total time derivative of a smooth function $\mathcal{F}$, that is,

$$\mathcal{L}\mathcal{D}_t(\tau) + \Xi^{(1)}(\mathcal{L}) - \mathcal{D}_t(\mathcal{F}) = 0.$$

Now we have all the ingredients for formulating the Noether theorem.

Theorem 7.3 (Noether) *Let $\mathcal{L}$ be a Lagrangian, and suppose we know a variational symmetry, that is a vector field Ξ such that*

$$\mathcal{L}\mathcal{D}_t(\tau) + \Xi^{(1)}(\mathcal{L}) = 0;$$

then, the relation

$$\tau\mathcal{L} + \sum_{k=1}^{n}(\eta_k - \tau\dot{q}_k)\frac{\partial\mathcal{L}}{\partial\dot{q}_k} = c,$$

where c is a constant, yields a first integral.

Proof It is

$$\begin{aligned}
0 &= \mathcal{L}\mathcal{D}_t(\tau) + \Xi^{(1)}(\mathcal{L}) \\
&= \mathcal{L}\mathcal{D}_t(\tau) + \tau\frac{\partial\mathcal{L}}{\partial t} + \sum_{k=1}^{n}\left(\eta_k\frac{\partial\mathcal{L}}{\partial q_k} + (\mathcal{D}_t(\eta_k) - \mathcal{D}_t(\tau)\dot{q}_k)\,\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right) \\
&= \mathcal{L}\mathcal{D}_t(\tau) + \tau\left(\mathcal{D}_t(\mathcal{L}) - \sum_{k=1}^{n}\left(\frac{\partial\mathcal{L}}{\partial q_k}\dot{q}_k + \frac{\partial\mathcal{L}}{\partial\dot{q}_k}\ddot{q}_k\right)\right) \\
&+ \sum_{k=1}^{n}\left(\eta_k\frac{\partial\mathcal{L}}{\partial q_k} + (\mathcal{D}_t(\eta_k) - \mathcal{D}_t(\tau)\dot{q}_k)\,\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right) \\
&= \mathcal{D}_t(\tau\mathcal{L}) - \tau\sum_{k=1}^{n}\left(\mathcal{D}_t\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right)\dot{q}_k + \frac{\partial\mathcal{L}}{\partial\dot{q}_k}\ddot{q}_k\right) \\
&+ \sum_{k=1}^{n}\left(\eta_k\mathcal{D}_t\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right) + (\mathcal{D}_t(\eta_k) - \mathcal{D}_t(\tau)\dot{q}_k)\,\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right) \\
&= \mathcal{D}_t(\tau\mathcal{L}) + \sum_{k=1}^{n}\left(\mathcal{D}_t\left(\eta_k\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right) - \tau\mathcal{D}_t\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\dot{q}_k\right) - \mathcal{D}_t(\tau)\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\dot{q}_k\right)\right) \\
&= \mathcal{D}_t\left(\tau\mathcal{L} + \sum_{k=1}^{n}\left((\eta_k - \tau\dot{q}_k)\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right)\right),
\end{aligned}$$

whereupon

$$\tau\mathcal{L} + \sum_{k=1}^{n}\left((\eta_k - \tau\dot{q}_k)\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right) = c$$

gives a first integral. □

Remark 7.3 Poisson and Jacobi first integrals are special cases obtainable from Noether theorem.

- Poisson integral: if for some i $(1 \leq i \leq n)$ $\dfrac{\partial\mathcal{L}}{\partial q_i} = 0$, then the Lie operator

$$\Xi = \frac{\partial}{\partial q_i} \qquad \text{(translation of the } q_i \text{ coordinate)}$$

  is a variational symmetry, and Noether theorem yields the classical Poisson first integral

$$\frac{\partial\mathcal{L}}{\partial\dot{q}_i} = c.$$

- Jacobi integral: if $\dfrac{\partial\mathcal{L}}{\partial t} = 0$, the Lie operator

$$\Xi = \frac{\partial}{\partial t} \qquad \text{(time translation)}$$

  is a variational symmetry, and Noether theorem gives the first integral

$$\mathcal{L} - \sum_{k=1}^{n}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\dot{q}_k = c,$$

  namely, the conservation of the Hamiltonian function.

In analytical mechanics, first integrals are very important as they allow for the order reduction of the Euler–Lagrange equations. Moreover, a sufficient number of first integrals can make the problem integrable (think of the equations for Lagrange spin top where we have three first integrals allowing for analytically solving the problem).

Remark 7.4 Using divergence symmetries, the invariance condition of Lagrangian action functional reads

$$\mathcal{L}\mathcal{D}_t(\tau) + \Xi^{(1)}(\mathcal{L}) - \mathcal{D}_t(\mathcal{F}) = 0,$$

and Noether theorem gives the first integral

$$\tau\mathcal{L}+\sum_{k=1}^{n}\left((\eta_k-\tau\dot{q}_k)\frac{\partial\mathcal{L}}{\partial\dot{q}_k}\right)-\mathcal{F}=c,$$

where c is a constant.

7.3 Lagrangian Field Theory

Variational formulations are important also in field theory, where we have n independent variables that we denote as usual with $\mathbf{x}\equiv(x_1,\ldots,x_n)$, m dependent variables $\mathbf{u}\equiv(u_1,\ldots,u_m)$, and a Lagrangian function $\mathcal{L}(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(r)})$ not necessarily of first order.

Consider the action integral $\mathcal{S}$ of the Lagrangian $\mathcal{L}$, involving independent variables $\mathbf{x}\in\mathbb{R}^n$, dependent variables $\mathbf{u}\in\mathbb{R}^m$, and their partial derivatives up to order r, on a domain $\Omega\subseteq\mathbb{R}^n$,

$$\mathcal{S}=\int_\Omega\mathcal{L}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(r)}\right)d\mathbf{x}.$$

Take an infinitesimal variation of $\mathbf{u}(\mathbf{x})$ given by $\mathbf{u}(\mathbf{x})\to\mathbf{u}(\mathbf{x})+a\mathbf{v}(\mathbf{x})$, where a is a small parameter, and $\mathbf{v}(\mathbf{x})$ is any function vanishing together with its derivatives up to order $(r-1)$ on the boundary $\partial\Omega$ of the domain Ω. The corresponding variation in the Lagrangian $\mathcal{L}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(r)}\right)$ is given by

$$\begin{aligned}\delta\mathcal{L}&=\mathcal{L}\left(\mathbf{x},\mathbf{u}+a\mathbf{v},\mathbf{u}^{(1)}+a\mathbf{v}^{(1)},\ldots,\mathbf{u}^{(r)}+a\mathbf{v}^{(r)}\right)-\mathcal{L}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(r)}\right)\\&=a\left(\frac{\partial\mathcal{L}}{\partial u_\alpha}v_\alpha+\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1}}v_{\alpha,j_1}+\cdots+\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1\ldots j_r}}v_{\alpha,j_1\ldots j_r}\right)+O(a^2),\end{aligned}$$

where $u_{\alpha,j_1}=\dfrac{\partial u_\alpha}{\partial x_{j_1}}$, $u_{\alpha,j_1\ldots j_r}=\dfrac{\partial^r u_\alpha}{\partial x_{j_1}\cdots\partial x_{j_r}}$, and the Einstein convention of sums over repeated indices is used.

Setting

$$\mathcal{X}_{\alpha,j_1}=\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1}},\quad\ldots,\quad\mathcal{X}_{\alpha,j_1\ldots j_r}=\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1\ldots j_r}},$$

repeated integrations by parts produce

$$
\begin{aligned}
\mathcal{X}_{\alpha,j_1} v_{\alpha,j_1} &= \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1} v_\alpha\right) - \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1}\right) v_\alpha,\\
\mathcal{X}_{\alpha,j_1 j_2} v_{\alpha,j_1 j_2} &= \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1 j_2} v_{\alpha,j_2}\right) - \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1 j_2}\right) v_{\alpha,j_2}\\
&= \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1 j_2} v_{\alpha,j_2}\right) - \mathcal{D}_{j_1 j_2}\left(\mathcal{X}_{\alpha,j_1 j_2} v_\alpha\right) + \mathcal{D}_{j_1 j_2}\left(\mathcal{X}_{\alpha,j_1 j_2}\right) v_\alpha\\
&= \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1 j_2} v_{\alpha,j_2} - \mathcal{D}_{j_2}\left(\mathcal{X}_{\alpha,j_1 j_2} v_\alpha\right)\right) + \mathcal{D}_{j_1 j_2}\left(\mathcal{X}_{\alpha,j_1 j_2}\right) v_\alpha,\\
\mathcal{X}_{\alpha,j_1 j_2 j_3} v_{\alpha,j_1 j_2 j_3} &= \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3} v_{\alpha,j_2 j_3}\right) - \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3}\right) v_{\alpha,j_2 j_3}\\
&= \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3} v_{\alpha,j_2 j_3}\right) - \mathcal{D}_{j_1 j_2}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3} v_{\alpha,j_3}\right)\\
&\quad + \mathcal{D}_{j_1 j_2}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3}\right) v_{\alpha,j_3}\\
&= \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3} v_{\alpha,j_2 j_3}\right) - \mathcal{D}_{j_1 j_2}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3} v_{\alpha,j_3}\right)\\
&\quad + \mathcal{D}_{j_1 j_2 j_3}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3} v_\alpha\right) - \mathcal{D}_{j_1 j_2 j_3}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3}\right) v_\alpha\\
&= \mathcal{D}_{j_1}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3} v_{\alpha,j_2 j_3} - \mathcal{D}_{j_2}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3} v_{\alpha,j_3}\right) + \mathcal{D}_{j_2 j_3}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3} v_\alpha\right)\right)\\
&\quad - \mathcal{D}_{j_1 j_2 j_3}\left(\mathcal{X}_{\alpha,j_1 j_2 j_3}\right) v_\alpha,
\end{aligned}
$$

and, in general,

$$
\begin{aligned}
\mathcal{X}_{\alpha,j_1\ldots j_r} v_{\alpha,j_1\ldots j_r} &= D_{j_1}\left(\mathcal{X}_{\alpha,j_1\ldots j_r} v_{\alpha,j_2\ldots j_r} - \mathcal{D}_{j_2}\left(\mathcal{X}_{\alpha,j_1\ldots j_r} v_{\alpha,j_3\ldots j_r}\right)\right.\\
&\quad + \mathcal{D}_{j_2 j_3}\left(\mathcal{X}_{\alpha,j_1\ldots j_r} v_{\alpha,j_4\ldots j_r}\right) + \cdots + (-1)^{r-1}\mathcal{D}_{j_2\ldots j_r}\left(\mathcal{X}_{\alpha,j_1\ldots j_r} v_\alpha\right)\Big)\\
&\quad + (-1)^r \mathcal{D}_{j_1\ldots j_r}\left(\mathcal{X}_{\alpha,j_1\ldots j_r}\right) v_\alpha
\end{aligned}
$$

where, as usual, $\mathcal{D}_{j_1}$ is the total derivative operator with respect to x_{j_1}, and $\mathcal{D}_{j_1\ldots j_r} = \mathcal{D}_{j_1}(\mathcal{D}_{j_2\ldots j_r})$. Thus, by defining the Euler operator relative to u_α,

$$
\begin{aligned}
E_{u_\alpha}(\cdot) &= \frac{\partial(\cdot)}{\partial u_\alpha} - \mathcal{D}_{j_1}\left(\frac{\partial(\cdot)}{\partial u_{\alpha,j_1}}\right) + \mathcal{D}_{j_1 j_2}\left(\frac{\partial(\cdot)}{\partial u_{\alpha,j_1 j_2}}\right)\\
&\quad - \mathcal{D}_{j_1 j_2 j_3}\left(\frac{\partial(\cdot)}{\partial u_{\alpha,j_1 j_2 j_3}}\right) + \cdots + (-1)^r \mathcal{D}_{j_1\ldots j_r}\left(\frac{\partial(\cdot)}{\partial u_{\alpha,j_1\ldots j_r}}\right),
\end{aligned}
$$

and the quantities

$$
\begin{aligned}
\Psi_{j_1} &= \frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1}} v_\alpha + \frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1 j_2}} v_{\alpha,j_2} - \mathcal{D}_{j_2}\left(\frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1 j_2}} v_\alpha\right)\\
&\quad + \frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1 j_2 j_3}} v_{\alpha,j_2 j_3} - \mathcal{D}_{j_2}\left(\frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1 j_2 j_3}} v_{\alpha,j_3}\right)\\
&\quad + \mathcal{D}_{j_2 j_3}\left(\frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1 j_2 j_3}} v_\alpha\right) + \cdots + \frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1\ldots j_r}} v_{\alpha,j_2\ldots j_r}\\
&\quad - \mathcal{D}_{j_2}\left(\frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1\ldots j_r}} v_{\alpha,j_3\ldots j_r}\right) + \mathcal{D}_{j_2 j_3}\left(\frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1\ldots j_r}} v_{\alpha,j_4\ldots j_r}\right)\\
&\quad + \cdots + (-1)^{r-1}\mathcal{D}_{j_2\ldots j_r}\left(\frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1\ldots j_r}} v_\alpha\right)
\end{aligned}
\tag{7.2}
$$

$(j_1 = 1, \ldots, n)$, we can write

$$\delta\mathcal{L} = a\left(\sum_{\alpha=1}^{m} v_\alpha E_{u_\alpha}(\mathcal{L}) + \sum_{k=1}^{n} \mathcal{D}_k(\Psi_k)\right) + O(a^2).$$

Therefore, using the divergence theorem,

$$\begin{aligned}\delta\mathcal{S} &= \int_\Omega \delta\mathcal{L}d\mathbf{x} = a\int_\Omega \left(\sum_{\alpha=1}^{m} v_\alpha E_{u_\alpha}(\mathcal{L}) + \sum_{k=1}^{n} \mathcal{D}_k(\Psi_k)\right) d\mathbf{x} + O(a^2)\\ &= a\left(\int_\Omega \sum_{\alpha=1}^{m} v_\alpha E_{u_\alpha}(\mathcal{L})d\mathbf{x} + \int_{\partial\Omega} \boldsymbol{\Psi}\cdot\boldsymbol{\nu}\, dS\right) + O(a^2),\end{aligned}$$

where $\int_{\partial\Omega}$ is the surface integral over the boundary $\partial\Omega$ of the domain Ω, $\boldsymbol{\nu}$ being the outward unit normal vector to $\partial\Omega$.

Since $\mathbf{v}(\mathbf{x})$ and its derivatives up to order $(r-1)$ are vanishing on the boundary $\partial\Omega$, whence $\boldsymbol{\Psi} \equiv \mathbf{0}$ on $\partial\Omega$, it is

$$\delta\mathcal{S} = a\left(\int_\Omega \sum_{\alpha=1}^{m} v_\alpha E_{u_\alpha}(\mathcal{L})\, d\mathbf{x}\right) + O(a^2).$$

Finally, the arbitrariness of v_α implies that the extremum $\mathbf{u}(\mathbf{x})$ of the action integral $\mathcal{S}$ has to satisfy the Euler–Lagrange equations

$$E_{u_\alpha}(\mathcal{L}) = 0,$$

that is,

$$\frac{\partial\mathcal{L}}{\partial u_\alpha} - \mathcal{D}_{j_1}\left(\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1}}\right) + \mathcal{D}_{j_1 j_2}\left(\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1 j_2}}\right) - \cdots + (-1)^r \mathcal{D}_{j_1\ldots j_r}\left(\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1\ldots j_r}}\right) = 0,$$

$\alpha = 1, \ldots, m$. From a Lagrangian of order r, the resulting Euler–Lagrange equations are of order $2r$, unless the Lagrangian is linear in the highest order derivatives.

Example 7.3 (*Lagrangian of minimal surfaces in* $\mathbb{R}^3$) Let $u(x_1, x_2)$ be a field, and consider the Lagrangian function

$$\mathcal{L} = \sqrt{1 + (u_{,1})^2 + (u_{,2})^2},$$

introduced by Lagrange in 1762 for determining the surface of least area stretched across a given closed contour. The corresponding Euler–Lagrange equation is the equation of minimal surfaces in $\mathbb{R}^3$ (see Sect. 8.5), namely

$$\left(1 + (u_{,2})^2\right) u_{,11} - 2u_{,1}u_{,2}u_{,12} + \left(1 + (u_{,1})^2\right) u_{,22} = 0.$$

Lagrange was not able to find any solution of the latter equation beyond the one describing a plane. In 1776, Meusnier discovered that the helicoid and catenoid satisfy the equation, and that the differential equation of minimal surfaces describes surfaces with zero mean curvature.

Example 7.4 Let $u(x_1, x_2, x_3)$ be a field, and consider the Lagrangian function

$$\mathcal{L} = \sqrt{(u_{,1})^2 + (u_{,2})^2 + (u_{,3})^2}.$$

The Euler–Lagrange equation is

$$u_{,11} + u_{,22} + u_{,33} = 0,$$

i.e., the Laplace equation.

For a field $u(x_0, x_1, x_2, x_3)$ (x_0 denotes the time), the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left((u_{,0})^2 - c^2\left((u_{,1})^2 + (u_{,2})^2 + (u_{,3})^2\right)\right)$$

produces the linear wave equation

$$u_{,00} - c^2(u_{,11} + u_{,22} + u_{,33}) = 0.$$

7.3.1 Equivalent Lagrangians

Also in Lagrangian field theory, it is possible to define an equivalence relation between Lagrangians.

Definition 7.5 (*Null Lagrangian*) A Lagrangian $\mathcal{L}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$ such that the corresponding Euler–Lagrange equations are identically satisfied is called a *null Lagrangian*.

Theorem 7.4 *$\mathcal{L}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$ is a null Lagrangian if it can be written in divergence form, namely:*

$$\mathcal{L}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}) = \sum_{i=1}^{n} \mathcal{D}_i \left(\mathcal{F}_i \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r-1)}\right)\right),$$

where $\mathcal{F}_i$ $(i = 1, \ldots, n)$ are arbitrary smooth functions.

Proof Assume

$$\mathcal{L}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}\right) = \sum_{i=1}^{n} \mathcal{D}_i \left(\mathcal{F}_i \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r-1)}\right)\right)$$

for some smooth functions $\mathcal{F}_i$; then,

$$\mathcal{S} = \int_{\Omega} \sum_{i=1}^{n} \mathcal{D}_i \left(\mathcal{F}_i \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r-1)}\right)\right) d\mathbf{x} = \int_{\partial\Omega} \sum_{i=1}^{n} \mathcal{F}_i \nu_i \, dS,$$

$(\nu_1, \ldots, \nu_n)$ being the outward unit normal vector to $\partial\Omega$.

Since the variations of $\mathbf{u}$, $\mathbf{u}^{(1)}$, …, $\mathbf{u}^{(r-1)}$ are vanishing on $\partial\Omega$, it is $\delta\mathcal{S} = 0$, and the Euler–Lagrange equations are identically vanishing. □

Finally, we can define the equivalence between Lagrangians also in the framework of Lagrangian field theory.

Definition 7.6 (*Equivalent Lagrangians*) The Lagrangians $\mathcal{L}_1\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}\right)$ and $\mathcal{L}_2\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}\right)$ are *equivalent* if their difference can be written in divergence form, i.e.,

$$\mathcal{L}_2 = \mathcal{L}_1 + \sum_{i=1}^{n} \mathcal{D}_i \left(\mathcal{F}_i(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r-1)})\right),$$

for some smooth functions $\mathcal{F}_i$.

Remark 7.5 Of course, two equivalent Lagrangians produce the same Euler–Lagrange equations.

7.3.2 Variational Symmetries

Let $\mathcal{L}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}\right)$ be a Lagrangian function, and consider a one-parameter Lie group of transformations

$$\mathbf{x}' = \mathbf{X}(\mathbf{x}, \mathbf{u}; a), \qquad \mathbf{u}' = \mathbf{U}(\mathbf{x}, \mathbf{u}; a)$$

together with its infinitesimal generator

$$\Xi = \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\alpha}.$$

Let us impose the invariance of the Lagrangian action, that is,

$$0 = \int_{\Omega'} \mathcal{L}\left(\mathbf{x}', \mathbf{u}', \mathbf{u}^{(1)\prime}, \ldots, \mathbf{u}^{(r)\prime}\right) d\mathbf{x}' - \int_{\Omega} \mathcal{L}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}) d\mathbf{x}, \tag{7.3}$$

where Ω' is the image of Ω under the point transformation.

The infinitesimal transformation prolonged to the jet space of order r reads

$$\begin{aligned}
&x_i' = x_i + a\xi_i + O(a^2),\\
&u_\alpha' = u_\alpha + a\eta_\alpha + O(a^2),\\
&u_{\alpha,j_1}' = u_{\alpha,j_1} + a\eta_{[\alpha,j_1]} + O(a^2),\\
&\cdots\\
&u_{\alpha,j_1\ldots j_r}' = u_{\alpha,j_1\ldots j_r} + a\eta_{[\alpha,j_1\ldots j_r]} + O(a^2).
\end{aligned}$$

Taking into account that

$$d\mathbf{x}' = J d\mathbf{x} = (1 + a \det(\|\mathcal{D}_j(\xi_i)\|) + O(a^2)) d\mathbf{x},$$

expanding condition (7.3) around $a = 0$, and neglecting higher order terms, we get

$$\begin{aligned}
0 &= \int_{\Omega} a \left(\mathcal{L}\mathcal{D}_i(\xi_i) + \xi_i \frac{\partial \mathcal{L}}{\partial x_i} + \eta_\alpha \frac{\partial \mathcal{L}}{\partial u_\alpha} \right.\\
&\quad \left. + \eta_{[\alpha,j_1]} \frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1}} + \cdots + \eta_{[\alpha,j_1\ldots j_r]} \frac{\partial \mathcal{L}}{\partial u_{\alpha,j_1\ldots j_r}} \right) d\mathbf{x} + O(a^2)\\
&= \int_{\Omega} a \left(\mathcal{L}\mathcal{D}_i(\xi_i) + \Xi^{(r)}(\mathcal{L}) \right) d\mathbf{x} + O(a^2).
\end{aligned}$$

Therefore, the Lagrangian action is invariant with respect to the symmetry generated by the vector field Ξ provided that

$$\mathcal{L} \sum_{i=1}^{n} \mathcal{D}_i(\xi_i) + \Xi^{(r)}(\mathcal{L}) = 0.$$

The definition of *divergence symmetry* can be also given in this context.

Definition 7.7 (*Divergence symmetry*) A *divergence symmetry* is a symmetry preserving the Lagrangian action up to a divergence expression, that is,

$$\mathcal{L}\sum_{i=1}^{n}\mathcal{D}_i(\xi_i)+\Xi^{(r)}\mathcal{L}-\sum_{i=1}^{n}\mathcal{D}_i(\mathcal{F}_i)=0.$$

Remark 7.6 The generators of the Lie point symmetries of the Lagrangian action leave invariant the extrema of the Lagrangian action. Hence, they are admitted as generators of symmetries of the corresponding Euler–Lagrange equations. However, in general, it is not guaranteed that Lie point symmetries admitted by the Euler–Lagrange equations are variational symmetries.

7.3.3 Conservation Laws

Here, we briefly review some elementary concepts about conservation laws, and emphasize them in the context of Lagrangian field theory.

Definition 7.8 (*Local conservation law*) A *local conservation law* of an sth-order system $\mathbf{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(s)}\right)=\mathbf{0}$ of partial differential equations is a divergence expression

$$\sum_{i=1}^{n}\mathcal{D}_i\left(\Phi_i\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(s-1)}\right)\right)=0 \tag{7.4}$$

holding for all solutions of the differential system, where Φ_i $(i=1,\ldots,n)$ are called the *fluxes* of the conservation law.

Definition 7.9 (*Trivial conservation law*) A conservation law (7.4) is *trivial* if each of its fluxes vanishes identically on the solutions of the given system of differential equations, or the conservation law vanishes identically as a differential identity.

A trivial conservation law contains no information about a given system of differential equations.

The notion of trivial conservation law allows for the introduction of equivalent conservation laws and linearly dependent conservation laws.

Definition 7.10 (*Equivalent conservation laws*) Two conservation laws

$$\sum_{i=1}^{n}\mathcal{D}_i(\Phi_i)=0 \quad \text{and} \quad \sum_{i=1}^{n}\mathcal{D}_i(\Psi_i)=0$$

are *equivalent* if

$$\sum_{i=1}^{n} \mathcal{D}_i \left(\Phi_i - \Psi_i\right) = 0$$

is a trivial conservation law.

Definition 7.11 (*Linearly dependent conservation laws*) A set of ℓ conservation laws

$$\left\{ \sum_{i=1}^{n} \mathcal{D}_i(\Phi_i^k) = 0 \colon k = 1, \ldots, \ell \right\}$$

is *linearly dependent* if there exists a set of constants $\{a_k : k = 1, \ldots, \ell\}$, not all zero, such that the linear combination

$$\sum_{i=1}^{n} \mathcal{D}_i \left(a_k \Phi_i^k\right) = 0$$

is a trivial conservation law. In this case, up to equivalence, at least one of the conservation laws in the set can be expressed as a linear combination of the remaining ones.

Actually, one is interested in finding linearly independent sets of conservation laws of a given system of differential equations.

In evolutionary problems, when one of the independent variables (say, x_n) is the time t, the conservation law has the form:

$$\mathcal{D}_t \left(\Psi \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(s-1)}\right)\right) + \sum_{i=1}^{n-1} \mathcal{D}_i \left(\Phi_i \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(s-1)}\right)\right) = 0,$$

where Ψ is referred to as the density, and Φ_i as spatial fluxes, of the conservation law.

We can now prove the Noether theorem for general Lagrangian functions $\mathcal{L}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$.

Theorem 7.5 (Noether) *Let $\mathcal{L}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)})$ be a Lagrangian function and Ξ the generator of a variational symmetry. Then, the conservation law*

$$\sum_{j=1}^{n} \mathcal{D}_j(\xi_j \mathcal{L} + \widetilde{\Psi}_j[\mathbf{x}, \mathbf{u}, \widetilde{\boldsymbol{\eta}}]) = 0,$$

depending on the Lagrangian and the infinitesimal generators of the Lie symmetry, can be constructed.

Proof Setting

$$\widetilde{\eta}_\alpha = \eta_\alpha - \xi_i u_{\alpha,i},$$

whence

$$\begin{aligned}
&\eta_{[\alpha,j_1]} - \xi_i u_{\alpha,ij_1} = \mathcal{D}_{j_1}(\eta_\alpha - \xi_i u_{\alpha,i}) = \mathcal{D}_{j_1}(\widetilde{\eta}_\alpha),\\
&\eta_{[\alpha,j_1j_2]} - \xi_i u_{\alpha,ij_1j_2} = \mathcal{D}_{j_1j_2}(\widetilde{\eta}_\alpha),\\
&\cdots\\
&\eta_{[\alpha,j_1\dots j_r]} - \xi_i u_{\alpha,ij_1\dots j_r} = \mathcal{D}_{j_1\dots j_r}(\widetilde{\eta}_\alpha),
\end{aligned}$$

we deduce

$$\begin{aligned}
0 &= \mathcal{L}\mathcal{D}_i(\xi_i) + \Xi^{(r)}(\mathcal{L})\\
&= \mathcal{D}_i(\xi_i\mathcal{L}) + (\eta_\alpha - \xi_i u_{\alpha,i})\frac{\partial\mathcal{L}}{\partial u_\alpha} + (\eta_{[\alpha,j_1]} - \xi_i u_{\alpha,ij_1})\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1}}\\
&\quad + \cdots + (\eta_{[\alpha,j_1\dots j_r]} - \xi_i u_{\alpha,ij_1\dots j_r})\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1\dots j_r}}\\
&= \mathcal{D}_i(\xi_i\mathcal{L}) + \widetilde{\eta}_\alpha\frac{\partial\mathcal{L}}{\partial u_\alpha} + \mathcal{D}_{j_1}(\widetilde{\eta}_\alpha)\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1}} + \cdots + \mathcal{D}_{j_1\dots j_r}(\widetilde{\eta}_\alpha)\frac{\partial\mathcal{L}}{\partial u_{\alpha,j_1\dots j_r}}\\
&\qquad \text{[repeatedly integrating by parts and using Euler–Lagrange equations]}\\
&= \mathcal{D}_{j_1}(\xi_{j_1}\mathcal{L} + \widetilde{\Psi}_{j_1}),
\end{aligned}$$

where $\widetilde{\Psi}_{j_1}$ is obtained from Ψ_{j_1} in (7.2) with the replacements

$$\begin{aligned}
&v_\alpha \to \widetilde{\eta}_\alpha,\\
&v_{\alpha,j_2} \to \mathcal{D}_{j_2}(\widetilde{\eta}_\alpha),\\
&\cdots,\\
&v_{\alpha,j_2\dots j_r} \to \mathcal{D}_{j_2\dots j_r}(\widetilde{\eta}_\alpha).
\end{aligned}$$

□

Remark 7.7 Using divergence symmetries, the invariance condition of Lagrangian action functional reads

$$\mathcal{L}\sum_{i=1}^{n}\mathcal{D}_i(\Xi_i) + \Xi^{(r)}\mathcal{L} - \sum_{i=1}^{n}\mathcal{D}_i(\mathcal{F}_i) = 0,$$

and, using Noether theorem, we get the conservation law

$$\sum_{j=1}^{n}\mathcal{D}_j\left(\xi_j\mathcal{L} + \widetilde{\Psi}_j[\mathbf{x},\mathbf{u},\widetilde{\boldsymbol{\eta}}] - \mathcal{F}_j\right) = 0.$$

Example 7.5 (*Klein–Gordon equation (see* [4]) The class of Klein–Gordon wave equations,

$$u_{,12} + g(u) = 0, \tag{7.5}$$

with a general nonlinear interaction term $g(u)$, can be derived from a variational principle given by the action functional with Lagrangian

$$\mathcal{L} = -\frac{1}{2}u_{,1}u_{,2} + h(u), \qquad \frac{dh}{du} = g(u).$$

For an arbitrary function $g(u)$, Eq. (7.5) is invariant with respect to translations of the independent variables, and a scaling group of x_1 and x_2. The vector fields of these symmetries are

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2},$$

and it is straightforward to verify that they are variational symmetries of the action functional. Applying Noether theorem, the following conservation laws are easily derived:

$$\begin{aligned}
&\mathcal{D}_1\left(\frac{1}{2}u_{,2}^2\right) + \mathcal{D}_2\left(h(u)\right) = 0,\\
&\mathcal{D}_1\left(h(u)\right) + \mathcal{D}_2\left(\frac{1}{2}u_{,1}^2\right) = 0,\\
&\mathcal{D}_1\left(\frac{1}{2}x_2u_{,2}^2 - x_1h(u)\right) + \mathcal{D}_2\left(-\frac{1}{2}x_1u_{,1}^2 + x_2h(u)\right) = 0.
\end{aligned}$$

Example 7.6 (*Hunter–Saxton equation* [9]) Let $u(x_1, x_2)$ be a field, and consider the Lagrangian function

$$\mathcal{L} = u_{,1}u_{,2} + uu_{,2}^2.$$

The Euler–Lagrange equation is

$$\left(u_{,1} + uu_{,2}\right)_{,2} - \frac{1}{2}u_{,2}^2 = 0,$$

describing asymptotically weakly unidirectional waves that satisfy a simplified model equation for the evolution of the director field of a nematic liquid crystal [9].

There are four variational symmetries generated by the vector fields

$$\Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_2}, \quad \Xi_3 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}, \quad \Xi_4 = x_1\frac{\partial}{\partial x_2} + \frac{\partial}{\partial u}.$$

Applying Noether theorem, the following conservation laws are easily derived:

$$\begin{aligned}
&\mathcal{D}_1\left(-uu_{,2}^2\right)+\mathcal{D}_2\left(u_{,1}\left(u_{,1}+2uu_{,2}\right)\right)=0,\\
&\mathcal{D}_1\left(u_{,2}^2\right)+\mathcal{D}_2\left(uu_{,2}^2\right)=0,\\
&\mathcal{D}_1\left((x_2-x_1u)u_{,2}^2\right)+\mathcal{D}_2\left(x_1(u_{,1}^2+2uu_{,1}u_{,2})+x_2uu_{,2}^2\right)=0,\\
&\mathcal{D}_1\left(u_{,2}(x_1u_{,2}-1)\right)+\mathcal{D}_2\left(x_1uu_{,2}^2-2uu_{,2}-u_{,1}\right)=0.
\end{aligned}$$

Example 7.7 (*Variational symmetries of minimal surfaces in* $\mathbb{R}^3$) The equation of minimal surfaces in $\mathbb{R}^3$,

$$\left(1+(u_{,2})^2\right)u_{,11}-2u_{,1}u_{,2}u_{,12}+\left(1+(u_{,1})^2\right)u_{,22}=0,$$

as shown in Example 7.3, derives from the Lagrangian

$$\mathcal{L}=\sqrt{1+(u_{,1})^2+(u_{,2})^2}.$$

The admitted variational symmetries are

$$\begin{aligned}
&\Xi_1=\frac{\partial}{\partial x_1},\qquad \Xi_2=\frac{\partial}{\partial x_2},\qquad \Xi_3=\frac{\partial}{\partial u},\\
&\Xi_4=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2},\\
&\Xi_5=u\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial u},\\
&\Xi_6=u\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial u},
\end{aligned}$$

and the application of Noether theorem yields the following conservation laws:

$$\begin{aligned}
C_1&\equiv\mathcal{D}_1\left(\frac{1+(u_{,2})^2}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)+\mathcal{D}_2\left(-\frac{u_{,1}u_{,2}}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)=0,\\
C_2&\equiv\mathcal{D}_1\left(\frac{u_{,1}u_{,2}}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)+\mathcal{D}_2\left(-\frac{1+(u_{,1})^2}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)=0,\\
C_3&\equiv\mathcal{D}_1\left(\frac{u_{,1}}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)+\mathcal{D}_2\left(\frac{u_{,2}}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)=0,\\
C_4&\equiv\mathcal{D}_1\left(\frac{u_{,2}(x_1u_{,1}+x_2u_{,2})+x_2}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)+\mathcal{D}_2\left(-\frac{u_{,1}(x_1u_{,1}+x_2u_{,2})+x_1}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)=0,
\end{aligned}$$

$$C_5 \equiv \mathcal{D}_1\left(\frac{u(1+(u_{,2})^2)-x_1u_{,1}}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)+\mathcal{D}_2\left(-\frac{u_{,2}(uu_{,1}+x_1)}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)=0,$$

$$C_6 \equiv \mathcal{D}_1\left(\frac{u_{,1}(uu_{,2}+x_2)}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)+\mathcal{D}_2\left(-\frac{u(1+(u_{,1})^2)-x_2u_{,2}}{\sqrt{1+(u_{,1})^2+(u_{,2})^2}}\right)=0.$$

These conservation laws are linearly independent. In fact, taking their linear combination, we get

$$\sum_{k=1}^{6}\lambda_k C_k \equiv \big(\lambda_3-\lambda_5x_1+\lambda_6x_2+(\lambda_2+\lambda_4u)u_{,2}-(\lambda_1+\lambda_4x_2+\lambda_5u)u_{,1}\big)$$
$$\times\ \big((1+(u_{,2})^2)u_{,11}-2u_{,1}u_{,2}u_{,12}+(1+(u_{,1})^2)u_{,22}\big)=0,$$

that identically vanishes when $\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=\lambda_6=0$.

Equation of minimal surfaces in $\mathbb{R}^3$ admits also the symmetry generated by

$$\Xi = x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+u\frac{\partial}{\partial u},$$

but this is *not* a variational symmetry.

7.4 Concluding Remarks

Noether theorem, despite its elegance and relevance, has an important limitation, for it permits the determination of local conservation laws only for variational problems. Therefore, for partial differential equations which are not the Euler–Lagrange equations of a functional, Noether theorem cannot be used. In the last years, various methods have been proposed for determining conservation laws also in non-variational problems. For general systems of differential equations, a divergence expression providing a conservation law may be found through a *direct approach*: by taking a linear combinations of the given differential equations with multipliers that are functions of the independent and dependent variables, as well as derivatives (up to some finite order) of the dependent variables. In the direct approach, the dependent variables (and their derivatives) involved in the system of differential equations, or appearing in the multipliers, are replaced by arbitrary functions (and their derivatives). By their construction, such divergence expressions vanish on all solutions of the given system. The interested reader may find details about many important contributions concerned with the direct approach and the introduction of *formal Lagrangians* in the literature (see, for instance, [1, 2, 4, 5, 15]).

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Chapter 8
Inverse Problems in Lie Group Analysis of Differential Equations



8.1 Introduction

The inverse problem in Lie group analysis of differential equations addressed in this chapter consists in the determination of differential equations possessing a given algebra of Lie point symmetries. An algorithmic approach for this problem can be found in [5]. An interesting contribution has been given by Rosenhaus in [41], who posed the problem of the unique determination of a differential equation by its group. Therein, the author considered the projective algebra of $\mathbb{R}^3$ and its subalgebras, and was able to prove that the equation of surfaces in $\mathbb{R}^3$ with vanishing Gaussian curvature (which is a Monge–Ampère-type equation) is uniquely determined by its Lie point symmetries.

One possible approach to the problem of finding the most general form of a differential equation admitting a given Lie algebra of infinitesimal point symmetries is to classify all possible realizations of the given Lie algebra as algebra of vector fields on the base manifold [40]. Then, the theory of differential invariants of such realizations allows to find the most general form of the equation [16, 40]. In fact, it is known (see [37]) that, under suitable hypotheses of regularity, the most general differential equation admitting a given Lie algebra of point symmetries is locally given, up to a multiplicative function (nowhere vanishing) depending on the coordinates of the jet spaces (see, for instance, [12]), by $\mathbf{\Delta}(J_1, J_2, \ldots, J_s) = 0$ where $\mathbf{\Delta}$ is a general smooth function, and J_k $(k = 1, \ldots, s)$ are the differential invariants of the realization under consideration. However, in some cases it could be hard to compute the differential invariants, though it amounts to solve a linear first-order system of partial differential equations (a constructive algorithm is given in [4]). Then, in such cases, it is necessary to confine the investigation to a specified class of differential equations, i.e., to impose additional constraints to the form of $\mathbf{\Delta}$; interesting questions may arise in this context.

Hereafter, we shall introduce the notion of *Lie remarkable equations*, i.e., differential equations uniquely determined by their Lie point symmetries, prove some

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relevant theorems, and give various examples. The main results hereafter presented are contained in [21, 27–34].

Of course, many relevant differential equations are not Lie remarkable for lack of point symmetries (see Theorems 8.4 and 8.5). Among them we recall Korteweg–deVries equation, Burgers equation, Kepler equations. Several authors studied the problem of finding, for a given equation, an extension of the algebra of point symmetries for which the equation at hand is determined. For instance, in [3, 22], such a construction is performed in the case of ordinary differential equations by considering a nonlocal extension of the algebra of point symmetries. Also, in [41], it is shown that the algebra of contact symmetries characterizes the equation of minimal surfaces.

8.2 Basics of Differential Geometry

Differential geometry is the most natural framework where Lie groups, as well as differential equations viewed as submanifolds in a suitable extended manifold, can be discussed. Roughly speaking, manifolds are spaces that locally look like a Euclidean space; they generalize the familiar concepts of curves and surfaces.

For the sake of clarity, let us introduce some basic notions useful to fix the language and set the theoretical framework that will be used hereafter [11, 13, 14, 36, 37].

Definition 8.1 In $\mathbb{R}^n$, the projection functions are defined as

$$\begin{aligned} &\pi_k : \mathbb{R}^n \to \mathbb{R}, \qquad k = 1, \dots, n, \\ &(z_1, \dots, z_n) \mapsto z_k. \end{aligned}$$

A fundamental concept for studying manifolds is concerned with charts.

Definition 8.2 Given a topological space M, a chart (or local coordinate map) is a pair (U, ϕ), where U is an open subset of M and $\phi : U \to \Omega$ is a homeomorphism onto an open subset $\Omega = \phi(U)$ of $\mathbb{R}^n$ (for some $n \geq 1$). For any $\mathbf{z} \in M$, a chart (U, ϕ) is a chart at $\mathbf{z}$ if and only if $\mathbf{z} \in U$. If (U, ϕ) is a chart, then

$$z_k = \pi_k \circ \phi(\mathbf{z}), \qquad k = 1, \dots, n,$$

are called local coordinates; for every $\mathbf{z} \in U$, the tuple $(z_1, \dots, z_n)$ is the set of coordinates of $\mathbf{z}$ with respect to the chart. The inverse (Ω, ϕ^{-1}) of a chart is called a local parametrization. Given any two charts (U_j, ϕ_j) and (U_k, ϕ_k), if $U_j \cap U_k \neq \emptyset$ we can define the transition maps

$$\begin{aligned} &\phi_j^k : \phi_j(U_j \cap U_k) \to \phi_k(U_j \cap U_k), \\ &\phi_k^j : \phi_k(U_j \cap U_k) \to \phi_j(U_j \cap U_k), \end{aligned}$$

defined by

$$\phi_j^k = \phi_k \circ \phi_j^{-1}, \qquad \phi_k^j = \phi_j \circ \phi_k^{-1}.$$

Notice that $\phi_j^k = (\phi_k^j)^{-1}$.

Definition 8.3 Let M be a topological space, and $n \geq 1$ an integer. A C^k ($k \geq 1$ a given integer) n-atlas $\mathcal{A}$ is a family of charts $\{(U_j, \phi_j)\}$ such that

1. $\phi_j(U_j) \subseteq \mathbb{R}^n$ for all j;
2. $M = \cup U_j$, i.e., U_j's cover M;
3. if $U_i \cap U_j \neq \emptyset$, the transition maps ϕ_i^j and ϕ_j^i are C^k diffeomorphisms. In particular, we have a C^∞ n-atlas if the transition maps are C^∞ (smooth).

Definition 8.4 Let $\mathcal{A}$ be a C^k n-atlas on a topological space M. Any other chart (U, ϕ) is compatible with the atlas $\mathcal{A}$ if every map $\phi_j \circ \phi^{-1}$ and $\phi \circ (\phi_j)^{-1}$ is C^k whenever $U \cap U_j \neq \emptyset$. Two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are compatible if every chart of one is compatible with the other atlas. This is equivalent to saying that the union of the two atlases is an atlas too.

Definition 8.5 Let $n \geq 1$ be some integer. A C^k manifold of dimension n is a topological space M together with an equivalence class $\overline{\mathcal{A}}$ of C^k n-atlases on M. Any atlas $\mathcal{A} \in \overline{\mathcal{A}}$ defines a differentiable structure of class C^k and dimension n on M. We say that M is modeled on $\mathbb{R}^n$. If the atlases are C^∞, we say that M is a smooth manifold.

Hereafter, we assume all manifolds and maps to be C^∞.

Definition 8.6 Let M be a differentiable manifold of dimension n. A curve $\boldsymbol{\gamma}$ at a point $\mathbf{z} \in M$ is a differentiable map

$$\boldsymbol{\gamma} : S \to M,$$

where S is a subinterval of $\mathbb{R}$, such that $\boldsymbol{\gamma}(0) = \mathbf{z}$, $0 \in S$.

We use Definition 8.6 to construct the tangent space $T_{\mathbf{z}}M$ at a point $\mathbf{z}$ of a manifold M.

Let $\mathbf{z} \in M$ be any point on M, and let $\boldsymbol{\gamma}$ a C^∞ curve passing through $\mathbf{z}$, i.e., it is $\boldsymbol{\gamma}(0) = \mathbf{z}$. For any chart, (U, ϕ) at $\mathbf{z}$, the map $\phi \circ \gamma$ is a C^∞ curve in $\mathbb{R}^n$, and the tangent vector $\boldsymbol{\zeta} = (\phi \circ \boldsymbol{\gamma})'(0)$ is well defined. However, it is possible that the same $\boldsymbol{\zeta}$ arises from different curves. This problem can be overcome by defining an equivalence relation on curves passing through $\mathbf{z}$.

Definition 8.7 Given a smooth manifold M of dimension n, for any $\mathbf{z} \in M$, two C^∞ curves, $\boldsymbol{\gamma}_1 : S \to M$ and $\boldsymbol{\gamma}_2 : S \to M$, with $\boldsymbol{\gamma}_1(0) = \boldsymbol{\gamma}_2(0) = \mathbf{z}$, are equivalent if there is some chart (U, ϕ), at $\mathbf{z}$ such that

$$(\phi \circ \boldsymbol{\gamma}_1)'(0) = (\phi \circ \boldsymbol{\gamma}_2)'(0).$$

Remark 8.1 This definition does not depend on the choice of the chart. In fact, if (V, ψ) is another chart at $\mathbf{z}$, as $\mathbf{z}$ belongs both to U and V, it is $U \cap V \neq \emptyset$, so the transition function $\eta = \psi \circ \phi^{-1}$ is smooth. Using the chain rule, we have

$$\begin{aligned}(\psi \circ \boldsymbol{\gamma}_1)'(0) &= (\eta \circ \phi \circ \boldsymbol{\gamma}_1)'(0) \\ &= \eta'(\phi(\mathbf{z}))((\phi \circ \boldsymbol{\gamma}_1)'(0)) = \eta'(\phi(\mathbf{z}))((\phi \circ \boldsymbol{\gamma}_2)'(0)) \\ &= (\eta \circ \phi \circ \boldsymbol{\gamma}_2)'(0) = (\psi \circ \boldsymbol{\gamma}_2)'(0).\end{aligned}$$

Definition 8.8 Let M be a smooth manifold of dimension n. For any $\mathbf{z} \in M$, a tangent vector to M at $\mathbf{z}$ is any equivalence class $\boldsymbol{\zeta} = [\boldsymbol{\gamma}]$ of C^∞ curves $\boldsymbol{\gamma}$ through $\mathbf{z}$ on M, modulo the equivalence relation given in Definition 8.7. The set of all tangent vectors at $\mathbf{z}$ is denoted by $T_{\mathbf{z}}M$.

Definition 8.9 Let M be a smooth manifold of dimension n. The tangent bundle TM is the disjoint union of the tangent spaces $T_{\mathbf{z}}M$, for all $\mathbf{z} \in M$, i.e.,

$$TM = \{(\mathbf{z}, \boldsymbol{\zeta}) \colon \mathbf{z} \in M,\ \boldsymbol{\zeta} \in T_{\mathbf{z}}M\}.$$

Thus, a vector field $\boldsymbol{\zeta}$ of the manifold M is a C^∞-section of the tangent bundle TM, i.e., a C^∞ mapping from M to TM assigning a vector $\boldsymbol{\zeta} \in T_{\mathbf{z}}M$ to each point $\mathbf{z} \in M$. Introducing a local coordinate system $\mathbf{z} = (z_1, \ldots, z_n) \in M$, $\boldsymbol{\zeta}$ can be expressed as the first-order operator

$$\Xi = \sum_{k=1}^{n} \zeta_k(\mathbf{z}) \frac{\partial}{\partial z_k},$$

where $\zeta_k(\mathbf{z}) \in C^\infty(M)$, $k = 1, \ldots, n$.

An integral curve of the vector field at the point $\mathbf{z}$ is the curve $\boldsymbol{\gamma}$ at $\mathbf{z}$ whose tangent vector $\dfrac{d\boldsymbol{\gamma}(a)}{da}$ coincides with $\boldsymbol{\zeta}$ at the point $\mathbf{z} = \boldsymbol{\gamma}(a)$ and such that

$$\frac{d\boldsymbol{\gamma}(a)}{da} = \boldsymbol{\zeta}(\boldsymbol{\gamma}(a))$$

for each $a \in S$. Locally, the curve $\boldsymbol{\gamma}$ can be found by solving the autonomous system of first-order ordinary differential equations

$$\begin{aligned}&\frac{d\boldsymbol{\gamma}(a)}{da} = \boldsymbol{\zeta}(\boldsymbol{\gamma}(a)), \\ &\boldsymbol{\gamma}(0) = \mathbf{z}_0.\end{aligned}$$

The existence and uniqueness theorem for systems of ordinary differential equations ensures that there is a unique solution to the system with the initial datum $\boldsymbol{\gamma}(0) = \mathbf{z}_0$. So, we have the existence of a unique maximal integral curve $\boldsymbol{\gamma}(a)$ passing through the point $\mathbf{z}_0 = \boldsymbol{\gamma}(0) \in M$. Such a maximal integral curve is called the flow of $\boldsymbol{\zeta}$ at $\mathbf{z} = \boldsymbol{\gamma}(a)$; let us denote it with $\mathbf{Z}(\mathbf{z}; a)$ satisfying the properties

$$\mathbf{Z}(\mathbf{a};0)=\mathbf{z},\qquad \mathbf{Z}(\mathbf{Z}(\mathbf{z};a);b)=\mathbf{Z}(\mathbf{z};a+b), \tag{8.1a}$$

$$\frac{d\mathbf{Z}(\mathbf{z};a)}{da}=\Xi(\mathbf{Z}(\mathbf{z};a)),\quad \mathbf{z}\in M, \tag{8.1b}$$

for all sufficiently small $a, b \in S$. An alternative notation for the flow is

$$\mathbf{Z}(\mathbf{z};a)=\exp(a\,\Xi)(\mathbf{z})=\mathbf{z}+a\,\Xi(\mathbf{z})+\cdots+\frac{a^k}{k!}\,\Xi^k(\mathbf{z})+\cdots.$$

The property (8.1b) implies that $\mathbf{Z}(\mathbf{z};-a)=\mathbf{Z}^{-1}(\mathbf{z};a)$, or, equivalently,

$$(\exp(a\,\Xi))^{-1}(\mathbf{z})=\exp(-a\,\Xi)(\mathbf{z}).$$

From the infinitesimal viewpoint, the flow is expressed as

$$\exp(a\,\Xi)(\mathbf{z})=\mathbf{z}+a\,\Xi(\mathbf{z})+O(a^2).$$

The flow $\exp(a\,\Xi)(\mathbf{z})$ generated by the vector field Ξ corresponds to a one-parameter group of transformations as it arises as the action of the Lie group $\mathbb{R}$ on the manifold M.

Conversely, given a flow with the first two properties (8.1), the corresponding generating vector field $\boldsymbol{\zeta}(\mathbf{z})$ is found by differentiating the flow:

$$\boldsymbol{\zeta}(\mathbf{z})=\left.\frac{d(\exp(a\,\Xi)(\mathbf{z}))}{da}\right|_{a=0},\qquad \mathbf{z}\in M.$$

Rectification of a vector field $\boldsymbol{\zeta}(\mathbf{z})$ in a neighborhood of a regular point (a point $\mathbf{z}$ where $\boldsymbol{\zeta}(\mathbf{z})$ is nonvanishing) is always possible.

Theorem 8.1 *If $\mathbf{z}_0$ is a regular point of $\boldsymbol{\zeta}$, then there exist local rectifying coordinates $\mathbf{y}=(y_1,\ldots,y_n)$ in a neighborhood of $\mathbf{z}_0$ such that $\widetilde{\Xi}=\frac{\partial}{\partial y_n}$ generates the translational flow* $\exp(a\,\widetilde{\Xi})(\mathbf{y})=(y_1, y_2, \ldots, y_n+a)$.

The coordinates $(y_1, y_2, \ldots, y_n)$ are what we called *canonical variables* in Chap. 2.

Let Ξ be a vector field in the local coordinates

$$\Xi=\sum_{k=1}^{n}\zeta_k(\mathbf{z})\frac{\partial}{\partial z_k},$$

and $F\colon M\to\mathbb{R}$ a smooth function. Applying the vector field Ξ to the function F produces the infinitesimal rate of change in F as the parameter a of the induced flow $\exp(a\,\Xi)(\mathbf{z})$ varies:

$$\Xi(F(\mathbf{z})) = \sum_{k=1}^{n} \zeta_k(\mathbf{z}) \frac{\partial F}{\partial z_k} = \left. \frac{dF(\exp(a\,\Xi)(\mathbf{z}))}{da} \right|_{a=0}.$$

Vector fields act on functions as derivations, for they are linear and satisfy the Leibniz rule:

$$\Xi(F+G) = \Xi(F) + \Xi(G), \qquad \Xi(FG) = F\,\Xi(G) + \Xi(F)G.$$

If we introduce an invertible change of coordinates $\mathbf{y} = \mathbf{Y}(\mathbf{z})$, then, by the chain rule, the vector field Ξ in the new coordinates $\mathbf{y}$ is expressed as

$$\Xi = \sum_{j=1}^{n} \sum_{k=1}^{n} \zeta_k(\mathbf{Y}^{-1}(\mathbf{y})) \frac{\partial Y_j(\mathbf{Y}^{-1}(\mathbf{y}))}{\partial z_k} \frac{\partial}{\partial y_j} = \sum_{j=1}^{n} \Xi(Y_j(\mathbf{z}))\big|_{\mathbf{z}=\mathbf{Y}^{-1}(\mathbf{y})} \frac{\partial}{\partial y_j}.$$

The commutator or Lie bracket of two vector fields Ξ_1 and Ξ_2 is defined as the unique vector field satisfying

$$[\Xi_1, \Xi_2](F) = \Xi_1(\Xi_2(F)) - \Xi_2(\Xi_1(F))$$

for all smooth functions $F\colon M \to \mathbb{R}$. If, in local coordinates,

$$\Xi_1 = \sum_{k=1}^{n} \zeta_{1k} \frac{\partial}{\partial z_k}, \quad \Xi_2 = \sum_{k=1}^{n} \zeta_{2k} \frac{\partial}{\partial z_k},$$

then

$$[\Xi_1, \Xi_2] = \sum_{k=1}^{n} (\Xi_1(\zeta_{2k}) - \Xi_2(\zeta_{1k})) \frac{\partial}{\partial z_k}.$$

so that the Lie bracket of two vector fields is a vector field too.

The Lie bracket of two vector fields is skew-symmetric and satisfies Jacobi identity

$$[\Xi_1, [\Xi_2, \Xi_3]] + [\Xi_2, [\Xi_3, \Xi_1]] + [\Xi_3, [\Xi_1, \Xi_2]] = 0.$$

Thus, the set of vector fields of the manifold M constitutes a vector space which is a Lie algebra.

If two vector fields Ξ_1 and Ξ_2 of the manifold M are such that $[\Xi_1, \Xi_2] = 0$, we say that Ξ_1 and Ξ_2 commute.

Definition 8.10 Let M be a differentiable manifold of dimension n, and $r \leq n$. An r-dimensional distribution $\mathfrak{D}$ on M is a function associating to every point $\mathbf{z} \in M$ an r-dimensional vector subspace of $T_{\mathbf{z}}M$ that we denote $\mathfrak{D}_{\mathbf{z}}$:

$$\mathfrak{D} : M \to TM$$

$$\mathbf{z} \mapsto \mathfrak{D}_{\mathbf{z}} \subset T_{\mathbf{z}}M.$$

We have a C^∞ distribution $\mathfrak{D}_{\mathbf{z}}$ it exists a tuple of C^∞ vector fields $\Xi_1, \ldots, \Xi_r$ defined on a neighborhood U of $\mathbf{z}$ yielding a basis of $\mathfrak{D}_{\mathbf{z}'}$ for every $\mathbf{z}' \in U$.

Remark 8.2 The dimension of a distribution $\mathfrak{D}_{\mathbf{z}}$ may not be the same in all points $\mathbf{z} \in M$. For instance, the distribution $\left\{ \dfrac{\partial}{\partial z_1}, z_1 \dfrac{\partial}{\partial z_2} \right\}$ is two-dimensional only when $z_1 \neq 0$.

A distribution whose dimension is constant for all $\mathbf{z} \in M$ is said *regular*; if the dimension is not constant, then we have a *singular distribution*. In the remaining of this section, the distributions are assumed to be regular in M (or, at least, in $M' \subset M$). We observe that a distribution $\mathfrak{D}$ does not change if we replace its vector fields with a linear combination with coefficients depending on $\mathbf{z}$, as long as the linear transformation is not degenerate.

Definition 8.11 Let $\mathfrak{D}$ a C^∞ r-dimensional distribution. A submanifold N of M is an integral manifold of $\mathfrak{D}$ if $T_{\mathbf{z}}N = \mathfrak{D}_{\mathbf{z}}$ for all $\mathbf{z} \in N$.

Remark 8.3 If N is the integral manifold of $\mathfrak{D}$, its tangent space contains r linearly independent vectors.

Definition 8.12 Let $\mathfrak{D}$ be a distribution on M. If it exists for any $\mathbf{z} \in M$ the integral manifold of $\mathfrak{D}$, then the distribution $\mathfrak{D}$ is said *completely integrable*.

Remark 8.4 A one-dimensional C^∞ distribution is a vector field and is trivially completely integrable.

Example 8.1 In $\mathbb{R}^n$, the vector fields

$$\Xi_k = \frac{\partial}{\partial z_k}, \qquad k = 1, \ldots, n,$$

span an n-dimensional distribution possessing as integral manifold $\mathbb{R}^n$.

Definition 8.13 A C^∞ distribution $\mathfrak{D}$ on the manifold M is *involutive* (or, equivalently, the vector fields in $\mathfrak{D}$ are in involution) if for any $\Xi_i, \Xi_j \in \mathfrak{D}$, it is $[\Xi_i, \Xi_j] \in \mathfrak{D}$, i.e., the distribution is closed under the Lie bracket.

An important result is the Frobenius theorem [37], that we state for vector fields.

Theorem 8.2 (Frobenius theorem) *Let $\mathfrak{D}$ be a regular distribution spanned by the linearly independent vector fields $\{\Xi_1, \ldots, \Xi_r\}$ for any $\mathbf{z} \in M' \subseteq M$. The distribution $\mathfrak{D}$ has an integral manifold of dimension r (so, it is completely integrable) if and only if the distribution $\mathfrak{D}$ is involutive. Moreover, it is possible to choose a set of coordinates $(y_1, \ldots, y_n)$ in a neighborhood U of $\mathbf{z}$ such that the distribution writes as*

$$\left\{ \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_r} \right\}.$$

8.3 Geometry of Differential Equations

The notions recalled in the previous section are now used for giving a short account about the geometry of differential equations in order to have a framework where Lie remarkable equations [30] can be characterized [6, 36, 37, 48].

Let E be an $(n+m)$-dimensional smooth manifold. For simplicity, let us denote local charts as (x_i, u_α), $i = 1, \ldots, n$, $\alpha = 1 \ldots m$; then, we describe (locally) an n-dimensional submanifold $L \subset E$ as the graph of a vector function $u_\alpha = f_\alpha(x_i)$. Let $\chi(E)$ denote the Lie algebra of vector fields on E.

By considering two submanifolds

$$\iota \colon L \hookrightarrow E, \qquad \iota' \colon L' \hookrightarrow E,$$

and $\mathbf{z} \in L \cap L'$, we say that L and L' have a *contact of order* r at $\mathbf{z}$ if ι and ι' have a contact of order r at $\mathbf{z}$. Locally, this means that the Taylor expansion of $(\iota - \iota')$ around $\mathbf{z}$ in a chart which is compatible with both L and L' vanishes up to the order r. This property is invariant by coordinate transformations.

The above relation is an equivalence relation, and an equivalence class is denoted by $[L]^r_{\mathbf{z}}$. The set of such classes is said to be the *rth jet of n-dimensional submanifolds of* E (also known as extended jet bundle [36], or manifold of contact elements), and it is denoted by $J^r(E, n)$. The set $J^r(E, n)$, whose dimension is $n + m\binom{n+r}{r}$, has a natural manifold structure with local charts $(x_i, u_\alpha, u_{\alpha, j_1}, \ldots, u_{\alpha, j_1 \ldots j_r})$, where we recall that

$$u_{\alpha, j_1 \ldots j_k} = \frac{\partial^k u_\alpha}{\partial x_{j_1} \ldots \partial x_{j_k}} = \frac{\partial^k f_\alpha}{\partial x_{j_1} \ldots \partial x_{j_k}}, \qquad 1 \le k \le r.$$

We have the following natural maps:

1. the embedding
$$j_r L \colon L \to J^r(E, n), \quad \mathbf{z} \mapsto [L]^r_{\mathbf{z}};$$

2. the projection
$$\pi_{k,h} \colon J^k(E, n) \to J^h(E, n), \quad [L]^k_{\mathbf{z}} \mapsto [L]^h_{\mathbf{z}}, \quad k \ge h.$$

Let us denote by $L^{(r)}$ the image of $j_r L$, and call the tangent plane $T_{\boldsymbol{\theta}_r} L^{(r)}$ at $\boldsymbol{\theta}_r = [L]^r_{\mathbf{z}}$ an *R-plane*. It is easy to realize that to each point $\boldsymbol{\theta}_{r+1} \in J^{r+1}(E, n)$ such that $\pi_{r+1,r}(\boldsymbol{\theta}_{r+1}) = \boldsymbol{\theta}_r$ there corresponds the R-plane $R_{\boldsymbol{\theta}_{r+1}} = T_{\boldsymbol{\theta}_{r+1}} L^{(r+1)}$.

The span $C^r_{\boldsymbol{\theta}_r}$ of all R-planes at $\boldsymbol{\theta}_r$ is said to be the *contact plane* (or Cartan plane). The correspondence $\boldsymbol{\theta}_r \mapsto C^r_{\boldsymbol{\theta}_r}$ is said to be the *contact distribution*. The contact distribution on $J^r(E, n)$ is generated by the vectors

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + u_{\alpha,i}\frac{\partial}{\partial u_\alpha} + u_{\alpha,ij_1}\frac{\partial}{\partial u_{\alpha,j_1}} + \cdots + u_{\alpha,ij_1\ldots j_{r-1}}\frac{\partial}{\partial u_{\alpha,j_1\ldots j_{r-1}}},$$
$$\frac{\partial}{\partial u_{\alpha,j_1\ldots j_r}}.$$

As already introduced in Chap. 4, the vector fields $\mathcal{D}_i$ are the *total derivatives*, and are tangent to any prolonged n-dimensional submanifold of E.

A diffeomorphism of $J^r(E,n)$ preserving the contact distribution is called a *contact transformation*. Analogously, a vector field $\Xi \in \chi(J^r(E,n))$ is said to be a *contact vector field* if $[\Xi, Y]$ is a vector field lying in C^r whenever Y is a vector field lying in C^r.

We can lift a contact transformation $G\colon J^r(E,n) \to J^r(E,n)$ to a contact transformation $G^{(1)}\colon J^{r+1}(E,n) \to J^{r+1}(E,n)$ in the following way: take an element $\boldsymbol{\theta}_{r+1} \in J^{r+1}(E,n)$ and identifies it with $R_{\boldsymbol{\theta}_{r+1}}$. Then define $G^{(1)}$ as the map that takes $R_{\boldsymbol{\theta}_{r+1}}$ to $T_{\boldsymbol{\theta}_r}G(R_{\boldsymbol{\theta}_{r+1}})$. We call such an operation the *1-lift*. By induction, we can define the k-lift of a contact transformation. By using this reasoning, we can lift contact fields by lifting the corresponding one-parameter group of contact transformations. In coordinates, if $\Xi \in \chi(E)$ has the coordinate expression

$$\Xi = \sum_{i=1}^{n} \xi_i(x_j, u_\beta)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(x_j, u_\beta)\frac{\partial}{\partial u_\alpha},$$

then its k-lift $\Xi^{(k)}$ has the coordinate expression (sum over repeated indices)

$$\Xi^{(k)} = \Xi^{(k-1)} + \eta_{[\alpha,j_1\ldots j_k]}\frac{\partial}{\partial u_{\alpha,j_1\ldots j_k}},$$

where $\Xi^{(0)} = \Xi$, and

$$\eta_{[\alpha,j_1\ldots j_k]} = \mathcal{D}_{j_k}(\eta_{[\alpha,j_1\ldots j_{k-1}]}) - \sum_{j=1}^{n} u_{\alpha,j_1\ldots j_{k-1}j}\mathcal{D}_{j_k}(\xi_j).$$

According to a classical result by Lie and Bäcklund [2] (see also Sect. 4.5 in Chap. 4), any contact transformation on $J^r(E,n)$ is the lifting of:

1. a contact transformation of $J^1(E,n)$ if $m = 1$;
2. a diffeomorphism of $J^0(E,n) = E$ (which we call *point transformation*) if $m > 1$.

An analogous result holds for contact vector fields.

Remark 8.5 If E is endowed with a fibring $\pi : E \to M$, where $\dim(M) = n$, then we can reproduce all the above constructions (see [6, 36, 37, 45, 48]). The space $J^r\pi$ of rth jets of (local) sections $s\colon M \to E$ of π is an open dense subset of $J^r(E,n)$. In fact, it coincides with the subset of rth jets of submanifolds of the type $s(M)$, which

are transverse to the fibring. Jets of maps $f\colon N \to P$ between two manifolds N and P are easily shown to coincide with jets of the trivial fibring $N \times P \to N$.

A *differential equation* $\mathcal{E}$ *of order* r *on* n*-dimensional submanifolds of a manifold* E is a submanifold of $J^r(E,n)$. The manifold $J^r(E,n)$ is called the *trivial equation*.

The *1-prolongation* $\mathcal{E}^1$ of the equation $\mathcal{E}$ is the set of first-order *differential consequences* of $\mathcal{E}$. Geometrically:

$$\mathcal{E}^1 = \{\boldsymbol{\theta}_{r+1} \in J^{r+1}(E,n)\colon \pi_{r+1,r}(\boldsymbol{\theta}_{r+1}) \in \mathcal{E},\ R_{\boldsymbol{\theta}_{r+1}} \subset T_{\boldsymbol{\theta}_r}\mathcal{E}\}.$$

Iterating, we can define the k-prolongation $\mathcal{E}^k$.

The equation $\mathcal{E}$ is said to be *formally integrable* if $\mathcal{E}^{k+1} \to \mathcal{E}^k$ are smooth fiber bundles. Locally, if the equation $\mathcal{E}$ is described by $\{\Delta_i = 0\colon i = 1,\dots,q\}$, with $\Delta_i \in C^\infty(J^r(E,n))$, then $\mathcal{E}^\ell$ is described by

$$\left\{\Delta_i = 0, \mathcal{D}_{j_1\dots j_k}(\Delta_i) = 0\colon i = 1,\dots,q,\ j_k = 1,\dots,n,\ 1 \le k \le \ell\right\},$$

with $\mathcal{D}_{j_1\dots j_k} = \mathcal{D}_{j_1}\mathcal{D}_{j_2}\dots\mathcal{D}_{j_k}$.

Definition 8.14 (*Symmetry of a differential equation*) An *infinitesimal classical external symmetry* of the equation $\mathcal{E} \subset J^r(E,n)$ is a contact vector field on $J^r(E,n)$ which is tangent to $\mathcal{E}$.

An *infinitesimal point symmetry* of $\mathcal{E}$ is an infinitesimal classical external symmetry which is the prolongation on $J^r(E,n)$ of a vector field on E.

Definition 8.14, as proved in Chap. 4, means that the symmetries of a differential equation $\mathcal{E}$, locally described by $\{\Delta_i = 0\colon i = 1,\dots,q < \dim(J^r(E,n)\}$ requires to solve the condition

$$\Xi^{(r)}(\Delta_i)\big|_{\Delta_1=\dots=\Delta_q=0} = 0, \qquad i = 1,\dots,q$$

for some $\Xi \in \chi(E)$.

The Lie algebra of infinitesimal point symmetries of the equation $\mathcal{E}$ is denoted by $\mathrm{sym}(\mathcal{E})$.

An *rth-order differential invariant* of a Lie subalgebra $\mathfrak{s}$ of $\chi(E)$ is a smooth function $F\colon J^r(E,n) \to \mathbb{R}$ such that for all $\Xi \in \mathfrak{s}$ we have $\Xi^{(r)}(F) = 0$.

The problem of determining the Lie algebra $\mathrm{sym}(\mathcal{E})$ is said to be the *direct Lie problem*. Conversely, given a Lie subalgebra $\mathfrak{s} \subset \chi(J^r(E,n))$ of contact vector fields, we consider the *inverse Lie problem*, i.e., the problem of classifying the equations $\mathcal{E} \subset J^r(E,n)$ such that $\mathrm{sym}(\mathcal{E}) \supseteq \mathfrak{s}$.

8.4 Lie Remarkable Equations

Hereafter, we give the definition of *Lie remarkable equations* in the framework of jets of submanifolds. We assume to have an $(n+m)$-dimensional manifold E and an ℓ-dimensional differential equation $\mathcal{E} \subset J^r(E,n)$, with $\ell < \dim(J^r(E,n))$.

Definition 8.15 (*Weakly Lie remarkable equation*) Let E be a manifold, $\dim(E) = n+m$, and let $0 < r \in \mathbb{N}$. An ℓ-dimensional equation $\mathcal{E} \subset J^r(E,n)$ is said to be *weakly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) ℓ-dimensional equation in $J^r(E,n)$ passing at any $\boldsymbol{\theta} \in \mathcal{E}$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of its algebra of infinitesimal point symmetries.

Definition 8.16 (*Strongly Lie remarkable equation*) Let E be a manifold, $\dim(E) = n+m$, and let $0 < r \in \mathbb{N}$. An ℓ-dimensional equation $\mathcal{E} \subset J^r(E,n)$ is said to be *strongly Lie remarkable* if $\mathcal{E}$ is the only maximal (with respect to the inclusion) ℓ-dimensional equation in $J^r(E,n)$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of its algebra of infinitesimal point symmetries.

Remark 8.6 Of course, a strongly Lie remarkable equation is also weakly Lie remarkable.

Let us analyze some direct consequences of Definitions 8.15 and 8.16. For each $\boldsymbol{\theta} \in J^r(E,n)$, denote by $S_{\boldsymbol{\theta}}(\mathcal{E}) \subset T_{\boldsymbol{\theta}} J^r(E,n)$ the subspace generated by the values of infinitesimal point symmetries of $\mathcal{E}$ at $\boldsymbol{\theta}$. In particular, if $\boldsymbol{\theta} \in \mathcal{E}$, then $S_{\boldsymbol{\theta}}(\mathcal{E}) \subseteq T_{\boldsymbol{\theta}}\mathcal{E}$; let us set

$$S(\mathcal{E}) = \bigcup_{\boldsymbol{\theta} \in J^r(E,n)} S_{\boldsymbol{\theta}}(\mathcal{E}).$$

In general, $\dim(S_{\boldsymbol{\theta}}(\mathcal{E}))$ may change with $\boldsymbol{\theta} \in J^r(E,n)$. The action on $J^r(E,n)$ of the algebra of point symmetries is called *regular* if $S(\mathcal{E})$ is a regular distribution.

The following inequality holds:

$$\dim(\mathrm{sym}(\mathcal{E})) \geq S_{\boldsymbol{\theta}}(\mathcal{E}), \quad \forall\, \boldsymbol{\theta} \in J^r(E,n),$$

where $\dim(\mathrm{sym}(\mathcal{E}))$ is the dimension, as a real vector space, of the Lie algebra of infinitesimal point symmetries $\mathrm{sym}(\mathcal{E})$. If the rank of $S(\mathcal{E})$ at each $\boldsymbol{\theta} \in J^r(E,n)$ is the same, then $S(\mathcal{E})$ is an involutive (smooth) distribution.

A submanifold N of $J^r(E,n)$ is an *integral submanifold* of $S(\mathcal{E})$ if $T_{\boldsymbol{\theta}} N = S_{\boldsymbol{\theta}}(\mathcal{E})$ for each $\boldsymbol{\theta} \in N$. Of course, an integral submanifold of $S(\mathcal{E})$ is an equation in $J^r(E,n)$ which admits all elements in $\mathrm{sym}(\mathcal{E})$ as infinitesimal point symmetries. Moreover, due to the fact that the point symmetries of $\mathcal{E}$ are tangent to $\mathcal{E}$, we have $\dim(S_{\boldsymbol{\theta}}(\mathcal{E})) \leq \ell$.

Theorem 8.3 *The points of $J^r(E,n)$ of maximal rank of $S(\mathcal{E})$ form an open set of $J^r(E,n)$.*

Proof Let us consider a chart U at a point $\boldsymbol{\theta}$ of maximal rank, and a finite number of symmetries $\{\Xi_j\}$ spanning $S_{\boldsymbol{\theta}}(\mathcal{E})$ at $\boldsymbol{\theta}$. In the chart U let us consider the matrix (Ξ^i_j), where Ξ^i_j is the ith component of the jth point symmetry of $\mathcal{E}$. The rank of such a matrix decreases if the determinant of some submatrix vanishes. The points satisfying such a condition form a closed set of U which does not contain $\boldsymbol{\theta}$. If we remove it from U, we obtain a neighborhood of $\boldsymbol{\theta}$ of maximal rank. □

Corollary 8.1 *The equation $\mathcal{E}$ cannot coincide with the set of points of maximal rank of $S(\mathcal{E})$.*

Proof It follows from the condition $\dim(\mathcal{E}) < \dim(J^r(E,n))$. □

Theorem 8.4 *A necessary condition for $\mathcal{E}$ to be strongly Lie remarkable is that* $\dim(\mathrm{sym}(\mathcal{E})) > \dim(\mathcal{E})$.

Proof Let $U \subset J^r(E,n)$ be an open neighborhood at $\boldsymbol{\theta} \notin \mathcal{E}$ where the rank is maximal (see Theorem 8.3 and Corollary 8.1). Let q be this rank. Suppose that there exists an ℓ-dimensional submanifold of $J^r(E,n)$ passing at $\boldsymbol{\theta}$ made up by q-dimensional leaves of $S(\mathcal{E})|_U$, so that $\dim(\mathrm{sym}(\mathcal{E})) \leq \dim(\mathcal{E})$. This implies that the equation is not strongly Lie remarkable. □

Theorem 8.5 *A necessary condition for $\mathcal{E}$ to be weakly Lie remarkable is that* $\dim(\mathrm{sym}(\mathcal{E})) \geq \dim(\mathcal{E})$.

Proof If $q = \dim(\mathrm{sym}(\mathcal{E})) < \dim(\mathcal{E})$, then there would be several submanifolds of $J^r(E,n)$ passing at a point $\boldsymbol{\theta} \in \mathcal{E}$ which are tangent to the vector fields in $\mathrm{sym}(\mathcal{E})$, and this contradicts the weak Lie remarkability. □

Theorem 8.6 *If $S(\mathcal{E})|_{\mathcal{E}}$ is an ℓ-dimensional distribution on $\mathcal{E} \subset J^r(E,n)$, then $\mathcal{E}$ is a weakly Lie remarkable equation.*

Proof Of course, an equation $\tilde{\mathcal{E}}$ passing at $\boldsymbol{\theta} \in \mathcal{E}$ admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of its algebra of point symmetries and such that $T_{\boldsymbol{\theta}}\tilde{\mathcal{E}} \neq T_{\boldsymbol{\theta}}\mathcal{E}$ cannot exist. So, suppose that there is an ℓ-dimensional equation $\tilde{\mathcal{E}}$ passing at $\boldsymbol{\theta} \in \mathcal{E}$ with $T_{\boldsymbol{\theta}}\tilde{\mathcal{E}} = T_{\boldsymbol{\theta}}\mathcal{E}$. A symmetry $\Xi \in \mathrm{sym}(\mathcal{E})$ would be tangent to both $\tilde{\mathcal{E}}$ and $\mathcal{E}$, and, in particular, $\Xi_{\boldsymbol{\theta}} \in T_{\boldsymbol{\theta}}\tilde{\mathcal{E}} = T_{\boldsymbol{\theta}}\mathcal{E}$. Then, Ξ would have two distinct integral curves passing at $\boldsymbol{\theta}$, leading to a contradiction. □

Remark 8.7 Theorem 8.6 suggests a way to determine if an equation is weakly Lie remarkable or not. We easily realize that the points which play a key role in the computation are the singular points (that is, the points of the equation where the dimension of the distribution determined by the symmetries decreases). Then, in Sects. 8.5 and 8.6, our strategy will be the following: we will find the singular point of the equation under consideration, then we check (if possible) if there exists another equation admitting the same symmetries and passing through these submanifolds. In view of these considerations, we realize that if we remove all singular points from $\mathcal{E}$, then we get an equation which is uniquely determined by its symmetries in the weak or strong sense.

Theorem 8.7 *Let $S(\mathcal{E})$ be such that for any $\boldsymbol{\theta} \notin \mathcal{E}$ we have $\dim(S_{\boldsymbol{\theta}}(\mathcal{E})) > \ell$. Then $\mathcal{E}$ is a strongly Lie remarkable equation.*

Proof By contradiction, let $\tilde{\mathcal{E}}$ be another ℓ-dimensional equation passing at $\boldsymbol{\theta} \notin \mathcal{E}$ and admitting $\mathrm{sym}(\mathcal{E})$ as subalgebra of its algebra of point symmetries. Then $T_{\boldsymbol{\theta}}\tilde{\mathcal{E}} \subsetneq S_{\boldsymbol{\theta}}(\mathcal{E})$. This would imply that $\tilde{\mathcal{E}}$ would not have $\mathrm{sym}(\mathcal{E})$ as a subalgebra of its algebra of point symmetries. □

Remark 8.8 In the proofs of previous theorems we construct some submanifold of the jet space without caring if this submanifold is a formally integrable equation. However, we would like to stress that in the scalar case any submanifold of codimension 1 is always a formally integrable equation.

The fact that an equation $\mathcal{E}$ is uniquely determined by its point symmetry algebra has consequences on the local, and in some cases global, topological structure of $\mathcal{E}$ as a manifold.

We recall (see, e.g., [38]) that a *(left) action* of a Lie group G on a manifold M is a smooth map $a\colon G \times M \to M$ such that $a(g, a(h, x)) = a(gh, x)$ and $a(e, x) = x$ for all $g, h \in G$, $x \in M$, e being the identity in G. This concept has a local analogue (see, e.g., [36]). From [38, Theorems XI, p. 58 and IV, p. 98] we obtain the following theorem.

Theorem 8.8 *Let $\mathfrak{s}$ be an ℓ-dimensional subalgebra of* $\mathrm{sym}(\mathcal{E})$. *Let $\{\Xi_i^{(r)}\}_{1\le i\le \ell}$ be a basis of $\mathfrak{s}$. Let us suppose that* $\mathrm{span}\{\Xi_i^{(r)}(\boldsymbol{\theta})\}_{1\le i\le \ell} = S_{\boldsymbol{\theta}}(\mathcal{E}) = T_{\boldsymbol{\theta}}\mathcal{E}$ *for any $\boldsymbol{\theta} \in \mathcal{E}$. Then, $\mathcal{E}$ has the structure of a local Lie group.*

More precisely, there exists a Lie group S whose Lie algebra is $\mathfrak{s}$, and a local differentiable action $\varphi\colon S \times \mathcal{E} \to \mathcal{E}$ by point symmetries of $\mathcal{E}$ whose tangent map is the inclusion of $\mathfrak{s}$ into $\mathrm{sym}(\mathcal{E})$. *If the vector fields Ξ_i are complete (in particular if $\mathcal{E}$ is a compact manifold), then the action is global.*

Notice that if the hypotheses of the above corollary are satisfied, then, following the terminology of [38], the Lie algebra $\mathfrak{s}$ is a *Kobayashi Lie algebra*. For this reason, if the action of S on $\mathcal{E}$ is global, then a necessary and sufficient condition that the map $\varphi(\cdot, \boldsymbol{\theta})\colon S \to \mathcal{E}$ be a diffeomorphism for $\boldsymbol{\theta} \in \mathcal{E}$ is that the isotropy group of the action at $\boldsymbol{\theta}$ is trivial [38, p. 105].

The previous theorem deals with topological aspects of the equation. Hereafter, we will not study topological aspects of Lie remarkability, even if they are quite interesting.

Many differential equations are given as the zero set of just one differentiable function. Indeed, the former definition and examples of Lie remarkable equations have been formulated in [33] in this case (see also [28]). We may rediscover the former definition as a particular case of our present more comprehensive theory.

Proposition 8.1 *Let $\mathfrak{s}$ be a Lie subalgebra of $\chi(E)$. Let us suppose that the rth prolongation subalgebra of $\mathfrak{s}$ acts regularly on $J^r(E, n)$ and that the set of rth order functionally independent differential invariants of the rth prolongation of $\mathfrak{s}$ has a unique element $J \in C^\infty(J^r(E, n))$. Then, the submanifold of $J^r(E, n)$ described by $\Delta(J) = 0$ (in particular, $J = c$ for any $c \in \mathbb{R}$), with Δ an arbitrary smooth function, is a weakly Lie remarkable equation.*

Proof It follows in view of the fact that $\Delta(J) = 0$ (up to a multiplicative function nowhere vanishing) is the most general equation admitting $\mathfrak{s}$ as a Lie subalgebra of point symmetries [36, 37]. □

When talking about Lie remarkable equations, a natural question arises. Namely, about Lie remarkability of the first prolongation $\mathcal{E}^1$ of $\mathcal{E}$ whenever $\mathcal{E}$ is Lie remarkable. Then we immediately realize (see Theorems 8.4 and 8.5) that Lie remarkability is a property that, for dimensional reasons, generally does not hold on prolongations. But there is a special class of differential equations for which this is true. For instance, let an equation $\mathcal{E}$ be the image of a section $\nabla\colon J^{r-1}(E,n) \to J^r(E,n)$. Such equations can be thought as *generalized sprays* [39]. Examples of such equations are, for instance, the totally geodesic submanifold equation. With such an equation it is possible to associate the n-dimensional distribution $R \circ \nabla$ on $J^{r-1}(E,n)$ defined by $\boldsymbol{\theta} \mapsto R_{\nabla(\boldsymbol{\theta})}$ (re recall that $R_{\nabla(\boldsymbol{\theta})}$ is an R-plane, see Sect. 8.3). We call such equation *integrable* if this distribution is integrable.

Theorem 8.9 *Let $\mathcal{E}$ be the image of a section $\nabla\colon J^{r-1}(E,n) \to J^r(E,n)$. Let $\mathcal{E}$ be integrable. If $\mathcal{E}$ is weakly Lie remarkable, then also $\mathcal{E}^k$ is weakly Lie remarkable.*

Proof Of course, it is sufficient to prove the proposition for $k = 1$. Following the same reasoning as in [26], we see that the prolongation $(\nabla(\boldsymbol{\theta}))^{(1)}$ of a point $\nabla(\boldsymbol{\theta})$ is the point prolongation represented by the pair $\big(\nabla(\boldsymbol{\theta}), T_{\boldsymbol{\theta}}\nabla(R_{\nabla(\boldsymbol{\theta})})\big)$. Let us note that $T_{\boldsymbol{\theta}}\nabla(R_{\nabla(\boldsymbol{\theta})})$ is an R-plane in view of the integrability. If we define $\nabla^{(1)}$ by $\nabla^{(1)}(\boldsymbol{\theta}) = (\nabla(\boldsymbol{\theta}))^{(1)}$, then

$$C_{\nabla^{(1)}(\boldsymbol{\theta})}\left(\mathcal{E}^1\right) \simeq T_{\boldsymbol{\theta}}\nabla^{(1)}\left(R_{\nabla(\boldsymbol{\theta})}\right) \simeq R_{\nabla(\boldsymbol{\theta})}.$$

Therefore, $\mathcal{E}^1$ is completely identified with $\mathcal{E}$, and point symmetries of both equations are vector fields on E whose $(r-1)$-prolongations are symmetries of the distribution $R \circ \nabla$. □

We have the following interesting result.

Theorem 8.10 *Let $\mathcal{E}$ be the image of a section $\nabla\colon J^{r-1}(E,n) \to J^r(E,n)$. Let $\mathcal{E}$ be integrable. If it is weakly Lie remarkable, then any solution is invariant with respect to a one-dimensional subalgebra of its algebra of point symmetries.*

Proof In view of our previous discussion, we can identify $\mathcal{E}$ with $J^{r-1}(E,n)$, and a solution with an integrable submanifold of the distribution $R \circ \nabla$. Let $\boldsymbol{\theta} \in J^{r-1}(E,n)$ and L be the unique solution passing at $\boldsymbol{\theta}$. Let $\mathbf{v}_{\boldsymbol{\theta}} \in T_{\boldsymbol{\theta}}L$. By hypothesis, $\mathbf{v}_{\boldsymbol{\theta}} = a^i\,\Xi_i^{(r)}(\boldsymbol{\theta})$ with $a^i \in \mathbb{R}$ and $\Xi_i \in \chi(E)$. Let us consider the vector field $\Xi^{(r)} = a^i\,\Xi_i^{(r)}$. In view of the fact that $\Xi^{(r)}$ is still a point symmetry, and then sends solutions into solutions, we obtain that $\Xi^{(r)}$ is tangent to L. □

8.5 Smooth Surfaces in $\mathbb{R}^3$

Let us consider a smooth surface in $\mathbb{R}^3$ (with the Euclidean metrics), with coordinates $(x_1, x_2, u(x_1, x_2))$, where $(x_1, x_2) \in \mathbb{D}$, and $\mathbb{D}$ is an open connected set of $\mathbb{R}^2$. We can define two important objects: the Gaussian curvature,

$$G = \frac{u_{,11}u_{,22} - (u_{,12})^2}{\left(1 + (u_{,1})^2 + (u_{,2})^2\right)^2},$$

and the mean curvature,

$$H = \frac{\left(1 + (u_{,2})^2\right) u_{,11} - 2u_{,1}u_{,2}u_{,12} + \left(1 + (u_{,1})^2\right) u_{,22}}{2\left(1 + (u_{,1})^2 + (u_{,2})^2\right)^{3/2}}.$$

A minimal surface is a surface with vanishing mean curvature, so that the differential equation for minimal surfaces is

$$\left(1 + (u_{,2})^2\right) u_{,11} - 2u_{,1}u_{,2}u_{,12} + \left(1 + (u_{,1})^2\right) u_{,22} = 0. \tag{8.2}$$

This equation admits a seven-dimensional Lie algebra of point symmetries spanned by

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_2}, \quad \Xi_3 = \frac{\partial}{\partial u}, \\
&\Xi_4 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}, \quad \Xi_5 = u\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial u}, \\
&\Xi_6 = u\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial u}, \quad \Xi_7 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + u\frac{\partial}{\partial u},
\end{aligned} \tag{8.3}$$

corresponding to the isometries in $\mathbb{R}^3$ together with a homogeneous scaling of the coordinates (x_1, x_2, u); the latter is also called homothety. The matrix of second-order prolonged vector fields,

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
x_2 & -x_1 & 0 & u_{,2} & -u_{,1} & 2u_{,12} & -u_{,11} + u_{,22} & -2u_{,12} \\
u & 0 & -x_1 & -1 - u_{,1}^2 & -u_{,1}u_{,2} & -3u_{,1}u_{,11} & -2u_{,1}u_{,12} - u_{,11}u_{,2} & -2u_{,12}u_{,2} - u_{,1}u_{,22} \\
0 & u & -x_2 & -u_{,1}u_{,2} & -1 - u_{,2}^2 & -2u_{,1}u_{,12} - u_{,11}u_{,2} & -2u_{,12}u_{,2} - u_{,1}u_{,22} & -3u_{,2}u_{,22} \\
x_1 & x_2 & u & 0 & 0 & -u_{,11} & -u_{,12} & -u_{,22}
\end{pmatrix}$$

give rise to a distribution of rank 7 on the whole jet space provided that we exclude the six-dimensional submanifold of $J^2(\mathbb{R}^3, 2)$ locally described by the system

$$u_{,1}u_{,2}u_{,11} - (1 + (u_{,1})^2)u_{,12} = 0,$$
$$u_{,1}u_{,2}u_{,22} - (1 + (u_{,2})^2)u_{,12} = 0,$$

where the rank is 6, and the five-dimensional submanifold of geodesics described by

$$u_{,11} = u_{,12} = u_{,22} = 0, \tag{8.4}$$

where the rank reduces to 5.

Remark 8.9 The unique second-order differential invariant of the vector fields in (8.3) is

$$J = \frac{\left(\left(1 + (u_{,2})^2\right) u_{,11} - 2u_{,1}u_{,2}u_{,12} + \left(1 + (u_{,1})^2\right) u_{,22}\right)^2}{\left(1 + (u_{,1})^2 + (u_{,2})^2\right)\left(u_{,11}u_{,22} - (u_{,12})^2\right)} = \frac{(2H)^2}{G}.$$

Therefore, it follows that the most general second-order scalar partial differential equation left invariant by (8.3) must be given as $\Delta(J) = 0$.

We note that:

- the system $\{H = G = 0\}$ reduces to Eq. (8.4);
- the system $\{H = 0,\ G \neq 0\}$, that is $J = 0$, is Eq. (8.2);
- the system $\{H \neq 0,\ G = 0\}$, that is $1/J = 0$, is the equation of surfaces with vanishing Gaussian curvature (which is a Monge–Ampère equation, see Sect. 8.6).

According to Proposition 8.1, the above equations are weakly Lie remarkable.

If we consider in general the equation of minimal n-dimensional submanifolds in $\mathbb{R}^{n+m}$, whose expression is outside the scope of this book and so it is omitted, it seems reasonable to conjecture that their point symmetries are isometries and homotheties (homogeneous scalings of the coordinates). In view of Theorems 8.4 and 8.5, we infer that possible candidates of Lie remarkable minimal submanifold equations are characterized by having a dimension less than or equal to the dimension of the Lie algebra generated by infinitesimal isometries and homotheties.

Lemma 8.1 *Let d denote the dimension of the equation of minimal n-dimensional submanifolds in $\mathbb{R}^{n+m}$, with $n \geq 2$, $m \geq 1$, and let i be the dimension of the Lie algebra generated by infinitesimal isometries and homotheties in $\mathbb{R}^{n+m}$. Then,*

1. *the only integer solutions of the equation $d = i$ are $(n, m) = (2, 1)$ and $(n, m) = (2, 4)$;*
2. *if $n = 2$ the inequality $d > i$ holds for $m = 2$, $m = 3$;*
3. *if $m = 1$ the inequality $d > i$ holds for $n > 2$.*

Proof Taking into account that

$$d = \dim(J^2(E, n)) - m = n + m\binom{n+2}{2} - m,$$

and

$$i = \frac{(n+m)(n+m+1)}{2} + 1,$$

a direct computation proves the statements. □

Theorem 8.11 *The equation of minimal surfaces in* $\mathbb{R}^4$ *and* $\mathbb{R}^5$ *is neither strongly nor weakly Lie remarkable, whereas it is weakly Lie remarkable in* $\mathbb{R}^3$ *and* $\mathbb{R}^6$*, provided that we remove a singular equation.*

Proof The point symmetries of the equation of minimal surfaces in $\mathbb{R}^3$, $\mathbb{R}^4$, $\mathbb{R}^5$ and $\mathbb{R}^6$ are the isometries and homotheties. Then the first part of the theorem follows in view of the above lemma and Theorem 8.4. In fact, we have that if $n = m = 2$ then $d = 12$ and $i = 11$, while if $n = 2$, $m = 3$ then $d = 17$ and $i = 16$. □

8.6 Lie Remarkability of Second-Order Monge–Ampère Equations

Monge–Ampère equation for a surface in $\mathbb{R}^3$ with vanishing Gaussian curvature is a particular instance of the equation, introduced by Ampère in a paper published in 1815 [1] (see also [47]), having the form

$$H\frac{\partial^2 u}{\partial x_1^2} + 2K\frac{\partial^2 u}{\partial x_1 \partial x_2} + L\frac{\partial^2 u}{\partial x_2^2} + M + N\left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2} - \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2\right) = 0, \quad (8.5)$$

where the coefficients $H, K, L, M, N \neq 0$ depend on x_1, x_2, u, and first-order derivatives of u (x_1 denotes the time variable). A remarkable property of Eq. (8.5) has been proved by Boillat [8], who showed that this is the most general second-order nonlinear hyperbolic equation in $(1+1)$ dimensions that is completely exceptional [7, 23]. In 1973, Ruggeri [44], using the property of complete exceptionality, generalized Monge–Ampère equation to $(2+1)$ dimensions. Moreover, Donato, Ramgulam, and Rogers [15] characterized the $(3+1)$-dimensional Monge–Ampère equation. Boillat [9], with the same procedure, derived the general form of Monge–Ampère equation for a function $u(x_1, \ldots, x_n)$. All these second-order Monge–Ampère equations are given as a linear combination (with coefficients being arbitrary functions of the independent variables, the dependent variable, and its first order derivatives) of all minors of all orders extracted from the Hessian matrix. Some of these equations can be reduced to quasilinear or linear form [19, 20, 32]. The property of complete exceptionality has been used by Boillat [10] to characterize higher order Monge–Ampère equations for a function $u(x_1, x_2)$; in such a case, they are given as a linear combination of all minors of all orders extracted from an Hankel matrix [18] built with the highest order derivatives of $u(x_1, x_2)$.

In this section, we apply our results to various second-order Monge–Ampère equations. As we deal with local point symmetries of these equations, and we are

interested to local aspects, we will interpret them as submanifolds of jets of a trivial bundle $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$. The interested reader may see [25] for a geometric treatment of second order Monge–Ampère equations and their classification up to contact transformations of $J^2(E, n)$.

8.6.1 Second-Order Monge–Ampère Equation in Two Independent Variables

According to the result proved in 1968 by Boillat [8], let us consider the equation

$$\kappa_1 \left(u_{,11}u_{,22} - (u_{,12})^2\right) + \kappa_2 u_{,11} + \kappa_3 u_{,12} + \kappa_4 u_{,22} + \kappa_5 = 0, \tag{8.6}$$

where $\kappa_i = \kappa_i(x_1, x_2, u, u_{,1}, u_{,2})$, $i = 1, \ldots, 5$, are arbitrary functions.

Let us restrict ourselves to consider Eq. (8.6) in the case where the coefficients κ_i are constant (in particular, we assume $\kappa_1 \neq 0$ in order to have a nonlinear equation).

Through the replacement $u \to u + \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2$, where $\alpha_1 = -\kappa_4/(2\kappa_1)$, $\alpha_2 = \kappa_3/(2\kappa_1)$, $\alpha_3 = -\kappa_2/(2\kappa_1)$, Eq. (8.6) is mapped to

$$u_{,11}u_{,22} - (u_{,12})^2 = \kappa, \tag{8.7}$$

where $\kappa = (4\kappa_2\kappa_4 - 4\kappa_1\kappa_5 - \kappa_3^2)/(4\kappa_1^2)$.

If $\kappa = 0$ we have the homogeneous Monge–Ampère equation for the surface $u(x_1, x_2)$ with zero Gaussian curvature.

Theorem 8.12 *Equation* (8.7) *is weakly Lie remarkable if* $\kappa \neq 0$*, whereas it is strongly Lie remarkable if* $\kappa = 0$*.*

Proof If $\kappa \neq 0$, then Eq. (8.7) admits a nine-parameter group of point symmetries whose Lie algebra is spanned by the vector fields

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_2}, \quad \Xi_3 = \frac{\partial}{\partial u}, \\
&\Xi_4 = x_1 \frac{\partial}{\partial x_2}, \quad \Xi_5 = x_2 \frac{\partial}{\partial x_1}, \quad \Xi_6 = x_1 \frac{\partial}{\partial u}, \\
&\Xi_7 = x_2 \frac{\partial}{\partial u}, \quad \Xi_8 = x_1 \frac{\partial}{\partial x_1} + u \frac{\partial}{\partial u}, \quad \Xi_9 = x_2 \frac{\partial}{\partial x_2} + u \frac{\partial}{\partial u}.
\end{aligned} \tag{8.8}$$

The second-order prolonged vector fields (8.8) give rise to the distribution

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & -u_{,2} & 0 & -2u_{,12} & -u_{,22} & 0 \\ x_2 & 0 & 0 & 0 & -u_{,1} & 0 & -u_{,11} & -2u_{,12} \\ 0 & 0 & x_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 1 & 0 & 0 & 0 \\ x_1 & 0 & u & 0 & u_{,2} & -u_{,11} & 0 & u_{,22} \\ 0 & x_2 & u & u_{,1} & 0 & u_{,11} & 0 & -u_{,22} \end{pmatrix}$$

whose rank is 7 on the whole jet space provided that we exclude the five-dimensional submanifold of $J^2(\mathbb{R}^3, 2)$ locally described by

$$u_{,11} = u_{,12} = u_{,22} = 0 \tag{8.9}$$

(totally geodesic submanifold of $\mathbb{R}^3$) where the rank reduces to 5. We note that (8.9) do not represent a submanifold of (8.7); then, according to Theorem 8.6, Eq. (8.7) is weakly Lie remarkable.

On the contrary, if $\kappa = 0$, Eq. (8.7) admits a 15-parameter group of point symmetries whose Lie algebra is spanned by the vector fields

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = \frac{\partial}{\partial u}, \\
&\Xi_4 = x_1 \frac{\partial}{\partial x_1}, \qquad \Xi_5 = x_1 \frac{\partial}{\partial x_2}, \qquad \Xi_6 = x_1 \frac{\partial}{\partial u}, \\
&\Xi_7 = x_2 \frac{\partial}{\partial x_1}, \qquad \Xi_8 = x_2 \frac{\partial}{\partial x_2}, \qquad \Xi_9 = x_2 \frac{\partial}{\partial u}, \\
&\Xi_{10} = u \frac{\partial}{\partial x_1}, \qquad \Xi_{12} = u \frac{\partial}{\partial x_2}, \qquad \Xi_{12} = u \frac{\partial}{\partial u}, \\
&\Xi_{13} = x_1 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + u \frac{\partial}{\partial u} \right), \\
&\Xi_{14} = x_2 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + u \frac{\partial}{\partial u} \right), \\
&\Xi_{15} = u \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + u \frac{\partial}{\partial u} \right).
\end{aligned} \tag{8.10}$$

In this case, the second-order prolonged vector fields give rise to a distribution of rank 8 on the whole jet space provided that we exclude the seven-dimensional submanifold locally described by Eq. (8.7) for $\kappa = 0$, where the rank reduces to 7, and the five-dimensional submanifold (8.9), where the rank reduces to 5. Thus, according to Theorem 8.7, Eq. (8.7) for $\kappa = 0$ is strongly Lie remarkable. □

There are various seven-dimensional Lie subalgebras of the nine-dimensional Lie algebra spanned by (8.8) whose second-order prolonged vector fields give rise to a distribution of rank 7. For instance, we can consider the subalgebra whose generators are $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_6, \Xi_7, \Xi_8\}$. Then, in view of Theorem 8.8, this equation has the structure of a local Lie group diffeomorphic to a local subgroup of the group of its point symmetries.

Moreover, it is natural to ask if among the 15-dimensional algebra of point symmetries (8.10) it is possible to extract some subalgebra such that Eq. (8.7) for $\kappa = 0$ is still strongly Lie remarkable. In fact, in view of Theorems 8.5 and 8.7, it could be sufficient an eight-dimensional Lie subalgebra. We see that there are many eight-dimensional Lie subalgebras whose second-order prolonged vector fields generate a distribution of rank 8 (provided that we exclude, of course, Eq. (8.7) for $\kappa = 0$), even if we confine ourselves to consider the eight-dimensional subalgebras spanned by subsets containing 8 elements chosen in the set of basic vector fields given by (8.10). We limit ourselves to quote the Lie subalgebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5, \Xi_6, \Xi_8, \Xi_{12}\}$.

Remark 8.10 In [41], Rosenhaus showed that the dimension of the subalgebra of the algebra of point symmetries of Eq. (8.7) (with $\kappa = 0$) which determines this equation has to be at least 8; this is easily obtained from Theorem 8.5. Furthermore, in [41–43], it is claimed that the six-dimensional subalgebra

$$\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial u}, \quad x_1 \frac{\partial}{\partial x_1}, \quad x_1 \frac{\partial}{\partial x_2}, \quad x_1 \frac{\partial}{\partial u}$$

characterizes completely Eq. (8.7) with $\kappa = 0$ provided that we restrict our attention to fully symmetric systems. This last requirement is essential as the previous symmetries cannot determine Eq. (8.7) for $\kappa = 0$ in view of Theorem 8.5. For instance, the equation $\Delta(u_{,2}, u_{,22}) = 0$, where Δ is an arbitrary smooth function, admits the same six-dimensional Lie subalgebra.

Remark 8.11 We observe that $J = u_{,11}u_{,22} - (u_{,12})^2$ is a second-order differential invariant of (8.8), and then, in view of Lie remarkability, it is unique up to functional dependence. Therefore it follows that the most general second-order scalar partial differential equation invariant by (8.8) must be given as $\Delta(J) = 0$, where Δ is an arbitrary function of the invariants. For instance, $J = \kappa$, with $\kappa \in \mathbb{R}$, is again weakly Lie remarkable in view of Proposition 8.1.

8.6.2 Second-Order Monge–Ampère Equation in Three Independent Variables

A Monge–Ampère equation in three independent variables has been characterized in [44], imposing the condition of complete exceptionality used by Boillat [8] for deriving the classical Monge–Ampère equation. This equation is a linear combination of all minors extracted from the Hessian matrix, with coefficient arbitrary functions

of the independent variables, the unknown and its first-order derivatives. Here, we limit ourselves to consider the case of constant coefficients. Denoting with Δ the determinant of the Hessian matrix $\|u_{,ij}\|, i, j = 1, 2, 3$, the Monge–Ampère equation in $(2 + 1)$ dimensions can be written as

$$\begin{aligned}\kappa_1\Delta + \kappa_2\frac{\partial\Delta}{\partial u_{,11}} + \kappa_3\frac{\partial\Delta}{\partial u_{,12}} + \kappa_4\frac{\partial\Delta}{\partial u_{,13}} + \kappa_5\frac{\partial\Delta}{\partial u_{,22}} + \kappa_6\frac{\partial\Delta}{\partial u_{,23}} + \kappa_7\frac{\partial\Delta}{\partial u_{,33}}\\ + \kappa_8 u_{,11} + \kappa_9 u_{,12} + \kappa_{10}u_{,13} + \kappa_{11}u_{,22} + \kappa_{12}u_{,23} + \kappa_{,13}\kappa_{14} = 0.\end{aligned}$$

Linear terms in the second-order derivatives can be dropped out by means of the replacement

$$u \to u + \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_1 x_3 + \alpha_4 x_2^2 + \alpha_5 x_2 x_3 + \alpha_6 x3^2,$$

where $\alpha_1, \ldots, \alpha_6$ satisfy the conditions

$$\begin{aligned}&\kappa_8 + 2\kappa_7\alpha_4 - 2\kappa_6\alpha_5 - \kappa_1\alpha_5^2 + 2\kappa_5\alpha_6 + 4\kappa_1\alpha_4\alpha_6 = 0,\\ &\kappa_{10} + 2\kappa_6\alpha_2 - 2\kappa_5\alpha_3 - 4\kappa_4\alpha_4 - 4\kappa_1\alpha_3\alpha_4 + 2\kappa_3\alpha_5 + 2\kappa_1\alpha_2\alpha_5 = 0,\\ &\kappa_{10} + 2\kappa_6\alpha_2 - 2\kappa_5\alpha_3 - 4\kappa_4\alpha_4 - 4\kappa_1\alpha_3\alpha_4 + 2\kappa_3\alpha_5 + 2\kappa_1\alpha_2\alpha_5 = 0,\\ &\kappa_{11} + 2\kappa_7\alpha_1 - 2\kappa_4\alpha_3 - \kappa_1\alpha_3^2 + 2\kappa_2\alpha_6 + 4\kappa_1\alpha_1\alpha_6 = 0,\\ &\kappa_{12} - 4\kappa_6\alpha_1 + 2\kappa_4\alpha_2 + 2\kappa_3\alpha_3 + 2\kappa_1\alpha_2\alpha_3 - 2\kappa_2\alpha_5 - 4\kappa_1\alpha_1\alpha_5 = 0,\\ &\kappa_{13} + 2\kappa_5\alpha_1 - 2\kappa_3\alpha_2 - \kappa_1\alpha_2^2 + 2\kappa_2\alpha_4 + 4\kappa_1\alpha_1\alpha_4 = 0.\end{aligned}$$

Therefore, we can consider the equation

$$\begin{aligned}&\kappa_1\left(u_{,11}u_{,22}u_{,33} + 2u_{,12}u_{,13}u_{,23} - (u_{,13})^2u_{,22} - u_{,11}(u_{,23})^2 - (u_{,12})^2u_{,33}\right)\\ &\quad+\kappa_2\left(u_{,22}u_{,33} - (u_{,23})^2\right) + 2\kappa_3\left(u_{,13}u_{,23} - u_{,12}u_{,33}\right)\\ &\quad+2\kappa_4\left(u_{,12}u_{,23} - u_{,13}u_{,22}\right) + \kappa_5\left(u_{,11}u_{,33} - (u_{,13})^2\right)\\ &\quad+2\kappa_6\left(u_{,12}u_{,13} - u_{,11}u_{,23}\right) + \kappa_7\left(u_{,11}u_{,22} - (u_{,12})^2\right) + \kappa = 0,\end{aligned} \tag{8.11}$$

where κ is also a constant. In the following, let us choose the value $\kappa_1 = 1$. Lie symmetries of Eq. (8.11) for an arbitrary choice of the involved parameters span a Lie algebra of dimension 11, with basis given by the vector fields

$$\begin{aligned}&\Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_2}, \quad \Xi_3 = \frac{\partial}{\partial x_3},\\ &\Xi_4 = x_1\frac{\partial}{\partial u}, \quad \Xi_5 = x_2\frac{\partial}{\partial u}, \quad \Xi_6 = x_3\frac{\partial}{\partial u}, \quad \Xi_7 = \frac{\partial}{\partial u},\\ &\Xi_8 = (\kappa_3 x_1 + \kappa_5 x_2 + \kappa_6 x_3)\frac{\partial}{\partial x_1} - (\kappa_2 x_1 + \kappa_3 x_2 + \kappa_4 x_3)\frac{\partial}{\partial x_2},\\ &\Xi_9 = -(\kappa_5 x_2 + \kappa_6 x_3)\frac{\partial}{\partial x_1} + (\kappa_2 x_1 + 2\kappa_3 x_2 + \kappa_4 x_3)\frac{\partial}{\partial x_2} + \kappa_3 x_3\frac{\partial}{\partial x_3} + 2\kappa_3 u\frac{\partial}{\partial u},\end{aligned}$$

$$
\begin{aligned}
\Xi_{10} &= \left((2\kappa_3^2\kappa_6 - \kappa_3\kappa_4\kappa_5 - \kappa_2\kappa_5\kappa_6)x_2 + (2\kappa_3^2\kappa_7 - \kappa_2\kappa_6^2 - \kappa_3\kappa_4\kappa_6)x_3\right)\frac{\partial}{\partial x_1} \\
&+ \left((\kappa_2^2\kappa_6 - \kappa_2\kappa_3\kappa_4)x_1 + (\kappa_3\kappa_4^2 + \kappa_2\kappa_4\kappa_6 - 2\kappa_2\kappa_3\kappa_7)x_3\right)\frac{\partial}{\partial x_2} \\
&+ \left(2\kappa_3(\kappa_2\kappa_5 - \kappa_3^2)x_2 + 3\kappa_3(\kappa_2\kappa_6 - \kappa_3\kappa_4)x_3\right)\frac{\partial}{\partial x_3} \\
&+ 2\kappa_3(\kappa_2\kappa_6 - \kappa_3\kappa_4)u\frac{\partial}{\partial u}, \\
\Xi_{11} &= \left((\kappa_4\kappa_5^2 - \kappa_3\kappa_5\kappa_6)x_2 + (\kappa_4\kappa_5\kappa_6 + \kappa_3\kappa_6^2 - 2\kappa_3\kappa_5\kappa_7)x_3\right)\frac{\partial}{\partial x_1} \\
&+ \left((2\kappa_3^2\kappa_4 - \kappa_2\kappa_4\kappa_5 - \kappa_2\kappa_3\kappa_6)x_1 + (2\kappa_3^2\kappa_7 - \kappa_4^2\kappa_5 - \kappa_3\kappa_4\kappa_6)x_3\right)\frac{\partial}{\partial x_2} \\
&+ \left(2\kappa_3(\kappa_2\kappa_5 - \kappa_3^2)x_1 + 3\kappa_3(\kappa_4\kappa_5 - \kappa_3\kappa_6)x_3\right)\frac{\partial}{\partial x_3} \\
&+ 2\kappa_3(\kappa_4\kappa_5 - \kappa_3\kappa_6)u\frac{\partial}{\partial u},
\end{aligned}
$$

so that, in general, Eq. (8.11) is not Lie remarkable for dimensionality reasons.

Nevertheless, there are various choices of the parameters such that a higher number of symmetries is admitted. These cases may be discovered performing a group classification of Eq. (8.11). This task may be accomplished by using the program ReLie (see Chap. 12). In fact, in a Reduce session, the code

```
% 2nd order Monge-Amp\'{e}re equation in 3 independent variables
% Group classification
in "relie.red" $
xvar:={x1,x2,x3} $
uvar:={u} $
jetorder:=2 $
matrix a(3,3) $
a(1,1):=u_x1x1;
a(1,2):=u_x1x2;
a(1,3):=u_x1x3;
a(2,1):=u_x1x2;
a(2,2):=u_x2x2;
a(2,3):=u_x2x3;
a(3,1):=u_x1x3;
a(3,2):=u_x2x3;
a(3,3):=u_x3x3;
delta:=det(a);
diffeqs:={kappa1*delta+kappa2*df(delta,u_x1x1)
        +kappa3*df(delta,u_x1x2)+kappa4*df(delta,u_x1x3)
        +kappa5*df(delta,u_x2x2)+kappa6*df(delta,u_x2x3)
        +kappa7*df(delta,u_x3x3)+kappa} $
```

```
leaders:={u_x1x1} $

kappa1:= 1 $
freepars:={kappa2,kappa3,kappa4,kappa5,kappa6,kappa7,kappa} $
nonzeropars:={kappa2,kappa3,kappa4,kappa5,kappa6,kappa7} $
relie(4) $
;end;
```

is able to compute in a few seconds 22 sets of symmetries according to specific choices of the parameters. As a result, the following theorem can be stated.

Theorem 8.13 *The second-order Monge–Ampère equation in three independent variables*

$$\begin{aligned}&\left(u_{,11}u_{,22}u_{,33}+2u_{,12}u_{,13}u_{,23}-(u_{,13})^2u_{,22}-u_{,11}(u_{,23})^2-(u_{,12}^2)u_{,33}\right)\\&\quad+\kappa_2\left(u_{,22}u_{,33}-(u_{,23})^2\right)+2\kappa_3\left(u_{,13}u_{,23}-u_{,12}u_{,33}\right)\\&\quad+2\kappa_4\left(u_{,12}u_{,23}-u_{,13}u_{,22}\right)+\kappa_5\left(u_{,11}u_{,33}-(u_{,13})^2\right)\\&\quad+2\kappa_6\left(u_{,12}u_{,13}-u_{,11}u_{,23}\right)+\kappa_7\left(u_{,11}u_{,22}-(u_{,12})^2\right)+\kappa=0,\end{aligned}\tag{8.12}$$

where $\kappa_2,\ldots,\kappa_7$ *and* κ *are constants, is*

1. *weakly Lie remarkable when*
   (a) *the constants are such that*

$$\kappa_2=\frac{\kappa_3^2\kappa_7-2\kappa_3\kappa_4\kappa_6+\kappa_4^2\kappa_5}{\kappa_5\kappa_7-\kappa_6^2},\qquad\kappa=0,$$

   along with $\kappa_3\neq0$, $\kappa_4\neq0$, $\kappa_5\neq0$, $\kappa_6\neq0$, $\kappa_7\neq0$, *and* $\kappa_3\kappa_7-\kappa_4\kappa_6\neq0$;
   (b) *the constants are such that*

$$\kappa_3=\frac{\kappa_4\kappa_6}{\kappa_7},\qquad\kappa_5=\frac{\kappa_6^2}{\kappa_7},\qquad\kappa=0,$$

   along with $\kappa_2\neq0$, $\kappa_4\neq0$, $\kappa_6\neq0$, $\kappa_7\neq0$, *and* $\kappa_2\kappa_7-\kappa_4^2\neq0$;
   (c) *the constants are such that*

$$\kappa_2=\frac{\kappa_4^2}{\kappa_7},\qquad\kappa_3=\frac{\kappa_4\kappa_6}{\kappa_7},\qquad\kappa=0,$$

   along with $\kappa_4\neq0$, $\kappa_5\neq0$, $\kappa_6\neq0$, $\kappa_7\neq0$, *and* $\kappa_5\kappa_7-\kappa_6^2\neq0$;
   (d) *the constants are such that*

$$\kappa_2=\frac{\kappa_4^2\kappa_5}{\kappa_6},\qquad\kappa_3=\frac{\kappa_4(2\kappa_6^2-\kappa_5\kappa_7)}{\kappa_6\kappa_7},\qquad\kappa=0,$$

along with $\kappa_4 \neq 0, \kappa_5 \neq 0, \kappa_6 \neq 0, \kappa_7 \neq 0, \kappa_5\kappa_7 - \kappa_6^2 \neq 0$ *and* $\kappa_5\kappa_7 - 2\kappa_6^2 \neq 0$;

(e) *the constants are such that*

$$\kappa_2 = \frac{\kappa_4^2}{\kappa_7}, \qquad \kappa_3 = \frac{\kappa_4\kappa_6}{\kappa_7},$$

along with $\kappa_4 \neq 0, \kappa_6 \neq 0, \kappa_7 \neq 0, \kappa \neq 0$.

2. *strongly Lie remarkable when*

$$\kappa_2 = \frac{\kappa_4^2}{\kappa_7}, \qquad \kappa_3 = \frac{\kappa_4\kappa_6}{\kappa_7}, \qquad \kappa_5 = \frac{\kappa_6^2}{\kappa_7}, \qquad \kappa = 0,$$

along with $\kappa_4 \neq 0, \kappa_6 \neq 0, \kappa_7 \neq 0$.

Proof Equation (8.12) characterizes a 12-dimensional submanifold in the 13-dimensional jet space $J^2(\mathbb{R}^4, 3)$. In all the cases (a), (b), (c), (d) we have a 14-dimensional Lie algebra, whereas in the case (e) we have a 16-dimensional Lie algebra of symmetries; their second prolongations give distributions with rank 12 in the whole space. On the contrary, in the case 2, we have a 24-dimensional Lie algebra of point symmetries whose second-order prolongation gives a distribution with rank 13 in the whole space. □

Remark 8.12 A similar analysis can be exploited for the extension of Monge–Ampère equation in four independent variables; this equation has been derived in [15] by requiring once again the complete exceptionality condition, and a reciprocal transformation has been used to map it to linear form. Linearizable second-order Monge–Ampère equations have been also investigated in [32].

8.7 Lie Remarkability of Higher Order Monge–Ampère Equations

The third-order Monge–Ampère equation [10] for the unknown $u(x_1, x_2)$ is built by starting with the Hankel matrix

$$H_3 = \begin{pmatrix} u_{,111} & u_{,112} & u_{,122} \\ u_{,112} & u_{,122} & u_{,222} \end{pmatrix},$$

and taking a linear combination (with coefficients depending on independent variables, dependent variable and derivatives up to the order 2) of all minors extracted from the matrix H_3:

$$\kappa_1\left(u_{,112}u_{,222}-(u_{,122})^2\right)+\kappa_2\left(u_{,111}u_{,222}-u_{,112}u_{,122}\right)+\kappa_3\left(u_{,111}u_{,122}-(u_{,112})^2\right)$$
$$+\kappa_4 u_{,111}+\kappa_5 u_{,112}+\kappa_6 u_{,122}+\kappa_7 u_{,222}+\kappa_8=0.$$

By limiting ourselves to consider the coefficients $\kappa_1,\dots,\kappa_8$ to be constant and using the replacement

$$u\to u+\alpha_1 x_1^3+\alpha_2 x_1^2 x_2+\alpha_3 x_1 x_2^2+\alpha_4 x_2^3,$$

with suitable coefficients α_i, $i=1,\dots,4$, we are led to the equation

$$\kappa_1\left(u_{,112}u_{,222}-(u_{,122})^2\right)+\kappa_2\left(u_{,111}u_{,222}-u_{,112}u_{,122}\right)$$
$$+\kappa_3\left(u_{,111}u_{,122}-(u_{,112})^2\right)+\kappa=0, \tag{8.13}$$

where κ is constant.

For arbitrary choices of constants involved in (8.13), this equation admits a nine-dimensional Lie algebra of point symmetries; as Eq. (8.13) characterizes an 11-dimensional submanifold in the 12-dimensional jet space $J^3(\mathbb{R}^3,2)$, it follows (see Theorem 8.4) that Eq. (8.13) is not weakly Lie remarkable, and, a fortiori, it is not strongly Lie remarkable.

An interesting case is obtained by choosing $\kappa_1=\kappa_2^2/\kappa_3$, whereupon Eq. (8.13) becomes

$$\left(u_{,112}u_{,222}-(u_{,122})^2\right)+\lambda(u_{,111}u_{,222}-u_{,112}u_{,122})$$
$$+\lambda^2\left(u_{,111}u_{,122}-(u_{,112})^2\right)+\mu=0, \tag{8.14}$$

with $\lambda=\kappa_3/\kappa_2$, $\mu=\kappa\kappa_3/\kappa_2^2$. In such a case, we have an infinite-dimensional Lie algebra of symmetries spanned by

$$\begin{aligned}
&\Xi_1=\frac{\partial}{\partial x_1},\quad \Xi_2=\frac{\partial}{\partial x_2},\quad \Xi_3=\frac{\partial}{\partial u},\\
&\Xi_4=(x_1-\lambda x_2)\frac{\partial}{\partial x_1}+u\frac{\partial}{\partial u},\quad \Xi_5=(x_1-\lambda x_2)\frac{\partial}{\partial x_2}-\lambda u\frac{\partial}{\partial u},\\
&\Xi_6=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+3u\frac{\partial}{\partial u},\quad \Xi_7=x_1\frac{\partial}{\partial u},\\
&\Xi_8=x_2\frac{\partial}{\partial u},\quad \Xi_9=x_1^2\frac{\partial}{\partial u},\quad \Xi_{10}=x_1x_2\frac{\partial}{\partial u},\\
&\Xi_{11}=x_2^2\frac{\partial}{\partial u},\quad \Xi_f=f(x_1-\lambda x_2)\frac{\partial}{\partial u},
\end{aligned} \tag{8.15}$$

where $f(x_1-\lambda x_2)$ is an arbitrary function of its argument.

The following two theorems hold true.

Theorem 8.14 *Equation (8.14) is weakly Lie remarkable (provided that we remove a singular subset).*

Proof Choosing $f(x_1, x_2) = (x_1 - \lambda x_2)^3$, the third-order prolongation of (8.15) produces a distribution of rank 11 on the whole jet space, provided that we exclude:

- the 11-dimensional submanifold locally described by the equation

$$\lambda^3 u_{,111} + 3\lambda^2 u_{,112} + 3\lambda u_{,122} + u_{,222} = 0,$$

  where the rank of the distribution is 10;
- the nine-dimensional submanifolds locally described by systems:

$$\begin{aligned} &\lambda u_{,111} + u_{,112} = 0,\\ &u_{,122} + \lambda u_{,112} = 0,\\ &u_{,222} - \lambda^2 u_{,112} = 0, \end{aligned}$$

  where the rank of the distribution is 9.

□

Remark 8.13 The unique third-order differential invariant of (8.15), with $f(x_1, x_2) = (x_1 - \lambda x_2)^3$, up to functional dependence, is

$$J = \left(u_{,112}u_{,222} - (u_{,122})^2\right) + \lambda\left(u_{,111}u_{,222} - u_{,112}u_{,122}\right) + \lambda^2\left(u_{,111}u_{,122} - (u_{,112})^2\right).$$

So, the most general third-order scalar partial differential equation invariant by (8.15) must be given as $\Delta(J) = 0$, where Δ is an arbitrary function of the invariant.

Theorem 8.15 *Equation* (8.14) *with* $\mu = 0$, *namely*

$$\left(u_{,112}u_{,222} - (u_{,122})^2\right) + \lambda\left(u_{,111}u_{,222} - u_{,112}u_{,122}\right) + \lambda^2\left(u_{,111}u_{,122} - (u_{,112})^2\right) = 0$$

is strongly Lie remarkable.

Proof In fact, it admits an infinite-dimensional Lie algebra of symmetries spanned by

$$\begin{aligned} &\Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_2},\\ &\Xi_3 = \frac{\partial}{\partial u}, \quad \Xi_4 = (x_1 - \lambda x_2)\frac{\partial}{\partial x_1},\\ &\Xi_5 = \lambda^2 x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}, \quad \Xi_6 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2},\\ &\Xi_7 = x_1\frac{\partial}{\partial u}, \quad \Xi_8 = x_2\frac{\partial}{\partial u},\\ &\Xi_9 = x_1^2\frac{\partial}{\partial u}, \quad \Xi_{10} = x_1x_2\frac{\partial}{\partial u}, \end{aligned} \tag{8.16}$$

$$\Xi_{11} = x_2^2 \frac{\partial}{\partial u}, \quad \Xi_{12} = u \frac{\partial}{\partial u},$$
$$\Xi_{13} = (x_1 - \lambda x_2)\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 2u \frac{\partial}{\partial u}\right), \quad \Xi_f = f(x_1 - \lambda x_2) \frac{\partial}{\partial u},$$

where $f(x_1 - \lambda x_2)$ is an arbitrary function of its argument.

Choosing $f(x_1, x_2) = (x_1 - \lambda x_2)^3$, the third-order prolongation of (8.16) produces a distribution of rank 12 on the whole jet space, lowering to 11 on the equation at hand. □

8.7.1 Fourth-Order Monge–Ampère Equation

Finally, let us consider the fourth-order Monge–Ampère equation [10] in two independent variables; it is built by starting with the Hankel matrix

$$H_4 = \begin{pmatrix} u_{,1111} & u_{,1112} & u_{,1122} \\ u_{,1112} & u_{,1122} & u_{,1222} \\ u_{,1122} & u_{,1222} & u_{,2222} \end{pmatrix}$$

and taking a linear combination (with coefficients depending on independent variables, dependent variable, and derivatives up to the order 3) of all minors (including the determinant of H_4) extracted from the matrix H_4:

$$\begin{aligned} &\kappa_1 \det(H_4) + \kappa_2 \det(H_4^{(1,1)}) + \kappa_3 \det(H_4^{(1,2)}) + \kappa_4 \det(H_4^{(1,3)}) \\ &\quad + \kappa_5 \det(H_4^{(2,2)}) + \kappa_6 \det(H_4^{(2,3)}) + \kappa_7 \det(H_4^{(3,3)}) \\ &\quad + \kappa_8 u_{,1111} + \kappa_9 u_{,1112} + \kappa_{10} u_{,1122} + \kappa_{11} u_{,1222} + \kappa_{12} u_{,2222} + \kappa_{13} = 0, \end{aligned}$$

where $H_4^{(i,j)}$ denotes the 2×2 matrix obtained by removing from H_4 the ith row and the jth column. By limiting ourselves to consider the coefficients κ_i ($i = 1, \ldots, 13$) to be constant and using the replacement $u \to u + \alpha_1 x_1^4 + \alpha_2 x_1^3 x_2 + \alpha_3 x_1^2 x_2^2 + \alpha_4 x_1 x_2^3 + \alpha_5 x_2^4$, with suitable coefficients α_i, $i = 1, \ldots, 5$, we are led to

$$\begin{aligned} &\kappa_1 \det(H_4) + \kappa_2 \det(H_4^{(1,1)}) + \kappa_3 \det(H_4^{(1,2)}) + \kappa_4 \det(H_4^{(1,3)}) \\ &\quad + \kappa_5 \det(H_4^{(2,2)}) + \kappa_6 \det(H_4^{(2,3)}) + \kappa_7 \det(H_4^{(3,3)}) + \kappa = 0, \end{aligned} \tag{8.17}$$

where κ is constant.

The Lie algebra of point symmetries of Eq. (8.17) is 13-dimensional; as Eq. (8.17) defines in the jet space a submanifold of dimension 17, in general it is neither strongly nor weakly Lie remarkable. Nevertheless, if we consider the special case $\det(H_4) = 0$, then the following theorem holds true.

Theorem 8.16 *The equation*

$$\det(H_4) = 0 \tag{8.18}$$

is weakly Lie remarkable (provided that we remove a singular subset).

Proof The Lie algebra of the admitted point symmetries of Eq. (8.18) is 19-dimensional and is spanned by the vector fields

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_2}, \quad \Xi_3 = \frac{\partial}{\partial u}, \quad \Xi_4 = x_1\frac{\partial}{\partial u},\\
&\Xi_5 = x_2\frac{\partial}{\partial u}, \quad \Xi_6 = x_1^2\frac{\partial}{\partial u}, \quad \Xi_7 = x_1x_2\frac{\partial}{\partial u}, \quad \Xi_8 = x_2^2\frac{\partial}{\partial u},\\
&\Xi_9 = x_1^3\frac{\partial}{\partial u}, \quad \Xi_{10} = x_1^2x_2\frac{\partial}{\partial u}, \quad \Xi_{11} = x_1x_2^2\frac{\partial}{\partial u}, \quad \Xi_{12} = x_2^3\frac{\partial}{\partial u},\\
&\Xi_{13} = x_1\frac{\partial}{\partial x_1}, \quad \Xi_{14} = x_2\frac{\partial}{\partial x_1}, \quad \Xi_{15} = x_1\frac{\partial}{\partial x_2}, \quad \Xi_{16} = x_2\frac{\partial}{\partial x_2},\\
&\Xi_{17} = u\frac{\partial}{\partial u}, \quad \Xi_{18} = x_1\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 3u\frac{\partial}{\partial u}\right),\\
&\Xi_{19} = x_2\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + 3u\frac{\partial}{\partial u}\right).
\end{aligned}$$

The corresponding second-order prolonged vector fields yield a distribution of rank 16 (equal to the dimension of the equation) on the whole jet space, provided that we exclude

- the 15-dimensional submanifold

$$\begin{aligned}
&2u_{,1111}u_{,1222} = -3u_{,1112}u_{,1122} + \sqrt{(u_{,1112})^2(9(u_{,1122})^2 - 8u_{,1112}u_{,1222})},\\
&2(u_{,1112})^2u_{,2222} = u_{,1222}\left(-3u_{,1112}u_{,1122} + \sqrt{(u_{,1112})^2(9(u_{,1122})^2 - 8u_{,1112}u_{,1222})}\right);
\end{aligned}$$

- the 15-dimensional submanifold

$$\begin{aligned}
&2u_{,1111}u_{,1222} = 3u_{,1112}u_{,1122} + \sqrt{(u_{,1112})^2(9(u_{,1122})^2 - 8u_{,1112}u_{,1222})},\\
&2(u_{,1112})^2u_{,2222} = u_{,1222}\left(3u_{,1112}u_{,1122} + \sqrt{(u_{,1112})^2(9(u_{,1122})^2 - 8u_{,1112}u_{,1222})}\right);
\end{aligned}$$

- the 15-dimensional submanifold

$$\begin{aligned}
&u_{,1111}(u_{,1222})^2 = -2(u_{,1122})^3 + 3u_{,1112}u_{,1122}u_{,1222} - 2\sqrt{((u_{,1122})^2 - u_{,1112}u_{,1222})^3},\\
&(u_{,1112})^2u_{,2222} = -2(u_{,1122})^3 + 3u_{,1112}u_{,1122}u_{,1222} + 2\sqrt{((u_{,1122})^2 - u_{,1112}u_{,1222})^3};
\end{aligned}$$

- the 15-dimensional submanifold

$$u_{,1111}(u_{,1222})^2 = -2(u_{,1122})^3 + 3u_{,1112}u_{,1122}u_{,1222} + 2\sqrt{((u_{,1122})^2 - u_{,1112}u_{,1222})^3},$$
$$(u_{,1112})^2u_{,2222} = -2(u_{,1122})^3 + 3u_{,1112}u_{,1122}u_{,1222} - 2\sqrt{((u_{,1122})^2 - u_{,1112}u_{,1222})^3};$$

- the 14-dimensional submanifold

$$u_{,1111} = 0, \qquad u_{,1112} = 0,$$
$$3u_{,1122}u_{,2222} = 2(u_{,1222})^2;$$

- the 14-dimensional submanifold

$$u_{,1111} = 0, \qquad u_{,1112} = 0, \qquad u_{,1122} = 0;$$

- the 14-dimensional submanifold

$$u_{,1112} = 0, \qquad u_{,1222} = 0,$$
$$u_{,1111}u_{,2222} = 9(u_{,1122})^2;$$

- the 14-dimensional submanifold

$$u_{,1111}\sqrt{u_{,1222}} = -(u_{,1112})^{3/2},$$
$$u_{,1122} = -(\sqrt{u_{,1112}u_{,1222}}),$$
$$\sqrt{u_{,1112}}u_{,2222} = -(u_{,1222})^{3/2};$$

- the 14-dimensional submanifold

$$u_{,1111}\sqrt{u_{,1222}} = (u_{,1112})^{3/2},$$
$$u_{,1122} = \sqrt{u_{,1112}u_{,1222}},$$
$$\sqrt{u_{,1112}}u_{,2222} = (u_{,1222})^{3/2};$$

- the 14-dimensional submanifold

$$u_{,1111} = 0, \qquad \sqrt{3}u_{,1122} = -2\sqrt{u_{,1112}u_{,1222}},$$
$$3\sqrt{3}\sqrt{u_{,1112}}u_{,2222} = -4(u_{,1222})^{3/2};$$

- the 14-dimensional submanifold

$$u_{,1111} = 0, \qquad \sqrt{3}u_{,1122} = 2\sqrt{u_{,1112}u_{,1222}},$$
$$3\sqrt{3}\sqrt{u_{,1112}}u_{,2222} = 4(u_{,1222})^{3/2};$$

- the 14-dimensional submanifold

$$3u_{,1111}u_{,1122} = 2(u_{,1112})^2,$$
$$u_{,1222} = 0, \qquad u_{,2222} = 0;$$

- the 14-dimensional submanifold

$$4u_{,1111}u_{,1122} = 3(u_{,1112})^2,$$
$$u_{,1222} = 0, \qquad (u_{,1112})^2 u_{,2222} = -4(u_{,1122})^3;$$

- the 14-dimensional submanifold

$$u_{,1122} = 0, \qquad u_{,1222} = 0, \qquad u_{,2222} = 0;$$

- the 13-dimensional submanifold

$$u_{,1111} = 0, \qquad u_{,1112} = 0, \qquad u_{,1222} = 0, \qquad u_{,2222} = 0;$$

- the 13-dimensional submanifold

$$u_{,1111} = 0, \qquad u_{,1112} = 0, \qquad u_{,1122} = 0, \qquad u_{,1222} = 0;$$

- the 13-dimensional submanifold

$$u_{,1112} = 0, \qquad u_{,1122} = 0, \qquad u_{,1222} = 0, \qquad u_{,2222} = 0,$$

where the rank of the distribution of third-order prolonged vector fields lowers to 14 or 15. □

8.8 Lie Algebras and Corresponding Lie Remarkable Equations

Let us consider scalar partial differential equations, that is, according to our notations, restrict our attention to the case $m = 1$. We denote by $\mathfrak{i}(\mathbb{R}^{n+1})$, $\mathfrak{a}(\mathbb{R}^{n+1})$, $\mathfrak{p}(\mathbb{R}^{n+1})$ and $\mathfrak{c}(\mathbb{R}^{n+1})$, respectively, the isometric, affine, projective and conformal algebra of $\mathbb{R}^{n+1}$ with respect to the metrics $g = \sum_{i=1}^{n} k_i dx_i \otimes dx_i + du \otimes du$, where k_i $(i = 1, \ldots, n)$ are nonvanishing real constants. Even if the algebras $\mathfrak{i}(\mathbb{R}^{n+1})$ and $\mathfrak{c}(\mathbb{R}^{n+1})$ depend on the constants k_i, we shall continue to denote them by the same symbol. For instance, $\mathfrak{i}(\mathbb{R}^4)$ can represent both the Euclidean and Poincaré algebra for suitable values of the constants k_i. For each of the previous algebras we shall

determine strongly scalar partial differential equations of various order associated with them.

8.8.1 The Case of $\mathfrak{i}(\mathbb{R}^{n+1})$

It is easily ascertained that the Lie algebra $\mathfrak{i}(\mathbb{R}^{n+1})$ has dimension equal to $\frac{(n+1)(n+2)}{2}$; in fact, we have $(n+1)$ translations, and $\binom{n+1}{2}$ rotations. Then, in view of Theorem 8.5, strongly Lie remarkable equations can be only of order 1. In order to get them, we have to construct the matrix representing the distribution of first prolonged vector fields of isometries. If $n = 2$, we have the vector fields

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2},\\
&\Xi_3 = \frac{\partial}{\partial u}, \; \Xi_4 = k_2 x_2 \frac{\partial}{\partial x_1} - k_1 x_1 \frac{\partial}{\partial x_2},\\
&\Xi_5 = u\frac{\partial}{\partial x_1} - k_1 x_1 \frac{\partial}{\partial u}, \qquad \Xi_6 = u\frac{\partial}{\partial x_2} - k_2 x_2 \frac{\partial}{\partial u}.
\end{aligned} \tag{8.19}$$

The first prolongation of these vector fields gives the distribution on $J^1(\mathbb{R}^3, 2)$

$$M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0\\
y & -\frac{k_1}{k_2}x_1 & 0 & \frac{k_1}{k_2}u_{,2} & -u_{,1}\\
u & 0 & -k_1 x_1 & -k_1 - (u_{,1})^2 & -u_{,1}u_{,2}\\
0 & u & -k_2 x_2 & -u_{,1}u_{,2} & -k_2 - (u_{,2})^2
\end{pmatrix}$$

whose rank is 5 except for the four-dimensional submanifold locally described by

$$1 + \frac{(u_{,1})^2}{k_1} + \frac{(u_{,2})^2}{k_2} = 0;$$

the latter corresponds to the vanishing of the infinitesimal area element.

The strongly Lie remarkable equation characterized by $\mathfrak{i}(\mathbb{R}^3)$ in this case is nonempty provided that not all k_i are positive.

The generalization to arbitrary n is straightforward, and yields the strongly Lie remarkable equation

$$1 + \sum_{i=1}^{n} \frac{(u_{,i})^2}{k_i} = 0.$$

8.8.2 *The Case of* $\mathfrak{a}(\mathbb{R}^{n+1})$

The Lie algebra $\mathfrak{a}(\mathbb{R}^{n+1})$ of affine transformations has dimension $n^2 + 3n + 2$. By using Theorem 8.5, we see that a strongly Lie remarkable equation can be of order 2 or 3 in the case $n = 2$, and of order 2 if $n \geq 3$. The infinitesimal generators of this Lie algebra are

$$\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b},$$

$\forall\, a, b \in \{x_1, x_2, \ldots, x_n, u\}$.

We see that the strongly Lie remarkable second-order equation in $J^2(\mathbb{R}^3, 2)$ is the homogeneous Monge–Ampère equation

$$u_{,11}u_{,22} - (u_{,12})^2 = 0, \tag{8.20}$$

which describes surfaces of $\mathbb{R}^3$ with vanishing Gaussian curvature.

Moreover, there exists also a strongly Lie remarkable equation of third order in $J^3(\mathbb{R}^3, 2)$ which has the following local expression:

$$\begin{aligned}
&(u_{,11})^3(u_{,222})^2 + (u_{,111})^2(u_{,22})^3 + 6u_{,11}u_{,111}u_{,12}u_{,22}u_{,222} \\
&- 6u_{,111}u_{,112}u_{,12}(u_{,22})^2 - 6u_{,11}u_{,111}u_{,122}(u_{,22})^2 \\
&- 6(u_{,11})^2u_{,12}u_{,122}u_{,222} - 6(u_{,11})^2u_{,112}u_{,22}u_{,222} \\
&- 8u_{,111}(u_{,12}^3)u_{,222} + 9u_{,11}(u_{,112})^2(u_{,22})^2 \\
&+ 9(u_{,11})^2(u_{,122})^2u_{,22} + 12u_{,111}(u_{,12})^2u_{,122}u_{,22} \\
&+ 12u_{,11}u_{,112}(u_{,12})^2u_{,222} - 18u_{,11}u_{,112}u_{,12}u_{,122}u_{,22} = 0.
\end{aligned} \tag{8.21}$$

To the best of our knowledge, this equation has been described in literature for the first time in [31].

Now, it remains to find the equations that live in $J^2(\mathbb{R}^{n+1}, n)$ with $n \geq 3$. In this case, for each n, we still have a strongly Lie remarkable equation, namely

$$\det \begin{pmatrix} u_{,11} & u_{,12} & \cdots & u_{,1n} \\ u_{,12} & u_{,22} & \cdots & u_{,2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{,1n} & u_{,2n} & \cdots & u_{,nn} \end{pmatrix} = 0, \tag{8.22}$$

that is, the second-order homogeneous Monge–Ampère equation in n variables [9].

8.8.3 The Case of $\mathfrak{p}(\mathbb{R}^{n+1})$

The Lie algebra $\mathfrak{p}(\mathbb{R}^{n+1})$ of projective transformations has dimension n^2+4n+3. Its infinitesimal generators are

$$\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b}, \qquad a\left(\sum_{i=1}^{n} x^i\frac{\partial}{\partial x^i}+u\frac{\partial}{\partial u}\right), \tag{8.23}$$

for all $a, b \in \{x_1, x_2, \ldots, x_n, u\}$.

Then we easily realize that we have to look for strongly Lie remarkable equations in the same jet spaces we considered in the previous section. Furthermore, we have to discuss also the case of an equation that lives in $J^3(\mathbb{R}^4, 3)$. We realize that Eqs. (8.20), (8.21), and (8.22) are strongly Lie remarkable also in this case, and that there are no strongly Lie remarkable equations in $J^3(\mathbb{R}^4, 3)$. In fact, the prolongations of vector fields (8.23) span, at each point of $J^3(\mathbb{R}^4, 3)$, a subspace of dimension at most 20. As the dimension of the equation we are looking for is 22, in view of Theorem 8.4, we have no chance of finding a strongly Lie remarkable equation.

8.8.4 The Case of $\mathfrak{c}(\mathbb{R}^{n+1})$

The algebra $\mathfrak{c}(\mathbb{R}^{n+1})$ has dimension equal to $(n+2)(n+3)/2$.

We start with $n=2$. The infinitesimal generators of this algebra are the generators (8.19) together with

$$\begin{aligned}
\Xi_7 &= \frac{1}{2}\frac{k_1x_1^2-k_2x_2^2-u^2}{k_1}\frac{\partial}{\partial x_1}+x_1x_2\frac{\partial}{\partial x_2}+x_1u\frac{\partial}{\partial u},\\
\Xi_8 &= x_1x_2\frac{\partial}{\partial x_1}+\frac{1}{2}\frac{k_2x_2^2-k_1x_1^2-u^2}{k_1}\frac{\partial}{\partial x_2}+x_2u\frac{\partial}{\partial u},\\
\Xi_9 &= x_1u\frac{\partial}{\partial x_1}+x_2u\frac{\partial}{\partial x_2}+\frac{1}{2}(u^2-k_1x_1^2-k_2x_2^2)\frac{\partial}{\partial u},\\
\Xi_{10} &= x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+u\frac{\partial}{\partial u}.
\end{aligned} \tag{8.24}$$

Assuming the metrics $k_1dx_1\otimes dx_1+k_2dx_2\otimes dx_2+du\otimes du$ on $\mathbb{R}^3$, with k_i nonvanishing real constants, the (scalar) mean curvature of a generic surface $u=u(x_1,x_2)$ reads

$$H=\frac{1}{2}\frac{\left(k_2+(u_{,2})^2\right)u_{,11}-2u_{,1}u_{,2}u_{,12}+\left(k_1+(u_{,1})^2\right)u_{,22}}{\left(k_1k_2+k_2(u_{,1})^2+k_1(u_{,2})^2\right)^{3/2}},$$

and the Gaussian curvature is

$$G = \frac{u_{,11}u_{,22} - (u_{,12})^2}{\left(k_1k_2 + k_2(u_{,1})^2 + k_1(u_{,2})^2\right)^2}.$$

Then, by analyzing the rank of the matrix of second-order prolongations of the vector fields (8.19) and (8.24), we realize the unique second-order equation which is strongly Lie remarkable with respect to the conformal algebra is

$$G = H^2.$$

Remark 8.14 By a direct computation, we realize that the unique second-order scalar differential invariant J of the algebra formed by $\mathfrak{i}(\mathbb{R}^3)$ with the addition of homotheties of $\mathbb{R}^3$ is $J = H^2/G$. Then $J = k$, with $k \in \mathbb{R}$, is a weakly Lie remarkable equation in view of Theorem 8.1. Therefore we could look for strongly Lie remarkable equations among the equations $J = k$. In fact, from the above discussion, we realize that $J = 1$ is the strongly Lie remarkable equation we were looking for.

Finally, it remains to analyze the case $n = 3$. So, we have to search strongly Lie remarkable equations in the jet space $J^2(\mathbb{R}^4, 3)$. In this case, we realize that we do not have strongly Lie remarkable equations. In fact, the prolongations of vector fields forming the algebra $C(\mathbb{R}^4)$, at each point of $J^2(\mathbb{R}^4, 3)$, span a subspace of dimension at most 12. As the dimension of the equation we are looking for is 12, in view of Theorem 8.5, we cannot find a strongly Lie remarkable equation. The same negative result is achieved by considering the algebra $\mathfrak{c}(\mathbb{R}^{n+1})$ with $n > 3$.

8.9 Second-Order Lie Remarkable Systems of Partial Differential Equations

Now, let us consider the case of systems of partial differential equations that can be Lie remarkable.

The Lie algebra $\mathfrak{a}(\mathbb{R}^{n+m})$ of affine transformations of $\mathbb{R}^{n+m}$ is spanned by the vector fields

$$\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b},$$

where $a, b \in \{x_1, \ldots, x_n, u_1, \ldots, u_m\}$; it is

$$\dim(\mathfrak{a}(\mathbb{R}^{n+m})) = (m+n)(m+n+1);$$

on the contrary, as far as the second-order jet space is concerned, it is

$$\dim(J^2(\mathbb{R}^{n+m}, n)) = n + \frac{m(n+1)(n+2)}{2}.$$

Moreover, we have

$$\dim(\mathfrak{a}(\mathbb{R}^{n+m})) \geq \dim(J^2(\mathbb{R}^{n+m}, n))$$

for all m provided that $n \leq 5$. Nevertheless, in order to characterize strongly (weakly, respectively) Lie remarkable equations, it is necessary that the dimension of the Lie algebra be greater (greater or equal, respectively) than the dimension of the equation. In the following, we shall prove that strongly Lie remarkable equations corresponding to the Lie algebra of affine transformations of $\mathbb{R}^{n+m}$ do exist.

For $n = 2$, $m = 1$, we have already proved that the Monge–Ampère equation for a surface with vanishing Gaussian curvature is strongly Lie remarkable: the Lie algebra of affine transformations does suffice, even if the equation admits the Lie algebra of projective transformations of $\mathbb{R}^3$.

Moreover, we have shown that the unique second-order partial differential equation for the unknown $u(x_1, \ldots, x_n)$ $(n \geq 2)$, characterized by the Lie algebra of affine transformations of $\mathbb{R}^{n+1}$, is the homogeneous Monge–Ampère equation in n independent variables given by (8.22)

Now, let us consider the situation where $m > 1$, i.e., the case of a system of partial differential equations, and derive new strongly Lie remarkable equations.

Theorem 8.17 *The system of second-order partial differential equations locally described in* $J^2(\mathbb{R}^4, 2)$ *by*

$$\begin{cases} u_{1,11}u_{2,12} - u_{1,12}u_{2,11} = 0, \\ u_{1,11}u_{2,22} - u_{1,22}u_{2,11} = 0, \end{cases} \tag{8.25}$$

is strongly Lie remarkable.

Proof System (8.25) characterizes a 12-dimensional manifold in the second-order jet space $J^2(\mathbb{R}^4, 2)$, that has dimension 14. The dimension of the Lie algebra of affine transformations of $\mathbb{R}^4$ is 20, and, by a straightforward calculation, one can easily recognize that the rank of the distribution of its second-order prolongation is 14, reducing to 12 only on the 12-dimensional manifold of $J^2(\mathbb{R}^4, 2)$ characterized by (8.25); therefore, system (8.25) is strongly Lie remarkable. In passing, we note that system (8.25) admits also the symmetries generated by

$$a\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right),$$

where $a \in \{x_1, x_2, u_1, u_2\}$, i.e., it admits the 24-dimensional Lie algebra of projective transformations of $\mathbb{R}^4$. □

Remark 8.15 In a four-dimensional Euclidean space, the functions

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2) \tag{8.26}$$

characterize a two-dimensional surface. In [17], the authors introduced an invariant linear map of Weingarten type in the tangent space of the surface, which generates two invariants k and κ. These invariants divide the points of the surface into four types: flat, elliptic, parabolic, and hyperbolic. The two invariants (see [17] for the details) for the surface defined by (8.26) have the form

$$
\begin{aligned}
k &= \frac{4(u_{1,11}u_{2,12}-u_{1,12}u_{2,11})(u_{1,12}u_{2,22}-u_{1,22}u_{2,12})-(u_{1,11}u_{2,22}-u_{1,22}u_{2,11})^2}{(1+(u_{1,1})^2+(u_{1,2})^2+(u_{2,1})^2+(u_{2,2})^2+(u_{1,1}u_{2,2}+u_{1,2}u_{2,1})^2)^3},\\
\kappa &= \frac{(1+(u_{1,1})^2+(u_{2,1})^2)(u_{1,12}u_{2,22}-u_{1,22}u_{2,12})}{(1+(u_{1,1})^2+(u_{1,2})^2+(u_{2,1})^2+(u_{2,2})^2+(u_{1,1}u_{2,2}+u_{1,2}u_{2,1})^2)^2}\\
&\quad+\frac{(1+(u_{1,2})^2+(u_{2,2})^2)(u_{1,11}u_{2,12}-u_{1,12}u_{2,11})}{(1+(u_{1,1})^2+(u_{1,2})^2+(u_{2,1})^2+(u_{2,2})^2+(u_{1,1}u_{2,2}+u_{1,2}u_{2,1})^2)^2}\\
&\quad-\frac{(u_{1,1}u_{1,2}-u_{2,1}u_{2,2})(u_{1,11}u_{2,22}-u_{1,22}u_{2,11})}{(1+(u_{1,1})^2+(u_{1,2})^2+(u_{2,1})^2+(u_{2,2})^2+(u_{1,1}u_{2,2}+u_{1,2}u_{2,1})^2)^2}.
\end{aligned}
$$

It is immediately seen that both invariants vanish identically on Eq. (8.25), whereupon it remains proved that these equations characterize the surfaces consisting of flat points: such surfaces are either planar surfaces or developable ruled surfaces [17].

Second-order strongly Lie remarkable systems do exist also for $m > 2$ or $n > 2$. Before stating the theorem that identifies such systems for arbitrary values of m and n, let us consider some special cases.

Theorem 8.18 *The system of second-order partial differential equations locally described in* $J^2(\mathbb{R}^5, 3)$ *by*

$$
\begin{cases}
u_{1,11}u_{2,12}-u_{1,12}u_{2,11}=0,\\
u_{1,11}u_{2,13}-u_{1,13}u_{2,11}=0,\\
u_{1,11}u_{2,22}-u_{1,22}u_{2,11}=0,\\
u_{1,11}u_{2,23}-u_{1,23}u_{2,11}=0,\\
u_{1,11}u_{2,33}-u_{1,33}u_{2,11}=0,
\end{cases}
\tag{8.27}
$$

is strongly Lie remarkable.

Proof System (8.27) characterizes an 18-dimensional manifold in the second-order jet space $J^2(\mathbb{R}^5, 3)$, that has dimension 23. The dimension of the Lie algebra of affine transformations of $\mathbb{R}^5$ is 30, and, by a straightforward calculation, one can easily recognize that the rank of the distribution of its second-order prolongation is 23, reducing to 18 only on the 18-dimensional manifold of $J^2(\mathbb{R}^5, 3)$ characterized by (8.27), that, consequently, is strongly Lie remarkable. Actually, the system (8.27) admits the 35-dimensional Lie algebra of projective transformations of $\mathbb{R}^5$. □

Theorem 8.19 *The system of second-order partial differential equations locally described in* $J^2(\mathbb{R}^5, 2)$ *by*

$$\begin{cases} u_{1,11}u_{2,12} - u_{1,12}u_{2,11} = 0, \\ u_{1,11}u_{2,22} - u_{1,22}u_{2,11} = 0, \\ u_{2,11}u_{3,12} - u_{2,12}u_{3,11} = 0, \\ u_{2,11}u_{3,22} - u_{2,22}u_{3,11} = 0, \end{cases} \tag{8.28}$$

is strongly Lie remarkable.

Proof System (8.28) characterizes a 16-dimensional manifold in the second-order jet space $J^2(\mathbb{R}^5, 2)$, that has dimension 20. The dimension of the Lie algebra of affine transformations of $\mathbb{R}^5$ is 30, and, by a straightforward calculation, one can easily recognize that the rank of the distribution of its second-order prolongation is 20, reducing to 16 only on the 16-dimensional manifold of $J^2(\mathbb{R}^5, 2)$ characterized by (8.28), that, consequently, is strongly Lie remarkable. Actually, it is easily verified that system (8.28) admits the 35-dimensional Lie algebra of projective transformations of $\mathbb{R}^5$ too. □

For arbitrary values of m and n, we are able to state the following result.

Theorem 8.20 *The system of second-order partial differential equations in the unknowns* $(u_1(x_1, \ldots, x_n), \ldots, u_m(x_1, \ldots, x_n))$

$$\boldsymbol{\Delta} = \mathbf{0}, \tag{8.29}$$

with the components of $\boldsymbol{\Delta}$ *given by*

$$\Delta_{\alpha,\alpha+1;1,1,p,q} \equiv u_{\alpha,11}u_{\alpha+1,pq} - u_{\alpha,pq}u_{\alpha+1,11} = 0, \tag{8.30}$$

where $\alpha = 1, \ldots, m-1$ *and* $p, q = 1, \ldots, n$, *is strongly Lie remarkable.*

Proof System (8.29), made of $\dfrac{(m-1)(n^2+n-2)}{2}$ independent differential equations, characterizes a manifold with dimension d in the second-order jet space $J^2(\mathbb{R}^{n+m}, n)$, where

$$d = n^2 + 2n + 3m - 2 + \frac{mn(1-n)}{2}.$$

A simple analysis shows that $d < (n+m)(n+m+1)$ for all positive values of m and n, so that the number of symmetries in the affine Lie algebra of $\mathbb{R}^{n+m}$, according to Theorem 8.7, may suffice. Moreover, the rank of the distribution of the second-order prolongation of the generators of affine Lie algebra in $\mathbb{R}^{n+m}$ is maximal (the minimum between the dimension of affine Lie algebra and the dimension of jet space).

It is easy to recognize that from (8.30) it follows

$$\Delta_{\alpha,\beta;i,j,p,q} \equiv u_{\alpha,ij}u_{\beta,pq} - u_{\alpha,pq}u_{\beta+1,ij} = 0,$$

for all $\alpha, \beta \in \{1, \ldots, m\}$, and all $i, j, p, q \in \{1, \ldots, n\}$.

Now, let us prove that system (8.29), along with (8.30), admits as subalgebra of the algebra of its Lie point symmetries the Lie algebra of affine transformations. In the sequel, we use the Einstein summation convention over repeated indices.

Consider the infinitesimal generators of the affine Lie algebra in $\mathbb{R}^{n+m}$, say

$$\frac{\partial}{\partial x_i}, \quad \frac{\partial}{\partial u_\beta}, \quad x_j \frac{\partial}{\partial x_i}, \quad u_\beta \frac{\partial}{\partial x_i}, \quad x_i \frac{\partial}{\partial u_\beta}, \quad u_\gamma \frac{\partial}{\partial u_\beta}, \tag{8.31}$$

where $i, j = 1, \ldots, n$, and $\beta, \gamma = 1, \ldots, m$. We will prove that each of the vector fields in (8.31) is admitted by system (8.29).

As neither the independent variables nor the dependent ones appear explicitly in system (8.29), it is evident the invariance with respect to the $(n + m)$ translations of the independent and dependent variables. Let us consider the remaining vector fields.

1. $\Xi = x_j \dfrac{\partial}{\partial x_i}$. Its second-order prolongation is

$$\Xi^{(2)} = x_j \frac{\partial}{\partial x_i} - \delta_{jk} u_{\gamma,i} \frac{\partial}{\partial u_{\gamma,k}} - \left(\delta_{j\ell} u_{\gamma,ik} + \delta_{jk} u_{\gamma,i\ell}\right) \frac{\partial}{\partial u_{\gamma,k\ell}},$$

where δ_{ij} is the Kronecker symbol. It is

$$\begin{aligned}
\Xi^{(2)}\left(\Delta_{\alpha,\alpha+1;1,1,p,q}\right) &= 2\delta_{j1}\left(u_{\alpha,1i}u_{\alpha+1,pq} - u_{\alpha,pq}u_{\alpha+1,1i}\right) \\
&+ \delta_{jp}\left(u_{\alpha,11}u_{\alpha+1,iq} - u_{\alpha,iq}u_{\alpha+1,11}\right) \\
&+ \delta_{jq}\left(u_{\alpha,11}u_{\alpha+1,ip} - u_{\alpha,ip}u_{\alpha+1,11}\right) \\
&= 2\delta_{j1}\Delta_{\alpha,\alpha+1;1,i,p,q} + \delta_{jp}\Delta_{\alpha,\alpha+1;1,1,i,q} \\
&+ \delta_{jq}\Delta_{\alpha,\alpha+1;1,1,i,p},
\end{aligned}$$

that vanishes on $\boldsymbol{\Delta} = \mathbf{0}$;

2. $\Xi = u_\beta \dfrac{\partial}{\partial x_i}$. Its second-order prolongation is

$$\Xi^{(2)} = u_\beta \frac{\partial}{\partial x_i} - u_{\gamma,i}u_{\beta,k} \frac{\partial}{\partial u_{\gamma,k}} - \left(u_{\gamma,i}u_{\beta,k\ell} + u_{\beta,k}u_{\gamma,i\ell} + u_{\beta,\ell}u_{\gamma,ik}\right) \frac{\partial}{\partial u_{\gamma,k\ell}},$$

whereupon

$$\begin{aligned}
\Xi^{(2)}\left(\Delta_{\alpha,\alpha+1;1,1,p,q}\right) &= u_{\alpha,i}\left(u_{\beta,11}u_{\alpha+1,pq} - u_{\beta,pq}u_{\alpha+1,11}\right) \\
&+ u_{\alpha+1,i}\left(u_{\alpha,11}u_{\beta,pq} - u_{\alpha,pq}u_{\beta,11}\right) \\
&+ 2u_{\beta,1}\left(u_{\alpha,1i}u_{\alpha+1,pq} - u_{\alpha,pq}u_{\alpha+1,1i}\right) \\
&+ u_{\beta,p}\left(u_{\alpha,11}u_{\alpha+1,iq} - u_{\alpha,iq}u_{\alpha+1,11}\right) \\
&+ u_{\beta,q}\left(u_{\alpha,11}u_{\alpha+1,ip} - u_{\alpha,ip}u_{\alpha+1,11}\right)
\end{aligned}$$

$$
\begin{aligned}
&= u_{\alpha,i}\,\Delta_{\beta,\alpha+1;1,1,p,q} + u_{\alpha+1,i}\,\Delta_{\alpha,\beta;1,1,p,q}\\
&+ 2u_{\beta,1}\,\Delta_{\alpha,\alpha+1;1,i,p,q} + u_{\beta,p}\,\Delta_{\alpha,\alpha+1;1,1,i,q}\\
&+ u_{\beta,q}\,\Delta_{\alpha,\alpha+1;1,1,i,p},
\end{aligned}
$$

vanishing on $\mathbf{\Delta} = \mathbf{0}$;

3. $\Xi = x_i \dfrac{\partial}{\partial u_\beta}$. Its second-order prolongation reads

$$
\Xi^{(2)} = x_i \frac{\partial}{\partial u_\beta} + \delta_{\beta\gamma}\delta_{ik} \frac{\partial}{\partial u_{\gamma,k}},
$$

whereupon

$$
\Xi^{(2)} \left(\Delta_{\alpha,\alpha+1;1,1,p,q}\right) = 0;
$$

4. $\Xi = u_\gamma \dfrac{\partial}{\partial u_\beta}$. Its second-order prolongation is

$$
\Xi^{(2)} = u_\gamma \frac{\partial}{\partial u_\beta} + \delta_{\beta\mu} u_{\gamma,k} \frac{\partial}{\partial u_{\mu,k}} + \delta_{\beta\mu} u_{\gamma,k\ell} \frac{\partial}{\partial u_{\mu,k\ell}},
$$

whereupon

$$
\begin{aligned}
\Xi^{(2)} \left(\Delta_{\alpha,\alpha+1;1,1,p,q}\right) &= \delta_{\alpha\beta} \left(u_{\gamma,11} u_{\alpha+1,pq} - u_{\gamma,pq} u_{\alpha+1,11}\right)\\
&+ \delta_{(\alpha+1)\beta} \left(u_{\alpha,11} u_{\gamma,pq} - u_{\alpha,pq} u_{\gamma,11}\right)\\
&= \delta_{\alpha\beta}\,\Delta_{\gamma,\alpha+1;1,1,p,q} + \delta_{(\alpha+1)\beta}\,\Delta_{\alpha,\gamma;1,1,p,q},
\end{aligned}
$$

vanishing on $\mathbf{\Delta} = \mathbf{0}$.

□

We also observe that system (8.29), along with (8.30), is invariant with respect to the projective transformations of $\mathbb{R}^{n+m}$ too.

Proposition 8.2 *System* (8.29) *admits the symmetries generated by the vector fields*

$$
a \left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} u_\alpha \frac{\partial}{\partial u_\alpha} \right)
$$

with $a \in \{x_1, \ldots, x_n, u_1, \ldots, u_m\}$*, so that system* (8.29) *is strongly Lie remarkable also with respect to the Lie algebra of projective transformations of* $\mathbb{R}^{n+m}$.

Proof We need to prove that system (8.29) admits also the symmetries spanned by the $(n + m)$ vector fields

$$x_j\left(\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}+\sum_{\gamma=1}^{m} u_\gamma\frac{\partial}{\partial u_\gamma}\right),\qquad u_\beta\left(\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i}+\sum_{\gamma=1}^{m} u_\gamma\frac{\partial}{\partial u_\gamma}\right),$$

where $j=1,\dots,n$, and $\beta=1,\dots,m$.

1. $\Xi=x_j\left(x_i\dfrac{\partial}{\partial x_i}+u_\gamma\dfrac{\partial}{\partial u_\gamma}\right)$ (sum over repeated indices).
   Its second-order prolongation is

$$\begin{aligned}\Xi^{(2)}&=x_jx_i\frac{\partial}{\partial x_i}+x_ju_\gamma\frac{\partial}{\partial u_\gamma}+\delta_{jk}\left(u_\gamma-x_mu_{\gamma,m}\right)\frac{\partial}{\partial u_{\gamma,k}}\\&\quad-\left(\delta_{jk}u_{\gamma,m}u_{\gamma,\ell m}+\delta_{j\ell}x_mu_{\gamma,km}+x_ju_{\gamma,k\ell}\right)\frac{\partial}{\partial u_{\gamma,k\ell}}.\end{aligned}$$

   It is

$$\begin{aligned}\Xi^{(2)}\left(\Delta_{\alpha,\alpha+1;1,1,p,q}\right)&=\delta_{jp}x_m\left(u_{\alpha,mq}u_{\alpha+1,11}-u_{\alpha,11}u_{\alpha+1,mq}\right)\\&\quad+\delta_{jq}x_m\left(u_{\alpha,mp}u_{\alpha+1,11}-u_{\alpha,11}u_{\alpha+1,mp}\right)\\&\quad+2\delta_{j1}x_m\left(u_{\alpha,pq}u_{\alpha+1,1m}-u_{\alpha,1m}u_{\alpha+1,pq}\right)\\&\quad+2x_j\left(u_{\alpha,pq}u_{\alpha+1,11}-u_{\alpha,11}u_{\alpha+1,pq}\right)\\&=-x_m(\delta_{jp}\Delta_{\alpha,\alpha+1;1,1,m,q}\delta_{jq}+\Delta_{\alpha,\alpha+1;1,1,m,p}\\&\quad+2\delta_{j1}\Delta_{\alpha,\alpha+1;1,m,p,q})-2x_j\Delta_{\alpha,\alpha+1;1,1,p,q},\end{aligned}$$

   that vanishes on $\mathbf{\Delta}=\mathbf{0}$;
2. $\Xi=u_\beta\left(x_i\dfrac{\partial}{\partial x_i}+u_\gamma\dfrac{\partial}{\partial u_\gamma}\right)$ (sum over repeated indices).
   Its second-order prolongation is

$$\begin{aligned}\Xi^{(2)}&=u_\beta x_i\frac{\partial}{\partial x_i}+u_\beta u_\gamma\frac{\partial}{\partial u_\gamma}+u_{\beta,k}\left(u_\gamma-x_iu_{\gamma,i}\right)\frac{\partial}{\partial u_{\gamma,k}}\\&\quad+\left(u_\gamma u_{\beta,k\ell}-u_\beta u_{\gamma,k\ell}x_i(u_{\gamma,i}u_{\beta,k\ell}+u_{\beta,k}u_{\gamma,i\ell}+u_{\beta,l}u_{\gamma,ik})\right)\frac{\partial}{\partial u_{\gamma,k\ell}}.\end{aligned}$$

   It is

$$\begin{aligned}\Xi^{(2)}\left(\Delta_{\alpha,\alpha+1;1,1,p,q}\right)&=(u_\alpha-x_iu_{\alpha,i})\left(u_{\gamma,11}u_{\alpha+1,pq}-u_{\gamma,pq}u_{\alpha+1,11}\right)\\&\quad+(u_{\alpha+1}-x_iu_{\alpha+1,i})\left(u_{\alpha,11}u_{\gamma,pq}-u_{\alpha,pq}u_{\gamma,11}\right)\\&\quad-2u_\gamma\left(u_{\alpha,11}u_{\alpha+1,pq}-u_{\alpha,pq}u_{\alpha+1,11}\right)\\&\quad-2x_iu_{\gamma,1}\left(u_{\alpha,1i}u_{\alpha+1,pq}-u_{\alpha,pq}u_{\alpha+1,1i}\right)\\&\quad-x_iu_{\gamma,p}\left(u_{\alpha,11}u_{\alpha+1,iq}-u_{\alpha,iq}u_{\alpha+1,11}\right)\\&\quad-x_iu_{\gamma,q}\left(u_{\alpha,11}u_{\alpha+1,ip}-u_{\alpha,ip}u_{\alpha+1,11}\right)\end{aligned}$$

$$
\begin{aligned}
&= (u_\alpha - x_i u_{\alpha,i})\Delta_{\gamma,\alpha+1;1,1,p,q} \\
&+ (u_{\alpha+1} - x_i u_{\alpha+1,i})\Delta_{\alpha,\gamma;1,1,p,q} \\
&- 2u_\gamma \Delta_{\alpha,\alpha+1;1,1,p,q} - 2x_i u_{\gamma,1}\Delta_{\alpha,\alpha+1;1,i,p,q} \\
&- x_i u_{\gamma,p}\Delta_{\alpha,\alpha+1;1,1,i,q} - x_i u_{\gamma,q}\Delta_{\alpha,\alpha+1;1,1,i,p},
\end{aligned}
$$

that vanishes on $\boldsymbol{\Delta} = \mathbf{0}$.

□

8.10 Third-Order Lie Remarkable Systems of Partial Differential Equations

We already proved that for $n = 2$ and $m = 1$, Eq. (8.21), admitting the Lie algebra of projective transformations (even if the Lie algebra of affine transformations is sufficient to characterize it), is strongly Lie remarkable.

For $n = 3$ and $m = 1$, the dimension of Lie algebra of affine transformations is 20, whereas the dimension of the third-order jet space is 23, so that we cannot characterize a third-order Lie remarkable equation. Even if we take the Lie algebra of projective transformations (with dimension 24), neither strongly nor weakly Lie remarkable equations exist, because the rank of the distribution generated by the third-order prolongation of vector fields of projective transformations is 20.

For $n = 2$, the dimension of the Lie algebra of projective transformations in $\mathbb{R}^{n+m}$, $d = (m+2)(m+4)$, is greater than the dimension of the jet space, $j = 10m + 2$ for all values of $m > 0$; thus, we may try to construct third order strongly Lie remarkable equations involving two independent variables and an arbitrary number of dependent ones. Thus, in principle, we may conjecture that a hierarchy of Lie remarkable third-order systems involving the unknowns $u_\alpha(x_1, x_2)$ ($\alpha = 1, \ldots, m$, $m \geq 2$) may exist.

Taking $m = 2$, the computation can be easily done, and the following result can be stated.

Theorem 8.21 *The third-order system of partial differential equations*

$$
\begin{aligned}
&(u_{1,22}u_{2,12} - u_{1,12}u_{2,22})((u_{1,22}u_{2,11} - u_{1,12}u_{2,12})^2 \\
&\quad - (u_{1,11}u_{1,22} - (u_{1,12})^2)(u_{2,11}u_{2,22} - (u_{2,12})^2))u_{1,111} \\
&\quad + 3(u_{1,11}u_{2,12} - u_{1,12}u_{2,11})(u_{1,12}u_{2,12} - u_{1,22}u_{2,11}) \\
&\quad \times (u_{1,12}u_{2,22} - u_{1,22}u_{2,12})u_{1,112} \\
&\quad + 3(u_{1,11}u_{2,12} - u_{1,12}u_{2,11})^2(u_{1,22}u_{2,12} - u_{1,12}u_{2,22})u_{1,122} \\
&\quad + (u_{1,11}u_{2,12} - u_{1,12}u_{2,11})^2(u_{1,11}u_{2,22} - u_{1,12}u_{2,12})u_{1,222} \\
&\quad + (u_{1,11}u_{1,22} - (u_{1,12})^2)(u_{1,22}u_{2,11} - u_{1,11}u_{2,22})(u_{1,12}u_{2,22} - u_{1,22}u_{2,12})u_{2,111} \\
&\quad + 3((u_{1,12})^2 - u_{1,11}u_{1,22})(u_{1,12}u_{2,11} - u_{1,11}u_{2,12})(u_{1,12}u_{2,22} - u_{1,22}u_{2,12})u_{2,112}
\end{aligned}
$$

$$+((u_{1,12})^2-u_{1,11}u_{1,22})(u_{1,12}u_{2,11}-u_{1,11}u_{2,12})^2u_{2,222}=0, \tag{8.32a}$$

$$\begin{aligned}
&3(u_{1,22}u_{2,12}-u_{1,12}u_{2,22})^2(u_{2,11}u_{2,22}-(u_{2,12})^2)u_{1,112}\\
&+3(u_{1,11}u_{2,22}-u_{1,22}u_{2,11})(u_{1,22}u_{2,12}-u_{1,12}u_{2,22})(u_{2,11}u_{2,22}-(u_{2,12})^2)u_{1,122}\\
&+(u_{2,11}u_{2,22}-(u_{2,12})^2)((u_{1,11}u_{2,22}-u_{1,22}u_{2,11})^2\\
&+(u_{1,12}u_{2,22}-u_{1,22}u_{2,12})(u_{1,12}u_{2,11}-u_{1,11}u_{2,12}))u_{1,222}\\
&+(u_{1,22}u_{2,12}-u_{1,12}u_{2,22})^3u_{2,111}\\
&+3(u_{1,12}u_{2,12}-u_{1,22}u_{2,11})(u_{1,22}u_{2,12}-u_{1,12}u_{2,22})^2u_{2,112}\\
&+3(u_{1,22}u_{2,12}-u_{1,12}u_{2,22})((u_{1,22}u_{2,11}-u_{1,12}u_{2,12})^2\\
&-(u_{1,11}u_{1,22}-(u_{1,12})^2)(u_{2,11}u_{2,22}-(u_{2,12})^2))u_{2,122}\\
&+((u_{1,12}u_{2,12}-u_{1,22}u_{2,11})^3-(u_{1,11}u_{2,22}+u_{1,12}u_{2,12}-2u_{1,22}u_{2,11})\\
&\times(u_{1,11}u_{1,22}-(u_{1,12})^2)(u_{2,11}u_{2,22}-(u_{2,12}^2)))u_{2,222}=0,
\end{aligned} \tag{8.32b}$$

characterizing a 20-dimensional submanifold in $J^3(\mathbb{R}^4,2)$, is strongly Lie remarkable with respect to the Lie algebra of projective transformations of $\mathbb{R}^4$.

Proof In fact, the rank of the distribution of the third-order prolongations of the vector fields of Lie algebra of projective transformations of $\mathbb{R}^4$ is maximal (and equal to the dimension of third-order jet space). Moreover, the rank lowers to 20 when evaluated on the submanifold described by (8.32). □

For higher values of m, the amount of computation required to look for possible third order strongly Lie remarkable equations is rapidly increasing.

8.11 Lie Remarkable Ordinary Differential Equations

The definition of Lie remarkable equations of course applies also to ordinary differential equations [27]. In this case, there are important consequences related to the existence of first integrals (see Sect. 5.8 in Chap. 5) because Lie remarkability requires the existence of a sufficient number of symmetries.

8.11.1 Primitive Lie Algebras of Vector Fields in $\mathbb{R}^2$

Let us start considering the primitive Lie algebras of vector fields in $\mathbb{R}^2$ [37], and classify Lie remarkable scalar ordinary differential equations [27].

I $\quad \mathbb{R}\ltimes\mathbb{R}^2$

$$\frac{\partial}{\partial x},\quad \frac{\partial}{\partial u},\quad u\frac{\partial}{\partial x}-x\frac{\partial}{\partial u}+\alpha\left(x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}\right),\quad \alpha\in\mathbb{R}.$$

This Lie algebra is three-dimensional: possible Lie remarkable equation can be of first and second order. Anyway, it is easily ascertained that there are no Lie remarkable equations.

II $\mathfrak{sl}(2)$

$$\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad (x^2 - u^2)\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}.$$

No Lie remarkable equation does exist.

III $\mathfrak{so}(3)$

$$u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, \quad (1 + x^2 - u^2)\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u},$$
$$2xu\frac{\partial}{\partial x} + (1 - x^2 + u^2)\frac{\partial}{\partial u}.$$

No Lie remarkable equation does exist.

IV $\mathbb{R}^2 \ltimes \mathbb{R}^2$

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}.$$

The only Lie remarkable equation is $u^{(2)} = 0$.

V $\mathfrak{sa}(2)$

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial u}.$$

The only Lie remarkable equation is $u^{(2)} = 0$.

VI $\mathfrak{a}(2)$

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial x}, \quad u\frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial u}.$$

The Lie remarkable equation associated to this Lie algebra are $u^{(2)} = 0$ and the equation of the vanishing affine curvature [46]

$$3u^{(2)}u^{(4)} - 5(u^{(3)})^2 = 0,$$

also known as the generalized Kummer–Schwarz equation [24].

VII $\mathfrak{so}(3, 1)$

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},$$
$$(x^2 - u^2)\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}, \quad 2xu\frac{\partial}{\partial x} + (x^2 - u^2)\frac{\partial}{\partial u}.$$

This algebra gives the Lie remarkable equation of circles, namely

$$(1 + (u^{(1)})^2)u^{(3)} - 3u^{(1)}(u^{(2)})^2 = 0.$$

VIII $\mathfrak{sl}(3)$

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u}, \quad x\frac{\partial}{\partial x}, \quad u\frac{\partial}{\partial u}, \quad u\frac{\partial}{\partial x}, \quad x\frac{\partial}{\partial u},$$
$$x^2\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \quad xu\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u}.$$

This algebra (projective Lie algebra of $\mathbb{R}^2$ that sends a conic curve into a conic curve) characterizes as Lie remarkable equations once again $u^{(2)} = 0$, and the equation of conic sections, namely

$$9u^{(5)}(u^{(2)})^2 + 40(u^{(3)})^3 - 45u^{(2)}u^{(3)}u^{(4)} = 0.$$

8.11.2 Lie Remarkable Systems of Ordinary Differential Equations

Here, we consider the isometry $\mathfrak{i}(\mathbb{R}^3)$ (dimension 6), affine $\mathfrak{a}(\mathbb{R}^3)$ (dimension 12), conformal $\mathfrak{c}(\mathbb{R}^3)$ (dimension 10), and projective $\mathfrak{p}(\mathbb{R}^3)$ (dimension 15) Lie algebras of $\mathbb{R}^3$. Thus, we limit ourselves to systems of two ordinary differential equations for the unknowns $u_1(x)$ and $u_2(x)$.

8.11.2.1 Lie Algebra $\mathfrak{i}(\mathbb{R}^3)$

The generators of the Lie Algebra $\mathfrak{i}(\mathbb{R}^3)$ are

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial u_1}, \quad \frac{\partial}{\partial u_2}, \quad u_1\frac{\partial}{\partial x} - x\frac{\partial}{\partial u_1}, \quad u_2\frac{\partial}{\partial x} - x\frac{\partial}{\partial u_2}.$$

The Lie remarkable equations associated to this Lie algebra are

$$u_1^{(2)} = 0, \qquad u_2^{(2)} = 0.$$

We observe that for any number of unknowns, the Lie remarkable equations associated to the algebra $\mathfrak{i}(\mathbb{R}^{m+1})$ are

$$u_k^{(2)} = 0, \qquad k = 1, \ldots, m.$$

8.11.2.2 Lie Algebra $\mathfrak{a}(\mathbb{R}^3)$

The generators of the Lie Algebra $\mathfrak{a}(\mathbb{R}^3)$ are

$$\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b}, \qquad a, b \in \{x, u_1, u_2\},$$

and we are able to characterize the following fifth-order system of ordinary differential equations:

$$\begin{aligned}
u_1^{(5)} = &\Big(5(42(u_1^{(3)})^2(u_1^{(4)})^2(u_2^{(2)})^3 - u_1^{(2)}(u_1^{(4)}u_2^{(2)})^3 - 72(u_1^{(3)})^3u_1^{(4)}(u_2^{(2)})^2u_2^{(3)} \\
&- 78u_1^{(2)}u_1^{(3)}(u_1^{(4)})^2(u_2^{(2)})^2u_2^{(3)} + 144u_1^{(2)}(u_1^{(3)})^2u_1^{(4)}u_2^{(2)}(u_2^{(3)})^2 \\
&+ 36(u_1^{(2)})^2(u_1^{(4)})^2u_2^{(2)}(u_2^{(3)})^2 - 72(u_1^{(2)})^2u_1^{(3)}u_1^{(4)}(u_2^{(3)})^3 \\
&+ 72(u_1^{(3)})^4(u_2^{(2)})^2u_2^{(4)} - 48u_1^{(2)}(u_1^{(3)})^2u_1^{(4)}(u_2^{(2)})^2u_2^{(4)} \\
&+ 3(u_1^{(2)})^2(u_1^{(4)})^2(u_2^{(2)})^2u_2^{(4)} - 144u_1^{(2)}(u_1^{(3)})^3u_2^{(2)}u_2^{(3)}u_2^{(4)} \\
&+ 84(u_1^{(2)})^2u_1^{(3)}u_1^{(4)}u_2^{(2)}u_2^{(3)}u_2^{(4)} + 72(u_1^{(2)})^2(u_1^{(3)})^2(u_2^{(3)})^2u_2^{(4)} \\
&- 36(u_1^{(2)})^3u_1^{(4)}(u_2^{(3)})^2u_2^{(4)} + 6(u_1^{(2)})^2(u_1^{(3)})^2u_2^{(2)}(u_2^{(4)})^2 \\
&- 3(u_1^{(2)})^3u_1^{(4)}u_2^{(2)}(u_2^{(4)})^2 - 6(u_1^{(2)})^3u_1^{(3)}u_2^{(3)}(u_2^{(4)})^2 \\
&+ (u_1^{(2)})^4(u_2^{(4)})^3\Big)\Big(144(u_1^{(3)}u_2^{(2)} - u_1^{(2)}u_2^{(3)})^3\Big)^{-1}, \\
u_2^{(5)} = &(5(-(u_1^{(4)})^3(u_2^{(2)})^4 + 6u_1^{(3)}(u_1^{(4)})^2(u_2^{(2)})^3u_2^{(3)} - 72(u_1^{(3)})^2u_1^{(4)}(u_2^{(2)})^2(u_2^{(3)})^2 \\
&- 6u_1^{(2)}(u_1^{(4)})^2(u_2^{(2)})^2(u_2^{(3)})^2 + 144u_1^{(2)}u_1^{(3)}u_1^{(4)}u_2^{(2)}(u_2^{(3)})^3 \\
&- 72(u_1^{(2)})^2u_1^{(4)}(u_2^{(3)})^4 + 36(u_1^{(3)})^2u_1^{(4)}(u_2^{(2)})^3u_2^{(4)} \\
&+ 3u_1^{(2)}(u_1^{(4)})^2(u_2^{(2)})^3u_2^{(4)} + 72(u_1^{(3)})^3(u_2^{(2)})^2u_2^{(3)}u_2^{(4)} \\
&- 84u_1^{(2)}u_1^{(3)}u_1^{(4)}(u_2^{(2)})^2u_2^{(3)}u_2^{(4)} - 144u_1^{(2)}(u_1^{(3)})^2u_2^{(2)}(u_2^{(3)})^2u_2^{(4)} \\
&+ 48(u_1^{(2)})^2u_1^{(4)}u_2^{(2)}(u_2^{(3)})^2u_2^{(4)} + 72(u_1^{(2)})^2u_1^{(3)}(u_2^{(3)})^3u_2^{(4)} \\
&- 36u_1^{(2)}(u_1^{(3)})^2(u_2^{(2)})^2(u_2^{(4)})^2 - 3(u_1^{(2)})^2u_1^{(4)}(u_2^{(2)})^2(u_2^{(4)})^2 \\
&+ 78(u_1^{(2)})^2u_1^{(3)}u_2^{(2)}u_2^{(3)}(u_2^{(4)})^2 - 42(u_1^{(2)})^3(u_2^{(3)})^2(u_2^{(4)})^2 \\
&+ (u_1^{(2)})^3u_2^{(2)}(u_2^{(4)})^3))\Big(144(u_1^{(3)}u_2^{(2)} - u_1^{(2)}u_2^{(3)})^3\Big)^{-1}.
\end{aligned}$$

8.11.2.3 Lie Algebra $\mathfrak{c}(\mathbb{R}^3)$

Conformal vector fields of the Euclidean space $\mathbb{R}^n$ form a Lie algebra of dimension $\frac{1}{2}(n+1)(n+2)$ except for the case $n = 2$. In fact, in such a case, the dimension of the Lie algebra of conformal vector fields is infinite, as the latter are essentially identified with holomorphic functions of one complex variable. More precisely, if

$X_1 + iX_2$ is a holomorphic function, then $X_1\dfrac{\partial}{\partial x_1} + X_2\dfrac{\partial}{\partial x_2}$ is a conformal vector field of the Euclidean metrics. If $n \geq 3$, the conformal algebra is formed by translations, rotations, the dilatation, and special conformal vector fields. In $\mathbb{R}^3$ we have the vector fields

$$
\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = \frac{\partial}{\partial u_1}, \quad \Xi_3 = \frac{\partial}{\partial u_2},\\
&\Xi_4 = u_1\frac{\partial}{\partial x} - x\frac{\partial}{\partial u_1}, \quad \Xi_5 = u_2\frac{\partial}{\partial x} - x\frac{\partial}{\partial u_2},\\
&\Xi_6 = u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2}, \quad \Xi_7 = x\frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2},\\
&\Xi_8 = (x^2 - u_1^2 - u_2^2)\frac{\partial}{\partial x} + 2xu_1\frac{\partial}{\partial u_1} + 2x\frac{\partial}{\partial u_2},\\
&\Xi_9 = 2xu_1\frac{\partial}{\partial x} + (u_1^2 - x^2 - u_2^2)\frac{\partial}{\partial u_1} + 2u_1u_2\frac{\partial}{\partial u_2},\\
&\Xi_{10} = 2xu_2\frac{\partial}{\partial u_1} + 2u_1u_2\frac{\partial}{\partial u_1} + (u_2^2 - x^2 - u_1^2)\frac{\partial}{\partial u_2}.
\end{aligned}
$$

To the conformal Lie algebra in $\mathbb{R}^3$ it is associated the third-order system of circles

$$
\begin{aligned}
\left(1 + (u_1^{(1)})^2 + (u_1^{(2)})^2\right) u_1^{(3)} &= 3\left(u_1^{(1)}u_1^{(2)} + u_2^{(1)}u_2^{(2)}\right) u_1^{(2)},\\
\left(1 + (u_1^{(1)})^2 + (u_1^{(2)})^2\right) u_2^{(3)} &= 3\left(u_1^{(1)}u_1^{(2)} + u_2^{(1)}u_2^{(2)}\right) u_2^{(2)}.
\end{aligned}
$$

8.11.2.4 Lie Algebra $\mathfrak{p}(\mathbb{R}^3)$

Finally, to the algebra of projective transformations in $\mathbb{R}^3$ there are associated the equations

$$u_1^{(2)} = 0, \qquad u_2^{(2)} = 0,$$

and the sixth-order system

$$
\begin{aligned}
u_1^{(6)} = \Big(&-144(u_1^{(2)})^5(u_2^{(3)})^2(u_2^{(5)})^2 + 360(u_1^{(2)})^5u_2^{(3)}(u_2^{(4)})^2u_2^{(5)}\\
&- 225(u_1^{(2)})^5(u_2^{(4)})^4 + 288(u_1^{(2)})^4u_1^{(3)}u_2^{(2)}u_2^{(3)}(u_2^{(5)})^2\\
&- 360(u_1^{(2)})^4u_1^{(3)}u_2^{(2)}(u_2^{(4)})^2u_2^{(5)} - 480(u_1^{(2)})^4u_1^{(3)}(u_2^{(3)})^2u_2^{(4)}u_2^{(5)}\\
&+ 600(u_1^{(2)})^4u_1^{(3)}u_2^{(3)}(u_2^{(4)})^3 - 720(u_1^{(2)})^4u_1^{(4)}u_2^{(2)}u_2^{(3)}u_2^{(4)}u_2^{(5)}\\
&+ 900(u_1^{(2)})^4u_1^{(4)}u_2^{(2)}(u_2^{(4)})^3 + 960(u_1^{(2)})^4u_1^{(4)}(u_2^{(3)})^3u_2^{(5)}\\
&- 1200(u_1^{(2)})^4u_1^{(4)}(u_2^{(3)})^2(u_2^{(4)})^2 + 288(u_1^{(2)})^4u_1^{(5)}u_2^{(2)}(u_2^{(3)})^2u_2^{(5)}\\
&- 360(u_1^{(2)})^4u_1^{(5)}u_2^{(2)}u_2^{(3)}(u_2^{(4)})^2 + 240(u_1^{(2)})^4u_1^{(5)}(u_2^{(3)})^3u_2^{(4)}
\end{aligned}
$$

$$
\begin{aligned}
&- 144(u_1^{(2)})^3(u_1^{(3)})^2(u_2^{(2)})^2(u_2^{(5)})^2 + 960(u_1^{(2)})^3(u_1^{(3)})^2u_2^{(2)}u_2^{(3)}u_2^{(4)}u_2^{(5)}\\
&- 600(u_1^{(2)})^3(u_1^{(3)})^2u_2^{(2)}(u_2^{(4)})^3 - 640(u_1^{(2)})^3(u_1^{(3)})^2(u_2^{(3)})^3u_2^{(5)}\\
&+ 400(u_1^{(2)})^3(u_1^{(3)})^2(u_2^{(3)})^2(u_2^{(4)})^2 + 720(u_1^{(2)})^3u_1^{(3)}u_1^{(4)}(u_2^{(2)})^2u_2^{(4)}u_2^{(5)}\\
&- 2400(u_1^{(2)})^3u_1^{(3)}u_1^{(4)}u_2^{(2)}(u_2^{(3)})^2u_2^{(5)} + 600(u_1^{(2)})^3u_1^{(3)}u_1^{(4)}u_2^{(2)}u_2^{(3)}(u_2^{(4)})^2\\
&+ 400(u_1^{(2)})^3u_1^{(3)}u_1^{(4)}(u_2^{(3)})^3u_2^{(4)} - 576(u_1^{(2)})^3u_1^{(3)}u_1^{(5)}(u_2^{(2)})^2u_2^{(3)}u_2^{(5)}\\
&+ 360(u_1^{(2)})^3u_1^{(3)}u_1^{(5)}(u_2^{(2)})^2(u_2^{(4)})^2 - 240(u_1^{(2)})^3u_1^{(3)}u_1^{(5)}u_2^{(2)}(u_2^{(3)})^2u_2^{(4)}\\
&+ 640(u_1^{(2)})^3u_1^{(3)}u_1^{(5)}(u_2^{(3)})^4 + 360(u_1^{(2)})^3(u_1^{(4)})^2(u_2^{(2)})^2u_2^{(3)}u_2^{(5)}\\
&- 1350(u_1^{(2)})^3(u_1^{(4)})^2(u_2^{(2)})^2(u_2^{(4)})^2 + 2400(u_1^{(2)})^3(u_1^{(4)})^2u_2^{(2)}(u_2^{(3)})^2u_2^{(4)}\\
&- 800(u_1^{(2)})^3(u_1^{(4)})^2(u_2^{(3)})^4 + 720(u_1^{(2)})^3u_1^{(4)}u_1^{(5)}(u_2^{(2)})^2u_2^{(3)}u_2^{(4)}\\
&- 1200(u_1^{(2)})^3u_1^{(4)}u_1^{(5)}u_2^{(2)}(u_2^{(3)})^3 - 144(u_1^{(2)})^3(u_1^{(5)})^2(u_2^{(2)})^2(u_2^{(3)})^2\\
&- 480(u_1^{(2)})^2(u_1^{(3)})^3(u_2^{(2)})^2u_2^{(4)}u_2^{(5)} + 1920(u_1^{(2)})^2(u_1^{(3)})^3u_2^{(2)}(u_2^{(3)})^2u_2^{(5)}\\
&- 800(u_1^{(2)})^2(u_1^{(3)})^3u_2^{(2)}u_2^{(3)}(u_2^{(4)})^2 + 1920(u_1^{(2)})^2(u_1^{(3)})^2u_1^{(4)}(u_2^{(2)})^2u_2^{(3)}u_2^{(5)}\\
&+ 600(u_1^{(2)})^2(u_1^{(3)})^2u_1^{(4)}(u_2^{(2)})^2(u_2^{(4)})^2 - 2000(u_1^{(2)})^2(u_1^{(3)})^2u_1^{(4)}u_2^{(2)}(u_2^{(3)})^2u_2^{(4)}\\
&+ 288(u_1^{(2)})^2(u_1^{(3)})^2u_1^{(5)}(u_2^{(2)})^3u_2^{(5)} - 240(u_1^{(2)})^2(u_1^{(3)})^2u_1^{(5)}(u_2^{(2)})^2u_2^{(3)}u_2^{(4)}\\
&- 1920(u_1^{(2)})^2(u_1^{(3)})^2u_1^{(5)}u_2^{(2)}(u_2^{(3)})^3 - 360(u_1^{(2)})^2u_1^{(3)}(u_1^{(4)})^2(u_2^{(2)})^3u_2^{(5)}\\
&- 3000(u_1^{(2)})^2u_1^{(3)}(u_1^{(4)})^2(u_2^{(2)})^2u_2^{(3)}u_2^{(4)} + 2800(u_1^{(2)})^2u_1^{(3)}(u_1^{(4)})^2u_2^{(2)}(u_2^{(3)})^3\\
&- 720(u_1^{(2)})^2u_1^{(3)}u_1^{(4)}u_1^{(5)}(u_2^{(2)})^3u_2^{(4)} + 3120(u_1^{(2)})^2u_1^{(3)}u_1^{(4)}u_1^{(5)}(u_2^{(2)})^2(u_2^{(3)})^2\\
&+ 288(u_1^{(2)})^2u_1^{(3)}(u_1^{(5)})^2(u_2^{(2)})^3u_2^{(3)} + 900(u_1^{(2)})^2(u_1^{(4)})^3(u_2^{(2)})^3u_2^{(4)}\\
&- 1200(u_1^{(2)})^2(u_1^{(4)})^3(u_2^{(2)})^2(u_2^{(3)})^2 - 360(u_1^{(2)})^2(u_1^{(4)})^2u_1^{(5)}(u_2^{(2)})^3u_2^{(3)}\\
&- 1920u_1^{(2)}(u_1^{(3)})^4(u_2^{(2)})^2u_2^{(3)}u_2^{(5)} + 400u_1^{(2)}(u_1^{(3)})^4(u_2^{(2)})^2(u_2^{(4)})^2\\
&- 480u_1^{(2)}(u_1^{(3)})^3u_1^{(4)}(u_2^{(2)})^3u_2^{(5)} + 2800u_1^{(2)}(u_1^{(3)})^3u_1^{(4)}(u_2^{(2)})^2u_2^{(3)}u_2^{(4)}\\
&+ 240u_1^{(2)}(u_1^{(3)})^3u_1^{(5)}(u_2^{(2)})^3u_2^{(4)} + 1920u_1^{(2)}(u_1^{(3)})^3u_1^{(5)}(u_2^{(2)})^2(u_2^{(3)})^2\\
&+ 600u_1^{(2)}(u_1^{(3)})^2(u_1^{(4)})^2(u_2^{(2)})^3u_2^{(4)} - 3200u_1^{(2)}(u_1^{(3)})^2(u_1^{(4)})^2(u_2^{(2)})^2(u_2^{(3)})^2\\
&- 2640u_1^{(2)}(u_1^{(3)})^2u_1^{(4)}u_1^{(5)}(u_2^{(2)})^3u_2^{(3)} - 144u_1^{(2)}(u_1^{(3)})^2(u_1^{(5)})^2(u_2^{(2)})^4\\
&+ 1800u_1^{(2)}u_1^{(3)}(u_1^{(4)})^3(u_2^{(2)})^3u_2^{(3)} + 360u_1^{(2)}u_1^{(3)}(u_1^{(4)})^2u_1^{(5)}(u_2^{(2)})^4\\
&- 225u_1^{(2)}(u_1^{(4)})^4(u_2^{(2)})^4 + 640(u_1^{(3)})^5(u_2^{(2)})^3u_2^{(5)} - 1200(u_1^{(3)})^4u_1^{(4)}(u_2^{(2)})^3u_2^{(4)}\\
&- 640(u_1^{(3)})^4u_1^{(5)}(u_2^{(2)})^3u_2^{(3)} + 1200(u_1^{(3)})^3(u_1^{(4)})^2(u_2^{(2)})^3u_2^{(3)}\\
&+ 720(u_1^{(3)})^3u_1^{(4)}u_1^{(5)}(u_2^{(2)})^4 - 600(u_1^{(3)})^2(u_1^{(4)})^3(u_2^{(2)})^4\Big)\\
&\times\Big(160(u_1^{(2)})^4(u_2^{(3)})^4 + 160(u_1^{(3)})^4(u_2^{(2)})^4 - 640(u_1^{(2)})^3u_1^{(3)}u_2^{(2)}(u_2^{(3)})^3\\
&+ 960(u_1^{(2)})^2(u_1^{(3)})^2(u_2^{(2)})^2(u_2^{(3)})^2 - 640u_1^{(2)}(u_1^{(3)})^3(u_2^{(2)})^3u_2^{(3)}\Big)^{-1},
\end{aligned}
$$

$$
\begin{aligned}
u_2^{(6)} = \Big(&-144(u_1^{(2)})^4u_2^{(2)}(u_2^{(3)})^2(u_2^{(5)})^2 + 360(u_1^{(2)})^4u_2^{(2)}u_2^{(3)}(u_2^{(4)})^2u_2^{(5)}\\
&- 225(u_1^{(2)})^4u_2^{(2)}(u_2^{(4)})^4 + 720(u_1^{(2)})^4(u_2^{(3)})^3u_2^{(4)}u_2^{(5)}\\
&-v600(u_1^{(2)})^4(u_2^{(3)})^2(u_2^{(4)})^3 + 288(u_1^{(2)})^3u_1^{(3)}(u_2^{(2)})^2u_2^{(3)}(u_2^{(5)})^2\\
&- 360(u_1^{(2)})^3u_1^{(3)}(u_2^{(2)})^2(u_2^{(4)})^2u_2^{(5)} - 2640(u_1^{(2)})^3u_1^{(3)}u_2^{(2)}(u_2^{(3)})^2u_2^{(4)}u_2^{(5)}\\
&+ 1800(u_1^{(2)})^3u_1^{(3)}u_2^{(2)}u_2^{(3)}(u_2^{(4)})^3 - 640(u_1^{(2)})^3u_1^{(3)}(u_2^{(3)})^4u_2^{(5)}\\
&+ 1200(u_1^{(2)})^3u_1^{(3)}(u_2^{(3)})^3(u_2^{(4)})^2 - 720(u_1^{(2)})^3u_1^{(4)}(u_2^{(2)})^2u_2^{(3)}u_2^{(4)}u_2^{(5)}\\
&+ 900(u_1^{(2)})^3u_1^{(4)}(u_2^{(2)})^2(u_2^{(4)})^3 + 240(u_1^{(2)})^3u_1^{(4)}u_2^{(2)}(u_2^{(3)})^3u_2^{(5)}\\
&+ 600(u_1^{(2)})^3u_1^{(4)}u_2^{(2)}(u_2^{(3)})^2(u_2^{(4)})^2 - 1200(u_1^{(2)})^3u_1^{(4)}(u_2^{(3)})^4u_2^{(4)}\\
&+ 288(u_1^{(2)})^3u_1^{(5)}(u_2^{(2)})^2(u_2^{(3)})^2u_2^{(5)} - 360(u_1^{(2)})^3u_1^{(5)}(u_2^{(2)})^2u_2^{(3)}(u_2^{(4)})^2\\
&- 480(u_1^{(2)})^3u_1^{(5)}u_2^{(2)}(u_2^{(3)})^3u_2^{(4)} + 640(u_1^{(2)})^3u_1^{(5)}(u_2^{(3)})^5\\
&- 144(u_1^{(2)})^2(u_1^{(3)})^2(u_2^{(2)})^3(u_2^{(5)})^2 + 3120(u_1^{(2)})^2(u_1^{(3)})^2(u_2^{(2)})^2u_2^{(3)}u_2^{(4)}u_2^{(5)}\\
&- 1200(u_1^{(2)})^2(u_1^{(3)})^2(u_2^{(2)})^2(u_2^{(4)})^3 + 1920(u_1^{(2)})^2(u_1^{(3)})^2u_2^{(2)}(u_2^{(3)})^3u_2^{(5)}\\
&- 3200(u_1^{(2)})^2(u_1^{(3)})^2u_2^{(2)}(u_2^{(3)})^2(u_2^{(4)})^2 + 720(u_1^{(2)})^2u_1^{(3)}u_1^{(4)}(u_2^{(2)})^3u_2^{(4)}(u_2^{(5)})\\
&- 240(u_1^{(2)})^2u_1^{(3)}u_1^{(4)}(u_2^{(2)})^2(u_2^{(3)})^2u_2^{(5)} - 3000(u_1^{(2)})^2u_1^{(3)}u_1^{(4)}(u_2^{(2)})^2u_2^{(3)}(u_2^{(4)})^2\\
&+ 2800(u_1^{(2)})^2u_1^{(3)}u_1^{(4)}u_2^{(2)}(u_2^{(3)})^3u_2^{(4)} - 576(u_1^{(2)})^2u_1^{(3)}u_1^{(5)}(u_2^{(2)})^3u_2^{(3)}u_2^{(5)}\\
&+ 360(u_1^{(2)})^2u_1^{(3)}u_1^{(5)}(u_2^{(2)})^3(u_2^{(4)})^2 + 1920(u_1^{(2)})^2u_1^{(3)}u_1^{(5)}(u_2^{(2)})^2(u_2^{(3)})^2u_2^{(4)}\\
&- 1920(u_1^{(2)})^2u_1^{(3)}u_1^{(5)}u_2^{(2)}(u_2^{(3)})^4 + 360(u_1^{(2)})^2(u_1^{(4)})^2(u_2^{(2)})^3u_2^{(3)}u_2^{(5)}\\
&- 1350(u_1^{(2)})^2(u_1^{(4)})^2(u_2^{(2)})^3(u_2^{(4)})^2 + 600(u_1^{(2)})^2(u_1^{(4)})^2(u_2^{(2)})^2(u_2^{(3)})^2u_2^{(4)}\\
&+ 400(u_1^{(2)})^2(u_1^{(4)})^2u_2^{(2)}(u_2^{(3)})^4 + 720(u_1^{(2)})^2u_1^{(4)}u_1^{(5)}(u_2^{(2)})^3u_2^{(3)}u_2^{(4)}\\
&- 480(u_1^{(2)})^2u_1^{(4)}u_1^{(5)}(u_2^{(2)})^2(u_2^{(3)})^3 - 144(u_1^{(2)})^2(u_1^{(5)})^2(u_2^{(2)})^3(u_2^{(3)})^2\\
&- 1200u_1^{(2)}(u_1^{(3)})^3(u_2^{(2)})^3u_2^{(4)}u_2^{(5)} - 1920u_1^{(2)}(u_1^{(3)})^3(u_2^{(2)})^2(u_2^{(3)})^2u_2^{(5)}\\
&+ 2800u_1^{(2)}(u_1^{(3)})^3(u_2^{(2)})^2u_2^{(3)}(u_2^{(4)})^2 - 240u_1^{(2)}(u_1^{(3)})^2u_1^{(4)}(u_2^{(2)})^3u_2^{(3)}u_2^{(5)}\\
&+ 2400u_1^{(2)}(u_1^{(3)})^2u_1^{(4)}(u_2^{(2)})^3(u_2^{(4)})^2 - 2000u_1^{(2)}(u_1^{(3)})^2u_1^{(4)}(u_2^{(2)})^2(u_2^{(3)})^2u_2^{(4)}\\
&+ 288u_1^{(2)}(u_1^{(3)})^2u_1^{(5)}(u_2^{(2)})^4u_2^{(5)} - 2400u_1^{(2)}(u_1^{(3)})^2u_1^{(5)}(u_2^{(2)})^3u_2^{(3)}u_2^{(4)}\\
&+ 1920u_1^{(2)}(u_1^{(3)})^2u_1^{(5)}(u_2^{(2)})^2(u_2^{(3)})^3 - 360u_1^{(2)}u_1^{(3)}(u_1^{(4)})^2(u_2^{(2)})^4u_2^{(5)}\\
&+ 600u_1^{(2)}u_1^{(3)}(u_1^{(4)})^2(u_2^{(2)})^3u_2^{(3)}u_2^{(4)} - 800u_1^{(2)}u_1^{(3)}(u_1^{(4)})^2(u_2^{(2)})^2(u_2^{(3)})^3\\
&- 720u_1^{(2)}u_1^{(3)}u_1^{(4)}u_1^{(5)}(u_2^{(2)})^4u_2^{(4)} + 960u_1^{(2)}u_1^{(3)}u_1^{(4)}u_1^{(5)}(u_2^{(2)})^3(u_2^{(3)})^2\\
&+ 288u_1^{(2)}u_1^{(3)}(u_1^{(5)})^2(u_2^{(2)})^4u_2^{(3)} + 900u_1^{(2)}(u_1^{(4)})^3(u_2^{(2)})^4u_2^{(4)}\\
&- 600u_1^{(2)}(u_1^{(4)})^3(u_2^{(2)})^3(u_2^{(3)})^2 - 360u_1^{(2)}(u_1^{(4)})^2u_1^{(5)}(u_2^{(2)})^4u_2^{(3)}\\
&+ 640(u_1^{(3)})^4(u_2^{(2)})^3u_2^{(3)}u_2^{(5)} - 800(u_1^{(3)})^4(u_2^{(2)})^3(u_2^{(4)})^2\\
&+ 240(u_1^{(3)})^3u_1^{(4)}(u_2^{(2)})^4u_2^{(5)} + 400(u_1^{(3)})^3u_1^{(4)}(u_2^{(2)})^3u_2^{(3)}u_2^{(4)}
\end{aligned}
$$

$$
\begin{aligned}
&+960(u_1^{(3)})^3u_1^{(5)}(u_2^{(2)})^4u_2^{(4)} - 640(u_1^{(3)})^3u_1^{(5)}(u_2^{(2)})^3(u_2^{(3)})^2\\
&-1200(u_1^{(3)})^2(u_1^{(4)})^2(u_2^{(2)})^4u_2^{(4)} + 400(u_1^{(3)})^2(u_1^{(4)})^2(u_2^{(2)})^3(u_2^{(3)})^2\\
&-480(u_1^{(3)})^2u_1^{(4)}u_1^{(5)}(u_2^{(2)})^4u_2^{(3)} - 144(u_1^{(3)})^2(u_1^{(5)})^2(u_2^{(2)})^5\\
&+600u_1^{(3)}(u_1^{(4)})^3(u_2^{(2)})^4u_2^{(3)} + 360u_1^{(3)}(u_1^{(4)})^2u_1^{(5)}(u_2^{(2)})^5\\
&-\ 225(u_1^{(4)})^4(u_2^{(2)})^5\Big)\\
&\times\Big(160(u_1^{(2)})^4(u_2^{(3)})^4 + 160(u_1^{(3)})^4(u_2^{(2)})^4 - 640(u_1^{(2)})^3u_1^{(3)}u_2^{(2)}(u_2^{(3)})^3\\
&+\ 960(u_1^{(2)})^2(u_1^{(3)})^2(u_2^{(2)})^2(u_2^{(3)})^2 - 640u_1^{(2)}(u_1^{(3)})^3(u_2^{(2)})^3u_2^{(3)}\Big)^{-1}.
\end{aligned}
$$

Remark 8.16 There should be no doubt about the great help that a Computer Algebra System can offer in order to find Lie remarkable equations characterized by a Lie algebra of symmetries! As an example, the sixth-order ordinary differential system characterized by Lie algebra $\mathfrak{p}(\mathbb{R}^3)$ is quickly found with the following ReLie code (see Chap. 12 for the description of ReLie [35]):

```
in "relie.red" $
jetorder:= 6 $
xvar:={x} $
uvar:={u1,u2} $
generators:=
  {{1,0,0},{0,1,0},{0,0,1},
   {x,0,0},{u1,0,0},{u2,0,0},
   {0,x,0},{0,u1,0},{0,u2,0},
   {0,0,x},{0,0,u1},{0,0,u2},
   {x^2,x*u1,x*u2},
   {x*u1,u1^2,u1*u2},
   {x*u2,u1*u2,u2^2}} $
relieinit() $
reliedistrib(generators) $
m:=minors(distribution,14) $
sol:=solve(m,{u1_xxxxxx,u2_xxxxxx}) $
;end
```

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Chapter 9
Lie Symmetries and Equivalent Differential Equations



9.1 Introduction

In this chapter, we will discuss another relevant application of Lie symmetries of differential equations, i.e., their use to algorithmically construct invertible point mappings suitable to transform a given differential equation into an equivalent, and often more convenient, form: as a consequence, the *source* differential equation and *target* one are (at least locally) equivalent. In particular, we prove some theorems stating necessary and sufficient conditions for:

- the transformation of differential equations of arbitrary order, involving an arbitrary number of independent and dependent variables, to autonomous form [14, 15];
- the transformation of first-order systems of partial differential equations, involving an arbitrary number of independent and dependent variables, to linear form [13];
- the transformation of nonhomogeneous and/or nonautonomous quasilinear first-order systems of partial differential equations to homogeneous and autonomous form both in the case of 2×2 systems [10], and in the case of general systems involving an arbitrary number of independent and dependent variables [34];
- the transformation of first-order nonlinear differential equations polynomial in the derivatives into autonomous first-order differential equations polynomially homogeneous in the derivatives [18, 19].

Notably, the proofs of the theorems are constructive, and the desired transformation can be algorithmically derived. In all cases, the key role is played by the canonical variables associated to a set of Lie symmetries spanning a Lie algebra.

The mappings we shall construct (whose existence is strictly related to Lie symmetries having a suitable algebraic structure) need not be group transformations. However, if such mappings do exist, then necessarily any generator admitted by the source equation has to be mapped to a generator admitted by the target equation. As we shall consider invertible mappings, then a one-to-one correspondence between

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generators of the source and target equation is established. Moreover, since the Lie bracket of two generators of symmetries is not affected by an invertible change of coordinates (see Theorem 2.14), the Lie algebra of the symmetries of the target equation has to be isomorphic to the Lie algebra of symmetries of the source equation.

If the mapping from the source equation to the target equation is allowed to be noninvertible, then not necessarily there is a one-to-one correspondence between Lie algebras of symmetries of source and target equation. But such a noninvertible mapping must take any generator admitted by the source equation into a generator (which could be a null operator) admitted by the target equation. More precisely, the mapping must establish a homomorphism between the two Lie algebras.

It is worth of being remarked that transforming a system of partial differential equations to an equivalent form may be used to find solutions; in fact, we may determine simple solutions of the target equation and obtain (hopefully) less simple solutions of the source equation. Many of the results hereafter presented are contained in [10, 12–15, 18, 19, 33, 34].

9.2 Abelian Lie Algebras and Canonical Variables

A theorem allowing to introduce a special set of coordinates as a consequence of the existence of a suitable Abelian Lie algebra of generators is presented. This will be the main ingredient in proving the theorems concerned with the transformation of partial differential equations to equivalent forms with special features.

Theorem 9.1 *Let $\mathbb{D} \subseteq \mathbb{R}^n$ an n-dimensional manifold, and $\mathbf{z} \equiv (z_1, \ldots, z_n) \in \mathbb{D}$; let*

$$\Xi_i = \sum_{j=1}^{n} \zeta_{ij}(\mathbf{z}) \frac{\partial}{\partial z_j}, \qquad i = 1, \ldots, s \leq n$$

be the generators of an s-parameter Lie group of point transformations. If the vector fields $\{\Xi_1, \ldots, \Xi_s\}$ generate a distribution of rank s, i.e.,

$$\operatorname{rank} \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{s1} & \zeta_{s2} & \cdots & \zeta_{sn} \end{pmatrix} = s,$$

and

$$\left[\Xi_i, \Xi_j\right] = 0, \qquad 1 \leq i < j \leq s,$$

i.e., we have an s-dimensional Abelian Lie algebra, then it is possible to construct new variables

$$\mathbf{v} = \mathbf{V}(\mathbf{z}),$$

such that the operators $\{\Xi_1, \ldots, \Xi_s\}$ *characterize translations of s variables chosen among the components of* $\mathbf{v}$*, and leave the remaining variables invariant.*

Proof Let us introduce the canonical variables associated to Ξ_1,

$$\Xi_1(y_1) = 1, \qquad \Xi_1(y^1_{i_1}) = 0, \qquad i_1 = 2, \ldots, n;$$

as a consequence, Ξ_1 takes the form

$$\Xi_1 = \frac{\partial}{\partial y_1},$$

i.e., it corresponds to a translation in the variable y_1.

From the relation $[\Xi_1, \Xi_2] = 0$ it follows

$$\begin{aligned} &\Xi_1(\Xi_2(y_1)) = \Xi_2(\Xi_1(y_1)) = 0, \\ &\Xi_1(\Xi_2(y^1_{i_1})) = \Xi_2(\Xi_1(y^1_{i_1})) = 0. \end{aligned}$$

Thus, representing Ξ_2 in terms of the canonical variables of Ξ_1, its components may depend at most upon the invariants of Ξ_1, i.e.,

$$\Xi_2 = \Theta^2_1(y^1_{j_1})\frac{\partial}{\partial y_1} + \sum_{i_1=2}^{n} \Theta^2_{i_1}(y^1_{j_1})\frac{\partial}{\partial y^1_{i_1}}, \qquad j_1 = 2, \ldots, n.$$

If $\Theta^2_1 \neq 0$, we introduce the variable

$$y^1_1 = y_1 + \varphi_1(y^1_{i_1}),$$

where φ_1 satisfies

$$\Theta^2_1 + \sum_{i_1=2}^{n} \Theta^2_{i_1}\frac{\partial \varphi_1}{\partial y^1_{i_1}} = 0,$$

otherwise we take $y^1_1 = y_1$.

That enables us to write Ξ_1 and Ξ_2 as follows:

$$\begin{aligned} \Xi_1 &= \frac{\partial}{\partial y^1_1}, \\ \Xi_2 &= \sum_{i_1=2}^{n} \Theta^2_{i_1}(y^1_{j_1})\frac{\partial}{\partial y^1_{i_1}}, \qquad j_1 = 2, \ldots, n; \end{aligned}$$

so doing, y^1_1, for which Ξ_1 induces a translation, is an invariant for Ξ_2.

Now, let us introduce the canonical variables associated to the operator Ξ_2,

$$\Xi_2(y_2) = 1, \qquad \Xi_2(y_1^1) = \Xi_2(y_{i_2}^2) = 0, \qquad i_2 = 3, \ldots, n,$$

so that

$$\Xi_2 = \frac{\partial}{\partial y_2}.$$

Since Ξ_3 commutes with Ξ_1 and Ξ_2, we can write

$$\Xi_3 = \Theta_1^3(y_{j_2}^2)\frac{\partial}{\partial y_1^1} + \Theta_2^3(y_{j_2}^2)\frac{\partial}{\partial y_2} + \sum_{i_2=2}^{n} \Theta_{i_2}^3(y_{j_2}^2)\frac{\partial}{\partial y_{i_2}^2},$$

where $j_2 = 3, \ldots, n$.

If $\Theta_1^3 \neq 0$, we introduce the variable

$$y_1^2 = y_1^1 + \varphi_1^1(y_{i_2}^2)$$

with φ_1^1 such that

$$\Theta_1^3 + \sum_{i_2=2}^{n} \Theta_{i_2}^3 \frac{\partial \varphi_1^1}{\partial y_{i_2}^2} = 0,$$

otherwise we take $y_1^2 = y_1^1$; moreover, if $\Theta_2^3 \neq 0$ we introduce the variable

$$y_2^2 = y_2 + \varphi_2^1(y_{i_2}^2)$$

with φ_2^1 such that

$$\Theta_2^3 + \sum_{i_2=2}^{n} \Theta_{i_2}^3 \frac{\partial \varphi_2^1}{\partial y_{i_2}^2} = 0,$$

otherwise we take $y_2^2 = y_2$.

Thus, Ξ_1, Ξ_2 and Ξ_3 become

$$\Xi_1 = \frac{\partial}{\partial y_1^2}, \qquad \Xi_2 = \frac{\partial}{\partial y_2^2}, \qquad \Xi_3 = \sum_{i_2=3}^{n} \Theta_{i_2}^3(y_{j_2}^2)\frac{\partial}{\partial y_{i_2}^2},$$

where $j_2 = 3, \ldots, n$.

By iterating the procedure for the remaining operators, at the generic kth step we introduce the canonical variables

$$\Xi_k(y_k) = 1, \qquad \Xi_k(y_1^{k-1}) = \cdots = \Xi_k(y_{k-1}^{k-1}) = \Xi_k(y_{i_k}^k) = 0,$$

where $i_k = k+1, \dots, n$, whence

$$\Xi_1 = \frac{\partial}{\partial y_1^{k-1}}, \qquad \Xi_2 = \frac{\partial}{\partial y_2^{k-1}}, \qquad \dots, \qquad \Xi_k = \frac{\partial}{\partial y_k},$$
$$\Xi_{k+1} = \Theta_1^{k+1}(y_{j_k}^k)\frac{\partial}{\partial y_1^{k-1}} + \dots + \Theta_{k-1}^{k+1}(y_{j_k}^k)\frac{\partial}{\partial y_{k-1}^{k-1}}$$
$$+ \Theta_k^{k+1}(y_{j_k}^k)\frac{\partial}{\partial y_k} + \sum_{i_k=k+1}^{n} \Theta_{i_k}^{k+1}(y_{j_k}^k)\frac{\partial}{\partial y_{i_k}^k},$$

where $j_k = k+1, \dots, n$.

Finally, after the complete application of the described algorithm, the composition of the various variable transformations produces new variables $\mathbf{v} \equiv (v_1, \dots, v_n)$, where

$$v_i = \begin{cases} y_i^{s-1}, & i = 1, \dots, s-1, \\ y_i, & i = s, \\ y_i^s, & i = s+1, \dots, n \end{cases}$$

such that

$$\Xi_i(v_j) = \delta_{ij}, \qquad i, j = 1, \dots, s,$$

and

$$\Xi_i(v_k) = 0, \qquad i = 1, \dots, s, \quad k = s+1, \dots, n.$$

Thus, in terms of the variables $\mathbf{v}$, the generators become

$$\Xi_i = \frac{\partial}{\partial v_i}, \qquad i = 1, \dots, s.$$

□

Remark 9.1 Theorem 9.1 is a special case of Frobenius theorem [36] (see Theorem 8.2); in fact, we require the most restrictive condition that the vector fields $\{\Xi_1, \dots, \Xi_s\}$ are commuting. This condition is essential because we will use this theorem in the case where the vector fields correspond to point symmetries of a differential equation, and only a linear combination of symmetries with constant coefficients is a symmetry.

9.3 Reduction to Autonomous Form

Here, we consider the case of the transformation of a system of partial differential equations to autonomous form, i.e., a form in which the independent variables do not appear explicitly.

Theorem 9.2 *The system of differential equations of order r*

$$\mathbf{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}\right) = \mathbf{0}, \tag{9.1}$$

where $\mathbf{x} \in \mathbb{D}_n \subseteq \mathbb{R}^n$ *and* $\mathbf{u} \in \mathbb{D}_m \subseteq \mathbb{R}^m$*, can be transformed by the (at least locally) invertible point transformation*

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}) \tag{9.2}$$

to the autonomous equivalent form

$$\widetilde{\mathbf{\Delta}}\left(\mathbf{w}, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(r)}\right) = \mathbf{0}, \tag{9.3}$$

if and only if it admits n Lie symmetries whose generators Ξ_i $(i = 1, \ldots, n)$ *give a distribution of rank n and satisfy the conditions*

$$[\Xi_i, \Xi_j] = 0, \qquad i, j = 1, \ldots, n,$$

that is, $\{\Xi_1, \ldots, \Xi_n\}$ *span an n-dimensional Abelian Lie algebra.*

Proof Using Theorem 9.1, we can construct a variable transformation like (9.2) such that the generators become

$$\Xi_i = \frac{\partial}{\partial z_i}, \qquad i = 1, \ldots, n,$$

and system (9.1) takes a form where all the independent variables do not explicitly appear, i.e., it is autonomous.

Conversely, if the nonautonomous system (9.1) can be mapped by the invertible point transformation (9.2) to the autonomous system (9.3), then, as the latter admits the n generators $\dfrac{\partial}{\partial z_i}$, spanning an $n-$dimensional Abelian Lie algebra and generating a distribution of rank n, then it follows that also the system (9.1) must admit n symmetries spanning an n-dimensional Abelian Lie algebra generating a distribution of rank n. □

Remark 9.2 It may happen in some applications that the generators of the independent variables are independent of the dependent variables, and that the rank of the $n \times n$ matrix, whose entries are the infinitesimals of the independent variables only, is n. In such a case, the new independent variables may be chosen as functions of the old independent variables only.

9.3.1 Applications

Let us give some examples of physical interest where Theorem 9.2 can be applied.

Example 9.1 (*Cylindrical Korteweg–deVries equation*) Consider the cylindrical Korteweg–deVries equation [22]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \frac{u}{2t} = 0, \tag{9.4}$$

which is said to describe the "cylindrical solitons"; it is known [22] that it can be reduced to the classical Korteweg–deVries equation. We will derive algorithmically this transformation by means of Theorem 9.2.

Equation (9.4) is invariant with respect to the Lie groups generated by the following vector fields:

$$\begin{aligned}
\Xi_1 &= 2t^{1/2}\frac{\partial}{\partial x} + t^{-1/2}\frac{\partial}{\partial u},\\
\Xi_2 &= 8t^{3/2}\frac{\partial}{\partial t} + 4xt^{1/2}\frac{\partial}{\partial x} + 2(xt^{-1/2} - 4ut^{1/2})\frac{\partial}{\partial u},\\
\Xi_3 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u},\\
\Xi_4 &= \frac{\partial}{\partial x},
\end{aligned}$$

spanning a four-dimensional Lie algebra. From the commutator table

	Ξ_1	Ξ_2	Ξ_3	Ξ_4
Ξ_1	0	0	$(-1/2)\Xi_1$	0
Ξ_2	0	0	$(-3/2)\Xi_2$	$-2\Xi_1$
Ξ_3	$(1/2)\Xi_1$	$(3/2)\Xi_2$	0	$-\Xi_4$
Ξ_4	0	$2\Xi_1$	Ξ_4	0

we easily recognize that the two-dimensional Lie subalgebra spanned by $\{\Xi_1, \Xi_2\}$ is Abelian and gives a distribution of rank 2. It is also

$$\operatorname{rank}\begin{pmatrix} 0 & 2t^{1/2} \\ 8t^{3/2} & 4xt^{1/2} \end{pmatrix} = 2,$$

so that the new independent variables can be chosen to depend only on t and x. The variables

$$t, \qquad \xi = \frac{x}{2t^{1/2}}, \qquad U = 4tu - 2x,$$

satisfying

$$\Xi_1(t) = 0, \qquad \Xi_1(\xi) = 1, \qquad \Xi_1(U) = 0,$$

are such that Ξ_1 and Ξ_2 become

$$\Xi_1 = \frac{\partial}{\partial \xi}, \qquad \Xi_2 = 8t^{3/2}\frac{\partial}{\partial t};$$

by introducing

$$\tau = -\frac{1}{4}t^{-1/2}$$

we obtain

$$\Xi_2 = \frac{\partial}{\partial \tau},$$

and Eq. (9.4) becomes

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial \xi} + \frac{\partial^3 U}{\partial \xi^3} = 0, \tag{9.5}$$

that is, we have the classical Korteweg–deVries equation. Equation (9.5) has the well-known "soliton" solution

$$U = 12\,\mathrm{sech}^2(\xi - 4\tau),$$

yielding the solution

$$u = \frac{x}{2t} + \frac{3}{t}\,\mathrm{sech}^2\left(\frac{x}{2\sqrt{t}} + \frac{1}{\sqrt{t}}\right)$$

to the cylindrical Korteweg–deVries equation.

A different reduction of cylindrical Korteweg–deVries equation to autonomous form can be obtained using the two-dimensional Lie subalgebra spanned by $\{\Xi_1, \Xi_4\}$. In such a case, the new independent variables will depend also on u; in fact, the variables

$$\tau = t^{1/2}u, \qquad \xi = x - 2tu, \qquad U = t,$$

allow to write

$$\Xi_1 = \frac{\partial}{\partial \tau}, \qquad \Xi_4 = \frac{\partial}{\partial \xi},$$

and the cylindrical Korteweg–deVries equation gains the following autonomous (but not simple) structure:

$$\begin{aligned}
&\left(\frac{\partial U}{\partial \tau} - 2\sqrt{U}\frac{\partial U}{\partial \xi}\right)\left(\left(\frac{\partial U}{\partial \tau}\right)^3\frac{\partial^3 U}{\partial \xi^3} - \left(\frac{\partial U}{\partial \xi}\right)^3\frac{\partial^3 U}{\partial \tau^3}\right.\\
&\quad\left. +3\frac{\partial U}{\partial \tau}\frac{\partial U}{\partial \xi}\left(\frac{\partial U}{\partial \xi}\frac{\partial^3 U}{\partial \tau^2 \partial \xi} - \frac{\partial U}{\partial \tau}\frac{\partial^3 U}{\partial \tau \partial \xi^2}\right)\right)\\
&\quad +3\left(\frac{\partial U}{\partial \xi}\right)^3\left(\frac{\partial^2 U}{\partial \tau^2}\right)^2 - 3\left(\frac{\partial U}{\partial \xi}\right)^2\left(3\frac{\partial U}{\partial \tau} + 2\sqrt{U}\frac{\partial U}{\partial \xi}\right)\frac{\partial^2 U}{\partial \tau^2}\frac{\partial^2 U}{\partial \tau \partial \xi}
\end{aligned}$$

$$+3\frac{\partial U}{\partial \tau}\frac{\partial U}{\partial \xi}\left(\frac{\partial U}{\partial \tau}+2\sqrt{U}\frac{\partial U}{\partial \xi}\right)\left(\frac{\partial^2 U}{\partial \tau^2}\frac{\partial^2 U}{\partial \xi^2}+2\left(\frac{\partial^2 U}{\partial \tau \partial \xi}\right)^2\right)$$
$$-3\left(\frac{\partial U}{\partial \tau}\right)^2\left(\frac{\partial U}{\partial \tau}+6\sqrt{U}\frac{\partial U}{\partial \xi}\right)\frac{\partial^2 U}{\partial \tau \partial \xi}\frac{\partial^2 U}{\partial \xi^2}+6\sqrt{U}\left(\frac{\partial U}{\partial \tau}\right)^3\left(\frac{\partial^2 U}{\partial \xi^2}\right)^2$$
$$-\left(\frac{\partial U}{\partial \tau}-2\sqrt{U}\frac{\partial U}{\partial \xi}\right)^4=0.$$

Example 9.2 (*Axi-symmetric gas-dynamics equations*) Consider the Euler equations for a polytropic gas subject to no external forces,

$$\frac{\partial \rho}{\partial t}+\sum_{k=1}^{3}\frac{\partial(\rho v_k)}{\partial x_k}=0,$$
$$\rho\left(\frac{\partial v_i}{\partial t}+\sum_{k=1}^{3}v_k\frac{\partial v_i}{\partial x_k}\right)+\frac{\partial p}{\partial x_i}=0,\qquad i=1,2,3,$$
$$\frac{\partial p}{\partial t}+\sum_{k=1}^{3}v_k\frac{\partial p}{\partial x_k}+\gamma p\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_k}=0,$$

where t is the time, $\mathbf{x}\equiv(x_1,x_2,x_3)$ the Cartesian space coordinates, $\rho(t,\mathbf{x})$ the mass density, $(v_1(t,\mathbf{x}),v_2(t,\mathbf{x}),v_3(t,\mathbf{x}))$ the components of velocity, $p(t,\mathbf{x})$ the pressure, and γ the adiabatic index. In some situations, particular flows, possessing some special geometrical symmetries, can be considered:

1. plane symmetry; setting $x=x_1$, and assuming $\rho=\rho(t,x)$, $p=p(t,x)$, $(v_1,v_2,v_3)=(1,0,0)u(t,x)$, i.e., the velocity directed along the x_1-axis;
2. cylindrical symmetry; setting $x=\sqrt{x_1^2+x_2^2}$, and assuming $\rho=\rho(t,x)$, $p=p(t,x)$, $(v_1,v_2,v_3)=\left(\frac{x_1}{x},\frac{x_2}{x},0\right)u(t,x)$, i.e., the velocity directed along the radial direction in cylindrical coordinates;
3. spherical symmetry; setting $x=\sqrt{x_1^2+x_2^2+x_3^2}$, and assuming $\rho=\rho(t,x)$, $p=p(t,x)$, $(v_1,v_2,v_3)=\left(\frac{x_1}{x},\frac{x_2}{x},\frac{x_3}{x}\right)u(t,x)$, i.e., the velocity directed along the radial direction in spherical coordinates.

In all these three cases, the equations may be written in a unified form, i.e.,

$$\begin{aligned}&\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\nu \rho u}{x}=0,\\&\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial p}{\partial x}=0,\\&\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\gamma p\frac{\partial u}{\partial x}+\frac{\gamma \nu p u}{x}=0,\end{aligned}\tag{9.6}$$

where ν is a parameter depending on the geometric symmetry of the flow ($\nu = 0, 1, 2$ for plane, cylindrical, and spherical symmetry, respectively).

It is easy to verify that system (9.6) is left invariant by the Lie group of transformations having the following generators:

$$\begin{aligned}
\Xi_1 &= \rho\frac{\partial}{\partial\rho} + p\frac{\partial}{\partial p},\\
\Xi_2 &= t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2p\frac{\partial}{\partial p},\\
Xi_3 &= x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2p\frac{\partial}{\partial p},\\
\Xi_4 &= \frac{\partial}{\partial t}.
\end{aligned}$$

Hence, system (9.6) admits the invariance with respect to three independent scaling (corresponding to the arbitrary choices for the scales of mass, time, and space), and with respect to time translation. The only nonzero Lie bracket is $[\Xi_2, \Xi_4] = -\Xi_4$. Let us consider the two-dimensional Lie subalgebra spanned by the generators

$$\begin{aligned}
\widehat{\Xi}_1 &= \Xi_2 + \alpha\Xi_1 = t\frac{\partial}{\partial t} + \alpha\rho\frac{\partial}{\partial\rho} - u\frac{\partial}{\partial u} + (\alpha-2)p\frac{\partial}{\partial p},\\
\widehat{\Xi}_2 &= \Xi_3 + \beta\Xi_1 = x\frac{\partial}{\partial x} + \beta\rho\frac{\partial}{\partial\rho} + u\frac{\partial}{\partial u} + (\beta+2)p\frac{\partial}{\partial p},
\end{aligned}$$

where α and β are arbitrary constants. These symmetries verify the conditions required by Theorem 9.2, and we may construct the transformation

$$\begin{aligned}
\tau &= \log t,\\
\eta &= \log x,\\
\rho(t,x) &= x^\beta t^\alpha R(\tau,\eta),\\
u(t,x) &= xt^{-1}U(\tau,\eta),\\
p(t,x) &= x^{\beta+2}t^{\alpha-2}P(\tau,\eta),
\end{aligned} \tag{9.7}$$

in terms of which the system (9.6) reduces to the following autonomous form:

$$\begin{aligned}
&\frac{\partial R}{\partial\tau} + \frac{\partial(RU)}{\partial\eta} + (\beta+\nu+1)RU + \alpha R = 0,\\
&R\left(\frac{\partial U}{\partial\tau} + U\frac{\partial U}{\partial\eta}\right) + \frac{\partial P}{\partial\eta} + RU(U-1) + (\beta+2)P = 0,\\
&\frac{\partial P}{\partial\tau} + U\frac{\partial P}{\partial\eta} + \gamma P\frac{\partial U}{\partial\eta} + (\beta+2+\gamma(\nu+1))PU + (\alpha-2)P = 0.
\end{aligned}$$

The form of this system allows for the possibility of looking for a constant solution

$$R = R_0, \qquad U = U_0, \qquad P = P_0, \tag{9.8}$$

where the constants R_0, U_0 and P_0 satisfy the system of algebraic equations

$$\begin{aligned} &\alpha + (\beta + \nu + 1)U_0 = 0, \\ &R_0 U_0(U_0 - 1) + (\beta + 2)P_0 = 0, \\ &(\alpha - 2) + (\beta + 2 + \gamma(\nu + 1))U_0 = 0. \end{aligned} \tag{9.9}$$

To the constant solution (9.8)–(9.9) there corresponds the following nonconstant solution of the original system:

$$\begin{aligned} \rho(t, x) &= R_0 x^\beta t^\alpha, \\ u(t, x) &= -\frac{\alpha}{\beta + \nu + 1}\frac{x}{t}, \\ p(t, x) &= R_0 \frac{-\alpha(\alpha + \beta + \nu + 1)}{(\beta + \nu + 1)^2(\beta + 2)} x^{\beta+2} t^{\alpha-2}, \end{aligned} \tag{9.10}$$

where R_0 and β are arbitrary constants, whereas

$$\alpha = \frac{-2(\beta + \nu + 1)}{(\gamma + 1) + \nu(\gamma - 1)}.$$

Note that the solution (9.10) still exists when $\nu = 0$ (plane symmetry). In such a case, the transformation (9.7) maps an autonomous system to a different autonomous system possessing nontrivial constant solutions which are nonconstant in the original physical variables!

Solution (9.10) generalizes the classical Sedov solution [40] for the point explosion problem, the latter being recovered when

$$\alpha = -\beta = \frac{2}{1 - \gamma} \quad \text{and} \quad \nu = 2.$$

Example 9.3 (*Navier–Stokes–Fourier equations*) Let us consider a gas in rotation around a vertical axis with a constant angular velocity $\boldsymbol{\omega}$; it is convenient to introduce explicitly the inertial frame $\{x_1', x_3', x_3'\}$ with respect to which the noninertial system (with coordinates $\{x_1, x_2, x_3\}$) is rotating around the x_3' axis (where x_3 and the x_3' axes will be assumed coincident).

The ordinary thermodynamics of viscous and heat conducting monatomic gases can be obtained starting from the balance equations for the first 13 moments and using a formal iterative scheme analogous to the Maxwellian iteration [20, 32]. In this case, we have only 5 scalar fields (mass density ρ, velocity $\mathbf{v} \equiv (v_1, v_2, v_3)$, and temperature T) governed by the balance equations for mass, momentum, and energy:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_k)}{\partial x_k} = 0, \tag{9.11a}$$

$$\frac{\partial (\rho v_i)}{\partial t} + \frac{\partial}{\partial x_k}\left(\rho v_i v_k - \frac{2}{\alpha}\left(\frac{K}{m}\right) T\left(\frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) - \frac{1}{3}\frac{\partial v_r}{\partial x_r}\delta_{ik}\right)\right.$$
$$\left. + \rho\left(\frac{K}{m}\right) T\delta_{ik}\right) = \rho(F_i^{(e)} + F_i^{(i)}), \qquad i = 1, 2, 3, \tag{9.11b}$$

$$\frac{\partial (\rho T)}{\partial t} + \frac{\partial}{\partial x_k}\left(\rho T v_k - \frac{5}{2\alpha}\left(\frac{K}{m}\right) T\frac{\partial T}{\partial x_k}\right)$$
$$+ \left(\frac{2}{3}\rho T\delta_{\ell k} - \frac{4}{3\alpha} T\left(\frac{1}{2}\left(\frac{\partial v_\ell}{\partial x_k} + \frac{\partial v_k}{\partial x_\ell}\right) - \frac{1}{3}\frac{\partial v_r}{\partial x_r}\delta_{\ell k}\right)\right)\frac{\partial v_\ell}{\partial x_k} = 0, \tag{9.11c}$$

where the Einstein convention of sums over repeated indices is used, $F_i^{(i)}$ are the components of the *specific inertial forces*, $F_i^{(e)}$ the components of the *specific external forces* acting on the gas, K the Boltzmann constant, m the mass of a single particle, and the coefficient α an appropriate constant which describes the interaction between the Maxwellian molecules. If the origins of both frames coincide, as we are considering a time independent rotation with angular velocity $\boldsymbol{\omega} = (0, 0, \omega)$, we have only the contributions of Coriolis and centrifugal forces, so that

$$\mathbf{F}^{(i)} = (2\omega v_2 + \omega^2 x_1, -2\omega v_1 + \omega^2 x_2, 0).$$

Moreover, assume that $\mathbf{F}^{(e)} = (0, 0, -g)$, where g is the gravitational acceleration.

System (9.11) is invariant with respect to a 12-parameter Lie group whose Lie algebra is spanned by the following generators (see Remark 12.1 for the ReLie code performing the computation):

$$\Xi_1 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + v_2\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial v_2},$$

$$\Xi_2 = t\frac{\partial}{\partial t} + (x_1 + \omega x_2 t)\frac{\partial}{\partial x_1} + (x_2 - \omega x_1 t)\frac{\partial}{\partial x_2} + \left(x_3 - \frac{gt^2}{2}\right)\frac{\partial}{\partial x_3}$$
$$- \rho\frac{\partial}{\partial \rho} + \omega(v_2 t + x_2)\frac{\partial}{\partial v_1} - \omega(v_1 t + x_1)\frac{\partial}{\partial v_2} - gt\frac{\partial}{\partial v_3},$$

$$\Xi_3 = t\frac{\partial}{\partial t} + \omega x_2 t\frac{\partial}{\partial x_1} - \omega x_1 t\frac{\partial}{\partial x_2} - gt^2\frac{\partial}{\partial x_3} - \rho\frac{\partial}{\partial \rho} + (-v_1 + \omega(v_2 t + x_2))\frac{\partial}{\partial v_1}$$
$$- (v_2 + \omega(v_1 t + x_1))\frac{\partial}{\partial v_2} - (v_3 + 2gt)\frac{\partial}{\partial v_3} - 2T\frac{\partial}{\partial T},$$

$$\Xi_4 = \frac{\partial}{\partial x_3}, \qquad \Xi_5 = t\frac{\partial}{\partial x_3} + \frac{\partial}{\partial v_3}, \qquad \Xi_6 = \frac{\partial}{\partial t},$$

$$\Xi_7 = \sin(\omega t)\frac{\partial}{\partial x_1} + \cos(\omega t)\frac{\partial}{\partial x_2} + \omega\cos(\omega t)\frac{\partial}{\partial v_1} - \omega\sin(\omega t)\frac{\partial}{\partial v_2},$$

$$\Xi_8 = t\sin(\omega t)\frac{\partial}{\partial x_1} + t\cos(\omega t)\frac{\partial}{\partial x_2}$$

$$
\begin{aligned}
&+ (\omega t\cos(\omega t) + \sin(\omega t))\frac{\partial}{\partial v_1} + (-\omega t\sin(\omega t) + \cos(\omega t)\frac{\partial}{\partial v_2},\\
\Xi_9 &= -\cos(\omega t)\frac{\partial}{\partial x_1} + \sin(\omega t)\frac{\partial}{\partial x_2} + \omega\sin(\omega t)\frac{\partial}{\partial v_1} + \omega\cos(\omega t)\frac{\partial}{\partial v_2},\\
\Xi_{10} &= -t\cos(\omega t)\frac{\partial}{\partial x_1} + t\sin(\omega t)\frac{\partial}{\partial x_2}\\
&+ (\omega t\sin(\omega t) - \cos(\omega t))\frac{\partial}{\partial v_1} + (\omega t\cos(\omega t) + \sin(\omega t))\frac{\partial}{\partial v_2},\\
\Xi_{11} &= (gt^2 + 2x_3)\sin(\omega t)\frac{\partial}{\partial x_1} + (gt^2 + 2x_3)\cos(\omega t)\frac{\partial}{\partial x_2}\\
&- 2(x_1\sin(\omega t) + x_2\cos(\omega t))\frac{\partial}{\partial x_3}\\
&+ (\omega(gt^2 + 2x_3)\cos(\omega t) + 2(gt + v_3)\sin(\omega t))\frac{\partial}{\partial v_1}\\
&+ (-\omega(gt^2 + 2x_3)\sin(\omega t) + 2(gt + v_3)\cos(\omega t))\frac{\partial}{\partial v_2}\\
&+ 2((v_1 - \omega x_2)\sin(\omega t) - (v_2 + \omega x_1)\cos(\omega t))\frac{\partial}{\partial v_3},\\
\Xi_{12} &= (gt^2 + 2x_3)\cos(\omega t)\frac{\partial}{\partial x_1} - (gt^2 + 2x_3)\sin(\omega t)\frac{\partial}{\partial x_2}\\
&- 2(x_1\cos(\omega t) - x_2\sin(\omega t))\frac{\partial}{\partial x_3}\\
&- (\omega(gt^2 + 2x_3)\sin(\omega t) + 2(gt + v_3)\cos(\omega t))\frac{\partial}{\partial v_1}\\
&- (\omega(gt^2 + 2x_3)\cos(\omega t) + 2(gt + v_3)\sin(\omega t))\frac{\partial}{\partial v_2}\\
&- 2((v_1 - \omega x_2)\cos(\omega t) + (v_2 + \omega x_1)\sin(\omega t))\frac{\partial}{\partial v_3}.
\end{aligned}
$$

The governing system (9.11) involves four independent variables (the time t and the three spatial coordinates x_i), so that its reduction to autonomous form through the application of Theorem 9.2 requires the existence of a four-dimensional Abelian Lie subalgebra. By analyzing the commutator table, it is easily seen that either the Lie subalgebra

$$\mathbb{L}' = \{\Xi_2, \Xi_5, \Xi_8, \Xi_{10}\},$$

or the Lie subalgebra

$$\mathbb{L}'' = \{\Xi_3, \Xi_4, \Xi_7, \Xi_9\}$$

are Abelian.

By considering the Lie subalgebra $\mathbb{L}'$, we may construct the following invertible point transformation

$$
\begin{aligned}
\widehat{x}_1 &= \frac{x_1}{t}\cos(\omega t) - \frac{x_2}{t}\sin(\omega t),\\
\widehat{x}_2 &= \frac{x_1}{t}\sin(\omega t) + \frac{x_2}{t}\cos(\omega t),\\
\widehat{x}_3 &= \frac{x_3}{t} + \frac{gt}{2}, \qquad \widehat{t} = \ln(t),\\
\rho &= \frac{1}{t}\widehat{\rho}, \qquad T = \widehat{T},\\
v_1 &= \widehat{v}_1\cos(\omega t) + \widehat{v}_2\sin(\omega t) + \frac{x_1}{t} + \omega x_2,\\
v_2 &= -\widehat{v}_1\sin(\omega t) + \widehat{v}_2\cos(\omega t) + \frac{x_2}{t} - \omega x_1,\\
v_3 &= \widehat{v}_3 + \frac{x_3}{t} - \frac{gt}{2},
\end{aligned}
$$

where the new dependent variables $\widehat{\rho}$, $\widehat{T}$, $\widehat{v}_1$, $\widehat{v}_2$ and $\widehat{v}_3$ are functions of the new independent variables $\widehat{t}$, $\widehat{x}_1$, $\widehat{x}_2$ and $\widehat{x}_3$.

In the new variables, the system (9.11) reduces to autonomous form, i.e.,

$$
\frac{\partial\widehat{\rho}}{\partial\widehat{t}} + \frac{\partial(\widehat{\rho}\widehat{v}_k)}{\partial\widehat{x}_k} + 2\widehat{\rho} = 0, \tag{9.12a}
$$

$$
\begin{aligned}
&\frac{\partial(\widehat{\rho}\widehat{v}_i)}{\partial\widehat{t}} + \frac{\partial}{\partial\widehat{x}_k}\left(\widehat{\rho}\widehat{v}_i\widehat{v}_k - \frac{2}{\alpha}\left(\frac{K}{m}\right)\widehat{T}\left(\frac{1}{2}\left(\frac{\partial\widehat{v}_i}{\partial\widehat{x}_k} + \frac{\partial\widehat{v}_k}{\partial\widehat{x}_i}\right) - \frac{1}{3}\frac{\partial\widehat{v}_r}{\partial\widehat{x}_r}\delta_{ik}\right)\right.\\
&\quad\left. + \widehat{\rho}\left(\frac{K}{m}\right)\widehat{T}\delta_{ik}\right) + 3\widehat{\rho}\widehat{v}_i = 0, \qquad i = 1,2,3,
\end{aligned} \tag{9.12b}
$$

$$
\begin{aligned}
&\frac{\partial(\widehat{\rho}\widehat{T})}{\partial\widehat{t}} + \frac{\partial}{\partial\widehat{x}_k}\left(\widehat{\rho}\widehat{T}\widehat{v}_k - \frac{5}{2\alpha}\left(\frac{K}{m}\right)\widehat{T}\frac{\partial\widehat{T}}{\partial\widehat{x}_k}\right) + 4\widehat{\rho}\widehat{T}\\
&\quad + \left(\frac{2}{3}\widehat{\rho}\widehat{T}\delta_{\ell k} - \frac{4}{3\alpha}\widehat{T}\left(\frac{1}{2}\left(\frac{\partial\widehat{v}_\ell}{\partial\widehat{x}_k} + \frac{\partial\widehat{v}_k}{\partial\widehat{x}_\ell}\right) - \frac{1}{3}\frac{\partial\widehat{v}_r}{\partial\widehat{x}_r}\delta_{\ell k}\right)\right)\frac{\partial\widehat{v}_\ell}{\partial\widehat{x}_k} = 0.
\end{aligned} \tag{9.12c}
$$

It is worth of noticing that this system has the same form as the original system with the exception of the inhomogeneity terms.

A direct inspection of (9.12) allows us for the determination of some simple classes of solutions to the system (9.11). For instance, by searching for the solutions of the transformed system (9.12) depending only on $\widehat{t}$, one gets the following solution to the original system (9.11):

$$
\begin{aligned}
\rho &= \frac{\rho_0}{t^3}, \qquad T = \frac{T_0}{t^2},\\
v_1 &= \frac{v_{1_0}\cos(\omega t) + v_{2_0}\sin(\omega t)}{t} + \frac{x_1}{t} + \omega x_2,
\end{aligned}
$$

$$v_2 = \frac{-v_{1_0}\sin(\omega t) + v_{2_0}\cos(\omega t)}{t} + \frac{x_2}{t} - \omega x_1,$$
$$v_3 = \frac{v_{3_0}}{t} + \frac{x_3}{t} - \frac{gt}{2},$$

where $\rho_0 > 0$, v_{1_0}, v_{2_0}, v_{3_0} and $T_0 > 0$ are arbitrary constants.

Also, by assuming in (9.12) $\widehat{v}_1 = \widehat{v}_2 = \widehat{v}_3 = 0$, we get the following solution to the original system:

$$\rho = \frac{1}{t^3\sqrt{\psi(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)}}, \qquad T = \frac{T_0}{t^2}\sqrt{\psi(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)},$$
$$v_1 = \frac{x_1}{t} + \omega x_2, \qquad v_2 = \frac{x_2}{t} - \omega x_1, \qquad v_3 = \frac{x_3}{t} - \frac{gt}{2},$$

where T_0 is a positive constant, and ψ is a positive solution of the Laplace equation

$$\frac{\partial^2\psi}{\partial\widehat{x}_1^2} + \frac{\partial^2\psi}{\partial\widehat{x}_2^2} + \frac{\partial^2\psi}{\partial\widehat{x}_3^2} = 0.$$

In the case of the subalgebra $\mathbb{L}''$, we are able to introduce the invertible point transformation

$$\widehat{x}_1 = x_1\cos(\omega t) - x_2\sin(\omega t),$$
$$\widehat{x}_2 = x_1\sin(\omega t) + x_2\cos(\omega t),$$
$$\widehat{x}_3 = x_3 + \frac{gt^2}{2}, \qquad \widehat{t} = \ln(t),$$
$$\rho = \frac{1}{t}\widehat{\rho}, \qquad T = \frac{1}{t^2}\widehat{T},$$
$$v_1 = \frac{\widehat{v}_1}{t}\cos(\omega t) + \frac{\widehat{v}_2}{t}\sin(\omega t) + \omega x_2,$$
$$v_2 = -\frac{\widehat{v}_1}{t}\sin(\omega t) + \frac{\widehat{v}_2}{t}\cos(\omega t) - \omega x_1,$$
$$v_3 = \frac{\widehat{v}_3}{t} - gt,$$

where the new dependent variables $\widehat{\rho}$, $\widehat{T}$, $\widehat{v}_1$, $\widehat{v}_2$ and $\widehat{v}_3$ are functions of the new independent variables $\widehat{t}$, $\widehat{x}_1$, $\widehat{x}_2$ and $\widehat{x}_3$.

In these new variables, system (9.11) is autonomous, i.e.,

$$\frac{\partial\widehat{\rho}}{\partial\widehat{t}} + \frac{\partial(\widehat{\rho}\widehat{v}_k)}{\partial\widehat{x}_k} - \widehat{\rho} = 0, \tag{9.13a}$$

$$\frac{\partial(\widehat{\rho}\widehat{v}_i)}{\partial\widehat{t}}+\frac{\partial}{\partial\widehat{x}_k}\left(\widehat{\rho}\widehat{v}_i\widehat{v}_k-\frac{2}{\alpha}\left(\frac{K}{m}\right)\widehat{T}\left(\frac{1}{2}\left(\frac{\partial\widehat{v}_i}{\partial\widehat{x}_k}+\frac{\partial\widehat{v}_k}{\partial\widehat{x}_i}\right)-\frac{1}{3}\frac{\partial\widehat{v}_r}{\partial\widehat{x}_r}\delta_{ik}\right)\right.$$
$$\left.+\,\widehat{\rho}\left(\frac{K}{m}\right)\widehat{T}\delta_{ik}\right)-2\widehat{\rho}\widehat{v}_i=0,\qquad i=1,2,3, \tag{9.13b}$$
$$\frac{\partial(\widehat{\rho}\widehat{T})}{\partial\widehat{t}}+\frac{\partial}{\partial\widehat{x}_k}\left(\widehat{\rho}\widehat{T}\widehat{v}_k-\frac{5}{2\alpha}\left(\frac{K}{m}\right)\widehat{T}\frac{\partial\widehat{T}}{\partial\widehat{x}_k}\right)-3\widehat{\rho}\widehat{T}$$
$$+\left(\frac{2}{3}\widehat{\rho}\widehat{T}\delta_{\ell k}-\frac{4}{3\alpha}\widehat{T}\left(\frac{1}{2}\left(\frac{\partial\widehat{v}_\ell}{\partial\widehat{x}_k}+\frac{\partial\widehat{v}_k}{\partial\widehat{x}_\ell}\right)-\frac{1}{3}\frac{\partial\widehat{v}_r}{\partial\widehat{x}_r}\delta_{\ell k}\right)\right)\frac{\partial\widehat{v}_\ell}{\partial\widehat{x}_k}=0, \tag{9.13c}$$

and also in this case the transformed system has the same form as the original system with the exception of the inhomogeneity terms. The solution of the transformed system (9.13) depending only on $\widehat{t}$ gives the following solution to the original system:

$$\begin{aligned}
&\rho=\rho_0,\qquad T=T_0,\\
&v_1=v_{1_0}\cos(\omega t)+v_{2_0}\sin(\omega t)+\omega x_2,\\
&v_2=-v_{1_0}\sin(\omega t)+v_{2_0}\cos(\omega t)-\omega x_1,\\
&v_3=v_{3_0}-gt,
\end{aligned}$$

where $\rho_0>0$, v_{1_0}, v_{2_0}, v_{3_0} and $T_0>0$ are arbitrary constants.

Also, by assuming in (9.13) $\widehat{v}_1=\widehat{v}_2=\widehat{v}_3=0$, we get the solution

$$\rho=\frac{1}{\sqrt{\psi(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3)}},\qquad T=T_0\sqrt{\psi(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3)},$$
$$v_1=\omega x_2,\qquad v_2=-\omega x_1,\qquad v_3=-gt,$$

where T_0 is a positive constant, and ψ is again a positive solution of the Laplace equation

$$\frac{\partial^2\psi}{\partial\widehat{x}_1^2}+\frac{\partial^2\psi}{\partial\widehat{x}_2^2}+\frac{\partial^2\psi}{\partial\widehat{x}_3^2}=0.$$

Remark 9.3 Reduction to autonomous form of differential equations has some important applications in the study of the propagation of weak discontinuity waves compatible with hyperbolic first-order quasilinear systems [5]. In 1987 (see [1]), it has been shown how to study the propagation of weak discontinuities in a nonconstant state, characterized by a self-similar solution, as a propagation problem in a constant state for an equivalent autonomous system. The same approach has been applied in [28] in the case of axi-symmetric equations of magnetohydrodynamics in cylindrical symmetry, and in [16], where, using the equations for a rarefied gas with spherical symmetry in the context of extended thermodynamics, the point explosion problem has been considered.

9.4 Transformation to Linear Form

An important problem one can consider is whether a first-order system of partial differential equations can be transformed to linear form, and, also in this case, Theorem 9.1 plays a role. Notice that this problem is rather different from that addressed when one *linearizes* some equations around a special solution.

Before stating the main theorem, let us prove the following lemma.

Lemma 9.1 *Let us consider a first-order linear system of partial differential equations for the unknown* $\mathbf{u} \equiv (u_1(\mathbf{x}), \ldots, u_m(\mathbf{x}))$ *with* $\mathbf{x} \equiv (x_1, \ldots, x_n)$*, say*

$$\boldsymbol{\Delta} \equiv \sum_{i=1}^{n} \mathsf{A}_i(\mathbf{x}) \frac{\partial \mathbf{u}}{\partial x_i} - \mathsf{B}(\mathbf{x})\mathbf{u} - \mathbf{C}(\mathbf{x}) = \mathbf{0}, \tag{9.14}$$

where $\mathsf{A}_i(\mathbf{x})$ $(i = 1, \ldots, n)$ *and* $\mathsf{B}(\mathbf{x})$ *are* $m \times m$ *matrices with entries depending at most on the independent variables, and* $\mathbf{C}(\mathbf{x})$ *a vector function.*

Whatever the functional dependence of the coefficients in (9.14) *is, the Lie symmetry generated by*

$$\Xi = \sum_{\alpha=1}^{m} F_\alpha(\mathbf{x}) \frac{\partial}{\partial u_\alpha}, \tag{9.15}$$

where $\mathbf{F}(\mathbf{x}) \equiv (F_1(\mathbf{x}), \ldots, F_m(\mathbf{x}))$ *is a solution of*

$$\sum_{i=1}^{n} \mathsf{A}_i(\mathbf{x}) \frac{\partial \mathbf{F}}{\partial x_i} - \mathsf{B}(\mathbf{x})\mathbf{F} = \mathbf{0},$$

is admitted by (9.14).

Proof The first prolongation of (9.15) is

$$\Xi^{(1)} = \sum_{\alpha=1}^{m} F_\alpha(\mathbf{x}) \frac{\partial}{\partial u_\alpha} + \sum_{\alpha=1}^{m} \sum_{i=1}^{n} \frac{\partial F_\alpha(\mathbf{x})}{\partial x_i} \frac{\partial}{\partial u_{\alpha,i}},$$

whereupon

$$\Xi^{(1)}(\boldsymbol{\Delta})\big|_{\boldsymbol{\Delta}=\mathbf{0}} = \mathbf{0}.$$

□

Remark 9.4 The operator (9.15) corresponds to the linear superposition principle of solutions of linear equations: adding to the solution of a linear nonhomogeneous system a solution of the associated linear homogeneous system gives another solution of the linear nonhomogeneous system.

Now, we can state a theorem giving necessary and sufficient conditions for the existence of a one-to-one transformation suitable to map a nonlinear first-order system

of partial differential equations to a linear one. Necessary and sufficient conditions for the existence of invertible mappings relating a nonlinear first-order system of partial differential equations to a linear system of partial differential equations have been given by Kumei and Bluman [26] (see also [4]). However, the proof they give does not seem to involve *natural* conditions. A more *natural* proof, given in [13], relies on the introduction of the canonical variables related to some operators (in general not admitted by the source system of partial differential equations) whose linear combination (with coefficients given by arbitrary functions that are solution of a linear system of partial differential equations) is an admitted symmetry of the source system.

Theorem 9.3 *The nonlinear first-order system of partial differential equations*

$$\mathbf{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}\right) = \mathbf{0}, \tag{9.16}$$

where $\mathbf{x} \in \mathbb{D}_n \subseteq \mathbb{R}^n$, $\mathbf{u} \in \mathbb{D}_m \subseteq \mathbb{R}^m$, *can be transformed to the linear form*

$$\sum_{i=1}^{n} \mathsf{A}_i(\mathbf{z}) \frac{\partial \mathbf{w}}{\partial z_i} - \mathsf{B}(\mathbf{z})\mathbf{w} - \mathbf{C}(\mathbf{z}) = \mathbf{0}, \tag{9.17}$$

by means of the invertible point transformation

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{9.18}$$

if and only if it is left invariant by a Lie group of point transformations whose infinitesimal operator has the form

$$\Xi = \sum_{\alpha=1}^{m} F_\alpha(\mathbf{z}) \Xi_\alpha, \tag{9.19}$$

where the vector fields

$$\Xi_\alpha = \sum_{i=1}^{n} \xi_{\alpha i}(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\beta=1}^{m} \eta_{\alpha\beta}(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\beta}, \qquad \alpha = 1, \ldots, m$$

generate a distribution of rank m, $\xi_{\alpha i}(\mathbf{x}, \mathbf{u})$ *and* $\eta_{\alpha\beta}(\mathbf{x}, \mathbf{u})$ *being specific functions of their arguments (their form depends on the nonlinear system we are dealing with), and the functions* $F_\alpha(\mathbf{z})$ *satisfy the linear system*

$$\sum_{i=1}^{n} \mathsf{A}_i(\mathbf{z}) \frac{\partial \mathbf{F}}{\partial z_i} - \mathsf{B}(\mathbf{z})\mathbf{F} = \mathbf{0},$$

along with the conditions

$$\Xi_\alpha(\mathbf{z}) = \mathbf{0}, \qquad [\Xi_\alpha, \Xi_\beta] = 0, \qquad 1 \le \alpha < \beta \le m.$$

The vector fields Ξ_α do not need to correspond to admitted symmetries of the source system (in some cases this is so), but only their linear combination Ξ does.

Proof Suppose that system (9.16) admits the symmetry generated by (9.19), and that the conditions of the theorem are satisfied. By using Theorem 9.1, let us construct the canonical variables $\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u})$ (playing the role of the new independent variables) and $\mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u})$ (playing the role of the new dependent variables) such that the operators Ξ_α become

$$\Xi_\alpha = \frac{\partial}{\partial w_\alpha}, \qquad \alpha = 1, \ldots, m.$$

Consequently, the symmetry generated by Ξ, admitted by the source system (9.16), becomes

$$\Xi = \sum_{\alpha=1}^{m} F_\alpha(\mathbf{z}) \frac{\partial}{\partial w_\alpha},$$

which is responsible for the linear superposition of solutions.

This allows us to write the source system (9.16) in the form

$$\widetilde{\mathbf{\Delta}}\left(\mathbf{z}, \mathbf{w}, \mathbf{w}^{(1)}\right) = \mathbf{0},$$

that must be linear, i.e.,

$$\widetilde{\mathbf{\Delta}}\left(\mathbf{z}, \mathbf{w}, \mathbf{w}^{(1)}\right) \equiv \sum_{i=1}^{n} \mathsf{A}_i(\mathbf{z}) \frac{\partial \mathbf{w}}{\partial z_i} - \mathsf{B}(\mathbf{z})\mathbf{w} - \mathbf{C}(\mathbf{z}) = \mathbf{0}.$$

Conversely, if the nonlinear system (9.16) can be mapped by the invertible point transformation (9.18) to the linear system (9.17), then, as the latter admits the symmetry (9.15), then it follows that also the system (9.16) must admit a symmetry like (9.19). □

Remark 9.5 It is worth of being recalled a theorem by Müller and Matschat [31], who proved that for first-order systems of partial differential equations a linearizing transformation, if it exists, is a point transformation like (9.18).

9.4.1 Applications

Various examples, where Theorem 9.3 applies, are given below.

Example 9.4 (*Hodograph transformation*) It is well known (see [4]) that the 2×2 quasilinear first-order homogeneous system

$$\mathsf{A}_1(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_1} + \mathsf{A}_2(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{0}, \tag{9.20}$$

where A_1 and A_2 are 2×2 matrices with entries depending on $\mathbf{u} \equiv (u_1, u_2)$, by introducing the hodograph transformation (interchanging the role of dependent and independent variables), i.e.,

$$x_1 = x_1(u_1, u_2), \quad x_2 = x_2(u_1, u_2),$$

assumes the *linear* form

$$-\mathsf{A}_2(\mathbf{u})\mathsf{J}\begin{pmatrix} \dfrac{\partial x_1}{\partial u_1} \\ \dfrac{\partial x_1}{\partial u_2} \end{pmatrix} + \mathsf{A}_1(\mathbf{u})\mathsf{J}\begin{pmatrix} \dfrac{\partial x_2}{\partial u_1} \\ \dfrac{\partial x_2}{\partial u_2} \end{pmatrix} = \mathbf{0},$$

where $\mathsf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The hodograph transformation works for the 2×2 system (9.20) because the latter admits the symmetry [4]

$$\Xi = F_1(u_1, u_2)\frac{\partial}{\partial x_1} + F_2(u_1, u_2)\frac{\partial}{\partial x_2},$$

where F_1 and F_2 satisfy the linear system

$$-\mathsf{A}_2(\mathbf{u})\mathsf{J}\begin{pmatrix} \dfrac{\partial F_1}{\partial u_1} \\ \dfrac{\partial F_1}{\partial u_2} \end{pmatrix} + \mathsf{A}_1(\mathbf{u})\mathsf{J}\begin{pmatrix} \dfrac{\partial F_2}{\partial u_1} \\ \dfrac{\partial F_2}{\partial u_2} \end{pmatrix} = \mathbf{0},$$

and the application of Theorem 9.3 implies that the new independent variables are the old dependent variables, and the new dependent variables are the old independent variables.

The hodograph transformation is one-to-one if

$$\det\begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} \end{pmatrix} \neq 0.$$

Example 9.5 In [24], it has been considered the nonlinear wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = c(\psi)\frac{\partial}{\partial x}\left(c(\psi)\frac{\partial \psi}{\partial x}\right) \tag{9.21}$$

as a simplified model equation ruling the evolution of the director field of a nematic liquid crystal [39], showing that weakly nonlinear unidirectional waves satisfying (9.21) are described asymptotically by the equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2, \tag{9.22}$$

where $u(t,x)$ is the perturbation of $\psi(t,x)$ around some constant value $\psi=\psi_0$ and x is a space variable in a reference frame moving with the linearized wave velocity. This equation has a variational formulation (see Example 7.6 in Chap. 7).

Let us write Eq. (9.22) as a first-order system:

$$\frac{\partial u}{\partial x}=v, \qquad \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+\frac{1}{2}v^2=0.$$

In compact form, we have

$$u_{1,2}=u_2, \qquad u_{2,1}+u_1u_{2,2}+\frac{1}{2}u_2^2=0, \tag{9.23}$$

where $u_1=u$, $u_2=v$, x_1 denotes the time, and x_2 the space variable.

System (9.23) admits the Lie symmetry generated by

$$\Xi=F_1(x_1,u_2)\Xi_1+F_2(x_1,u_2)\Xi_2,$$

where the vector fields

$$\Xi_1=\frac{\partial}{\partial x_2}, \qquad \Xi_2=\frac{\partial}{\partial u_1}$$

are commuting, whereas $F_1(x_1,u_2)$ and $F_2(x_1,u_2)$ are arbitrary solutions of the linear system

$$\frac{\partial F_1}{\partial x_1}-\frac{u_2^2}{2}\frac{\partial F_1}{\partial u_2}-F_2=0, \qquad \frac{\partial F_2}{\partial u_2}-u_2\frac{\partial F_1}{\partial u_2}=0.$$

According to Theorem 9.3, introduce the new variables

$$z_1=x_1, \qquad z_2=u_2, \qquad w_1=u_1, \qquad w_2=x_2$$

(note that this change of variables corresponds to a semi-hodograph transformation) so that (9.23) transforms to linear form

$$w_{1,2}-z_2w_{2,2}=0,$$
$$w_{2,1}-\frac{z_2^2}{2}w_{2,2}-w_1=0.$$

Latter system is also equivalent to the following linear second-order equation:

$$w_{2,12} - 2z_2 w_{2,2} - \frac{z_2^2}{2} w_{2,22} = 0. \tag{9.24}$$

Introducing

$$w = w_{2,2},$$

Equation (9.24) reduces to first order,

$$w_{,1} - \frac{z_2^2}{2} w_{,2} = 2z_2 w,$$

that, integrated by the method of characteristics, gives

$$w(z_1, z_2) = z_2^{-4} \psi_1 \left(z_1 - \frac{2}{z_2} \right)$$

where ψ_1 is an arbitrary function of its argument. Another integration with respect to z_2 yields the general solution for $w_2(z_1, z_2)$:

$$w_2(z_1, z_2) = \int z_2^{-4} \psi_1 \left(z_1 - \frac{2}{z_2} \right) dz_2 + \psi_2(z_1),$$

$\psi_2(z_1)$ being an arbitrary function of z_1.

Notice that Ξ_1 (and of course Ξ) is a symmetry of (9.23), but Ξ_2 is a vector field not generating a symmetry of (9.23).

Example 9.6 (*Burgers equation*) The Burgers equation

$$\frac{\partial u}{\partial x_1} + u \frac{\partial u}{\partial x_2} - \frac{\partial}{\partial x_2} \left(\frac{\partial u}{\partial x_2} \right) = 0,$$

written in compact form under the form of a first-order system, namely

$$u_{2,2} = 2u_1, \qquad u_{2,1} = 2u_{1,2} - u_1^2, \tag{9.25}$$

where $u_1 = u$, admits the Lie symmetry generated by

$$\Xi = F_1(x_1, x_2)\Xi_1 + F_2(x_1, x_2)\Xi_2,$$

with

$$\Xi_1 = \exp\left(\frac{u_2}{4}\right) \left(u_1 \frac{\partial}{\partial u_1} + 4 \frac{\partial}{\partial u_2} \right), \qquad \Xi_2 = 2 \exp\left(\frac{u_2}{4}\right) \frac{\partial}{\partial u_1}, \tag{9.26}$$

where F_1 and F_2 satisfy the linear system

$$\frac{\partial F_1}{\partial x_2} = F_2, \qquad \frac{\partial F_2}{\partial x_2} = \frac{\partial F_1}{\partial x_1}.$$

Applying Theorem 9.3, and introducing

$$z_1 = x_1, \qquad z_2 = x_2,$$
$$w_1 = -\exp\left(-\frac{u_2}{4}\right), \qquad w_2 = \frac{1}{2}u_1 \exp\left(-\frac{u_2}{4}\right),$$

we have

$$\Xi = F_1(z_1, z_2)\frac{\partial}{\partial w_1} + F_2(z_1, z_2)\frac{\partial}{\partial w_2},$$

and (9.25) is mapped into the linear system

$$w_{1,2} = w_2, \qquad w_{2,2} = w_{1,1},$$

that corresponds to the linear heat equation for the field $w_1(z_1, z_2)$

$$w_{1,1} - w_{1,22} = 0.$$

Notice that in this case, system (9.25) does not admit the symmetry Ξ_2 in (9.26), whereas Ξ_1 is admitted; of course, the symmetry generated by Ξ is admitted.

Example 9.7 The equation

$$\frac{\partial u}{\partial x_1} - \frac{\partial}{\partial x_2}\left(\frac{1}{u^2}\frac{\partial u}{\partial x_2}\right) - \frac{\alpha}{u^2}\frac{\partial u}{\partial x_2} = 0, \quad \alpha \text{ constant},$$

occurring in two-phase flows in porous media [17], can be written in compact form as a first-order system, namely

$$u_{2,2} = u_1, \qquad u_{2,1} = \frac{1}{u_1^2}u_{1,2} + \frac{\alpha}{u_1},$$

where $u_1 = u$. This system admits the Lie symmetry

$$\Xi = F_1(x_1, u_2)\Xi_1 + F_2(x_1, u_2)\Xi_2,$$

where

$$\Xi_1 = \exp(-\alpha x_2)\left(\frac{1}{\alpha}\frac{\partial}{\partial x_2} + u_1\frac{\partial}{\partial u_1}\right),$$
$$\Xi_2 = -\frac{u_1^2 \exp(-\alpha x_2)}{\alpha}\frac{\partial}{\partial u_1},$$

with F_1 and F_2 solutions of the linear system

$$\frac{\partial F_1}{\partial u_2} = F_2, \qquad \frac{\partial F_2}{\partial u_2} = \frac{\partial F_1}{\partial x_1}.$$

The vector fields Ξ_1 and Ξ_2 are commuting. Thus, by introducing the canonical variables with respect to Ξ_1 and Ξ_2,

$$\begin{aligned} &z_1 = x_1, \qquad z_2 = u_2, \\ &w_1 = \exp(\alpha x_2), \qquad w_2 = \frac{\alpha}{u_1}\exp(\alpha x_2), \end{aligned}$$

we obtain

$$\Xi = F_1(z_1, z_2)\frac{\partial}{\partial w_1} + F_2(z_1, z_2)\frac{\partial}{\partial w_2},$$

and the source system assumes the linear form

$$w_{1,2} = w_2, \qquad w_{2,2} = w_{1,1};$$

the latter system is once again reducible to the linear heat equation

$$w_{1,1} - w_{1,22} = 0.$$

Also in this case, the generator Ξ_2 is not admitted, whereas Ξ_1 does.

Example 9.8 The first-order system (corresponding to a nonlinear telegraph equation)

$$u_{1,1} - a u_{2,2} + a u_1 = 0, \qquad u_{2,1} - \frac{b}{u_1^2} u_{1,2} = 0$$

(a and b constants), admits the Lie symmetry generated by

$$\Xi = F_1(z_1, z_2)\Xi_1 + F_2(z_1, z_2)\Xi_2,$$

where

$$\Xi_1 = \exp(-\alpha x_1)\left(\frac{\partial}{\partial x_1} - a u_1 \frac{\partial}{\partial u_1}\right), \qquad \Xi_2 = \frac{\partial}{\partial x_2},$$

with F_1 and F_2 solutions of the linear system

$$a\frac{\partial F_1}{\partial z_1} - \frac{\partial F_2}{\partial z_2} = 0, \qquad b\frac{\partial F_1}{\partial z_2} - z_1^2\frac{\partial F_2}{\partial z_1} = 0,$$

and

$$z_1 = u_1 \exp(a x_1), \qquad z_2 = u_2.$$

Theorem 9.3 allows us to introduce the new independent and dependent variables

$$z_1 = u_1 \exp(ax_1), \qquad z_2 = u_2,$$
$$w_1 = \frac{\exp(ax_1)}{a}, \qquad w_2 = x_2,$$

whereupon

$$\Xi = F_1(z_1, z_2)\frac{\partial}{\partial w_1} + F_2(z_1, z_2)\frac{\partial}{\partial w_2},$$

and the source system is rewritten in the form

$$aw_{1,1} - w_{2,2} = 0, \qquad bw_{1,2} - z_1^2 w_{2,1} = 0.$$

From the latter system, the following linear wave equation can be derived:

$$az_1^2 w_{1,11} - bw_{1,22} = 0.$$

In this case, both vector fields Ξ_1 and Ξ_2 are admitted symmetries.

Example 9.9 The system [43]

$$u_{1,1} - u_{2,2} = 0,$$
$$u_{2,1} - (1 + u_2)^2 u_{1,2} + (1 + u_2)u_2 = 0$$

admits the symmetry generated by

$$\Xi = F_1(z_1, z_2)\Xi_1 + F_2(z_1, z_2)\Xi_2,$$

where

$$\Xi_1 = \exp(-x_1)\left(\frac{\partial}{\partial x_1} + (1 + u_2)\frac{\partial}{\partial u_2}\right),$$
$$\Xi_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u_1},$$

with F_1 and F_2 solutions of the linear system

$$\frac{\partial F_1}{\partial z_1} + \frac{\partial F_2}{\partial z_2} = 0, \qquad z_2^2\frac{\partial F_1}{\partial z_2} + \frac{\partial F_2}{\partial z_1} = 0,$$

and

$$z_1 = x_2 - u_1, \qquad z_2 = \exp(-x_1)(1 + u_2).$$

Applying also in this case Theorem 9.3, we have the new variables

$$z_1 = x_2 - u_1, \qquad z_2 = \exp(-x_1)(1 + u_2),$$
$$w_1 = \exp(x_1), \qquad w_2 = x_2,$$

and the source system is mapped into the linear system

$$w_{1,1} + w_{2,2} = 0, \qquad z_2^2 w_{1,2} + w_{2,1} = 0,$$

that corresponds to the nonautonomous linear wave equation

$$w_{2,11} - z_2^2 w_{2,22} = 0.$$

9.5 Reduction of First-Order Quasilinear Systems to Homogeneous and Autonomous Form

Quasilinear first-order systems of partial differential equations are relevant in many problems of nonlinear wave propagation [5, 11]. In some cases, they can be nonautonomous, nonhomogeneous, or both, i.e., they have the general form

$$\sum_{i=1}^{n} \mathsf{A}_i\,(\mathbf{x}, \mathbf{u})\,\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}(\mathbf{x}, \mathbf{u}), \tag{9.27}$$

where $\mathbf{u}(\mathbf{x})$ is the unknown column vector with m components, $\mathsf{A}_1, \ldots, \mathsf{A}_n$ are $m \times m$ matrices and $\mathbf{B}$ a column vector, all with components depending at most on $\mathbf{x}$ and $\mathbf{u}$, respectively. The aim is to find the necessary and sufficient conditions for the existence of an invertible transformation

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}) \tag{9.28}$$

mapping (9.27) to autonomous and homogeneous first-order quasilinear systems, namely

$$\sum_{i=1}^{n} \widehat{\mathsf{A}}_i\,(\mathbf{w})\,\frac{\partial \mathbf{w}}{\partial z_i} = \mathbf{0}. \tag{9.29}$$

The invertible transformation (9.28) we consider is such that the new independent variables $\mathbf{z}$ are functions of the old independent variables $\mathbf{x}$ only. This choice ensures us that the target system is still in quasilinear form (see Lemma 9.2). On the contrary, if the new independent variables $\mathbf{z}$ are allowed to depend also on the old dependent variables $\mathbf{u}$, then in general it may occur that the target system is not in quasilinear form.

Lemma 9.2 *Given the quasilinear system* (9.27)*, an invertible mapping of the form* (9.28) *produces a system still in quasilinear form.*

Proof Straightforward, through the use of the chain rule. □

We distinguish the case where $m = n = 2$ (2×2 systems) from the general case where $m > 2$ and/or $n > 2$.

9.5.1 2 × 2 First-Order Quasilinear Systems

2×2 systems belonging to the class (9.27) describe models arising from different physical contexts such as viscoelastic materials [9], nonlinear elastic rods with variable cross section [25], nonlinear heat conduction [38], as well as problems with special geometric assumptions.

The mathematical structure of the system (9.27) does not allow for the direct linearization via the hodograph transformation (always possible in the homogeneous and autonomous case). Some of these systems can be mapped by an invertible point transformation to autonomous and homogeneous form, i.e., to a system like (9.29). The key idea of such a possibility is that any 2×2 homogeneous and autonomous first-order quasilinear system is left invariant by an infinite-parameter Lie group of point transformations (see Example 9.4) [4]. Therefore, if a nonautonomous and/or nonhomogeneous 2×2 quasilinear system can be reduced to autonomous and homogeneous form by an invertible point transformation, an infinite-parameter Lie group has to be admitted by the original system. Conversely, if a nonautonomous and/or nonhomogeneous 2×2 quasilinear system admits a suitable infinite-parameter Lie group of point symmetries, then an invertible map exists transforming it to autonomous and homogeneous form.

Theorem 9.4 *The nonautonomous and/or nonhomogeneous* 2×2 *quasilinear system*

$$\mathsf{A}_1(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_1} + \mathsf{A}_2(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_2} = \mathbf{B}(\mathbf{x}, \mathbf{u}) \tag{9.30}$$

transforms, under the action of the one-to-one variable transformation

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{9.31}$$

to the autonomous and homogeneous form

$$\widehat{\mathsf{A}}_1(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial z_1} + \widehat{\mathsf{A}}_2(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial z_2} = \mathbf{0}, \tag{9.32}$$

if and only if the system (9.30) *is left invariant by the Lie group of point symmetries*

$$\Xi = F_1(\mathbf{w}) \Xi_1 + F_2(\mathbf{w}) \Xi_2 \tag{9.33}$$

where the vector fields

$$\Xi_\alpha = \xi_{\alpha 1}(\mathbf{x})\frac{\partial}{\partial x_1} + \xi_{\alpha 2}(\mathbf{x})\frac{\partial}{\partial x_2} + \eta_{\alpha 1}(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_1} + \eta_{\alpha 2}(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_2}$$

($\alpha = 1, 2$) are commuting, i.e., $[\Xi_1, \Xi_2] = 0$, the infinitesimals, at most functions of the indicated arguments, depend on the specific system we are dealing with, whereas $F_1(\mathbf{w})$ and $F_2(\mathbf{w})$ are solutions of the linear system of partial differential equations

$$-\widehat{\mathsf{A}}_2(\mathbf{w})\,\mathsf{J}\begin{pmatrix}\dfrac{\partial F_1}{\partial w_1}\\[2mm] \dfrac{\partial F_1}{\partial w_2}\end{pmatrix} + \widehat{\mathsf{A}}_1(\mathbf{w})\,\mathsf{J}\begin{pmatrix}\dfrac{\partial F_2}{\partial w_1}\\[2mm] \dfrac{\partial F_2}{\partial w_2}\end{pmatrix} = \mathbf{0}, \tag{9.34}$$

where $\mathsf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof Let us assume that there exists a one-to-one point variable transformation (9.31) mapping (9.30) to (9.32). Since the homogeneous and autonomous 2×2 quasilinear system (9.32) admits the infinite-parameter Lie group of point symmetries generated by

$$\Xi = F_1(\mathbf{w})\frac{\partial}{\partial z_1} + F_2(\mathbf{w})\frac{\partial}{\partial z_2}, \tag{9.35}$$

with $F_1(\mathbf{w})$ and $F_2(\mathbf{w})$ solution of the linear system (9.34) (see Example 9.4 [4]), and the mapping (9.31) is invertible, to (9.35) it must correspond an infinite-parameter Lie group of point symmetries admitted by the source system.

Conversely, let us assume the system (9.30) to be left invariant by the Lie group generated by (9.33); by introducing the canonical variables associated both to Ξ_1 and Ξ_2,

$$\Xi_1(z_1) = \Xi_2(z_2) = 1, \quad \Xi_1(z_2) = \Xi_2(z_1) = 0, \quad \Xi_1(\mathbf{w}) = \Xi_2(\mathbf{w}) = \mathbf{0}$$

(this is possible because the two generators commute), we are able to write (9.33) in the form (9.35). Hence, the target system must be autonomous and homogeneous. □

Some examples where Theorem 9.4 works are discussed below.

Example 9.10 (*Nonlinear elastic rod*) Let us consider a nonlinear elastic incompressible rod with a variable cross section which is assumed everywhere symmetrical with respect to the axis of the rod and two fixed mutually orthogonal axes that are normal to it. Under the assumptions that the cross-sectional area is a (slowly) varying function of the distance measured along the rod, we are led to the following system of governing equations considered in [25]:

$$\begin{aligned} &u_{1,1} - u_{2,2} = 0, \\ &u_{2,1} - \frac{1}{\rho}\frac{d\mathcal{T}}{du_1}u_{1,2} = \frac{\mathcal{T}(u_1)}{\rho}\frac{S'(x_2)}{S(x_2)}, \end{aligned} \tag{9.36}$$

where x_1 is the time, x_2 the space coordinate along the rod, $u_1 = \dfrac{\partial u}{\partial x_2}$, $u_2 = \dfrac{\partial u}{\partial x_1}$, u being the displacement of the particle along the rod with respect to the axis; ρ is the constant density, $\mathcal{T}(u_1)$ denotes the stress, and $S(x_2)$ is the cross-sectional area.

By carrying on a full group analysis for the system (9.36), owing to the arbitrariness of the functions $\mathcal{T}(u_1)$ and $S(x_2)$, several cases arise. In particular, we find that (9.36) is left invariant by the Lie group generated by

$$\Xi = F_1(w_1, w_2)\,\Xi_1 + F_2(w_1, w_2)\,\Xi_2,$$

where

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{1}{S(x_2)}\left(\frac{\partial}{\partial x_2} + \frac{S'(x_2)}{S(x_2)}(u_1 + \beta)\frac{\partial}{\partial u_1}\right),$$

with $F_1(w_1, w_2)$ and $F_2(w_1, w_2)$ satisfying the following linear system of partial differential equations

$$\frac{\partial F_2}{\partial w_2} - \frac{\partial F_1}{\partial w_1} = 0, \qquad \rho w_1^2\frac{\partial F_2}{\partial w_1} + \gamma\frac{\partial F_1}{\partial w_2} = 0,$$

whereas β and γ are arbitrary constants, $S(x_2)$ is arbitrary, $\mathcal{T}(u_1) = \dfrac{\gamma}{\beta + u_1}$, and

$$w_1 = \frac{u_1 + \beta}{S(x_2)}, \qquad w_2 = u_2.$$

Theorem 9.4 can be used, whereupon, introducing the new independent variables

$$z_1 = x_1, \qquad z_2 = \int S(x_2)\,dx_2,$$

and taking as new dependent variables w_1 and w_2, system (9.36) reduces to autonomous and homogeneous form, namely

$$w_{1,1} - w_{2,2} = 0, \qquad \rho w_1^2 w_{2,1} + \gamma w_{2,2} = 0.$$

Example 9.11 (*Rate-type materials*) Within the framework of viscoelasticity, in order to describe the experimental behavior of special classes of materials under loading-unloading one-dimensional processes, the following quasilinear first-order system has been widely investigated [9]:

$$\begin{aligned} &u_{1,1} - u_{2,2} = 0, \\ &u_{2,1} - \phi\,(x_1, u_2)\, u_{1,2} = \psi\,(x_1, u_2)\,, \end{aligned} \tag{9.37}$$

where x_1 is the time, x_2 the space variable, u_1 denotes the Lagrangian velocity and u_2 the stress, whereas the functions $\phi\,(x_1, u_2)$ and $\psi\,(x_1, u_2)$ measure the noninstantaneous and the instantaneous response of the material to an increment of the stress, respectively. The functional forms of $\phi\,(x_1, u_2)$ and $\psi\,(x_1, u_2)$ are usually suggested by experimental investigations. In the particular case where

$$\phi = u_2^2, \qquad \psi = u_2(1 - u_2), \tag{9.38}$$

the system (9.37) has been linearized by means of a hodograph-like transformation [43].

Because of the arbitrariness of the constitutive functions $\phi\,(x_1, u_2)$ and $\psi\,(x_1, u_2)$, looking for the symmetries of (9.37), several cases may be distinguished. In particular, provided that

$$\begin{aligned} &\phi(x_1, u_2) = \frac{\varphi(x_1, u_2)}{g^2(x_1)}, \\ &\psi(x_1, u_2) = -\phi(x_1, u_2) - \frac{u_2 g'(x_1) + h'(x_1)}{g(x_1)}, \end{aligned} \tag{9.39}$$

where the function φ depends upon its arguments through the combination $(u_2 g(x_1) + h(x_1))$, system (9.37) admits the symmetry generated by

$$\Xi = F_1\,(w_1, w_2)\,\Xi_1 + F_2\,(w_1, w_2)\,\Xi_2,$$

where

$$\begin{aligned} \Xi_1 &= g(x_1)\left(\frac{\partial}{\partial x_1} - \left(h'(x_1) + u_2 g'(x_1)\right)\frac{\partial}{\partial u_2}\right), \\ \Xi_2 &= \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u_1}, \end{aligned}$$

with $g(x_1)$ and $h(x_1)$ arbitrary functions of time, F_1 and F_2 being solutions of the linear system of partial differential equations

$$\begin{aligned} &\frac{\partial F_2}{\partial w_2} - \frac{\partial F_1}{\partial w_1} = 0, \\ &\frac{\partial F_2}{\partial w_1} - \varphi(w_2)\frac{\partial F_1}{\partial w_2} = 0, \end{aligned}$$

where

$$w_1 = u_1 - x_2, \qquad w_2 = u_2 g(x_1) + h(x_1).$$

The conditions of Theorem 9.4 are satisfied, whereupon, introducing the new independent variables

$$z_1 = \int \frac{dx_1}{g(x_1)}, \qquad z_2 = x_2,$$

the system (9.37) reduces to autonomous and homogeneous form:

$$\begin{aligned} &w_{1,1} - w_{2,2} = 0, \\ &w_{2,1} - \varphi(w_2) w_{1,2} = 0. \end{aligned}$$

We remark that the choices

$$g(x_1) = \exp(-x_1), \qquad h(x_1) = 0, \qquad \varphi(w_2) = w_2^2$$

in (9.39) allow us to obtain the constitutive relations (9.38).

Example 9.12 (*Nonlinear hyperbolic heat equation*) Let us consider the hyperbolic heat equation proposed in [38]

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1} + \frac{u}{\tau_0}\right) - \frac{\partial}{\partial x_2}\left(\frac{\chi_0^2}{u^2}\frac{\partial u}{\partial x_2}\right) = 0, \tag{9.40}$$

where χ_0 and τ_0 are positive constants; this equation has been reduced [38] to the standard linear telegraph equation by means of a reciprocal Bäcklund transformation. Furthermore, by setting $u = 1/v$ and choosing $\tau_0 = \chi_0 = 1$, Eq. (9.40) specializes to the nonlinear telegraph equation linearized in [4] within the framework of potential symmetries. Actually, by setting $u = u_1$ and introducing the potential function $u_2\,(x_1, x_2)$, we are led to the following 2×2 quasilinear system

$$u_{1,1} - u_{2,2} = -\frac{u_1}{\tau_0}, \qquad u_{2,1} - \frac{\chi_0^2}{u_1^2} u_{1,2} = 0. \tag{9.41}$$

System (9.41) is invariant with respect to the Lie group generated by

$$\Xi = F_1\,(w_1, w_2)\,\Xi_1 + F_2\,(w_1, w_2)\,\Xi_2,$$

where

$$\Xi_1 = \exp\left(-\frac{x_1}{\tau_0}\right)\left(\tau_0 \frac{\partial}{\partial x_1} - u_1 \frac{\partial}{\partial u_1}\right), \qquad \Xi_2 = \frac{\partial}{\partial x_2},$$

along with

$$w_1 = \exp\left(\frac{x_1}{\tau_0}\right) u_1, \qquad w_2 = u_2,$$

where the functions $F_1(w_1, w_2)$ and $F_2(w_1, w_2)$ satisfy the linear system

$$\frac{\partial F_2}{\partial w_1} - \frac{\chi_0^2 \tau_0}{w_1^2} \frac{\partial F_1}{\partial w_2} = 0,$$
$$\frac{\partial F_2}{\partial w_2} - \tau_0 \frac{\partial F_1}{\partial w_1} = 0.$$

Using Theorem 9.4, we introduce the new independent variables

$$z_1 = \exp\left(\frac{x_1}{\tau_0}\right), \qquad z_2 = x_2,$$

whereupon system (9.41) takes the form

$$w_{1,1} - \tau_0 w_{2,2} = 0, \qquad w_{2,1} - \frac{\chi_0^2 \tau_0}{w_1^2} w_{1,2} = 0.$$

9.5.2 The General Case

Contrarily to what happens for 2×2 quasilinear first-order systems of partial differential equations, where the reduction to homogeneous and autonomous form is related to the possibility of their transformation to linear form [10], for general first-order quasilinear systems involving more than two independent variables and/or more than two dependent variables, this link does not exist. Nevertheless, also in this case, it is possible to recover the necessary and sufficient conditions allowing for the transformation to homogeneous and autonomous form within the framework of Lie group analysis of differential equations [34].

Given the first-order quasilinear system

$$\sum_{i=1}^{n} \mathsf{A}_i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}(\mathbf{x}, \mathbf{u}) \tag{9.42}$$

for the unknown field $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$, we are interested to the derivation of an invertible mapping, like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{9.43}$$

such that system (9.42) is transformed to the homogeneous and autonomous form

$$\sum_{i=1}^{n} \widehat{\mathsf{A}}_i(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial z_i} = \mathbf{0}. \tag{9.44}$$

Before stating the main theorem, let us make some preliminary considerations. Every system of the form (9.44) is invariant with respect to the n independent translations and to a uniform scaling of all the n independent variables, i.e., a system of the form (9.44) admits (whatever the functional form of the entries of matrices $\widehat{\mathsf{A}}^i$ is) the Lie point symmetries generated by the following vector fields:

$$\Xi_i = \frac{\partial}{\partial z_i} \quad (i = 1 \ldots, n), \qquad \Xi_{n+1} = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}.$$

These vector fields span an $(n+1)$-dimensional solvable Lie algebra [23] where the only nonzero commutators are

$$\left[\Xi_i, \Xi_{n+1}\right] = \Xi_i, \qquad i = 1, \ldots, n.$$

Moreover, this Lie Algebra contains an n-dimensional Abelian Lie subalgebra generating a distribution of rank n. Consequently, if a system of the form (9.42) can be mapped by (9.43) to the form (9.44), it has to admit, as subalgebra of the Lie algebra of its point symmetries, an $(n+1)$-dimensional Lie algebra with a suitable algebraic structure.

Now we have all the ingredients to prove the main theorem.

Theorem 9.5 ([34]) *A nonhomogeneous and/or nonautonomous first-order quasilinear system of partial differential equations*

$$\sum_{i=1}^{n} \mathsf{A}_i \left(\mathbf{x}, \mathbf{u}\right) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B} \left(\mathbf{x}, \mathbf{u}\right) \tag{9.45}$$

can be transformed by the invertible map

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}) \tag{9.46}$$

into an autonomous and homogeneous first-order quasilinear system,

$$\sum_{i=1}^{n} \widehat{\mathsf{A}}_i \left(\mathbf{w}\right) \frac{\partial \mathbf{w}}{\partial z_i} = \mathbf{0}, \tag{9.47}$$

if and only if it admits as subalgebra of the algebra of its Lie point symmetries an $(n+1)$-dimensional Lie algebra spanned by the vector fields

$$\Xi_i = \sum_{j=1}^{n} \xi_{ij}(\mathbf{x}) \frac{\partial}{\partial x_j} + \sum_{\alpha=1}^{m} \eta_{i\alpha}(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\alpha}, \qquad i = 1, \ldots, n+1,$$

such that

$$\left[\Xi_i, \Xi_j\right] = 0, \qquad \left[\Xi_i, \Xi_{n+1}\right] = \Xi_i, \qquad 1 \le i < j \le n.$$

Furthermore, the $n \times n$ matrix whose (i, j)-entry is ξ_{ij} $(i, j = 1, \ldots, n)$ must be nonsingular, and the variables $\mathbf{w}$*, which by construction are invariants of* $\Xi_1, \ldots, \Xi_n$*, must be invariant with respect to* Ξ_{n+1} *too.*

Proof Suppose the conditions of the theorem are satisfied. By using the commuting generators $\Xi_1, \ldots, \Xi_n$, under the hypothesis that the infinitesimals of the independent variables are independent of the dependent variables, Theorem 9.1 allows for the construction of the variable transformation

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}),$$

such that

$$\Xi_i = \frac{\partial}{\partial z_i}, \qquad i = 1, \ldots, n.$$

Therefore, what we have obtained is a variable transformation like (9.46) allowing to write the system (9.45) in autonomous form.

Finally, since $[\Xi_i, \Xi_{n+1}] = \Xi_i$ $(i = 1, \ldots, n)$, it is

$$\begin{aligned}
&\Xi_i(\Xi_{n+1}(z_k)) = \Xi_{n+1}(\Xi_i(z_k)) + \Xi_i(z_k) = \delta_{ik}, \qquad i, k = 1, \ldots, n,\\
&\Xi_i(\Xi_{n+1}(w_\alpha)) = \Xi_{n+1}(\Xi_j(w_\alpha)) + \Xi_i(w_\alpha) = 0, \qquad i = 1, \ldots, n,\ \alpha = 1, \ldots, m,
\end{aligned}$$

where δ_{ik} is the Kronecker symbol; these relations, together with the hypothesis that the variables w_α $(\alpha = 1, \ldots, m)$ are invariant with respect to Ξ_{n+1}, allow the vector field Ξ_{n+1} to gain the representation

$$\Xi_{n+1} = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}.$$

As a consequence, the system, written in terms of the variables $\mathbf{z}$, $\mathbf{w}$, must be homogeneous too, i.e., it has the form (9.47).

Conversely, if the nonautonomous and/or nonhomogeneous system (9.45) can be mapped by the invertible point transformation (9.46) to the autonomous and homogeneous system (9.47), then, since the latter admits the n vector fields $\dfrac{\partial}{\partial z_i}$, spanning an n-dimensional Abelian Lie algebra, and the vector field $\displaystyle\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$, then it follows that also the system (9.45) must admit $(n + 1)$ Lie point symmetries with the requested algebraic structure. □

Remark 9.6 The first part of the proof is quite similar to the proof of Theorem 9.2, the only difference being that here the new independent variables depend only on the old independent variables.

Of course, Theorem 9.5 may be used also for 2×2 quasilinear systems, that, as proved in Theorem 9.4, can be mapped to autonomous and homogeneous first-order quasilinear form provided that they admit an infinite-dimensional Lie algebra of symmetries. As a consequence, the following result holds true.

Corollary 9.1 *If a nonautonomous and/or nonhomogeneous first-order quasilinear 2×2 system admits as subalgebra of the algebra of Lie symmetries a three-dimensional Lie algebra with the properties required by Theorem 9.5, then it must admit an infinite-dimensional algebra of Lie symmetries.*

From an algorithmic point of view, given a system

$$\sum_{i=1}^{n} \mathsf{A}_i \left(\mathbf{x}, \mathbf{u}\right) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}\left(\mathbf{x}, \mathbf{u}\right),$$

the following steps have to be taken:

- determine the Lie algebra $\mathbb{L}$ of point symmetries (various computer algebra packages are available [2, 21, 35]);
- if $\dim(\mathbb{L}) \geq n + 1$:

1. determine the $(n + 1)$-dimensional Lie subalgebras;
2. check if among the Lie subalgebras there is a Lie algebra having the structure required by Theorem 9.5, and then reduce the system to homogeneous and autonomous form.

Hereafter, we present some examples where Theorem 9.5 can be applied.

Example 9.13 (*Rate-type materials revisited*) The system in Example 9.11 can be mapped to autonomous and homogeneous form using Theorem 9.5. In fact, it is easily recognized that system (9.37), along with (9.39), admits the three-dimensional Lie algebra spanned by

$$\begin{aligned}
\Xi_1 &= \exp(-x_1)\left(\frac{\partial}{\partial x_1} + \exp(-x_1)u_2\frac{\partial}{\partial u_2}\right),\\
\Xi_2 &= \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u_1},\\
\Xi_3 &= \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial u_1} + \exp(-x_1)u_2\frac{\partial}{\partial u_2}.
\end{aligned}$$

Example 9.14 (*Nonlinear hyperbolic heat equation revisited*) The system in Example 9.12 can be mapped to autonomous and homogeneous form using Theorem 9.5

by observing that system (9.41) admits the three-dimensional Lie algebra spanned by

$$\begin{aligned}
\Xi_1 &= \exp\left(-\frac{x_1}{\tau_0}\right)\left(\tau_0\frac{\partial}{\partial x_1} - u_1\frac{\partial}{\partial u_1}\right),\\
\Xi_2 &= \frac{\partial}{\partial x_2},\\
\Xi_3 &= \tau_0\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - u_1\frac{\partial}{\partial u_1}.
\end{aligned}$$

Example 9.15 (*Shallow water equations*) Let us consider the (2+1)-dimensional system of the shallow water equations describing the behavior of an inviscid fluid flowing over a bottom flat topography, incorporating Coriolis terms due to the horizontal components of the Earth rotation vector [42],

$$\begin{aligned}
&\frac{\partial h}{\partial t} + u_1\frac{\partial h}{\partial x_1} + u_2\frac{\partial h}{\partial x_2} + h\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = 0,\\
&\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + g\frac{\partial h}{\partial x_1} = 2\omega u_2,\\
&\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + g\frac{\partial h}{\partial x_2} = -2\omega u_1,
\end{aligned} \tag{9.48}$$

where h is the height of the fluid, (u_1, u_2) the components of its velocity, g the gravitational constant, and ω the constant angular velocity of the fluid around the x_3-axis responsible for the Coriolis force.

System (9.48) is autonomous, nonhomogeneous, and involves three independent variables; hence, in order to map it to autonomous and homogeneous form by means of Theorem 9.5, we need a four-dimensional Lie algebra of point symmetries. The Lie point symmetries admitted by system (9.48) span a nine-dimensional Lie algebra generated by the following vector fields:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x_1}, \qquad \Xi_3 = \frac{\partial}{\partial x_2},\\
&\Xi_4 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2},\\
&\Xi_5 = \cos(2\omega t)\frac{\partial}{\partial x_1} - \sin(2\omega t)\frac{\partial}{\partial x_2} - 2\omega\sin(2\omega t)\frac{\partial}{\partial u_1} - 2\omega\cos(2\omega t)\frac{\partial}{\partial u_2},\\
&\Xi_6 = \sin(2\omega t)\frac{\partial}{\partial x_1} + \cos(2\omega t)\frac{\partial}{\partial x_2} + 2\omega\cos(2\omega t)\frac{\partial}{\partial u_1} - 2\omega\sin(2\omega t)\frac{\partial}{\partial u_2},\\
&\Xi_7 = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + 2h\frac{\partial}{\partial h},\\
&\Xi_8 = \sin(2\omega t)\frac{\partial}{\partial t} + \omega(x_1\cos(2\omega t) + x_2\sin(2\omega t))\frac{\partial}{\partial x_1}
\end{aligned}$$

$$
\begin{aligned}
&+ \omega(x_2\cos(2\omega t) - x_1\sin(2\omega t))\frac{\partial}{\partial x_2} - 2\omega h\cos(2\omega t)\frac{\partial}{\partial h}\\
&+ \omega((2\omega x_2 - u_1)\cos(2\omega t) + (-2\omega x_1 + u_2)\sin(2\omega t))\frac{\partial}{\partial u_1}\\
&- \omega((2\omega x_1 + u_2)\cos(2\omega t) + (2\omega x_2 + u_1)\sin(2\omega t))\frac{\partial}{\partial u_2},\\
\Xi_9 = {}&\cos(2\omega t)\frac{\partial}{\partial t} + \omega(x_2\cos(2\omega t) - x_1\sin(2\omega t))\frac{\partial}{\partial x_1}\\
&- \omega(x_1\cos(2\omega t) + x_2\sin(2\omega t))\frac{\partial}{\partial x_2} - 2\omega h\sin(2\omega t)\frac{\partial}{\partial h}\\
&- \omega((2\omega x_1 - u_2)\cos(2\omega t) + (2\omega x_2 - u_1)\sin(2\omega t))\frac{\partial}{\partial u_1}\\
&- \omega((2\omega x_2 + u_1)\cos(2\omega t) - (2\omega x_1 + u_2)\sin(2\omega t))\frac{\partial}{\partial u_2}.
\end{aligned}
$$

Hence, we have the invariance with respect to time and space translations, rotation, helical rotations, scaling, and conformal transformations.

By considering the four-dimensional subalgebra spanned by

$$
\begin{aligned}
&\widehat{\Xi}_1 = \Xi_1 + \omega\Xi_4 - \Xi_9, \qquad \widehat{\Xi}_2 = \Xi_3 - \Xi_6,\\
&\widehat{\Xi}_3 = -\Xi_2 + \Xi_5, \qquad \widehat{\Xi}_4 = \frac{1}{2}\left(\Xi_7 - \frac{1}{\omega}\Xi_8\right),
\end{aligned}
$$

the nonzero commutators are

$$
\left[\widehat{\Xi}_i, \widehat{\Xi}_4\right] = \widehat{\Xi}_i, \qquad i = 1, \ldots, 3.
$$

The hypotheses of Theorem 9.5 are satisfied; hence, we may introduce the new independent (τ, z_1, z_2) and dependent $(\widehat{h}, \widehat{u}_1, \widehat{u}_2)$ variables

$$
\begin{aligned}
\tau &= -\frac{1}{2\omega}\cot(\omega t),\\
z_1 &= \frac{1}{2}(x_2 - x_1\cot(\omega t)),\\
z_2 &= -\frac{1}{2}(x_1 + x_2\cot(\omega t)),\\
\widehat{h} &= \frac{1 - \cos(2\omega t)}{2}h,\\
\widehat{u}_1 &= -\frac{1}{2}(u_1\sin(2\omega t) - u_2(1 - \cos(2\omega t)) - 2\omega x_1),\\
\widehat{u}_2 &= -\frac{1}{2}(u_1(1 - \cos(2\omega t)) + u_2\sin(2\omega t) - 2\omega x_2),
\end{aligned}
$$

whereupon

$$\widehat{\Xi}_1 = \frac{\partial}{\partial \tau}, \qquad \widehat{\Xi}_2 = \frac{\partial}{\partial z_1}, \qquad \widehat{\Xi}_3 = \frac{\partial}{\partial z_2},$$
$$\widehat{\Xi}_4 = \tau \frac{\partial}{\partial \tau} + z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2},$$

and the system (9.48) reduces to homogeneous and autonomous form:

$$\frac{\partial \widehat{h}}{\partial \tau} + \widehat{u}_1 \frac{\partial \widehat{h}}{\partial z_1} + \widehat{u}_2 \frac{\partial \widehat{h}}{\partial z_2} + \widehat{h}\left(\frac{\partial \widehat{u}_1}{\partial z_1} + \frac{\partial \widehat{u}_2}{\partial z_2}\right) = 0,$$
$$\frac{\partial \widehat{u}_1}{\partial \tau} + \widehat{u}_1 \frac{\partial \widehat{u}_1}{\partial z_1} + \widehat{u}_2 \frac{\partial \widehat{u}_1}{\partial z_2} + g \frac{\partial \widehat{h}}{\partial z_1} = 0,$$
$$\frac{\partial \widehat{u}_2}{\partial \tau} + \widehat{u}_1 \frac{\partial \widehat{u}_2}{\partial z_1} + \widehat{u}_2 \frac{\partial \widehat{u}_2}{\partial z_2} + g \frac{\partial \widehat{h}}{\partial z_2} = 0.$$

It should be noticed that a similar transformation has been derived in [7] through the comparison of the Lie algebra of point symmetries admitted by the rotating shallow water equations with the Lie algebra of point symmetries of the nonrotating shallow water equations. Nevertheless, the transformation reported in [7] exists because of Theorem 9.5.

Example 9.16 (*Perfect gas in rotation and subject to gravity*) Considering the system (9.11) where we neglect viscous effects and heat conduction, we have a first-order quasilinear system of partial differential equations,

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} \frac{\partial (\rho v_k)}{\partial x_k} = 0,$$
$$\rho\left(\frac{\partial v_i}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial v_i}{\partial x_k}\right) + \frac{K}{m} \frac{\partial (\rho T)}{\partial x_i} = \rho (F_i^{(e)} + F_i^{(i)}), \qquad i = 1, 2, 3, \tag{9.49}$$
$$\frac{\partial T}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial T}{\partial x_k} + \frac{2}{3} T \sum_{k=1}^{3} \frac{\partial v_k}{\partial x_k} = 0.$$

In such a case, we have a nonautonomous and nonhomogeneous system involving four independent variables; hence, a five-dimensional Lie algebra of point symmetries is needed in order to apply Theorem 9.5.

Looking for the Lie point symmetries admitted by system (9.49), we obtain a 14-dimensional Lie algebra generated by the following vector fields:

$$
\begin{aligned}
\Xi_1 &= \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x_3}, \qquad \Xi_3 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + v_2\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial v_2},\\
\Xi_4 &= t\frac{\partial}{\partial x_3} + \frac{\partial}{\partial v_3}, \qquad \Xi_5 = \rho\frac{\partial}{\partial \rho},\\
\Xi_6 &= x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \left(x_3 + \frac{1}{2}gt^2\right)\frac{\partial}{\partial x_3} + v_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial v_2}\\
&\quad + (v_3 + gt)\partial_{v_3} + 2T\frac{\partial}{\partial T},\\
\Xi_7 &= t\frac{\partial}{\partial t} + \omega t x_2\frac{\partial}{\partial x_1} - \omega t x_1\frac{\partial}{\partial x_2} - gt^2\frac{\partial}{\partial x_3} + (-v_1 + \omega t v_2 + \omega x_2)\frac{\partial}{\partial v_1}\\
&\quad + (-\omega t v_1 - v_2 - \omega x_1)\frac{\partial}{\partial v_2} + (-v_3 - 2gt)\frac{\partial}{\partial v_3} - 2T\frac{\partial}{\partial T},\\
\Xi_8 &= t^2\frac{\partial}{\partial t} + t(x_1 + \omega t x_2)\frac{\partial}{\partial x_1} - t(\omega t x_1 - x_2)\frac{\partial}{\partial x_2} - \frac{1}{2}t(gt^2 - 2x_3)\frac{\partial}{\partial x_3}\\
&\quad - 3\rho t\frac{\partial}{\partial \rho} + (-tv_1 + \omega t^2 v_2 + x_1 + 2\omega t x_2)\frac{\partial}{\partial v_1}\\
&\quad + (-\omega t^2 v_1 - tv_2 - 2\omega t x_1 + x_2)\frac{\partial}{\partial v_2} + \frac{1}{2}(-3gt^2 - 2tv_3 + 2x_3)\frac{\partial}{\partial v_3} - 2Tt\frac{\partial}{\partial T},\\
\Xi_9 &= \cos(\omega t)\frac{\partial}{\partial x_1} - \sin(\omega t)\frac{\partial}{\partial x_2} - \omega\sin(\omega t)\frac{\partial}{\partial v_1} - \omega\cos(\omega t)\frac{\partial}{\partial v_2},\\
\Xi_{10} &= \sin(\omega t)\frac{\partial}{\partial x_1} + \cos(\omega t)\frac{\partial}{\partial x_2} + \omega\cos(\omega t)\frac{\partial}{\partial v_1} - \omega\sin(\omega t)\frac{\partial}{\partial v_2},\\
\Xi_{11} &= -t\cos(\omega t)\frac{\partial}{\partial x_1} + t\sin(\omega t)\frac{\partial}{\partial x_2} + (-\cos(\omega t) + \omega t\sin(\omega t))\frac{\partial}{\partial v_1}\\
&\quad + (\sin(\omega t) + \omega t\cos(\omega t))\frac{\partial}{\partial v_2},\\
\Xi_{12} &= t\sin(\omega t)\frac{\partial}{\partial x_1} + t\cos(\omega t)\frac{\partial}{\partial x_2} + (\sin(\omega t) + \omega t\cos(\omega t))\frac{\partial}{\partial v_1}\\
&\quad + (\cos(\omega t) - \omega\sin(\omega t))\frac{\partial}{\partial v_2},\\
\Xi_{13} &= (-gt^2 - 2x_3)\cos(\omega t)\frac{\partial}{\partial x_1} + (gt^2 + 2x_3)\sin(\omega t)\frac{\partial}{\partial x_2}\\
&\quad + 2(x_1\cos(\omega t) - x_2\sin(\omega t))\frac{\partial}{\partial x_3}\\
&\quad + (-2gt\cos(\omega t) - 2v_3\cos(\omega t) + g\omega t^2\sin(\omega t) + 2\omega x_3\sin(\omega t))\frac{\partial}{\partial v_1}\\
&\quad + (g\omega t^2\cos(\omega t) + 2\omega x_3\cos(\omega t) + 2gt\sin(\omega t) + 2v_3\sin(\omega t))\frac{\partial}{\partial v_2}\\
&\quad + 2(v_1\cos(\omega t) - \omega x_2\cos(\omega t) - v_2\sin(\omega t) - \omega x_1\sin(\omega t))\frac{\partial}{\partial v_3},\\
\Xi_{14} &= (gt^2 + 2x_3)\sin(\omega t)\frac{\partial}{\partial x_1} + (gt^2 + 2x_3)\cos(\omega t)\frac{\partial}{\partial x_2}
\end{aligned}
$$

$$
\begin{aligned}
&- 2(x_2\cos(\omega t) + x_1\sin(\omega t))\frac{\partial}{\partial x_3}\\
&+ (g\omega t^2\cos(\omega t) + 2\omega x_3\cos(\omega t) + 2gt\sin(\omega t) + 2v_3\sin(\omega t))\frac{\partial}{\partial v_1}\\
&+ (2gt\cos(\omega t) + 2v_3\cos(\omega t) - g\omega t^2\sin(\omega t) - 2\omega x_3\sin(\omega t))\frac{\partial}{\partial v_2}\\
&+ 2(-v_2\cos(\omega t) - \omega x_1\cos(\omega t) - v_1\sin(\omega t) + \omega x_2\sin(\omega t))\frac{\partial}{\partial v_3}.
\end{aligned}
$$

By taking the five-dimensional Lie subalgebra of Lie point symmetries spanned by the vector fields

$$
\begin{aligned}
&\widehat{\Xi}_1 = \Xi_1 + \omega\Xi_3 - g\,\Xi_4, \quad \widehat{\Xi}_2 = \Xi_9,\\
&\widehat{\Xi}_3 = \Xi_{10}, \quad \widehat{\Xi}_4 = \Xi_2,\\
&\widehat{\Xi}_5 = \Xi_5 + \Xi_7,
\end{aligned}
$$

the nonzero commutators are

$$
\left[\widehat{\Xi}_i, \widehat{\Xi}_5\right] = \widehat{\Xi}_i, \qquad i = 1, \dots, 4.
$$

All the hypotheses of Theorem 9.5 are satisfied, thus we may introduce the new independent (τ, z_1, z_2, z_3) and dependent $(\widehat{\rho}, \widehat{v}_1, \widehat{v}_2, \widehat{v}_3, \widehat{T})$ variables

$$
\begin{aligned}
&z_1 = x_1\cos(\omega t) - x_2\sin(\omega t),\\
&z_2 = x_1\sin(\omega t) + x_2\cos(\omega t),\\
&z_3 = x_3 + \frac{1}{2}gt^2, \qquad \tau = t,\\
&\widehat{\rho} = \rho, \qquad \widehat{T} = T,\\
&\widehat{v}_1 = (v_1 - \omega x_2)\cos(\omega t) - (v_2 + \omega x_1)\sin(\omega t),\\
&\widehat{v}_2 = (v_2 + \omega x_1)\cos(\omega t) + (v_1 - \omega x_2)\sin(\omega t),\\
&\widehat{v}_3 = v_3 + gt,
\end{aligned}
$$

whereupon

$$
\begin{aligned}
&\widehat{\Xi}_1 = \frac{\partial}{\partial\tau}, \qquad \widehat{\Xi}_2 = \frac{\partial}{\partial z_1}, \qquad \widehat{\Xi}_3 = \frac{\partial}{\partial z_2}, \qquad \widehat{\Xi}_4 = \frac{\partial}{\partial z_3},\\
&\widehat{\Xi}_5 = \tau\frac{\partial}{\partial\tau} + z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3},
\end{aligned}
$$

and the system (9.49) reduces to homogeneous and autonomous form:

$$\begin{aligned}
&\frac{\partial\widehat{\rho}}{\partial\tau}+\sum_{k=1}^{3}\frac{\partial(\widehat{\rho}\widehat{v}_k)}{\partial z_k}=0,\\
&\widehat{\rho}\left(\frac{\partial\widehat{v}_i}{\partial\tau}+\sum_{k=1}^{3}\widehat{v}_k\frac{\partial\widehat{v}_i}{\partial z_k}\right)+\frac{K}{m}\frac{\partial(\widehat{\rho}\widehat{T})}{\partial z_i}=0, \qquad i=1,2,3,\\
&\frac{\partial\widehat{T}}{\partial\tau}+\sum_{k=1}^{3}\widehat{v}_k\frac{\partial\widehat{T}}{\partial z_k}+\frac{2}{3}\widehat{T}\sum_{k=1}^{3}\frac{\partial\widehat{v}_k}{\partial z_k}=0.
\end{aligned} \tag{9.50}$$

From a physical point of view, system (9.50) represents the system (9.49) in a reference frame where *apparent* forces (gravity, Coriolis and centrifugal forces) disappear.

Remark 9.7 The reduction of a nonautonomous and/or nonhomogeneous quasilinear system to autonomous and homogeneous form, besides its intrinsic interest, may reveal useful in investigating a well-known problem connected with a system of hyperbolic conservation laws, i.e., the Riemann problem [11, 41], where one takes a piecewise constant initial datum with a single discontinuity. As well known, there is an existence and uniqueness theorem for the Riemann problem for a system of conservation laws; on the contrary, analogous results have not been obtained yet for a system of balance laws, even for a generalized (i.e., piecewise nonconstant initial data) Riemann problem [3]. In the case in which we have a system of balance laws, and the application of the theorem leads to a system of conservation laws, then one may investigate a Riemann problem (classical or generalized) for the original system of balance laws by studying an associated Riemann problem (which can be classical or generalized) for a system of conservation laws. Once the latter problem has been solved, thanks to the inverse transformation, it is possible to obtain the corresponding solution of the original system. An application can be found in [8].

9.6 Reduction of Nonlinear First-Order PDEs to Autonomous Form Polynomially Homogeneous in the Derivatives

Let us now consider a general nonlinear system of first-order partial differential equations involving the derivatives of the unknown variables in polynomial (of degree greater than the first) form, and establish a theorem giving necessary and sufficient conditions in order to map it to an autonomous system which is polynomially homogeneous in the derivatives. Differently from Theorem 9.5 (concerned with the transformation of a general first-order quasilinear system of partial differential equations into a first-order quasilinear homogeneous and autonomous system), we consider now the possibility of admitting an invertible point transformation like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{9.51}$$

i.e., a mapping where the new independent variables $\mathbf{z}$ are allowed to depend also on the old dependent ones. According to Theorem 9.5, this choice works very well in some relevant situations, e.g., Monge–Ampère systems, where the target system turns out to be quasilinear; however, there are cases where the system we obtain is polynomially homogeneous in the derivatives but not quasilinear. This fact means that the conditions of Theorem 9.5, using the invertible point transformation (9.51), are only necessary for the reduction of a *nonlinear* first-order system to autonomous and homogeneous *quasilinear* form [18]. Thence, necessary and sufficient conditions can be guaranteed if the derivatives of the unknown variables in the source system appear only in polynomial form [19].

Let us consider a general system of first-order partial differential equations

$$\mathbf{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}\right) = \mathbf{0}, \tag{9.52}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{u}^{(1)} \in \mathbb{R}^{mn}$ are the independent variables, the dependent variables, and the first-order partial derivatives, respectively. In particular, in the following we consider a system of the form (9.52) made of equations which are polynomial in the derivatives, with coefficients depending at most on $\mathbf{x}$ and $\mathbf{u}$, i.e.,

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=1}^{N_s} A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x}, \mathbf{u}) \prod_{k=1}^{|\boldsymbol{\alpha}|} \frac{\partial u_{\alpha_k}}{\partial x_{j_k}} + B^s(\mathbf{x}, \mathbf{u}) = 0, \qquad s = 1, \ldots, m, \tag{9.53}$$

where $\boldsymbol{\alpha}$ is the multi-index $(\alpha_1, \ldots, \alpha_r)$, $\mathbf{j}$ the multi-index $(j_1, \ldots, j_r)$, $\alpha_k = 1, \ldots, m$, $j_k = 1, \ldots, n$, N_s are integers, and $A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x}, \mathbf{u})$, $B^s(\mathbf{x}, \mathbf{u})$ smooth functions of their arguments.

The aim is to determine necessary and sufficient conditions for the construction of an invertible point transformation

$$\mathbf{z} = \mathbf{Z}\left(\mathbf{x}, \mathbf{u}\right), \qquad \mathbf{w} = \mathbf{W}\left(\mathbf{x}, \mathbf{u}\right), \tag{9.54}$$

mapping (9.53) into an equivalent autonomous system which is homogeneous polynomial in the derivatives $\mathbf{w}^{(1)}$, i.e., made by equations of the form

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=\overline{N}_s} \widetilde{A}^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{w}) \prod_{k=1}^{\overline{N}_s} \frac{\partial w_{\alpha_k}}{\partial z_{j_k}} = 0, \qquad s = 1, \ldots, m,$$

for some integers $\overline{N}_s$; of course, it may occur that the target system turns out to be linear in the derivatives, i.e., $\overline{N}_s = 1$ $(s = 1, \ldots, m)$, whereupon we have an autonomous and homogeneous quasilinear system.

The following lemma guarantees that an invertible point transformation like (9.54) preserves the polynomial structure in the derivatives.

Lemma 9.3 *Given a first-order system of partial differential equations like* (9.53) *which is polynomial in the derivatives, then an invertible point transformation like* (9.54) *produces a first-order system which is still polynomial in the derivatives.*

Proof Straightforward, by using the chain rule. □

Then, we can prove the following theorem [19].

Theorem 9.6 *The nonlinear first-order system of partial differential equations polynomial in the derivatives*

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=1}^{N_s} A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x},\mathbf{u}) \prod_{k=1}^{|\boldsymbol{\alpha}|} \frac{\partial u_{\alpha_k}}{\partial x_{j_k}} + B^s(\mathbf{x},\mathbf{u}) = 0, \qquad s = 1,\ldots,m \tag{9.55}$$

is mapped by an invertible point transformation

$$\mathbf{z} = \mathbf{Z}(\mathbf{x},\mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x},\mathbf{u}) \tag{9.56}$$

to the equivalent nonlinear first-order autonomous system having homogeneous polynomial form

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=\overline{N}_s} \widetilde{A}^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{w}) \prod_{k=1}^{\overline{N}_s} \frac{\partial w_{\alpha_k}}{\partial z_{j_k}} = 0, \qquad s = 1,\ldots,m, \tag{9.57}$$

for some integers $\overline{N}_s$*, if and only if there exists an* $(n+1)$*-dimensional subalgebra of the Lie algebra of point symmetries, admitted by system* (9.55)*, spanned by the vector fields*

$$\Xi_i = \sum_{j=1}^{n} \xi_{ij}(\mathbf{x},\mathbf{u}) \frac{\partial}{\partial x_j} + \sum_{\alpha=1}^{m} \eta_{i\alpha}(\mathbf{x},\mathbf{u}) \frac{\partial}{\partial u_\alpha}, \qquad i = 1,\ldots,n+1,$$

generating a distribution of rank $(n+1)$*, and such that*

$$\left[\Xi_i, \Xi_j\right] = 0, \quad \left[\Xi_i, \Xi_{n+1}\right] = \Xi_i, \qquad 1 \le i < j \le n. \tag{9.58}$$

Moreover, the variables $\mathbf{w}$*, which by construction are invariants of* $\Xi_1, \ldots, \Xi_n$*, have to be invariant with respect to* Ξ_{n+1} *too.*

Proof Suppose the conditions of Theorem 9.6 are satisfied, and so the system (9.55) admits an $(n+1)$-dimensional algebra as subalgebra of the algebra of its Lie point

symmetries generating a distribution of rank $(n+1)$ and verifying the structure conditions (9.58). Using Theorem 9.1, we are able to construct a set of canonical variables $\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u})$ (independent variables) and $\mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u})$ (dependent variables) such that the vector fields Ξ_i are written as

$$\Xi_i = \frac{\partial}{\partial z_i}, \qquad i = 1, \dots, n.$$

Therefore, what we have obtained is a variable transformation like (9.56) allowing to write the system (9.55) in autonomous form.

Finally, since $[\Xi_i, \Xi_{n+1}] = \Xi_i$ $(i = 1, \dots, n)$, it is

$$\begin{aligned}
&\Xi_i(\Xi_{n+1}(z_j)) = \Xi_{n+1}(\Xi_i(z_j)) + \Xi_i(z_j) = \delta_{ij},\\
&\Xi_i(\Xi_{n+1}(w_\alpha)) = \Xi_{n+1}(\Xi_i(w_\alpha)) + \Xi_i(w_\alpha) = 0;
\end{aligned}$$

where δ_{ij} is the Kronecker symbol; these relations, together with the hypothesis that the variables w_α $(\alpha = 1, \dots, m)$ are invariant with respect to Ξ_{n+1}, allow the vector field Ξ_{n+1} to gain the representation

$$\Xi_{n+1} = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}.$$

As a consequence, since the system written in the variables $\mathbf{z}$ and $\mathbf{w}$ is autonomous and polynomial in the derivatives, and is invariant with respect to a uniform scaling of all independent variables, then it necessarily must be polynomially homogeneous in the derivatives, i.e., it has the form (9.57).

The condition that the symmetries generate a distribution of rank $(n+1)$, whence the vector fields spanning the n-dimensional Abelian Lie subalgebra generate a distribution of rank n, ensures that we may construct the complete set of the new independent variables $\mathbf{z}$.

Conversely, if the nonautonomous and/or nonhomogeneous system (9.55) can be mapped by the invertible point transformation (9.56) to the autonomous system polynomially homogeneous in the derivatives (9.57), then, since the latter admits the n vector fields $\dfrac{\partial}{\partial z_i}$, spanning an n-dimensional Abelian Lie algebra, and the vector field $\displaystyle\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$, then it follows that also the system (9.55) must admit an $(n+1)$-dimensional Lie algebra of symmetries with the requested algebraic structure. □

Example 9.17 Let us consider the first-order system made of the equations

$$u_{1,2} - u_{2,1} = 0, \tag{9.59a}$$
$$\kappa_1 \left(u_{1,2}\right)^4 + \left(\kappa_2 u_{1,1} u_{2,2} + \kappa_3 \left(u_{1,2}\right)^2\right) u_{1,1} u_{2,2}$$

$$+\left(\kappa_4 u_{1,1}u_{2,2}+\kappa_5\left(u_{1,2}\right)^2\right)u_{1,1}+\left(\kappa_6 u_{1,1}u_{2,2}+\kappa_7\left(u_{1,2}\right)^2\right)u_{1,2}$$
$$+\left(\kappa_8 u_{1,1}u_{2,2}+\kappa_9\left(u_{1,2}\right)^2\right)u_{2,2}+\kappa_{10}\left(u_{1,1}\right)^2+\kappa_{11}u_{1,1}u_{1,2}$$
$$+\kappa_{12}u_{1,1}u_{2,2}+\kappa_{13}\left(u_{1,2}\right)^2+\kappa_{14}u_{1,2}u_{2,2}+\kappa_{15}\left(u_{2,2}\right)^2=0, \tag{9.59b}$$

with $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ scalar functions, and κ_i (u_1, u_2) $(i = 1, \ldots, 15)$ arbitrary smooth functions of the indicated arguments.

It can be easily ascertained that system (9.59) admits the Lie point symmetries

$$\Xi_1=\frac{\partial}{\partial x_1},\qquad \Xi_2=\frac{\partial}{\partial x_2},$$
$$\Xi_3=(x_1-au_1-bu_2)\frac{\partial}{\partial x_1}+(x_2-bu_1-cu_2)\frac{\partial}{\partial x_2},$$

a, b, c being constants, provided that the conditions

$$\kappa_1-c^2\kappa_{10}+bc\kappa_{11}-ac\kappa_{12}-b^2\kappa_{13}+ab\kappa_{14}-a^2\kappa_{15}=0,$$
$$\kappa_2-c^2\kappa_{10}+bc\kappa_{11}-ac\kappa_{12}-b^2\kappa_{13}+ab\kappa_{14}-a^2\kappa_{15}=0,$$
$$\kappa_3+2(c^2\kappa_{10}-bc\kappa_{11}+ac\kappa_{12}+b^2\kappa_{13}-ab\kappa_{14}+a^2\kappa_{15})=0,$$
$$\kappa_4+2c\kappa_{10}-b\kappa_{11}+a\kappa_{12}=0,$$
$$\kappa_5-2c\kappa_{10}+b\kappa_{11}-a\kappa_{12}=0,$$
$$\kappa_6+c\kappa_{11}-2b\kappa_{13}+a\kappa_{14}=0,$$
$$\kappa_7-c\kappa_{11}+2b\kappa_{13}-a\kappa_{14}=0,$$
$$\kappa_8+c\kappa_{12}-b\kappa_{14}+2a\kappa_{15}=0,$$
$$\kappa_9-c\kappa_{12}+b\kappa_{14}-2a\kappa_{15}=0$$

are satisfied. Since

$$[\Xi_1, \Xi_2]=0,\qquad [\Xi_1, \Xi_3]=\Xi_1,\qquad [\Xi_2, \Xi_3]=\Xi_2,$$

applying Theorem 9.6, we introduce the new independent and dependent variables

$$z_1=x_1-au_1-bu_2,\qquad z_2=x_2-bu_1-cu_2,$$
$$w_1=u_1,\qquad w_2=u_2,$$

and the nonlinear system (9.59) reduces to

$$w_{1,2}-w_{2,1}=0, \tag{9.60a}$$
$$\kappa_{10}\left(w_{1,1}\right)^2+\kappa_{11}w_{1,1}w_{1,2}+\kappa_{12}w_{1,1}w_{2,2}$$
$$+\kappa_{13}\left(w_{1,2}\right)^2+\kappa_{14}w_{1,2}w_{2,2}+\kappa_{15}\left(w_{2,2}\right)^2=0, \tag{9.60b}$$

i.e., an autonomous system polynomially homogeneous in the derivatives. We notice that system (9.60), by specializing the functions $\kappa_{10}, \ldots, \kappa_{15}$ as follows,

$$\begin{aligned}
\kappa_{10} &= -\kappa(1+w_2^2)^2,\\
\kappa_{11} &= 4\kappa w_1 w_2(1+w_2^2),\\
\kappa_{12} &= 2((2-\kappa)(1+w_1^2+w_2^2)-\kappa w_1^2 w_2^2),\\
\kappa_{13} &= -4(1+w_1^2+w_2^2+\kappa w_1^2 w_2^2),\\
\kappa_{14} &= 4\kappa w_1 w_2(1+w_1^2),\\
\kappa_{15} &= -\kappa(1+w_1^2)^2,
\end{aligned}$$

is equivalent to the second-order partial differential equation

$$\begin{aligned}
&\kappa\left(1+\left(w_{,2}\right)^2\right)^2\left(w_{,11}\right)^2-4\kappa w_{,1}w_{,2}\left(1+\left(w_{,2}\right)^2\right)w_{,11}w_{,12}\\
&-2\left((2-\kappa)\left(1+\left(w_{,1}\right)^2+\left(w_{,2}\right)^2\right)-\kappa\left(w_{,1}w_{,2}\right)^2\right)w_{,11}w_{,22}\\
&+4\left(1+\left(w_{,1}\right)^2+\left(w_{,2}\right)^2+\kappa\left(w_{,1}w_{,2}\right)^2\right)\left(w_{,12}\right)^2\\
&-4\kappa w_{,1}w_{,2}\left(\left(1+w_{,1}\right)^2\right)w_{,12}w_{,22}+\kappa\left(1+\left(w_{,1}\right)^2\right)^2\left(w_{,22}\right)^2=0, \quad (9.61)
\end{aligned}$$

where $\kappa \equiv \kappa\left(w_{,1}, w_{,2}\right)$ is an arbitrary function of the indicated arguments.

Considering a smooth surface in $\mathbb{R}^3$ with the metric $ds^2 = dz_1^2 + dz_2^2 + dw^2$, and its Gaussian and mean curvature,

$$\begin{aligned}
G &= \frac{w_{,11}w_{,22}-\left(w_{,12}\right)^2}{\left(1+\left(w_{,1}\right)^2+\left(w_{,2}\right)^2\right)^2},\\
H &= \frac{1}{2}\frac{\left(1+\left(w_{,2}\right)^2\right)w_{,11}-2w_{,1}w_{,2}w_{,12}+\left(1+\left(w_{,1}\right)^2\right)w_{,22}}{\left(1+\left(w_{,1}\right)^2+\left(w_{,2}\right)^2\right)^{3/2}},
\end{aligned}$$

respectively, Eq. (9.61) can be written as

$$G = \kappa H^2, \quad (9.62)$$

whereupon it should be $\kappa \leq 1$. In the limit case $\kappa = 1$, Eq. (9.62) characterizes a surface with all its points umbilic; it is known that a surface with all its point umbilic is a (open) domain of a plane or a sphere [37]. It is worth of being remarked that Eq. (9.62) with $\kappa \equiv 1$ is *strongly Lie remarkable* [30], since it is the unique second order partial differential equation uniquely characterized by the conformal Lie algebra in $\mathbb{R}^3$ [29] (see Sect. 8.8.4 in Chap. 8).

Remark 9.8 Notice that Example 9.17 yields a system polynomial in the derivatives which is transformed into a system polynomially homogeneous of degree 2 in the

derivatives. Now, we analyze various first-order Monge–Ampère systems, and show that they can be transformed to quasilinear form.

9.6.1 Applications: Monge Systems

Here, we give some examples of first-order nonlinear systems polynomial in the derivatives whose Lie symmetries satisfy the conditions of Theorem 9.6, and prove that they can be transformed under suitable conditions to autonomous first-order systems having homogeneous polynomial form; the systems that will be considered are of Monge–Ampère type, and, remarkably, they are reduced to quasilinear (or linear) form.

In particular, we are concerned with the nonlinear first-order systems of Monge equations for the unknowns $u_\alpha(x_i)$ $(\alpha = 1, \ldots, m,\ i = 1, \ldots, n)$. These systems have been characterized by Boillat [6] by looking for the nonlinear first-order systems possessing, as the quasilinear systems, the property of the linearity of the Cauchy problem. These systems are also completely exceptional [5, 27], and are made by equations which are expressed as linear combinations (with coefficients depending at most on the independent and the dependent variables) of all minors of all orders extracted from the gradient matrix of $u_\alpha(x_i)$. In the following, we limit ourselves to consider the coefficients of the Monge systems generic functions depending on the field variables.

Example 9.18 (*Monge system: case* $m = n = 2$) Let us consider the nonlinear first-order Monge system made of the equations

$$\kappa_0^i \left(u_{1,1}u_{2,2} - u_{1,2}u_{2,1}\right) + \kappa_1^i u_{1,1} + \kappa_2^i u_{1,2} + \kappa_3^i u_{2,1} + \kappa_4^i u_{2,2} + \kappa_5^i = 0 \tag{9.63}$$

$(i = 1, 2)$, with $u_1(x_1, x_2), u_2(x_1, x_2)$ scalar functions, and $\kappa_j^i(u_1, u_2)$ $(i = 1, 2;\ j = 0, \ldots, 5)$ arbitrary smooth functions of the indicated arguments.

The replacements

$$u_i \to u_i + \alpha_{i1}x_1 + \alpha_{i2}x_2, \qquad i = 1, 2,$$

where α_{ij} are arbitrary constants, produce a system with $\kappa_5^i = 0$ $(i = 1, 2)$ provided that

$$\kappa_0^i(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) + \kappa_1^i\alpha_{11} + \kappa_2^i\alpha_{12} + \kappa_3^i\alpha_{21} + \kappa_4^i\alpha_{22} + \kappa_5^i = 0 \tag{9.64}$$

$(i = 1, 2)$. Conditions (9.64) give two constraints on the functional form of the coefficients so that not all systems can be written in a form where $\kappa_5^i = 0$; however, if the coefficients κ_j^i are constant, due to the arbitrariness of the constants α_{ij}, then (9.64) can always be satisfied.

It is easily recognized that system (9.63), now taken with $\kappa_5^i = 0$, admits the Lie point symmetries spanned by the operators

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = (x_1 - f_1)\frac{\partial}{\partial x_1} + (x_2 - f_2)\frac{\partial}{\partial x_2},$$

where $f_k(u_1, u_2)$ $(k = 1, 2)$ are arbitrary smooth functions of their arguments, provided that

$$\kappa_0^i + \kappa_1^i \frac{\partial f_2}{\partial u_2} - \kappa_2^i \frac{\partial f_1}{\partial u_2} - \kappa_3^i \frac{\partial f_2}{\partial u_1} + \kappa_4^i \frac{\partial f_1}{\partial u_1} = 0, \qquad i = 1, 2. \tag{9.65}$$

The constraints (9.65), once we assign the 10 functions $k_j^i(u_1, u_2)$ $(i = 1, 2,\ j = 0, \ldots, 4)$, are the differential equations providing us the functional form of $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$.

In the case where all the coefficients κ_j^i are constant, then the functions f_1 and f_2 are forced to be linear, i.e.,

$$f_i = \beta_{i1} u_1 + \beta_{i2} u_2, \qquad i = 1, 2,$$

β_{ij} being constants whose values are determined by the coefficients κ_j^i.

Since

$$[\Xi_1, \Xi_2] = 0, \qquad [\Xi_1, \Xi_3] = \Xi_1, \qquad [\Xi_2, \Xi_3] = \Xi_2,$$

we introduce the new variables

$$z_1 = x_1 - f_1, \qquad z_2 = x_2 - f_2, \qquad w_1 = u_1, \qquad w_2 = u_2, \tag{9.66}$$

and the Lie point symmetries become

$$\Xi_1 = \frac{\partial}{\partial z_1}, \qquad \Xi_2 = \frac{\partial}{\partial z_2}, \qquad \Xi_3 = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}.$$

In terms of the new variables (9.66), the nonlinear system (9.63) becomes

$$\kappa_1^i w_{1,1} + \kappa_2^i w_{1,2} + \kappa_3^i w_{2,1} + \kappa_4^i w_{2,2} = 0, \qquad i = 1, 2,$$

i.e., it is an autonomous and homogeneous quasilinear system. This system is linear if all the coefficients κ_j^i are constant; nevertheless, since it is a 2×2 homogeneous and autonomous quasilinear system, it can be written in linear form by means of the hodograph transformation also when the coefficients κ_j^i depend on u_1 and u_2.

In conclusion, all Monge systems with $m = n = 2$ can be reduced to a linear system when the coefficients κ_j^i are constant; on the contrary, when the coefficients κ_j^i depend upon u_1 and u_2, the reduction to linear form is possible provided that the constraints (9.64) are satisfied.

Example 9.19 (*Monge system: case* $m = 2,\ n = 3$) By considering the gradient matrix of $u_\alpha(x_i)$, with entries $\dfrac{\partial u_\alpha}{\partial x_i}$ $(\alpha = 1, 2, i = 1, 2, 3)$, and all its extracted minors, the nonlinear first-order Monge system is made of equations like

$$\begin{aligned}&\kappa_1^i \left(u_{1,1}u_{2,2} - u_{1,2}u_{2,1}\right) + \kappa_2^i \left(u_{1,1}u_{2,3} - u_{1,3}u_{2,1}\right) + \kappa_3^i \left(u_{1,2}u_{2,3} - u_{1,3}u_{2,2}\right)\\ &\quad + \kappa_4^i u_{1,1} + \kappa_5^i u_{1,2} + \kappa_6^i u_{1,3} + \kappa_7^i u_{2,1} + \kappa_8^i u_{2,2} + \kappa_9^i u_{2,3} + \kappa_{10}^i = 0,\end{aligned} \tag{9.67}$$

with $\kappa_j^i\,(u_1, u_2)$ $(i = 1, 2,\ \ j = 1, \ldots, 10)$ arbitrary smooth functions of the indicated arguments.

The replacements

$$u_i \to u_i + \alpha_{i1}x_1 + \alpha_{i2}x_2 + \alpha_{i3}x_3, \qquad i = 1, 2,$$

where α_{ij} are arbitrary constants, produce a system with $\kappa_{10}^i = 0$ $(i = 1, 2)$ provided that

$$\begin{aligned}&\kappa_1^i(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) + \kappa_2^i(\alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21}) + \kappa_3^i(\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22})\\ &\quad + \kappa_4^i\alpha_{11} + \kappa_5^i\alpha_{12} + \kappa_6^i\alpha_{13} + \kappa_7^i\alpha_{21} + \kappa_8^i\alpha_{22} + \kappa_9^i\alpha_{23} + \kappa_{10}^i = 0\end{aligned} \tag{9.68}$$

$(i = 1, 2)$. Actually, conditions (9.68) can always be satisfied when the coefficients κ_j^i are constant because of the arbitrariness of the constants α_{ij}.

The nonlinear system (9.67), now taken with $\kappa_{10}^i = 0$, admits the Lie point symmetries spanned by the operators

$$\begin{aligned}&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = \frac{\partial}{\partial x_3},\\ &\Xi_4 = (x_1 - f_1)\,\frac{\partial}{\partial x_1} + (x_2 - f_2)\,\frac{\partial}{\partial x_2} + (x_3 - f_3)\,\frac{\partial}{\partial x_3},\end{aligned} \tag{9.69}$$

where $f_k(u_1, u_2)$ $(k = 1, 2, 3)$ are arbitrary smooth functions of their arguments, provided that

$$\begin{aligned}&\kappa_1^i + \kappa_4^i\frac{\partial f_2}{\partial u_2} - \kappa_5^i\frac{\partial f_1}{\partial u_2} - \kappa_7^i\frac{\partial f_2}{\partial u_1} + \kappa_8^i\frac{\partial f_1}{\partial u_1} = 0,\\ &\kappa_2^i + \kappa_4^i\frac{\partial f_3}{\partial u_2} - \kappa_6^i\frac{\partial f_1}{\partial u_2} - \kappa_7^i\frac{\partial f_3}{\partial u_1} + \kappa_9^i\frac{\partial f_1}{\partial u_1} = 0,\\ &\kappa_3^i + \kappa_5^i\frac{\partial f_3}{\partial u_2} - \kappa_6^i\frac{\partial f_2}{\partial u_2} - \kappa_8^i\frac{\partial f_3}{\partial u_1} + \kappa_9^i\frac{\partial f_2}{\partial u_1} = 0\end{aligned} \tag{9.70}$$

$(i = 1, 2)$. The six conditions (9.70) cannot be fulfilled for an arbitrary choice of the coefficients κ_j^i. In the simplest case, where the coefficients κ_j^i are constant, they can be always satisfied, and the functions f_k must be linear:

$$f_k = \beta_{k1} u_1 + \beta_{k2} u_2, \qquad k = 1, 2, 3,$$

β_{ij} being arbitrary constants.

The symmetries (9.69) span a four-dimensional solvable Lie algebra,

$$\left[\Xi_i, \Xi_j\right] = 0, \qquad [\Xi_i, \Xi_4] = \Xi_i, \qquad i, j = 1, 2, 3,$$

whereupon we may introduce the new variables

$$\begin{array}{l} z_1 = x_1 - f_1,\ z_2 = x_2 - f_2,\ z_3 = x_3 - f_3, \\ w_1 = u_1, \qquad w_2 = u_2. \end{array} \tag{9.71}$$

So,

$$\Xi_1 = \frac{\partial}{\partial z_1}, \quad \Xi_2 = \frac{\partial}{\partial z_2}, \quad \Xi_3 = \frac{\partial}{\partial z_3}, \quad \Xi_4 = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3}.$$

In terms of the new variables (9.71), the nonlinear system (9.67) reduces to

$$\kappa_4^i w_{1,1} + \kappa_5^i w_{1,2} + \kappa_6^i w_{1,3} + \kappa_7^i w_{2,1} + \kappa_8^i w_{2,2} + \kappa_9^i w_{2,3} = 0,$$

i.e., it becomes an autonomous and homogeneous quasilinear system.

Example 9.20 (*Monge system: case* $m = 3$, $n = 2$) By considering the gradient matrix of $u_\alpha(x_i)$, with entries $\dfrac{\partial u_\alpha}{\partial x_i}$ $(\alpha = 1, 3, i = 1, 2)$, and all its extracted minors, the nonlinear first-order Monge system is made of equations like

$$\begin{array}{l} \kappa_1^i \left(u_{1,1} u_{2,2} - u_{1,2} u_{2,1}\right) + \kappa_2^i \left(u_{1,1} u_{3,2} - u_{1,2} u_{3,1}\right) + \kappa_3^i \left(u_{2,1} u_{3,2} - u_{2,2} u_{3,1}\right) \\ \quad + \kappa_4^i u_{1,1} + \kappa_5^i u_{1,2} + \kappa_6^i u_{2,1} + \kappa_7^i u_{2,2} + \kappa_8^i u_{3,1} + \kappa_9^i u_{3,2} + \kappa_{10}^i = 0, \end{array} \tag{9.72}$$

with $\kappa_j^i\ (u_1, u_2, u_3)$ $(i = 1, \ldots, 3,\ \ j = 1, \ldots, 10)$ arbitrary smooth functions of the indicated arguments.

The replacements

$$u_i \to u_i + \alpha_{i1} x_1 + \alpha_{i2} x_2, \qquad i = 1, \ldots, 3,$$

where α_{ij} are arbitrary constants, produce a system with $\kappa_{10}^i = 0$ $(i = 1, \ldots, 3)$ provided that

$$\begin{array}{l} \kappa_1^i(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) + \kappa_2^i(\alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31}) + \kappa_3^i(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}) \\ \quad + \kappa_4^i \alpha_{11} + \kappa_5^i \alpha_{12} + \kappa_6^i \alpha_{21} + \kappa_7^i \alpha_{22} + \kappa_8^i \alpha_{31} + \kappa_9^i \alpha_{32} + \kappa_{10}^i = 0 \end{array} \tag{9.73}$$

$(i = 1, \dots, 3)$. Also in this case, conditions (9.73) can always be satisfied when the coefficients κ_j^i are constant because of the arbitrariness of the constants α_{ij}.

This nonlinear system (9.72), now taken with $\kappa_{10}^i = 0$, admits the Lie point symmetries

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = (x_1 - f_1)\frac{\partial}{\partial x_1} + (x_2 - f_2)\frac{\partial}{\partial x_2}, \tag{9.74}$$

where $f_k(u_1, u_2, u_3)$ $(k = 1, 2)$ are arbitrary smooth functions of their arguments, provided that

$$\begin{aligned}
&\kappa_1^i + \kappa_4^i \frac{\partial f_2}{\partial u_2} - \kappa_5^i \frac{\partial f_1}{\partial u_2} - \kappa_6^i \frac{\partial f_2}{\partial u_1} + \kappa_7^i \frac{\partial f_1}{\partial u_1} = 0,\\
&\kappa_2^i + \kappa_4^i \frac{\partial f_2}{\partial u_3} - \kappa_5^i \frac{\partial f_1}{\partial u_3} - \kappa_8^i \frac{\partial f_2}{\partial u_1} + \kappa_9^i \frac{\partial f_1}{\partial u_1} = 0,\\
&\kappa_3^i + \kappa_6^i \frac{\partial f_2}{\partial u_3} - \kappa_7^i \frac{\partial f_1}{\partial u_3} - \kappa_8^i \frac{\partial f_2}{\partial u_2} + \kappa_9^i \frac{\partial f_1}{\partial u_2} = 0.
\end{aligned} \tag{9.75}$$

The conditions (9.75) cannot be fulfilled for an arbitrary choice of the coefficients κ_j^i. In the simplest case, where the coefficients κ_j^i are constant, they can be always satisfied, and the functions f_k must be linear:

$$f_k = \beta_{k1} u_1 + \beta_{k2} u_2 + \beta_{k3} u_3, \qquad k = 1, 2,$$

β_{ij} being arbitrary constants.

The symmetries (9.74) span a three-dimensional solvable Lie algebra,

$$\big[\Xi_i, \Xi_j\big] = 0, \qquad [\Xi_i, \Xi_3] = \Xi_i, \qquad i, j = 1, 2,$$

and we may introduce the new variables

$$\begin{aligned}
&z_1 = x_1 - f_1, \ z_2 = x_2 - f_2,\\
&w_1 = u_1, \qquad w_2 = u_2, \qquad w_3 = u_3.
\end{aligned} \tag{9.76}$$

Therefore,

$$\Xi_1 = \frac{\partial}{\partial z_1}, \qquad \Xi_2 = \frac{\partial}{\partial z_2}, \qquad \Xi_4 = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}.$$

In terms of the new variables (9.76), the nonlinear system (9.72) reduces to

$$\kappa_4^i w_{1,1} + \kappa_5^i w_{1,2} + \kappa_6^i w_{2,1} + \kappa_7^i w_{2,2} + \kappa_8^i w_{3,1} + \kappa_9^i w_{3,2} = 0, \qquad i = 1, 2, 3,$$

i.e., as an autonomous and homogeneous quasilinear system.

Example 9.21 (*Monge system: case* $m = n = 3$) By considering the gradient matrix of $u_\alpha(x_i)$ with entries $\dfrac{\partial u_\alpha}{\partial x_i}$ $(\alpha, i = 1, 2, 3)$, and all its extracted minors, the nonlinear first-order Monge system is made of equations like

$$\begin{aligned}
&\kappa_0^i \left(u_{1,1}u_{2,2}u_{3,3} - u_{1,1}u_{2,3}u_{3,2} + u_{1,2}u_{2,3}u_{3,1} - u_{1,2}u_{2,1}u_{3,3}\right.\\
&\quad \left. + u_{1,3}u_{2,1}u_{3,2} - u_{1,3}u_{2,2}u_{3,1}\right) + \kappa_1^i \left(u_{2,2}u_{3,3} - u_{2,3}u_{3,2}\right)\\
&\quad + \kappa_2^i \left(u_{2,1}u_{3,3} - u_{2,3}u_{3,1}\right) + \kappa_3^i \left(u_{2,1}u_{3,2} - u_{2,2}u_{3,1}\right)\\
&\quad + \kappa_4^i \left(u_{1,2}u_{3,3} - u_{1,3}u_{3,2}\right) + \kappa_5^i \left(u_{1,1}u_{3,3} - u_{1,3}u_{3,1}\right)\\
&\quad + \kappa_6^i \left(u_{1,1}u_{3,2} - u_{1,2}u_{3,1}\right) + \kappa_7^i \left(u_{1,2}u_{2,3} - u_{1,3}u_{2,2}\right)\\
&\quad + \kappa_8^i \left(u_{1,1}u_{2,3} - u_{1,3}u_{2,1}\right) + \kappa_9^i \left(u_{1,1}u_{2,2} - u_{1,2}u_{2,1}\right)\\
&\quad + \kappa_{10}^i u_{1,1} + \kappa_{11}^i u_{1,2} + \kappa_{12}^i u_{1,3} + \kappa_{13}^i u_{2,1} + \kappa_{14}^i u_{2,2}\\
&\quad + \kappa_{15}^i u_{2,3} + \kappa_{16}^i u_{3,1} + \kappa_{17}^i u_{3,2} + \kappa_{18}^i u_{3,3} + \kappa_{19}^i = 0,
\end{aligned} \tag{9.77}$$

with $\kappa_j^i\,(u_1, u_2, u_3)$ $(i = 1, \ldots, 3,\ \ j = 0, \ldots, 19)$ arbitrary smooth functions of the indicated arguments.

Also in this case, the substitutions

$$u_i \to u_i + \alpha_{i1}x_1 + \alpha_{i2}x_2 + \alpha_{i3}x_3, \qquad i = 1, \ldots, 3,$$

where α_{ij} are arbitrary constants, allow us to obtain a system with $\kappa_{19}^i = 0$ provided that $u_{i,j} = \alpha_{ij}$ is a solution of Eqs.(9.77). This requirement implies some constraints on the coefficients κ_j^i in the general case; on the contrary, no limitation to the values of the coefficients exists if they are assumed to be constant.

The system (9.77), with $\kappa_{19}^i = 0$, admits the Lie point symmetries spanned by the operators

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = \frac{\partial}{\partial x_3},\\
&\Xi_4 = (x_1 - f_1)\frac{\partial}{\partial x_1} + (x_2 - f_2)\frac{\partial}{\partial x_2} + (x_3 - f_3)\frac{\partial}{\partial x_3},
\end{aligned} \tag{9.78}$$

where $f_k(u_1, u_2, u_3)$ $(k = 1, \ldots, 3)$ are arbitrary smooth functions of their arguments, provided that

$$\begin{aligned}
&\kappa_0^i - \kappa_{10}^i \left(\frac{\partial f_2}{\partial u_2}\frac{\partial f_3}{\partial u_3} - \frac{\partial f_2}{\partial u_3}\frac{\partial f_3}{\partial u_2}\right) - \kappa_{11}^i \left(\frac{\partial f_1}{\partial u_3}\frac{\partial f_3}{\partial u_2} - \frac{\partial f_1}{\partial u_2}\frac{\partial f_3}{\partial u_3}\right)\\
&\quad - \kappa_{12}^i \left(\frac{\partial f_1}{\partial u_2}\frac{\partial f_2}{\partial u_3} - \frac{\partial f_1}{\partial u_3}\frac{\partial f_2}{\partial u_2}\right) - \kappa_{13}^i \left(\frac{\partial f_2}{\partial u_3}\frac{\partial f_3}{\partial u_1} - \frac{\partial f_2}{\partial u_1}\frac{\partial f_3}{\partial u_3}\right)\\
&\quad - \kappa_{14}^i \left(\frac{\partial f_1}{\partial u_1}\frac{\partial f_3}{\partial u_3} - \frac{\partial f_1}{\partial u_3}\frac{\partial f_3}{\partial u_1}\right) - \kappa_{15}^i \left(\frac{\partial f_1}{\partial u_3}\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_1}\frac{\partial f_2}{\partial u_3}\right)\\
&\quad - \kappa_{16}^i \left(\frac{\partial f_2}{\partial u_1}\frac{\partial f_3}{\partial u_2} - \frac{\partial f_2}{\partial u_2}\frac{\partial f_3}{\partial u_1}\right) - \kappa_{17}^i \left(\frac{\partial f_1}{\partial u_2}\frac{\partial f_3}{\partial u_1} - \frac{\partial f_1}{\partial u_1}\frac{\partial f_3}{\partial u_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&-\kappa_{18}^i\left(\frac{\partial f_1}{\partial u_1}\frac{\partial f_2}{\partial u_2}-\frac{\partial f_1}{\partial u_2}\frac{\partial f_2}{\partial u_1}\right)=0,\\
&\kappa_1^i+\kappa_{14}^i\frac{\partial f_3}{\partial u_3}-\kappa_{15}^i\frac{\partial f_2}{\partial u_3}-\kappa_{17}^i\frac{\partial f_3}{\partial u_2}+\kappa_{18}^i\frac{\partial f_2}{\partial u_2}=0,\\
&\kappa_2^i+\kappa_{13}^i\frac{\partial f_3}{\partial u_3}-\kappa_{15}^i\frac{\partial f_1}{\partial u_3}-\kappa_{16}^i\frac{\partial f_3}{\partial u_2}+\kappa_{18}^i\frac{\partial f_1}{\partial u_2}=0,\\
&\kappa_3^i+\kappa_{13}^i\frac{\partial f_2}{\partial u_3}-\kappa_{14}^i\frac{\partial f_1}{\partial u_3}-\kappa_{16}^i\frac{\partial f_2}{\partial u_2}+\kappa_{17}^i\frac{\partial f_1}{\partial u_2}=0,\\
&\kappa_4^i+\kappa_{11}^i\frac{\partial f_3}{\partial u_3}-\kappa_{12}^i\frac{\partial f_2}{\partial u_3}-\kappa_{17}^i\frac{\partial f_3}{\partial u_1}+\kappa_{18}^i\frac{\partial f_2}{\partial u_1}=0,\\
&\kappa_5^i+\kappa_{10}^i\frac{\partial f_3}{\partial u_3}-\kappa_{12}^i\frac{\partial f_1}{\partial u_3}-\kappa_{16}^i\frac{\partial f_3}{\partial u_1}+\kappa_{18}^i\frac{\partial f_1}{\partial u_1}=0,\\
&\kappa_6^i+\kappa_{10}^i\frac{\partial f_2}{\partial u_3}-\kappa_{11}^i\frac{\partial f_1}{\partial u_3}-\kappa_{16}^i\frac{\partial f_2}{\partial u_1}+\kappa_{17}^i\frac{\partial f_1}{\partial u_1}=0,\\
&\kappa_7^i+\kappa_{11}^i\frac{\partial f_3}{\partial u_2}-\kappa_{12}^i\frac{\partial f_2}{\partial u_2}-\kappa_{14}^i\frac{\partial f_3}{\partial u_1}+\kappa_{15}^i\frac{\partial f_2}{\partial u_1}=0,\\
&\kappa_8^i+\kappa_{10}^i\frac{\partial f_3}{\partial u_2}-\kappa_{12}^i\frac{\partial f_1}{\partial u_2}-\kappa_{13}^i\frac{\partial f_3}{\partial u_1}+\kappa_{15}^i\frac{\partial f_1}{\partial u_1}=0,\\
&\kappa_9^i+\kappa_{10}^i\frac{\partial f_2}{\partial u_2}-\kappa_{11}^i\frac{\partial f_1}{\partial u_2}-\kappa_{13}^i\frac{\partial f_2}{\partial u_1}+\kappa_{14}^i\frac{\partial f_1}{\partial u_1}=0.
\end{aligned}
\tag{9.79}
$$

The vector fields (9.78) span a four-dimensional solvable Lie algebra,

$$
\left[\Xi_i,\Xi_j\right]=0,\qquad [\Xi_i,\Xi_4]=\Xi_i,\qquad i,j=1,\dots,3;
$$

introducing the new variables

$$
\begin{aligned}
&z_1=x_1-f_1,\ z_2=x_2-f_2,\ z_3=x_3-f_3,\\
&w_1=u_1,\qquad w_2=u_2,\qquad w_3=u_3,
\end{aligned}
\tag{9.80}
$$

we obtain

$$
\begin{aligned}
&\Xi_1=\frac{\partial}{\partial z_1},\qquad \Xi_2=\frac{\partial}{\partial z_2},\qquad \Xi_3=\frac{\partial}{\partial z_3},\\
&\Xi_4=z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}+z_3\frac{\partial}{\partial z_3}.
\end{aligned}
$$

In terms of the new variables (9.80), Eq. (9.77) become

$$
\begin{aligned}
&\kappa_{10}^i w_{1,1}+\kappa_{11}^i w_{1,2}+\kappa_{12}^i w_{1,3}+\kappa_{13}^i w_{2,1}+\kappa_{14}^i w_{2,2}\\
&\quad+\kappa_{15}^i w_{2,3}+\kappa_{16}^i w_{3,1}+\kappa_{17}^i w_{3,2}+\kappa_{18}^i w_{3,3}=0,\qquad i=1,2,3,
\end{aligned}
$$

i.e., they are in autonomous and homogeneous quasilinear (linear, if the coefficients are constant) form.

Conditions (9.79) play severe restrictions to the expression of the coefficients κ_j^i. When these coefficients are assumed to be constant, we are forced to take

$$f_i = \beta_{i1}u_1 + \beta_{i2}u_2 + \beta_{i3}u_3, \qquad i = 1, \ldots, 3,$$

where β_{ij} are arbitrary constants; also in such a simple case, the reduction to linear form is not always possible due to (9.79).

Example 9.22 (*Monge system: case m and n arbitrary*) It is easily recognized that a general Monge system with m dependent variables and n independent variables, provided that some suitable conditions on the coefficients (at most depending on the field variables) are satisfied, is invariant with respect to the Lie groups generated by the vector fields

$$\Xi_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n, \qquad \Xi_{n+1} = \sum_{i=1}^{n} (x_i - f_i(u_\alpha)) \frac{\partial}{\partial x_i},$$

where $f_i(u_\alpha)$ are smooth functions of $(u_1, \ldots, u_m)$ which have to be linear in their arguments when the coefficients of the Monge system are constant.

As one expects, for $m > 3$ or $n > 3$, we have a situation similar to the case $m = n = 3$, i.e., even in the case of constant coefficients, not all Monge systems can be reduced to (quasi)linear form.

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Chapter 10
Conditional Symmetries and Equivalence Transformations



10.1 Introduction

The term *conditional symmetries* is nowadays used for a generalization of a class of symmetries introduced in 1969 by Bluman and Cole [8] investigating the linear heat equation. In that paper, the authors studied the symmetries (referred to as *nonclassical symmetries*) of the solutions of Fourier equation constrained to be also solutions of the invariant surface condition. Today, conditional symmetries are mainly used to determine meaningful solutions of partial differential equations of interest in the applications that, because of phenomenologically included terms, admit very few symmetries.

Typical examples where conditional symmetries reveal fruitful are reaction–diffusion equations. The latter of course do not exhaust the models for which conditional symmetries are effective [2, 4, 5, 11–15, 17, 18, 22, 25, 34, 35, 38, 42, 43, 49, 52, 58–60, 62–64, 69, 70]. Notably, the solutions that are recovered very often cannot be obtained using classical symmetries.

On the contrary, equivalence transformations, extensively studied by Ovsiannikov [53], constitute a useful tool to study the invariance of families of differential equations containing arbitrary elements [37]. Equivalence transformations have been also used in group classification problems [30, 31]. Sophus Lie himself [36] used equivalence transformations for classifying a large class of second-order partial differential equations with two independent variables.

10.2 The Nonclassical Method by Bluman and Cole

Lie symmetry theory gives general and powerful methods for obtaining solutions to partial differential equations. Nevertheless, some differential equations of interest in applications have very few Lie symmetries; so, symmetry reductions are not always

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applicable, though there are ad hoc reduction methods yielding nontrivial solutions obtainable independently of the existence of Lie symmetries.

In 1969, Bluman and Cole [8], in their seminal paper on the general similarity solution of the heat equation, introduced an essential generalization of the Lie symmetry approach, named *nonclassical method*. The key idea was that of using in the invariance condition an additional equation produced by the symmetry itself, namely, the invariant surface condition. So, they studied the continuous symmetries mapping the manifold characterized by the linear heat equation and the invariant surface condition (together with its differential consequences) into itself. Let us briefly recall this approach.

Consider the system made of the linear heat equation and the invariant surface condition:

$$u_{,1} - u_{,22} = 0, \tag{10.1a}$$

$$\xi_1(x_1, x_2, u)u_{,1} + \xi_2(x_1, x_2, u)u_{,2} - \eta(x_1, x_2, u) = 0. \tag{10.1b}$$

Assuming $\xi_1 \neq 0$, setting $\xi = \xi_2/\xi_1$, and using the replacement $\eta/\xi_1 \to \eta$, (10.1b) becomes

$$u_{,1} + \xi u_{,2} - \eta = 0; \tag{10.2}$$

the latter is considered together with its differential consequence, i.e.:

$$u_{,12} + \left(\frac{\partial \xi}{\partial x_2} - \frac{\partial \eta}{\partial u}\right) u_{,2} + \frac{\partial \xi}{\partial u} u_{,2}^2 + \xi u_{,22} - \frac{\partial \eta}{\partial x_2} = 0 \tag{10.3}$$

(the other differential consequence, obtained differentiating (10.2) with respect to x_1, is not needed in this case). Taking the infinitesimal operator

$$\Xi = \frac{\partial}{\partial x_1} + \xi \frac{\partial}{\partial x_2} + \eta \frac{\partial}{\partial u},$$

it is easily to ascertain that it is identically

$$\Xi^{(1)} \left(u_{,1} + \xi u_{,2} - \eta\right)\big|_{u_{,1}+\xi u_{,2}-\eta=0} = 0,$$

and an analogous result holds for the differential consequences of (10.2). Moreover, requiring

$$\Xi^{(2)} \left(u_{,1} - u_{,22}\right)\big|_{\mathcal{M}} = 0,$$

where $\mathcal{M} \subseteq J^2(\mathbb{R}^3, 2)$ is the manifold characterized by (10.1), (10.2) and (10.3), we get a quadratic polynomial in $u_{,2}$, whence the determining equations

$$\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\xi}{\partial x_2 \partial u} + 2\xi\frac{\partial\xi}{\partial u} = 0,$$
$$\frac{\partial\xi}{\partial x_1} - 2\eta\frac{\partial\xi}{\partial u} + 2\frac{\partial^2\eta}{\partial x_2\partial u} - \frac{\partial^2\xi}{\partial x_2^2} + 2\xi\frac{\partial\xi}{\partial x_2} = 0,$$
$$\frac{\partial^2\eta}{\partial x_2^2} - 2\eta\frac{\partial\xi}{\partial x_2} - \frac{\partial\eta}{\partial x_1} = 0$$

are obtained. These determining equations for ξ and η are *nonlinear*, contrarily to what happens when looking for classical Lie symmetries. They may be partially solved, yielding

$$\xi = A(x_1, x_2), \qquad \eta = C(x_1, x_2)u + D(x_1, x_2),$$

where the functions $A(x_1, x_2)$, $C(x_1, x_2)$, and $D(x_1, x_2)$ satisfy the coupled nonlinear system

$$\begin{aligned}
&\frac{\partial A}{\partial x_1} + 2A\frac{\partial A}{\partial x_2} - \frac{\partial^2 A}{\partial x_2^2} + 2\frac{\partial C}{\partial x_2} = 0,\\
&\frac{\partial C}{\partial x_1} + 2C\frac{\partial A}{\partial x_2} - \frac{\partial^2 C}{\partial x_2^2} = 0,\\
&\frac{\partial D}{\partial x_1} + 2D\frac{\partial A}{\partial x_2} - \frac{\partial^2 D}{\partial x_2^2} = 0.
\end{aligned} \tag{10.4}$$

So, any solution of Eq. (10.4) yields a symmetry reduction of the linear heat equation.

For less simple partial differential equations, the nonlinear determining equations are hard to solve; probably, this was the main reason why the nonclassical method was forgotten for almost 20 years, if we exclude the book by Ames [1], where the nonclassical symmetries of Burgers equation have been considered.

Only in 1999, Mansfield [38] (see also [2, 3]) solved Eq. (10.4). In fact, introducing the matrix

$$\Omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ D & C & A \end{pmatrix},$$

system (10.4) can be written as a matrix Burgers equation,

$$\frac{\partial\Omega}{\partial x_1} + 2\frac{\partial\Omega}{\partial x_2}\Omega - \frac{\partial^2\Omega}{\partial x_2^2} = 0,$$

that is linearized by the matrix Cole–Hopf transform [33],

$$\Omega = -\frac{\partial\Phi}{\partial x_2}\Phi^{-1},$$

where the matrix Φ satisfies the linear matrix equation

$$\frac{\partial \Phi}{\partial x_1} - \frac{\partial^2 \Phi}{\partial x_2^2} = 0.$$

Nonclassical symmetries are now part of *conditional symmetries*, i.e., symmetries of differential equations where some additional differential conditions are imposed to restrict the set of solutions. In many cases, the results allow for the introduction of nontrivial reductions of partial differential equations leading to the determination of wide classes of exact solutions in many applied problems.

10.3 Conditional Symmetries

The revival of interest toward the nonclassical method started in the 1980s after the papers by Olver and Rosenau [51], and, independently, by Fushchych and Tsyfra [21].

In particular, in [51], it is shown that every solution of a given partial differential equation is indeed an invariant solution under the action of at least a Lie group that in general is not a symmetry group of the equation at hand. These groups do not leave invariant the whole set of solutions but only a proper subset of the solutions, and so are called *weak symmetries* [51] (see also [10, 59]). The definition of weak symmetries relies on the analytic properties of the overdetermined system composed of the given partial differential equation and the characteristic system associated with the basis vector fields of the local Lie group. The local Lie group is a weak symmetry of the given differential equation if and only if this overdetermined system is compatible.

In [50], the authors, with the aid of some examples, underline that the specialized methods for finding explicit solutions can all be viewed as originating by appending to a differential equation some auxiliary conditions called *side conditions*. This approach implies that the *unifying theme behind finding special solutions to partial differential equations is not, as is commonly supposed, group theory, but rather the more analytic subject of overdetermined systems of partial differential equations* [50]. A question now arises: what is the criterion for identifying which side conditions are admissible and good for obtaining useful solutions?

A similar approach by Fushchych and Tsyfra [21] introduced the general concept of *conditional invariance*, replacing the conditions for the invariance of a differential equation by the weaker condition of the invariance of the combined system made of the original equation, and some differential constraints. When the differential constraints are given by the invariance surface conditions and their differential consequences (the nonclassical symmetries introduced by Bluman and Cole [8]), we have what now are usually referred to as Q-conditional symmetries. The latter are conditional symmetries, but conditional symmetries are not necessarily Q-conditional symmetries. Some authors [32] do not recognize the status of symmetry to nonclassical symmetries, and prefer to use the term *reduction operators*.

A meaningful example of the concept of conditional symmetry is illustrated in [23]. The first pair of Maxwell equations,

$$\frac{\partial \mathbf{E}}{\partial t} - \nabla \wedge \mathbf{H} = \mathbf{0}, \qquad \frac{\partial \mathbf{H}}{\partial t} + \nabla \wedge \mathbf{E} = \mathbf{0}, \tag{10.5}$$

admits the symmetries generated by

$$\begin{aligned}
&\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x_k}, \quad \frac{\partial}{\partial E_k}, \quad \frac{\partial}{\partial H_k}, \quad k = 1, 2, 3, \\
&x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} + E_j \frac{\partial}{\partial E_k} - E_k \frac{\partial}{\partial E_j} + H_j \frac{\partial}{\partial H_k} - H_k \frac{\partial}{\partial H_j}, \quad 1 \le j < k \le 3, \\
&t \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}.
\end{aligned}$$

Thus, Eq. (10.5) is invariant under the four-dimensional translations, translations of electric and magnetic fields, the rotations, scale transformations of independent variables, but are not invariant under the Lorentz boosts. However, adding the conditions

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{H} = 0, \tag{10.6}$$

and so taking the full set of Maxwell equations, we have the invariance under the Lorenz boosts

$$x_k \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_k}, \qquad k = 1, 2, 3.$$

So, Lorenz boosts can be considered as *conditional symmetries* of the first pair of Maxwell equations provided that we add the *side conditions* (10.6).

Some interesting applications of conditional symmetries in ideal gas dynamics and ideal magneto gas dynamics are discussed in Sects. 10.6 and 10.7.

The rapid explosion in the number of papers on nonclassical symmetries of partial differential equations starts around 1990. Then, a lot of papers were published investigating nonclassical symmetries of several evolution equations [2, 4, 5, 11–15, 17, 18, 22, 25, 35, 38, 42, 43, 49, 52, 58–60, 62–64, 69, 70].

As a result, many nontrivial reductions of partial differential equations were possible, and wide classes of exact solutions in many applied problems have been found.

10.3.1 Clarkson and Kruskal Direct Method

Another approach, called *direct method*, not using group theory, has been proposed in 1989 by Clarkson and Kruskal [18]. Given a differential equation

$$\Delta(x_1, x_2, u, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}) = 0 \tag{10.7}$$

for the scalar unknown $u(x_1, x_2)$, the direct method assumes the solution to have the form:

$$u(x_1, x_2) = F(x_1, x_2, w(z)), \quad z = z(x_1, x_2). \tag{10.8}$$

Then, the functions $F(x_1, x_2, w(z))$ and $z(x_1, x_2)$ are determined in such a way the substitution of (10.8) into (10.7) produces an ordinary differential equation in the variables $w(z)$ and z. As an example, this method revealed successful for obtaining new symmetry reductions of Boussinesq equation [18]. Further, in [19], Clarkson and Winternitz generalized the method to determine solutions to Kadomtsev–Petviashvili equation. The solutions found by using the direct method can be also obtained by means of the nonclassical method by Bluman and Cole. However, the two methods are not equivalent, as proved in [43], where it is given an exact solution to the FitzHugh–Nagumo equation [20, 41] that is obtained by the nonclassical method but is not achievable with the direct approach.

10.4 Q-Conditional Symmetries

Hereafter, we shall *not* give meaningful examples of exact solutions found using conditional symmetries. Instead, after shortly reviewing the theoretical setting, we will focus on the classical Burgers equation, and derive an infinite hierarchy of systems that, because of their structure, we call *coupled systems of Burgers-like equations*. This hierarchy is found by suitably looking for Q-conditional symmetries.

Given an rth-order differential equation

$$\boldsymbol{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{[1]}, \ldots, \mathbf{u}^{(r)}) = 0,$$

for the unknown $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^m$, where $\mathbf{x} \in \mathbb{R}^n$, and considering the invariant surface conditions $\mathbf{Q} = \mathbf{0}$, where

$$\mathbf{Q} = \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} - \boldsymbol{\eta}(\mathbf{x}, \mathbf{u}), \qquad \sum_{i=1}^{n} \xi_i^2 \neq 0,$$

the Q-conditional symmetries are found by requiring

$$\Xi^{(r)}(\boldsymbol{\Delta})\big|_{\mathcal{M}} = \mathbf{0},$$

where $\mathcal{M}$ is the manifold of the jet space $J^r(\mathbb{R}^{m+n}, n)$ defined by the system of equations

$$\begin{aligned} &\boldsymbol{\Delta} = \mathbf{0}, \qquad \mathbf{Q} = \mathbf{0}, \\ &\mathcal{D}_{j_1}(\mathbf{Q}) = \mathcal{D}_{j_1 j_2}(\mathbf{Q}) = \cdots = \mathcal{D}_{j_1 \ldots j_{r-1}}(\mathbf{Q}) = \mathbf{0}, \end{aligned}$$

where $1 \leq j_k \leq n$, $\mathcal{D}_{j_1 j_2 \ldots j_{r-1}} = \mathcal{D}_{j_1}(\mathcal{D}_{j_2 \ldots j_{r-1}})$, and $\mathcal{D}_j$ denotes the total derivative with respect to x_j.

It is easily proved that a (classical) Lie symmetry is a nonclassical symmetry. However, differently from Lie symmetries, all possible nonclassical symmetries of a differential equation form a set which is neither a Lie algebra nor a vector space in the general case. Moreover, if the vector field of a conditional symmetry is multiplied by an arbitrary smooth function $f(\mathbf{x}, \mathbf{u})$, we have still a conditional symmetry. This means that, setting $k = \min\{i \in \{1, \ldots, n\} : \xi_i \neq 0\}$, we can look for conditional symmetries such that the invariant surface conditions are divided by ξ_k.

When $m > 1$ (more than one dependent variable), one can look for conditional symmetries of qth type ($1 \leq q \leq m$) [12], by requiring the condition

$$\Xi^{(r)}(\Delta)\big|_{\mathcal{M}_q} = 0,$$

where $\mathcal{M}_q$ is the manifold of the jet space defined by the system of equations

$$\begin{aligned} &\mathbf{\Delta} = 0, \qquad Q_{i_1} = \cdots = Q_{i_q} = 0, \\ &\mathcal{D}_{j_1}(Q_i) = \mathcal{D}_{j_1 j_2}(Q_i) = \cdots = \mathcal{D}_{j_1 \ldots j_{r-1}}(Q_i) = 0, \end{aligned}$$

where $i \in \{i_1, \ldots, i_q\}$. Of course, for $q = 0$, conditional symmetries of qth type are classical Lie symmetries. We do not need to impose the condition that $\mathbf{Q} = 0$ and its differential consequences are invariant with respect to the prolongation of Ξ, since these conditions are identically satisfied. Moreover, when $q < m$, it may be easier to find solutions to the nonlinear determining equations. Conditional symmetries allow for symmetry reductions of differential equations and allow us to determine explicit solutions not obtainable with classical symmetries.

10.5 Conditional Symmetries of Burgers-Like Equations

In [52], it was proved that the Q-conditional symmetries of Burgers equation are expressed in terms of three functions that are solutions of a system of coupled Burgers-like equations. In what follows, we prove that the latter system of coupled Burgers-like equations admits Q-conditional symmetries expressed in terms of five functions satisfying a new system of coupled Burgers-like equations. This process can be repeatedly used, and a hierarchy of systems involving an odd number of unknowns arises. Moreover, starting with a pair of coupled Burgers-like equations, another hierarchy of systems involving an even number of coupled Burgers-like equations is generated. Both hierarchies can be unified [26], and each member written as a matrix Burgers equation that is linearized by means of the matrix Hopf–Cole transformation.

10.5.1 Hierarchy Originating from Burgers Equation

Consider the Burgers equation

$$\Delta_1 \equiv u_{,1} + uu_{,2} - u_{,22} = 0 \tag{10.9}$$

for the unknown $u(x_1, x_2)$, the vector field

$$\Xi = \frac{\partial}{\partial x_1} + \xi(x_1, x_2, u)\frac{\partial}{\partial x_2} + \eta(x_1, x_2, u)\frac{\partial}{\partial u} \tag{10.10}$$

together with the invariant surface condition $Q = 0$, where

$$Q = u_{,1} + \xi(x_1, x_2, u)u_{,2} - \eta(x_1, x_2, u).$$

In order to compute the Q-conditional symmetries of (10.9) associated to (10.10), let us define the manifold $\mathcal{M}_1$ as

$$\Delta_1 = 0, \quad Q = 0, \quad \mathcal{D}_1(Q) = 0, \quad \mathcal{D}_2(Q) = 0.$$

So, the conditional symmetries are found by requiring

$$\Xi^{(2)}(\Delta_1)\big|_{\mathcal{M}_1} = 0.$$

The latter yields the following polynomial of third degree in the derivative $u_{,2}$:

$$\begin{aligned}
&\frac{\partial^2\xi}{\partial u^2}\left(u_{,2}\right)^3 + \left(2\frac{\partial^2\xi}{\partial x_2\partial u} - \frac{\partial^2\eta}{\partial u^2} + 2\frac{\partial\xi}{\partial u}u - 2\xi\frac{\partial\xi}{\partial u}\right)\left(u_{,2}\right)^2 \\
&\quad + \left(\frac{\partial^2\xi}{\partial x_2^2} - 2\frac{\partial^2\eta}{\partial x_2\partial u} - \frac{\partial\xi}{\partial x_1} + \frac{\partial\xi}{\partial x_2}u - 2\xi\frac{\partial\xi}{\partial x_2} + 2\frac{\partial\xi}{\partial u}\eta + \eta\right)u_{,2} \\
&\quad - \frac{\partial^2\eta}{\partial x_2^2} + 2\frac{\partial\xi}{\partial x_2}\eta + \frac{\partial\eta}{\partial x_1} + \frac{\partial\eta}{\partial x_2}u = 0.
\end{aligned}$$

Annihilating the coefficients of this polynomial, after simple algebra, we get

$$\begin{aligned}
\xi &= \kappa u + \frac{1}{2}v_1, \\
\eta &= \frac{\kappa(1-\kappa)}{3}u^3 - \frac{\kappa}{2}v_1u^2 + \frac{1}{4}uv_2 + \frac{1}{4}v_3,
\end{aligned}$$

where κ is a constant such that $\kappa(\kappa - 1)(2\kappa + 1) = 0$, whereas $v_1(x_1, x_2)$, $v_2(x_1, x_2)$, and $v_3(x_1, x_2)$ are functions depending on the indicated arguments.

Three cases must be distinguished, say $\kappa = 0$, $\kappa = 1$, and $\kappa = -1/2$, the latter being the most interesting one. In such a case, the functions $v_1(x_1, x_2)$, $v_2(x_1, x_2)$, and $v_3(x_1, x_2)$ satisfy the system

$$\boldsymbol{\Delta}_3 \equiv \begin{cases} v_{1,1} + v_1 v_{1,2} - v_{1,22} + v_{2,2} = 0, \\ v_{2,1} + v_2 v_{1,2} - v_{2,22} + v_{3,2} = 0, \\ v_{3,1} + v_3 v_{1,2} - v_{3,22} = 0. \end{cases} \tag{10.11}$$

As already remarked, this result has been obtained in [52].

Then, it can be interesting to explore the Q-conditional symmetries admitted by the system (10.11). As a result, the following proposition is proved.

Proposition 10.1 *The vector field*

$$\Xi = \frac{\partial}{\partial x_1} + \xi(x_1, x_2, v_1, v_2, v_3)\frac{\partial}{\partial x_2} + \sum_{\alpha=1}^{3} \eta_\alpha(x_1, x_2, v_1, v_2, v_3)\frac{\partial}{\partial v_\alpha}$$

gives a Q-conditional symmetry of the system (10.11) provided that

$$\begin{aligned} \xi &= \frac{1}{2}(-v_1 + w_1), \\ \eta_1 &= \frac{1}{4}\left(-v_1^3 - 2v_1 v_2 + w_1 v_1^2 + w_2 v_1 + w_1 v_2 - v_3 + w_3\right), \\ \eta_2 &= \frac{1}{4}\left(-v_1^2 v_2 - v_1 v_3 - v_2^2 + w_1 v_1 v_2 + w_2 v_2 + w_1 v_3 + w_4\right), \\ \eta_3 &= \frac{1}{4}\left(-v_1^2 v_3 - v_2 v_3 + w_1 v_1 v_3 + w_2 v_3 + w_5\right), \end{aligned}$$

where the functions $w_\alpha(x_1, x_2)$ ($\alpha = 1, \ldots, 5$) satisfy the constraints

$$\boldsymbol{\Delta}_5 \equiv \begin{cases} w_{1,1} + w_1 w_{1,2} - w_{1,22} + w_{2,2} = 0, \\ w_{2,1} + w_2 w_{1,2} - w_{2,22} + w_{3,2} = 0, \\ w_{3,1} + w_3 w_{1,2} - w_{3,22} + w_{4,2} = 0, \\ w_{4,1} + w_4 w_{1,2} - w_{4,22} + w_{5,2} = 0, \\ w_{5,1} + w_5 w_{1,2} - w_{5,22} = 0. \end{cases} \tag{10.12}$$

Proof The proof immediately follows by requiring

$$\Xi^{(2)}(\boldsymbol{\Delta}_3)\big|_{\mathcal{M}_3} = \mathbf{0},$$

where $\mathcal{M}_3$ is the manifold of the jet space defined by the system (10.11) together with the invariant surface conditions and their differential consequences, say

$$\begin{cases} \boldsymbol{\Delta}_3 = \mathbf{0}, \\ Q_\alpha \equiv v_{\alpha,1} + \xi(x_1, x_2, v_1, v_2, v_3)v_{\alpha,2} - \eta_\alpha(x_1, x_2, v_1, v_2, v_3) = 0, \quad \alpha = 1, 2, 3, \\ \mathcal{D}_1(Q_\alpha) = 0, \quad \mathcal{D}_2(Q_\alpha) = 0. \end{cases}$$

The lengthy computations can be done by using the program ReLie [46] written in the Computer Algebra System Reduce [28]. So, we obtain three polynomials of third degree in the derivatives $v_{\alpha,2}$, whose integration yields

$$\xi = \kappa v_1 + \frac{1}{2} w_1,$$

$w_1(x_1, x_2)$ being a function of the indicated arguments, and κ is a constant that has to satisfy the condition

$$\kappa(2\kappa + 1) = 0.$$

The most interesting case corresponds again to the choice $\kappa = -1/2$. After straightforward though tedious computations, all the determining equations can be solved; so, the claim is proved. □

Remark 10.1 We note that the system (10.12) has the same structure as (10.11), even if it involves two more unknowns.

Nothing prevents us to repeat the procedure and look for Q-conditional symmetries of (10.12); in fact, we find Q-conditional symmetries of (10.12) expressed in terms of seven functions satisfying a similar coupled system of Burgers-like equations [26].

Summarizing, we can say:

- there are Q-conditional symmetries of the Burgers equation expressed in terms of three functions representing arbitrary solutions of the system $\mathbf{\Delta}_3$ made of three coupled Burgers-like equations;
- there are Q-conditional symmetries of $\mathbf{\Delta}_3$ expressed in terms of five functions representing arbitrary solutions of the system $\mathbf{\Delta}_5$ made of five coupled Burgers-like equations;
- there are Q-conditional symmetries of $\mathbf{\Delta}_5$ expressed in terms of seven functions representing arbitrary solutions of a system made of seven coupled Burgers-like equations;
- and so on.

It seems natural to conjecture that repeatedly looking for Q-conditional symmetries, and starting from the classical Burgers equation, a hierarchy of systems made of an odd number of Burgers-like equations may arise.

In the next subsection, we shall consider the case of a coupled system made of an even number of Burgers-like equations. In particular, the starting point will be the system of two Burgers-like equations whose structure is deduced from (10.11) where we set $v_3 \equiv 0$.

10.5.2 Hierarchy Originating from a Pair of Coupled Burgers-Like Equations

Let us consider the following system made of two coupled Burgers-like equations

$$\mathbf{\Delta}_2 \equiv \begin{cases} u_{1,1} + u_1 u_{1,2} - u_{1,22} + u_{2,2} = 0, \\ u_{2,1} + u_2 u_{1,2} - u_{2,22} = 0. \end{cases} \tag{10.13}$$

Looking for Q-conditional symmetries of (10.13) in correspondence to the vector field

$$\Xi = \frac{\partial}{\partial x_1} + \xi(x_1, x_2, u_1, u_2)\frac{\partial}{\partial x_2} + \sum_{\alpha=1}^{2} \eta_\alpha(x_1, x_2, u_1, u_2)\frac{\partial}{\partial u_\alpha}, \tag{10.14}$$

and requiring that

$$\Xi^{(2)}(\mathbf{\Delta}_2)\big|_{\mathcal{M}_2} = \mathbf{0},$$

where $\mathcal{M}_2$ is the manifold of the jet space defined by the system (10.13) together with the invariant surface conditions and their differential consequences, say

$$\begin{cases} \mathbf{\Delta}_2 = \mathbf{0}, \\ Q_\alpha \equiv u_{\alpha,1} + \xi(x_1, x_2, u_1, u_2)u_{\alpha,2} - \eta_\alpha(x_1, x_2, u_1, u_2) = 0, \qquad \alpha = 1, 2, \\ \mathcal{D}_1(Q_\alpha) = 0, \quad \mathcal{D}_2(Q_\alpha) = 0, \end{cases}$$

we obtain the following conditions:

$$\begin{aligned} &\frac{\partial^2\xi}{\partial u_1^2}\left(u_{1,2}\right)^3 + 2\frac{\partial^2\xi}{\partial u_1 \partial u_2}\left(u_{1,2}\right)^2 u_{2,2} + \frac{\partial^2\xi}{\partial u_2^2}u_{1,2}\left(u_{2,2}\right)^2 \\ &\quad + \left(2\frac{\partial^2\xi}{\partial x_2 \partial u_1} - \frac{\partial^2\eta_1}{\partial u_1^2} + 2\frac{\partial\xi}{\partial u_1}u_1 - 2\xi\frac{\partial\xi}{\partial u_1} + \frac{\partial\xi}{\partial u_2}u_2\right)\left(u_{1,2}\right)^2 \\ &\quad + \left(2\frac{\partial^2\xi}{\partial x_2 \partial u_2} - 2\frac{\partial^2\eta_1}{\partial u_1 \partial u_2} + 2\frac{\partial\xi}{\partial u_1} - 2\xi\frac{\partial\xi}{\partial u_2} + \frac{\partial\xi}{\partial u_2}u_1\right)u_{1,2}u_{2,2} \\ &\quad + \left(\frac{\partial\xi}{\partial u_2} - \frac{\partial^2\eta_1}{\partial u_2^2}\right)\left(u_{2,2}\right)^2 + \left(\frac{\partial^2\xi}{\partial x_2^2} - 2\frac{\partial^2\eta_1}{\partial x_2 \partial u_1} - \frac{\partial\xi}{\partial x_1} + \frac{\partial\xi}{\partial x}u_1\right. \\ &\quad \left. - 2\xi\frac{\partial\xi}{\partial x_2} + 2\eta_1\frac{\partial\xi}{\partial u_1} - \frac{\partial\eta_1}{\partial u_2}u_2 + \frac{\partial\eta_2}{\partial u_1} + \eta_1\right)u_{1,2} \\ &\quad + \left(-2\frac{\partial\eta_1}{\partial x_2 \partial u_2} + \frac{\partial\xi}{\partial x_2} + 2\eta_1\frac{\partial\xi}{\partial u_2} - \frac{\partial\eta_1}{\partial u_1} + \frac{\partial\eta_1}{\partial u_2}u_1 + \frac{\partial\eta_2}{\partial u_2}\right)u_{2,2} \\ &\quad - \frac{\partial^2\eta_1}{\partial x_2^2} + 2\eta_1\frac{\partial\xi}{\partial x_2} + \frac{\partial\eta_1}{\partial x_1} + \frac{\partial\eta_1}{\partial x_2}u_1 + \frac{\partial\eta_2}{\partial x_2} = 0, \end{aligned}$$

$$
\begin{aligned}
&\frac{\partial^2\xi}{\partial u_1^2}\left(u_{1,2}\right)^2 u_{2,2} + 2\frac{\partial^2\xi}{\partial u_1\partial u_2}u_{1,2}\left(u_{2,2}\right)^2 + \frac{\partial^2\xi}{\partial u_2^2}\left(u_{2,2}\right)^3\\
&\quad + \left(\frac{\partial\xi}{\partial u_1}u_2 - \frac{\partial^2\eta_2}{\partial u_1^2}\right)\left(u_{1,2}\right)^2 + \left(2\frac{\partial^2\xi}{\partial x_2\partial u_1} - 2\frac{\partial^2\eta_2}{\partial u_1\partial u_2} + \frac{\partial\xi}{\partial u_1}u_1\right.\\
&\quad \left. -2\xi\frac{\partial\xi}{\partial u_1} + 2\frac{\partial\xi}{\partial u_2}u_2\right)u_{1,2}u_{2,2} + \left(2\frac{\partial^2\xi}{\partial x_2\partial u_2} - \frac{\partial^2\eta_2}{\partial u_2^2} + \frac{\partial\xi}{\partial u_1} - 2\xi\frac{\partial\xi}{\partial u_2}\right)\left(u_{2,2}\right)^2\\
&\quad + \left(-2\frac{\partial^2\eta_2}{\partial x_2\partial u_1} + \frac{\partial\xi}{\partial x_2}u_2 + 2\eta_2\frac{\partial\xi}{\partial u_1} + \frac{\partial\eta_1}{\partial u_1}u_2 - \frac{\partial\eta_2}{\partial u_1}u_1 - \frac{\partial\eta_2}{\partial u_2}u_2 + \eta_2\right)u_{1,2}\\
&\quad + \left(\frac{\partial^2\xi}{\partial x_2^2} - 2\frac{\partial^2\eta_2}{\partial x_2\partial u_2} - \frac{\partial\xi}{\partial x_1} - 2\xi\frac{\partial\xi}{\partial x_2} + 2\eta_2\frac{\partial\xi}{\partial u_2} + \frac{\partial\eta_1}{\partial u_2}u_2 - \frac{\partial\eta_2}{\partial u_1}\right)u_{2,2}\\
&\quad - \frac{\partial^2\eta_2}{\partial x_2^2} + 2\eta_2\frac{\partial\xi}{\partial x_2} + \frac{\partial\eta_2}{\partial x_1} + \frac{\partial\eta_1}{\partial x_2}u_2 = 0.
\end{aligned}
$$

So, we have a couple of polynomials of third degree in the derivatives $u_{1,2}$ and $u_{2,2}$, whose coefficients must be vanishing. After simple algebra, we get

$$\xi = \kappa u_1 + \frac{1}{2}v_1,$$

where $v_1(x_1, x_2)$ is a function of the indicated arguments, and κ is a constant that has to satisfy the constraint $\kappa(2\kappa + 1) = 0$. Again, looking for the Q-conditional symmetries of (10.13), we choose $\kappa = -1/2$. Thence, integrating the determining equations, we obtain the Q-conditional symmetries characterized by the vector field (10.14), with

$$
\begin{aligned}
\xi &= \frac{1}{2}\left(-u_1 + v_1\right),\\
\eta_1 &= \frac{1}{4}\left(-\left(u_1\right)^3 - 2u_1u_2 + v_1u_1^2 + v_2u_1 + v_1u_2 + v_3\right),\\
\eta_2 &= \frac{1}{4}\left(-u_1^2u_2 - u_2^2 + v_1u_1u_2 + v_2u_2 + v_4\right),
\end{aligned}
$$

and the functions $v_\alpha(x_1, x_2)$ $(\alpha = 1, \ldots, 4)$ satisfying the constraints

$$\mathbf{\Delta}_4 \equiv \begin{cases} v_{1,1} + v_1v_{1,2} - v_{1,22} + v_{2,2} = 0,\\ v_{2,1} + v_2v_{1,2} - v_{2,22} + v_{3,2} = 0,\\ v_{3,1} + v_3v_{1,2} - v_{3,22} + v_{4,2} = 0,\\ v_{4,1} + v_4v_{1,2} - v_{4,22} = 0. \end{cases} \tag{10.15}$$

Repeating the same algorithm for the latter system of four coupled Burgers-like equations, the admitted Q-conditional symmetries are expressed in terms of six arbitrary functions depending on x_1 and x_2, and so on.

The results heretofore obtained can be summarized as follows:

- there are Q-conditional symmetries of the system $\mathbf{\Delta}_2$ made of two coupled Burgers-like equations expressed in terms of four functions representing arbitrary solutions of the system $\mathbf{\Delta}_4$ made of four coupled Burgers-like equations;
- there are Q-conditional symmetries of $\mathbf{\Delta}_4$ expressed in terms of six functions representing arbitrary solutions of the system $\mathbf{\Delta}_6$ made of six coupled Burgers-like equations;
- and so on.

These results suggest to conjecture that repeatedly searching for Q-conditional symmetries, and starting from a pair of coupled Burgers-like equations, a hierarchy of systems made of an even number of coupled Burgers-like equations arises.

Indeed, the latter conjecture and the one made in the previous subsection can be unified and proved to be true, as it will be shown below.

10.5.3 The General Hierarchy of Coupled Burgers-Like Equations

The existence of both hierarchies of Burgers-like equations, arising by searching at each step nontrivial Q-conditional symmetries, can be proved in general. In fact, we have an infinite hierarchy of systems made of an odd number of coupled Burgers-like equations, or made of an even number of coupled Burgers-like equations, depending on the starting point. The results obtained in Subsects. 10.5.1 and 10.5.2 for the first members of the two hierarchies are useful for guessing the structure of the vector fields of the Q-conditional symmetries for the generic element of the hierarchy. Thus, we can state the following result.

Theorem 10.1 *Let m be a positive integer. The system of coupled Burgers-like equations*

$$\mathbf{\Delta}_m \equiv \begin{cases} u_{\alpha,1} + u_\alpha u_{1,2} - u_{\alpha,22} + u_{\alpha+1,2} = 0, & \alpha = 1, \ldots, m-1, \\ u_{m,1} + u_m u_{1,2} - u_{m,22} = 0 \end{cases} \tag{10.16}$$

admits the Q-conditional symmetries associated to the vector field

$$\Xi = \frac{\partial}{\partial x_1} + \xi(x_1, x_2, u_1, \ldots, u_m)\frac{\partial}{\partial x_2} + \sum_{\alpha=1}^{m} \eta_\alpha(x_1, x_2, u_1, \ldots, u_m)\frac{\partial}{\partial u_\alpha}, \tag{10.17}$$

where

$$\begin{aligned}
\xi &= \frac{1}{2}(-u_1 + v_1),\\
\eta_\alpha &= \frac{1}{4}\left(-u_1^2 u_\alpha - u_1 u_{\alpha+1} - u_2 u_\alpha + v_1 u_1 u_\alpha + v_2 u_\alpha + v_1 u_{\alpha+1} - u_{\alpha+2} + v_{\alpha+2}\right),\\
\eta_{m-1} &= \frac{1}{4}\left(-u_1^2 u_{m-1} - u_1 u_m - u_2 u_{m-1} + v_1 u_1 u_{m-1} + v_2 u_{m-1} + v_1 u_m + v_{m+1}\right),\\
\eta_m &= \frac{1}{4}\left(-(u_1)^2 u_m - (1-\delta_{1m}) u_2 u_m + v_1 u_1 u_m + v_2 u_m + v_{m+2}\right)
\end{aligned}$$

($\alpha = 1, \dots, m-2$), δ_{1m} being the Kronecker symbol, provided that the functions $v_1(x_1, x_2), \dots, v_{m+2}(x_1, x_2)$ satisfy the system

$$\mathbf{\Delta}_{m+2} \equiv \begin{cases} v_{\alpha,1} + v_\alpha v_{1,2} - v_{\alpha,22} + v_{\alpha+1,2} = 0, \quad \alpha = 1, \dots, m+1,\\ v_{m+2,1} + v_{m+2} v_{1,2} - v_{m+2,22} = 0. \end{cases} \tag{10.18}$$

Proof It must be verified that the vector field (10.17) is admitted by the system (10.16) if the constraints (10.18) are satisfied. In fact, requiring

$$\Xi^{(2)}(\mathbf{\Delta}_m)\big|_{\mathcal{M}_m} = \mathbf{0},$$

where the manifold $\mathcal{M}_m$ of the jet space is defined by

$$\begin{cases} \mathbf{\Delta}_m = \mathbf{0},\\ Q_\alpha \equiv u_{\alpha,1} + \xi(x_1, x_2, u_1, \dots, u_m) u_{\alpha,2} - \eta_\alpha(x_1, x_2, u_1, \dots, u_m) = 0,\\ \mathcal{D}_1(Q_\alpha) = 0, \quad \mathcal{D}_2(Q_\alpha) = 0, \qquad \alpha = 1, \dots, m, \end{cases}$$

we get the following polynomial system of m equations in the variables u_α and $u_{\alpha,2}$:

$$\begin{aligned}
&\left(v_{1,1} + v_1 v_{1,2} - v_{1,22} + v_{2,2}\right) u_1 u_\alpha + \left(v_{2,1} + v_2 v_{1,2} - v_{2,22} + v_{3,2}\right) u_\alpha\\
&\quad + \left(v_{1,1} + v_1 v_{1,2} - v_{1,22} + v_{2,2}\right) u_{\alpha+1} - 2\left(v_{1,1} + v_1 v_{1,2} - v_{1,22} + v_{2,2}\right) u_{\alpha,2}\\
&\quad + v_{\alpha+2,1} + v_{\alpha+2} v_{1,2} - v_{\alpha+2,22} + v_{\alpha+3,2} = 0,\\
&\left(v_{1,1} + v_1 v_{1,2} - v_{1,22} + v_{2,2}\right) u_1 u_m + \left(v_{2,1} + v_2 v_{1,2} - v_{2,22} + v_{3,2}\right) u_m\\
&\quad - 2\left(v_{1,1} + v_1 v_{1,2} - v_{1,22} + v_{2,2}\right) u_{m,2} + v_{m+2,1} + v_{m+2} v_{1,2} - v_{m+2,22} = 0,
\end{aligned} \tag{10.19}$$

where $\alpha = 1, \dots, m-1$.

Due to the arbitrariness of u_α and $u_{\alpha,2}$, the system (10.19) is satisfied if and only if

$$\begin{cases} v_{1,1} + v_1 v_{1,2} - v_{1,22} + v_{2,2} = 0,\\ v_{2,1} + v_2 v_{1,2} - v_{2,22} + v_{3,2} = 0,\\ v_{\alpha+2,1} + v_{\alpha+2} v_{1,2} - v_{\alpha+2,22} + v_{\alpha+3,2} = 0, \quad \alpha = 1, \dots, m-1,\\ v_{m+2,1} + v_{m+2} v_{1,2} - v_{m+2,22} = 0, \end{cases}$$

i.e., the system $\mathbf{\Delta}_{m+2} = \mathbf{0}$ has to be satisfied. □

Remark 10.2 Note that if m is odd (even, respectively), a hierarchy of systems with an odd (even, respectively) number of equations is generated.

10.5.4 Matrix Burgers Equation

Each element of the infinite hierarchy of coupled Burgers-like equations can be written in the form of a matrix Burgers equation [3, 38] that can be linearized by means of the matrix Hopf–Cole transformation [33].

In fact, defining the $m \times m$ matrix Ω as

$$\Omega = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ u_m & u_{m-1} & \cdots & u_2 & u_1 \end{pmatrix},$$

the system (10.16) can be written in the form of a matrix Burgers equation, say

$$\Omega_{,1} + \Omega_{,2}\Omega - \Omega_{,22} = 0. \tag{10.20}$$

The matrix Hopf–Cole transformation [3, 33],

$$\Omega = -2\Phi_{,2}\Phi^{-1}, \tag{10.21}$$

Φ being an invertible $m \times m$ matrix with entries depending on x_1 and x_2, maps (10.20) to a matrix heat equation,

$$\Phi_{,1} - \Phi_{,22} = 0;$$

moreover, from (10.21), it is $\Phi_{,2} = -\frac{1}{2}\Omega\Phi$, whereupon, computing the entries of $\Omega\Phi$, the solution of the system (10.16) is achieved from the linear algebraic system

$$\begin{cases} u_m w_1 + \displaystyle\sum_{j=1}^{m-1} (-2)^j u_{m-j} \frac{\partial^j w_1}{\partial x_2^j} = (-2)^m \frac{\partial^m w_1}{\partial x_2^m}, \\ u_m w_2 + \displaystyle\sum_{j=1}^{m-1} (-2)^j u_{m-j} \frac{\partial^j w_2}{\partial x_2^j} = (-2)^m \frac{\partial^m w_2}{\partial x_2^m}, \\ \cdots, \\ u_m w_m + \displaystyle\sum_{j=1}^{m-1} (-2)^j u_{m-j} \frac{\partial^j w_m}{\partial x_2^j} = (-2)^m \frac{\partial^m w_m}{\partial x_2^m}, \end{cases}$$

where $w_\alpha(x_1, x_2)$ ($\alpha = 1, \ldots, m$) are m solutions of linear heat equations, i.e.:

$$w_{\alpha,1} - w_{\alpha,22} = 0, \qquad \alpha = 1, \ldots, m.$$

Therefore, similar to what happens for classical Burgers equation, also the solutions of each element of the hierarchy of systems of coupled Burgers-like equations, because of the matrix Hopf–Cole transformation, can be obtained by the solutions of linear heat equation.

As a last comment, we observe that the classical Lie point symmetries of a generic element of this infinite hierarchy span a five-dimensional Lie algebra, as shown below.

Theorem 10.2 *Let m be a positive integer. The system of Burgers-like equations*

$$\mathbf{\Delta}_m \equiv \begin{cases} u_{\alpha,1} + u_\alpha u_{1,2} - u_{\alpha,22} + u_{\alpha+1,2} = 0, & \alpha = 1, \ldots, m-1, \\ u_{m,1} + u_m u_{1,2} - u_{m,22} = 0, \end{cases}$$

for $m = 1$ (classical Burgers equation) admits the Lie point symmetries generated by

$$\begin{aligned} \Xi_1 &= \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \\ \Xi_3 &= 2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - u_1 \frac{\partial}{\partial u_1}, \\ \Xi_4 &= x_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u_1}, \\ \Xi_5 &= x_1^2 \frac{\partial}{\partial t} + x_1 x_2 \frac{\partial}{\partial x_2} + (x_2 - x_1 u_1) \frac{\partial}{\partial u_1}; \end{aligned}$$

for $m = 2$ the Lie point symmetries generated by

$$\begin{aligned} \Xi_1 &= \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \\ \Xi_3 &= 2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - u_1^{(1)} \frac{\partial}{\partial u_1} - 2u_2 \frac{\partial}{\partial u_2}, \\ \Xi_4 &= x_1 \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2}, \\ \Xi_5 &= x_1^2 \frac{\partial}{\partial x_1} + x_1 x_2 \frac{\partial}{\partial x_2} + (2x_2 - x_1 u_1) \frac{\partial}{\partial u_1} - (x_2 u_1 + 2x_1 u_2 + 2) \frac{\partial}{\partial u_2}; \end{aligned}$$

for $m \geq 3$ the Lie point symmetries generated by

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2},$$

$$\Xi_3 = 2x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - \sum_{\alpha=1}^{m} \alpha u_\alpha \frac{\partial}{\partial u_\alpha},$$

$$\Xi_4 = x_1\frac{\partial}{\partial x_2} + m\frac{\partial}{\partial u_1} + \sum_{\alpha=2}^{m}(\alpha - m - 1)u_{\alpha-1}\frac{\partial}{\partial u_\alpha},$$

$$\begin{aligned}\Xi_5 = {} & x_1^2\frac{\partial}{\partial x_1} + x_1x_2\frac{\partial}{\partial x_2} + (mx_2 - x_1u_1)\frac{\partial}{\partial u_1} \\ & - ((m-1)(x_2u_1 + m) + 2x_1u_2)\frac{\partial}{\partial u_2} \\ & - \sum_{\alpha=3}^{m}(\alpha x_1u_\alpha + (m - \alpha + 1)(x_2u_{\alpha-1} - (m - \alpha + 2)u_{\alpha-2}))\frac{\partial}{\partial u_\alpha}.\end{aligned}$$

Whatever the number m of coupled equations is, we have always a five-dimensional Lie algebra (time and space translation, scaling, Galilean and projective transformation, respectively); these Lie algebras, although realized in terms of vector fields on manifolds with different dimensionality, share the same structure constants and so they are all isomorphic.

10.6 Substitution Principles in Ideal Gas Dynamics

In this Section, it is shown that some classical results in ideal gas dynamics, known as substitution principles [40, 57, 65], can be viewed as conditional symmetries.

Consider the equations governing flows of an isentropic ideal gas

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{10.22a}$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) + \nabla p = \mathbf{0}, \tag{10.22b}$$

$$\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s = 0, \tag{10.22c}$$

where $\rho(t, \mathbf{x})$ represents the mass density, $\mathbf{v} = (v_1(t, \mathbf{x}), v_2(t, \mathbf{x}), v_3(t, \mathbf{x}))$ the velocity vector, $p(t, \mathbf{x})$ the pressure, and $s(t, \mathbf{x})$ (a function of) the entropy. The system (10.22) must be supplemented by an equation of state that it is assumed in separable form:

$$\rho = P(p)s, \tag{10.23}$$

$P(p)$ being a suitable constitutive function. Equation of state (10.23) corresponds to that of a Prim gas [57], which reduces to that of a perfect gas when $P(p) = p^{1/\gamma}$, γ being the adiabatic exponent. Using (10.23), Eqs. (10.22) can be written as

$$P'(p)\left(\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p\right) + P(p)\nabla\cdot\mathbf{v} = 0. \tag{10.24a}$$

$$P(p)s\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) + \nabla p = \mathbf{0}, \tag{10.24b}$$

$$\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s = 0. \tag{10.24c}$$

Munk and Prim [40] (see also [57]) derived the remarkable result named *substitution principle* and stated by the following theorem.

Theorem 10.3 ([40, 57]) *The steady version of Eqs. (10.24) (i.e., no time dependence) of an inviscid, thermally nonconducting gas is invariant with respect to the transformation*

$$p^\star(\mathbf{x}) = p(\mathbf{x}), \quad \mathbf{v}^\star(\mathbf{x}) = \frac{\mathbf{v}(\mathbf{x})}{m(\mathbf{x})}, \quad s^\star(\mathbf{x}) = (m(\mathbf{x}))^2\, s(\mathbf{x}),$$

where $m(\mathbf{x})$ represents a smooth scalar function of the space variables such that

$$\mathbf{v}\cdot\nabla m = 0,$$

i.e., $m(\mathbf{x})$ is constant along each individual streamline.

This result has been extended in 1964 by Smith [66] to the class of unsteady flows having steady pressure.

Theorem 10.4 ([66]) *If*

$$\{p(\mathbf{x}),\ \mathbf{v}(t,\mathbf{x}),\ s(t,\mathbf{x})\}$$

represents a solution to Eqs. (10.24) having the pressure steady, then another solution is

$$\big\{p(\mathbf{x}),\ m(\mathbf{x})\mathbf{v}(mt + h(m),\mathbf{x}),\ (m(\mathbf{x}))^{-2}s(mt + h(m),\mathbf{x})\big\},$$

provided that $\mathbf{v}\cdot\nabla m = 0$ for all t, where $m(\mathbf{x})$ is a smooth scalar function of $\mathbf{x}$, and $h(m)$ an arbitrary function of its argument.

Proof Introduce the variables

$$\mathbf{x}^\star = \mathbf{x}, \qquad t^\star = f\,(t, m(\mathbf{x})),$$

where $m(\mathbf{x})$ is a scalar function such that $\mathbf{v} \cdot \nabla m = 0$, and let

$$\mathbf{x} = \mathbf{x}^\star, \quad t = g\left(t^\star, m(\mathbf{x}^\star)\right)$$

be the inverse transformation. For any scalar smooth function $G(t, \mathbf{x})$, the partial derivatives of G transform as follows:

$$\frac{\partial G}{\partial t} = \frac{\partial G}{\partial t^\star}\frac{\partial f}{\partial t}, \qquad \frac{\partial G}{\partial x_i} = \frac{\partial G}{\partial x_i^\star} + \frac{\partial G}{\partial t^\star}\frac{\partial f}{\partial m}\frac{\partial m}{\partial x_i}. \tag{10.25}$$

If $\mathbf{X} = (X_1, X_2, X_3)$ and t are material variables, taking into account (10.25), we have

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{x}(t, \mathbf{X})}{\partial t} = \frac{\partial \mathbf{x}\left(g\left(t^\star, m(\mathbf{x}^\star)\right), \mathbf{X}\right)}{\partial t^\star}\frac{\partial f}{\partial t}. \tag{10.26}$$

From

$$\mathbf{x}^\star = \mathbf{x}\left(g\left(t^\star, m(\mathbf{x}^\star)\right), \mathbf{X}\right),$$

it follows

$$\mathbf{v}^\star = \frac{d\mathbf{x}^\star}{dt^\star} = \frac{\partial \mathbf{x}\left(g\left(t^\star, m(\mathbf{x}^\star)\right), \mathbf{X}\right)}{\partial t^\star} + \frac{\partial g}{\partial m}\mathbf{v}\frac{d\mathbf{x}^\star}{dt^\star} \cdot \nabla^\star m. \tag{10.27}$$

From relations (10.26) and (10.27) one deduces

$$\mathbf{v} = \frac{\partial f}{\partial t}\left(\mathbf{v}^\star - \frac{\partial g}{\partial m}\mathbf{v}\mathbf{v}^\star \cdot \nabla^\star m\right),$$

whence

$$\mathbf{v}^\star \cdot \nabla^\star m = 0.$$

Thus, we get the following transformation for the velocity vector

$$\mathbf{v}^\star = \left(\frac{\partial f}{\partial t}\right)^{-1}\mathbf{v},$$

as long as $\dfrac{\partial f}{\partial t} \neq 0$.

Using (10.25) and (10.26), and assuming $\dfrac{\partial p}{\partial t} = 0$, Eqs. (10.24) become

$$\nabla^\star(P(p)\mathbf{v}) + \frac{\partial f}{\partial m}\frac{\partial (P(p)\mathbf{v})}{\partial t^\star} \cdot \nabla m = 0. \tag{10.28a}$$

$$P(p)s\left(\frac{\partial f}{\partial t}\frac{\partial \mathbf{v}}{\partial t^\star} + (\mathbf{v} \cdot \nabla^\star)\mathbf{v}\right) + \nabla^\star p = \mathbf{0}, \tag{10.28b}$$

$$\frac{\partial f}{\partial t}\frac{\partial s}{\partial t^\star} + \mathbf{v} \cdot \nabla^\star s = 0. \tag{10.28c}$$

If we introduce, in addition to $\mathbf{v}^\star$, also the following new field variables:

$$
\begin{aligned}
&p^\star(\mathbf{x}^\star) = p(\mathbf{x}^\star),\\
&\rho^\star(t^\star, \mathbf{x}^\star) = \left(\frac{\partial f}{\partial t}\right)^2 \rho(g\left(t^\star, m(\mathbf{x}^\star)\right), \mathbf{x}^\star),\\
&s^\star(t^\star, \mathbf{x}^\star) = \left(\frac{\partial f}{\partial t}\right)^2 s(g\left(t^\star, m(\mathbf{x}^\star)\right), \mathbf{x}^\star),
\end{aligned}
$$

then Eqs. (10.28) become

$$
\begin{aligned}
&\nabla^\star(P(p^\star)\mathbf{v}^\star) = 0,\\
&P(p^\star)s^\star\left(\frac{\partial \mathbf{v}^\star}{\partial t^\star} + (\mathbf{v}^\star \cdot \nabla^\star)\mathbf{v}^\star\right) + \nabla^\star p^\star + P(p)s\frac{\partial^2 f}{\partial t^2} = \mathbf{0},\\
&\frac{\partial f}{\partial t}\left(\frac{\partial s^\star}{\partial t^\star} + \mathbf{v}^\star \cdot \nabla^\star s^\star\right) - 2\frac{\partial^2 f}{\partial t^2}s^\star = 0,
\end{aligned}
$$

where we have taken into account that $\dfrac{\partial \mathbf{v}}{\partial t} \cdot \nabla m = 0$. Now, it is immediate to observe that the condition $\dfrac{\partial^2 f}{\partial t^2} = 0$ ensures that the equations at hand do not change their form when they are expressed in terms of the asterisked variables. Moreover, since $m(\mathbf{x})$ is an arbitrary function, and so can be substituted to an arbitrary function of itself, it follows that

$$
f\left(t, m(\mathbf{x})\right) = m(\mathbf{x})t + h\left(m(\mathbf{x})\right).
$$

$h\left(m(\mathbf{x})\right)$ being an arbitrary differentiable function of its argument. Finally, it is a simple matter to show that also the equation of state remains invariant, i.e., $\rho^\star = P(p^\star)S(s^\star)$. □

We observe that the substitution principle for unsteady flows can be thought as a combination of a *generalized* scaling group and a *generalized* time translation; we use the term *generalized* to stress that the parameters entering the transformation are not constant but depend on functions of the space variables. In such a framework, a slight generalization of Theorem 10.4 can be found [44].

Let us supplement Eqs. (10.22) with the following general equation of state

$$
\rho = R(p, s), \tag{10.29}
$$

where $R(p, s)$ is a physically admissible constitutive law. Then, assume that (10.22) and (10.29) are left invariant by the family of finite transformations

$$
\begin{aligned}
&t^\star = m(\mathbf{x})t + h(m), \quad \mathbf{x}^\star = \mathbf{x},\\
&\mathbf{v}^\star = (m(\mathbf{x}))^\alpha \mathbf{v}, \qquad p^\star = (m(\mathbf{x}))^\beta p,\\
&s^\star = (m(\mathbf{x}))^\delta s, \qquad R^\star(p^\star, s^\star) = (m(\mathbf{x}))^\theta R(p, s),
\end{aligned} \tag{10.30}
$$

where α, β, δ, and θ are suitable constants to be further determined, whereas $m(\mathbf{x})$ represents a scalar differentiable function of space variables and $h(m)$ an arbitrary function of m. From (10.30), it easily follows that the derivatives transform according to the rules

$$\frac{\partial}{\partial t^\star} = \frac{1}{m(\mathbf{x})}\frac{\partial}{\partial t}, \qquad \nabla^\star = \nabla - \frac{t+h'}{m(\mathbf{x})}\nabla m\frac{\partial}{\partial t}, \tag{10.31}$$

where $h' = \dfrac{dh}{dm}$; in particular, it is $\nabla^\star m = \nabla m$. Moreover, using dimensional analysis,

$$R(p,s) = s^{\theta/\delta}\widehat{R}(p\cdot s^{-\beta/\delta}), \tag{10.32}$$

provided that $\delta \neq 0$. Notice that the constitutive equation (10.32) includes in particular that of a perfect gas when

$$\widehat{R}\left(p\cdot s^{-\beta/\delta}\right) = \left(p\cdot s^{-\beta/\delta}\right)^{1/\gamma}.$$

By using (10.30)–(10.32) in the asterisked version of Eqs. (10.22), we get

$$\begin{aligned}
&m^{\theta-1}\frac{\partial\rho}{\partial t} + m^{\alpha+\theta}\nabla\cdot(\rho\mathbf{v}) + m^{\alpha+\theta-1}\left((\alpha+\theta)\rho\mathbf{v}\cdot\nabla m - \frac{\partial(\rho\mathbf{v})}{\partial t}\nabla m(t+h')\right) = 0,\\
&m^{\theta}\rho\left(m^{\alpha-1}\frac{\partial\mathbf{v}}{\partial t} + m^{2\alpha}(\mathbf{v}\cdot\nabla)\mathbf{v}\right) + m^{\beta}\nabla p\\
&\quad + m^{2\alpha-1}\left(\alpha\mathbf{v}(\mathbf{v}\cdot\nabla m) - \frac{\partial\mathbf{v}}{\partial t}(\mathbf{v}\cdot\nabla m)(t+h')\right)\\
&\quad + m^{\beta-1}\left(\beta p\nabla m - \frac{\partial p}{\partial t}\nabla m(t+h')\right) = 0,\\
&m^{\delta-1}\frac{\partial s}{\partial t} + m^{\alpha+\delta}\mathbf{v}\cdot\nabla s + m^{\alpha+\delta-1}\left(\delta s\mathbf{v}\cdot\nabla m - \frac{\partial p}{\partial t}\nabla m(t+h')\right) = 0.
\end{aligned}$$

Powers of $m(\mathbf{x})$ are balanced if

$$\alpha = -1,\ \beta = \theta - 2,\ \delta = 2;$$

furthermore, invariance of Eqs. (10.22) and (10.32) under (10.30) is ensured as long as the following additional constraints are satisfied:

1. if $\theta = 2$:

$$\frac{\partial p}{\partial t} = 0, \quad \mathbf{v}\cdot\nabla m = 0;$$

2. if $\theta \neq 2$:

$$(\theta - 2)p + (t + h'(m))\frac{\partial p}{\partial t} = 0, \quad \mathbf{v} \cdot \nabla m = 0, \quad h(m) = h_0(m - 1),$$

where h_0 is an arbitrary constant.

The first case corresponds exactly to the substitution principle in the form stated by Smith [65].

Thus, the following theorem can be stated [44].

Theorem 10.5 ([44]) *If*

$$\{p(t, \mathbf{x}), \mathbf{v}(t, \mathbf{x}), s(t, \mathbf{x})\}$$

represents a solution to Eqs. (10.22) and (10.32), then another solution is

$$\left\{(m(\mathbf{x}))^{-\beta}\, p(mt + h(m), \mathbf{x}),\ m(\mathbf{x})\mathbf{v}(mt + h(m), \mathbf{x}),\ (m(\mathbf{x}))^{-2}\, s(mt + h(m), \mathbf{x})\right\},$$

provided that

$$\mathbf{v} \cdot \nabla m = 0, \qquad \frac{\partial p}{\partial t} - \frac{\beta p}{t + h'(m)} = 0,$$

where $m(\mathbf{x})$ is a smooth scalar function of $\mathbf{x}$, and $h(m)$ an arbitrary function of its argument if $\beta = 0$, or $h(m) = h_0 m - h_0$ (with h_0 constant) if $\beta \neq 0$.

Further, a *mixed* substitution principle relating steady solutions to unsteady ones can be stated. Equations (10.22), supplemented by the constitutive relation $\rho = p^{1/\gamma} s$, admit a projective group of transformations when $\gamma = \dfrac{5}{3}$ (see Example 6.4 in Chap. 6). The corresponding invariant solutions yield a reduced system that is formally the steady version of the governing equations. Combining this result with the substitution principle for steady equations of gas dynamics, we can state the following result [44].

Theorem 10.6 ([44]) *If*

$$\{p(\mathbf{x}),\ \mathbf{v}(\mathbf{x}),\ s(\mathbf{x})\}$$

represents a solution set of the equations governing the steady flow of a perfect gas whose adiabatic exponent is $\dfrac{5}{3}$, and the scalar differentiable function $m(\mathbf{x})$ is such that $\mathbf{v} \cdot \nabla m = 0$, then

$$\left\{t^{-5} p\left(\frac{\mathbf{x}}{t}\right),\ \frac{\mathbf{x}}{t} + \frac{1}{t} m\left(\frac{\mathbf{x}}{t}\right)\mathbf{v}\left(\frac{\mathbf{x}}{t}\right),\ \left(m\left(\frac{\mathbf{x}}{t}\right)\right)^{-2} s\left(\frac{\mathbf{x}}{t}\right)\right\}$$

represents a new solution describing unsteady flows.

As a final comment, we mention a recent paper [61] where a substitution principle for the general relativistic system of gas dynamics has been considered.

10.6.1 Substitution Principle and Conditional Symmetries

The substitution principle in the form given in Theorem 10.4 [66] can be derived using infinitesimal methods appending to Eqs. (10.22), supplemented by (10.23), and the constraints

$$\frac{\partial p}{\partial t} = 0, \quad \mathbf{v} \cdot \nabla m = 0, \quad \frac{\partial m}{\partial t} = 0.$$

In particular, assuming without loss of generality $\dfrac{\partial m}{\partial x_3} \neq 0$, we can eliminate from Eq. (10.22) the third component of the velocity, so obtaining

$$\begin{aligned}
&P(p)\left(\frac{\partial m}{\partial x_3}\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\right)-\left(\frac{\partial m}{\partial x_1}\frac{\partial v_1}{\partial x_3}+\frac{\partial m}{\partial x_2}\frac{\partial v_2}{\partial x_3}\right)-\left(v_1\frac{\partial^2 m}{\partial x_1\partial x_3}+v_2\frac{\partial^2 m}{\partial x_2\partial x_3}\right)\right.\\
&\quad\left.+\left(v_1\frac{\partial m}{\partial x_1}+v_2\frac{\partial m}{\partial x_2}\right)\frac{\partial^2 m}{\partial x_3^2}\left(\frac{\partial m}{\partial x_3}\right)^{-1}\right)+P(p)\left(\frac{\partial m}{\partial x_3}\left(v_1\frac{\partial m}{\partial x_1}+v_2\frac{\partial m}{\partial x_2}\right)\right.\\
&\quad\left.-\left(v_1\frac{\partial m}{\partial x_1}+v_2\frac{\partial m}{\partial x_2}\right)\frac{\partial p}{\partial x_3}\right)=0,\\
&P(p)s\left(\frac{\partial m}{\partial x_3}\left(\frac{\partial v_1}{\partial t}+v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}\right)-\left(v_1\frac{\partial m}{\partial x_1}+v_2\frac{\partial m}{\partial x_2}\right)\frac{\partial v_1}{\partial x_3}\right)+\frac{\partial m}{\partial x_3}\frac{\partial p}{\partial x_1}=0,\\
&P(p)s\left(\frac{\partial m}{\partial x_3}\left(\frac{\partial v_2}{\partial t}+v_1\frac{\partial v_2}{\partial x_1}+v_2\frac{\partial v_2}{\partial x_2}\right)-\left(v_1\frac{\partial m}{\partial x_1}+v_2\frac{\partial m}{\partial x_2}\right)\frac{\partial v_2}{\partial x_3}\right)+\frac{\partial m}{\partial x_3}\frac{\partial p}{\partial x_2}=0,\\
&P(p)s\left(v_1^2\left(\left(\frac{\partial m}{\partial x_3}\right)^2\frac{\partial^2 m}{\partial x_1^2}-2\frac{\partial m}{\partial x_1}\frac{\partial m}{\partial x_3}\frac{\partial^2 m}{\partial x_1\partial x_3}+\left(\frac{\partial m}{\partial x_1}\right)^2\frac{\partial^2 m}{\partial x_3^2}\right)\right.\\
&\quad+2v_1v_2\left(\frac{\partial m}{\partial x_3}\left(\frac{\partial m}{\partial x_3}\frac{\partial^2 m}{\partial x_1\partial x_2}-\frac{\partial m}{\partial x_1}\frac{\partial^2 m}{\partial x_2\partial x_3}-\frac{\partial m}{\partial x_2}\frac{\partial^2 m}{\partial x_1\partial x_3}\right)+\frac{\partial m}{\partial x_1}\frac{\partial m}{\partial x_2}\frac{\partial^2 m}{\partial x_3^2}\right)\\
&\quad\left.+v_2^2\left(\left(\frac{\partial m}{\partial x_3}\right)^2\frac{\partial^2 m}{\partial x_2^2}-2\frac{\partial m}{\partial x_2}\frac{\partial m}{\partial x_3}\frac{\partial^2 m}{\partial x_2\partial x_3}+\left(\frac{\partial m}{\partial x_2}\right)^2\frac{\partial^2 m}{\partial x_3^2}\right)\right)\\
&\quad-\left(\frac{\partial m}{\partial x_3}\right)^2\left(\frac{\partial m}{\partial x_1}\frac{\partial p}{\partial x_1}+\frac{\partial m}{\partial x_2}\frac{\partial p}{\partial x_2}+\frac{\partial m}{\partial x_3}\frac{\partial p}{\partial x_3}\right)+\frac{\partial m}{\partial x_3}\frac{\partial p}{\partial x_3}=0,\\
&\frac{\partial m}{\partial x_3}\left(\frac{\partial s}{\partial t}+v_1\frac{\partial s}{\partial x_1}+v_2\frac{\partial s}{\partial x_2}\right)-\left(v_1\frac{\partial m}{\partial x_1}+v_2\frac{\partial m}{\partial x_2}\right)\frac{\partial s}{\partial x_3}=0
\end{aligned}$$

for the unknowns p, v_1, v_2, and s, whereas m is considered an arbitrary function.

Using Lie infinitesimal criterion, the previous system admits the Lie symmetry generated by the vector field

$$\Xi = (mt + h(m))\frac{\partial}{\partial t} - mv_1\frac{\partial}{\partial v_1} - mv_2\frac{\partial}{\partial v_2} + 2ms\frac{\partial}{\partial s},$$

where $h(m)$ is an arbitrary function of m. From the infinitesimal transformation for v_1 and v_2,

$$v_1' = v_1 - amv_1 + O(a^2), \qquad v_2' = v_2 - amv_2 + O(a^2),$$

and using the constraint $\mathbf{v} \cdot \nabla m = 0$, it is immediate to obtain the infinitesimal transformation for v_3, i.e.:

$$v_3' = v_3 - amv_3 + O(a^2).$$

It is not difficult to recognize that, integrating the corresponding Lie equations, we obtain the transformation of the substitution principle in the form stated by Smith [66].

We can verify the above results by using the ReLie code

```
in "relie.red" $
jetorder:=1 $
xvar:={t,x1,x2,x3} $
uvar:={s,v1,v2,p} $
depend fun,p $. % P(p) in the formulae
depend m,x1,x2,x3 $
diffeqs := { - df(fun,p)*df(m,x1)*df(m,x3)*p_x3*v1
- df(fun,p)*df(m,x2)*df(m,x3)*p_x3*v2
+ df(fun,p)*df(m,x3)^2*p_t + df(fun,p)*df(m,x3)^2*p_x1*v1
+ df(fun,p)*df(m,x3)^2*p_x2*v2 - df(m,x1,x3)*df(m,x3)*fun*v1
+ df(m,x1)*df(m,x3,2)*fun*v1 - df(m,x1)*df(m,x3)*fun*v1_x3
- df(m,x2,x3)*df(m,x3)*fun*v2 + df(m,x2)*df(m,x3,2)*fun*v2
- df(m,x2)*df(m,x3)*fun*v2_x3 + df(m,x3)^2*fun*v1_x1
+ df(m,x3)^2*fun*v2_x2,

- df(m,x1)*fun*s*v1*v1_x3 - df(m,x2)*fun*s*v1_x3*v2
+ df(m,x3)*fun*s*v1*v1_x1 + df(m,x3)*fun*s*v1_t
+ df(m,x3)*fun*s*v1_x2*v2 + df(m,x3)*p_x1,
- df(m,x1)*fun*s*v1*v2_x3 - df(m,x2)*fun*s*v2*v2_x3
+ df(m,x3)*fun*s*v1*v2_x1 + df(m,x3)*fun*s*v2*v2_x2
+ df(m,x3)*fun*s*v2_t + df(m,x3)*p_x2,

- 2*df(m,x1,x2)*df(m,x3)^2*fun*s*v1*v2
+ 2*df(m,x1,x3)*df(m,x1)*df(m,x3)*fun*s*v1^2
+ 2*df(m,x1,x3)*df(m,x2)*df(m,x3)*fun*s*v1*v2
```

```
- df(m,x1,2)*df(m,x3)^2*fun*s*v1^2
- df(m,x1)^2*df(m,x3,2)*fun*s*v1^2
+ df(m,x1)^2*df(m,x3)*fun*s*v1*v1_x3
+ 2*df(m,x1)*df(m,x2,x3)*df(m,x3)*fun*s*v1*v2
- 2*df(m,x1)*df(m,x2)*df(m,x3,2)*fun*s*v1*v2
+ df(m,x1)*df(m,x2)*df(m,x3)*fun*s*v1*v2_x3
+ df(m,x1)*df(m,x2)*df(m,x3)*fun*s*v1_x3*v2
- df(m,x1)*df(m,x3)^2*fun*s*v1*v1_x1
- df(m,x1)*df(m,x3)^2*fun*s*v1_t
- df(m,x1)*df(m,x3)^2*fun*s*v1_x2*v2
+ 2*df(m,x2,x3)*df(m,x2)*df(m,x3)*fun*s*v2^2
- df(m,x2,2)*df(m,x3)^2*fun*s*v2^2
- df(m,x2)^2*df(m,x3,2)*fun*s*v2^2
+ df(m,x2)^2*df(m,x3)*fun*s*v2*v2_x3
- df(m,x2)*df(m,x3)^2*fun*s*v1*v2_x1
- df(m,x2)*df(m,x3)^2*fun*s*v2*v2_x2
- df(m,x2)*df(m,x3)^2*fun*s*v2_t + df(m,x3)^3*p_x3,


 - df(m,x1)*s_x3*v1 - df(m,x2)*s_x3*v2 + df(m,x3)*s_t
 + df(m,x3)*s_x1*v1 + df(m,x3)*s_x2*v2,

p_t}$

leadders:={p_x3,v1_t,v2_t,v2_x3,s_t,p_t} $

relie(3) $
% Infinitesimal generators
depend h,x1,x2,x3 $
xi_t:=m*t+h $
xi_x1:=0 $
xi_x2:=-0 $
xi_x3:=0 $
eta_p:=0 $
eta_v1:=-m*v1 $
eta_v2:=-m*v2 $
eta_s:=2*m*s $
df(h,x1):=df(h,m)*df(m,x1) $
df(h,x2):=df(h,m)*df(m,x2) $
df(h,x3):=df(h,m)*df(m,x3) $
deteqs:=delzero(deteqs) $
;end;
```

and all determining equations are satisfied.

10.7 Substitution Principles in Ideal Magneto Gas Dynamics

Substitution principles for the equations of ideal magneto gas dynamics have been established by Smith [65] in the steady case, and by Power and Rogers [56] in the unsteady one.

The equations governing the three-dimensional flow of an inviscid, thermally nonconducting fluid of infinite electrical conductivity in the presence of a magnetic field and subject to no extraneous force are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{10.33a}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) + \nabla p + \mu \mathbf{H} \times (\nabla \times \mathbf{H}) = 0, \tag{10.33b}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{10.33c}$$

$$\frac{\partial \mathbf{H}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{H}) = 0, \tag{10.33d}$$

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0, \tag{10.33e}$$

where $\rho(t, \mathbf{x})$ is the mass density, $p(t, \mathbf{x})$ is the pressure, $s(t, \mathbf{x})$ (a function of) is the entropy, $\mathbf{v}(t, \mathbf{x})$ is the velocity, and $\mathbf{H}(t, \mathbf{x})$ is the magnetic field; moreover, μ is the constant magnetic permeability. The system (10.33) is supplemented by the separable equation of state (10.23).

The theorem proved by Smith [65] and Power and Rogers [56] is stated below.

Theorem 10.7 ([65]) *If*

$$\{p(\mathbf{x}),\ \mathbf{v}(\mathbf{x}),\ s(\mathbf{x}),\ \mathbf{H}(\mathbf{x})\}$$

is a solution set of steady Eqs. (10.33) (i.e., the field variables are independent of t), then another solution is given by the substituted flow

$$\left\{p(\mathbf{x}),\ m(\mathbf{x})\mathbf{v}(\mathbf{x}),\ (m(\mathbf{x}))^{-2} s(\mathbf{x}),\ \mathbf{H}(\mathbf{x})\right\},$$

where $m(\mathbf{x})$ is a steady scalar function which is constant along each individual gas streamline and along each individual magnetic line, that is,

$$\mathbf{v} \cdot \nabla m = \mathbf{H} \cdot \nabla m = 0. \tag{10.34}$$

Theorem 10.8 ([56]) *If*

$$\{p(t, \mathbf{x}),\ \mathbf{v}(t, \mathbf{x}),\ s(t, \mathbf{x}),\ \mathbf{H}(t, \mathbf{x})\}$$

constitutes a solution set of the basic equations of unsteady magneto gas dynamics when the total magnetic pressure

$$p + \frac{\mu}{2}\mathbf{H}\cdot\mathbf{H}$$

is steady, then

$$\begin{aligned}&\left\{p(mt + h(m), \mathbf{x}),\ m(\mathbf{x})\mathbf{v}(mt + h(m), \mathbf{x}),\right.\\&\left.(m(\mathbf{x}))^{-2}\, s(mt + h(m), \mathbf{x}),\ \mathbf{H}(mt + h(m), \mathbf{x})\right\}\end{aligned}$$

also comprises a solution set, provided $m(\mathbf{x})$ is a steady differentiable scalar function such that (10.34) hold for all t, and $h(m)$ is an arbitrary function of the indicated argument.

Interpreting the transformation of the substituted flow as in previous section, and using the same arguments of the proof of Theorem 10.5 we can state the following result [45].

Theorem 10.9 ([45]) *If*

$$\{p(t, \mathbf{x}),\ \mathbf{v}(t, \mathbf{x}),\ s(t, \mathbf{x}),\ \mathbf{H}(t, \mathbf{x})\}$$

represents a solution of the equations of unsteady magneto gas dynamics with the equation of state of the form:

$$\rho = s^{\theta/\delta}\widehat{R}(p\cdot s^{-\beta/\delta}),$$

then another solution is

$$\begin{aligned}&\left\{m^{2-\theta}\, p(mt + h(m), \mathbf{x}), m(\mathbf{x})\mathbf{v}(mt + h(m), \mathbf{x}),\right.\\&\left.m^{-\delta}(x)s(mt + h(m), \mathbf{x}), m^{1-\theta/2}\mathbf{H}(mt + h(m), \mathbf{x})\right\},\end{aligned}$$

provided that

$$\begin{aligned}&\frac{\partial}{\partial t}\left(p + \frac{\mu}{2}\mathbf{H}\cdot\mathbf{H}\right) - \frac{\theta - 2}{t + h'(m)}\left(p + \frac{\mu}{2}\mathbf{H}\cdot\mathbf{H}\right) = 0,\\&\mathbf{v}\cdot\nabla m = 0, \qquad \mathbf{H}\cdot\nabla m = 0,\end{aligned}$$

for all t, where $m(\mathbf{x})$ is a steady differentiable scalar function of $\mathbf{x}$ and $h(m)$ is an arbitrary function of the indicated argument if $\theta = 2$ (whereupon $p + \frac{\mu}{2}\mathbf{H}\cdot\mathbf{H}$ must be steady), or $h(m) = h_0 m - h_0$, with h_0 constant, if $\theta \neq 2$.

The substitution principle for steady incompressible magneto gas dynamics [65] is a special subcase of the family of transformations found by Bogoyavlenskij [9]

for the magnetohydrodynamics equilibrium equations in the case where the subsonic dynamics of plasma is considered (in such a case the condition $\nabla \cdot \mathbf{v} = 0$ is satisfied with high accuracy, and this implies, by using the continuity equation, that the plasma density ρ is constant along the streamlines). The main result proved in [9] is stated as follows.

Theorem 10.10 ([9]) *Let $\rho(\mathbf{x})$, $\mathbf{v}(\mathbf{x})$, $p(\mathbf{x})$, $\mathbf{H}(\mathbf{x})$ be an arbitrary solution of ideal magneto gas dynamics for which $\nabla \cdot \mathbf{v} = 0$. A new solution is given by*

$$\begin{aligned}
\rho_1(\mathbf{x}) &= a^2(\mathbf{x})\rho(\mathbf{x}),\\
\mathbf{v}_1(\mathbf{x}) &= \frac{c(\mathbf{x})}{a(\mathbf{x})}\sqrt{\frac{\mu}{\rho(\mathbf{x})}}\mathbf{H}(\mathbf{x}) + \frac{b(\mathbf{x})}{a(\mathbf{x})}\mathbf{v}(\mathbf{x}),\\
p_1(\mathbf{x}) &= Cp(\mathbf{x}) + \frac{\mu}{2}(C\mathbf{H}^2(\mathbf{x}) - \mathbf{H}_1^2(\mathbf{x})),\\
\mathbf{H}_1(\mathbf{x}) &= b(\mathbf{x})\mathbf{H}(\mathbf{x}) + c(\mathbf{x})\sqrt{\frac{\rho(\mathbf{x})}{\mu}}\mathbf{v}(\mathbf{x}),
\end{aligned}$$

where $a(\mathbf{x})$, $b(\mathbf{x})$, $c(\mathbf{x})$, and $\rho(\mathbf{x}) > 0$ are arbitrary smooth functions that are constant along the magnetic field lines and on the plasma streamlines. Moreover, the functions $b(\mathbf{x})$ and $c(\mathbf{x})$ satisfy the condition

$$b^2(\mathbf{x}) - c^2(\mathbf{x}) = C = const.$$

It is easy to recognize that the substitution principle as stated in Theorem 10.7, also true in the case of nonsolenoidal velocity fields, is recovered by choosing $a(\mathbf{x}) = m(\mathbf{x})$, $b(\mathbf{x}) \equiv 1$, and $c(\mathbf{x}) \equiv 0$. Moreover, Cheviakov [16] proved that these Bogoyavlenskij symmetries can be derived within the context of Lie group analysis.

10.7.1 A Mixed Substitution Principle for Plane Flows Subjected to a Transverse Magnetic Field

The case of magneto gas dynamics in two space dimensions subjected to a transverse magnetic field, when the adiabatic exponent is $\gamma = 2$, is particularly interesting. The governing equations are obtained from (10.33) assuming the velocity vector as $\mathbf{v} = (v_1, v_2, 0)$; the magnetic field $\mathbf{H} = (0, 0, h)$; and all field variables depending upon t, x_1, and x_2.

$$\frac{\partial p}{\partial t} + v_1 \frac{\partial p}{\partial x_1} + v_2 \frac{\partial p}{\partial x_2} + 2p \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) = 0, \tag{10.35a}$$

$$p^{1/2} s \left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} \right) + \frac{\partial p}{\partial x_1} + h \frac{\partial h}{\partial x_1} = 0, \tag{10.35b}$$

$$p^{1/2} s \left(\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} \right) + \frac{\partial p}{\partial x_1} + h \frac{\partial h}{\partial x_1} = 0, \tag{10.35c}$$

$$\frac{\partial h}{\partial t} + \left(\frac{\partial (h v_1)}{\partial x_1} - \frac{\partial (h v_2)}{\partial x_2} \right) = 0, \tag{10.35d}$$

$$\frac{\partial s}{\partial t} + v_1 \frac{\partial s}{\partial x_1} + v_2 \frac{\partial s}{\partial x_2} = 0. \tag{10.35e}$$

System (10.35) admits the symmetry generated by the vector field

$$\Xi = t^2 \frac{\partial}{\partial t} + x_1 t \frac{\partial}{\partial x_1} + x_2 t \frac{\partial}{\partial x_2} + (x_1 - v_1 t) \frac{\partial}{\partial v_1} + (x_2 - v_2 t) \frac{\partial}{\partial v_2} - 4pt + \frac{\partial}{\partial p} - 2ht \frac{\partial}{\partial h},$$

and the corresponding invariant solutions are

$$p = t^{-4} \widehat{p}(\omega_1, \omega_2), \qquad h = t^{-2} \widehat{h}(\omega_1, \omega_2), \qquad s = \widehat{s}(\omega_1, \omega_2),$$
$$v_1 = \omega_1 + t^{-1} \widehat{v}_1(\omega_1, \omega_2), \qquad v_2 = \omega_2 + t^{-1} \widehat{v}_2(\omega_1, \omega_2),$$

where $\omega_k = \dfrac{x_k}{t}$ $(k = 1, 2)$, and $\widehat{p}, \widehat{h}, \widehat{s}, \widehat{v}_1$, and $\widehat{v}_2$ satisfy the reduced system

$$\widehat{v}_1 \frac{\partial \widehat{p}}{\partial \omega_1} + \widehat{v}_2 \frac{\partial \widehat{p}}{\partial \omega_2} + 2\widehat{p} \left(\frac{\partial \widehat{v}_1}{\partial \omega_1} + \frac{\partial \widehat{v}_2}{\partial \omega_2} \right) = 0, \tag{10.36a}$$

$$\widehat{p}^{1/2} \widehat{s} \left(\widehat{v}_1 \frac{\partial \widehat{v}_1}{\partial \omega_1} + \widehat{v}_2 \frac{\partial \widehat{v}_1}{\partial \omega_2} \right) + \frac{\partial \widehat{p}}{\partial \omega_1} + \widehat{h} \frac{\partial \widehat{h}}{\partial \omega_1} = 0, \tag{10.36b}$$

$$\widehat{p}^{1/2} \widehat{s} \left(\widehat{v}_1 \frac{\partial \widehat{v}_2}{\partial \omega_1} + \widehat{v}_2 \frac{\partial \widehat{v}_2}{\partial \omega_2} \right) + \frac{\partial \widehat{p}}{\partial \omega_2} + \mu \widehat{h} \frac{\partial \widehat{h}}{\partial \omega_2} = 0, \tag{10.36c}$$

$$\frac{\partial (\widehat{h} \widehat{v}_1)}{\partial \omega_1} - \frac{\partial (\widehat{h} \widehat{v}_2)}{\partial \omega_2} = 0, \tag{10.36d}$$

$$\widehat{v}_1 \frac{\partial \widehat{s}}{\partial \omega_1} + \widehat{v}_2 \frac{\partial \widehat{s}}{\partial \omega_2} = 0. \tag{10.36e}$$

We observe that system (10.36), identifying ω_1 with x_1 and ω_2 with x_2, has the same form as the steady version of system (10.35). This implies that a steady solution allows us to generate an unsteady solution. Using the substitution principle for steady magneto gas dynamics, we can state the following mixed substitution principle [45].

Theorem 10.11 *If*

$$\{p(x_1, x_2), v_1(x_1, x_2), v_2(x_1, x_2), s(x_1, x_2), h(x_1, x_2)\}$$

constitutes a solution set of the steady equations of plane magneto gas dynamics with a magnetic field transverse to the plane of motion and the adiabatic exponent $\gamma = 2$, *and* $m(x_1, x_2)$ *is a scalar function such that*

$$v_1 \frac{\partial m}{\partial x_1} + v_2 \frac{\partial m}{\partial x_2} = 0,$$

then

$$\left\{ t^{-4} p(x_1/t, x_2/t),\ \frac{x_1}{t} + t^{-1} \frac{v_1(x_1/t, x_2/t)}{m(x_1/t, x_2/t)},\ \frac{x_2}{t} + t^{-1} \frac{v_2(x_1/t, x_2/t)}{m(x_1/t, x_2/t)}, \right.$$
$$\left. m^2 s(x_1/t, x_2/t),\ t^{-2} h(x_1/t, x_2/t) \right\}$$

gives a solution set to the unsteady equations.

10.7.2 Substitution Principle and Conditional Symmetries

Hereafter, we append to Eqs. (10.33) the conditions that the velocity vector $\mathbf{v}(t, \mathbf{x})$ and the magnetic field $\mathbf{H}(t, \mathbf{x})$ result for all t tangent to the surface $m(\mathbf{x}) = \text{constant}$, and the condition ensuring the steadiness of the total magnetic pressure. As in the case of ideal gas dynamics [44], we explicit the above constraints, eliminating the third components of velocity and magnetic field from Eqs. (10.33) with the constitutive relation $\rho = p^{1/\gamma} s$. These constraints are "side conditions" [50, 51] that restrict the set of solutions: as a consequence, a different (usually larger) group of transformations leaving these solutions invariant may arise. This will be shown in the sequel.

Without loss of generality, let us assume $\dfrac{\partial m}{\partial x_3} \neq 0$; thus, we may solve the constraints (10.34) with respect to v_3 and H_3:

$$v_3 = -\frac{v_1 \dfrac{\partial m}{\partial x_1} + v_2 \dfrac{\partial m}{\partial x_2}}{\dfrac{\partial m}{\partial x_3}}, \qquad H_3 = -\frac{H_1 \dfrac{\partial m}{\partial x_1} + H_2 \dfrac{\partial m}{\partial x_2}}{\dfrac{\partial m}{\partial x_3}}. \tag{10.37}$$

By using (10.37) into (10.33), the third component of (10.33d) is identically satisfied; therefore, we get a system of eight scalar partial differential equations involving the unknowns $p(t, \mathbf{x})$, $v_1(t, \mathbf{x})$, $v_2(t, \mathbf{x})$, $H_1(t, \mathbf{x})$, $H_2(t, \mathbf{x})$, and $s(t, \mathbf{x})$, whereas m is an arbitrary function of space variables (we do not write the expression of the transformed system that, despite its straightforward derivation, is quite long).

Using Lie infinitesimal criterion, this system admits the symmetry generated by the vector field

$$\Xi = (mt + h(m))\frac{\partial}{\partial t} - mv_1\frac{\partial}{\partial v_1} - mv_2\frac{\partial}{\partial v_2} + 2ms\frac{\partial}{\partial s},$$

where $h(m)$ is an arbitrary function of m.

From the infinitesimal transformation for v_1, v_2, H_1, and H_2,

$$v_1' = v_1 - amv_1 + O(a^2), \quad v_2' = v_2 - amv_2 + O(a^2), \quad H_1' = H_1, \quad H_2' = H_2,$$

and using the constraints $\mathbf{v} \cdot \nabla m = 0$ and $\mathbf{H} \cdot \nabla m = 0$, it is immediate to obtain the infinitesimal transformation for v_3 and H_3, i.e.:

$$v_3' = v_3 - amv_3 + O(a^2), \qquad H_3' = H_3.$$

It is not difficult to recognize that, integrating the corresponding Lie equations, we obtain the transformation of the substitution principle in the form stated in [56].

10.8 Equivalence Transformations

There are situations where the differential equations at hand contain unspecified functions (*arbitrary elements*); in such cases, it is convenient to consider a class $\mathcal{E}(\mathbf{p})$ of differential equations involving some arbitrary continuously differentiable functions p_j ($j = 1, \ldots, \ell$). The infinitesimal arguments used to study the invariance of a differential equation, suitably adapted, can be used to study the invariance of a class of differential equations containing arbitrary elements [53].

Let us consider a class of differential equations

$$\mathbf{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}; \mathbf{p}, \mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(s)}\right) = 0, \tag{10.38}$$

whose elements are given once we fix the functions p_j, where $\mathbf{p}^{(k)}$, $k = 1, \ldots, s$, denotes the set of all derivatives of order k of the $\mathbf{p}$'s with respect to their arguments; these arguments may be the independent variables, the dependent ones and the derivatives of the latter with respect to the former up to a fixed order q: let us denote the arguments of the arbitrary elements $\mathbf{p}$ as $\mathbf{z} \equiv (z_1, \ldots, z_N)$.

The study of the invariance of (10.38) can be done by performing what is called a group classification problem, consisting in determining the symmetries for arbitrary functions $\mathbf{p}$, and also the existence of additional symmetries for special forms of the elements $\mathbf{p}$ (for instance, see [6, 7, 24, 29, 31]).

Another approach consists in using *equivalence transformations*, i.e., transformations that preserve the differential structure of the equations in the system but may

change the form of the constitutive functions and/or parameters therein involved [30, 31, 37, 53, 54, 67, 68].

A one-parameter Lie group of equivalence transformations [53] of a family $\mathcal{E}(\mathbf{p})$ of partial differential equations is a one-parameter Lie group of transformations given by

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, \mathbf{u}, \mathbf{p}; a), \qquad \mathbf{U} = \mathbf{U}(\mathbf{x}, \mathbf{u}, \mathbf{p}; a), \qquad \mathbf{P} = \mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{p}; a),$$

a being the parameter, which is locally a C^∞ diffeomorphism mapping a class $\mathcal{E}(\mathbf{p})$ of differential equations into itself.

A common assumption consists in considering transformations of the independent and dependent variables independent of the arbitrary elements $\mathbf{p}$. This allows for the *projection* of the transformation in the space of independent and dependent variables, and will be the framework we will consider in the following.

In an *augmented space* $\mathcal{A} \equiv \mathcal{X} \times \mathcal{U} \times \mathcal{P} \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell$ [37, 53], where the independent variables, the dependent variables, and the arbitrary functions live, respectively, the generator of the equivalence transformation,

$$\Xi = \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\alpha} + \sum_{j=1}^{\ell} \mu_j(\mathbf{x}, \mathbf{u}, \mathbf{p}) \frac{\partial}{\partial p_j},$$

involves also the infinitesimals $\mu_j(\mathbf{x}, \mathbf{u}, \mathbf{p})$ accounting for the arbitrary functions p_j. The search for continuous equivalence transformations can be exploited by using the Lie infinitesimal criterion, as proposed by Ovsiannikov [53]. Of course, we need prolongations.

The first prolongation of Ξ is

$$\Xi^{(1)} = \Xi + \sum_{\alpha=1}^{m} \sum_{i=1}^{n} \eta_{[\alpha,i]} \frac{\partial}{\partial u_{\alpha,i}} + \sum_{j=1}^{\ell} \sum_{\beta=1}^{N} \mu_{[j,\beta]} \frac{\partial}{\partial p_{j,\beta}},$$

with

$$\eta_{[\alpha,i]} = \mathcal{D}_i(\eta_\alpha) - \sum_{j=1}^{n} u_{\alpha,j} \mathcal{D}_i(\xi_j), \qquad \mu_{[j,\beta]} = \widehat{\mathcal{D}}_\beta(\mu_j) - \sum_{\gamma=1}^{N} p_{j,\gamma} \widehat{\mathcal{D}}_\beta(\zeta_\gamma),$$

where $p_{j,\gamma} = \dfrac{\partial p_j}{\partial z_\gamma}$ and ζ_γ is the infinitesimal generator of z_γ, along with the standard total derivatives $\mathcal{D}_i$ already defined, and

$$\widehat{\mathcal{D}}_\beta = \frac{\partial}{\partial z_\beta} + \sum_{k=1}^{\ell} p_{k,\beta} \frac{\partial}{\partial p_k}.$$

Higher order prolongations are immediately obtained by recursion,

$$\Xi^{(k)} = \Xi^{(k-1)} + \sum_{\alpha=1}^{m} \sum_{i_1=1}^{n} \sum_{i_2=i_1}^{n} \cdots \sum_{i_k=i_{k-1}}^{n} \eta_{[\alpha,i_1\ldots i_k]} \frac{\partial}{\partial u_{\alpha,i_1\ldots i_k}} + \sum_{j=1}^{\ell} \sum_{\beta_1=1}^{N} \sum_{\beta_2=\beta_1}^{N} \cdots \sum_{\beta_k=\beta_{k-1}}^{N} \mu_{[j,\beta_1\ldots\beta_k]} \frac{\partial}{\partial p_{j,\beta_1\ldots\beta_k}},$$

where

$$\mu_{[j,\beta_1\ldots\beta_k]} = \widehat{\mathcal{D}}_{\beta_k}(\mu_{[j,\beta_1\ldots\beta_{k-1}]}) - \sum_{\gamma=1}^{N} p_{j,\gamma\beta_1\ldots\beta_{k-1}} \widehat{\mathcal{D}}_{\beta_k}(\zeta_\gamma).$$

In the augmented space $\mathcal{A}$, the arbitrary functions determining the class of differential equations are assumed as dependent variables, and we require the invariance of the class in this augmented space via Lie infinitesimal criterion [53]:

$$\Xi^{(max(r,s))}(\mathbf{\Delta})\big|_{\mathbf{\Delta}=0} = \mathbf{0}.$$

If we project the symmetries on the space $\mathcal{Z} \equiv \mathcal{X} \times \mathcal{U}$ of the independent and dependent variables (this is always possible if the infinitesimals of independent and dependent variables are assumed to be independent of $\mathbf{p}$), we obtain a transformation changing an element of the class of differential equations to another element in the same class (same differential structure but in general different arbitrary elements) [27, 47, 48]. Such projected transformations map solutions of a system in the class to solutions of a transformed system in the same class.

If some arbitrary elements do not depend upon some variables, the differential equations at hand have to be supplemented with *auxiliary conditions*. For instance, the auxiliary conditions for the equivalence transformations of the class of differential equations [31]

$$\frac{\partial^2 u}{\partial t^2} - f\left(x, \frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} - g\left(x, \frac{\partial u}{\partial x}\right) = 0,$$

where f and g are unspecified functions of their arguments, are

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial(\partial u/\partial t)} = \frac{\partial g}{\partial t} = \frac{\partial g}{\partial u} = \frac{\partial g}{\partial(\partial u/\partial t)} = 0.$$

More generally, nothing prevents to consider equivalence transformations where the arbitrary elements enter the transformation of independent and dependent variables [39]; in such a case, the total derivative $\mathcal{D}_i$ in the expression of prolongation reads

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} u_{\alpha,i} \frac{\partial}{\partial u_\alpha} + \sum_{k=1}^{\ell} \left(\frac{\partial p_k}{\partial x_i} + \sum_{\beta=n+1}^{N} \frac{\partial p_k}{\partial z_\beta} \frac{\partial z_\beta}{\partial x_i} \right) \frac{\partial}{\partial p_k}.$$

In the next section we give some applications.

10.9 Equivalence Transformations of Balance Laws in (2 + 1) Dimensions

Consider the class $\mathcal{E}(\mathbf{p})$ with $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5, p_6)$ of systems [48]

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_2} = 0, \tag{10.39a}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial p_1}{\partial x_1} + \frac{\partial p_2}{\partial x_2} = p_5, \tag{10.39b}$$

$$\frac{\partial u_3}{\partial t} + \frac{\partial p_3}{\partial x_1} + \frac{\partial p_4}{\partial x_2} = p_6, \tag{10.39c}$$

where t, x_1, and x_2 are the independent variables; u_1, u_2, and u_3 are the dependent variables, whereas $p_i \equiv p_i(t, x_1, x_2, u_1, u_2, u_3)$ $(i = 1, \ldots, 6)$ stand for arbitrary continuously differentiable functions of the indicated arguments.

By imposing the invariance condition to the class $\mathcal{E}(\mathbf{p})$ in the augmented space $\mathcal{A} \equiv \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^6$, we determine nine symmetry operators of equivalence

$$\begin{aligned}
\Xi_1^e = {} & f^{(1)} \frac{\partial}{\partial x_1} - \frac{\partial f^{(1)}}{\partial x_1} u_1 \frac{\partial}{\partial u_1} + \left(\frac{\partial f^{(1)}}{\partial t} u_1 + \frac{\partial f^{(1)}}{\partial x_2} u_3 \right) \frac{\partial}{\partial u_2} - \frac{\partial f^{(1)}}{\partial x_1} u_3 \frac{\partial}{\partial u_3} \\
& + \left(2 \frac{\partial f^{(1)}}{\partial t} u_2 + \frac{\partial f^{(1)}}{\partial x_1} p_1 + \frac{\partial f^{(1)}}{\partial x_2} (p_2 + p_3) \right) \frac{\partial}{\partial p_1} \\
& + \left(\frac{\partial f^{(1)}}{\partial t} u_3 + \frac{\partial f^{(1)}}{\partial x_2} p_4 \right) \left(\frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3} \right) - \frac{\partial f^{(1)}}{\partial x_2} p_5 \frac{\partial}{\partial p_4} \\
& + \left(\frac{\partial^2 f^{(1)}}{\partial t^2} u_1 + 2 \frac{\partial^2 f^{(1)}}{\partial t \partial x_1} u_2 + 2 \frac{\partial^2 f^{(1)}}{\partial t \partial x_2} u_3 + \frac{\partial^2 f^{(1)}}{\partial x_1^2} p_1 \right. \\
& \left. + \frac{\partial^2 f^{(1)}}{\partial x_1 \partial x_2} (p_2 + p_3) + \frac{\partial^2 f^{(1)}}{\partial x_2^2} p_4 + \frac{\partial f^{(1)}}{\partial x_2} p_6 \right) \frac{\partial}{\partial p_5} - \frac{\partial f^{(1)}}{\partial x_1} p_6 \frac{\partial}{\partial p_6},
\end{aligned}$$

$$
\begin{aligned}
\Xi^e_2 &= f^{(2)}\frac{\partial}{\partial x_2} - \frac{\partial f^{(2)}}{\partial x_2}u_1\frac{\partial}{\partial u_1} - \frac{\partial f^{(2)}}{\partial x_2}u_2\frac{\partial}{\partial u_2} + \left(\frac{\partial f^{(2)}}{\partial t}u_1 + \frac{\partial f^{(2)}}{\partial x_1}u_2\right)\frac{\partial}{\partial u_3}\\
&\quad - \frac{\partial f^{(2)}}{\partial x_2}p_1\frac{\partial}{\partial p_1} + \left(\frac{\partial f^{(2)}}{\partial t}u_2 + \frac{\partial f^{(2)}}{\partial x_1}p_1\right)\left(\frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3}\right)\\
&\quad + \left(2\frac{\partial f^{(2)}}{\partial t}u_3 + \frac{\partial f^{(2)}}{\partial x_1}(p_2 + p_3) + \frac{\partial f^{(2)}}{\partial x_2}p_4 - \frac{\partial f^{(2)}}{\partial x_2}p_5\right)\frac{\partial}{\partial p_4}\\
&\quad - \frac{\partial f^{(2)}}{\partial x_2}p_5\frac{\partial}{\partial p_5} + \left(\frac{\partial^2 f^{(2)}}{\partial t^2}u_1 + 2\frac{\partial^2 f^{(2)}}{\partial t\partial x_1}u_2 + 2\frac{\partial^2 f^{(2)}}{\partial t\partial x_2}u_3 + \frac{\partial^2 f^{(2)}}{\partial x_1^2}p_1\right.\\
&\quad \left. + \frac{\partial^2 f^{(2)}}{\partial x_1\partial x_2}(p_2 + p_3) + \frac{\partial^2 f^{(2)}}{\partial x_2^2}p_4 + \frac{\partial f^{(2)}}{\partial x_1}p_5\right)\frac{\partial}{\partial p_6},\\
\Xi^e_3 &= f(t)\frac{\partial}{\partial t} - f'(t)\left(u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3}\right)\\
&\quad - 2f'(t)\left(p_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial p_2} + p_3\frac{\partial}{\partial p_3} + p_4\frac{\partial}{\partial p_4}\right)\\
&\quad - \left(f''(t)u_2 + 2g'(t)p_5\right)\frac{\partial}{\partial p_5} - \left(f''(t)u_3 + 2f'(t)p_6\right)\frac{\partial}{\partial p_6},\\
\Xi^e_4 &= f^{(3)}\frac{\partial}{\partial u_1} + f^{(4)}\frac{\partial}{\partial u_2} + f^{(5)}\frac{\partial}{\partial u_3} + \frac{\partial f^{(4)}}{\partial t}\frac{\partial}{\partial p_5} + \frac{\partial f^{(5)}}{\partial t}\frac{\partial}{\partial p_6},\\
\Xi^e_5 &= \sum_{i=1}^{3} u_i\frac{\partial}{\partial u_i} + \sum_{i=1}^{6} p_i\frac{\partial}{\partial p_i},\\
\Xi^e_6 &= f^{(6)}\frac{\partial}{\partial p_1} + \frac{\partial f^{(6)}}{\partial x_1}\frac{\partial}{\partial p_5}, \qquad \Xi^e_7 = f^{(7)}\frac{\partial}{\partial p_2} + \frac{\partial f^{(7)}}{\partial x_2}\frac{\partial}{\partial p_5},\\
\Xi^e_8 &= f^{(8)}\frac{\partial}{\partial p_3} + \frac{\partial f^{(8)}}{\partial x_1}\frac{\partial}{\partial p_5}, \qquad \Xi^e_9 = f^{(9)}\frac{\partial}{\partial p_4} + \frac{\partial f^{(9)}}{\partial x_2}\frac{\partial}{\partial p_6},
\end{aligned}
$$

where $f^{(i)}$ $(i = 1, \dots 9)$ are arbitrary functions of the independent variables t, x_1, x_2 with $f^{(3)}$, $f^{(4)}$, and $f^{(5)}$ subjected to the constraint

$$\frac{\partial f^{(3)}}{\partial t} + \frac{\partial f^{(4)}}{\partial x_1} + \frac{\partial f^{(5)}}{\partial x_2} = 0,$$

and $f(t)$ is an arbitrary function of its argument (see the results provided by ReLie in Example 12.14).

By projecting the admitted equivalence generators on the space $\mathcal{Z} \equiv \mathbb{R}^3 \times \mathbb{R}^3$, and computing from Lie equations the finite corresponding transformations, say

$$
\begin{aligned}
&T = T(t, x_1, x_2, u_1, u_2, u_3; a), \quad && X_1 = X_1(t, x_1, x_2, u_1, u_2, u_3; a),\\
&X_2 = X_2(t, x_1, x_2, u_1, u_2, u_3; a), && U_1 = U_1(t, x_1, x_2, u_1, u_2, u_3; a),\\
&U_2 = U_2(t, x_1, x_2, u_1, u_2, u_3; a), && U_3 = U_3(t, x_1, x_2, u_1, u_2, u_3; a),
\end{aligned}
\tag{10.40}
$$

the original system (10.39) is mapped to a different system with the same differential structure: in particular, we are interested to the case where the target system is an autonomous system of conservation laws:

$$\begin{aligned}
&\frac{\partial U_1}{\partial T} + \frac{\partial U_2}{\partial X_1} + \frac{\partial U_3}{\partial X_2} = 0,\\
&\frac{\partial U_2}{\partial T} + \frac{\partial P_1}{\partial X_1} + \frac{\partial P_2}{\partial X_2} = 0,\\
&\frac{\partial U_3}{\partial T} + \frac{\partial P_3}{\partial X_1} + \frac{\partial P_4}{\partial X_2} = 0,
\end{aligned} \tag{10.41}$$

where $P_i \equiv P_i(U_1, U_2, U_3)$. To simplify the computation, we exchange the source and target system; in fact, taking the inverse transformation of (10.40) (which is obtained by exchanging lower and capital letters and replacing a with $-a$), and starting from the autonomous system (10.41) of conservation laws, we are able to obtain the equivalent nonautonomous system of balance laws.

After projecting on $\mathcal{Z}$ the generators, there remain

$$\begin{aligned}
\Xi_1 &= f^{(1)}\frac{\partial}{\partial x_1} - \frac{\partial f^{(1)}}{\partial x_1}u_1\frac{\partial}{\partial u_1} + \left(\frac{\partial f^{(1)}}{\partial t}u_1 + \frac{\partial f^{(1)}}{\partial x_2}u_3\right)\frac{\partial}{\partial u_2} - \frac{\partial f^{(1)}}{\partial x_1}u_3\frac{\partial}{\partial u_3},\\
\Xi_2 &= f^{(2)}\frac{\partial}{\partial x_2} - \frac{\partial f^{(2)}}{\partial x_2}u_1\frac{\partial}{\partial u_1} - \frac{\partial f^{(2)}}{\partial x_2}u_2\frac{\partial}{\partial u_2} + \left(\frac{\partial f^{(2)}}{\partial t}u_1 + \frac{\partial f^{(2)}}{\partial x_1}u_2\right)\frac{\partial}{\partial u_3},\\
\Xi_3 &= f(t)\frac{\partial}{\partial t} - f'(t)\left(u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3}\right),\\
\Xi_4 &= f^{(3)}\frac{\partial}{\partial u_1} + f^{(4)}\frac{\partial}{\partial u_2} + f^{(5)}\frac{\partial}{\partial u_3},\\
\Xi_5 &= \sum_{i=1}^{3} u_i\frac{\partial}{\partial u_i}.
\end{aligned}$$

Now, since we start from an autonomous system of conservation laws to arrive to a nonautonomous system of balance laws, let us write the operators in terms of the capital letters; then, we build the finite transformations corresponding to the operators projected on $\mathcal{Z}$. The most general finite transformation can be recovered by the composition of the finite transformations induced by each generator.

10.9.1 Equivalence Transformations Generated by Ξ_1

By considering the operator Ξ_1,

$$\Xi_1 = f\frac{\partial}{\partial X_1} - \frac{\partial f}{\partial X_1}U_1\frac{\partial}{\partial U_1} + \left(\frac{\partial f}{\partial T}U_1 + \frac{\partial f}{\partial X_2}U_3\right)\frac{\partial}{\partial U_2} - \frac{\partial f}{\partial X_1}U_3\frac{\partial}{\partial U_3},$$

where we set $f(T, X_1, X_2) = f^{(1)}(T, X_1, X_2)$, we get the following finite transformation:

$$
\begin{aligned}
&t = T, \qquad x_1 = \tilde{x}_1(T, X_1, X_2; a), \qquad x_2 = X_2,\\
&u_1 = U_1 \frac{f(T, X_1, X_2)}{f(T, x_1, X_2)}, \qquad u_3 = U_3 \frac{f(T, X_1, X_2)}{f(T, x_1, X_2)},\\
&u_2 = U_2 + \frac{f(T, X_1, X_2)}{f(T, x_1, X_2)} \left(U_1 \frac{\partial \tilde{x}_1}{\partial T} + U_3 \frac{\partial \tilde{x}_1}{\partial X_2} \right),
\end{aligned}
$$

$\tilde{x}_1(T, X_1, X_2; a)$ being such that $\tilde{x}_1(T, X_1, X_2; 0) = X_1$, and

$$
\begin{aligned}
&\frac{\partial \tilde{x}_1}{\partial T} = f(T, x_1, X_2) \int_0^a \frac{1}{f(T, x_1, X_2)} \frac{\partial f(T, x_1, X_2)}{\partial T} da,\\
&\frac{\partial \tilde{x}_1}{\partial X_1} = \frac{f(T, x_1, X_2)}{f(T, X_1, X_2)},\\
&\frac{\partial \tilde{x}_1}{\partial X_2} = f(T, x_1, X_2) \int_0^a \frac{1}{f(T, x_1, X_2)} \frac{\partial f(T, x_1, X_2)}{\partial X_2} da,
\end{aligned}
$$

whereupon

$$
U_1 = u_1 \frac{\partial \tilde{x}_1}{\partial X_1}, \qquad U_2 = u_2 - u_1 \frac{\partial \tilde{x}_1}{\partial T} - u_3 \frac{\partial \tilde{x}_1}{\partial X_2}, \qquad U_3 = u_3 \frac{\partial \tilde{x}_1}{\partial X_1}.
$$

Thus, system (10.41) is equivalent to system (10.39) with

$$
\begin{aligned}
p_1 =& P_1 \frac{\partial \tilde{x}_1}{\partial X_1} + (P_1 + P_3) \frac{\partial \tilde{x}_1}{\partial X_2} + P_4 \left(\frac{\partial \tilde{x}_1}{\partial X_2} \right)^2 \left(\frac{\partial \tilde{x}_1}{\partial X_1} \right)^{-1} - \left(\frac{\partial \tilde{x}_1}{\partial T} \right)^2 u_1 + 2 \frac{\partial \tilde{x}_1}{\partial T} u_2,\\
p_2 =& P_2 + P_4 \frac{\partial \tilde{x}_1}{\partial X_2} \left(\frac{\partial \tilde{x}_1}{\partial X_1} \right)^{-1} + \frac{\partial \tilde{x}_1}{\partial T} u_3,\\
p_3 =& P_3 + P_4 \frac{\partial \tilde{x}_1}{\partial X_2} \left(\frac{\partial \tilde{x}_1}{\partial X_1} \right)^{-1} + \frac{\partial \tilde{x}_1}{\partial T} u_3,\\
p_4 =& P_4 \left(\frac{\partial \tilde{x}_1}{\partial X_1} \right)^{-1},\\
p_5 =& \left(\partial_{U_2} \left(P_3 \frac{\partial^2 \tilde{x}_1}{\partial T \partial X_1} + P_4 \frac{\partial^2 \tilde{x}_1}{\partial T \partial X_2} \right) - \partial_{U_1} \left(P_3 \frac{\partial^2 \tilde{x}_1}{\partial X_1^2} + P_4 \frac{\partial^2 \tilde{x}_1}{\partial X_1 \partial X_2} \right) \right) \left(\frac{\partial \tilde{x}_1}{\partial X_1} \right)^{-1} u_1\\
&+ \left(\partial_{U_2} \left(P_3 \frac{\partial^2 \tilde{x}_1}{\partial X_1 \partial X_2} + P_4 \frac{\partial^2 \tilde{x}_1}{\partial X_2^2} \right) - \partial_{U_3} \left(P_3 \frac{\partial^2 \tilde{x}_1}{\partial X_1^2} + P_4 \frac{\partial \tilde{x}_1}{\partial X_1 \partial X_2} \right) \right) \left(\frac{\partial \tilde{x}_1}{\partial X_1} \right)^{-1} u_3,\\
&- \frac{\partial^2 \tilde{x}_1}{\partial T \partial X_1} \left(\frac{\partial \tilde{x}_1}{\partial X_1} \right)^{-1} u_3
\end{aligned}
$$

$$p_6 = \left(\partial_{U_2}\left(P_1\frac{\partial^2\tilde{x}_1}{\partial T\partial X_1} + P_2\frac{\partial^2\tilde{x}_1}{\partial T\partial X_2}\right) - \partial_{U_1}\left(P_1\frac{\partial^2\tilde{x}_1}{\partial X_1^2} + P_2\frac{\partial^2\tilde{x}_1}{\partial X_1\partial X_2}\right) + \frac{\partial^2\tilde{x}_1}{\partial T^2}\right)u_1$$
$$+\left(\partial_{U_2}\left(P_1\frac{\partial^2\tilde{x}_1}{\partial X_1\partial X_2} + P_2\frac{\partial^2\tilde{x}_1}{\partial X_2^2}\right) - \partial_{U_3}\left(P_1\frac{\partial^2\tilde{x}_1}{\partial X_1^2} + P_2\frac{\partial^2\tilde{x}_1}{\partial X_1\partial X_2}\right) + \frac{\partial^2\tilde{x}_1}{\partial T\partial X_2}\right)u_3$$
$$+p_5\frac{\partial\tilde{x}_1}{\partial X_2},$$

where $P_k = P_k\left(u_1\frac{\partial\tilde{x}_1}{\partial X_1}, u_3\frac{\partial\tilde{x}_1}{\partial X_1}, u_2 - u_1\frac{\partial\tilde{x}_1}{\partial T} - u_3\frac{\partial\tilde{x}_1}{\partial X_2}\right), k = 1, \ldots, 4.$

10.9.2 Equivalence Transformations Generated by Ξ_2

By taking the operator Ξ_2,

$$\Xi_2 = f\frac{\partial}{\partial X_2} - \frac{\partial f}{\partial X_2}U_1\frac{\partial}{\partial U_1} - \frac{\partial f}{\partial X_2}U_2\frac{\partial}{\partial U_2} + \left(\frac{\partial f}{\partial T}U_1 + \frac{\partial f}{\partial X_1}U_2\right)\frac{\partial}{\partial U_3},$$

where we set $f(T, X_1, X_2) = f^{(2)}(T, X_1, X_2)$, we get from Lie equations the general finite transformation

$$t = T, \quad x_1 = X_1, \quad x_2 = \tilde{x}_2(T, X_1, X_2; a),$$
$$u_1 = U_1\frac{f(T, X_1, X_2)}{f(T, X_1, x_2)}, \quad u_2 = U_2\frac{f(T, X_1, X_2)}{f(T, X_1, x_2)},$$
$$u_3 = U_3 + \frac{f(T, X_1, X_2)}{f(T, X_1, x_2)}\left(U_1\frac{\partial\tilde{x}_2}{\partial T} + U_2\frac{\partial\tilde{x}_2}{\partial X_1}\right),$$

where $\tilde{x}_2(T, X_1, X_2; a)$ is such that $\tilde{x}_2(T, X_1, X_2; 0) = X_2$, and

$$\frac{\partial\tilde{x}_2}{\partial T} = f(T, X_1, x_2)\int_0^a \frac{1}{f(T, X_1, x_2)}\frac{\partial f(T, X_1, x_2)}{\partial T}da,$$
$$\frac{\partial\tilde{x}_2}{\partial X_1} = f(T, X_1, x_2)\int_0^a \frac{1}{f(T, X_1, x_2)}\frac{\partial f(T, X_1, x_2)}{\partial X_1}da,$$
$$\frac{\partial\tilde{x}_2}{\partial X_2} = \frac{f(T, X_1, x_2)}{f(T, X_1, X_2)}.$$

Therefore, we may write

$$U_1 = u_1\frac{\partial\tilde{x}_2}{\partial X_2}, \quad U_2 = u_2\frac{\partial\tilde{x}_2}{\partial X_2}, \quad U_3 = u_3 - u_1\frac{\partial\tilde{x}_2}{\partial T} - u_2\frac{\partial\tilde{x}_2}{\partial X_1},$$

whence the system (10.41) is mapped to (10.39) with $p_i(t, x_1, x_2, u_1, u_2, u_3)$, $i = 1, \ldots, 6$, given by

$$
\begin{aligned}
p_1 &= P_1 \left(\frac{\partial \tilde{x}_2}{\partial X_2}\right)^{-1},\\
p_2 &= P_2 + P_1 \frac{\partial \tilde{x}_2}{\partial X_1}\left(\frac{\partial \tilde{x}_2}{\partial X_2}\right)^{-1} + \frac{\partial \tilde{x}_2}{\partial T}u_2,\\
p_3 &= P_3 + P_1 \frac{\partial \tilde{x}_2}{\partial X_1}\left(\frac{\partial \tilde{x}_2}{\partial X_2}\right)^{-1} + \frac{\partial \tilde{x}_2}{\partial T}u_2,\\
p_4 &= P_4 \frac{\partial \tilde{x}_2}{\partial X_2} + (P_2 + P_3)\frac{\partial \tilde{x}_2}{\partial X_1} + P_1 \left(\frac{\partial \tilde{x}_2}{\partial X_1}\right)^2 \left(\frac{\partial \tilde{x}_2}{\partial X_2}\right)^{-1} - \left(\frac{\partial \tilde{x}_2}{\partial T}\right)^2 u_2 + 2\frac{\partial \tilde{x}_2}{\partial T}u_3,\\
p_5 &= \left(\partial_{U_3}\left(P_1 \frac{\partial^2 \tilde{x}_2}{\partial T \partial X_1} + P_2 \frac{\partial^2 \tilde{x}_2}{\partial T \partial X_2}\right) - \partial_{U_1}\left(P_1 \frac{\partial^2 \tilde{x}_2}{\partial X_1 \partial X_2} + P_2 \frac{\partial^2 \tilde{x}_2}{\partial X_2^2}\right)\right)\left(\frac{\partial \tilde{x}_2}{\partial X_2}\right)^{-1} u_1,\\
&\quad + \left(\partial_{U_3}\left(P_1 \frac{\partial^2 \tilde{x}_2}{\partial X_1^2} + P_2 \frac{\partial^2 \tilde{x}_2}{\partial X_1 \partial X_2}\right) - \partial_{U_2}\left(P_1 \frac{\partial^2 \tilde{x}_2}{\partial X_1 \partial X_2} + P_2 \frac{\partial^2 \tilde{x}_2}{\partial X_2^2}\right)\right)\left(\frac{\partial \tilde{x}_2}{\partial X_2}\right)^{-1} u_3\\
&\quad - \frac{\partial^2 \tilde{x}_2}{\partial T \partial X_2}\left(\frac{\partial \tilde{x}_2}{\partial X_2}\right)^{-1} u_3\\
p_6 &= \left(\partial_{U_3}\left(P_3 \frac{\partial^2 \tilde{x}_2}{\partial T \partial X_1} + P_4 \frac{\partial^2 \tilde{x}_2}{\partial T \partial X_2}\right) - \partial_{U_1}\left(P_3 \frac{\partial^2 \tilde{x}_2}{\partial X_1 \partial X_2} + P_4 \frac{\partial^2 \tilde{x}_2}{\partial X_2^2}\right) + \frac{\partial^2 \tilde{x}_2}{\partial T^2}\right)u_1\\
&\quad + \left(\partial_{U_3}\left(P_3 \frac{\partial^2 \tilde{x}_2}{\partial X_1^2} + P_4 \frac{\partial^2 \tilde{x}_2}{\partial X_1 \partial X_2}\right) - \partial_{U_2}\left(P_3 \frac{\partial^2 \tilde{x}_2}{\partial X_1 \partial X_2} + P_4 \frac{\partial^2 \tilde{x}_2}{\partial X_2^2}\right) + \frac{\partial^2 \tilde{x}_2}{\partial T \partial X_1}\right)u_2\\
&\quad + p_5 \frac{\partial \tilde{x}_2}{\partial X_1},
\end{aligned}
$$

where $P_k = P_k\left(u_1 \dfrac{\partial \tilde{x}_2}{\partial X_2}, u_2 \dfrac{\partial \tilde{x}_2}{\partial X_2}, u_3 - u_1 \dfrac{\partial \tilde{x}_2}{\partial T} - u_2 \dfrac{\partial \tilde{x}_2}{\partial X_1}\right)$, $k = 1, \ldots 4$.

10.9.3 Equivalence Transformations Generated by Ξ_3

By considering the operator Ξ_3,

$$\Xi_3 = f\frac{\partial}{\partial T} - \frac{\partial f}{\partial T}\left(U_2 \frac{\partial}{\partial U_2} + U_3 \frac{\partial}{\partial U_3}\right),$$

we get from Lie equations the finite transformation

$$
\begin{aligned}
&x_1 = X_1, \qquad x_2 = X_2, \qquad t = \tilde{t}(T; a),\\
&u_1 = U_1, \qquad u_2 = U_2 \frac{f(T)}{f(t)}, \qquad u_3 = U_3 \frac{f(T)}{f(t)},
\end{aligned}
$$

with $\tilde{t}(T;0) = T$, and $\dfrac{\partial \tilde{t}}{\partial T} = \dfrac{f(t)}{f(T)}$, whereupon we obtain

$$U_1 = u_1, \qquad U_2 = u_2 \frac{\partial \tilde{t}}{\partial T}, \qquad U_3 = u_3 \frac{\partial \tilde{t}}{\partial T},$$

and system (10.41) is equivalent to (10.39) with

$$p_1 = P_1 \left(\frac{\partial \tilde{t}}{\partial T}\right)^{-2}, \qquad p_2 = P_2 \left(\frac{\partial \tilde{t}}{\partial T}\right)^{-2}, \qquad p_3 = P_3 \left(\frac{\partial \tilde{t}}{\partial T}\right)^{-2},$$
$$p_4 = P_4 \left(\frac{\partial \tilde{t}}{\partial T}\right)^{-2}, \qquad p_5 = -u_2 \frac{\partial^2 \tilde{t}}{\partial T^2} \left(\frac{\partial \tilde{t}}{\partial T}\right)^{-2}, \qquad p_6 = -u_3 \frac{\partial^2 \tilde{t}}{\partial T^2} \left(\frac{\partial \tilde{t}}{\partial T}\right)^{-2},$$

where $P_k = P_k \left(u_1, u_2 \dfrac{\partial \tilde{t}}{\partial T}, u_3 \dfrac{\partial \tilde{t}}{\partial T}\right), k = 1, \ldots, 4.$

10.9.4 Equivalence Transformations Generated by Ξ_4

By considering the operator Ξ_4,

$$\Xi_4 = f^{(3)} \frac{\partial}{\partial U_1} + f^{(4)} \frac{\partial}{\partial U_2} + f^{(5)} \frac{\partial}{\partial U_3},$$

where the functions $f^{(3)}$, $f^{(4)}$, and $f^{(5)}$ are subjected to the constraint

$$\frac{\partial f^{(3)}}{\partial t} + \frac{\partial f^{(4)}}{\partial x_1} + \frac{\partial f^{(5)}}{\partial x_2} = 0,$$

the finite transformation is

$$t = T, \qquad x_1 = X_1, \qquad x_2 = X_2,$$
$$U_k = u_k - a f^{(k+2)}(T, X_1, X_2), \qquad k = 1, 2, 3,$$

and system (10.41) is equivalent to system (10.39) with

$$p_1 = P_1, \qquad p_2 = P_2, \qquad p_3 = P_3, \qquad p_4 = P_4,$$
$$p_5 = a \left(\frac{\partial}{\partial U_1} \left(P_1 \frac{\partial f^{(3)}}{\partial X_1} + P_2 \frac{\partial f^{(3)}}{\partial X_2} \right) + \frac{\partial}{\partial U_2} \left(P_1 \frac{\partial f^{(4)}}{\partial X_1} + P_2 \frac{\partial f^{(4)}}{\partial X_2} \right) \right.$$
$$\left. + \frac{\partial}{\partial U_3} \left(P_1 \frac{\partial f^{(5)}}{\partial X_1} + P_2 \frac{\partial f^{(5)}}{\partial X_2} \right) \right) + \frac{\partial f^{(4)}}{\partial T},$$

$$p_6 = a\left(\frac{\partial}{\partial U_1}\left(P_3\frac{\partial f^{(3)}}{\partial X_1} + P_4\frac{\partial f^{(3)}}{\partial X_2}\right) + \frac{\partial}{\partial U_2}\left(P_3\frac{\partial f^{(4)}}{\partial X_1} + P_4\frac{\partial f^{(4)}}{\partial X_2}\right) + \frac{\partial}{\partial U_3}\left(P_3\frac{\partial f^{(5)}}{\partial X_1} + P_4\frac{\partial f^{(5)}}{\partial X_2}\right)\right) + \frac{\partial f^{(5)}}{\partial T},$$

where $P_k = P_k(u_1 - af^{(3)}, u_2 - af^{(4)}, u_3 - af^{(5)})$, $k = 1, \ldots, 4$.

10.9.5 Equivalence Transformations Generated by Ξ_5

Using Ξ_5, the situation is very simple: the finite transformation consists of a uniform scaling of the dependent variables,

$$t = T, \quad x_1 = X_1, \quad x_2 = X_2,$$
$$U_1 = \exp(a)u_1, \quad U_2 = \exp(a)u_2, \quad U_3 = \exp(a)u_3.$$

Therefore, if the source system is autonomous, so is the target one; the target system coincides with the source system if the coefficients p_1, p_2, p_3, and p_4 are homogeneous functions of degree zero.

10.10 Applications

The results of previous section can be applied to physically relevant systems of differential equations. This goal is achieved by constructing the finite transformations corresponding to linear combinations of the infinitesimal generators Ξ_1, Ξ_2, and Ξ_3 where some assumptions on the functional dependence of the involved arbitrary functions are made.

10.10.1 First Case

As a first case let us consider the operator $\Xi_1 + \Xi_2 + \Xi_3$, where $f^{(1)}$, $f^{(2)}$ are assumed to be arbitrary functions depending only on T. The integration of Lie equations yields the following finite transformation:

$$t = \tilde{t}(T; a), \quad \text{with} \quad \tilde{t}(T; 0) = T, \qquad \frac{d\tilde{t}}{dT} = \frac{f(t)}{f(T)},$$
$$x_1 = \tilde{x}_1(T, X_1; a) = X_1 + \int_T^t \frac{f^{(1)}(t)}{f(t)}dt,$$

$$x_2 = \tilde{x}_2(T, X_2; a) = X_2 + \int_T^t \frac{f^{(2)}(t)}{f(t)} dt,$$
$$u_1 = U_1, \qquad u_2 = \frac{f(T)}{f(t)} U_2 + \frac{U_1}{f(t)}(f^{(1)}(t) - f^{(1)}(T)),$$
$$u_3 = \frac{f(T)}{f(t)} U_3 + \frac{U_1}{f(t)}(f^{(2)}(t) - f^{(2)}(T)).$$

Because of the relations

$$\frac{\partial \tilde{x}_1}{\partial T} = \frac{f^{(1)}(t) - f^{(1)}(T)}{f(T)}, \qquad \frac{\partial \tilde{x}_2}{\partial T} = \frac{f^{(2)}(t) - f^{(2)}(T)}{f(T)},$$

we may write

$$U_2 = \frac{d\tilde{t}}{dT} u_2 - \frac{\partial \tilde{x}_1}{\partial T} u_1,$$
$$U_3 = \frac{d\tilde{t}}{dT} u_3 - \frac{\partial \tilde{x}_2}{\partial T} u_1.$$

As a consequence, system (10.41) is transformed to (10.39), where

$$p_1 = \left(P_1 - \left(\frac{\partial \tilde{x}_1}{\partial T} \right)^2 u_1 + 2 \frac{\partial \tilde{x}_1}{\partial T} \frac{d\tilde{t}}{dT} u_2 \right) \left(\frac{d\tilde{t}}{dT} \right)^{-2},$$
$$p_2 = \left(P_2 - \frac{\partial \tilde{x}_1}{\partial T} \frac{\partial \tilde{x}_2}{\partial T} u_1 + \frac{\partial \tilde{x}_1}{\partial T} \frac{d\tilde{t}}{dT} u_3 + \left(\frac{d\tilde{t}}{dT} \right)^2 u_2 \right) \left(\frac{d\tilde{t}}{dT} \right)^{-2},$$
$$p_3 = \left(P_3 - \frac{\partial \tilde{x}_1}{\partial T} \frac{\partial \tilde{x}_2}{\partial T} u_1 + \frac{\partial \tilde{x}_1}{\partial T} \frac{d\tilde{t}}{dT} u_3 + \left(\frac{d\tilde{t}}{dT} \right)^2 u_2 \right) \left(\frac{d\tilde{t}}{dT} \right)^{-2},$$
$$p_4 = \left(P_4 - \left(\frac{\partial \tilde{x}_2}{\partial T} \right)^2 u_1 + 2 \frac{\partial \tilde{x}_2}{\partial T} \frac{d\tilde{t}}{dT} u_3 \right) \left(\frac{d\tilde{t}}{dT} \right)^{-2},$$
$$p_5 = \left(u_1 \frac{\partial^2 \tilde{x}_1}{\partial T^2} - u_2 \frac{d^2 \tilde{t}}{dT^2} \right) \left(\frac{d\tilde{t}}{dT} \right)^{-2},$$
$$p_6 = \left(u_1 \frac{\partial^2 \tilde{x}_2}{\partial T^2} - u_3 \frac{d^2 \tilde{t}}{dT^2} \right) \left(\frac{d\tilde{t}}{dT} \right)^{-2},$$

with $P_k = P_k \left(u_1, u_2 \frac{d\tilde{t}}{dT} - u_1 \frac{\partial \tilde{x}_1}{\partial T}, u_3 \frac{d\tilde{t}}{dT} - u_1 \frac{\partial \tilde{x}_2}{\partial T} \right), k = 1, \ldots, 4.$

Example 10.1 (*2D Euler equations*) System (10.41), with the assignments

$$U_1 = \rho, \qquad U_2 = \rho u_1, \qquad U_3 = \rho u_2,$$
$$P_1 = \frac{U_2^2}{U_1} + p(U_1), \qquad P_2 = P_3 = \frac{U_2 U_3}{U_1}, \qquad P_4 = \frac{U_3^2}{U_1} + p(U_1),$$

where ρ is the mass density, (u_1, u_2) is the components of velocity, and $p(\rho)$ is the pressure, represents the two-dimensional Euler model of a perfect barotropic gas. By considering the finite transformation generated by the previous equivalence group, this system is mapped to a nonautonomous and nonhomogeneous system of the form (10.39) with

$$p_1 = \frac{u_2^2}{u_1} + p(u_1)\left(\frac{d\tilde{t}}{dT}\right)^{-2}, \qquad p_2 = p_3 = \frac{u_2 u_3}{u_1}, \qquad p_4 = \frac{u_3^2}{u_1} + p(u_1)\left(\frac{d\tilde{t}}{dT}\right)^{-2},$$
$$p_5 = \left(u_1 \frac{\partial^2 \tilde{x}_1}{\partial T^2} - u_2 \frac{d^2 \tilde{t}}{dT^2}\right)\left(\frac{d\tilde{t}}{dT}\right)^{-2}, \qquad p_6 = \left(u_1 \frac{\partial^2 \tilde{x}_2}{\partial T^2} - u_3 \frac{d^2 \tilde{t}}{dT^2}\right)\left(\frac{d\tilde{t}}{dT}\right)^{-2}.$$

The choice $\dfrac{d\tilde{t}}{dT} = 1$ (i.e., assuming the function f to be a constant that, without loss of generality, can be set equal to 1) produces $t = T + a$. So, $p_k(u_1, u_2, u_3) = P_k(U_1, U_2, U_3)$, $k = 1, \ldots, 4$, and the Euler equations are mapped to

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_2} = 0,\\
&\frac{\partial u_2}{\partial t} + \frac{\partial}{\partial x_1}\left(\frac{u_2^2}{u_1} + p(u_1)\right) + \frac{\partial}{\partial x_2}\left(\frac{u_2 u_3}{u_1}\right) = u_1 \frac{\partial^2 \tilde{x}_1}{\partial T^2},\\
&\frac{\partial u_3}{\partial t} + \frac{\partial}{\partial x_1}\left(\frac{u_2 u_3}{u_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{u_3^2}{u_1} + p(u_1)\right) = u_1 \frac{\partial^2 \tilde{x}_2}{\partial T^2},
\end{aligned} \tag{10.42}$$

i.e., the Euler equations involving source terms (linear in the mass density) in the momentum equations. If x_2 denotes the vertical coordinate, taking $\dfrac{\partial^2 \tilde{x}_1}{\partial T^2} = 0$ and $\dfrac{\partial^2 \tilde{x}_2}{\partial T^2} = g$, where g is the gravitational acceleration, Eq. (10.42) gives the unsteady 2D Euler equations of a barotropic fluid in a vertical plane and subject to the gravity.

10.10.2 Second Case

Now, let us consider again the operator $\Xi_1 + \Xi_2 + \Xi_3$ with

$$f^{(1)} = n_1(T)X_1 + n_2(T)X_2, \qquad f^{(2)} = -n_2(T)X_1 + n_1(T)X_2,$$

where $n_1(T)$ and $n_2(T)$ are arbitrary functions of T.

Integration of Lie equations yields

$$t = \tilde{t}(T; a), \quad \text{with} \quad \tilde{t}(T; 0) = T, \qquad \frac{d\tilde{t}}{dT} = \frac{f(t)}{f(T)},$$

$$x_1 = \tilde{x}_1(T, X_1, X_2; a) == \exp(m_1(T; a))(X_1 \cos(m_2(T; a)) + X_2 \sin(m_2(T; a)),$$

$$x_2 = \tilde{x}_2(T, X_1, X_2; a) = \exp(m_1(T; a))(-X_1 \sin(m_2(T; a)) + X_2 \cos(m_2(T; a)),$$

$$u_1 = \exp(-2m_1(T; a))U_1,$$

$$\begin{aligned} u_2 = {} & \frac{\exp(-m_1(T; a))}{f(t)} (f(T) (U_2 \cos(m_2(T; a)) + U_3 \sin(m_2(T; a))) \\ & + (X_1\tilde{n}_1(T; a) + X_2\tilde{n}_2(T; a)) \cos(m_2(T; a))U_1 \\ & + (X_2\tilde{n}_1(T; a) - X_1\tilde{n}_2(T; a)) \sin(m_2(T; a))U_1), \end{aligned}$$

$$\begin{aligned} u_3 = {} & \frac{\exp(-m_1(T; a))}{f(t)} (f(T) (-U_2 \sin(m_2(T; a)) + U_3 \cos(m_2(T; a))) \\ & + (X_2\tilde{n}_1(T; a) - X_1\tilde{n}_2(T; a)) \cos(m_2(T; a))U_1 \\ & - (X_1\tilde{n}_1(T; a) + X_2\tilde{n}_2(T; a)) \sin(m_2(T; a))U_1), \end{aligned}$$

where

$$m_i(T; a) = \int_T^t \frac{n_i(t)}{f(t)} dt, \qquad \tilde{n}_i(T; a) = n_i(t) - n_i(T), \qquad i = 1, 2.$$

Moreover,

$$\frac{\partial \tilde{x}_1}{\partial T} = \frac{\tilde{n}_1(T; a)x_1 + \tilde{n}_2(T; a)x_2}{f(T)}, \qquad \frac{\partial \tilde{x}_2}{\partial T} = \frac{-\tilde{n}_2(T; a)x_1 + \tilde{n}_1(T; a)x_2}{f(T)},$$

whence

$$
\begin{aligned}
U_1 &= \exp(2m_1(T;a))u_1,\\
U_2 &= \exp(m_1(T;a))\left((u_2\cos(m_2(T;a)) - u_3\sin(m_2(T;a)))\,\frac{d\tilde{t}}{dT}\right.\\
&\quad \left. - u_1\left(\frac{\partial\tilde{x}_1}{\partial T}\cos(m_2(T;a)) - \frac{\partial\tilde{x}_2}{\partial T}\sin(m_2(T;a))\right)\right),\\
U_3 &= \exp(m_1(T;a))\left((u_2\sin(m_2(T;a)) + u_3\cos(m_2(T;a)))\,\frac{d\tilde{t}}{dT}\right.\\
&\quad \left. - u_1\left(\frac{\partial\tilde{x}_1}{\partial T}\sin(m_2(T;a)) + \frac{\partial\tilde{x}_2}{\partial T}\cos(m_2(T;a))\right)\right).
\end{aligned}
$$

We get the following expressions for the functions p_k, $k = 1,\dots 6$:

$$
\begin{aligned}
p_1 &= \left(\left(P_1\cos^2 m_2 + P_4\sin^2 m_2\right) + (P_2+P_3)\sin m_2\cos m_2\right)\left(\frac{d\tilde{t}}{dT}\right)^{-2}\\
&\quad - \left(x_1\frac{dm_2}{dT} + x_2\frac{dm_1}{dT}\right)\left(\frac{d\tilde{t}}{dT}\right)^{-2}\left(\left(x_1\frac{dm_1}{dT} + x_2\frac{dm_2}{dT}\right)u_1 - 2\frac{d\tilde{t}}{dT}u_2\right),\\
p_2 &= ((P_4-P_1)\sin m_2\cos m_2 + (P_2\cos^2 m_2 - P_3\sin^2 m_2))\left(\frac{d\tilde{t}}{dT}\right)^{-2}\\
&\quad + \left(\left(x_1\frac{dm_2}{dT} - x_2\frac{dm_1}{dT}\right)\left(x_1\frac{dm_1}{dT} + x_2\frac{dm_2}{dT}\right)\right)\left(\frac{d\tilde{t}}{dT}\right)^{-2}u_1\\
&\quad - \left(x_1\frac{dm_2}{dT} - x_2\frac{dm_1}{dT}\right)\left(\frac{d\tilde{t}}{dT}\right)^{-1}u_2 + \left(x_1\frac{dm_2}{dT} + x_2\frac{dm_1}{dT}\right)\frac{d\tilde{t}}{dT}u_3,\\
p_3 &= ((P_4-P_1)\sin m_2\cos m_2 - (P_2\sin^2 m_2 - P_3\cos^2 m_2))\left(\frac{d\tilde{t}}{dT}\right)^{-2}\\
&\quad + \left(\left(x_1\frac{dm_2}{dT} - x_2\frac{dm_1}{dT}\right)\left(x_1\frac{dm_1}{dT} + x_2\frac{dm_2}{dT}\right)\right)\left(\frac{d\tilde{t}}{dT}\right)^{-2}u_1\\
&\quad - \left(x_1\frac{dm_2}{dT} - x_2\frac{dm_1}{dT}\right)\left(\frac{d\tilde{t}}{dT}\right)^{-1}u_2 + \left(x_1\frac{dm_2}{dT} + x_2\frac{dm_1}{dT}\right)\left(\frac{d\tilde{t}}{dT}\right)^{-1}u_3,\\
p_4 &= ((P_1\sin^2 m_2 + P_4\cos^2 m_2) - (P_2+P_3)\sin m_2\cos m_2)\left(\frac{\partial\tilde{t}}{\partial T}\right)^{-2}\\
&\quad - \left(x_1\frac{dm_2}{dT} - x_2\frac{dm_1}{dT}\right)\left(\frac{\partial\tilde{t}}{\partial T}\right)^{-2}\left(\left(x_1\frac{dm_2}{dT} - x_2\frac{dm_1}{dT}\right)u_1 + 2\frac{d\tilde{t}}{dT}u_3\right),
\end{aligned}
$$

$$
\begin{aligned}
p_5 = {} & \left(\frac{dm_2}{dT}u_3 - \frac{dm_1}{dT}u_2\right)\left(\frac{d\tilde{t}}{dT}\right)^{-1} - \frac{d^2\tilde{t}}{dT^2}\left(\frac{d\tilde{t}}{dT}\right)^{-2}u_2 \\
& + \left(x_1\left(\frac{d^2m_1}{dT^2} + 2\left(\frac{dm_1}{dT}\right)^2\right) + x_2\left(\frac{d^2m_1}{dT^2} + 2\frac{dm_1}{dT}\frac{dm_2}{dT}\right)\right)\left(\frac{d\tilde{t}}{dT}\right)^{-2}u_1 \\
& + \exp(m_1)\left(\frac{d\tilde{t}}{dT}\right)^{-2}\partial_{U_3}\left(\frac{dm_1}{dT}(P_2\cos m_2 + P_4\sin m_2)\right. \\
& \left. - \frac{dm_2}{dT}(P_1\cos m_2 + P_3\sin m_2)\right)u_1 \\
& + \exp(m_1)\left(\frac{d\tilde{t}}{dT}\right)^{-2}\partial_{U_2}\left(\frac{dm_2}{dT}(P_1\cos m_2 + P_3\sin m_2)\right. \\
& \left. + \frac{dm_1}{dT}(P_2\cos m_2 + P_4\sin m_2)\right)u_1, \\
p_6 = {} & \left(-\frac{dm_2}{dT}u_2 - \frac{dm_1}{dT}u_3\right)\left(\frac{d\tilde{t}}{dT}\right)^{-1} - \frac{d^2\tilde{t}}{dT^2}\left(\frac{d\tilde{t}}{dT}\right)^{-2}u_3 \\
& - \left(x_1\left(\frac{d^2m_2}{dT^2} + 2\frac{dm_1}{dT}\frac{dm_2}{dT}\right) + x_2\left(\frac{d^2m_1}{dT^2} + 2\left(\frac{dm_1}{dT}\right)^2\right)\right)\left(\frac{d\tilde{t}}{dT}\right)^{-2}u_1 \\
& - \exp(m_1)\left(\frac{d\tilde{t}}{dT}\right)^{-2}\partial_{U_3}\left(\frac{dm_1}{dT}(P_2\sin m_2 - P_4\cos m_2)\right. \\
& \left. - \frac{dm_2}{dT}(P_1\sin m_2 - P_3\cos m_2)\right)u_1 \\
& - \exp(m_1)\left(\frac{d\tilde{t}}{dT}\right)^{-2}\partial_{U_2}\left(\frac{dm_1}{dT}(P_1\sin m_2 - P_3\cos m_2)\right. \\
& \left. + \frac{dm_2}{dT}(P_2\sin m_2 - P_4\cos m_2)\right)u_1,
\end{aligned}
$$

where $P_i = P_i(U_1, U_2, U_3)$ $(i = 1, \ldots, 4)$.

A physical application of the results in this subsection concerns the shallow water equations both in the case of bottom flat topography and bottom parabolic topography; the transformations linking the nonrotating equations to rotating equations (where Coriolis forces are taken into account) are derived.

Example 10.2 (*2D Shallow Water Equations*) Consider the 2D shallow water equations [55] of an inviscid fluid with constant density flowing over a flat bottom topography; by neglecting the Coriolis forces, along with the assignments

$$
\begin{aligned}
& U_1 = h, \qquad U_2 = hu_1, \qquad U_3 = hu_2, \\
& P_1 = \frac{U_2^2}{U_1} + g\frac{U_1^2}{2}, \quad P_2 = P_3 = \frac{U_2U_3}{U_1}, \quad P_4 = \frac{U_3^2}{U_1} + g\frac{U_1^2}{2},
\end{aligned}
$$

u_1 and u_2 being the velocity components in the plane (X_1, X_2), h is the height of the fluid layer, and g is the gravitational constant, the governing equations for such a model have the form (10.41). In such a case the finite transformation we obtained in the second case produces a system like (10.39) with

$$
\begin{aligned}
p_1 &= \frac{u_2^2}{u_1} + \exp(4m_1)\left(\frac{d\tilde{t}}{dT}\right)^{-2} g\frac{u_1^2}{2}\\
p_2 &= p_3 = \frac{u_2 u_3}{u_1},\\
p_4 &= \frac{u_3^2}{u_1} + \exp(4m_1)\left(\frac{d\tilde{t}}{dT}\right)^{-2} g\frac{u_1^2}{2},\\
p_5 &= 2\left(\frac{dm_2}{dT}u_3 - \frac{dm_1}{dT}u_2\right)\left(\frac{d\tilde{t}}{dT}\right)^{-1} + \frac{d^2\tilde{t}}{dT^2}\left(\frac{d\tilde{t}}{dT}\right)^{-2} u_2\\
&\quad + \left(x_1\left(\frac{d^2m_1}{dT^2} - \left(\frac{dm_1}{dT}\right)^2 + \left(\frac{dm_2}{dT}\right)^2\right) + x_2\left(\frac{d^2m_2}{dT^2} - 2\frac{dm_1}{dT}\frac{dm_2}{dT}\right)\right)\left(\frac{d\tilde{t}}{dT}\right)^{-2} u_1,\\
p_6 &= -2\left(\frac{dm_2}{dT}u_2 - \frac{dm_1}{dT}u_3\right)\left(\frac{d\tilde{t}}{dT}\right)^{-1} - \frac{d^2\tilde{t}}{dT^2}\left(\frac{d\tilde{t}}{dT}\right)^{-2} u_3\\
&\quad + \left(x_1\left(2\frac{dm_1}{dT}\frac{dm_2}{dT} - \frac{d^2m_2}{dT^2}\right) + x_2\left(\frac{d^2m_1}{dT^2} + \left(\frac{dm_2}{dT}\right)^2 - \left(\frac{dm_1}{dT}\right)^2\right)\right)\left(\frac{d\tilde{t}}{dT}\right)^{-2} u_1.
\end{aligned}
$$

If

$$\frac{d\tilde{t}}{dT} = \exp(2m_1), \tag{10.43}$$

we have $p_k(u_1, u_2, u_3) = P_k(u_1, u_2, u_3), k = 1\ldots, 4$, and the source terms become

$$
\begin{aligned}
p_5 &= 2\exp(-2m_1)\frac{dm_2}{dT}u_3\\
&\quad + \exp(-4m_1)\left(x_1\left(\frac{d^2m_1}{dT^2} + \left(\frac{dm_2}{dT}\right)^2 - \left(\frac{dm_1}{dT}\right)^2\right) + x_2\left(\frac{d^2m_2}{dT^2} - 2\frac{dm_1}{dT}\frac{dm_2}{dT}\right)\right)u_1,\\
p_6 &= -2\exp(2m_1)\frac{dm_2}{dT}u_2\\
&\quad - \exp(-4m_1)\left(x_1\left(\frac{d^2m_2}{dT^2} - 2\frac{dm_1}{dT}\frac{dm_2}{dT}\right) - x_2\left(\frac{d^2m_1}{dT^2} + \left(\frac{dm_2}{dT}\right)^2 - \left(\frac{dm_1}{dT}\right)^2\right)\right)u_1.
\end{aligned}
$$

Now, let us assume

$$\frac{d^2m_2}{dT^2} - 2\frac{dm_1}{dT}\frac{dm_2}{dT} = 0, \tag{10.44a}$$

$$\exp(-4m_1)\left(\left(\frac{dm_2}{dT}\right)^2 + \frac{d^2m_1}{dT^2} - \left(\frac{dm_1}{dT}\right)^2\right) = k_1(a), \tag{10.44b}$$

where $k_1(a)$ is a function of a.

Solving the constraints (10.43) and (10.44), one obtains

$$m_1(T;a) = \frac{1}{2}\log\left|\frac{k_3(a)}{(k_2^2(a)-k_1(a))+k_3^2(a)(T+k_4(a))^2}\right|,$$
$$m_2(T;a) = \frac{k_2(a)}{\sqrt{k_2^2(a)-k_1(a)}}\arctan\frac{k_3(a)(T+k_4(a))}{\sqrt{k_2^2(a)-k_1(a)}}+k_5(a),$$
$$\tilde{t}(T;a) = \frac{1}{\sqrt{k_2^2(a)-k_1(a)}}\arctan\frac{k_3(a)(T+k_4(a))}{\sqrt{k_2^2(a)-k_1(a)}}+k_6(a),$$

where $k_i(a), i = 1,\dots,6$, are arbitrary functions of parameter a. Taking into account that $m_1(T;0) = m_2(T;0) = 0$ and $\tilde{t}(T;0) = T$, we may choose

$$k_1(a) = k_2^2(a) - a^2, \quad k_3(a) = a^2, \quad k_4(a) = k_5(a) = k_6(a) = 0,$$

whereupon

$$\tilde{t}(T;a) = \frac{1}{a}\arctan(aT),$$
$$m_1(T;a) = -\frac{1}{2}\log(1+a^2T^2),$$
$$m_2(T;a) = \frac{k_2(a)}{a}\arctan(aT),$$

where $k_2(a)$ has to be such that $\lim_{a\to 0} k_2(a) = 0$.

Then, the source terms in the transformed system become

$$p_5 = 2k_2(a)u_3 + (k_2^2(a)-a^2)u_1x_1, \qquad p_6 = -2k_2(a)u_2 + (k_2^2(a)-a^2)u_1x_2.$$

By choosing the value $\overline{a}$ of the parameter a in such a way $k_2(\overline{a}) = \omega$, where ω is the constant angular velocity of the fluid around the vertical axis responsible for the Coriolis force, and setting $\kappa = \omega^2 - \overline{a}^2$, we recover the source terms involved in the shallow water equations for a fluid flowing over a bottom parabolic topography described by the depth function

$$H(x_1,x_2) = h_0 + \frac{\kappa}{2}(x_1^2+x_2^2),$$

with h_0 arbitrary constant, where Coriolis forces are incorporated.

Therefore, the shallow water equations over a parabolic bottom topography and incorporating Coriolis forces are linked by the transformation

$$
\begin{aligned}
t =& \frac{1}{a}\arctan(aT),\\
x_1 =& \frac{X_1 + k_2(a)\tau X_2}{\sqrt{(1+a^2T^2)(1+k_2^2(a)\tau^2)}}, \quad x_2 = \frac{-k_2(a)\tau X_1 + X_2}{\sqrt{(1+a^2T^2)(1+k_2^2(a)\tau^2)}},\\
u_1 =& (1+a^2T^2)U_1,\\
u_2 =& \sqrt{\frac{1+a^2\,T^2}{1+k_2^2(a)\tau^2}}\\
&\times\Big((1+a^2\,T^2)(U_2+k_2(a)\tau U_3) - ((a^2T+k_2^2(a)\tau)X_1 - k_2(a)(1-a^2\,T\tau)X_2)U_1\Big),\\
u_3 =& \sqrt{\frac{1+a^2\,T^2}{1+k_2^2(a)\tau^2}}\\
&\times\Big((1+a^2\,T^2)(U_3-k_2(a)\tau U_2) - ((a^2T+k_2^2(a)\tau)X_2 + k_2(a)(1-a^2\,T\tau)X_1)U_1\Big),
\end{aligned}
$$

where $\tau = \dfrac{\tan(k_2(a)t)}{k_2(a)}$, to the equations of shallow water equations for a fluid flowing over a bottom flat topography where Coriolis forces are neglected.

In particular, by choosing $k_2(a) = a$, for $a = \omega$, where ω is the Coriolis parameter, we have the transformation

$$
\begin{aligned}
t =& \frac{1}{a}\arctan(aT), \quad x_1 = \frac{X_1 + aTX_2}{1+a^2T^2}, \quad x_2 = \frac{-aTX_1 + X_2}{1+a^2T^2},\\
u_1 =& (1+a^2T^2)U_1,\\
u_2 =& (1+a^2T^2)(U_2+aTU_3) - a(2aTX_1 - (1-a^2\,T^2)X_2))U_1,\\
u_3 =& (1+a^2\,T^2)(U_3-aTU_2) - a(2aTX_2 + (1-a^2\,T^2)X_1)U_1,
\end{aligned}
$$

linking the nonrotating shallow water equations of a fluid flowing over a bottom flat topography with the rotating shallow water equations over a bottom flat topography.

Another solution to the constraints (10.43) and (10.44) is

$$
\begin{aligned}
&m_1(T;a) = k_2(a), \qquad m_2(T;a) = \sqrt{k_1(a)}\exp(2k_2(a))T + k_3(a),\\
&\tilde{t}(T;a) = \exp(k_2(a))T + k_4(a),
\end{aligned}
$$

where the functions $k_i(a)$ $(i = 1,\dots,4)$ are such that

$$
k_1(a) \geq 0, \qquad \lim_{a\to 0} k_i(a) = 0, \quad i = 1,\dots,4.
$$

Without loss of generality, we may choose

$$
k_1(a) = k_3(a) = k_4(a) = 0, \qquad k_2(a) = a^2,
$$

whereupon the transformation linking the nonrotating shallow water equations with bottom flat topography with the rotating shallow water equations with bottom parabolic topography becomes

$$\begin{aligned} t &= T, \\ x_1 &= X_1 \cos(aT) + X_2 \sin(aT), \\ x_2 &= -X_1 \sin(aT) + X_2 \cos(aT), \\ U_1 &= u_1, \\ U_2 &= u_2 \cos(aT) + u_3 \sin(aT) + a(X_2 \cos(aT) - X_1 \sin(aT))U_1, \\ U_3 &= u_3 \cos(aT) - u_2 \sin(aT) - a(X_2 \sin(aT) + X_1 \cos(aT))U_1. \end{aligned}$$

The source terms specialize to

$$\begin{aligned} p_5 &= 2au_3 + a^2 x_1 u_1, \\ p_6 &= -2au_2 + a^2 x_2 u_1; \end{aligned}$$

taking $a = \omega$, where ω is the angular velocity, the fluid around the vertical axis responsible for the Coriolis force, the bottom topography is given by

$$H = h_0 + \frac{\omega^2}{2}(x_1^2 + x_2^2),$$

with h_0 constant.

Remark 10.3 The approach described in Sect. 10.9 has been investigated in [47] for a class of quasilinear first-order systems, involving three arbitrary elements, in $(1 + 1)$ dimensions, and in [27] for a class of quasilinear first-order systems, involving 16 arbitrary elements, in $(3 + 1)$ dimensions.

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Chapter 11
Approximate Symmetries



11.1 Introduction

Lie symmetry analysis yields a general algorithmic approach to the investigation of differential equations that, besides its intrinsic theoretical interest (e.g., for classifying differential equations according to the admitted symmetries), plays a fundamental role from a geometrical or analytical viewpoint. In fact, the exploitation of symmetries of differential equations often leads to their simplification.

Unfortunately, any small perturbation in the structure of a differential equation has the—often dramatic—effect of destroying many useful symmetries, and this restricts the applicability of Lie group methods to concrete problems where equations involving terms of different order of magnitude occur. To overcome this inconvenient, some *approximate symmetry theories* have been proposed in order to deal with differential equations involving small terms, and the notion of *approximate invariance* has been introduced.

The first, illuminating, paper was by Baikov, Gazizov and Ibragimov [1] (see also [23, 26]), who in 1988 proposed to expand in a perturbation series the Lie generator in order to have an approximate generator. The constructed theory is quite elegant, and all the useful properties of *exact* Lie symmetries are moved to the *approximate world*: reduction of order of ordinary differential equations, approximate invariant solutions, approximate conservation laws, etc. Since its introduction, this approach has been applied in many physical situations [3, 7, 10, 15–17, 24, 27, 28, 33, 36, 37]. Nevertheless, the expanded generator is not consistent with the principles of perturbative analysis [30], essentially because the dependent variables are not expanded. This implies that in some examples the approximate invariant solutions that are found with this method are not the most general ones.

In 1989, Fushchych and Shtelen [14] proposed a different approach. The dependent variables are expanded in a series as done in usual perturbative analysis; terms are then separated at each order of approximation, and a system of equations to be solved in a hierarchy is obtained. Moreover, this system is assumed to be coupled,

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and the approximate symmetries of the original equation are defined as the *exact symmetries* of the equations obtained from perturbation. This approach has an obvious, simple, and coherent basis. Per contra, a lot of algebra (especially for higher order perturbations) is needed; moreover, the basic assumption of a fully coupled system is too strong and not easily justifiable, for the equations at a level should not be influenced by those at higher levels. In addition, there is no possibility to work in a hierarchy: this means that if one computes first-order approximate symmetries and then wants second-order approximate symmetries, all the work must be done from the very beginning. Applications of this method to various equations can be found, for instance, in [9, 11–13, 37].

In 2004, the two different approaches have been compared [33], and a third method as a variation of Fushchych–Shtelen one by removing the assumption of a fully coupled system has been proposed. Anyway, this method applies only to differential equations having the form

$$\mathcal{L}(\mathbf{u}) + \varepsilon \mathcal{N}(\mathbf{u}) = 0, \tag{11.1}$$

where $\mathcal{L}(\mathbf{u})$ and $\mathcal{N}(\mathbf{u})$ are linear and nonlinear differential operators, respectively. Expanding the dependent variables,

$$\mathbf{u} = \mathbf{u}_{(0)} + \varepsilon \mathbf{u}_{(1)} + \varepsilon^2 \mathbf{u}_{(2)} + \cdots,$$

Equation (11.1) gives

$$\begin{aligned} &\mathcal{L}(\mathbf{u}_{(0)}) = 0 = h_0(\mathbf{x}), \\ &\mathcal{L}(\mathbf{u}_{(1)}) = -\mathcal{N}(\mathbf{u}_{(0)}) = h_1(\mathbf{x}), \\ &\mathcal{L}(\mathbf{u}_{(2)}) = -\mathcal{N}(\mathbf{u}_{(0)}, \mathbf{u}_{(1)}) = h_2(\mathbf{x}). \\ &\cdots \end{aligned} \tag{11.2}$$

Due to the circumstance that the differential operators in the left-hand sides of all the equations are the same, and the right-hand sides can be considered as arbitrary functions of the independent variables which are determined sequentially starting from the first equation, the approximate symmetries of Eq. (11.1) are defined as the exact symmetries of the linear nonhomogeneous equation

$$\mathcal{L}(\mathbf{u}) = h(\mathbf{x}), \tag{11.3}$$

with $h(\mathbf{x})$ considered as an arbitrary function. Actually, the functions $h_i(\mathbf{x})$ in (11.2) are known functions, so that, substituting these forms to the general symmetry of Eq. (11.3), one obtains the symmetries at each level of approximation. The algebra involved is much less than that required by Fushchich–Shtelen method, and it is possible to work in a hierarchy; nevertheless, the method is not general.

Another variant of Fushchich–Shtelen method, proposed in [35], has been applied to the third-order partial differential equation

$$\frac{\partial^2 w}{\partial t^2} - f\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2} = \varepsilon \frac{\partial^3 w}{\partial t \partial x^2}, \tag{11.4}$$

where $f\left(\dfrac{\partial w}{\partial x}\right)$ is an arbitrary function, describing one-dimensional wave propagation in nonlinear dissipative media. Inserting the expansion of the dependent variable in power series of ε,

$$w(t, x, \varepsilon) = w_{(0)}(t, x) + \varepsilon w_{(1)}(t, x) + O(\varepsilon^2),$$

into (11.4), and separating terms with different powers in ε, an approximate system is found. Then, the approximate symmetries of (11.4) are defined as the (exact) symmetries of the approximate system through the Lie group generated by

$$\begin{aligned}\Xi = {} & \xi_0^1(t, x, w_{(0)})\frac{\partial}{\partial t} + \xi_0^2(t, x, w_{(0)})\frac{\partial}{\partial x} + \eta_0(t, x, w_{(0)})\frac{\partial}{\partial w_{(0)}} \\ & + \left(\eta_{10}(t, x, w_{(0)}) + \eta_{11}(t, x, w_{(0)})w_{(1)}\right)\frac{\partial}{\partial w_{(1)}}.\end{aligned}$$

This approach allows to work in a hierarchy; per contra, there is not a full mix between Lie group methods and perturbative analysis (as in Fushchich–Shtelen method).

In the remainder of this chapter, we present an approximate symmetry theory which is consistent with perturbative analysis and such that the relevant properties of exact Lie symmetries of differential equations are inherited [8]. More precisely, we expand the dependent variables in power series of the small parameter as done in classical perturbative analysis; then, instead of considering the approximate symmetries as the exact symmetries of the approximate system (as done in Fushchich–Shtelen method), we construct the consequent expansion of the Lie operator, as in Baikov–Gazizov–Ibragimov method.

11.2 Approximate One-Parameter Lie Groups

We start by introducing the necessary notation in order to define approximate one-parameter Lie groups of transformations.

Let $f(\mathbf{z}, \varepsilon)$ be a C^∞ function depending on $\mathbf{z} \equiv (z_1, \ldots, z_n) \in \mathbb{D} \subseteq \mathbb{R}^n$ and the small parameter $\varepsilon \in \mathbb{R}$; in the following, we will consider such kind of functions locally in a neighborhood of $\varepsilon = 0$.

Let us expand $\mathbf{z}$ in power series of ε,

$$\mathbf{z} = \mathbf{z}_{(0)} + \varepsilon \mathbf{z}_{(1)} + \cdots + \varepsilon^p \mathbf{z}_{(p)} + O(\varepsilon^{p+1}),$$

with $\mathbf{z}_{(k)} \equiv (z_{(k)1}, \ldots, z_{(k)n})$, whereupon

$$f(\mathbf{z}, \varepsilon) = \sum_{k=0}^{p} \sum_{|\sigma|=k} \frac{\varepsilon^{\sigma_0}}{\sigma_0!} \left(\prod_{i=1}^{n} \frac{(z_i - z_{(0)i})^{\sigma_i}}{\sigma_i!} \right) \frac{\partial^{|\sigma|} f(\mathbf{z}, \varepsilon)}{\partial \varepsilon^{\sigma_0} \partial z_1^{\sigma_1} \cdots \partial z_n^{\sigma_n}} \bigg|_{\varepsilon=0} + O(\varepsilon^{p+1}),$$

with σ the multi-index $(\sigma_0, \sigma_1, \ldots, \sigma_n)$, and $|\sigma| = \sigma_0 + \sigma_1 + \cdots + \sigma_n$.

By means of the assignments

$$\begin{aligned} f_{(0)}(\mathbf{z}_{(0)}) &= f(\mathbf{z}, \varepsilon)|_{\varepsilon=0}\,, \\ f_{(k)}(\mathbf{z}_{(0)}) &= \frac{\partial^k f(\mathbf{z}, \varepsilon)}{\partial \varepsilon^k} \bigg|_{\varepsilon=0}\,, \end{aligned}$$

for it is

$$\frac{\partial^{|\sigma|} f(\mathbf{z}, \varepsilon)}{\partial \varepsilon^{\sigma_0} \partial z_1^{\sigma_1} \cdots \partial z_n^{\sigma_n}} \bigg|_{\varepsilon=0} = \frac{\partial^{|\sigma|-\sigma_0} f_{(\sigma_0)}(\mathbf{z}_{(0)})}{\partial z_{(0)1}^{\sigma_1} \cdots \partial z_{(0)n}^{\sigma_n}},$$

we have

$$f(\mathbf{z}, \varepsilon) = \sum_{k=0}^{p} \sum_{|\sigma|=k} \frac{\varepsilon^{\sigma_0}}{\sigma_0!} \left(\prod_{i=1}^{n} \frac{(z_i - z_{(0)i})^{\sigma_i}}{\sigma_i!} \right) \frac{\partial^{|\sigma|-\sigma_0} f_{(\sigma_0)}(\mathbf{z}_{(0)})}{\partial z_{(0)1}^{\sigma_1} \cdots \partial z_{(0)n}^{\sigma_n}} + O(\varepsilon^{p+1}).$$

So, the expansion in power series of ε is characterized (up to the order p in ε) by $p+1$ functions of $\mathbf{z}_{(0)}$. Such an expansion can be written as

$$f(\mathbf{z}, \varepsilon) = \sum_{k=0}^{p} \varepsilon^k \widetilde{f}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}) + O(\varepsilon^{p+1}),$$

where $\widetilde{f}_{(k)}$ $(k > 0)$ are suitable polynomials in $\mathbf{z}_{(1)}, \ldots, \mathbf{z}_{(k)}$ with coefficients given by $f_{(0)}(\mathbf{z}_{(0)}), \ldots, f_{(k)}(\mathbf{z}_{(0)})$ and their derivatives with respect to $\mathbf{z}_{(0)}$. More precisely, the functions $\widetilde{f}_{(k)}$ are defined as follows:

$$\begin{aligned} \widetilde{f}_{(0)} &= f_{(0)}, \\ \widetilde{f}_{(k+1)} &= \frac{1}{k+1} \mathcal{R}[\widetilde{f}_{(k)}], \end{aligned}$$

$\mathcal{R}$ being a *linear* recursion operator satisfying *product rule* of derivatives and such that

$$\begin{aligned} \mathcal{R}\left[\frac{\partial^{|\tau|} f_{(k)}(\mathbf{z}_{(0)})}{\partial z_{(0)1}^{\tau_1} \ldots \partial z_{(0)n}^{\tau_n}} \right] &= \frac{\partial^{|\tau|} f_{(k+1)}(\mathbf{z}_{(0)})}{\partial z_{(0)1}^{\tau_1} \ldots \partial z_{(0)n}^{\tau_n}} \\ &\quad + \sum_{i=1}^{n} \frac{\partial}{\partial z_{(0)i}} \left(\frac{\partial^{|\tau|} f_{(k)}(\mathbf{z}_{(0)})}{\partial z_{(0)1}^{\tau_1} \ldots \partial z_{(0)n}^{\tau_n}} \right) z_{(1)i}, \qquad (11.5a) \\ \mathcal{R}[z_{(k)j}] &= (k+1) z_{(k+1)j}, \qquad (11.5b) \end{aligned}$$

where $k \geq 0$, $j = 1, \ldots, n$, $|\tau| = \tau_1 + \cdots + \tau_n$.

Definition 11.1 Given two smooth functions $f(\mathbf{z}, \varepsilon)$ and $g(\mathbf{z}, \varepsilon)$, and fixing a positive integer p, we will write

$$f(\mathbf{z}, \varepsilon) \overset{p}{\approx} g(\mathbf{z}, \varepsilon)$$

when it is

$$f(\mathbf{z}, \varepsilon) = g(\mathbf{z}, \varepsilon) + O(\varepsilon^{p+1}).$$

This means that $g(\mathbf{z}, \varepsilon)$ has the same Taylor expansion as $f(\mathbf{z}, \varepsilon)$ up to the order p in ε. Hereafter, we shall use the simplest notation $\approx$ instead of $\overset{p}{\approx}$.

Definition 11.2 (*Approximate one-parameter Lie groups*) Let $\mathbb{D} \subseteq \mathbb{R}^n$ be a domain, $E \subseteq \mathbb{R}$ an interval containing zero, $S \subseteq \mathbb{R}$ another interval (bounded or unbounded) containing zero such that $\forall\, a, b \in S,\ a + b \in S$. The set of transformations

$$\begin{aligned} &\mathbf{Z}\colon D \times E \times S \to D, \\ &(\mathbf{z}, \varepsilon, a) \mapsto \mathbf{z}' = \mathbf{Z}(\mathbf{z}, \varepsilon; a), \end{aligned}$$

depending on the parameter a, where $\varepsilon \in E$ is *small*, is a one-parameter (a) Lie group of transformations if

1. $\mathbf{Z}(\mathbf{Z}(\mathbf{z}, \varepsilon; a), \varepsilon; b) = \mathbf{Z}(\mathbf{z}, \varepsilon; a + b)$;
2. $\mathbf{Z}(\mathbf{z}, \varepsilon; 0) = \mathbf{z}$;
3. for each value of the parameter $a \in S$ and each $\varepsilon \in E$ the transformations are one-to-one onto D;
4. $\mathbf{Z}$ is C^∞ with respect to $\mathbf{z} \in D$ and $\varepsilon \in E$, and analytic with respect to the parameter a.

Let us fix a positive integer p, and expand $\mathbf{z}$ in power series of ε,

$$\mathbf{z} = \mathbf{z}_{(0)} + \varepsilon \mathbf{z}_{(1)} + \cdots + \varepsilon^p \mathbf{z}_{(p)} + O(\varepsilon^{p+1});$$

by defining

$$\begin{aligned} \mathbf{Z}_{(0)}(\mathbf{z}_{(0)}; a) &= \mathbf{Z}(\mathbf{z}, \varepsilon; a)|_{\varepsilon=0}\,, \\ \mathbf{Z}_{(k)}(\mathbf{z}_{(0)}; a) &= \left.\frac{\partial^k \mathbf{Z}(\mathbf{z}, \varepsilon; a)}{\partial \varepsilon^k}\right|_{\varepsilon=0}, \end{aligned}$$

we can write

$$\mathbf{Z}(\mathbf{z}, \varepsilon; a) = \sum_{k=0}^{p} \varepsilon^k \widetilde{\mathbf{Z}}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}; a) + O(\varepsilon^{p+1}),$$

along with

$$\widetilde{\mathbf{Z}}_{(0)} = \mathbf{Z}_{(0)},$$
$$\widetilde{\mathbf{Z}}_{(k+1)} = \frac{1}{k+1}\mathcal{R}[\widetilde{\mathbf{Z}}_{(k)}], \qquad k \geq 0,$$

where $\mathcal{R}$ is the recursion operator defined in (11.5).

The set of transformations

$$\mathbf{z}' = \mathbf{Z}(\mathbf{z}, \varepsilon; a) \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\mathbf{Z}}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}; a)$$

is an approximate one-parameter Lie group with respect to the parameter a if

$$\mathbf{Z}(\mathbf{z}, \varepsilon; 0) \approx \mathbf{z}, \qquad \mathbf{Z}(\mathbf{Z}(\mathbf{z}, \varepsilon; a), \varepsilon; b) \approx \mathbf{Z}(\mathbf{z}, \varepsilon; a + b),$$

and the condition $\mathbf{Z}(\mathbf{z}, \varepsilon; a) \approx \mathbf{z}$ for all $\mathbf{z}$ implies $a = 0$.

Hereafter, we show how all the relevant features of classical Lie groups of transformations can be extended in the approximate sense.

Definition 11.3 (*Generator of an approximate Lie group*) The generator of the Lie group of transformations

$$\mathbf{z}' = \mathbf{Z}(\mathbf{z}, \varepsilon; a) \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\mathbf{Z}}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}; a)$$

is given by

$$\Xi = \sum_{i=1}^{n} \zeta_i(\mathbf{z}, \varepsilon) \frac{\partial}{\partial z_i},$$

where

$$\zeta_i(\mathbf{z}, \varepsilon) = \left. \frac{\partial Z_i(\mathbf{z}, \varepsilon; a)}{\partial a} \right|_{a=0}.$$

Inserting the expansion of $\mathbf{z}$, we have

$$\zeta_i(\mathbf{z}, \varepsilon) \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\zeta}_{(k)i}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}),$$

where

$$\widetilde{\zeta}_{(k)i}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}) = \left. \frac{\partial \widetilde{Z}_{(k)i}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}; a)}{\partial a} \right|_{a=0}.$$

Setting

$$\zeta_{(0)i}(\mathbf{z}_{(0)}) = \zeta_i(\mathbf{z}, \varepsilon)|_{\varepsilon=0},$$
$$\zeta_{(k)i}(\mathbf{z}_{(0)}) = \left.\frac{\partial^k \zeta_i(\mathbf{z}, \varepsilon)}{\partial \varepsilon^k}\right|_{\varepsilon=0},$$

it is

$$\widetilde{\zeta}_{(0)i} = \zeta_{(0)i},$$
$$\widetilde{\zeta}_{(k+1)i} = \frac{1}{k+1}\mathcal{R}[\widetilde{\zeta}_{(k)i}],$$

where $\mathcal{R}$ is the recursion operator defined in (11.5). Therefore, we may write

$$\Xi \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\Xi}_{(k)},$$

with obvious meaning of terms, that represents the approximate Lie generator.

Definition 11.4 (*pproximate Lie equations*) Let

$$\Xi \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\Xi}_{(k)}$$

be an approximate generator; the corresponding finite transformation

$$\mathbf{z}' \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\mathbf{Z}}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}; a)$$

is recovered by solving the approximate Lie equations

$$\frac{d\mathbf{z}'_{(0)}}{da} = \widetilde{\boldsymbol{\zeta}}_{(0)}(\mathbf{z}'_{(0)}), \quad \mathbf{z}'_{(0)}(0) = \mathbf{z}_{(0)},$$
$$\frac{d\mathbf{z}'_{(k)}}{da} = \widetilde{\boldsymbol{\zeta}}_{(k)}(\mathbf{z}'_{(0)}, \ldots, \mathbf{z}'_{(k)}), \quad \mathbf{z}'_{(k)}(0) = \mathbf{z}_{(k)}, \qquad k = 1, \ldots, p.$$

Example 11.1 If $p = 1$, the generator of the approximate Lie group of transformations reads

$$\Xi \approx \sum_{i=1}^{n} \left(\zeta_{(0)i}(\mathbf{z}_{(0)}) + \varepsilon \left(\zeta_{(1)i}(\mathbf{z}_{(0)}) + \sum_{j=1}^{n} \frac{\partial \zeta_{(0)i}(\mathbf{z}_{(0)})}{\partial z_{(0)j}} z_{(1)j} \right) \right) \frac{\partial}{\partial z_i}.$$

Definition 11.5 (*Invariance of a function)* A smooth function

$$F(\mathbf{z}, \varepsilon) \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{F}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)})$$

is approximately invariant with respect to

$$\mathbf{z}' \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\mathbf{Z}}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}; a) \tag{11.6}$$

if

$$F(\mathbf{z}', \varepsilon) \approx F(\mathbf{z}, \varepsilon).$$

By using the same arguments as those in classical Lie theory, the following theorem about the approximate invariance of a function can be stated.

Theorem 11.1 *Let*

$$\Xi \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\Xi}_{(k)}$$

be an approximate Lie generator of (11.6). Then, the function

$$F(\mathbf{z}, \varepsilon) \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{F}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)})$$

is approximately invariant with respect to (11.6) if and only if

$$\Xi(F(\mathbf{z}, \varepsilon)) \approx 0. \tag{11.7}$$

Some details for deriving the constraints arising from condition (11.7) are worth of being explicitly given. It is:

$$\begin{aligned}
\Xi(F(\mathbf{z}, \varepsilon)) &\approx \sum_{j=0}^{p} \varepsilon^j \left(\sum_{i=1}^{n} \widetilde{\zeta}_{(j)i}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(j)}) \frac{\partial F(\mathbf{z}, \varepsilon)}{\partial z_i} \right) \\
&= \sum_{j,k=0}^{p} \varepsilon^{j+k} \left(\sum_{i=1}^{n} \widetilde{\zeta}_{(j)i}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(j)}) \frac{\partial \widetilde{F}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)})}{\partial z_{(0)i}} \right) \\
&= \sum_{k=0}^{p} \varepsilon^k \left(\sum_{j=0}^{k} \sum_{i=1}^{n} \widetilde{\zeta}_{(j)i}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(j)}) \frac{\partial \widetilde{F}_{(k-j)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k-j)})}{\partial z_{(0)i}} \right).
\end{aligned} \tag{11.8}$$

From (11.7), taking into account (11.8), and separating the various powers in ε, we easily obtain the conditions

$$\begin{aligned}
&\sum_{j=0}^{k} \Xi_{(j)}(\widetilde{F}_{(k-j)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k-j)})) \\
&\quad = \sum_{j=0}^{k} \sum_{i=1}^{n} \widetilde{\zeta}_{(j)i}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(j)}) \frac{\partial \widetilde{F}_{(k-j)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k-j)})}{\partial z_{(0)i}} = 0,
\end{aligned} \tag{11.9}$$

for $k = 0, \ldots, p$, where we introduced the *reduced operators*

$$\Xi_{(j)} = \sum_{i=1}^{n} \widetilde{\zeta}_{(j)i}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(j)}) \frac{\partial}{\partial z_{(0)i}}, \qquad j = 0, \ldots, p.$$

Therefore, the approximate invariance condition of a function $F(\mathbf{z}, \varepsilon)$ yields the $p + 1$ conditions (11.9) for the $p + 1$ functions characterizing the expansion (up to the order p in ε) of the function F.

Example 11.2 Let $p = 1$, and $\mathbf{z} \equiv (z_1, z_2)$; a smooth function

$$\begin{aligned} F(\mathbf{z}, \varepsilon) \approx & F_{(0)}(z_{(0)1}, z_{(0)2}) \\ & + \varepsilon \left(F_{(1)}(z_{(0)1}, z_{(0)2}) + \frac{\partial F_{(0)}(z_{(0)1}, z_{(0)2})}{\partial z_{(0)1}} z_{(1)1} + \frac{\partial F_{(0)}(z_{(0)1}, z_{(0)2})}{\partial z_{(0)2}} z_{(1)2} \right) \end{aligned}$$

is approximately invariant with respect to the Lie generator

$$\Xi \approx \left(z_{(0)2} + \varepsilon z_{(1)2}\right) \frac{\partial}{\partial z_1} - \left(z_{(0)1} + \varepsilon z_{(1)1}\right) \frac{\partial}{\partial z_2},$$

provided that

$$\begin{aligned} z_{(0)2} \frac{\partial F_{(0)}(z_{(0)1}, z_{(0)2})}{\partial z_{(0)1}} - z_{(0)1} \frac{\partial F_{(0)}(z_{(0)1}, z_{(0)2})}{\partial z_{(0)2}} = 0, \\ z_{(0)2} \frac{\partial F_{(1)}(z_{(0)1}, z_{(0)2})}{\partial z_{(0)1}} - z_{(0)1} \frac{\partial F_{(1)}(z_{(0)1}, z_{(0)2})}{\partial z_{(0)2}} = 0, \end{aligned}$$

whereupon

$$F(\mathbf{z}, \varepsilon) \approx \widehat{F}_{(0)}(r_0^2) + \varepsilon \left(\widehat{F}_{(1)}(r_0^2) + 2(z_{(0)1} z_{(1)1} + z_{(0)2} z_{(1)2}) \widehat{F}'_{(0)}(r_0^2) \right),$$

$\widehat{F}_{(0)}$ and $\widehat{F}_{(1)}$ being arbitrary functions of $r_0^2 = z_{(0)1}^2 + z_{(0)2}^2$, and the prime ($'$) denoting the differentiation with respect to the argument.

Definition 11.6 (*Invariance of an equation*) The equation

$$F(\mathbf{z}, \varepsilon) \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{F}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}) = 0$$

is approximately invariant with respect to (11.6) if

$$F(\mathbf{z}', \varepsilon) \approx 0 \text{ when } F(\mathbf{z}, \varepsilon) \approx 0.$$

Theorem 11.2 *Let*

$$\Xi \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\Xi}_{(k)}$$

be the approximate Lie generator of (11.6).

Then, the equation

$$F(\mathbf{z}, \varepsilon) \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{F}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)}) = 0$$

is approximately invariant with respect to (11.6) if and only if

$$\Xi(F(\mathbf{z}, \varepsilon)) \approx 0 \text{ when } F(\mathbf{z}, \varepsilon) \approx 0,$$

i.e.,

$$\Xi(F(\mathbf{z}, \varepsilon)) \approx \lambda(\mathbf{z}, \varepsilon) F(\mathbf{z}, \varepsilon),$$

$\lambda(\mathbf{z}, \boldsymbol{\varepsilon})$ *being a multiplier nowhere vanishing.*

Definition 11.7 (*Approximate canonical variables*) Let

$$\Xi \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\Xi}_{(k)}$$

be an approximate Lie generator. Then

$$\mathbf{w}(\mathbf{z}, \varepsilon) \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\mathbf{w}}_{(k)}(\mathbf{z}_{(0)}, \ldots, \mathbf{z}_{(k)})$$

gives a set of approximate canonical variables if

$$\begin{aligned} &\Xi(w_i) \approx 0, \qquad i = 1, \ldots, n-1, \\ &\Xi(w_n) \approx 1. \end{aligned}$$

In terms of the approximate canonical variables, the approximate Lie generator becomes

$$\Xi \approx \frac{\partial}{\partial w_n},$$

and the approximate Lie group of transformations corresponds to the translation of the variable w_n only.

Example 11.3 Let $\mathbf{z} \equiv (z_1, z_2) \in \mathbb{R}^2$, $p = 1$ and

$$\Xi \approx \left(z_{(0)1}^2 + \varepsilon(2z_{(0)1}z_{(1)1})\right)\frac{\partial}{\partial z_1} + \left(z_{(0)1}z_{(0)2} + \varepsilon(z_{(0)1}z_{(1)2} + z_{(0)2}z_{(1)1})\right)\frac{\partial}{\partial z_2}.$$

The first-order approximate canonical variables

$$\mathbf{w}(\mathbf{z}, \varepsilon) \approx \mathbf{w}_{(0)}(\mathbf{z}_{(0)}) + \varepsilon\left(\mathbf{w}_{(1)}(\mathbf{z}_{(0)}) + \sum_{i=1}^{2}\frac{\partial \mathbf{w}_{(0)}(\mathbf{z}_{(0)})}{\partial z_{(0)i}}z_{(1)i}\right)$$

are determined by solving the following system of partial differential equations:

$$z_{(0)1}^2\frac{\partial \widetilde{w}_{(0)1}}{\partial z_{(0)1}} + z_{(0)1}z_{(0)2}\frac{\partial \widetilde{w}_{(0)1}}{\partial z_{(0)2}} = 0, \quad z_{(0)1}^2\frac{\partial \widetilde{w}_{(0)2}}{\partial z_{(0)1}} + z_{(0)1}z_{(0)2}\frac{\partial \widetilde{w}_{(0)2}}{\partial z_{(0)2}} = 1,$$
$$z_{(0)1}^2\frac{\partial \widetilde{w}_{(1)1}}{\partial z_{(0)1}} + z_{(0)1}z_{(0)2}\frac{\partial \widetilde{w}_{(1)1}}{\partial z_{(0)2}} = 0, \quad z_{(0)1}^2\frac{\partial \widetilde{w}_{(1)2}}{\partial z_{(0)1}} + z_{(0)1}z_{(0)2}\frac{\partial \widetilde{w}_{(1)2}}{\partial z_{(0)2}} = 0,$$

whereupon

$$w_1 \approx \frac{z_{(0)2}}{z_{(0)1}} + \varepsilon\left(\frac{z_{(0)2}}{z_{(0)1}} - \frac{z_{(0)2}z_{(1)1}}{z_{(0)1}^2} + \frac{z_{(1)2}}{z_{(0)1}}\right),$$
$$w_2 \approx -\frac{1}{z_{(0)1}} + \varepsilon\left(\frac{z_{(0)2}}{z_{(0)1}} + \frac{z_{(1)1}}{z_{(0)1}^2}\right),$$

and the Lie operator reduces to

$$\Xi \approx \frac{\partial}{\partial w_2}.$$

Definition 11.8 (*Approximate Lie bracket*) Let

$$\Xi_1 \approx \sum_{k=0}^{p}\varepsilon^k\widetilde{\Xi}_{1(k)}, \qquad \Xi_2 \approx \sum_{k=0}^{p}\varepsilon^k\widetilde{\Xi}_{2(k)},$$

where

$$\widetilde{\Xi}_{1(k)} = \sum_{j=1}^{N}\widetilde{\zeta}_{1(k)j}\frac{\partial}{\partial z_j}, \qquad \widetilde{\Xi}_{2(k)} = \sum_{j=1}^{N}\widetilde{\zeta}_{2(k)j}\frac{\partial}{\partial z_j},$$

be two approximate Lie operators. Their approximate Lie bracket is defined as follows:

$$
\begin{aligned}
[\Xi_1, \Xi_2] &\approx \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=0}^{k} \left[\widetilde{\Xi}_{1(i)}, \widetilde{\Xi}_{2(k-i)} \right] \right) = \\
&= \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=0}^{k} \left(\widetilde{\Xi}_{1(i)} \, \widetilde{\Xi}_{2(k-i)} - \widetilde{\Xi}_{2(k-i)} \, \widetilde{\Xi}_{1(i)} \right) \right) = \\
&= \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=0}^{k} \left(\sum_{\ell=1}^{n} \left(\sum_{j=1}^{n} \left(\widetilde{\zeta}_{1(i)j} \frac{\partial \widetilde{\zeta}_{2(k-i)\ell}}{\partial z_{(0)j}} - \widetilde{\zeta}_{2(k-i)j} \frac{\partial \widetilde{\zeta}_{1(i)\ell}}{\partial z_{(0)j}} \right) \frac{\partial}{\partial z_\ell} \right) \right) \right).
\end{aligned}
$$

Remark 11.1 The approximate Lie bracket possesses the usual properties:

1. linearity: $[\alpha \Xi_1 + \beta \Xi_2, \Xi_3] \approx \alpha [\Xi_1, \Xi_3] + \beta [\Xi_2, \Xi_3]$, α, β constants;
2. skew-symmetry: $[\Xi_1, \Xi_2] \approx -[\Xi_2, \Xi_1]$;
3. the Jacobi identity: $[[\Xi_1, \Xi_2], \Xi_3] + [[\Xi_2, \Xi_3], \Xi_1] + [[\Xi_3, \Xi_1], \Xi_2] \approx 0$.

The definition of approximate Lie bracket allows us to introduce a structure of approximate Lie algebra.

Though the approximate theory here considered, is different from the one in the Baikov–Gazivoz–Ibragimov approach, some definitions are still valid [26].

Definition 11.9 A vector space $\mathbb{L}$ of approximate operators is an approximate Lie algebra of operators if it is closed (in the approximation of order p) under the approximate Lie bracket, i.e.,

$$[\Xi_1, \Xi_2] \in \mathbb{L}, \quad \forall \, \Xi_1, \Xi_2 \in \mathbb{L}.$$

Definition 11.10 The elements of the set $\{\Xi_\alpha : \alpha = 1, \ldots, r\}$ of approximate operators are linearly independent if

$$\sum_{\alpha=1}^{r} c_\alpha \, \Xi_\alpha \approx 0,$$

with constants c_α not depending on ε, imply

$$c_1 = c_2 = \cdots = c_r = 0.$$

Definition 11.11 A basis of an approximate Lie algebra $\mathbb{L}$ is a set of linearly independent approximate operators spanning the vector space $\mathbb{L}$. If the basis has a finite number of approximate operators, then we adopt the usual notation $\mathbb{L}_r$ for the r-dimensional approximate Lie algebra.

Let $\mathbb{L}$ be an approximate Lie algebra. Fixing the order p of approximation, let $\{\Xi_\alpha : 1, \ldots, m_1\}$ be a set of linearly independent approximate operators belonging

to $\mathbb{L}$. Denote with $\left\{\Xi'_\alpha : \alpha = 1, \ldots, m_2\right\}$ the set of approximate operators obtained by multiplying each Ξ_α by $\varepsilon, \varepsilon^2, \ldots, \varepsilon^p$ and dropping the terms of order $O(\varepsilon^{p+1})$. If the set

$$\left\{\Xi_\alpha, \Xi'_\beta : \alpha = 1, \ldots, m_1, \beta = 1, \ldots, m_2\right\}$$

gives a basis of $\mathbb{L}$, then the operators Ξ_α are called *essential operators* of the approximate Lie algebra $\mathbb{L}$.

Using (11.8) along with (11.9), and the definitions of approximate Lie bracket and approximate Lie algebra, the following results are easily proved.

Theorem 11.3 *The set of approximate Lie generators leaving a function approximately invariant is an approximate Lie algebra.*

Theorem 11.4 *The set of approximate Lie generators leaving a manifold approximately invariant is an approximate Lie algebra.*

11.3 Approximate Symmetries of Differential Equations

Here, we use the previous results in order to define and compute the approximate Lie symmetries of differential equations involving small terms. Within this framework, we distinguish the independent variables $\mathbf{x}$ from the dependent ones $\mathbf{u}$, and limit ourselves to expand only the dependent variables in powers of ε.

Let

$$\boldsymbol{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}; \varepsilon) = \mathbf{0}$$

be a differential equation of order r, where $\mathbf{u}^{(k)}$, $k = 1, \ldots, r$, denotes the set of all derivatives of the dependent variables $\mathbf{u} \in \mathbb{D}_m \subseteq \mathbb{R}^m$ with respect to the independent variables $\mathbf{x} \in \mathbb{D}_n \subseteq \mathbb{R}^n$ up to the order r, involving a small parameter ε.

If one looks for classical Lie point symmetries, in general it is not guaranteed that the infinitesimal generators depend on the parameter ε. Nevertheless, the occurrence of terms involving ε has dramatic effects because one loses some symmetries admitted by the unperturbed equation

$$\boldsymbol{\Delta}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}; 0) = \mathbf{0},$$

as the following examples clearly show.

Example 11.4 The second-order ordinary differential equation

$$\frac{d^2u}{dx^2} + u = 0$$

admits an eight-dimensional Lie algebra of point symmetries spanned by the vector fields

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = u\frac{\partial}{\partial u}, \quad \Xi_3 = \sin(x)\frac{\partial}{\partial u}, \quad \Xi_4 = \cos(x)\frac{\partial}{\partial u},$$
$$\Xi_5 = \sin(x)\cos(x)\frac{\partial}{\partial x} - \sin^2(x)u\frac{\partial}{\partial u}, \quad \Xi_6 = \cos(2x)\frac{\partial}{\partial x} - \sin(2x)\frac{\partial}{\partial u},$$
$$\Xi_7 = \cos(x)u\frac{\partial}{\partial x} - \sin(x)u^2\frac{\partial}{\partial u}, \quad \Xi_8 = \sin(x)u\frac{\partial}{\partial x} + \cos(x)u\frac{\partial}{\partial u}.$$

On the contrary, the equation

$$\frac{d^2u}{dx^2} + u + \varepsilon u^3 = 0$$

admits only the exact symmetry generated by Ξ_1.

Example 11.5 The Korteweg–deVries equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

admits a four-dimensional Lie algebra of exact point symmetries spanned by the vector fields

$$\Xi_1 = \frac{\partial}{\partial t}, \quad \Xi_2 = \frac{\partial}{\partial x}, \quad \Xi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \Xi_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}.$$

On the contrary, by considering the Korteweg–deVries–Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \varepsilon\frac{\partial^2 u}{\partial x^2} = 0,$$

we lose the scaling group, and have only the three exact symmetries generated by Ξ_1, Ξ_2 and Ξ_3.

Example 11.6 The 2×2 first-order quasilinear system

$$\frac{\partial u_1}{\partial t} + a_{11}(u_1, u_2)\frac{\partial u_1}{\partial x} + a_{12}(u_1, u_2)\frac{\partial u_2}{\partial x} = \varepsilon b_1(u_1, u_2),$$
$$\frac{\partial u_2}{\partial t} + a_{21}(u_1, u_2)\frac{\partial u_1}{\partial x} + a_{22}(u_1, u_2)\frac{\partial u_2}{\partial x} = \varepsilon b_2(u_1, u_2),$$

where a_{ij} and b_i $(i, j = 1, 2)$ are arbitrary functions of the indicated arguments, does not admit an infinite-dimensional Lie algebra of point symmetries unless it is $\varepsilon = 0$. Moreover, also the invariance with respect to a homogeneous scaling of the independent variables

$$t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$$

is lost when $\varepsilon \neq 0$.

In perturbation theory [30], a differential equation involving small terms is often studied by looking for solutions in the form

$$\mathbf{u}(\mathbf{x};\varepsilon) = \sum_{k=0}^{p} \varepsilon^k \mathbf{u}_{(k)}(\mathbf{x}) + O(\varepsilon^{p+1}). \tag{11.10}$$

Thus, the differential equation can be written as

$$\begin{aligned}&\Delta(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \dots, \mathbf{u}^{(r)}; \varepsilon)\\ &\quad\approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\Delta}_{(k)}\left(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}^{(1)}_{(0)}, \dots, \mathbf{u}^{(r)}_{(0)}, \dots, \mathbf{u}_{(k)}, \mathbf{u}^{(1)}_{(k)}, \dots, \mathbf{u}^{(r)}_{(k)}\right) = 0.\end{aligned}$$

Now, let us consider a Lie generator

$$\Xi = \sum_{i=1}^{n} \xi_i(\mathbf{x}, \mathbf{u}; \varepsilon)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(\mathbf{x}, \mathbf{u}; \varepsilon)\frac{\partial}{\partial u_\alpha},$$

and assume that the infinitesimals depend on the small parameter ε.

According to the results of previous section, by using the expansion (11.10) of the dependent variables only, we have for the infinitesimals,

$$\xi_i \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\xi}_{(k)i}, \qquad \eta_\alpha \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\eta}_{(k)\alpha},$$

where

$$\begin{aligned}&\widetilde{\xi}_{(0)i} = \xi_{(0)i} = \xi_i(\mathbf{x}, \mathbf{u}; \varepsilon)|_{\varepsilon=0}, \qquad \widetilde{\eta}_{(0)\alpha} = \eta_{(0)\alpha} = \eta_\alpha(\mathbf{x}, \mathbf{u}; \varepsilon)|_{\varepsilon=0,}\\ &\widetilde{\xi}_{(k+1)i} = \frac{1}{k+1}\mathcal{R}[\widetilde{\xi}_{(k)i}], \qquad \widetilde{\eta}_{(k+1)\alpha} = \frac{1}{k+1}\mathcal{R}[\widetilde{\eta}_{(k)\alpha}],\end{aligned}$$

$\mathcal{R}$ being the recursion operator defined by (11.5).

Thence, we have an approximate Lie generator

$$\Xi \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\Xi}_{(k)}, \tag{11.11}$$

where

$$\widetilde{\Xi}_{(k)} = \sum_{i=1}^{n} \widetilde{\xi}_{(k)i}(\mathbf{x}, \mathbf{u}_{(0)}, \dots, \mathbf{u}_{(k)})\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \widetilde{\eta}_{(k)\alpha}(\mathbf{x}, \mathbf{u}_{(0)}, \dots, \mathbf{u}_{(k)})\frac{\partial}{\partial u_\alpha}.$$

We have to deal with differential equations, and so we need to prolong the Lie generator to account for the transformation of derivatives. This is done as in classical Lie group analysis of differential equations, i.e., the derivatives are transformed in such a way the contact conditions relating differentials are preserved. Of course, in the expression of prolongations, we need to take into account the expansions of ξ_i, η_α and u_α, and drop the $O(\varepsilon^{p+1})$ terms.

Therefore, we have the prolongations

$$\begin{aligned}
\Xi^{(0)} &= \Xi, \\
\Xi^{(r)} &= \Xi^{(r-1)} + \sum_{\alpha=1}^{m} \sum_{i_1=1}^{n} \cdots \sum_{i_r=1}^{n} \eta_{[\alpha,i_1\ldots i_r]} \frac{\partial}{\partial u_{\alpha,i_1\ldots i_r}}, \qquad r > 0,
\end{aligned}$$

where

$$\eta_{[\alpha,i_1\ldots i_r]} = \widetilde{\mathcal{D}}_{i_r} \eta_{[\alpha,i_1\ldots i_{r-1}]} - \sum_{j=1}^{n} u_{\alpha,i_1\ldots i_{r-1}j} \widetilde{\mathcal{D}}_{i_r} \xi_j,$$

along with the *approximate total derivative* now defined as

$$\widetilde{\mathcal{D}}_i = \frac{\partial}{\partial x_i} + \sum_{k=0}^{p} \sum_{\alpha=1}^{m} \left(u_{(k)\alpha,i} \frac{\partial}{\partial u_{(k)\alpha}} + \sum_{j=1}^{n} u_{(k)\alpha,ij} \frac{\partial}{\partial u_{(k)\alpha,j}} + \cdots \right).$$

Example 11.7 Let $p = 1$, and consider the approximate Lie generator

$$\begin{aligned}
\Xi \approx &\sum_{i=1}^{n} \left(\xi_{(0)i} + \varepsilon \left(\xi_{(1)i} + \sum_{\beta=1}^{m} \frac{\partial \xi_{(0)i}}{\partial u_{(0)\beta}} u_{(1)\beta} \right) \right) \frac{\partial}{\partial x_i} \\
&+ \sum_{\alpha=1}^{m} \left(\eta_{(0)\alpha} + \varepsilon \left(\eta_{(1)\alpha} + \sum_{\beta=1}^{m} \frac{\partial \eta_{(0)\alpha}}{\partial u_{(0)\beta}} u_{(1)\beta} \right) \right) \frac{\partial}{\partial u_\alpha},
\end{aligned} \tag{11.12}$$

where $\xi_{(0)i}$, $\xi_{(1)i}$, $\eta_{(0)\alpha}$ and $\eta_{(1)\alpha}$ depend on $(\mathbf{x}, \mathbf{u}_{(0)})$. The first-order prolongation is

$$\Xi^{(1)} \approx \Xi + \sum_{\alpha=1}^{m} \sum_{i=1}^{n} \eta_{[\alpha,i]} \frac{\partial}{\partial u_{\alpha,i}},$$

where

$$\eta_{[\alpha,i]} = \widetilde{\mathcal{D}}_i \left(\eta_{(0)\alpha} + \varepsilon \left(\eta_{(1)\alpha} + \sum_{\beta=1}^{m} \frac{\partial \eta_{(0)\alpha}}{\partial u_{(0)\beta}} u_{(1)\beta} \right) \right) - \sum_{j=1}^{n} \widetilde{\mathcal{D}}_i \left(\xi_{(0)j} + \varepsilon \left(\xi_{(1)j} + \sum_{\beta=1}^{m} \frac{\partial \xi_{(0)j}}{\partial u_{(0)\beta}} u_{(1)\beta} \right) \right) \left(u_{(0)\alpha,j} + \varepsilon u_{(1)\alpha,j} \right).$$

Similar reasonings lead to higher order prolongations.

The approximate (at the order p) invariance condition of a differential equation reads

$$\sum_{k=0}^{p} \varepsilon^k \sum_{\ell=0}^{k} \widetilde{\Xi}_{(\ell)}^{(r)} \widetilde{\Delta}_{(k-\ell)} \Big|_{\Delta \approx 0} \approx 0.$$

The Lie generator $\widetilde{\Xi}_{(0)}$ is always a symmetry of the unperturbed equations ($\varepsilon = 0$); the *correction* terms $\sum_{k=1}^{p} \varepsilon^k \widetilde{\Xi}_{(k)}$ give the deformation of the symmetry due to the terms involving ε.

Remark 11.2 Not all symmetries of the unperturbed equations are admitted as the zeroth terms of approximate symmetries; the symmetries of the unperturbed equations that are the zeroth terms of approximate symmetries are called *stable symmetries* [1].

Remark 11.3 If Ξ is the generator of a Lie point symmetry of a differential equation, $\varepsilon \Xi$ is a generator of a Lie point symmetry too.

By the same arguments as in classical Lie theory of differential equations it remains proved the following result.

Theorem 11.5 *The approximate Lie point symmetries of a differential equation are the elements of an approximate Lie algebra.*

Let us show some simple examples of first-order approximate Lie symmetries admitted by differential equations.

Example 11.8 The second-order ordinary differential equation

$$\frac{d^2u}{dx^2} + u + \varepsilon u^3 = 0$$

admits a nine-dimensional approximate Lie algebra of first-order approximate point symmetries spanned by the vector fields:

$$\begin{aligned}
\Xi_1 &= \frac{\partial}{\partial x},\\
\Xi_2 &= \varepsilon u_{(0)} \frac{\partial}{\partial u},\\
\Xi_3 &= \varepsilon \sin(x) \frac{\partial}{\partial u},\\
\Xi_4 &= \varepsilon \cos(x) \frac{\partial}{\partial u},\\
\Xi_5 &= \varepsilon \left(\cos(2x) \frac{\partial}{\partial x} - \sin(2x) u_{(0)} \frac{\partial}{\partial u} \right),\\
\Xi_6 &= \varepsilon \left(\sin(2x) \frac{\partial}{\partial x} + \cos(2x) u_{(0)} \frac{\partial}{\partial u} \right),\\
\Xi_7 &= \varepsilon \left(u_{(0)} \sin(x) \frac{\partial}{\partial x} + u_{(0)}^2 \cos(x) \frac{\partial}{\partial u} \right),\\
\Xi_8 &= \varepsilon \left(u_{(0)} \cos(x) \frac{\partial}{\partial x} - u_{(0)}^2 \sin(x) \frac{\partial}{\partial u} \right),\\
\Xi_9 &= \varepsilon \frac{\partial}{\partial x}.
\end{aligned}$$

Example 11.9 (*Korteweg–deVries–Burgers equations*) Let us consider the Korteweg–deVries–Burgers equation, obtained by adding a small dissipative term to the Korteweg–deVries equation,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \varepsilon \frac{\partial^2 u}{\partial x^2} = 0.$$

The first-order approximate symmetries are spanned by the following vector fields:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u},\\
&\Xi_4 = \varepsilon \frac{\partial}{\partial t}, \qquad \Xi_5 = \varepsilon \frac{\partial}{\partial x}, \qquad \Xi_6 = \left(t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right),\\
&\Xi_7 = \varepsilon \left(3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u_{(0)} \frac{\partial}{\partial u} \right).
\end{aligned}$$

Analogously, we can consider the Burgers equation perturbed with the addition of a small dispersive term,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial^3 u}{\partial x^3} = 0.$$

The latter admits the first-order approximate symmetries spanned by the following vector fields:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\\
&\Xi_4 = \varepsilon\frac{\partial}{\partial t}, \qquad \Xi_5 = \varepsilon\frac{\partial}{\partial x}, \qquad \Xi_6 = \varepsilon\left(t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right),\\
&\Xi_7 = \varepsilon\left(2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u_{(0)}\frac{\partial}{\partial u}\right),\\
&\Xi_8 = \varepsilon\left(t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - tu_{(0)})\frac{\partial}{\partial u}\right).
\end{aligned} \tag{11.13}$$

Example 11.10 (*Perturbed linear telegraph equation*) Let us consider the linear telegraph equation in $(1+1)$ dimensions perturbed with the addition of a small dissipative term,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \varepsilon\frac{\partial^2 u}{\partial t^2} = 0.$$

The first-order approximate symmetries are spanned by the following vector fields:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x}, \qquad \Xi_3 = \varepsilon\frac{\partial}{\partial t}, \qquad \Xi_4 = \varepsilon\frac{\partial}{\partial x},\\
&\Xi_5 = t\frac{\partial}{\partial x} - \frac{xu_{(0)}}{2}\frac{\partial}{\partial u} + \varepsilon\left(x\frac{\partial}{\partial t} - \frac{xu_{(1)}}{2}\frac{\partial}{\partial u}\right), \qquad \Xi_6 = \left(u_{(0)} + \varepsilon u_{(1)}\right)\frac{\partial}{\partial u},\\
&\Xi_7 = \varepsilon\left(t\frac{\partial}{\partial x} - \frac{xu_{(0)}}{2}\frac{\partial}{\partial u}\right), \qquad \Xi_8 = \varepsilon u_{(0)}\frac{\partial}{\partial u},\\
&\Xi_9 = \varepsilon\left(2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right), \qquad \Xi_{10} = \varepsilon\left(t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - \left(\frac{t}{2} + \frac{x^2}{4}\right)u_{(0)}\frac{\partial}{\partial u}\right),\\
&\Xi_{11} = f_1(t,x)\frac{\partial}{\partial u}, \qquad \Xi_{12} = \varepsilon f_2(t,x)\frac{\partial}{\partial u},
\end{aligned}$$

where $f_i(t,x)$ $(i = 1, 2)$ are functions of the indicated arguments, along with the constraints

$$\begin{aligned}
&\frac{\partial f_1(t,x)}{\partial t} - \frac{\partial^2 f_1(t,x)}{\partial x^2} = 0,\\
&\frac{\partial^2 f_1(t,x)}{\partial t^2} - \frac{\partial^2 f_2(t,x)}{\partial x^2} + \frac{\partial f_2(t,x)}{\partial t} = 0.
\end{aligned}$$

It is well known that the search of Lie symmetries of differential equations involves a lot of lengthy and tedious, though straightforward, calculations. Nevertheless, many symbolic packages do exist doing often automatically the required work (see [22], and references therein, [4, 25]). The computation of approximate Lie symmetries of differential equations requires additional computational costs. The REDUCE [21]

package ReLie [32], described in Chap. 12, is able to perform much of the work for computing approximate Lie symmetries as introduced here.

The approximate Lie symmetries can be used to lower the order of ordinary differential equations or to compute approximate invariant solutions of partial differential equations.

Definition 11.12 (*Approximate invariant solutions*) Given a rth-order differential equation

$$\mathbf{\Delta} \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\Delta}_{(k)} \left(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}^{(1)}_{(0)}, \ldots, \mathbf{u}^{(r)}_{(0)}, \ldots, \mathbf{u}_{(k)}, \mathbf{u}^{(1)}_{(k)}, \ldots, \mathbf{u}^{(r)}_{(k)} \right) = \mathbf{0}, \tag{11.14}$$

and an approximate Lie generator

$$\Xi \approx \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} \widetilde{\xi}_{(k)i}(\mathbf{x}, \mathbf{u}_{(0)}, \ldots, \mathbf{u}_{(k)}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \widetilde{\eta}_{(k)\alpha}(\mathbf{x}, \mathbf{u}_{(0)}, \ldots, \mathbf{u}_{(k)}) \frac{\partial}{\partial u_\alpha} \right),$$

the *approximate invariant surface conditions* are

$$\mathbf{Q} \approx \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} \widetilde{\xi}_{(k)i}(\mathbf{x}, \mathbf{u}_{(0)}, \ldots, \mathbf{u}_{(k)}) \frac{\partial \mathbf{u}}{\partial x_i} - \widetilde{\boldsymbol{\eta}}_{(k)}(\mathbf{x}, \mathbf{u}_{(0)}, \ldots, \mathbf{u}_{(k)}) \right) = \mathbf{0}, \tag{11.15}$$

where we have to insert the expansion of $\mathbf{u}$; then, by separating (11.15) at the various order in ε, and appending them to the Eq. (11.14), approximate invariant solutions can be determined.

For simplicity, in the following examples we take $p = 1$, i.e., we consider first-order approximate symmetries. For higher values of p what is needed is only a larger amount of computations.

11.4 Order Lowering of an Ordinary Differential Equation

Here we show how a solvable approximate Lie algebra of symmetries can be used to lower the order of an ordinary differential equation.

Example 11.11 (*Perturbed Blasius equation*) Let us consider the equation

$$\frac{d^3u}{dx^3} + \frac{1}{2} u \frac{d^2u}{dx^2} + \varepsilon u \frac{du}{dx} = 0, \tag{11.16}$$

that for $\varepsilon = 0$ is the well-known Blasius equation. For $\varepsilon \neq 0$, Eq. (11.16) admits only one exact Lie point symmetry, say

$$\Xi = \frac{\partial}{\partial x}.$$

On the contrary, looking for first-order approximate symmetries, Eq. (11.16) admits a four-dimensional approximate Lie algebra spanned by the vector fields:

$$\Xi_1 = \frac{\partial}{\partial x}, \qquad \Xi_2 = \varepsilon \frac{\partial}{\partial x}, \qquad \Xi_3 = \varepsilon \left(x \frac{\partial}{\partial x} - u_{(0)} \frac{\partial}{\partial u} \right),$$
$$\Xi_4 = x \frac{\partial}{\partial x} - u_{(0)} \frac{\partial}{\partial u} + \varepsilon \left(\frac{x^2}{3} \frac{\partial}{\partial x} + \left(8 - \frac{2}{3} u_{(0)} x - u_{(1)} \right) \frac{\partial}{\partial u} \right).$$

Let us consider the approximate operators $\widetilde{\Xi}_1 = \Xi_1 + \frac{2}{3}\Xi_3$ and $\widetilde{\Xi}_2 = \Xi_4$; it is $\left[\widetilde{\Xi}_1, \widetilde{\Xi}_2\right] \approx \widetilde{\Xi}_1$, that means that they span an approximate (at first order) solvable two-dimensional Lie subalgebra. Let us introduce the canonical variables for the operator $\widetilde{\Xi}_1$, say

$$\widetilde{\Xi}_1(v) = 0, \qquad \widetilde{\Xi}_1(t) = 1,$$

whereupon

$$v = u_{(0)} + \varepsilon \left(\frac{2}{3} x u_{(0)} + u_{(1)} \right), \qquad t = x - \varepsilon \frac{x^2}{3},$$

and Eq. (11.16) becomes

$$\frac{d^3 v}{dt^3} + \frac{1}{2} v \frac{d^2 v}{dt^2} - 4\varepsilon \frac{d^2 v}{dt^2} = 0.$$

The standard substitution $\dfrac{dv}{dt} = w(v)$ yields, if $w \neq 0$,

$$w \frac{d^2 w}{dv^2} + \left(\frac{dw}{dv} \right)^2 + \frac{1}{2} v \frac{dw}{dv} - 4\varepsilon \frac{dw}{dv} = 0. \tag{11.17}$$

In terms of the new variables, the first-order prolonged operator $\widetilde{\Xi}_2$ assumes the form

$$\widetilde{\Xi}_2 = (v - 8\varepsilon) \frac{\partial}{\partial v} + 2w \frac{\partial}{\partial w};$$

by computing the new independent and dependent variables r and s, respectively, such that

$$\widetilde{\Xi}_2(r) = 1, \qquad \widetilde{\Xi}_2(s) = 0,$$

we have

$$\begin{aligned} r &= \log v_{(0)} + \varepsilon \left(\frac{v_{(1)}}{v_{(0)}} - \frac{8}{v_{(0)}} \right), \\ s &= \frac{w_{(0)}}{v_{(0)}^2} + \varepsilon \left(\frac{w_{(1)}}{v_{(0)}^2} - 2\frac{(v_{(1)} - 8)w_{(0)}}{v_{(0)}^3} \right). \end{aligned} \tag{11.18}$$

By inserting (11.18) into (11.17), we obtain

$$s\frac{d^2 s}{dr^2} + 7s\frac{ds}{dr} + \left(\frac{ds}{dr}\right)^2 + \frac{1}{2}\frac{ds}{dr} + 6s^2 + s = 0.$$

Finally, by setting $\dfrac{ds}{dr} = p(s)$, the following first-order ordinary differential equation

$$ps\frac{dp}{ds} + p^2 + 7ps + \frac{1}{2}p + 6s^2 + s = 0$$

is recovered. We notice that the small parameter ε does not explicitly appear, but it is hidden inside the involved variables.

Remark 11.4 The reduction of perturbed Blasius equation to a first-order ordinary differential equation where ε does not appear explicitly, as well as the form of admitted approximate Lie symmetries, suggests us the possibility of transforming Eq. (11.16) into the standard Blasius equation. Let us introduce the new independent and dependent variables t and v, respectively, such that

$$\Xi_4(t) \approx t, \qquad \Xi_4(v) \approx -v,$$

whence

$$v = u + \varepsilon \left(\frac{2}{3}xu - 8 \right), \qquad t = x - \varepsilon\frac{x^2}{3},$$

and Eq. (11.16) reduces to

$$\frac{d^3 v}{dt^3} + \frac{1}{2}v\frac{d^2 v}{dt^2} = 0,$$

i.e., the classical Blasius equation.

11.5 Approximate Invariant Solutions

In this section, we consider some examples of partial differential equations, and compute first-order approximate invariant solutions.

For first-order approximate Lie symmetries like (11.12), the approximate invariant surface conditions (11.15) become

$$\sum_{i=1}^{n}\left(\xi_{(0)i}\frac{\partial u_{(0)\alpha}}{\partial x_i}+\varepsilon\left(\left(\xi_{(1)i}+\sum_{\beta=1}^{m}\frac{\partial \xi_{(0)i}}{\partial u_{(0)\beta}}u_{(1)\beta}\right)\frac{\partial u_{(0)\alpha}}{\partial x_i}+\xi_{(0)i}\frac{\partial u_{(1)\alpha}}{\partial x_i}\right)\right)$$
$$-\left(\eta_{(0)\alpha}+\varepsilon\left(\eta_{(1)\alpha}+\sum_{\beta=1}^{m}\frac{\partial \eta_{(0)\alpha}}{\partial u_{(0)\beta}}u_{(1)\beta}\right)\right)\approx 0,$$

where $\alpha = 1, \ldots, m$.

Example 11.12 Consider the nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right)+\varepsilon\frac{\partial u}{\partial t}=0. \tag{11.19}$$

The first-order approximate symmetries are generated by the following vector fields:

$$\Xi_1=\frac{\partial}{\partial t},\qquad \Xi_2=\frac{\partial}{\partial x},$$
$$\Xi_3=\left(t+\varepsilon\frac{t^2}{6}\right)\frac{\partial}{\partial t}-\left(u_{(0)}+\varepsilon\left(u_{(1)}+\frac{tu_{(0)}}{3}\right)\right)\frac{\partial}{\partial u},$$
$$\Xi_4=x\frac{\partial}{\partial x}+(u_{(0)}+\varepsilon u_{(1)})\frac{\partial}{\partial u},\qquad \Xi_5=\varepsilon\frac{\partial}{\partial t},\qquad \Xi_6=\varepsilon\frac{\partial}{\partial x},$$
$$\Xi_7=\varepsilon\left(t\frac{\partial}{\partial t}-u_{(0)}\frac{\partial}{\partial u}\right),\qquad \Xi_8=\varepsilon\left(x\frac{\partial}{\partial x}+u_{(0)}\frac{\partial}{\partial u}\right).$$

The solutions approximately invariant with respect to the Lie generator Ξ_3 are such that

$$\left(t+\varepsilon\frac{t^2}{6}\right)\frac{\partial u}{\partial t}=-u_{(0)}-\varepsilon\left(u_{(1)}+\frac{tu_{(0)}}{3}\right);$$

whereupon, insertion of $u = u_{(0)} + \varepsilon u_{(1)} + O(\varepsilon^2)$ and separation of the coefficients of different powers of ε yields the system

$$\begin{cases} t\dfrac{\partial u_{(0)}}{\partial t}=-u_{(0)},\\ t\dfrac{\partial u_{(1)}}{\partial t}+\dfrac{t^2}{6}\dfrac{\partial u_{(0)}}{\partial t}=-u_{(1)}-\dfrac{tu_{(0)}}{3}, \end{cases}$$

whose solution is

$$u_{(0)}(t,x)=\frac{U_0(x)}{t},\qquad u_{(1)}(t,x)=\frac{U_1(x)}{t}-\frac{U_0(x)}{6}. \tag{11.20}$$

Substitution of (11.20) into Eq. (11.19) gives the following reduced system of ordinary differential equations:

$$\begin{aligned}(U_0^2U_0')' - 2U_0 = 0,\\ (U_0^2U_1)'' - 2U_1 = 0,\end{aligned} \tag{11.21}$$

where $U_0(x)$ and $U_1(x)$ are functions to be determined, and the prime ($'$) denotes differentiation with respect to x. A solution to the system (11.21) is

$$U_0(x) = \pm x, \qquad U_1(x) = \frac{k_1}{x^3} + k_2,$$

providing the following approximately invariant solution of (11.19):

$$u(t, x; \varepsilon) = \pm\frac{x}{t} + \varepsilon\left(\frac{k_1}{tx^3} + \frac{k_2}{t} \mp \frac{x}{6}\right). \tag{11.22}$$

Remark 11.5 In [23], Eq. (11.19) has been analyzed by means of the Baikov–Gazizov–Ibragimov approach for approximate symmetries. The solution approximately invariant with respect to the first-order approximate symmetry generated by

$$X = \left(t + \varepsilon\frac{t^2}{6}\right)\frac{\partial}{\partial t} - \left(u + \varepsilon\frac{tu}{3}\right)\frac{\partial}{\partial u} \tag{11.23}$$

there obtained was

$$u(t, x; \varepsilon) = \pm\left(\frac{x}{t} - \varepsilon\frac{x}{6}\right),$$

which is less general than (11.22). The approximate Lie generator (11.23) has the invariants

$$J_1 = x, \qquad J_2 = tu + \varepsilon\frac{t^2u}{6},$$

and the approximate invariant solutions are sought by setting $J_2 = \varphi(J_1)$, where the function φ is determined by satisfying Eq. (11.19). The solution (11.22) can be recovered by setting $J_2 = \varphi(J_1, \varepsilon)$ and then taking into account the expansion of u.

Example 11.13 Consider the Korteweg–deVries–Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + \varepsilon\frac{\partial^3 u}{\partial x^3} = 0,$$

and compute the solutions approximately invariant with respect to the Lie generator $\Xi = \Xi_1 + c\Xi_2 + \Xi_7$ (see (11.13)), where c is a constant, i.e.,

$$(1 + 2\varepsilon t)\frac{\partial u}{\partial t} + (c + \varepsilon x)\frac{\partial u}{\partial x} = -\varepsilon u_{(0)}.$$

Therefore, inserting $u = u_{(0)} + \varepsilon u_{(1)} + O(\varepsilon^2)$, we get the system

$$\begin{cases} \dfrac{\partial u_{(0)}}{\partial t} + c\dfrac{\partial u_{(0)}}{\partial x} = 0, \\ \dfrac{\partial u_{(1)}}{\partial t} + c\dfrac{\partial u_{(1)}}{\partial x} + 2t\dfrac{\partial u_{(0)}}{\partial t} + x\dfrac{\partial u_{(0)}}{\partial x} = -u_{(0)}, \end{cases}$$

whose integration gives

$$\begin{cases} u_{(0)}(t,x) = U_0(\omega), \\ u_{(1)}(t,x) = \left(\dfrac{3ct^2}{2} - tx\right) U_0'(\omega) - tU_0(\omega) + U_1(\omega), \end{cases}$$

where $\omega = x - ct$, and $U_0(\omega)$, $U_1(\omega)$ satisfy the following reduced system of ordinary differential equations:

$$\begin{aligned} &U_0'' - (U_0 - c)\, U_0' = 0, \\ &\left(U_1' + (c - U_0)(U_1 + U_0') + \omega U_0\right)' = 0, \end{aligned} \tag{11.24}$$

the prime ($'$) denoting the differentiation with respect to ω; by solving (11.24), we find the following first-order approximate invariant solution

$$\begin{aligned} u(t,x;\varepsilon) = {} & c - 2k_1 \tanh\left(k_1(x-ct) + k_2\right) \\ & + \varepsilon \left(\left(\frac{4\kappa_1^2 t - c(x-ct) + \kappa_3 - 1}{4\kappa_1} \sinh(2(\kappa_1(x-ct) + \kappa_2)) \right.\right. \\ & + 4\kappa_1^2 \log(\cosh(\kappa_1(x-ct) + \kappa_2)) - \frac{c(8\kappa_1^4 t^2 + 2\kappa_1^2 (x-ct)^2 + 1)}{8\kappa_1^2} \\ & \left.\left. + \frac{(4\kappa_1^2 t + \kappa_3 - 1)(x-ct)}{2} + \kappa_4 \right) \operatorname{sech}^2(\kappa_1(x-ct) + \kappa_2) \right. \\ & \left. - \frac{c(8\kappa_1^2 t - 1)}{4\kappa_1^2} + x \right), \end{aligned}$$

where k_1, k_2, k_3 and k_4 are constants.

Remark 11.6 The search of approximate solutions requires a careful analysis about their range of validity. This problem is not addressed here, as we limited ourselves to illustrate only the procedure for obtaining approximate invariant solutions.

11.6 Approximate Q-Conditional Symmetries

It is known that many equations of interest in concrete applications possess poor Lie symmetries [6]. More rich reductions leading to wide classes of exact solutions are possible by using the conditional symmetries, which are obtained by appending to the equations at hand some differential constraints. An important class is represented by Q-conditional symmetries, where the constraints to be added to the differential

equation at hand are the invariant surface conditions and their differential consequences. Remarkably, Q-conditional symmetries allow for symmetry reductions of differential equations and give explicit solutions not always obtainable with classical symmetries.

Now, combining the approximate symmetry theory introduced in [8] with the approach of conditional symmetries, we can define approximate Q-conditional symmetries [19].

Definition 11.13 (*Approximate Q-conditional symmetries*) Given the differential equation

$$\boldsymbol{\Delta} \approx \sum_{k=0}^{p} \varepsilon^k \widetilde{\boldsymbol{\Delta}}_{(k)} \left(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}^{(1)}_{(0)}, \dots, \mathbf{u}^{(r)}_{(0)}, \dots, \mathbf{u}_{(k)}, \mathbf{u}^{(1)}_{(k)}, \dots, \mathbf{u}^{(r)}_{(k)} \right) = \mathbf{0},$$

and the approximate invariant surface conditions

$$\mathbf{Q} \approx \sum_{k=0}^{p} \varepsilon^k \left(\sum_{i=1}^{n} \widetilde{\xi}_{(k)i}(\mathbf{x}, \mathbf{u}_{(0)}, \dots, \mathbf{u}_{(k)}) \frac{\partial \mathbf{u}}{\partial x_i} - \widetilde{\boldsymbol{\eta}}_{(k)}(\mathbf{x}, \mathbf{u}_{(0)}, \dots, \mathbf{u}_{(k)}) \right) = \mathbf{0},$$

the *approximate Q-conditional symmetries* of order p are found by requiring

$$\Xi^{(r)}(\boldsymbol{\Delta})\big|_{\mathcal{M}} = O(\varepsilon^{p+1}), \tag{11.25}$$

where $\mathcal{M}$ is the manifold of the jet space defined by the system of equations

$$\begin{aligned} &\Delta = O(\varepsilon^{p+1}), \qquad \mathbf{Q} = O(\varepsilon^{p+1}), \\ &\mathcal{D}_{j_1}(\mathbf{Q}) = \mathcal{D}_{j_1 j_2}(\mathbf{Q}) = \cdots = \mathcal{D}_{j_1 \dots j_{r-1}}(\mathbf{Q}) = O(\varepsilon^{p+1}), \end{aligned}$$

where $1 \le j_k \le n$, $\mathcal{D}_{j_1 j_2 \dots j_{r-1}} = \mathcal{D}_{j_1}(\mathcal{D}_{j_2 \dots j_{r-1}})$, and $\mathcal{D}_j$ denotes the total derivative with respect to x_j.

For the multiplication of the infinitesimals of a Q-conditional symmetry by an arbitrary smooth function $f(\mathbf{x}, \mathbf{u})$ gives still a Q-conditional symmetry, setting $k = \min\{i \in \{1, \dots, n\} : \xi_i \neq 0\}$, we can look for conditional symmetries such that the invariant surface conditions are divided by ξ_k.

For simplicity, hereafter we will consider the case $p = 1$; for higher values of p things are similar but at the cost of an increased amount of computations. If we want to determine the first-order approximate Q-conditional symmetries, by imposing (11.25), we obtain a very long invariance condition from which we can extract the determining equations (that in general are *nonlinear*); it is easy to recognize that the use of a computer algebra package may help in order to perform the lengthy and tedious calculations.

11.7 Approximate Conditionally Invariant Solutions

Consider the equation

$$\varepsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right) - \alpha u \frac{\partial u}{\partial x} + \beta u(1 - \gamma u) = 0, \tag{11.26}$$

where ε is a small constant parameter, that represents a hyperbolic version of the reaction–diffusion–convection equation [29]

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right) - \alpha u \frac{\partial u}{\partial x} + \beta u(1 - \gamma u) = 0, \tag{11.27}$$

α, β and γ being real positive constants. Equation (11.27) can model a population moving to domains with lower density faster than the overpopulation is reached and contains a logistic reaction term; it also includes another transport phenomenon, namely convection. The introduction of a term with a second-order time derivative yields the hyperbolic equation (11.26). Conditional symmetries as well as some exact solutions of Eq. (11.27) have been considered in [34]. A special class of approximate conditional symmetries of Eq. (11.26), and some associated approximate solutions have been obtained in [19].

It can be easily proved that Eq. (11.26) possesses only the exact symmetries corresponding to space and time translations. Also the admitted approximate symmetries of (11.26) reduce to the translation of independent variables. This means that either exact or approximate invariant solutions are only those depending on just one independent variable (t or x) as well as the ones describing traveling waves. More complex approximate symmetry reductions can be obtained by looking for approximate Q-conditional symmetries. We observe that the zeroth terms of approximate Q-conditional symmetries of (11.26) yield particular exact Q-conditional symmetries of (11.27).

Expanding $u(t, x; \varepsilon)$ at first order in ε,

$$u(t, x; \varepsilon) = u_{(0)}(t, x) + \varepsilon u_{(1)}(t, x) + O(\varepsilon^2),$$

we consider the approximate Q-conditional symmetries of (11.26) corresponding to the approximate generator [8]

$$\begin{aligned}\Xi \approx & \frac{\partial}{\partial t} + \left(\xi_{(0)}(t, x, u_{(0)}) + \varepsilon\left(\xi_{(1)}(t, x, u_{(0)}) + \frac{\partial \xi_{(0)}(t, x, u_{(0)})}{\partial u_{(0)}} u_{(1)}\right)\right) \frac{\partial}{\partial x} \\ & + \left(\eta_{(0)}(t, x, u_{(0)}) + \varepsilon\left(\eta_{(1)}(t, x, u_{(0)}) + \frac{\partial \eta_{(0)}(t, x, u_{(0)})}{\partial u_{(0)}} u_{(1)}\right)\right) \frac{\partial}{\partial u}.\end{aligned}$$

By using the approach so far described, a set of determining equations is obtained, and various approximate Q-conditional symmetries can be recovered. Hereafter, we

limit ourselves to list those leading to representations of the approximate solutions which can be defined for all t. All the required computations have been done with the help of the REDUCE [21] program ReLie [32].

The representations of the approximate solutions have to satisfy the approximate invariant surface conditions; by separating terms at each order of ε, we have:

$$\begin{aligned} &\frac{\partial u_{(0)}}{\partial t} + \xi_{(0)} \frac{\partial u_{(0)}}{\partial x} - \eta_{(0)} = 0, \\ &\frac{\partial u_{(1)}}{\partial t} + \xi_{(0)} \frac{\partial u_{(1)}}{\partial x} + \xi_{(1)} \frac{\partial u_{(0)}}{\partial x} + \left(\frac{\partial \xi_{(0)}}{\partial u_{(0)}} \frac{\partial u_{(0)}}{\partial x} - \frac{\partial \eta_{(0)}}{\partial u_{(0)}} \right) u_{(1)} = \eta_{(1)}. \end{aligned} \tag{11.28}$$

Then, by inserting the solutions of (11.28) into Eq. (11.26), and solving the associated reduced systems of ordinary differential equations, some approximate Q-conditional solutions, for every set of approximate generators, can be explicitly determined.

In order to classify the approximate Q-conditional symmetries admitted by Eq. (11.26) and the corresponding approximate solutions, three cases need to be distinguished:

(i) $8\beta\gamma - \alpha^2 = -\delta^2$;
(ii) $8\beta\gamma - \alpha^2 = 0$;
(iii) $8\beta\gamma - \alpha^2 = \delta^2$.

For each case, because the determining equations are nonlinear, various sets of approximate Q-conditional symmetries are recovered, as shown below, where the generators of approximate Q-conditional symmetries we found, the consequent representations of the approximate solutions, and the result of the integration of the reduced ordinary differential equations are listed. Often we are able to give the most general solution of the reduced equations. In passing, we observe that for $\varepsilon = 0$ the obtained solutions satisfy the unperturbed parabolic reaction–diffusion–convection Eq. (11.27).

The Case (i): $8\beta\gamma - \alpha^2 = -\delta^2$

In this case, six different sets of approximate Q-conditional symmetries of (11.26) have been determined.

Using the infinitesimals

$$\begin{aligned} &\xi_{(0)} = 0, \qquad \eta_{(0)} = -\frac{\beta\kappa_1 \exp(\beta t)}{\kappa_1 \exp(\beta t) + 1} u_{(0)}, \qquad \xi_{(1)} = \frac{4\kappa_2}{\kappa_1 \exp(\beta t) + 1}, \\ &\eta_{(1)} = \frac{\exp(-\beta t - \frac{\alpha+\delta}{2} x)}{(\kappa_1 \exp(\beta t) + 1)^2} \frac{\kappa_3 \exp(\delta x) + \kappa_4}{u_{(0)}} - \frac{\exp(\beta t)}{(\kappa_1 \exp(\beta t) + 1)^2} \\ &\quad \times \left(2\beta^2 \kappa_1 \log(\kappa_1 \exp(\beta t) + 1) + \beta^2 \kappa_1 (\kappa_1 \exp(\beta t) - \beta t) + \kappa_1 \kappa_2 \alpha - \kappa_5 \right) u_{(0)}, \end{aligned}$$

and integrating the approximate invariant surface conditions (11.28), we get the following representation of the solution:

$$
\begin{aligned}
u_{(0)}(t,x) &= \frac{U_0(x)}{\kappa_1 \exp(\beta t)+1},\\
u_{(1)}(t,x) &= \frac{1}{\kappa_1 \exp(\beta t)+1}\left(-\frac{\exp\left(-\beta t-\frac{\alpha+\delta}{2}x\right)}{\beta}\frac{\kappa_3\exp(\delta x)+\kappa_4}{U_0(x)}\right.\\
&\quad -\frac{\beta^2\kappa_1(\kappa_1\exp(\beta t)(2\log(\kappa_1\exp(\beta t)+1)-\beta t)-1)-\kappa_1\kappa_2\alpha+\kappa_5}{\beta\kappa_1(\kappa_1\exp(\beta t)+1)}U_0(x)\\
&\quad \left.+4\,\frac{\kappa_2}{\beta}(\log(\kappa_1\exp(\beta t)+1)-\beta t)U_0'(x)+U_1(x)\right),
\end{aligned}
$$

where $U_0(x)$ and $U_1(x)$ are unknown functions to be determined.

Now, by inserting this solution into the system (11.26), and solving the corresponding reduced system, the associated approximate Q-conditional solution has the form:

$$
\begin{aligned}
u(t,x;\varepsilon) &= \frac{8\beta}{(\alpha^2-\delta^2)(\kappa_1\exp(\beta t)+1)}\\
&\quad +\varepsilon\left(-\frac{\alpha^2-\delta^2}{8\beta^2}\exp\left(-\beta t-\frac{\alpha+\delta}{2}x\right)(\kappa_3\exp(\delta x)+\kappa_4)\right.\\
&\quad -\frac{8\exp(\beta t)\left(\beta^2\kappa_1(2\log(\kappa_1\exp(\beta t)+1)-\beta t+1)+\kappa_1\kappa_2\alpha-\kappa_5\right)}{(\alpha^2-\delta^2)(\kappa_1\exp(\beta t)+1)^2}\\
&\quad \left.+\frac{\exp\left(-(\alpha+\theta)x/2\right)(c_1\exp(\theta x)+c_2)}{\kappa_1\exp(\beta t)+1}\right),
\end{aligned}
$$

where $\theta=\sqrt{\dfrac{\alpha^2+\delta^2}{2}}$, whereas c_1 and c_2 are arbitrary constants.

By integrating the approximate invariant surface conditions (11.28), along with

$$
\begin{aligned}
\xi_{(0)} &= 0, \qquad \eta_{(0)} = -\beta u_{(0)},\\
\xi_{(1)} &= 4\exp(-\beta t-\delta x)\left(\kappa_1\exp(2\delta x)+\kappa_2\exp(\delta x)+\kappa_3\right)\\
&\quad +4\exp\left(\beta t+\frac{\alpha-\delta}{2}x\right)(\kappa_4\exp(\delta x)+\kappa_5)u_{(0)}^2,\\
\eta_{(1)} &= \exp\left(-3\beta t-\frac{\alpha+\delta}{2}x\right)\frac{\kappa_6\exp(\delta x)+\kappa_7}{u_{(0)}}-\exp(-\beta t-\delta x)\\
&\quad \times\left(\kappa_1(\alpha-\delta)\exp(2\delta x)+(\kappa_2\alpha+\kappa_8)\exp(\delta x)+\kappa_3(\alpha+\delta)\right)u_{(0)}\\
&\quad -\beta^2 u_{(0)}-\exp\left(\beta t+\frac{\alpha-\delta}{2}x\right)(\kappa_4(\alpha-\delta)\exp(\delta x)+\kappa_5(\alpha+\delta))\,u_{(0)}^3,
\end{aligned}
$$

we get

$$
\begin{aligned}
u_{(0)}(t,x) &= \exp(-\beta t)U_0(x),\\
u_{(1)}(t,x) &= -\frac{\exp\left(-2\beta t-\frac{\alpha+\delta}{2}x\right)}{\beta}\frac{\kappa_6\exp(\delta x)+\kappa_7}{U_0(x)}\\
&\quad+\frac{\exp(-2\beta t)}{\beta}\left(\kappa_1(\alpha-\delta)\exp(\delta x)+\kappa_3(\alpha+\delta)\exp(-\delta x)\right.\\
&\quad\left.+\kappa_2\alpha+\kappa_8-\exp(\beta t)\beta^3 t\right)U_0(x)+\exp(-\beta t)U_1(x)\\
&\quad+\frac{\exp\left(-2\beta t+\frac{\alpha-\delta}{2}x\right)}{\beta}\left(\kappa_4(\alpha-\delta)\exp(\delta x)+\kappa_5(\alpha+\delta)\right)U_0^3(x)\\
&\quad+4\frac{\exp(-2\beta t-\delta x)}{\beta}\left(\kappa_1\exp(2\delta x)+\kappa_2\exp(\delta x)+\kappa_3\right)U_0'(x)\\
&\quad+4\frac{\exp\left(-2\beta t+\frac{\alpha-\delta}{2}x\right)}{\beta}(\kappa_4\exp(\delta x)+\kappa_5)U_0^2(x)U_0'(x),
\end{aligned}
$$

and the corresponding approximate Q-conditional solution is written as

$$
\begin{aligned}
u(t,x;\varepsilon) &= c_1\exp\left(-\beta t-\frac{\alpha+\delta}{4}x\right)\sqrt{\exp(\delta x)+c_2}\\
&\quad+\varepsilon\left(\frac{\exp(-\beta t)}{c_1^2\sqrt{c_2}\delta\sqrt{\exp(\delta x)+c_2}}\left(-\frac{4}{(\alpha-3\delta)(\alpha-\delta)}\right.\right.\\
&\quad\times\left(2c_1^2c_2^2\delta(\kappa_1(\alpha-3\delta)-2c_1^2\kappa_4\delta)+(\alpha-\delta)(2c_1^2\kappa_3\delta-\kappa_7)\right.\\
&\quad\left.+\ c_2(2c_1^2\delta(2(\kappa_2+c_1^2\kappa_5)\delta-\kappa_2\alpha+\kappa_8)+\kappa_6(\alpha-3\delta))\right)\\
&\quad\times{}_2F_1\left(\frac{1}{2},\frac{\alpha-3\delta}{4\delta},\frac{\alpha+\delta}{4\delta};-\frac{\exp(\delta x)}{c_2}\right)-\frac{4}{\alpha+\delta}\\
&\quad\times\left(c_1^2((c_2(2c_1^2(c_2\kappa_4-\kappa_5)+\kappa_2)-2\kappa_3)\delta-c_2\kappa_8)+\kappa_7\right)\\
&\quad\times{}_2F_1\left(\frac{1}{2},\frac{\alpha+\delta}{4\delta},\frac{\alpha+5\delta}{4\delta};-\frac{\exp(\delta x)}{c_2}\right)-4\frac{\exp(\delta x)}{\alpha+5\delta}\\
&\quad\times\left(c_1^2((2c_1^2(c_2\kappa_4-\kappa_5)-\kappa_2+2c_2\kappa_1)\delta-\kappa_8)+\kappa_6\right)\\
&\quad\left.\times\ {}_2F_1\left(\frac{1}{2},\frac{\alpha+5\delta}{4\delta},\frac{\alpha+9\delta}{4\delta};-\frac{\exp(\delta x)}{c_2}\right)\right)\\
&\quad+\frac{\exp\left(-2\beta t-\frac{\alpha+\delta}{4}x\right)}{c_1\beta\sqrt{\exp(\delta x)+c_2}}\\
&\quad\times\left(-\exp(\delta x)\left(c_1^2((2c_1^2(c_2\kappa_4-\kappa_5)+2c_2\kappa_1-\kappa_2)\delta-\kappa_8)+\kappa_6\right)\right.\\
&\quad\left.-\ c_1^2((c_2(2c_1^2(c_2\kappa_4-\kappa_5)+\kappa_2)-2\kappa_3)\delta-c_2\kappa_8)-\kappa_7\right)\\
&\quad+\frac{\exp\left(-\beta t-\frac{\alpha+\delta}{4}x\right)}{c_1^2\delta(\alpha-\delta)(\exp(\delta x)+c_2)}\left(c_1^2(\alpha-\delta)((c_4-c_1\delta\beta^2 t)\exp(\delta x)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+(c_3 - c_1c_2\beta^2 t)\delta)\sqrt{\exp(\delta x) + c_2} + 4\exp\left(\frac{\alpha + 5\delta}{4}x\right)\\
&\times \left(c_1^2((2c_1^2(c_2\kappa_4 - \kappa_5) + 2c_2\kappa_1 - \kappa_2)\delta - \kappa_8) + \kappa_6\right)\\
&+\frac{4c_2\exp(-\beta t)}{\exp(\delta x) + c_2}\ \frac{c_1^2((2(c_1^2(c_2\kappa_4 - \kappa_5) + c_2\kappa_1) - \kappa_2)\delta - \kappa_8) + \kappa_6}{c_1^2\delta(\alpha - \delta)}\Bigg),
\end{aligned}
$$

${}_2F_1$ being the *hypergeometric function* defined as

$$
{}_2F_1(a, b, c; x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!},
$$

where

$$
(a)_0 = 1, \quad (a)_k = \prod_{i=0}^{k-1}(i + a).
$$

Taking

$$
\begin{aligned}
\xi_{(0)} &= -\frac{4\beta}{\alpha + \delta}, \qquad \eta_{(0)} = 0,\\
\xi_{(1)} &= 4\left(\kappa_1\exp(\beta t) + \kappa_2\exp\left(\frac{\beta(\alpha - 7\delta)}{\alpha + \delta}t - \delta x\right) + \kappa_3\right),\\
\eta_{(1)} &= -(\alpha + \delta)\exp\left(\frac{(\alpha^2 - 8\alpha\delta - \delta^2)\beta t - \alpha^2\delta x}{\alpha^2 - \delta^2}\right)\\
&\quad\times\left(\kappa_1\exp\left(\frac{\alpha\delta(8\beta t + \alpha x)}{\alpha^2 - \delta^2}\right) + \kappa_2\exp\left(\frac{\delta^2(8\beta t + \delta x)}{\alpha^2 - \delta^2}\right)\right)u_{(0)},
\end{aligned}
$$

the integration of the conditions (11.28) gives the following representation of the solution:

$$
\begin{aligned}
u_{(0)}(t, x) &= U_0(\omega),\\
u_{(1)}(t, x) &= -\left(\frac{\kappa_1}{\beta}\exp(\beta t) + \frac{\alpha + \delta}{\beta(\alpha - 3\delta)}\kappa_2\exp\left(\frac{\beta(\alpha - 3\delta)}{\alpha + \delta}t - \delta\omega\right)\right)\\
&\quad\times\left((\alpha + \delta)U_0(\omega) + 4U_0'(\omega)\right) - 4\kappa_3 t U_0'(\omega) + U_1(\omega),
\end{aligned}
$$

with $\omega = \dfrac{4\beta}{\alpha + \delta}t + x$.

The associated approximate Q-conditional solution is

$$
\begin{aligned}
u(t, x; \varepsilon) &= \frac{8\beta}{\alpha^2 - \delta^2}\\
&\quad+\varepsilon\left(-8\frac{(3\delta - \alpha)\kappa_1\exp(\beta t) - (\alpha + \delta)\kappa_2\exp\left(\frac{\beta(\alpha - 7\delta)}{\alpha + \delta}t - \delta x\right)}{(3\delta - \alpha)(\alpha - \delta)}\right.
\end{aligned}
$$

$$+\exp\left(-\frac{\alpha+\delta+\sqrt{(3\delta-\alpha)(\alpha+\delta)}}{4}\left(\frac{4\beta}{\alpha+\delta}t+x\right)\right)$$
$$\times\left(c_1\exp\left(\frac{\sqrt{(3\delta-\alpha)(\alpha+\delta)}}{2}\left(\frac{4\beta}{\alpha+\delta}t+x\right)\right)+c_2\right)\Bigg).$$

Using the infinitesimals

$$\begin{aligned}&\xi_{(0)}=\kappa_1,\qquad \eta_{(0)}=0,\\&\xi_{(1)}=\kappa_2\exp(\beta t)+\kappa_3,\qquad \eta_{(1)}=\frac{\kappa_2}{\kappa_1}\beta\exp(\beta t)u_{(0)},\end{aligned}\tag{11.29}$$

with $\kappa_1\neq 0, -\frac{4\beta}{\alpha+\delta}, \frac{4\beta}{\alpha-\delta}$, the integration of the conditions (11.28) yields

$$\begin{aligned}&u_{(0)}(t,x)=U_0(\omega),\\&u_{(1)}(t,x)=\tfrac{\kappa_2}{\kappa_1}\exp(\beta t)U_0(\omega)-\left(\tfrac{\kappa_2}{\beta}\exp(\beta t)+\kappa_3 t\right)U_0'(\omega)+U_1(\omega),\end{aligned}\tag{11.30}$$

with $\omega=-\kappa_1 t+x$. The corresponding approximate Q-conditional solution we find turns out to be

$$\begin{aligned}u(t,x;\varepsilon)=&\frac{8\beta}{\alpha^2-\delta^2}+\varepsilon\left(\frac{8\beta}{\alpha^2-\delta^2}\frac{\kappa_2}{\kappa_1}\exp(\beta t)+\exp\left(\left(\frac{\kappa_1(\alpha^2-\delta^2)}{8\beta}\right.\right.\right.\\&\left.+\alpha+\sqrt{\frac{(\kappa_1(\alpha^2-\delta^2)+8\alpha\beta)^2}{64\beta^2}}-\frac{\alpha^2-\delta^2}{2}\right)\frac{\kappa_1 t-x}{2}\Bigg)\\&\times\left(c_1\exp\left(\sqrt{\frac{(\kappa_1(\alpha^2-\delta^2)+8\alpha\beta)^2}{64\beta^2}}-\frac{\alpha^2-\delta^2}{2}(-\kappa_1 t+x)\right)+c_2\right)\Bigg).\end{aligned}$$

Moreover, along with the choice

$$\begin{aligned}\xi_{(0)}=&-\frac{4\beta}{\alpha-\delta},\qquad \eta_{(0)}=0,\\\xi_{(1)}=&4\left(\kappa_1\exp(\beta t)+\kappa_2\exp\left(\frac{\beta(\alpha+7\delta)}{\alpha-\delta}t+\delta x\right)+\kappa_3\right),\\\eta_{(1)}=&-(\alpha-\delta)\exp\left(-\frac{\delta^2}{\alpha-\delta}x\right)\\&\times\left(\kappa_1\exp\left(\beta t+\frac{\delta^2}{\alpha-\delta}x\right)+\kappa_2\exp\left(\frac{\beta(\alpha+7\delta)t+\alpha\delta x}{\alpha-\delta}\right)\right)u_{(0)},\end{aligned}$$

we obtain the following representation of the solution:

$$
\begin{aligned}
u_{(0)}(t,x) &= U_0(\omega),\\
u_{(1)}(t,x) &= \left(\frac{\kappa_1(\delta-\alpha)}{\beta}\exp(\beta t) - \frac{\kappa_2(\alpha-\delta)^2}{\beta(\alpha+3\delta)}\exp\left(\frac{\beta(\alpha+3\delta)}{\alpha-\delta}t+\delta\omega\right)\right)U_0(\omega)\\
&\quad - 4\left(\frac{\kappa_1}{\beta}\exp(\beta t) + \frac{\alpha-\delta}{\beta(\alpha+3\delta)}\kappa_2\exp\left(\frac{\beta(\alpha+3\delta)}{\alpha-\delta}t+\delta\omega\right)+\kappa_3 t\right)U_0'(\omega)\\
&\quad + U_1(\omega),
\end{aligned}
$$

with $\omega = \dfrac{4\beta}{\alpha-\delta}t + x$. An approximate Q-conditional solution is given by

$$
u(t,x;\varepsilon) = \frac{8\beta}{\alpha^2-\delta^2} - 8\varepsilon\frac{(3\delta+\alpha)\kappa_1\exp(\beta t) + (\alpha-\delta)\kappa_2\exp\left(\frac{\beta(\alpha+7\delta)}{\alpha-\delta}t+\delta x\right)}{(3\delta+\alpha)(\alpha+\delta)}.
$$

Finally, with the approximate infinitesimals

$$
\begin{aligned}
\xi_{(0)} &= \beta\kappa_1\exp(-\beta t), \qquad \eta_{(0)} = -\beta u_{(0)},\\
\xi_{(1)} &= (\kappa_2 - \kappa_1\beta^3 t)\exp(-\beta t) + \kappa_1(\kappa_3 - \kappa_1\alpha\beta^2)\exp(-2\beta t)\\
&\quad - \beta^2\frac{\alpha^2-\delta^2}{8}\kappa_1^3\exp(-3\beta t),\\
\eta_{(1)} &= -\beta^3\kappa_1^2\exp(-2\beta t) + \left(\frac{\beta^2(\alpha^2-\delta^2)}{4}\kappa_1^2\exp(-2\beta t) - \kappa_3\exp(-\beta t) - \beta^2\right)u_{(0)},
\end{aligned}
$$

we have the representation of the solution in the form

$$
u_{(0)}(t,x) = \exp(-\beta t)U_0(\omega), \tag{11.31a}
$$

$$
\begin{aligned}
u_{(1)}(t,x) &= -\frac{\exp(-3\beta t)}{\beta}\left(\beta^2\kappa_1^2\frac{\alpha^2-\delta^2}{8} - \kappa_3\exp(\beta t) + \exp(2\beta t)\beta^3 t\right)U_0(\omega)\\
&\quad - \frac{\exp(-4\beta t)}{\beta}\Big(\exp(2\beta t)(\beta^2\kappa_1(\beta t+1)-\kappa_2)\\
&\quad + \frac{\kappa_1}{2}(\kappa_1\alpha\beta^2-\kappa_3)\exp(\beta t) + \beta^2\kappa_1^3\frac{\alpha^2-\delta^2}{24}\Big)U_0'(\omega)\\
&\quad + \beta^2\kappa_1^2\exp(-2\beta t) + \exp(-\beta t)U_1(\omega),
\end{aligned} \tag{11.31b}
$$

with $\omega = \kappa_1\exp(-\beta t) + x$. Such a representation does not allow us to solve in general the corresponding reduced system of ordinary differential equations; however, by choosing $\kappa_1 = 0$, we are able to get the following approximate Q-conditional solution:

$$
\begin{aligned}
u(t,x;\varepsilon) = & c_1 \exp\left(-\beta t - \frac{\alpha+\delta}{4}x\right)\sqrt{\exp(\delta x)+c_2} \\
& + \varepsilon\left(\frac{\sqrt{c_2}\exp(-\beta t)}{\sqrt{\exp(\delta x)+c_2}}\right. \\
& \times\left(\frac{4(\kappa_2(\alpha-\delta)-2\kappa_3)}{\alpha^2-4\alpha\delta+3\delta^3}\,{}_2F_1\left(\frac{1}{2},\frac{\alpha-3\delta}{4\delta},\frac{\alpha+\delta}{4\delta};-\frac{\exp(\delta x)}{c_2}\right)\right. \\
& - \frac{\kappa_2(\alpha+\delta)-4\kappa_3}{\delta(\alpha+\delta)}\,{}_2F_1\left(\frac{1}{2},\frac{\alpha+\delta}{4\delta},\frac{\alpha+5\delta}{4\delta};-\frac{\exp(\delta x)}{c_2}\right) \\
& \left.\left. - \frac{\exp(\delta x)(\kappa_2(\alpha-\delta)-4\kappa_3)}{c_2\delta(\alpha+5\delta)}\,{}_2F_1\left(\frac{1}{2},\frac{\alpha+5\delta}{4\delta},\frac{\alpha+9\delta}{4\delta};-\frac{\exp(\delta x)}{c_2}\right)\right)\right. \\
& - c_1\frac{\exp\left(-2\beta t-\frac{\alpha+\delta}{4}x\right)}{4\beta\sqrt{\exp(\delta x)+c_2}}\left(\exp(\delta x)(\kappa_2(\alpha-\delta)-4\kappa_3)+c_2(\kappa_2(\alpha+\delta)-4\kappa_3)\right) \\
& - \frac{\exp\left(-\beta t-\frac{\alpha+\delta}{4}x\right)}{\delta\sqrt{\exp(\delta x)+c_2}}\left(\exp(\delta x)(c_1\delta\beta^2 t-c_4)+(c_1c_2\beta^2 t-c_3)\delta\right) \\
& \left. + \frac{\exp(-\beta t)}{\delta(\alpha-\delta)}(\kappa_2(\alpha-\delta)-4\kappa_3)\right).
\end{aligned}
$$

The Case (ii): $8\beta\gamma - \alpha^2 = 0$

In this case, we are able to recover five different sets of approximate Q-conditional symmetries of (11.26).

Using

$$
\begin{aligned}
\xi_{(0)} &= 0, \qquad \eta_{(0)} = -\frac{\beta\kappa_1\exp(\beta t)}{\kappa_1\exp(\beta t)+1}u_{(0)}, \\
\xi_{(1)} &= \frac{4\kappa_2}{\kappa_1\exp(\beta t)+1}, \\
\eta_{(1)} &= \frac{\exp(-\beta t-\frac{\alpha}{2}x)}{(\kappa_1\exp(\beta t)+1)^2}\frac{\kappa_3 x+\kappa_4}{u_{(0)}} - \frac{\exp(\beta t)}{(\kappa_1\exp(\beta t)+1)^2}\Big(\beta^2\kappa_1(2\log(\kappa_1\exp(\beta t)+1) \\
&\quad + \kappa_1\exp(\beta t)-\beta t)+\kappa_1\kappa_2\alpha-\kappa_5\Big)u_{(0)},
\end{aligned}
$$

and integrating the approximate invariant surface conditions (11.28), we get:

$$
\begin{aligned}
u_{(0)}(t,x) &= \frac{U_0(x)}{\kappa_1\exp(\beta t)+1}, \\
u_{(1)}(t,x) &= \frac{1}{\kappa_1\exp(\beta t)+1}\left(-\frac{\exp\left(-\beta t-\frac{\alpha}{2}x\right)}{\beta}\frac{\kappa_3 x+\kappa_4}{U_0(x)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\beta^2\kappa_1(\kappa_1\exp(\beta t)(2\log(\kappa_1\exp(\beta t)+1)-\beta t)-1)-\kappa_1\kappa_2\alpha+\kappa_5}{\beta\kappa_1(\kappa_1\exp(\beta t)+1)}U_0(x)\\
&+ 4\,\frac{\kappa_2}{\beta}(\log(\kappa_1\exp(\beta t)+1)-\beta t)U_0'(x)+U_1(x)\Big).
\end{aligned}
$$

Solving the reduced ordinary differential equations, the approximate Q-conditional solution has the form:

$$
\begin{aligned}
u(t,x;\varepsilon) = {}& \frac{8\beta}{\alpha^2(\kappa_1\exp(\beta t)+1)}\\
&+\varepsilon\left(-\frac{\alpha^2}{8\beta^2}\exp\left(-\beta t-\frac{\alpha}{2}x\right)(\kappa_3 x+\kappa_4)\right.\\
&-\frac{8\exp(\beta t)\left(\beta^2\kappa_1(2\log(\kappa_1\exp(\beta t)+1)-\beta t+1)+\kappa_1\kappa_2\alpha-\kappa_5\right)}{\alpha^2(\kappa_1\exp(\beta t)+1)^2}\\
&\left.+\frac{\exp\left(-\left(1+\frac{\sqrt{2}}{2}\right)\frac{\alpha x}{2}\right)}{\kappa_1\exp(\beta t)+1}\left(c_1\exp\left(\frac{\sqrt{2}}{2}\alpha x\right)+c_2\right)\right),
\end{aligned}
$$

with c_1 and c_2 arbitrary constants.

Taking

$$
\begin{aligned}
&\xi_{(0)} = 0, \qquad \eta_{(0)} = -\beta u_{(0)},\\
&\xi_{(1)} = 4\exp(-\beta t)\left(\kappa_1 x^2+\kappa_2 x+\kappa_3\right)+4\exp\left(\beta t+\frac{\alpha}{2}x\right)(\kappa_4 x+\kappa_5)u_{(0)}^2,\\
&\eta_{(1)} = \exp\left(-3\beta t-\frac{\alpha}{2}x\right)\frac{\kappa_6 x+\kappa_7}{u_{(0)}}\\
&\quad -\exp(-\beta t)(\kappa_1(\alpha x-2)x+\kappa_2(\alpha x-1)+\kappa_3\alpha+\kappa_8)u_{(0)}\\
&\quad -\beta^2 u_{(0)}-\exp\left(\beta t+\frac{\alpha}{2}x\right)(\kappa_4(\alpha x-2)+\kappa_5\alpha)u_{(0)}^3,
\end{aligned}
$$

the integration of the approximate invariant surface conditions (11.28) gives

$$
\begin{aligned}
u_{(0)}(t,x) &= \exp(-\beta t)U_0(x),\\
u_{(1)}(t,x) &= -\frac{\exp\left(-2\beta t-\frac{\alpha}{2}x\right)(\kappa_6 x+\kappa_7)}{\beta U_0(x)}\\
&\quad+\frac{\exp(-2\beta t)}{\beta}(\kappa_1(\alpha x-2)x+\kappa_2(\alpha x-1)+\kappa_3\alpha+\kappa_8\\
&\quad-\exp(\beta t)\beta^3 t)U_0(x)+\frac{\exp\left(-2\beta t+\frac{\alpha x}{2}\right)}{\beta}(\kappa_4(\alpha x-2)+\kappa_5\alpha)U_0^3(x)
\end{aligned}
$$

$$+ 4\frac{\exp(-2\beta t)}{\beta}\left(\kappa_1 x^2 + \kappa_2 x + \kappa_3\right) U_0'(x)$$
$$+ 4\frac{\exp\left(-2\beta t + \frac{\alpha x}{2}\right)}{\beta}(\kappa_4 x + \kappa_5) U_0^2(x) U_0'(x) + \exp(-\beta t) U_1(x),$$

and the corresponding approximate Q-conditional solution turns out to be

$$\begin{aligned}
u(t,x;\varepsilon) = {} & c_1 \exp\left(-\beta t - \frac{\alpha x}{4}\right)\sqrt{x} \\
& + \varepsilon\left(\frac{\exp\left(-2\beta t - \alpha x/4\right)}{c_1\beta\sqrt{x}}\left((c_1^2(\kappa_2 + 2c_1^2\kappa_5 + \kappa_8) - \kappa_6)x + 2c_1^2\kappa_3 - \kappa_7\right)\right. \\
& + \exp\left(-\beta t - \frac{\alpha x}{4}\right)\left(\frac{(c_3 - c_1\beta^2 t)x + c_2}{\sqrt{x}}\right. \\
& + \frac{2\sqrt{\pi}}{c_1^2\alpha^2\sqrt{\alpha}}\left((2c_1^2(\kappa_2 - \kappa_3\alpha + 2c_1^2\kappa_5 + \kappa_8) - 2\kappa_6 + \kappa_7\alpha)\alpha x\right. \\
& + \left.\left. 4c_1^2(3(\kappa_2 + 2c_1^2\kappa_5 + \kappa_8) - \kappa_3\alpha) - 12\kappa_6 + 2\kappa_7\alpha\right)\frac{\operatorname{erfi}\left(\sqrt{\alpha x}/2\right)}{\sqrt{x}}\right) \\
& + \left.\left.\frac{4\exp(-\beta t)}{c_1^2\alpha^2}\left(2c_1^2(\kappa_3\alpha - 3(\kappa_2 + 2c_1^2\kappa_5 + \kappa_8)) + 6\kappa_6 - \kappa_7\alpha\right)\right),
\end{aligned}$$

where $\operatorname{erfi}\left(\frac{\sqrt{\alpha x}}{2}\right)$ is the *imaginary error function*, defined as

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(t^2\right) dt.$$

Choosing

$$\begin{aligned}
\xi_{(0)} &= -\frac{4\beta}{\alpha}, \qquad \eta_{(0)} = 0, \\
\xi_{(1)} &= 4\left(\exp(\beta t)(\kappa_1\alpha + \kappa_2\beta(8\beta t + \alpha x - 8)) + \kappa_3\right), \\
\eta_{(1)} &= -\alpha\exp(\beta t)(\kappa_1\alpha + \kappa_2\beta(8\beta t + \alpha x - 8))u_{(0)},
\end{aligned}$$

and integrating the conditions (11.28), we get the following representation of the solution:

$$\begin{aligned}
u_{(0)}(t,x) =& U_0(\omega), \\
u_{(1)}(t,x) =& -\exp(\beta t)\left(\kappa_1\frac{\alpha}{\beta} + \kappa_2(4\beta t + \alpha\omega - 12)\right) \\
& \times\left(\alpha U_0(\omega) + 4U_0'(\omega)\right) - 4\kappa_3 t U_0'(\omega) + U_1(\omega),
\end{aligned}$$

with $\omega = \dfrac{4\beta}{\alpha}t + x$. The associated approximate Q-conditional solution is

$$u(t,x;\varepsilon) = \frac{8\beta}{\alpha^2} + \varepsilon\left(-8\exp(\beta t)\left(\kappa_1 + \kappa_2\frac{\beta}{\alpha}(8\beta t + \alpha x - 12)\right)\right.$$
$$\left. + \exp\left(-\beta t - \frac{\alpha x}{4}\right)\left(c_1\sin\left(\beta t + \frac{\alpha x}{4}\right) + c_2\cos\left(\beta t + \frac{\alpha x}{4}\right)\right)\right).$$

Another set of the approximate generators writes as (11.29), and the corresponding approximate Q-conditional solution is given by

$$u(t,x;\varepsilon) = \frac{8\beta}{\alpha^2} + \varepsilon\left(\frac{8\beta}{\alpha^2}\frac{\kappa_2}{\kappa_1}\exp(\beta t)\right.$$
$$+\exp\left(\left(\frac{\kappa_1\alpha^2}{8\beta^2} + \alpha + \sqrt{\frac{\alpha^2(\kappa_1\alpha + 8\beta)^2}{64\beta^2} - \frac{\alpha^2}{2}}\right)\frac{\kappa_1 t - x}{2}\right)$$
$$\left.\times\left(c_1\exp\left(\sqrt{\frac{\alpha^2(\kappa_1\alpha + 8\beta)^2}{64\beta^2} - \frac{\alpha^2}{2}}(-\kappa_1 t + x)\right) + c_2\right)\right).$$

Finally, using

$$\xi_{(0)} = \beta\kappa_1\exp(-\beta t), \qquad \eta_{(0)} = -\beta u_{(0)},$$
$$\xi_{(1)} = (\kappa_2 - \kappa_1\beta^3 t)\exp(-\beta t) + \kappa_1(\kappa_3 - \kappa_1\alpha\beta^2)\exp(-2\beta t) - \kappa_1^3\frac{\alpha^2\beta^2}{8}\exp(-3\beta t),$$
$$\eta_{(1)} = -\beta^3\kappa_1^2\exp(-2\beta t) + \left(\kappa_1^2\frac{\alpha^2\beta^2}{4}\exp(-2\beta t) - \kappa_3\exp(-\beta t) - \beta^2\right)u_{(0)}$$

we recover a representation of the solution obtainable from (11.31) by setting $\delta = 0$, i.e.,

$$u_{(0)}(t,x) = \exp(-\beta t)U_0(\omega),$$
$$u_{(1)}(t,x) = -\frac{\exp(-3\beta t)}{\beta}\left(\kappa_1^2\frac{\alpha^2\beta^2}{8} - \kappa_3\exp(\beta t) + \exp(2\beta t)\beta^3 t\right)U_0(\omega)$$
$$-\frac{\exp(-4\beta t)}{\beta}\left(\exp(2\beta t)(\beta^2\kappa_1(\beta t + 1) - \kappa_2)\right.$$
$$\left.+\frac{\kappa_1}{2}(\kappa_1\alpha\beta^2 - \kappa_3)\exp(\beta t) + \kappa_1^3\frac{\alpha^2\beta^2}{24}\right)U_0'(\omega)$$
$$+\beta^2\kappa_1^2\exp(-2\beta t) + \exp(-\beta t)U_1(\omega),$$

where $\omega = \kappa_1\exp(-\beta t) + x$.

Also in this case, we cannot solve in general the corresponding reduced system; nevertheless, by taking $\kappa_1 = 0$, we obtain the following approximate Q-conditional solution:

$$\begin{aligned} u(t,x;\varepsilon) = {} & c_1 \exp\left(-\beta t - \frac{\alpha x}{4}\right)\sqrt{x} \\ & + \varepsilon\left(\frac{c_1}{4\beta}\exp\left(-2\beta t - \frac{\alpha x}{4}\right)\frac{(4\kappa_3 - \kappa_2\alpha)x + 2\kappa_2}{\sqrt{x}}\right. \\ & + \frac{\exp\left(-\beta t - \frac{\alpha x}{4}\right)}{\alpha^2}\left(\alpha^2((c_2 - c_1\beta^2 t)x + c_3)\right. \\ & + \frac{2\sqrt{\pi}}{\sqrt{a}}\left((2\kappa_3 - \kappa_2\alpha)\alpha x + 4(3\kappa_3 - \kappa_2\alpha)\right)\operatorname{erfi}\left(\frac{\sqrt{\alpha x}}{2}\right)\Big)\frac{1}{\sqrt{x}} \\ & \left. + 8\frac{\exp(-\beta t)}{\alpha^2}(\kappa_2\alpha - \kappa_3)\right). \end{aligned}$$

The Case (iii): $8\beta\gamma - \alpha^2 = \delta^2$

In this case, we obtain four different sets of approximate Q-conditional symmetries of (11.26), but in only two cases we obtain general real valued solutions.

By using

$$\begin{aligned} \xi_{(0)} &= 0, \qquad \eta_{(0)} = -\frac{\beta\kappa_1\exp(\beta t)}{\kappa_1\exp(\beta t) + 1}u_{(0)}, \\ \xi_{(1)} &= \frac{4\kappa_2}{\kappa_1\exp(\beta t) + 1}, \\ \eta_{(1)} &= \frac{\exp(-\beta t - \frac{\alpha}{2}x)}{(\kappa_1\exp(\beta t) + 1)^2}\frac{\kappa_3\sin\left(\frac{\delta}{2}x\right) + \kappa_4\cos\left(\frac{\delta}{2}x\right)}{u_{(0)}} \\ & \quad - \frac{\exp(\beta t)}{(\kappa_1\exp(\beta t) + 1)^2}\left(2\beta^2\kappa_1\log(\kappa_1\exp(\beta t) + 1)\right. \\ & \quad \left. + \beta^2\kappa_1(\kappa_1\exp(\beta t) - \beta t) + \kappa_1\kappa_2\alpha - \kappa_5\right)u_{(0)}, \end{aligned}$$

and integrating the approximate invariant surface conditions (11.28), we get the following representation of the solution:

$$\begin{aligned} u_{(0)}(t,x) &= \frac{U_0(x)}{\kappa_1\exp(\beta t) + 1}, \\ u_{(1)}(t,x) &= \frac{1}{\kappa_1\exp(\beta t) + 1}\left(-\frac{\exp\left(-\beta t - \frac{\alpha}{2}x\right)}{\beta}\frac{\kappa_3\sin\left(\frac{\delta}{2}x\right) + \kappa_4\cos\left(\frac{\delta}{2}x\right)}{U_0(x)}\right. \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\beta\kappa_1(\kappa_1\exp(\beta t)+1)}\left(\beta^2\kappa_1(\kappa_1\exp(\beta t)(2\log(\kappa_1\exp(\beta t)+1)\right.\\
&- \beta t) - 1) - \kappa_1\kappa_2\alpha + \kappa_5)\,U_0(x)\\
&\left.+ 4\,\frac{\kappa_2}{\beta}(\log(\kappa_1\exp(\beta t)+1) - \beta t)U_0'(x) + U_1(x)\right).
\end{aligned}
$$

The associated approximate Q-conditional solution has the form:

$$
\begin{aligned}
u(t,x;\varepsilon) = &\frac{8\beta}{(\alpha^2+\delta^2)(\kappa_1\exp(\beta t)+1)}\\
&+\varepsilon\left(-\frac{\alpha^2+\delta^2}{8\beta^2}\exp\left(-\beta t-\frac{\alpha}{2}x\right)\left(\kappa_3\sin\left(\frac{\delta}{2}x\right)+\kappa_4\cos\left(\frac{\delta}{2}x\right)\right)\right.\\
&-\frac{8\exp(\beta t)}{(\alpha^2+\delta^2)(\kappa_1\exp(\beta t)+1)^2}(\beta^2\kappa_1(2\log(\kappa_1\exp(\beta t)+1)\\
&-\beta t+1)+\kappa_1\kappa_2\alpha-\kappa_5)\\
&\left.+\frac{\exp\left(-\left(\alpha+\sqrt{\frac{\alpha^2-\delta^2}{2}}\right)\frac{x}{2}\right)}{\kappa_1\exp(\beta t)+1}\left(c_1\exp\left(\sqrt{\frac{\alpha^2-\delta^2}{2}}x\right)+c_2\right)\right).
\end{aligned}
$$

Another case of admitted approximate Q-conditional symmetries has the approximate infinitesimals (and, consequently, the representation of the solution) with the same expression as (11.29) and (11.30). The approximate Q-conditional solution we find is

$$
\begin{aligned}
u(t,x;\varepsilon) = &\frac{8\beta}{\alpha^2+\delta^2}+\varepsilon\left(\frac{8\beta}{\alpha^2+\delta^2}\frac{\kappa_2}{\kappa_1}\exp(\beta t)+\exp\left(\left(\frac{\kappa_1(\alpha^2+\delta^2)}{8\beta}\right.\right.\right.\\
&\left.\left.+\alpha+\sqrt{\frac{(\kappa_1(\alpha^2+\delta^2)+8\alpha\beta)^2}{64\beta^2}-\frac{\alpha^2+\delta^2}{2}}\right)\frac{\kappa_1 t-x}{2}\right)\\
&\left.\times\left(c_1\exp\left(\sqrt{\frac{(\kappa_1(\alpha^2+\delta^2)+8\alpha\beta)^2}{64\beta^2}-\frac{\alpha^2+\delta^2}{2}}(-\kappa_1 t+x)\right)+c_2\right)\right).
\end{aligned}
$$

Finally, for the approximate infinitesimals

$$
\begin{aligned}
\xi_{(0)} &= 0, \qquad \eta_{(0)} = -\beta u_{(0)},\\
\xi_{(1)} &= 4\exp(-\beta t)\,(\kappa_1\sin(\delta x)+\kappa_2\cos(\delta x)+\kappa_3)\\
&\quad +4\exp\left(\beta t+\frac{\alpha}{2}x\right)\left(\kappa_4\sin\left(\frac{\delta}{2}x\right)+\kappa_5\cos\left(\frac{\delta}{2}x\right)\right)u_{(0)}^2,
\end{aligned}
$$

$$\eta_{(1)} = \exp\left(-3\beta t - \frac{\alpha}{2}x\right)\frac{\kappa_6 \sin\left(\frac{\delta}{2}x\right) + \kappa_7 \cos\left(\frac{\delta}{2}x\right)}{u_{(0)}}$$
$$- \exp(-\beta t)\Big(\,(\kappa_1\alpha + \kappa_2\delta)\sin(\delta x) - (\kappa_1\delta - \kappa_2\alpha)\cos(\delta x)$$
$$+\ \kappa_3\alpha - \kappa_8\Big)u_{(0)} - \beta^2 u_{(0)} - \exp\left(\beta t + \frac{\alpha}{2}x\right)$$
$$\times\Big(\,(\kappa_4\alpha + \kappa_5\delta)\sin(\delta x) - (\kappa_4\delta - \kappa_5\alpha)\cos(\delta x)\Big)u_{(0)}^3,$$

and

$$\xi_{(0)} = \beta\kappa_1\exp(-\beta t), \qquad \eta_{(0)} = -\beta u_{(0)},$$
$$\xi_{(1)} = (\kappa_2 - \kappa_1\beta^3 t)\exp(-\beta t) + \kappa_1(\kappa_3 - \kappa_1\alpha\beta^2)\exp(-2\beta t)$$
$$- \beta^2\frac{\alpha^2+\delta^2}{8}\kappa_1^3\exp(-3\beta t),$$
$$\eta_{(1)} = -\beta^3\kappa_1^2\exp(-2\beta t) + \left(\frac{\beta^2(\alpha^2+\delta^2)}{4}\kappa_1^2\exp(-2\beta t) - \kappa_3\exp(-\beta t) - \beta^2\right)u_{(0)},$$

after computing the corresponding representations of the solutions and obtaining the reduced ordinary differential equations, in general we do not find real valued solution, so that we omit them.

Remark 11.6 still applies to approximate conditional invariant solutions.

11.8 Approximate Noether Theorem

The approach for approximate Lie symmetries heretofore described of course can be used in variational problems. This leads to state an approximate Noether theorem for constructing approximate conservation laws of equations admitting a Lagrangian function containing small terms.

Definition 11.14 (*Approximate conservation law*) Given a system of differential equations,

$$\mathbf{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}; \varepsilon\right)$$
$$\equiv \sum_{k=0}^{p}\varepsilon^k\widetilde{\mathbf{\Delta}}_{(k)}\left(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}_{(0)}^{(1)}, \ldots, \mathbf{u}_{(0)}^{(r)}, \ldots, \mathbf{u}_{(k)}, \mathbf{u}_{(k)}^{(1)}, \ldots, \mathbf{u}_{(k)}^{(r)}\right) \approx \mathbf{0}, \tag{11.32}$$

an *approximate conservation law* of order r compatible with the system (11.32) is an approximate divergence expression, say

$$\sum_{k=0}^{p}\varepsilon^k\left(\sum_{i=1}^{n}\widetilde{\mathcal{D}}_i\left(\widetilde{\Phi}^i_{(k)}\left(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}_{(0)}^{(1)}, \ldots, \mathbf{u}_{(0)}^{(r-1)}, \ldots, \mathbf{u}_{(k)}, \mathbf{u}_{(k)}^{(1)}, \ldots, \mathbf{u}_{(k)}^{(r-1)}\right)\right)\right) \approx 0,$$

holding for all solutions of system (11.32), where

$$\sum_{k=0}^{p} \varepsilon^k \widetilde{\Phi}^i_{(k)} \left(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}^{(1)}_{(0)}, \mathbf{u}^{(r-1)}_{(0)}, \ldots\ldots, \mathbf{u}_{(k)}, \mathbf{u}^{(1)}_{(k)}, \ldots, \mathbf{u}^{(r-1)}_{(k)} \right), \qquad i = 1, \ldots, n$$

are the expansions at order p of the fluxes $\Phi^i \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r-1)}; \varepsilon \right)$ of the approximate conservation law, according to (11.10).

Some properties about exact conservation laws can be formulated in the approximate context:

- an approximate conservation law is *trivial* if each of its approximate fluxes is $O(\varepsilon^{p+1})$ on the solutions of the given system of differential equations, or the approximate conservation law is $O(\varepsilon^{p+1})$ identically;
- two approximate conservation laws are said *equivalent* if their linear combination is a trivial approximate conservation law;
- a finite set $\mathcal{S}$ of approximate conservation laws is *linearly dependent* if there exists a set of constants, not all zero, such that the linear combination of the elements in $\mathcal{S}$ is trivial; in this case, at least one of the approximate conservation laws in $\mathcal{S}$ can be expressed as a linear combination of the remaining ones.

It is well known that for variational problems, i.e., differential equations derived from a Lagrangian function, the determination of conservation laws is ruled by Noether theorem [31], establishing a correspondence between symmetries of the action integral and conservation laws through an explicit formula involving the infinitesimal generator of the Lie symmetry and the Lagrangian itself.

In what follows, let us limit ourselves to Lagrangian functions involving first-order derivatives, albeit the procedure can be straightforwardly extended in the case of higher order Lagrangians (see [18] for details).

Let

$$\mathcal{L} \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}; \varepsilon \right) \tag{11.33}$$

be a first-order Lagrangian function involving a small parameter ε. After inserting the expansion (11.10) in (11.33), we have

$$\begin{aligned} \mathcal{L} \left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}; \varepsilon \right) \approx & \widetilde{\mathcal{L}}_{(0)} \left(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}^{(1)}_{(0)} \right) \\ & + \sum_{k=1}^{p} \varepsilon^k \widetilde{\mathcal{L}}_{(k)} \left(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}^{(1)}_{(0)}, \ldots, \mathbf{u}_{(k)}, \ldots, \mathbf{u}^{(1)}_{(k)} \right), \end{aligned} \tag{11.34}$$

where $\widetilde{\mathcal{L}}_{(0)} = \mathcal{L} \left(\mathbf{x}, \mathbf{u}_{(0)}, \mathbf{u}^{(1)}_{(0)}; 0 \right)$ is the unperturbed Lagrangian and the remaining terms in (11.34) represent the perturbation up to the order p in ε.

Let us consider the corresponding approximate Lagrangian action

$$\mathcal{S}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)};\varepsilon\right)=\int_{\Omega}\mathcal{L}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)};\varepsilon\right)d\mathbf{x}$$
$$\approx\int_{\Omega}\left(\sum_{k=0}^{p}\varepsilon^{k}\widetilde{\mathcal{L}}_{(k)}\left(\mathbf{x},\mathbf{u}_{(0)},\ldots,\mathbf{u}_{(k)},\mathbf{u}^{(1)}_{(0)},\ldots,\mathbf{u}^{(1)}_{(k)}\right)\right)d\mathbf{x},\qquad(11.35)$$

over a domain Ω. By requiring the first variation of the approximate Lagrangian action (11.35) to be $O(\varepsilon^{p+1})$ under variations of order $O(\varepsilon^{p+1})$ at the boundary of Ω, we obtain the approximate Euler–Lagrange equations

$$\sum_{k=0}^{p}\varepsilon^{k}\left(\frac{\partial\widetilde{\mathcal{L}}_{(k)}}{\partial u_{(0)\alpha}}-\sum_{i=1}^{n}\widetilde{\mathcal{D}}_{i}\left(\frac{\partial\widetilde{\mathcal{L}}_{(k)}}{\partial u_{(0)\alpha,i}}\right)\right)\approx 0,\qquad\alpha=1,\ldots,m.$$

Approximate variational and divergence symmetries of an approximate Lagrangian action are defined as in the exact case, and an approximate Noether theorem can be stated. We limit ourselves to first-order Lagrangians.

Theorem 11.6 (Approximate Noether theorem) *Let us consider a variational system of differential equations arising from a first-order Lagrangian function. Let (11.11) be the generator of an approximate variational symmetry of the approximate Lagrangian action (11.35), say*

$$\sum_{k=0}^{p}\varepsilon^{k}\left(\sum_{j=0}^{k}\left(\widetilde{\Xi}^{(1)}_{(j)}\widetilde{\mathcal{L}}_{(k-j)}+\widetilde{\mathcal{L}}_{(k-j)}\sum_{i=1}^{n}\widetilde{\mathcal{D}}_{i}\widetilde{\xi}_{(j)i}\right)\right)\approx 0.\qquad(11.36)$$

Then, there exists the approximate conservation law

$$\sum_{k=0}^{p}\varepsilon^{k}\left(\sum_{i=1}^{n}\widetilde{\mathcal{D}}_{i}\widetilde{\Phi}^{i}_{(k)}\right)\approx 0,\qquad(11.37)$$

where

$$\widetilde{\Phi}^{i}_{(k)}=\sum_{\ell=0}^{k}\left(\sum_{\alpha=1}^{m}\left(\left(\widetilde{\eta}_{(\ell)\alpha}-\sum_{j=1}^{n}\widetilde{\xi}_{(\ell)j}u_{(\ell)\alpha,j}\right)\sum_{q=0}^{k-\ell}\frac{\partial\widetilde{\mathcal{L}}_{(k-\ell)}}{\partial u_{(q)\alpha,i}}\right)+\widetilde{\xi}_{(\ell)i}\widetilde{\mathcal{L}}_{(k-\ell)}\right).$$

If we consider an approximate divergence symmetry, i.e.,

$$\sum_{k=0}^{p}\varepsilon^{k}\left(\sum_{j=0}^{k}\left(\widetilde{\Xi}^{(1)}_{(j)}\widetilde{\mathcal{L}}_{(k-j)}+\widetilde{\mathcal{L}}_{(k-j)}\sum_{i=1}^{n}\widetilde{\mathcal{D}}_{i}\widetilde{\xi}_{(j)i}\right)-\sum_{i=1}^{n}\widetilde{\mathcal{D}}_{i}\phi^{i}_{(k)}\right)\approx 0,\qquad(11.38)$$

where $\phi^i_{(k)}\left(\mathbf{x}, \mathbf{u}_{(0)}, \ldots, \mathbf{u}_{(k)}\right)$ $(i = 1, \ldots, n)$ *are functions of their arguments to be suitably determined, in the approximate conservation law (11.37) it is*

$$\widetilde{\Phi}^i_{(k)} = \sum_{\ell=0}^{k} \left(\sum_{\alpha=1}^{m} \left(\left(\widetilde{\eta}_{(\ell)\alpha} - \sum_{j=1}^{n} \widetilde{\xi}_{(\ell)j} u_{(\ell)\alpha,j} \right) \sum_{q=0}^{k-\ell} \frac{\partial \widetilde{\mathcal{L}}_{(k-\ell)}}{\partial u_{(q)\alpha,i}} \right) + \widetilde{\xi}_{(\ell)i} \widetilde{\mathcal{L}}_{(k-\ell)} \right) - \phi^i_{(k)}.$$

In the case of only one independent variable, say the time t*, the approximate Noether theorem yields an approximate first integral.*

Proof The proof is immediate and is simply an adaptation of the proof of classical Noether theorem in the approximate context. □

Remark 11.7 Approximate variational or divergence symmetries leave the Lagrangian action approximately invariant; consequently, they are admitted as approximate Lie symmetries of the approximate Euler–Lagrange equations. As in the exact case, the converse is not true, because not all the approximate Lie symmetries of the approximate Euler–Lagrange equations are approximate variational or divergence symmetries.

11.8.1 Applications

Let us consider some examples of physical interest where the approximate Noether theorem can be applied. The first two examples have been considered in literature and analyzed by means of the Baikov–Gazizov–Ibragimov approach to approximate Lie symmetries, the third one addresses the planar three-body problem where one of the material points has a mass of order ε. Moreover, we limit ourselves to first-order approximate variational symmetries of first-order perturbed Lagrangian functions, and determine the corresponding first-order approximate conservation laws. The computation of approximate variational (or divergence) symmetries and approximate conservation laws has been done by means of the program ReLie [32]. In the examples, for the independent variable is the time t, to light the notation we denote derivatives with superposed dots.

Example 11.14 (*Perturbed harmonic oscillator*) Consider the one-dimensional perturbed harmonic oscillator, say the differential equation

$$\ddot{u} + u + \varepsilon F(u) = 0,$$

derived from the Lagrangian

$$\mathcal{L}(u, \dot{u}; \varepsilon) = \frac{1}{2}\left(\dot{u}^2 - u^2\right) - \varepsilon \int F(u) du,$$

where $F(u)$ is an arbitrary function of u such that $\dfrac{d^2F}{du^2} \neq 0$. This model has been studied in [2] for the classification of the admitted approximate Lie symmetries, and in [20], where an analysis of the approximate Noether symmetries and the corresponding approximate conservation laws have been developed. In order to perform the algorithmic procedure for approximate variational Lie symmetries described in the previous section, let us expand $u(t;\varepsilon)$ at first order in ε, i.e.,

$$u(t;\varepsilon) = u_{(0)}(t) + \varepsilon u_{(1)}(t) + O(\varepsilon^2),$$

whereupon, we have the first-order approximate Lagrangian

$$\begin{aligned}\mathcal{L} \approx & \widetilde{\mathcal{L}}_{(0)}\left(u_{(0)}, \dot{u}_{(0)}\right) + \varepsilon \widetilde{\mathcal{L}}_{(1)}\left(u_{(0)}, u_{(1)}, \dot{u}_{(0)}, \dot{u}_{(1)}\right) \\ = & \frac{1}{2}\left(\dot{u}_{(0)}\right)^2 - u_{(0)}^2\Big) + \varepsilon\left(\dot{u}_{(0)}\dot{u}_{(1)} - u_{(0)}u_{(1)} - \int F_{(0)}(u_{(0)})du_{(0)}\right),\end{aligned}$$

where the expansion $F(u) \approx F_{(0)}(u_{(0)}) + \varepsilon \dfrac{dF_{(0)}(u_{(0)})}{du_{(0)}} u_{(1)}$ has been used.

By solving the approximate determining equations, a classification is needed according to the functional form of $F(u)$:

(1) $F(u)$ arbitrary;
(2) $F(u) = (u+\delta)^2$, δ constant;
(3) $F(u) = \dfrac{\kappa}{u^3}$, κ constant.

For $F(u)$ arbitrary, we determine the following generators of approximate divergence symmetries together with the expansion of the associated function ϕ entering the invariance condition (11.38):

$$\begin{aligned}&\Xi_1 = \frac{\partial}{\partial t} \text{ with } \phi_{(0)} = \phi_{(1)} = 0; \\ &\Xi_2 = \varepsilon \sin(t)\frac{\partial}{\partial u} \text{ with } \phi_{(0)} = 0,\ \phi_{(1)} = -\cos(t)u_{(0)}; \\ &\Xi_3 = \varepsilon \cos(t)\frac{\partial}{\partial u} \text{ with } \phi_{(0)} = 0,\ \phi_{(1)} = \sin(t)u_{(0)}; \\ &\Xi_4 = \varepsilon\left(\sin(2t)\frac{\partial}{\partial t} + \cos(2t)u_{(0)}\frac{\partial}{\partial u}\right) \text{ with } \phi_{(0)} = 0,\ \phi_{(1)} = \sin(2t)u_{(0)}^2; \\ &\Xi_5 = \varepsilon\left(\cos(2t)\frac{\partial}{\partial t} - \sin(2t)u_{(0)}\frac{\partial}{\partial u}\right) \text{with } \phi_{(0)} = 0,\ \phi_{(1)} = \cos(2t)u_{(0)}^2; \\ &[0.25cm]\Xi_6 = \varepsilon\,\Xi_1 \text{ with } \phi_{(0)} = \phi_{(1)} = 0;\end{aligned} \tag{11.39}$$

the application of Theorem 11.6 yields the following approximate conserved quantities:

$$\mathcal{I}_1 = \frac{1}{2}\left(\dot{u}_{(0)} + u_{(0)}^2\right) + \varepsilon\left(\dot{u}_{(0)}\dot{u}_{(1)} + u_{(0)}u_{(1)} + \int F_{(0)}(u_{(0)})du_{(0)}\right),$$

$$
\begin{aligned}
\mathcal{I}_2 &= \varepsilon \left(\sin(t)\dot{u}_{(0)} - \cos(t)u_{(0)}\right),\\
\mathcal{I}_3 &= \varepsilon \left(\cos(t)\dot{u}_{(0)} + \sin(t)u_{(0)}\right),\\
\mathcal{I}_4 &= \varepsilon \left(\left(\sin(t)\dot{u}_{(0)} - \cos(t)u_{(0)}\right)\left(\cos(t)\dot{u}_{(0)} + \sin(t)u_{(0)}\right)\right),\\
\mathcal{I}_5 &= \varepsilon \left(\sin(t)\dot{u}_{(0)} - \cos(t)u_{(0)}\right)^2,\\
\mathcal{I}_6 &= \varepsilon \mathcal{I}_1.
\end{aligned}
$$

For $F(u) = (u+\delta)^2$, in addition to (11.39), we have two further approximate divergence symmetries, namely

$$
\begin{aligned}
\Xi_{7a} =& 4\varepsilon \sin(t)\frac{\partial}{\partial t} + \left(-3\cos(t) + \varepsilon\left(3\delta t \sin(t) + 2\cos(t)u_{(0)}\right)\right)\frac{\partial}{\partial u},\\
&\text{with } \phi_{(0)} = -3\sin(t)u_{(0)},\\
&\quad \phi_{(1)} = \sin(t)u_{(0)}^2 - 3\delta(t\cos(t) + \sin(t))u_{(0)} - 3\sin(t)u_{(1)} - 3\delta^2\sin(t);\\
\Xi_{8a} =& 4\varepsilon \cos(t)\frac{\partial}{\partial t} + \left(3\sin(t) + \varepsilon\left(3\delta t\cos(t) - 2\sin(t)u_{(0)}\right)\right)\frac{\partial}{\partial u},\\
&\text{with } \phi_{(0)} = -3\cos(t)u_{(0)},\\
&\quad \phi_{(1)} = \cos(t)u_{(0)}^2 + 3\delta(t\sin(t) - \cos(t))u_{(0)} - 3\cos(t)u_{(1)} - 3\delta^2\cos(t).
\end{aligned}
$$

The associated approximate conserved quantities turn out to be

$$
\begin{aligned}
\mathcal{I}_{7a} &= \cos(t)\dot{u}_{(0)} + \sin(t)u_{(0)}\\
&\quad + \varepsilon\left(\frac{2}{3}\sin(t)\dot{u}_{(0)}^2 - \left(\frac{2}{3}\cos(t)u_{(0)} + \delta t\sin(t)\right)\dot{u}_{(0)}\right.\\
&\quad \left. + \frac{\sin(t)}{3}u_{(0)}^2 + \delta(t\cos(t) + \sin(t))u_{(0)} + \cos(t)\dot{u}_{(1)} + \sin(t)u_{(1)} + \delta^2\sin(t)\right),\\
\mathcal{I}_{8a} &= \sin(t)\dot{u}_{(0)} - \cos(t)u_{(0)}\\
&\quad + \varepsilon\left(-\frac{2}{3}\cos(t)\dot{u}_{(0)} - \left(\frac{2}{3}\sin(t)u_{(0)} - \delta t\cos(t)\right)\dot{u}_{(0)}\right.\\
&\quad \left. - \frac{\cos(t)}{3}u_{(0)}^2 + \delta(t\sin(t) - \cos(t))u_{(0)} + \sin(t)\dot{u}_{(1)} - \cos(t)u_{(1)} - \delta^2\cos(t)\right).
\end{aligned}
$$

Finally, for $F(u) = \dfrac{\kappa}{u^3}$, we have, besides generators (11.39), the following generators of approximate divergence symmetries, say

$$
\begin{aligned}
\Xi_{7b} =& \cos(2t)\frac{\partial}{\partial t} - \sin(2t)\left(u_{(0)} + \varepsilon u_{(1)}\right)\frac{\partial}{\partial u}\\
&\text{with } \phi_{(0)} = \cos(2t)u_{(0)}^2,\ \phi_{(1)} = 2\cos(2t)u_{(0)}u_{(1)};\\
\Xi_{8b} =& \sin(2t)\frac{\partial}{\partial t} + \cos(2t)\left(u_{(0)} + \varepsilon u_{(1)}\right)\frac{\partial}{\partial u},\\
&\text{with } \phi_{(0)} = \sin(2t)u_{(0)}^2,\ \phi_{(1)} = 2\sin(2t)u_{(0)}u_{(1)}.
\end{aligned}
$$

Thus, the following approximate conserved quantities are obtained:

$$\begin{aligned}
\mathcal{I}_{7b} &= \cos(2t)\frac{(u_{(0),1})^2 - u_{(0)}^2}{2} + \sin(2t)u_{(0),1}u_{(0)} \\
&\quad + \varepsilon\left(\left(\cos(2t)u_{(1),1} + \sin(2t)u_{(1)}\right)u_{(0),1}\right. \\
&\quad \left. + \left(\sin(2t)u_{(1),1} - \cos(2t)u_{(1)}\right)u_{(0)} - \kappa\frac{\cos(2t)}{2u_{(0)}^2}\right), \\
\mathcal{I}_{8b} &= \sin(2t)\frac{(u_{(0),1})^2 - u_{(0)}^2}{2} - \cos(2t)u_{(0),1}u_{(0)} \\
&\quad + \varepsilon\left(\left(\sin(2t)u_{(1),1} - \cos(2t)u_{(1)}\right)u_{(0),1}\right. \\
&\quad \left. - \left(\cos(2t)u_{(1),1} + \sin(2t)u_{(1)}\right)u_{(0)} - \kappa\frac{\sin(2t)}{2u_{(0)}^2}\right).
\end{aligned}$$

Remark 11.8 The above approximate Noether symmetries are just the expansion of those determined in [20]; this explains essentially the concept that our approach, compared with the Baikov–Gazizov–Ibragimov method, is consistent with the principles of perturbation analysis; moreover, the corresponding first-order approximate conserved quantities here obtained are somehow different from those reported in [20] (maybe some formulae of the latter paper contain some typos). We stress that, albeit in [20] first-order approximate conserved quantities are considered, the results there obtained include also higher order terms in ε).

Example 11.15 (*A second-order system of ordinary differential equations*) Let us consider the following perturbed second-order system of ordinary differential equations investigated in [5],

$$\begin{aligned}
&\ddot{u} + \frac{\alpha}{u^3} - \varepsilon\frac{1}{u^3}\frac{dF(v)}{dv} = 0, \\
&[0.25cm]\ddot{v} - 4\alpha\frac{v}{u^4} + 2\frac{1}{u}\dot{u}\dot{v} + 2\varepsilon\frac{1}{u^4}\frac{d}{dv}\left(vF(v)\right) = 0;
\end{aligned} \tag{11.40}$$

system (11.40) can be derived from the Lagrangian

$$\mathcal{L}(u, v, \dot{u}, \dot{v}; \varepsilon) = v\dot{u}^2 + u\dot{u}\dot{v} - \alpha\frac{v}{u^2} + \varepsilon\frac{F(v)}{u^2},$$

where $u \equiv u(t;\varepsilon)$, $v \equiv v(t;\varepsilon)$, $F(v)$ arbitrary smooth function of its argument, and α constant.

In [5], system (11.40) was analyzed as an approximate variational problem, and some approximate conserved quantities for arbitrary $F(v)$ have been determined. As

our consistent procedure requires, expanding $u(t;\varepsilon)$ and $v(t;\varepsilon)$ at first order in ε, we get the first-order perturbed Lagrangian

$$\begin{aligned}\mathcal{L} &\approx \widetilde{\mathcal{L}}_{(0)}\left(u_{(0)}, v_{(0)}, \dot{u}_{(0)}, \dot{v}_{(0)}\right) + \varepsilon\widetilde{\mathcal{L}}_{(1)}\left(u_{(0)}, v_{(0)}, u_{(1)}, v_{(1)}, \dot{u}_{(0)}, \dot{v}_{(0)}, \dot{u}_{(1)}, \dot{v}_{(1)}\right)\\ &= v_{(0)}\dot{u}_{(0)}^2 + u_{(0)}\dot{u}_{(0)}\dot{v}_{(0)} - \alpha\frac{v_{(0)}}{u_{(0)}^2}\\ &+ \varepsilon\left(2v_{(0)}\dot{u}_{(0)}\dot{u}_{(1)} + v_{(1)}\dot{u}_{(0)}^2 + u_{(0)}(\dot{u}_{(0)}\dot{v}_{(1)} + \dot{u}_{(1)}\dot{v}_{(0)}) + u_{(1)}\dot{u}_{(0)}\dot{v}_{(0)}\right.\\ &\left. - \alpha\left(\frac{v_{(1)}}{u_{(0)}^2} - 2\frac{v_{(0)}}{u_{(0)}^3}u_{(1)}\right) + \frac{F_{(0)}(v_{(0)})}{u_{(0)}^2}\right),\end{aligned}$$

where the expansion $F(v) \approx F_{(0)}(v_{(0)}) + \varepsilon\dfrac{dF_{(0)}(v_{(0)})}{dv_{(0)}}v_{(1)}$ has been used; by imposing the approximate invariance of the action integral, we are able to obtain the following approximate generators together with the expansion of the associated function ϕ entering condition (11.38):

$$\begin{aligned}\Xi_1 &= \frac{\partial}{\partial t}, \qquad \phi_{(0)} = \phi_{(1)} = 0;\\ \Xi_2 &= t^2\frac{\partial}{\partial t} + t\left(u_{(0)} + \varepsilon u_{(1)}\right)\frac{\partial}{\partial u},\\ &\qquad \text{with } \phi_{(0)} = -u_{(0)}^2 v_{(0)},\ \phi_{(1)} = -u_{(0)}\left(u_{(0)}v_{(1)} + 2v_{(0)}u_{(1)}\right);\\ \Xi_3 &= 2t\frac{\partial}{\partial t} + \left(u_{(0)} + \varepsilon u_{(1)}\right)\frac{\partial}{\partial u},\ \text{with } \phi_{(0)} = \phi_{(1)} = 0;\\ \Xi_4 &= \varepsilon\Xi_1,\ \text{with } \phi_{(0)} = \phi_{(1)} = 0;\\ \Xi_5 &= \varepsilon\Xi_2,\ \text{with } \phi_{(0)} = 0, \quad \phi_{(1)} = -u_{(0)}^2 v_{(0)};\\ \Xi_6 &= \varepsilon\Xi_3,\ \text{with } \phi_{(0)} = \phi_{(1)} = 0;\end{aligned}$$

these generators give the following approximate conserved quantities:

$$\begin{aligned}\mathcal{I}_1 &= \dot{u}_{(0)}v_{(0)} + \dot{u}_{(0)}\dot{v}_{(0)}u_{(0)} + \alpha u_{(0)}^{-2}v_{(0)}\\ &+ \varepsilon\left(\dot{u}_{(0)}v_{(1)} + \dot{u}_{(0)}\dot{v}_{(0)}u_{(1)} + 2\dot{u}_{(0)}\dot{u}_{(1)}v_{(0)}\right.\\ &\left. + u_{(0)}(\dot{u}_{(0)}\dot{v}_{(1)} + \dot{v}_{(0)}\dot{u}_{(1)}) + \alpha u_{(0)}^{-3}\left(u_{(0)}v_{(1)} - 2v_{(0)}u_{(1)}\right) - u_{(0)}^{-2}F_{(0)}(v_{(0)})\right),\\ \mathcal{I}_2 &= t^2\left(\dot{u}_{(0)}v_{(0)} + \dot{u}_{(0)}\dot{v}_{(0)}u_{(0)} + \alpha u_{(0)}^{-2}v_{(0)}\right)\\ &- t\left(2\dot{u}_{(0)}u_{(0)}v_{(0)} + \dot{v}_{(0)}u_{(0)}^2\right) + u_{(0)}^2 v_{(0)}\\ &+ \varepsilon\left(t^2\left(\dot{u}_{(0)}^2 v_{(1)} + \dot{u}_{(0)}\dot{v}_{(0)}u_{(1)} + 2\dot{u}_{(0)}\dot{u}_{(1)}v_{(0)}\right.\right.\\ &\left.\left. + u_{(0)}(\dot{u}_{(0)}\dot{v}_{(1)} + \dot{v}_{(0)}\dot{u}_{(1)}) + \alpha u_{(0)}^{-3}\left(u_{(0)}v_{(1)} - 2v_{(0)}u_{(1)}\right) - u_{(0)}^{-2}F_{(0)}(v_{(0)})\right)\right.\end{aligned}$$

$$
\begin{aligned}
&- t\left(2\dot{u}_{(0)}\left(u_{(0)}v_{(1)} + u_{(1)}v_{(0)}\right)\right.\\
&+ \left.u_{(0)}(2\dot{v}_{(0)}u_{(1)} + 2\dot{u}_{(1)}v_{(0)} + \dot{v}_{(1)}u_{(0)})\right) + u_{(0)}^2 v_{(1)} + 2u_{(0)}v_{(0)}u_{(1)}\Big),\\
\mathcal{I}_3 &= 2t\left(\dot{u}_{(0)}^2 v_{(0)} + \dot{u}_{(0)}\dot{v}_{(0)}u_{(0)} + \alpha u_{(0)}^{-2}v_{(0)}\right) - 2\dot{u}_{(0)}u_{(0)}v_{(0)} - \dot{v}_{(0)}u_{(0)}^2\\
&+ \varepsilon\left(2t\left(\dot{u}_{(0)}v_{(1)} + \dot{u}_{(0)}\dot{v}_{(0)}u_{(1)} + 2\dot{u}_{(0)}\dot{u}_{(1)}v_{(0)} + \dot{u}_{(0)}v_{(1),1}u_{(0)} + \dot{v}_{(0)}\dot{u}_{(1)}u_{(0)}\right.\right.\\
&+ \left.\alpha u_{(0)}^{-3}(u_{(0)}v_{(1)} - 2v_{(0)}u_{(1)}) - u_{(0)}^{-2}F_{(0)}(v_{(0)})\right) - \Big(2\dot{u}_{(0)}\left(u_{(0)}v_{(1)} + u_{(1)}v_{(0)}\right)\\
&+ \left.2\dot{v}_{(0)}u_{(0)}u_{(1)} + 2\dot{u}_{(1)}u_{(0)}v_{(0)} + \dot{v}_{(1)}u_{(0)}^2\right)\Big),\\
\mathcal{I}_4 &= \varepsilon\mathcal{I}_1, \qquad \mathcal{I}_5 = \varepsilon\mathcal{I}_2, \qquad \mathcal{I}_6 = \varepsilon\mathcal{I}_3.
\end{aligned}
$$

Remark 11.9 In this application, we recovered a set of approximate conserved quantities different from those reported in [5]; furthermore, there is no need to consider special instances of the arbitrary function $F(v)$.

Example 11.16 (*The three-body problem*) Let us now focus on the problem of three bodies with masses m_α ($\alpha = 1, 2, 3$) moving under their mutual gravitational attraction. We assume that the mass m_3 of the third body is much smaller than m_1 and m_2, and that the three point masses move in a fixed plane. Therefore, we consider a planar restricted problem without neglecting the gravitational action of the third body on the two main bodies. Let $\mathbf{r}_i \equiv (x_i, y_i, 0)$ ($i = 1, 2, 3$) be the position vectors of the three point masses in a fixed frame reference, and $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r_j}$ ($1 \le i < j \le 3$). Under these hypotheses, the motion equations are:

$$
\begin{aligned}
&\ddot{\mathbf{r}}_1 + Gm_2\frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} + \varepsilon Gm_3\frac{\mathbf{r}_{13}}{|\mathbf{r}_{13}|^3} = \mathbf{0},\\
&[0.25cm]\ddot{\mathbf{r}}_2 - Gm_1\frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} + \varepsilon Gm_3\frac{\mathbf{r}_{23}}{|\mathbf{r}_{23}|^3} = \mathbf{0},\\
&\ddot{\mathbf{r}}_3 - Gm_1\frac{\mathbf{r}_{13}}{|\mathbf{r}_{13}|^3} - Gm_2\frac{\mathbf{r}_{23}}{|\mathbf{r}_{23}|^3} = \mathbf{0},
\end{aligned}
\tag{11.41}
$$

where G is the gravitational constant. Equation (11.41) is derived from the Lagrangian function

$$
\mathcal{L} = \frac{1}{2}\left(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2\right) + \frac{Gm_1m_2}{|\mathbf{r}_{12}|} + \varepsilon\left(m_3\dot{\mathbf{r}}_3^2 + \frac{Gm_1m_3}{|\mathbf{r}_{13}|} + \frac{Gm_2m_3}{|\mathbf{r}_{23}|}\right).
$$

Let us expand the dependent variables at first order in ε, i.e.,

$$
\begin{aligned}
\mathbf{r}_i &= \mathbf{r}_{(0)i} + \varepsilon\mathbf{r}_{(1)i} + O(\varepsilon^2)\\
&\equiv \left(x_{(0)i}(t) + \varepsilon x_{(1)i}(t) + O(\varepsilon^2), y_{(0)i}(t) + \varepsilon y_{(1)i}(t) + O(\varepsilon^2), 0\right), \qquad i = 1, 2, 3,
\end{aligned}
$$

whereupon

$$\mathbf{r}_{ij} = \mathbf{r}_{(0)ij} + \varepsilon \mathbf{r}_{(1)ij} + O(\varepsilon^2) = \mathbf{r}_{(0)i} - \mathbf{r}_{(0)j} + \varepsilon \left(\mathbf{r}_{(1)i} - \mathbf{r}_{(1)j}\right) + O(\varepsilon^2),$$

and

$$\begin{aligned}
\mathcal{L} &\approx \widetilde{\mathcal{L}}_{(0)} + \varepsilon \widetilde{\mathcal{L}}_{(1)} \\
&= \frac{1}{2}\left(m_1 \dot{\mathbf{r}}_{(0)1}^2 + m_2 \dot{\mathbf{r}}_{(0)2}^2\right) + \frac{G m_1 m_2}{|\mathbf{r}_{(0)12}|} \\
&+ \varepsilon \left(m_1 \dot{\mathbf{r}}_{(0)1} \dot{\mathbf{r}}_{(1)1} + m_2 \dot{\mathbf{r}}_{(0)2} \cdot \dot{\mathbf{r}}_{(1)2} + m_3 \dot{\mathbf{r}}_{(0)3}^2 \right. \\
&\left. - \frac{G m_1 m_2}{|\mathbf{r}_{(0)12}|^3} \mathbf{r}_{(0)12} \cdot \mathbf{r}_{(1)12} + \frac{G m_1 m_3}{|\mathbf{r}_{(0)13}|} + \frac{G m_2 m_3}{|\mathbf{r}_{(0)23}|} \right);
\end{aligned}$$

by searching for the approximate divergence symmetries, we are able to determine the following approximate generators together with the expansion of the associated function ϕ entering the invariance condition (11.38):

$$\Xi_1 = \frac{\partial}{\partial t} \text{ with } \phi_{(0)} = \phi_{(1)} = 0;$$

$$\Xi_{2a} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}, \quad \Xi_{2b} = \sum_{i=1}^{3} \frac{\partial}{\partial y_i} \text{ with } \phi_{(0)} = \phi_{(1)} = 0;$$

$$\Xi_{3a} = t \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \text{ with } \phi_{(0)} = -\sum_{i=1}^{2} m_i x_{(0)i}, \ \phi_{(1)} = -\sum_{i=1}^{2} m_i x_{(1)i} - m_3 x_{(0)3};$$

$$\Xi_{3b} = t \sum_{i=1}^{3} \frac{\partial}{\partial y_i} \text{ with } \phi_{(0)} = -\sum_{i=1}^{2} m_i y_{(0)i}, \quad \phi_{(1)} = -\sum_{i=1}^{2} m_i y_{(1)i} - m_3 y_{(0)3};$$

$$\Xi_4 = \sum_{i=1}^{3} \left(\left(y_{(0)i} + \varepsilon y_{(1)i}\right) \frac{\partial}{\partial x_i} - \left(x_{(0)i} + \varepsilon x_{(1)i}\right) \frac{\partial}{\partial y_i} \right) \text{ with } \phi_{(0)} = \phi_{(1)} = 0;$$

$$\begin{aligned}
\Xi_5 = &\sum_{j=1}^{3} \sum_{i=1}^{2} \left(m_i y_{(0)i} + \varepsilon m_i y_{(1)i}\right) \frac{\partial}{\partial x_j} + \varepsilon m_3 y_{(0)3} \sum_{j=1}^{2} \frac{\partial}{\partial x_j} \\
&- \sum_{j=1}^{3} \sum_{i=1}^{2} \left(m_i x_{(0)i} + \varepsilon m_i x_{(1)i}\right) \frac{\partial}{\partial y_j} - \varepsilon m_3 x_{(0)3} \sum_{j=1}^{2} \frac{\partial}{\partial y_j} \text{ with } \phi_{(0)} = \phi_{(1)} = 0;
\end{aligned}$$

$$\begin{aligned}
\Xi_6 = \varepsilon &\left(\left(y_{(0)2} - y_{(0)1}\right) \left(m_2 \frac{\partial}{\partial x_1} - m_1 \frac{\partial}{\partial x_2} \right) \right. \\
&\left. + \left(x_{(0)1} - x_{(0)2}\right) \left(m_2 \frac{\partial}{\partial y_1} - m_1 \frac{\partial}{\partial y_2} \right) \right) \text{ with } \phi_{(0)} = \phi_{(1)} = 0;
\end{aligned}$$

$$\Xi_7 = \varepsilon \Xi_1 \text{ with } \phi_{(0)} = \phi_{(1)} = 0;$$

$$\Xi_{8a} = \varepsilon \Xi_{2a}, \quad \Xi_{8b} = \varepsilon \Xi_{2b} \text{ with } \phi_{(0)} = \phi_{(1)} = 0;$$

$$\Xi_{9a} = \varepsilon t \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \text{ with } \phi_{(0)} = 0, \phi_{(1)} = -\sum_{i=1}^{2} m_i x_{(0)i};$$

$$\Xi_{9b} = \varepsilon t \sum_{i=1}^{2} \frac{\partial}{\partial y_i} \text{ with } \phi_{(0)} = 0, \phi_{(1)} = -\sum_{i=1}^{2} m_i y_{(0)i};$$

$$\Xi_{10} = \varepsilon \sum_{i=1}^{2} \left(y_{(0)i} \frac{\partial}{\partial x_i} - x_{(0)i} \frac{\partial}{\partial y_i} \right) \text{ with } \phi_{(0)} = \phi_{(1)} = 0;$$

$$\Xi_{11} = \varepsilon \frac{\partial}{\partial x_3}, \quad \Xi_{12} = \varepsilon \frac{\partial}{\partial y_3} \text{ with } \phi_{(0)} = \phi_{(1)} = 0.$$

As far as the nontrivial conservation laws are concerned, the results that we obtain are as follows:

- from generator Ξ_1, we have the first-order approximate conservation of total energy

$$\begin{aligned} \mathcal{I}_1 = {} & \frac{1}{2} \left(m_1 \dot{\mathbf{r}}_{(0)1}^2 + m_2 \dot{\mathbf{r}}_{(0)2}^2 \right) - \frac{G m_1 m_2}{|\mathbf{r}_{(0)12}|} \\ & + \varepsilon \left(\frac{1}{2} m_3 \dot{\mathbf{r}}_{(0)3}^2 + m_1 \dot{\mathbf{r}}_{(0)1} \cdot \dot{\mathbf{r}}_{(1)1} + m_2 \dot{\mathbf{r}}_{(0)2} \cdot \dot{\mathbf{r}}_{(1)2} \right. \\ & \left. - \frac{G m_1 m_3}{|\mathbf{r}_{(0)13}|} - \frac{G m_2 m_3}{|\mathbf{r}_{(0)23}|} + \frac{G m_1 m_2}{|\mathbf{r}_{(0)23}|^3} \mathbf{r}_{(0)12} \cdot \mathbf{r}_{(1)12} \right); \end{aligned}$$

- from generators Ξ_{2a} and Ξ_{2b}, we have the approximate conservation of total linear momentum

$$\mathcal{I}_2 = m_1 \dot{\mathbf{r}}_{(0)1} + m_2 \dot{\mathbf{r}}_{(0)2} + \varepsilon \left(m_1 \dot{\mathbf{r}}_{(1)1} + m_2 \dot{\mathbf{r}}_{(1)2} + m_3 \dot{\mathbf{r}}_{(0)3} \right);$$

- from the generators Ξ_{3a} and Ξ_{3b}, we have

$$\begin{aligned} \mathcal{I}_3 = {} & m_1 (t \dot{\mathbf{r}}_{(0)1} - \mathbf{r}_{(0)1}) + m_2 (t \dot{\mathbf{r}}_{(0)2} - \mathbf{r}_{(0)2}) \\ & + \varepsilon \left(m_1 (t \dot{\mathbf{r}}_{(1)1} - \mathbf{r}_{(1)1}) + m_2 (t \dot{\mathbf{r}}_{(1)2} - \mathbf{r}_{(1)2}) + m_3 (t \dot{\mathbf{r}}_{(0)3} - \mathbf{r}_{(0)3}) \right), \end{aligned}$$

  expressing that the approximate (at order ε) barycenter of the system has a uniform and rectilinear motion;
- from the generator Ξ_4, we have the approximate conservation of total angular momentum

$$\begin{aligned} \mathcal{I}_4 = {} & m_1 \mathbf{r}_{(0)1} \wedge \dot{\mathbf{r}}_{(0)1} + m_2 \mathbf{r}_{(0)2} \wedge \dot{\mathbf{r}}_{(0)2} \\ & + \varepsilon \left(m_1 \left(\mathbf{r}_{(0)1} \wedge \dot{\mathbf{r}}_{(1)1} + \mathbf{r}_{(1)1} \wedge \dot{\mathbf{r}}_{(0)1} \right) + m_2 \left(\mathbf{r}_{(0)2} \wedge \dot{\mathbf{r}}_{(1)2} + \mathbf{r}_{(1)2} \wedge \dot{\mathbf{r}}_{(0)2} \right) \right. \\ & \left. + m_3 \mathbf{r}_{(0)3} \wedge \dot{\mathbf{r}}_{(0)3} \right). \end{aligned}$$

Therefore, we recovered the approximate conservation laws we expect. The remaining generators do not yield new independent conserved quantities. In fact:

- the generator Ξ_5 produces

$$\begin{aligned}\mathcal{I}_5 &= \left(m_1\mathbf{r}_{(0)1} + m_2\mathbf{r}_{(0)2}\right) \wedge \left(m_1\dot{\mathbf{r}}_{(0)1} + m_2\dot{\mathbf{r}}_{(0)2} + \varepsilon\left(m_1\dot{\mathbf{r}}_{(1)1} + m_2\dot{\mathbf{r}}_{(1)2} + m_3\dot{\mathbf{r}}_{(0)3}\right)\right)\\ &\quad + \varepsilon\left(m_1\mathbf{r}_{(1)1} + m_2\mathbf{r}_{(1)2} + m_3\mathbf{r}_{(0)3}\right) \wedge \left(m_1\dot{\mathbf{r}}_{(0)1} + m_2\dot{\mathbf{r}}_{(0)2}\right)\\ &= \left(\int \mathcal{I}_2 dt\right) \wedge \mathcal{I}_2;\end{aligned}$$

- the generator Ξ_6 produces

$$\mathcal{I}_6 = \varepsilon\left(\mathbf{r}_{(0)1} - \mathbf{r}_{(0)2}\right) \wedge \left(\dot{\mathbf{r}}_{(0)1} - \dot{\mathbf{r}}_{(0)2)}\right) = \varepsilon\frac{(m_1 + m_2)\mathcal{I}_4 - \mathcal{I}_5}{m_1 m_2};$$

- the generator Ξ_7 produces $\varepsilon\mathcal{I}_1$;
- the generators Ξ_{8a} and Ξ_{8b} produce $\varepsilon\mathcal{I}_2$;
- the generators Ξ_{9a} and Ξ_{9b} produce $\varepsilon\mathcal{I}_3$;
- the generator Ξ_{10} produces $\varepsilon\mathcal{I}_5$;
- the generators Ξ_{11} and Ξ_{12} produce only trivial conserved quantities.

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Chapter 12
Computer Algebra Programs



12.1 Introduction

Though the application of Lie theory and its modern extensions to differential equations is completely algorithmic, it usually involves a lot of cumbersome and tedious calculations. Today, many powerful Computer Algebra Systems (either commercial or open source) are available, and the needed algebraic manipulations can be rapidly done almost automatically. In fact, many specific packages for performing symmetry analysis of differential equations are currently available in the literature [4, 9, 10, 12, 13, 17, 19, 20, 22, 23, 39–43, 45]. In this Chapter, two different packages will ne discussed. The first one, ReLie [31], written in the Computer Algebra System Reduce [18], is able to find Lie point symmetries, conditional symmetries, contact symmetries, variational or divergence symmetries of differential equations, both exact and approximate. The second one, SymboLie [1, 2], written in Wolfram Mathematica™[46], is devoted to compute optimal systems of Lie subalgebras. As far as this last problem is concerned, we quote a recently introduced SageMath algorithm [47].

12.2 The Program ReLie

The package ReLie [31], written in the open source Computer Algebra System Reduce [18], and freely available, can help researchers to perform with a little effort most of the computations described in this book.

Reduce is a general purpose Computer Algebra System whose development was started by Anthony Hearn in 1963. It is written in a Lisp dialect (Portable Standard Lisp); since December 2008 it is an open source program (available for all platforms at the url http://www.reduce-algebra.com). The source code of ReLie, as well as the user's manual, can be found at the url https://mat521.unime.it/oliveri. Therefore,

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ReLie can be freely used by researchers who do not have licenses of commercial programs like Wolfram Mathematica™[46] or Maple™ [27].

ReLie requires the user to have a minimal experience in working with the algebraic mode of Reduce. The origin of this package dates back to 1994 when the author developed some routines useful to manage the lengthy expressions needed to determine the Lie point symmetries of differential equations; this set of routines constantly grew through the years providing new capabilities, and now constitutes a package able to perform most part of the tedious work of looking for Lie symmetries of differential equations. Moreover, the program can be used also in interactive mode mimicking the steps one has to take with pencil and paper but with the benefits of using a computer with a dedicated program. This can represent a useful support in a higher course on symmetry analysis of differential equations, or in all those situations where the determining equations cannot be automatically solved and some special assumptions are needed, or a special pre-processing of input, or post-processing of the results, are needed; in such cases, the standard functions made available by Reduce, or some routines written by the users, could be necessary.

What the package ReLie can do is hereafter presented by considering some simple examples; nevertheless, more complex applications can be efficiently addressed as well, and some of these cases are discussed. It is worth of being noticed that almost all the computations shown in this book have been carried out using ReLie.

The package ReLie is a collection of (algebraic) routines that allow the user to investigate ordinary and partial differential equations within the framework of Lie symmetry analysis. By using the program, it is possible to compute almost automatically (only input data of the problem are requested): Lie point symmetries, conditional symmetries, contact symmetries, variational or divergence symmetries with their associated conservation laws (all these symmetries may be either exact or approximate), and equivalence transformations of classes of differential equations containing arbitrary elements. Furthermore, the program implements functions for computing Lie brackets, the commutator table of a list of Lie operators, and the distribution of an algebra of Lie symmetries (useful in the context of inverse Lie problem [16, 24–26]).

After entering Reduce, assuming that we are in the directory containing the source file `relie.red`, the set of routines implemented in the package ReLie becomes available after the statement

```
in "relie.red" $
```

The loading of ReLie is successful if the output

```
ReLie, version 3.0
A REduce program for Lie group analysis
of differential equations
(c) Francesco Oliveri (foliveri@unime.it) - 2020
Last update August 9, 2020
https://mat521.unime.it/oliveri/
```

is displayed. Alternatively, one may include the package in the Reduce image by issuing in a Reduce session the statements

```
faslout 'relie $
in "relie.red" $
faslend $
```

If all works, ReLie is loaded through the command

```
load_package relie $
```

According to the task we are interested to, some input data are required. The *minimal* required set of data is

- a positive integer value for the global variable `jetorder` (the maximum order of prolongation of the Lie infinitesimal group);
- the list `xvar` of the independent variables;
- the list `uvar` of the dependent variables.

Example 12.1 The assignments

```
jetorder:=2 $
xvar:={x1,x2} $
uvar:={u} $
```

refer to a second-order partial differential equation for the unknown $u(x_1, x_2)$.

The number of independent as well as dependent variables is not a priori bounded; moreover, the identifiers of independent and dependent variables are arbitrary provided that no conflict arises with the reserved words of Reduce, or identifiers used by ReLie (see Sect. 12.2.10 for a list of internal global variables used by ReLie). Also the value of `jetorder` is *not* bounded by ReLie. Of course, as `jetorder` and/or the number of independent and dependent variables increase, the performance in terms of memory and/or execution time decreases.

The first function one has to call is `relieinit()`, that defines and initializes all the needed objects required to perform Lie group analysis. It checks the input and possibly displays some warnings. If only `jetorder`, `xvar` and `uvar` have been assigned, then the call `relieinit()` produces the output:

```
The list 'diffeqs' of differential equation(s) is missing!
The list 'leadders' of leading derivative(s) is missing!
Check 'diffeqs' and/or 'leadders'!
You may only call relieprol() or
generatealgebra(k) (k=1,2,3).
```

In fact, we did not define the list `diffeqs` of differential equations and the list `leadders` of leading derivatives. Nevertheless, we can compute the `jetorder`–prolongation of the vector field generating a Lie group of point transformation. The infinitesimal generators of the independent and dependent variables are automatically defined by the program (therefore, the user is not requested to set them), and stored in the list `allinfinitesimals`. The infinitesimals of the independent variables are denoted by `xi_` followed by the identifiers in `xvar`; analogously, the infinitesimals of the dependent variables are denoted by `eta_` followed by the identifiers in `uvar`. According to the Example 12.1, the list `allinfinitesimals` is a list of two elements, each element being the list `{xi_x1, xi_x2, eta_u}` (this redundancy is used in order to have a unified coding for dealing also with other kinds of symmetries, see below). The dependencies of the elements in `allinfinitesimals` upon the independent and dependent variables (the elements in `xvar` and `uvar`, respectively) are automatically set by ReLie.

By calling the ReLie function `relieprol()`, we get the list `prolongation`. This list has two elements: the first one is a list containing the *coordinates* of the jet space, the second one the list of the corresponding infinitesimal generators. Thus, the second element of `prolongation` contains the components of the `jetorder`–prolonged vector field.

ReLie internally represents the derivatives of the dependent variables upon the independent variables by appending to the identifiers of the dependent variables the underscore `_` followed by the identifiers of the involved independent variables. For mixed derivatives the order of the independent variables in the internal representation of derivatives reflects the order in the list `xvar`. For instance, if `xvar` is `{x1,x2}` and `uvar` is `{u}`, the left-hand side of the equation (Benjamin–Bona–Mahoney equation [5])

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u\frac{\partial u}{\partial x_2} - \frac{\partial^3 u}{\partial x_2^2 \partial x_1} = 0$$

has to be represented as

```
u_x1 + u_x2 + u*u_x2 - u_x1x2x2
```

12.2.1 Computing Lie Point Symmetries of Differential Equations

To compute the Lie point symmetries admitted by differential equations, in addition to `jetorder`, `xvar` and `uvar`, we have to define:

1. the list `diffeqs` of the left-hand sides of the differential equations to be studied which are assumed with zero right-hand sides;

2. the list `leadders` of some derivatives appearing in the differential equations: when computing the invariance conditions, the elements in the list `leadders` are removed by solving the differential equations with respect to them.

Both lists, `diffeqs` and `leadders`, need to have the same length, otherwise a warning is displayed; moreover, the differential equations must be solvable with respect to the leading derivatives. The user should be careful in the choice of `leadders`, especially in the case of systems of differential equations, in order to guarantee that the differential equations stored in the list `diffeqs` can be solved with respect to the elements in `leadders`.

Example 12.2 Once RELIE has been loaded, the statements

```
in "relie.red" $
jetorder:=3 $
xvar:={x} $
uvar:={u} $
diffeqs:={u_xxx+u*u_xx/2} $
leadders:={u_xxx} $
relieinit() $
relieinv() $
reliedet() $
reliesolve() $
;end;
```

allow to compute the Lie point symmetries of Blasius equation [6]

$$\frac{d^3u}{dx^3} + \frac{1}{2}u\frac{d^2u}{dx^2} = 0.$$

The call to the function `relieinit()` initializes the needed objects, then the call to the function `relieinv()` gives the invariance condition of the differential equation defined in `diffeqs`; in other words, the `jetorder`–prolongation of the vector field of the Lie group is applied to the differential equation and the equation itself is used to eliminate $\frac{d^3u}{dx^3}$. Then, the list `invcond` is obtained:

```
invcond ->
   {6*df(eta_u,u,x,2)*u_x + 2*df(eta_u,u,x)*u*u_x
    + 6*df(eta_u,u,x)*u_xx + 2*df(eta_u,u,3)*u_x^3
    + 6*df(eta_u,u,2,x)*u_x^2 + df(eta_u,u,2)*u*u_x^2
    + 6*df(eta_u,u,2)*u_x*u_xx + 2*df(eta_u,x,3)
    + df(eta_u,x,2)*u - 6*df(xi_x,u,x2)*u_x^2
    - 2*df(xi_x,u,x)*u*u_x^2 - 18*df(xi_x,u,x)*u_x*u_xx
    - 2*df(xi_x,u,3)*u_x^4 - 6*df(xi_x,u,2,x)*u_x^3
    - df(xi_x,u,2)*u*u_x^3 - 12*df(xi_x,u,2)*u_x^2*u_xx
```

```
    + df(xi_x,u)*u*u_x*u_xx - 6*df(xi_x,u)*u_xx^2
    - 2*df(xi_x,x,3)*u_x - df(xi_x,x,2)*u*u_x
    - 6*df(xi_x,x,2)*u_xx + df(xi_x,x)*u*u_xx + eta_u*u_xx)}
```

Since we have a scalar ordinary differential equation, the list `invcond` contains just one element, which is a polynomial in the variables `u_x` and `u_xx`. Invoking the function `reliedet()`, one gets the list `deteqs` of the coefficients of such (list of) polynomial(s); thus, we have the *determining equations*, i.e., a system of linear partial differential equations for the infinitesimals `xi_x` and `eta_u`:

```
deteqs ->
   {2*df(eta_u,x,3) + df(eta_u,x,2)*u,
    6*df(eta_u,u,x) - 6*df(xi_x,x,2) + df(xi_x,x)*u + eta_u,
    - 6*df(xi_x,u),
    6*df(eta_u,u,x,2) + 2*df(eta_u,u,x)*u - 2*df(xi_x,x,3)
    - df(xi_x,x,2)*u,
    6*df(eta_u,u,2) - 18*df(xi_x,u,x) + df(xi_x,u)*u,
    6*df(eta_u,u,2,x) + df(eta_u,u,2)*u - 6*df(xi_x,u,x,2)
    - 2*df(xi_x,u,x)*u,
    - 12*df(xi_x,u,2),
    2*df(eta_u,u,3) - 6*df(xi_x,u,2,x) - df(xi_x,u,2)*u,
    - 2*df(xi_x,u,3)}
```

We may automatically solve this set of determining equations by calling the function `reliesolve()`, and the result turns out to be contained in the list `symmetries`:

```
symmetries ->
{
 {
  {},
  {eta_u= - k_1*u,
   xi_x=k_1*x + k_2},
  {k_1,k_2},
  {}
 }
}
```

The function `reliesolve()` uses the REDUCE package Crack by Wolf [45], specifically suitable to solve (overdetermined) systems of linear partial differential equations.

The list `symmetries` contains an element (when more than one solution set is possible for particular choices of the involved parameters this list has one element for each solution set, see later) which in turn is a list of four elements:

1. the first one is a list of conditions that are still unsolved (in this case the list is empty because no condition remains unsolved);
2. the second one is a list giving the solution to the determining equations, i.e., the expressions of the infinitesimals;
3. the third one is a list containing the parameters involved in the solution (in this case `k_1` and `k_2`);
4. the fourth one is a list of expressions which cannot vanish (in this case the list is empty).

The parameters involved in the symmetries of Blasius equation are constant; symmetries of partial differential equations may involve arbitrary functions that ReLie denotes as `f_1`, `f_2`, …; we can check the dependencies of such parameters by passing the name of the function to the procedure `fargs()` (described in Crack [45]).

The infinitesimals are solutions of linear homogeneous differential equations, so they are linear combinations of fundamental solutions. We can compute these fundamental solutions through the use of the function `reliegen()`. It requires two arguments: an integer and a list. If `reliesolve()` returns only one solution this integer is 1; when more than one solution is available (this may occur in group classification problems or in the search of conditional symmetries), the integer selects the solution. The second argument is a list of values. If the length of this list corresponds to the number of arbitrary (constant) parameters involved in the chosen symmetries, then a list of vector fields (`generators`) is returned where the kth parameter is replaced by the kth element in the list and the other parameters are replaced by 0; otherwise, the kth parameter is replaced by 1 and the other parameters are replaced by 0. For instance, invoking `reliegen(1,{1,1})` (in this case we can write also `reliegen(1,{})`), we get the list `generators`:

```
generators ->
{
 {x, -u},
 {1,0}
}
```

We have a list where each element is the list of the components of the vector field generating a symmetry (the components of the infinitesimals of the independent and dependent variables); for Blasius equation, the vector fields characterize a scaling and a translation of the independent variable, respectively:

$$\Xi_1 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, \qquad \Xi_2 = \frac{\partial}{\partial x}.$$

In the example above, we detailed all the steps leading to the computation of the Lie point symmetries of a differential equation. Nevertheless, we have a general function, called `relie()`, requiring an integer argument:

1. `relie(1)` is equivalent to calling `relieinit();`
2. `relie(2)` is equivalent to calling in sequence `relieinit()` and `relieinv();`
3. `relie(3)` is equivalent to calling in sequence `relieinit()`, `relieinv()` and `reliedet();`
4. `relie(4)` is equivalent to calling in sequence `relieinit()`, `relieinv()`, `reliedet()` and, finally, `reliesolve()`.

Example 12.3 After loading RELIE, the assignments

```
in "relie.red" $
jetorder:=2 $
xvar:={x1,x2} $
uvar:={u} $
diffeqs:={u_x1-u_x2x2} $
leadders:={u_x1} $
relie(4) $
;end;
```

allow one to immediately compute the Lie point symmetries of the linear heat equation

$$u_{,1} - u_{,22} = 0.$$

The output produced by RELIE is

```
symmetries ->
{
 {
  {df(f_1,x1)-df(f_1,x2,2)},
  {
   eta_u=(-8*f_1+2*k_1*u*x2+2*k_4*x1*u+k_4*u*x2^2+2*k_6*u)/8,
   xi_x2=(-k_1*x1-k_2-k_3*x2-k_4*x1*x2)/2,
   xi_x1=(-2*k_3*x1-k_4*x1^2-2*k_5)/2
  },
  {f_1, k_1, k_2, k_3, k_4, k_5, k_6},
  {}
 }
}
```

The infinitesimals of the point symmetries of the heat equation involve 7 parameters. The parameter `f_1` is a function, whereas the other parameters are constant. By calling `fargs(f_1)`, we have as a result `{x1,x2}`, denoting that the function `f_1` depends on x_1 and x_2; moreover, this function satisfies the heat equation (the first element of the list `symmetries`). The symmetry corresponding to this parameter is responsible for the linear superposition principle of the solutions (property shared by linear equations).

Calling the function `reliegen(1,{-2,-2,-2,-2,-1,4})`, we get as a result the list `generators` of vector fields of the finite subalgebra of the Lie algebra of symmetries of linear heat equation:

```
generators ->
{
 {0,x1,( - u*x2)/2},
 {0,1,0},
 {2*x1,x2,0},
 {x1^2,x1*x2,(u*( - 2*x1 - x2^2))/4},
 {1,0,0},
 {0,0,u}
}
```

We can sort the vector fields in `generators` by using the function `newordering()`

```
generators:=newordering(generators,{5,2,6,3,1,4}) $
```

so obtaining

```
generators ->
{
  {1,0,0},
  {0,1,0},
  {0,0, u}
  {2*x1,x2,0},
  {0, x1,-u*x2/2},
  {(x1^2, x1*x2,u*(-2*x1 - x2^2)/4},
}
```

i.e.,

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = u\frac{\partial}{\partial u},$$
$$\Xi_4 = 2x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}, \qquad \Xi_5 = x_1\frac{\partial}{\partial x_2} - \frac{x_2 u}{2}\frac{\partial}{\partial u},$$
$$\Xi_6 = x_1^2\frac{\partial}{\partial x_1} + x_1x_2\frac{\partial}{\partial x_2} - \frac{u(2x_1 + x_2^2)}{4}\frac{\partial}{\partial u},$$

spanning a six-dimensional Lie algebra. The symmetry corresponding to the function `f_1`,

$$\Xi_f = f(x_1, x_2)\frac{\partial}{\partial u} \qquad \text{with} \qquad \frac{\partial f}{\partial x_1} - \frac{\partial^2 f}{\partial x_2^2} = 0,$$

renders the algebra of symmetries of linear heat equation infinite-dimensional.

The function `reliedet()` assumes that the invariance conditions are polynomials in the derivatives: this is true if all derivatives appear polynomially in the differential equations at hand. In the case where the differential equations we are investigating contain some derivatives not in polynomial form, ReLie constructs correctly the determining equations if we inform the program about the derivatives not appearing in polynomial form. These derivatives have to be the elements of the list `nonpolyders`.

Example 12.4 (*Some derivatives do not appear polynomially*)
The following code allows the user to compute the Lie symmetries of the equation

$$u_{,12} + \exp\left(u_{,2}\right) - u = 0.$$

```
in "relie.red" $
jetorder:=2 $
xvar:={x1,x2} $
uvar:={u} $
diffeqs:={u_x1x2+exp(u_x2)-u} $
leadders:={u_x1x2} $
nonpolyders:={u_x2} $
relie(4) $
;end;
```

Remark 12.1 The use of ReLie is not limited to simple cases. For instance, we can easily determine the Lie point symmetries of the Navier–Stokes–Fourier equations of a gas in a rotating (with constant angular velocity ω along the vertical axis) frame reference and subject to gravity (see Example 9.3). In this case, we may use Reduce to produce the code to be managed by ReLie, and compute in a few seconds the twelve-dimensional Lie algebra of symmetries [30]. The code (with comments) we may use is:

```
in "relie.red" $
% Set the dependencies of the field variables:
% the density rho,
% the components of the velocity v1, v2, v3,
% and the temperature th

dependence({rho,v1,v2,v3,th},{t,x1,x2,x3}) $
v:={v1,v2,v3} $ % list of components of velocity
x:={x1,x2,x3} $ % list of space variables

% define the continuity equation
eq1:=df(rho,t)+df(rho*v1,x1)+df(rho*v2,x2)+df(rho*v3,x3) $

% define the fluxes in momentum equations
procedure flux(j,k) $
begin
  return
  rho*part(v,j)*part(v,k)
  -2*kk/(alpha*m)*th*((df(part(v,j),part(x,k))
  +df(part(v,k),part(x,j)))/2-(df(v1,x1)+df(v2,x2)
  +df(v3,x3))*kroneckerdelta(j,k)/3)
  +rho*kk/m*th*kroneckerdelta(j,k)$
end $

% define the list of the components of forces
forces:={2*omega*v2+omega^2*x1,-2*omega*v1+omega^2*x2,-g} $

% define the momentum equations
eq2:=df(rho*v1,t)+df(flux(1,1),x1)+df(flux(1,2),x2)
       +df(flux(1,3),x3)-rho*part(forces,1) $
eq3:=df(rho*v2,t)+df(flux(2,1),x1)+df(flux(2,2),x2)
        +df(flux(2,3),x3)-rho*part(forces,2) $
eq4:=df(rho*v3,t)+df(flux(3,1),x1)+df(flux(3,2),x2)
        +df(flux(3,3),x3)-rho*part(forces,3) $

% define the flux in energy equation
procedure fluxenergy(k) $
begin
  return
    rho*th*part(v,k)-5*kk/(2*alpha*m)*th*df(th,part(x,k)) $
end $

% define the remaining terms in energy equation
```

```
temp:=
  for k:=1:3 sum
    for j:=1:3 sum
      (2*rho*th*kroneckerdelta(j,k)/3
      -4/(3*alpha)*th*((df(part(v,j),part(x,k))
      +df(part(v,k),part(x,j)))/2-(df(v1,x1)+df(v2,x2)
      +df(v3,x3))*kroneckerdelta(j,k)/3))
      *df(part(v,j),part(x,k)) $

% define the energy equation
eq5:=df(rho*th,t)+df(fluxenergy(1),x1)
     +df(fluxenergy(2),x2)+df(fluxenergy(3),x3)+temp $

% solve field equations w.r.t. time derivatives
sol:=solve({eq1,eq2,eq3,eq4,eq5},
       {df(rho,t),df(v1,t),df(v2,t),df(v3,t),df(th,t)}) $

% input for ReLie
jetorder:=2 $
xvar:={t,x1,x2,x3} $
uvar:={rho,v1,v2,v3,th} $

% define the equations to be managed by ReLie
diffeqs:={df(rho,t)-sub(part(sol,1),df(rho,t)),
          df(v1,t)-sub(part(sol,1),df(v1,t)),
          df(v2,t)-sub(part(sol,1),df(v2,t)),
          df(v3,t)-sub(part(sol,1),df(v3,t)),
          df(th,t)-sub(part(sol,1),df(th,t))} $

% define the list of substitutions for writing derivatives
% according to the internal representation used by ReLie

subst:={
 df(rho,t)=rho_t,df(v1,t)=v1_t,df(v2,t)=v2_t,
 df(v3,t)=v3_t,df(th,t)=th_t,df(rho,x1)=rho_x1,
 df(v1,x1)=v1_x1,df(v2,x1)=v2_x1,df(v3,x1)=v3_x1,
 df(th,x1)=th_x1,df(rho,x2)=rho_x2,df(v1,x2)=v1_x2,
 df(v2,x2)=v2_x2,df(v3,x2)=v3_x2,df(th,x2)=th_x2,
 df(rho,x3)=rho_x3,df(v1,x3)=v1_x3,df(v2,x3)=v2_x3,
 df(v3,x3)=v3_x3,df(th,x3)=th_x3,df(th,x1,x1)=th_x1x1,
 df(th,x2,x2)=th_x2x2,df(th,x3,x3)=th_x3x3,
 df(v1,x1,x1)=v1_x1x1,df(v1,x1,x2)=v1_x1x2,
 df(v1,x1,x3)=v1_x1x3,df(v1,x2,x2)=v1_x2x2,
 df(v1,x2,x3)=v1_x2x3,df(v1,x3,x3)=v1_x3x3,
 df(v2,x1,x1)=v2_x1x1,df(v2,x1,x2)=v2_x1x2,
```

```
 df(v2,x1,x3)=v2_x1x3,df(v2,x2,x2)=v2_x2x2,
 df(v2,x2,x3)=v2_x2x3,df(v2,x3,x3)=v2_x3x3,
 df(v3,x1,x1)=v3_x1x1,df(v3,x1,x2)=v3_x1x2,
 df(v3,x1,x3)=v3_x1x3,df(v3,x2,x2)=v3_x2x2,
 df(v3,x2,x3)=v3_x2x3,df(v3,x3,x3)=v3_x3x3} $

diffeqs:=sub(subst,diffeqs) $
leadders:={rho_t,v1_t,v2_t,v3_t,th_t} $

% remove the dependencies (they are no more needed)
nodependence({rho,v1,v2,v3,th},{t,x1,x2,x3}) $

% compute the symmetries
relie(4) $
;end;
```

The results (and their consequent use) are described in Example 9.3.

12.2.2 An Example of Group Classification

ReLie is able to address the problem of group classification of differential equations. In fact, in certain cases it may be interesting to check if special instances of arbitrary constants and/or functions involved in the differential equations lead to different sets of Lie symmetries. This problem is addressed by including the parameters we want to check in the list `freepars`, as illustrated in the following example. Moreover, we may define the list `nonzeropars` containing parameters (constants or functions involved in the differential equations, and listed in `freepars`), or expressions that cannot be vanishing.

Example 12.5 Consider the three-dimensional Euler equations of ideal gas dynamics (6.5); we can perform a group classification according to the value of the adiabatic exponent γ.

The following code performs the required task:

```
jetorder:=1 $
xvar:={t,x1,x2,x3} $
uvar:={rho,v1,v2,v3,p} $
freepars:={gamma} $
nonzeropars:={gamma} $
diffeqs:={
 rho_t+v1*rho_x1+v2*rho_x2+v3*rho_x3
    +rho*(v1_x1+v2_x2+v3_x3),
 rho*(v1_t+v1*v1_x1+v2*v1_x2+v3*v1_x3)+p_x1,
```

```
 rho*(v2_t+v1*v2_x1+v2*v2_x2+v3*v2_x3)+p_x2,
 rho*(v3_t+v1*v3_x1+v2*v3_x2+v3*v3_x3)+p_x3,
 p_t+v1*p_x1+v2*p_x2+v3*p_x3+gamma*p*(v1_x1+v2_x2+v3_x3)} $
leadders:={rho_t,v1_t,v2_t,v3_t,p_t} $
relie(4) $
;end;
```

We get

```
symmetries ->
{
{
{},
{
 gamma=5/3,
 eta_p=5*k_10*p*t -k_14*p,
 eta_v3=k_10*t*v3-k_10*x3+k_12*v3+k_3*v2-k_5*v1-k_7-k_8*v3,
 eta_v2=-k_1+k_10*t*v2-k_10*x2+k_12*v2-k_2*v1-k_3*v3-k_8*v2,
 eta_v1=k_10*t*v1-k_10*x1-k_11+k_12*v1+k_2*v2+k_5*v3-k_8*v1,
 eta_rho=3*k_10*rho*t-2*k_12*rho-k_14*rho+2*k_8*rho,
 xi_x3=-k_10*t*x3+k_3*x2-k_5*x1-k_6-k_7*t-k_8*x3,
 xi_x2=-k_1*t-k_10*t*x2-k_2*x1-k_3*x3-k_4-k_8*x2,
 xi_x1=-k_10*t*x1-k_11*t+k_2*x2+k_5*x3-k_8*x1-k_9,
 xi_t=-k_10*t^2-k_12*t-k_13
},
{
 k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_10,
 k_11, k_12, k_13, k_14
},
{}
},
{
{},
{
 eta_p=-k_14*p,
 eta_v3=k_12*v3+k_3*v2-k_5*v1-k_7-k_8*v3,
 eta_v2=-k_1+k_12*v2-k_2*v1-k_3*v3-k_8*v2,
 eta_v1=-k_11+k_12*v1+k_2*v2+k_5*v3-k_8*v1,
 eta_rho=-2*k_12*rho-k_14*rho+2*k_8*rho,
 xi_x3=k_3*x2-k_5*x1-k_6-k_7*t-k_8*x3,
 xi_x2=-k_1*t-k_2*x1-k_3*x3-k_4-k_8*x2,
 xi_x1=-k_11*t+k_2*x2+k_5*x3-k_8*x1-k_9,
 xi_t=-k_12*t-k_13
},
{
```

```
 gamma, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9,
 k_11, k_12, k_13, k_14
},
{
 gamma, 3*gamma - 5
}
}
}
```

In this case, `symmetries` is a list of two elements, as we have different solutions to the determining equations according to the value of `gamma` (included in the list `freepars`). Looking at the results given by RELIE, the first solution set refers to $\gamma = 5/3$, the second one to a value of γ different from 0 and $5/3$. By calling

```
reliegen(2,{-1,1,-1,-1,1,-1,-1,-1,-1,-1,-1,-1,-1}) $
generators:=newordering(generators,{12,9,4,6,2,5,3,10,
     1,7,11,8,13}) $
```

we obtain

```
generators ->
{
 {1,0,0,0,0,0,0,0,0},
 {0,1,0,0,0,0,0,0,0},
 {0,0,1,0,0,0,0,0,0},
 {0,0,0,1,0,0,0,0,0},
 {0,x2, - x1,0,0,v2, - v1,0,0},
 {0,x3,0, - x1,0,v3,0, - v1,0},
 {0,0,x3, - x2,0,0,v3, - v2,0},
 {0,t,0,0,0,1,0,0,0},
 {0,0,t,0,0,0,1,0,0},
 {0,0,0,t,0,0,0,1,0},
 {t,0,0,0,2*rho,-v1,-v2,-v3,0},
 {0,x1,x2,x3,-2*rho,v1,v2,v3,0},
 {0,0,0,0,rho,0,0,0,p}
}
```

so that for γ unspecified we have the thirteen-dimensional Lie algebra of point symmetries spanned by the vector fields in (6.6). Calling

```
reliegen(1,{-1,1,-1,-1,1,-1,-1,-1,-1,-1,-1,-1,-1,-1}) $
generators:=newordering(generators,{13,9,4,6,2,5,3,11,
   1,7,12,8,14,10}) $
```

we have that for $\gamma = 5/3$ there is a fourteenth symmetry generated by the vector field (6.7).

Remark 12.2 The differential equations can contain arbitrary functions (in this case we need to declare in REDUCE the variables these functions depend on), and they can be elements of the list `freepars` and/or `nonzeropars`.

For instance, if we want to classify the symmetries admitted by the Korteweg–deVries equation with variable coefficients [29]

$$u_{,1} + uu_{,2} + u_{,222} + f(x_1)u = 0,$$

where $f(x_1)$ is an unspecified function of time, the input data for RELIE are the following ones:

```
in "relie.red" $
jetorder:=3 $
xvar:={x1,x2} $
uvar:={u} $
depend f,x1 $
freepars:={f} $
diffeqs:={u_x1+u*u_x2+u_x2x2x2+f*u} $
leadders:={u_x1} $
relie(4) $
;end;
```

12.2.3 Commutator Table

RELIE is able to compute the commutator table of a set of infinitesimal operators spanning a Lie algebra. Let us illustrate how to proceed with an example.

Example 12.6 Consider the Burgers equation

$$u_{,1} + uu_{,2} - u_{,22} = 0.$$

To compute its Lie symmetries we give to RELIE the following input:

```
in "relie.red" $
jetorder:=2 $
xvar:={x1,x2} $
uvar:={u} $
diffeqs:={u_x1+u*u_x2-u_x2x2} $
leadders:={u_x1} $
relie(4) $
```

We get quickly the output

```
symmetries ->
{
 {
  {},
  {
   {0, - k_1*x1, - k_1},
   {0, - k_2,0},
   { - k_3*x1,( - k_3*x2)/2,(k_3*u)/2},
   {( - k_4*x1^2)/2,( - k_4*x1*x2)/2,(k_4*(x1*u - x2))/2},
   { - k_5,0,0}
  },
  {}
 }
}
```

Therefore, we have a five-dimensional Lie algebra of Lie point symmetries. After extracting and reordering the components of the infinitesimal generators,

```
reliegen(1,{-1,-1,-2,-2,-1}) $
generators:=newordering(generators,{5,2,1,3,4}) $
```

we have

```
generators ->
  {
   {1, 0, 0},
   {0, 1,0},
   {0, x1, 1},
   {2*x1, x2, -u},
   {x1^2, x1*x2, -x1*u + x2)}
  }
```

that is, a five-dimensional Lie algebra spanned by

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = x_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u},\\
&\Xi_4 = 2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - u \frac{\partial}{\partial u},\\
&\Xi_5 = x_1^2 \frac{\partial}{\partial x_1} + x_1 x_2 \frac{\partial}{\partial x_2} + (x_2 - x_1 u) \frac{\partial}{\partial u}.
\end{aligned} \tag{12.1}$$

The commutator table of the generators (12.1) is obtained by calling

```
commutatortable(generators);
```

that produces the output

```
commtable ->
{
 {0, 0, -vf_1, -vf_5, 0},
 { 0, 0, -2*vf_2, -vf_3, -vf_1},
 {vf_1, 2*vf_2, 0, -2*vf_4, -vf_5},
 {vf_5, vf_3, 2*vf_4, 0, 0},
 {0, vf_1, vf_5, 0, 0}
}
```

where the five elements of the list `generators` have been referred to as `vf_1`, `vf_2`, `vf_3`, `vf_4`, `vf_5`, respectively.

12.2.4 Computation of Q–Conditional Symmetries

Q–conditional symmetries [3, 8, 11, 35] are computed by using the same functions as above; we only need to assign a nonzero value (in the range between 1 and the number of independent variables) to the variable `nonclassical` and define the list `qcond` (a list of distinct integers chosen in the range from 1 to the number of dependent variables). Suppose that `xvar:={x1,x2,x3}` and `uvar:={u,v}`. The components of the vector field of the conditional symmetries are

- `{1, xi_x2, xi_x3, eta_u, eta_v}` if `nonclassical = 1`;
- `{0, 1, xi_x3, eta_u, eta_v}` if `nonclassical = 2`;
- `{0, 0, 1, eta_u, eta_v}` if `nonclassical = 3`.

The list `qcond` tells the program which invariant surface conditions have to be used to restrict the solution manifold:

- `qcond:={1,2}` if the invariant surface conditions relative to both dependent variables have to be used;
- `qcond:={1}` if the invariant surface condition relative to the first dependent variable has to be used;
- `qcond:={2}` if the invariant surface condition relative to the second dependent variable has to be used.

Looking for conditional symmetries leads to *nonlinear* determining equations, whereupon it is not unusual that `reliesolve()` may fail to recover the solution. In such cases, it is necessary to make some special assumptions on the infinitesimals and/or try to solve interactively the determining equations.

Example 12.7 (*Nonclassical symmetries of linear heat equation*) The code

```
in "relie.red" $
jetorder:=2 $
xvar:={x1,x2} $
uvar:={u} $
nonclassical:=1 $
qcond:={1} $
diffeqs:={u_x1-u_x2x2} $
leadders:={u_x2x2} $
relie(4) $
;end;
```

immediately computes the conditional symmetries, corresponding to the vector field

$$\Xi = \frac{\partial}{\partial x_1} + \xi(x_1, x_2, u)\frac{\partial}{\partial x_2} + \eta(x_1, x_2, u)\frac{\partial}{\partial u},$$

of the linear heat equation

$$u_{,1} - u_{,22} = 0.$$

So, we get

```
symmetries ->
{
{
{
 df(f_1,x1) - df(f_1,x2,2) - 2*df(f_2,x2,2)*f_1,
 2*df(f_2,x1,x2,2) + 2*df(f_2,x1,x2)*df(f_2,x2)
 - df(f_2,x1,2) + 2*df(f_2,x1)*df(f_2,x2,2)
 - df(f_2,x2,4) - 2*df(f_2,x2,3)*df(f_2,x2)
 - 4*df(f_2,x2,2)^2 - 2*df(f_2,x2,2)*df(f_2,x2)^2
},
{
 eta_u=(df(f_2,x1)*u-df(f_2,x2,2)*u-df(f_2,x2)^2*u-2*f_1)/2,
 xi_x2= - df(f_2,x2)
},
{
 f_1,f_2
},
{}
}
}
```

Therefore, we have:

$$\xi = -\frac{\partial f_2}{\partial x_2}, \qquad \eta = -f_1 + \frac{1}{2}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial^2 f_2}{\partial x_2^2} - \left(\frac{\partial f_2}{\partial x_2}\right)^2\right) u,$$

where $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ satisfy the differential equations

$$\begin{aligned}
&\frac{\partial f_1}{\partial x_1} - \frac{\partial^2 f_1}{\partial x_2^2} - 2 f_1 \frac{\partial^2 f_2}{\partial x_2^2} = 0,\\
&2\frac{\partial^3 f_2}{\partial x_1 \partial x_2^2} + 2\frac{\partial^2 f_2}{\partial x_1 \partial x_2}\frac{\partial f_2}{\partial x_2} - \frac{\partial^2 f_2}{\partial x_1^2} + 2\frac{\partial f_2}{\partial x_1}\frac{\partial^2 f_2}{\partial x_2^2} - \frac{\partial^4 f_2}{\partial x_2^4} - 2\frac{\partial f_2}{\partial x_2}\frac{\partial^3 f_2}{\partial x_2^3}\\
&\quad -4\left(\frac{\partial^2 f_2}{\partial x_2^2}\right)^2 - 2\left(\frac{\partial f_2}{\partial x_2}\right)^2 \frac{\partial^2 f_2}{\partial x_2^2} = 0.
\end{aligned}$$

The assignments

$$A(x_1, x_2) = -\frac{\partial f_2}{\partial x_2}, \quad C(x_1, x_2) = \frac{1}{2}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial^2 f_2}{\partial x_2^2} - \left(\frac{\partial f_2}{\partial x_2}\right)^2\right), \quad D(x_1, x_2) = -f_1,$$

allow us to write the nonclassical symmetries of linear heat equation in the form given in [8].

Remark 12.3 When computing conditional symmetries we have to be careful in the choice of `leadders`, since also the invariant surface conditions and their differential consequences are solved with respect to some derivatives.

12.2.5 Computation of Contact Symmetries

For computing contact symmetries of a scalar differential equation we need to assign the value 1 to the internal variable `contact`. The functions that we have to call do not change, as the following example shows.

Example 12.8 The contact symmetries of the ordinary differential equation

$$\frac{d^3 u}{dx^3} = 0$$

can be computed in ReLie using the following code:

```
in "relie.red" $
jetorder:=3 $
xvar:={x} $
```

```
uvar:={u} $
contact:=1 $
diffeqs:={u_xxx} $
leadders:={u_xxx} $
relie(4) $
;end;
```

Then, we obtain:

```
symmetries ->
{
 {
  {},
  {omega=(-2*k_1*x^2-4*k_10*u_x-4*k_2*x-4*k_3*u-4*k_4
    +4*k_5*u*u_x -2*k_5*u_x^2*x-4*k_6*u^2+4*k_6*u*u_x*x
    -k_6*u_x^2*x^2 -2*k_7*u_x^2-4*k_8*u_x*x+4*k_9*u*x
    -2*k_9*u_x*x^2)/4},
  {k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8,k_9,k_10},
  {}
 }
}
```

i.e., the expression of the generating function $\Omega\left(x, u, \dfrac{du}{dx}\right)$ involves 10 arbitrary constants. A basis of the ten-dimensional Lie algebra of contact symmetries is obtained by invoking

```
reliegen(1,{-1,-1,-1,-1,-1,-1,1,1,1,1}) $
```

yielding

```
generators ->
{
 {0, x^2/2, x},
 {0, x, 1},
 {0, u, u_x},
 {0, 1, 0},
 {u - u_x*x, ( - u_x^2*x)/2, ( - u_x^2)/2},
 {(x*(2*u - u_x*x))/2, (4*u^2 - u_x^2*x^2)/4,
   (u_x*(2*u - u_x*x))/2},
 {u_x, u_x^2/2, 0},
 {x, 0, - u_x},
 {x^2/2, u*x, u},
 {1, 0, 0}
}
```

Note that only the fifth, sixth and seventh vector fields correspond to proper contact transformations, the remaining ones being first-order prolongations of point symmetries.

12.2.6 Computation of Variational or Divergence Symmetries and Associated Conservation Laws

For computing variational or divergence symmetries, besides assigning `jetorder`, `xvar` and `uvar` we need to assign the list `lagrangian` with only one element; if we want variational symmetries, we have to set the variable `variational` to 1; on the contrary, if we want divergence symmetries we have to set the variable `divergence` to 1.

The functions to be used do not change, as the following examples clearly show.

Example 12.9 The code

```
in "relie.red" $
jetorder:=1 $
variational:=1 $
xvar:={x} $
uvar:={u} $
lagrangian:={x^2*u_x^2/2-x^2*u^6/6} $
relie(4) $
;end;
```

allows the user to compute the variational symmetries of the Lagrangian

$$\mathcal{L} = \frac{x^2}{2}\left(\frac{du}{dx}\right)^2 - \frac{x^2u^6}{6},$$

generating the Emden–Fowler equation

$$\frac{d^2u}{dx^2} + \frac{2}{x}\frac{du}{dx} + u^5 = 0.$$

We get

```
symmetries ->
{
 {{},
 {eta_u=(k_1*u)/2, xi_x= - k_1*x},
 {k_1},
 {}}
}
```

Calling `reliegen(1,{})`, we obtain the list

```
generators ->
{
 { - x, u/2}
}
```

i.e., the (list of) vector fields of the variational symmetries. We have only a Lie operator.

Finally, calling `relieclaw(1)`, we get

```
fluxes ->
{
 (x^2*(x*u^6 + 3*x*u_x^2 + 3*u*u_x))/6
}
```

i.e., the first integral

$$\frac{1}{6}\left(x^2\left(xu^6+3x\left(\frac{du}{dx}\right)^2+3u\frac{du}{dx}\right)\right)=\text{constant}.$$

Example 12.10 (*Klein–Gordon equation* [7]) The class of Klein–Gordon wave equations for the unknown $u(x_1, x_2)$,

$$u_{,12}+g(u)=0$$

with a general nonlinear interaction term $g(u)$, can be derived from a variational principle given by the action functional with Lagrangian

$$\mathcal{L}=-\frac{u_{,1}u_{,2}}{2},\qquad h'(u)=g(u).$$

For an arbitrary $g(u)$, the generators of these symmetries are

$$\Xi_1=x_1\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial x_2},\qquad \Xi_2=\frac{\partial}{\partial x_1},\qquad \Xi_3=\frac{\partial}{\partial x_2},$$

and it is straightforward to verify that they are variational symmetries of the action functional. Applying Noether Theorem, the following conservation laws are easily derived:

$$\mathcal{D}_1\left(\frac{u_{,2}^2}{2}\right)+\mathcal{D}_2\left(h(u)\right)=0,$$

$$\mathcal{D}_1\left(h(u)\right)+\mathcal{D}_2\left(\frac{u_{,1}^2}{2}\right)=0,$$

$$\mathcal{D}_1\left(\frac{x_2u_{,2}^2}{2}-x_1h(u)\right)+\mathcal{D}_2\left(-\frac{x_1u_{,1}^2}{2}+x_2h(u)\right)=0.$$

The REDUCE code for this computation is

```
in "relie.red" $
jetorder:=1 $
xvar:={x1,x2} $
uvar:={u} $
depend h,u $
lagrangian:={-u_x1*u_x2/2+h} $
variational:=1 $
relie(4) $
reliegen(1,{-1,-1,-1}) $
;end;
```

The fluxes entering the conservation laws associated to the three Lie operators are obtained by issuing `relieclaw(1)`, `relieclaw(2)`, and `relieclaw(3)`, respectively.

Example 12.11 (*Variational Symmetries of Lagrangian for minimal surfaces*) The RELIE code for computing the variational symmetries of the Lagrangian of minimal surfaces in $\mathbb{R}^3$ (see Example 7.7) is:

```
in "relie.red" $
jetorder:=1 $
variational:=1 $
xvar:={x1,x2} $
uvar:={u} $
lagrangian:={sqrt(1+u_x1^2+u_x2^2)} $
relie(4) $
reliegen(1,{}) $
;end;
```

The fluxes of the associated conservation laws are obtained means of the function `relieclaw(n)`, with parameter $n \in \{1, 2, 3, 4, 5, 6\}$.

12.2.7 Computation of Approximate Symmetries

Approximate Lie point symmetries of differential equations containing small terms are computed according to the approach described in [15]. The small parameter must be denoted by `epsilon`. In addition to the input data described before, the user has to define the integer `approxorder` for the order of approximation, and a rule, say `let epsilon^(approxorder+1) = 0`. The same functions providing the necessary computations for exact Lie point symmetries work in this case too.

Example 12.12 The following code shows how to compute first-order approximate symmetries of the Korteweg–deVries–Burgers equation

$$u_{,1} + uu_{,2} + u_{,222} - \varepsilon u_{,22} = 0 \tag{12.2}$$

```
in "relie.red" $
jetorder:=3 $
approxorder:=1 $
let epsilon^2 = 0 $
xvar:={x1,x2} $
uvar:={u} $
diffeqs:={u_x1+u*u_x2+u_x2x2x2-epsilon*u_x2x2} $
leadders:={u_x1} $
relie(4) $
;end;
```

As a result, we get:

```
symmetries ->
{
 {
  {},
  {eta1_u=( - 3*k_4 + 2*k_6*u0)/3,
   eta0_u= - k_1,
   xi1_x2=( - 3*k_4*x1 - 3*k_5 - k_6*x2)/3,
   xi1_x1= - k_6*x1 - k_7,
   xi0_x2= - k_1*x1 - k_2,
   xi0_x1= - k_3},
  {k_1,k_2,k_3,k_4,k_5,k_6,k_7},
  {}
 }
}
```

Invoking `reliegen(1,{-1,-1,-1,-1,-1,-1,-1})` yields

```
generators ->
{
  {0,x1,1},
  {0,1,0},
  {1,0,0},
  {0,epsilon*x1,epsilon},
  {0,epsilon,0},
  {epsilon*x1,(epsilon*x2)/3,( - 2*epsilon*u0)/3},
  {epsilon,0,0}
}
```

so that a basis of the approximate Lie algebra of symmetries of Eq. (12.2) is (after a reordering)

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = x_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u},$$
$$\Xi_4 = \varepsilon \left(x_1 \frac{\partial}{\partial x_1} + \frac{x_2}{3} \frac{\partial}{\partial x_2} - \frac{2}{3} u_0 \frac{\partial}{\partial u} \right), \qquad \Xi_5 = \varepsilon \left(x_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u} \right) = \varepsilon \Xi_3,$$
$$\Xi_6 = \varepsilon \frac{\partial}{\partial x_1} = \varepsilon \Xi_1, \qquad \Xi_7 = \varepsilon \frac{\partial}{\partial x_2} = \varepsilon \Xi_2.$$

Remark 12.4 If the differential equation involves some of the dependent variables or some of their derivatives not in explicit and definite form, the user needs to expand these functions in power series of ε up to the desired order of approximation. For instance, to compute the first-order approximate symmetries of the equation

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial}{\partial x_2} \left(f(u) \frac{\partial u}{\partial x_2} \right) + \varepsilon \frac{\partial u}{\partial x_1} = 0,$$

we have to write

```
depend f,u0 $
diffeqs:={u_x1x1-(f+epsilon*df(f,u0)*u1)*u_x2x2-(df(f,u0)
              +epsilon*df(f,u0,2)*u1)*u_x2^2-epsilon*u_x1} $
```

Of course, the user may insert a differential equation completely expanded in power series of ε, but this is not strictly required.

To compute approximate conditional, contact or variational symmetries the only difference with respect to the corresponding *exact* case consists in setting a positive value to `approxorder` and defining the rule

```
let epsilon^(approxorder+1) = 0 $
```

12.2.8 Computation of Equivalence Transformations

If we consider a class of differential equations, and want to determine equivalence transformations some more input data are necessary. In particular, in addition to the data already discussed for the computation of Lie point symmetries, we need to set:

1. the list `arbelem` of the arbitrary elements involved in the differential equations;
2. the integer `arborder` denoting the highest order of the derivatives of the arbitrary elements with respect to their arguments;
3. the integer `zorder` characterizing the variables the arbitrary elements depend on; for instance, if `zorder` is 0, then the arbitrary elements depend at most on the independent and dependent variables; if `zorder` is 1, then the arbitrary elements depend at most on the independent, dependent variables and first-order derivatives, ….

If we want to remove the dependence of some arbitrary elements on some variables, we need to add *auxiliary conditions* to the differential equations at hand.

The infinitesimal generators of the arbitrary elements are automatically computed by the program (so the user is not requested to set them), and stored, together with the infinitesimals of independent and dependent variables in the list `allinfinitesimals`. The infinitesimal generator of an arbitrary element is denoted by `mu_` followed by the name of the arbitrary element. Let us illustrate with two different examples how ReLie works in such a situation.

Example 12.13 (*System of two balance laws* [32]) Consider the class $\mathcal{E}(\mathbf{p})$ with $\mathbf{p} = (p_1, p_2, p_3)$ of (2×2) systems

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial x} = 0, \\ &\frac{\partial u_2}{\partial t} + p_1(x, t, u_1, u_2)\frac{\partial u_1}{\partial x} + p_2(x, t, u_1, u_2)\frac{\partial u_2}{\partial x} = p_3(x, t, u_1, u_2), \end{aligned}$$

where $p_1(x, t, u_1, u_2)$, $p_2(x, t, u_1, u_2)$ and $p_3(x, t, u_1, u_2)$ are arbitrary continuously differentiable functions of the indicated arguments. The necessary input in ReLie to compute the equivalence transformations of such a system is:

```
in "relie.red" $
jetorder:=1 $
uvar:={u1,u2} $
xvar:={t,x} $
arbelem:={p1,p2,p3} $
arborder:=0 $
zorder:=0 $
diffeqs:=
    {
     u1_t-u2_x,
     u2_t+p1*u1_x+p2*u2_x-p3
    } $
leadders:={u1_t,u2_t}$
```

The functions to be called are the same as in the case of Lie point symmetries. Therefore, with the command

```
relie(4) $
```

the program calls in sequence `relieinit()`, `relieinv()`, `reliedet()`, and `reliesolve()`. As a result we have the generators:

```
symmetries ->
{
 {
  {},
  {mu_p3= - df(c_12,t,x)*p2 - df(c_12,t,2) - df(c_12,x,2)*p1
    - df(c_3,t,x)*p2*u1 - df(c_3,t,2)*u1 - df(c_3,x,2)*p1*u1
    - df(c_4,t,x)*p2*u2 - df(c_4,t,2)*u2 - 2*df(c_4,t)*p3
    - df(c_4,x,2)*p1*u2 - df(c_4,x)*p2*p3 + c_11*p3,
   mu_p2=2*df(c_3,t) + df(c_3,x)*p2 - df(c_4,t)*p2
     + 2*df(c_4,x)*p1 - df(c_4,x)*p2^2,
   mu_p1=df(c_3,t)*p2 + 2*df(c_3,x)*p1
     - 2*df(c_4,t)*p1 - df(c_4,x)*p1*p2,
   eta_u2= - df(c_12,t) - df(c_3,t)*u1
     - df(c_4,t)*u2 + c_11*u2,
   eta_u1= - df(c_12,x) - df(c_3,x)*u1
     - df(c_4,x)*u2 + c_11*u1,
   xi_x=c_3,
   xi_t=c_4},
  {c_3,c_4,c_12,c_11},
  {}
 }
}
```

The dependencies of the parameters involved in the infinitesimals can be determined calling the function `fargs()`.

Example 12.14 (*System of three balance laws* [33]) Consider the class $\mathcal{E}(\mathbf{p})$ with $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5, p_6)$ of systems

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial y} = 0,\\
&\frac{\partial u_2}{\partial t} + \frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial y} - p_5 = 0,\\
&\frac{\partial u_3}{\partial t} + \frac{\partial p_3}{\partial x} + \frac{\partial p_4}{\partial y} - p_6 = 0,
\end{aligned}$$

where t, x, and y are the independent variables, u_1, u_2, and u_3 the dependent variables, whereas $p_i \equiv p_i(t, x, y, u_1, u_2, u_3)$ $(i = 1, \ldots, 6)$ stand for arbitrary continuously differentiable functions of the indicated arguments (see Sect. 10.9).

ReLie determines the equivalence transformations with the code

```
in "relie.red" $
jetorder:=1 $
uvar:={u1,u2,u3} $
xvar:={t,x,y} $
arbelem:={p1,p2,p3,p4,p5,p6} $
arborder:=1 $
zorder:=0 $
diffeqs:={u1_t+u2_x+u3_y,
          u2_t+p1_x+p1_u1*u1_x+p1_u2*u2_x+p1_u3*u3_x
              +p2_y+p2_u1*u1_y+p2_u2*u2_y+p2_u3*u3_y-p5,
          u3_t+p3_x+p3_u1*u1_x+p3_u2*u2_x+p3_u3*u3_x
              +p4_y+p4_u1*u1_y+p4_u2*u2_y+p4_u3*u3_y-p6} $
leadders:={u1_t,u2_t,u3_t} $
relie(4) $
;end;
```

As a result we have the infinitesimals:

```
symmetries ->
{
 {
  {
   df(f_3,y)+df(f_6,t)+df(f_7,x)
  },
  {
   eta_u3=-df(f_1,t)*u1-df(f_1,x)*u2+df(f_10,t)*u3
     +df(f_2,x)*u3+f_3-k_1*u3,
```

```
    eta_u2=df(f_1,y)*u2+df(f_10,t)*u2-df(f_2,t)*u1
      -df(f_2,y)*u3+f_7-k_1*u2,
    eta_u1=df(f_1,y)*u1+df(f_2,x)*u1+f_6-k_1*u1,
    xi_y=-f_1,
    xi_x=-f_2,
    xi_t=-f_10,
    mu_p6=-2*df(f_1,t,x)*u2-2*df(f_1,t,y)*u3-df(f_1,t,2)*u1
      -df(f_1,x,y)*p2 -df(f_1,x,y)*p3-df(f_1,x,2)*p1
      -df(f_1,x)*p5-df(f_1,y,2)*p4+df(f_10,t,2)*u3
      +2*df(f_10,t)*p6+df(f_2,x)*p6+df(f_3,t)
       +df(f_4,x)-df(f_5,y)-k_1*p6,
    mu_p5=df(f_1,y)*p5+df(f_10,t,2)*u2+2*df(f_10,t)*p5
      -2*df(f_2,t,x)*u2-2*df(f_2,t,y)*u3-df(f_2,t,2)*u1
      -df(f_2,x,y)*p2-df(f_2,x,y)*p3-df(f_2,x,2)*p1
      -df(f_2,y,2)*p4-df(f_2,y)*p6+df(f_7,t)+df(f_8,y)
      +df(f_9,x)-k_1*p5,
    mu_p4=-2*df(f_1,t)*u3-df(f_1,x)*p2-df(f_1,x)*p3
      -df(f_1,y)*p4+2*df(f_10,t)*p4+df(f_2,x)*p4-f_5-k_1*p4,
    mu_p3=-df(f_1,t)*u2-df(f_1,x)*p1+2*df(f_10,t)*p3
      -df(f_2,t)*u3-df(f_2,y)*p4+f_4-k_1*p3,
    mu_p2=-df(f_1,t)*u2-df(f_1,x)*p1+2*df(f_10,t)*p2
      -df(f_2,t)*u3-df(f_2,y)*p4+f_8-k_1*p2,
    mu_p1=df(f_1,y)*p1+2*df(f_10,t)*p1-2*df(f_2,t)*u2
      -df(f_2,x)*p1-df(f_2,y)*p2-df(f_2,y)*p3+f_9-k_1*p1
   },
   {
     f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_10, k_1
   },
   {}
  }
 }
```

where `f_1`, …, `f_10` are functions of t, x, y, along with the constraint

$$\frac{\partial f_6}{\partial t} + \frac{\partial f_3}{\partial x} + \frac{\partial f_7}{\partial y} = 0.$$

Example 12.15 The equivalence transformations of the equation

$$\frac{\partial^2 u}{\partial t^2} - f\left(x, \frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} - g\left(x, \frac{\partial u}{\partial x}\right) = 0$$

are immediately found in ReLie with the following code:

```
in "relie.red" $
jetorder:=2 $
xvar:={t,x} $
uvar:={u} $
zorder:=1 $
arborder:=1 $
arbelem:={f,g} $
diffeqs:={u_tt-f*u_xx-g, f_t, g_t, f_u, g_u, f_u_t, g_u_t} $
leadders:={u_tt, f_t, g_t, f_u, g_u, f_u_t, g_u_t} $
relie(4) $
```

Notice that we have added in `diffeqs` the auxiliary conditions; the variable `zorder` has been set to 1 since the functions f and g depend on first-order derivatives; moreover, the variable `arborder` is set to 1 since in the auxiliary conditions the derivatives of f and g with respect to some of their arguments occur.

As a result we obtain:

```
symmetries ->
{
 {
  {},
  {eta_u=(2*f_1-2*k_1*u-k_6*t^2-2*k_7*t-2*k_8)/2,
   xi_x= -k_4*x-k_5,
   xi_t=-k_2*t-k_3,
   mu_g=-df(f_1,x,2)*f-g*k_1+2*g*k_2-k_6,
   mu_f=2*f*k_2-2*f*k_4
  },
  {f_1,k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8},
  {}
 }
}
```

where the function `f_1` is arbitrary and depends on x.

Remark 12.5 By default, the infinitesimals for the independent and dependent variables do not depend on arbitrary elements; if we are interested to general equivalence transformations where also the infinitesimal generators of the independent and dependent variables depend on the arbitrary elements [28], then we have to add the statement

```
generalequiv:=1 $
```

12.2.9 Inverse Lie Problem

Here we show with a simple example how RELIE can be used for investigating the Lie remarkability of differential equations [16, 24–26]. The following code shows how one can easily verify that the Monge–Ampère equation describing a surface in $\mathbb{R}^3$ with zero Gaussian curvature,

$$u_{,11}u_{,22} - u_{,12}^2 = 0,$$

is strongly Lie remarkable [26]:

```
in "relie.red" $
jetorder:=2 $
xvar:={x1,x2} $
uvar:={u} $
relieinit() $
generatealgebra(2) $
;end;
```

After setting the necessary objects (`jetorder`, `xvar` and `uvar`), and calling `relieinit()`, we generate the vector fields spanning the affine Lie algebra in $\mathbb{R}^3$. Then, calling `testrank(generators)`, we ascertain that the rank of the second-order distribution is maximal (equal to the dimension of second-order jet space); the call `inverselie(8)` computes the list `allminors` of all minors of order 8 of such a distribution; all minors vanish by evaluating them on the differential equation, that can be verified by calling

```
allminors:=sub(u_x2x2=u_x1x2^2/u_x1x1,allminors) $
```

As a last check,

```
rank(sub(u_x2x2=u_x1x2^2/u_x1x1,distribution);
```

returns 7, which is the dimension of the manifold in the second-order jet space characterized by the Monge–Ampère equation.

12.2.10 Inside RELIE: Global Variables and Routines

RELIE uses some global variables to perform the various tasks: they can be distinguished among *input variables* (that the user needs to set before starting computation), *output variables* (computed by RELIE and of interest to the user), and *intermediate variables* (computed by RELIE and in general not of interest to the common user).

12.2.10.1 Input Variables

- `approxorder`: maximum order of approximate symmetries of equations containing a small parameter; by default it is 0, i.e., exact symmetries; the small parameter involved in the approximate symmetries must be denoted by `epsilon`; when looking for approximate symmetries the user has to define the rule

```
let epsilon^(approxorder+1)=0;
```

- `arbelem`: list of the arbitrary elements (only for equivalence transformations); by default it is an empty list;
- `arborder`: maximum order of derivatives of arbitrary elements (only for equivalence transformations); by default it is -1, i.e., point symmetries;
- `contact`: set to 1 for contact symmetries; by default it is 0;
- `diffeqs`: list of the left-hand sides of differential equations (with vanishing right-hand sides);
- `divergence`: set to 1 if divergence symmetries of a Lagrangian are needed; by default it is 0;
- `freepars`: list of arbitrary constants or functions involved in the differential equations (for group classification problems); by default it is an empty list;
- `generalequiv`: set to 1 for general equivalence transformations where all infinitesimals depend on independent and dependent variables and arbitrary elements; the default value is 0, meaning that the infinitesimals of independent and dependent variables do not depend upon the arbitrary elements;
- `jetorder`: maximum order of derivatives in differential equations;
- `lagrangian`: a list with only one element corresponding to the Lagrangian (it is necessary to set `variational` to 1, or `divergence` to 1);
- `leadders`: list of the leading derivatives; `diffeqs` are solved with respect to them;
- `nonclassical`: set to a value between 1 and the number of independent variables (only for conditional symmetries); by default it is 0, i.e., classical symmetries;
- `nonpolyders`: list of derivatives not occurring in polynomial form in the differential equations; by default it is an empty list;
- `nonzeropars`: list of arbitrary constants or functions involved in the differential equations that cannot vanish; by default it is an empty list;
- `qcond`: list of indexes of dependent variables whose invariant surface conditions have to be used for computing conditional symmetries;
- `uvar`: list of the dependent variables;
- `variational`: set to 1 if variational symmetries of a Lagrangian are needed; by default it is 0;

- `xvar`: list of the independent variables;
- `zorder`: maximum order of derivatives of `uvar` with respect to `xvar` the elements in `arbelem` depend on (only for equivalence transformations); if `zorder` is set to 0, the arbitrary elements depend on `xvar` and `uvar`; `zorder` cannot exceed `jetorder`.

12.2.10.2 Output Variables

- `allinfinitesimals`: list of two lists; the first sublist is the list of the infinitesimals, in order, of the independent variables, dependent variables and arbitrary elements (the latter in the case of equivalence transformations); the second sublist is the list of various terms (constants and functions) involved in the expression of infinitesimals;
- `allminors`: list of minors of a given order extracted from a matrix; returned by the function `minors(m,k)`, where `m` is a matrix and `k` a positive integer, or by the function `inverselie(k)` that takes the `jetorder`-th distribution of `generators` as the matrix from which the minors of order k are extracted;
- `arbconst`: list of arbitrary constants involved in the expression of the symmetries;
- `arbfun`: list of arbitrary functions involved in the symmetries;
- `cogenerators`: list of the functions entering the definition of divergence symmetries and corresponding to the infinitesimal generators (produced by `reliegen()`);
- `commtable`: table of commutators of a list of vector fields;
- `deteqs`: list of the determining equations (produced by `reliedet()`);
- `distribution`: matrix of the `jetorder`-th distribution of a list of generators (produced by `reliedistrib()`), i.e., a matrix where each row is the prolonged vector field evaluated on one of the obtained infinitesimal generators;
- `fluxes`: list of the fluxes of the conservation law corresponding to a Lie generator (computed by `relieclaw()`);
- `generators`: list of the infinitesimal generators of the finite Lie algebra admitted by the differential equations at hand (produced by `reliegen()`); the list `generators` may also been obtained by calling `generatealgebra(k)`, where `k` can be 1 (algebra of isometries), 2 (algebra of affine transformations) or 3 (algebra of projective transformations); of course, it is necessary to set `jetorder`, `xvar` and `uvar` before calling `generatealgebra(k)`;
- `invcond`: list of the invariance conditions of the differential equations at hand (produced by `relieinv()`);
- `nzcomm`: list of nonzero commutators of a list of vector fields;
- `prolongation`: list of two lists: the first one is the list of the coordinates of the jet space, the second one the list of the corresponding infinitesimals (produced by `relieprol()`);
- `symmetries`: list of the infinitesimals admitted by the differential equations at hand (produced by `reliesolve()`).

12.2.10.3 Intermediate Variables

- `jet`: list of three lists: indices for computing the infinitesimals and their prolongations, coordinates of jet space and their internal representation;
- `jetapprox`: list of two lists: list of independent variables and expansions of dependent variables and their derivatives, and list of their internal representation (only for approximate symmetries);
- `jetequiv`: list of three lists: indices for computing the infinitesimals and their prolongations for arbitrary elements, arbitrary elements, and their internal representation (only for equivalence transformations);
- `jetsplit`: list of two lists: indices for computing the infinitesimals and their prolongations, list of independent variables, zeroth order dependent variables and their derivatives (only for approximate symmetries);
- `solutiondedv`: solution of the differential equations specified in `diffeqs` with respect to `leadders`; for conditional symmetries, the invariant surface conditions and their needed differential consequences are solved too;
- `steprelie`: integer that stores the status of the computation; 0: no computation done; 1: `relieinit()` has been called; 2: `relieinv()` has been called; 3: `reliedet()` has been called; 4: `reliesolve()` has been called;
- `zvar`: list of two lists: the first one is the list of the variables `arbelem` depend on, the second one the corresponding infinitesimals (only for equivalence transformations).

12.2.10.4 Functions

A short description of the main functions the user may call is the following:

- `abelian(gens)`: checks if the generators `gens` span an Abelian Lie algebra;
- `commutatortable(gens)`: returns the commutator table of the generators `gens`;
- `essentialpars(gens,vars)`: takes a list of generators `gens` of a multiparameter Lie group of transformations for the variables `vars`, and returns the generators which are not linearly independent;
- `generatealgebra(k)`: once `jetorder`, `xvar` and `uvar` have been properly assigned, this function returns a list of generators spanning the algebra of isometries (for `k = 1`), affine algebra (for `k = 2`), projective algebra (for `k = 3`);
- `inverselie(k)`: computes all the minors of order `k` of the `jetorder`-th distribution generated by the list of vector fields contained in `generators`;
- `liebracket(gen1,gen2)`: returns the Lie bracket of the generators `gen1` and `gen2`;
- `newordering(lis,ind)`: returns a list of the elements in `lis` reordered according to the permutation `ind` of the integers $\{1, \ldots, n\}$, where n is the length of list `lis`;
- `nonzerocommutators(gens)`: returns `nzcomm`, a list of nonzero commutators of generators `gens`;

- `offprintcrack()`: prevents `reliesolve()` to display the steps needed for solving determining equations (this is the default configuration);
- `onprintcrack()`: sets a variable used in Crack package (used in the function `reliesolve()`) in order to display the steps needed for solving determining equations;
- `relieclaw(k)`: returns `fluxes`, a list of the components of the fluxes of the conservation law corresponding to the `k`-th Lie generator (obtained after calling `reliegen()`);
- `reliedet()`: splits the invariant conditions providing the list `deteqs` of determining equations;
- `reliedistrib()`: returns the matrix `distribution`, i.e., a matrix whose rows are the prolonged vector fields evaluated in the list `generators` (computed by the function `reliegen()`, or by the function `generatealgebra()`, or suitably assigned by the user;
- `reliegen(k,lis)`: returns the list `generators`; k is an integer (less or equal to the length of `symmetries`) and `lis` a list that can be empty; if `lis` is made by as many values as the number of arbitrary constants occurring in `symmetries`, `generators` consists of a list of vector fields, where each vector field is obtained replacing the i-th parameter by the i-th element in the list `lis` (or 1 if `lis` is empty or its length is different from the number of arbitrary constants entering `symmetries`) and the other parameters are replaced by 0; if the list `lis` is `{-1}`, then `generators` is a list with containing the components of the infinitesimals in their general form, i.e., the linear combinations of all admitted generators;
- `relieinit()`: if input data have been correctly defined, the function initializes the objects for doing the computation;
- `relieinv()`: computes the invariance conditions; returns `invcond`;
- `relieprol()`: returns the prolongation of a general vector field;
- `reliesolve()`: solves the determining equations for the infinitesimals; returns `symmetries`; in group classification problems (but also in the case of conditional symmetries), the list `symmetries` may contain different solutions for the infinitesimals according to the values of `freepars`; as a default `reliesolve()` does not display the steps made to obtain the solution of determining equations; the user can see these steps by calling `onprintcrack()`; this is suggested when `reliesolve()` seems to use too much time to complete its execution; the list `symmetries` contains a list of the sets of solutions of determining equations; each element of this list in turn is a list of four elements: the first one is a list of conditions (possibly empty) that remained unsolved; the second one is a list giving the solution to the determining equations, i.e., the expressions of the infinitesimals; the third one is a list containing the parameters involved in the solution; the fourth one is a list of expressions which cannot vanish (this list can be empty);
- `solvable(gens)`: checks if the generators `gens` span a solvable Lie algebra;
- `testrank(gens)`: returns the rank of the `jetorder`-th distribution generated by the generators `gens`.

Notice that ReLie contains some companion functions, used by the main procedures. Here we list some of them that can be useful in interactive sessions.

- `allcoeffs(lis1,lis2)`: returns the list of coefficients of `lis1` (list of polynomyals) with respect to the variables in list `lis2`;
- `bincoeff(n,k)`: returns $\binom{n}{k}$;
- `combnorep(n,k)`: returns the combinations without repetition of k elements chosen in $\{1, 2, \ldots, n\}$;
- `combrep(n,k)`: returns the combinations with repetition of k elements chosen in $\{1, 2, \ldots, n\}$;
- `delzero(lis)`: returns a list containing all nonzero elements in the list `lis`;
- `dependence(lis1,lis2)`: declares that the elements in the list `lis1` depend on the variables in the list `lis2`;
- `dlie(obj,var)`: returns the usual Lie derivative of `obj` with respect to `var`;
- `dlieapprox(obj,var)`: returns the Lie derivative of `obj` with respect to `var` in the context of approximate symmetries;
- `dliestar(obj,var)`: returns the additional Lie derivative used for equivalence transformations;
- `kroneckerdelta(k1,k2)`: returns 1 if $k1 = k2$, 0 otherwise;
- `letterlist(obj,n)`: `obj` is a symbol, n a positive integer;
  for instance, `letterlist(x,4)` builds the list $\{x1, x2, x3, x4\}$;
- `letterlistvar(obj,lis)`: `obj` is a symbol, `lis` a list;
  for instance, `letterlistvar(xi_,{x,y})` builds the list `{xi_x,xi_y}`;
- `listletter(lis,ch)`: `lis` is a list, `ch` a symbol;
  for instance, `listletter({u_,v_},x)` builds the list `{u_x,v_x}`;
- `membership(elem,lis)`: returns the number of occurrences of the element `elem` in the list `lis`;
- `minors(m,k)`: `m` is a matrix and `k` is a positive integer: returns the list of minors of order `k` of the matrix `m`;
- `nodependence(lis1,lis2)`: removes the dependence of the objects in the list `lis1` upon the variables in the list `lis2`;
- `removeelement(lis,elem)`: removes the element `elem` from the list `lis`;
- `removemultiple(lis1,lis2)`: removes from the list `lis1` the elements of the list `lis2`;
- `scalarmult(obj,lis)`: returns a list whose k-th element is the product of the scalar `obj` and the k-th element of list `lis`;
- `scalarproduct(lis1,lis2)`: returns the sum of the products element by element of two lists with the same number of elements;
- `sumlist(lis1,lis2)`: returns a list summing element by element the two lists with the same length;
- `zerolist(n)`: returns a list of n zeros, the empty list if $n \leq 0$.

12.3 SymboLie

As shown in Sect. 6.8, Lie groups of symmetries of differential equations constitute a fundamental tool for constructing group invariant solutions. The number of subgroups is potentially infinite and so the number of invariant solutions; thus, it is crucial to obtain a classification of subgroups in order to have an *optimal system* of inequivalent solutions from which all other solutions can be derived by the action of the group itself. Since Lie groups are intimately connected to Lie algebras, a classification of inequivalent subgroups induces a classification of inequivalent Lie subalgebras, and vice versa (see also Sect. 3.6). A general method for classifying the Lie subalgebras of a finite-dimensional Lie algebra uses inner automorphisms obtained by exponentiating the adjoint groups. In this section, we present an effective algorithm able to automatically determine optimal systems of families of Lie subalgebras of a generic finite-dimensional Lie algebra abstractly assigned by means of its structure constants, or realized in terms of matrices or vector fields, or defined by a basis and the set of nonzero Lie brackets. The algorithm [1, 2] is implemented in the Computer Algebra System Wolfram Mathematica™[46]; some meaningful and nontrivial examples are also considered.

We introduce some new definitions useful for the algorithmic computation of optimal systems of Lie subalgebras, and clarify them by means of some examples. Then, we describe in detail the main algorithms allowing us to automatically determine optimal systems of an r-dimensional real Lie algebra $\mathbb{L}_r$. If not assigned, the program first computes the structure constants of the Lie algebra.

Let us first analyze the case of one-dimensional Lie subalgebras, and then the one of Lie subalgebras of higher dimension.

12.3.1 One-Dimensional Lie Subalgebras

As remarked in Sects. 3.6 and 6.8, a one-dimensional Lie subalgebra is completely defined by the coefficients of the linear combination of the generators of the basis of $\mathbb{L}_r$, say

$$X = f^1 \Xi_1 + f^2 \Xi_2 + \cdots + f^r \Xi_r,$$

and actions of inner automorphisms can be suitably transferred to the *coordinates* $(f^1, f^2, \ldots, f^r)$ of an r-dimensional vector; the method largely used in the literature for obtaining optimal systems of one-dimensional Lie subalgebras consists in using inner automorphisms to obtain from the most general r-dimensional vector the maximum possible number of zero coordinates, also using the invariants of the inner automorphisms. More in detail, the method takes a tuple $\{f^1, f^2, \ldots, f^r\}$, and, through *judicious* applications of inner automorphisms, simplifies as many of the coefficients f^α [34]. Though this approach is straightforward, it may imply technical difficulties when implemented in a Computer Algebra System, as one needs

to solve algebraic equations and make suitable choices during the process to distinguish cases; moreover, it is a relatively easy task only for low-dimensional Lie algebras, and the *simplicity* of the results obtained is not always clear. Typically, this is a top-down approach.

Our aim is to render the process of identifying similar subalgebras automatic; to achieve this result, we adopt a general algorithm with a bottom–up philosophy, and reduce to a minimum the need of solving algebraic equations. To this end, let us introduce some more definitions.

Definition 12.1 For any integer $r > 1$, let $\mathcal{S}_r$ be the set of all possible tuples with r components (not all zero) chosen in $\{0, 1\}$, i.e.,

$$\mathcal{S}_r = \{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (1, 1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1)\}\,.$$

Definition 12.2 (*p–family of one–dimensional Lie subalgebras*) Let $\mathbb{L}_r$ be an r-dimensional Lie algebra generated by $\{\Xi_1, \Xi_2, \ldots, \Xi_r\}$, and let $\mathbf{f} \equiv \{f^1, f^2, \ldots, f^r\}$ be a tuple of r functions depending on some variables belonging to a set $\mathcal{P}$. Then, the family of one-dimensional subalgebras

$$X = f^1 s_1 \Xi_1 + f^2 s_2 \Xi_2 + \cdots + f^r s_r \Xi_r,$$

where $\mathbf{s} \equiv (s_1, \ldots, s_r) \in \mathcal{S}_r$, is called a *p–family* of one-dimensional Lie subalgebras of $\mathbb{L}_r$ if the rank of the Jacobian matrix of $(f^1 s_1, f^2 s_2, \ldots, f^r s_r)$ with respect to the elements of $\mathcal{P}$ is equal to $p = \sum_{i=1}^{r} s_i$. This is equivalent to say that the decomposition of X in terms of the elements of the basis of the Lie algebra involves p functionally independent components.

Example 12.16 Let $\mathbb{L}_4$ be a four-dimensional Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$, and consider the family of one-dimensional subalgebras

$$X = f^1 \Xi_1 + f^2 \Xi_2,$$

with f^1 and f^2 arbitrary real numbers; in this case, it is $\mathcal{P} = \{f^1, f^2\}$, and we have a 2–family of one-dimensional Lie subalgebras. Also the family

$$X' = (f^1 \cos(t) - f^2 \sin(t))\Xi_1 + (f^1 \sin(t) + f^2 \cos(t))\Xi_2,$$

where $\mathcal{P} = \{f^1, f^2, t\}$, is a 2–family of one-dimensional Lie subalgebras which has to be considered indistinguishable from X. On the contrary, the family

$$Y = f^1 \Xi_1 + f^1 (f^2)^2 \Xi_2 + f^1 f^2 \Xi_3,$$

even if the vector $(f^1, f^1(f^2)^2, f^1 f^2, 0)$ possesses three nonzero components, does not correspond to a 3–family; in fact, we have $\mathcal{P} = \{f^1, f^2\}$, and the rank of the Jacobian matrix of the components of Y with respect to the elements of $\mathcal{P}$ is 2.

Remark 12.6 The p–family of one-dimensional Lie subalgebras of $\mathbb{L}_r$

$$X = f^1 s_1 \Xi_1 + f^2 s_2 \Xi_2 + \cdots + f^r s_r \Xi_r,$$

where $\mathbf{s} \equiv \{s_1, s_2, \ldots, s_r\} \in \mathcal{S}_r$, $p = \sum_{i=1}^r s_i$, in the following will be represented by the tuple

$$\left(f^1 s_1, f^2 s_2, \ldots, f^r s_r\right)$$

which in turn can be identified by the integer $c_X = \sum_{k=1}^{r} s_k 2^{k-1}$. Moreover, where there is no need to specify a value for p, we will write simply family instead of p–family.

Let us introduce a relation $\mathcal{R}$ between families of one-dimensional Lie subalgebras.

Definition 12.3 (*Relation between p–families of one–dimensional Lie subalgebras*) Let $\mathbb{L}_r$ be an r-dimensional Lie algebra generated by $\{\Xi_1, \ldots, \Xi_r\}$ and let $X \equiv (f^1 s_1, \ldots, f^r s_r)$ and $Y \equiv (\tilde{f}^1 \tilde{s}_1, \ldots, \tilde{f}^r \tilde{s}_r)$ two different families of Lie subalgebras. We write

$$X \mathcal{R} Y \quad \text{or} \quad (X, Y) \in \mathcal{R}$$

if there exists an inner automorphism $A \in \mathrm{Int}(\mathbb{L}_r)$ such that $Z^T = A X^T$ (^{T} denotes the transpose operator) is a family of one-dimensional Lie subalgebras such that $c_Z = c_Y$.

If it is also

$$Y \mathcal{R} X \quad \text{or} \quad (Y, X) \in \mathcal{R},$$

then we say that the two families X and Y are *equivalent*.

According to Definition 12.3, the relation $\mathcal{R}$ is clearly reflexive and transitive, but, in general, it is not necessarily a symmetric relation, as will be shown in Example 12.17. In such a case, $\mathcal{R}$ is a so-called *preorder*, and, it is well known that a preorder induces an equivalence relation, defined by forcing the symmetric property; nevertheless, we do not assume the symmetric property to be a priori satisfied by the relation $\mathcal{R}$.

Remark 12.7 For an r-dimensional Lie algebra, all $2^r - 1$ possible families of one-dimensional subalgebras, according to relation $\mathcal{R}$, can be represented by means of a suitable *directed* multigraph $\mathcal{G}(\mathbb{L}_r)$ (this because more than one automorphism can be such that the two families belong to the relation $\mathcal{R}$), where the vertices correspond to the various families of one-dimensional Lie subalgebras, and the edges to

the automorphisms connecting couples of subalgebras belonging to $\mathcal{R}$. Indeed, we represent this multigraph as a graph by means of its adjacency matrix whose (i, j)th entry is 1 if the ith family is mapped by some automorphism to jth family, and zero otherwise.

Example 12.17 Let $\mathbb{L}_4$ be a four-dimensional Lie algebra,

$$\mathcal{L}_4 = \langle \Xi_1, \Xi_2, \Xi_3, \Xi_4 \rangle ,$$

with nonzero Lie brackets

$$[\Xi_2, \Xi_4] = \Xi_1, \qquad [\Xi_3, \Xi_4] = \Xi_2.$$

As the inner automorphism $\exp(t_1 \operatorname{ad}_{\Xi_1})$ is the identity morphism, in the following we write the matrices associated to inner automorphisms $A_i = \exp(t_i \operatorname{ad}_{\Xi_i})$ $(i = 2, 3, 4)$:

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & -t_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -t_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 1 & t_4 & \frac{t_4^2}{2} & 0 \\ 0 & 1 & t_4 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Acting with the automorphism A_4 on the 2–family $\left(0, f^2, f^3, 0\right)$ yields

$$\left(f^2 t_4 + \frac{f^3 t_4^2}{2}, f^2 + f^3 t_4, f^3, 0\right),$$

which is a 3–family according to Definition 12.2; thus, we can say

$$\left(0, f^2, f^3, 0\right) \mathcal{R} \left(f^1, f^2, f^3, 0\right).$$

Vice versa, it is

$$\left(f^1, f^2, f^3, 0\right) \overset{A_4}{\mapsto} \left(f^1 + f^2 t_4 + \frac{f^3 t_4^2}{2}, f^2 + f^3 t_4, f^3, 0\right),$$

and the image cannot be reduced to $\left(0, f^2, f^3, 0\right)$ for all choices of f^1, f^2 and f^3, whence

$$\left(f^1, f^2, f^3, 0\right) \not\mathcal{R} \left(0, f^2, f^3, 0\right).$$

Moreover, choosing $t_4 = -\dfrac{f^2}{f^3}$, the 2–family $(0,\, f^2,\, f^3, 0)$ is mapped by A_4 to $\left(-\dfrac{(f^2)^2}{2f^3}, 0,\, f^3, 0\right)$, which is a 2-family. Hence, we can write

$$\left(0,\, f^2,\, f^3, 0\right) \mathcal{R} \left(f^1, 0,\, f^3, 0\right).$$

Finally, as

$$\left(f^1, 0,\, f^3, 0\right) \overset{A_4}{\mapsto} \left(f^1 + \frac{f^3 t_4^2}{2},\, f^3 t_4,\, f^3, 0\right),$$

we have

$$\left(f^1, 0,\, f^3, 0\right) \mathcal{R} \left(f^1,\, f^2,\, f^3, 0\right),$$

but

$$\left(f^1, 0,\, f^3, 0\right) \not\mathcal{R} \left(0,\, f^2,\, f^3, 0\right).$$

Definition 12.4 (*Ordering of p–families of one–dimensional Lie subalgebras*) The families of one-dimensional Lie subalgebras are sorted according to the short-lexicographical ordering (slex). In more detail, let $\mathbb{L}_r$ be an r-dimensional Lie algebra and let

$$X = \left(f^1 s_1,\, f^2 s_2, \ldots,\, f^r s_r\right),$$
$$Y = \left(\tilde{f}^1 \tilde{s}_1,\, \tilde{f}^2 \tilde{s}_2, \ldots,\, \tilde{f}^r \tilde{s}_r\right)$$

be two families. It is

$$X >_{\text{slex}} Y \text{ if } \begin{cases} \sum_{i=1}^r s_i > \sum_{i=1}^r \tilde{s}_i \\ \text{or} \\ \sum_{i=1}^r s_i = \sum_{i=1}^r \tilde{s}_i \quad \text{and} \quad c_X > c_Y. \end{cases}$$

Remark 12.8 When we represent the graph corresponding to the set of p–families of one-dimensional Lie subalgebras, the label we append to each vertex X is the position ind_X in the list of ordered families.

12.3.2 Multi-dimensional Lie Subalgebras

As in the case of one-dimensional Lie subalgebras, let us define a p–family of multi-dimensional Lie subalgebras.

Definition 12.5 (***p*–family of** *d–dimensional Lie subalgebras*) Let $\mathbb{L}_r$ be an r-dimensional Lie algebra spanned by $\{\Xi_1, \ldots, \Xi_r\}$, and let f_k^α $(k = 1, \ldots, d, \alpha = 1, \ldots, r)$ be $d \cdot r$ functions depending on some variables belonging to a set $\mathcal{P}$. A tuple of d elements $(X_1, \ldots, X_d)$, where

$$X_k = \sum_{\alpha=1}^{r} f_k^\alpha s_{k,\alpha} \Xi_\alpha, \qquad k = 1, \ldots, d,$$

and $\mathbf{s}_k \equiv (s_{k,1}, \ldots, s_{k,r}) \in \mathcal{S}_r$ for $k = 1, \ldots, d$, is called a p–family of d-dimensional Lie subalgebras of $\mathbb{L}_r$ if

(1) X_k $(k = 1, \ldots, d)$ is a p_k–family of one-dimensional Lie aubalgebras, and $p = \sum_{k=1}^{d} p_k$;
(2) the matrix

$$F = \begin{pmatrix} f_1^1 s_{1,1} & f_1^2 s_{1,2} & \cdots & f_1^r s_{1,r} \\ f_2^1 s_{2,1} & f_2^2 s_{2,2} & \cdots & f_2^r s_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ f_d^1 s_{d,1} & f_d^2 s_{d,2} & \cdots & f_d^r s_{d,r} \end{pmatrix}$$

has rank d;
(3) the rank of the Jacobian matrix of the list of all entries of F with respect to the elements of $\mathcal{P}$ is equal to p;
(4) the conditions

$$\sum_{\alpha,\beta=1}^{r} f_i^\alpha s_{i,\alpha} f_j^\beta s_{j,\beta} C_{\alpha\beta}^\gamma = \sum_{k=1}^{s} \lambda_{ij}^k f_k^\gamma s_{k,\gamma}, \quad i, j = 1, \ldots, d, \gamma = 1, \ldots, r$$

are satisfied for suitable constants λ_{ij}^k whatever the functions f_k^α are, $C_{\alpha\beta}^\gamma$ being the structure constants of the Lie algebra.

Remark 12.9 Definition 12.5 (in particular, item 4) implies a deep simplification of the conditions for the check that the set $\{X_1, \ldots, X_d\}$ is closed with respect to the Lie bracket; in fact, we need to determine the unknowns λ_{ij}^k whatever the values of f_k^α are, and this requires only simple elementary linear algebra tools.

Remark 12.10 We are aware that Definition 12.5 in some cases may leave out some Lie subalgebras. For instance, consider the six-dimensional Lie algebra spanned by $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5, \Xi_6\}$ [36] with the following nonzero Lie brackets:

$$[\Xi_1, \Xi_5] = -\Xi_2,\ [\Xi_2, \Xi_5] = \Xi_1,\ [\Xi_3, \Xi_5] = -\Xi_4,$$
$$[\Xi_3, \Xi_6] = -\Xi_1,\ [\Xi_4, \Xi_5] = \Xi_3,\ [\Xi_4, \Xi_6] = -\Xi_2.$$

The three-dimensional vector space spanned by

$$\{f_1^1 \Xi_1 + f_1^4 \Xi_4,\ f_2^2 \Xi_2 + f_2^3 \Xi_3,\ f_3^5 \Xi_5\}$$

is not a 5–family of three-dimensional Lie subalgebras because the closure with respect to the commutator is not guaranteed unless it is $f_1^1 f_2^3 + f_2^2 f_1^4 = 0$, so lowering the arbitrary parameters to 4. In our approach, such instances of Lie subalgebras are not considered, because this would require some more steps that may be cumbersome to tackle automatically. This limitation will be removed in a future version of the program. Nevertheless, we observe that in most cases (for instance, almost all real three- and four-dimensional Lie algebras [38]) this approach works correctly without losing anything.

Remark 12.11 Of course, because the basis of a d-dimensional Lie subalgebras of a Lie algebra $\mathbb{L}_r$ is not unique, we choose to adopt as the basis of a p–family of d-dimensional Lie subalgebras the one such that the matrix F is in row reduced echelon form (RREF); once the basis for $\mathbb{L}_r$ has been assigned, this matrix represents the Lie subalgebra.

A relation $\mathcal{R}$ linking two different families of d-dimensional Lie subalgebras can be defined as well.

Definition 12.6 (*Relation between p–families of d–dimensional Lie subalgebras*) Let $\mathbb{L}_r$ be an r-dimensional Lie algebra generated by $\{\Xi_1, \ldots, \Xi_r\}$ and let $\mathbf{X} \equiv (X_1, \ldots, X_d)$ and $\mathbf{Y} \equiv (Y_1, \ldots, Y_d)$ be two different families of d-dimensional Lie subalgebras. We write

$$\mathbf{X} \mathcal{R} \mathbf{Y} \quad \text{or} \quad (\mathbf{X}, \mathbf{Y}) \in \mathcal{R}$$

if there exists an inner automorphism $A \in \mathrm{Int}(\mathbb{L}_r)$ such that

$$\mathbf{Z}^T = (A X_1^T, \ldots, A X_d^T),$$

possibly reduced to row echelon form, is a family of d-dimensional Lie subalgebras such that $c_{\mathbf{Z}} = c_{\mathbf{Y}}$, where $c_{\mathbf{X}} \equiv (c_{X_1}, \ldots, c_{X_d})$.

If it is also

$$\mathbf{Y}\,\mathcal{R}\,\mathbf{X},$$

then we say that the two families are *equivalent*.

As in the case of one-dimensional families of Lie subalgebras, relation $\mathcal{R}$ is reflexive and transitive but in general not symmetric.

Definition 12.7 (*Ordering of p–families of d–dimensional Lie subalgebras*) The p–families of d-dimensional Lie subalgebras are sorted according to the short-lexicographical ordering. In more detail, let $\mathbb{L}_r$ be an r-dimensional Lie algebra and let $\mathbf{X} \equiv \{X_1, \ldots, X_d\}$ and $\mathbf{Y} \equiv \{Y_1, \ldots, Y_d\}$ be a p–family and q–family of different d-dimensional Lie subalgebras, respectively. It is

$$\mathbf{X} >_{\text{slex}} \mathbf{Y} \quad \text{if} \quad \begin{cases} p > q \\ \text{or} \\ p = q \quad \text{and} \quad X_k >_{\text{slex}} Y_k, \ \text{where } k = \min\{i : c_{X_i} \neq c_{Y_i}\}. \end{cases}$$

Remark 12.12 A family $\mathbf{X} \equiv (X_1, \ldots, X_d)$ where

$$X_k = \sum_{\alpha=1}^{r} f_k^{\alpha} s_{k,\alpha}\, \Xi_\alpha, \qquad k = 1, \ldots, d,$$

of d-dimensional Lie subalgebras of an r-dimensional Lie algebra $\mathbb{L}_r$ can be represented as well by the tuple $c_{\mathbf{X}}$.

Moreover, when we represent the graph corresponding to the set of families of d-dimensional Lie subalgebras, also in this case the label we append to each vertex is the position, $\text{ind}_{\mathbf{X}}$, in the list of sorted families.

Now we have all the elements for clarifying the details of the algorithms leading to the determination of optimal systems of families of Lie subalgebras.

12.3.3 Algorithms for Optimal Systems of Lie Subalgebras

Let $\mathbb{L}_r$ be an r-dimensional real Lie algebra assigned through its structure constants; the latter are computed if the Lie algebra is realized in terms of vector fields or matrices, or if the list of nonzero Lie brackets is obtained. The program SymboLie, written in Wolfram Mathematica™[46], makes available a set of functions devoted primarily to the construction of optimal systems of families of Lie subalgebras. If the Lie algebra is assigned by means of a suitable realization, or if the list of nonzero Lie brackets is assigned, then the method `StructureConstants[]` computes the structure constants that will be used in all the remaining functions.

Example 12.18 Take the three-dimensional Lie algebra of real 2×2 traceless matrices, generated by

$$\Xi_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \Xi_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \Xi_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

After loading the package SYMBOLIE, the following steps lead to the computation of the structure constants:

- define the generators `gens` of the basis:

```
gens={{{1,0},{0,-1}},{{0,1},{0,0}},{{0,0},{1,0}}};
```

- compute the structure constants:

```
cs = StructureConstants[gens];
```

The structure constants could be used for instance to display the commutator table:

```
CommutatorTable[cs] //MatrixForm
```

$$\begin{bmatrix} 0 & 2\,\Xi_2 & -2\,\Xi_3 \\ -2\,\Xi_2 & 0 & \Xi_1 \\ 2\,\Xi_3 & -\Xi_1 & 0 \end{bmatrix}.$$

The computation of the structure constants through the list of nonzero Lie brackets using a set of unassigned symbols is done as follows:

```
basis = {x1, x2, x3};
bracket = {{{x1, x2}, 2*x2},
           {{x1, x3}, -2*x3},
           {{x2, x3}, x1}
           };
cs = StructureConstants[bracket, basis];
```

Example 12.19 Consider the Lie algebra spanned by the vector fields

$$\Xi_1 = \frac{\partial}{\partial t}, \quad \Xi_2 = \frac{\partial}{\partial x}, \quad \Xi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \Xi_4 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}$$

generating the Lie point symmetries of Korteweg–deVries equation [34],

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

The structure constants are computed as follows:

```
vars = {t,x,u};
gens = {{1,0,0}, {0,1,0}, {0,t,1}, {3t,x,-2u}};
cs = StructureConstants[gens,vars];
```

The main function devoted to the determination of optimal systems of families of Lie subalgebras is the method `SubAlgebra[]` (see Algorithm 1) receiving as input the structure constants `cs` of the Lie algebra, a list `pars` made of two lists specifying arbitrary parameters and their constraints, respectively (this parameter is `{{},{}}` if no arbitrary parameter is involved in the structure constants), and the dimension `dim` of the required subalgebras.

Algorithm 1: `SubAlgebra(cs,pars,dim)`
Returns the `dim`–dimensional optimal subalgebras of the Lie algebra $\mathbb{L}_r$ with structure constants `cs` (involving parameters `pars`)

Input: structure constants `cs`, parameters `pars`, dimension `dim`
Output: adjacency matrix, candidates, structure constants `cs`
begin
  if `pars` *is not empty* **then**
    `cs` ← evaluation of `cs` on an instance of `pars`;
  end
  if `PrintDebug` **then**
    $file.log$ ← printout all the algorithm operations;
  end
  if $1 \leq$ `dim` $< r$ **then**
    return `FindAdjacency(cs,dim)`;
  end
end

Algorithm 2: `CheckAlgebra(alg,cs,dim)`
Whether `alg` generate a p–family of subalgebras of $\mathbb{L}_r$.

Input: `alg`, structure constants `cs`, dimension `dim`
Output: `isSub` boolean value
begin
  `isSub` $\leftarrow true$;
  foreach (s_1, s_2) in 2-element combinations of $\{1, \ldots,$ `dim`$\}$ **do**
    $X \leftarrow set[s_1]$;
    $Y \leftarrow set[s_2]$;
    $eqnI \leftarrow a_{dim+1} \sum_{\alpha,\beta,\gamma=1}^{r} \mathtt{cs}^{\gamma}_{\alpha\beta} X_\alpha Y_\beta$;
    $eqnC \leftarrow eqnI - \sum_{i=1}^{dim} b_i set[i]$;
    $matI \leftarrow$ `JacobianMatrix`$(eqnI, \{a_{dim+1}\})$;
    $matC \leftarrow$ `JacobianMatrix`$(eqnC, \{a_{dim+1}, b_i\})$;
    if `Rank`$(matI) \neq$ `Rank`$(matC)$ **then**
      `isSub` $\leftarrow false$;
      break foreach;
    end
  end
  return `isSub`;
end

Example 12.20 Let $\mathbb{L}_4$ be a real four-dimensional Lie algebra with the nonzero Lie brackets $[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = a\Xi_2$, where $0 < |a| < 1$ [38]. In this case, the argument `pars` of `SubAlgebra[]` has to be $\{\{a\}, \{-1 < a,\ a < 1,\ a \neq 0\}\}$.

If the argument `pars` is not `{{},{}}`, the `SubAlgebra[]` method evaluates the structure constants on an instance of the parameters satisfying the assigned constraints. Moreover, in the SymboLie package, there are several global variables, one of which is `PrintDebug`. By default, it is set to 0. When the user sets it to 1, a debug file is generated in the same directory as the Mathematica notebook file. The user can set the global string variable `logfile` to choose the name (the dimension of subalgebras is appended) of the text file where the trace of the computation is recorded; if `logfile` is not set by the user, then the trace of the computation is saved on the file `Debug-dim.log`.

The function `FindAdjacency[]` is the method doing all the work. Inside this function, first of all, the list C of possible candidates for families of d-dimensional Lie subalgebras is computed: for $d = 1$ we have $2^r - 1$ candidates, whereas for $d > 1$ the cardinality of the set of candidates depends on the structure constants of the Lie algebra. In other words, for $1 < d < r$, we build all the families of d-dimensional subspaces whose representative matrices are in RREF, and then select those verifying

condition (4) of Definition 12.5. This is done by the method `FindCandidates[]`, accepting as input the structure constants as well as the dimension of the subalgebras, and returning the list of candidates. For $d > 1$, to check whether a d-dimensional family of vector subspaces (a candidate) is a p–family of d-dimensional Lie subalgebras of $\mathbb{L}_r$ (Definition 12.5), the Algorithm 2 is used.

Algorithm 3: `FindAdjacency(cs,dim)`
Computation of the adjacency matrix of the multigraph $\mathcal{G}(\mathbb{L}_r)$.

Input: structure constants `cs` of $\mathcal{L}_r$
Output: `adj` adjacency matrix of the multigraph
begin
  $families \leftarrow$ `FindCandidates(cs,dim);`
  $auto \leftarrow$ `InnerAutomorphisms(cs);`
  $\mathscr{A} \leftarrow$ `FindAutomorphisms(`$auto$`);`
  `adj` $\leftarrow$ identity matrix of order #($families$);
  foreach X in $families$ **do** `// first scan`
    foreach A in $\mathscr{A}$ **do**
      $Y \leftarrow A \cdot X$;
      if `CheckPFamily(`Y`)` **then**
        `adj[`ind_X, ind_Y`]` $\leftarrow 1$;
        $Y \leftarrow families(ind_Y)$;
        $Z \leftarrow A^{-1} \cdot Y$;
        $Z_0 \leftarrow$ an evaluation of Z such that $c_{Z_0} = c_X$;
        if `CheckRelation(`Z, X`)` *and* `CheckPFamily(`Z_0`)` **then**
          `adj[`ind_Y, ind_X`]` $\leftarrow 1$;
        end
      end
    end
  end
  foreach pair (X, Y) of $families$ **do** `// second scan`
    foreach A in $\mathscr{A}$ **do**
      if `adj[`ind_X, ind_Y`]` $= 0$ **then**
        $X \leftarrow A \cdot X$;
        $X_0 \leftarrow$ an evaluation of X such that $c_{X_0} = c_Y$;
        if `CheckRelation(`X, Y`)` *and* `CheckPFamily(`X_0`)` **then**
          `adj[`ind_X, ind_Y`]` $\leftarrow 1$;
        end
      end
    end
  end
  return `adj;`
end

According to Definition 12.5, this algorithm requires to solve only linear equations.

As a second step, the inner automorphisms

$$\exp\left(t_\alpha \operatorname{ad}_{\Xi_\alpha}\right), \qquad \alpha = 1, \ldots, r,$$

where $\{\Xi_1, \ldots, \Xi_r\}$ denote the generators of a basis of $\mathbb{L}_r$, are computed.

Let A_0 be the inner automorphism made of the composition of the inner automorphisms represented by matrices in diagonal form; A_0 is the $r \times r$ identity matrix I_r if all inner automorphisms are not diagonal matrices. Moreover, let $\mathcal{A} = \{A_k,\ k = 1, \ldots r' \leq r\}$ be the set of inner automorphisms not in diagonal form; if $r' = 0$, we set $\mathcal{A} = \{I_r\}$.

Then, using the algorithm `FindCompositions[]`, the set made of the compositions of the permutations of all the inner automorphisms belonging to the nonempty subsets of $\mathcal{A}$ is computed. For the computation of this set, it is possible to use a flag variable, `FastRun`, within the package. By default this optional parameter is zero, and automorphisms are computed as previously described. If the user sets `FastRun` to 1, instead of considering permutations, we consider the compositions of all the inner automorphisms belonging to the nonempty proper subsets of $\mathcal{A}$ joined with the set of the compositions of the permutations of all inner automorphisms. Moreover, if `FastRun` is set to 2, then we only consider the compositions of the inner automorphisms belonging to the nonempty subsets of $\mathcal{A}$. The higher is `FastRun` the faster is the execution; anyway, we should be aware that there are cases where setting `FastRun` to 1 or 2 does not guarantee the completeness of the results.

Now, the process of identifying the pairs of candidates belonging to the relation $\mathcal{R}$ can start: the result is achieved by performing a double scan of the candidates.

Before analyzing the families, the adjacency matrix is set equal to the identity matrix of dimension equal to the number of candidates (this because every family is trivially equivalent to itself). In the first scan, we act on every p–family X (starting from the simplest ones, i.e., those with smallest values of p) with each of the automorphisms previously characterized: let A be one of these automorphisms acting on X. We need to check if the result is a family Y of subalgebras according to the definition. This process is quite fast and, when the check is successful, allows us to link a p–family with a q–family with $p < q$. Consequently the adjacency matrix is updated setting its (ind_X, ind_Y)–th entry to 1. If this occurs, the program checks if, acting on the family Y with the automorphism A^{-1}, there are suitable values of the parameters involved in the automorphism such that a family that can be represented by X can be obtained.

This first scan is in general not able to recognize all possible pairs of families belonging to the relation $\mathcal{R}$. Thus, the program performs a second scan on couples of families of subalgebras for which the corresponding entries in the adjacency matrix are zero. The used automorphisms are those belonging to the set of the compositions of the permutations of all inner automorphisms, and it is exploited the possibility that suitable values of the parameters involved in the automorphisms allow for recognizing other couples of Lie subalgebras belonging to the relation $\mathcal{R}$.

We observe that there are cases where the first scan is sufficient to find all possible relations between the subalgebras. Furthermore, during the process, the algorithm repeatedly uses the transitivity property to update the adjacency matrix. The pseudocode of this method is shown in the Algorithm 3.

The output returned by `SubAlgebra[]` is the input for other methods that display the results:

- `PrintGraph[]`, displaying the graph corresponding to the relation $\mathcal{R}$; each connected component of this graph may be represented by the simplest family whose indegree is equal to the number of nodes in the component;
- `PrintOptimal[]`, listing the families of Lie subalgebras of the optimal system; when possible, the optimal system which is returned is simplified by a suitable rescaling;
- `PrintClasses[]`, listing the families in each connected component of the graph.

Algorithm 4: `Adj2Classes(pfam,adj)`
Returns the connected components of the p-families in `pfam` according to the adjacency matrix `adj`

Input: list of p–families `pfam`, adjacency matrix `adj`
Output: list of connected components
begin
  $connected \leftarrow$ list of singletons of `pfam`;
  foreach (k_1, k_2) in 2-element combinations of $\{1, \ldots, \#\texttt{pfam}\}$ **do**
    if $\texttt{adj}[k_1, k_2] \neq 0$ **then**
      $connected[k_1] \leftarrow connected[k_1] \cup \{\texttt{pfam}[k_2]\}$;
    end
  end
  return `Compact`$(connected)$;
end

Several methods are employed to characterize the representatives of the optimal system of Lie subalgebras. The main method is `FindClasses[]` (Algorithm 5), where the `Adj2Classes[]` method (Algorithm 4) is initially invoked. It takes two arguments as input: the candidates (i.e., the p–families) and the adjacency matrix. First, a list is created made of lists, each containing one of the candidates. Then, the adjacency matrix is scanned and the candidates corresponding to the entries of the i–th row of the adjacency matrix not vanishing are joined to the i–th list. Finally, the `Compact[]` method is called, which is responsible for merging classes with common elements.

This process yields classes whose elements represent families of Lie subalgebras belonging to the relation $\mathcal{R}$; representing the adjacency matrix as a graph, each class corresponds to one of its connected component. The final step consists in

determining for each connected component its representative, i.e., if there exists at least one node whose indegree is equal to n, where n is the number of nodes in the connected component. The smallest representative with respect to the slex ordering gives the family able to represent the connected component. In some cases, it could be necessary to have more than one representative family for some connected component.

After the optimal system of families of Lie subalgebras has been obtained, the method `PrintOptimal[]` displays the result; this process is mediated by the `RescaleAlgebra[]` method (Algorithm 6) that rescales and writes the representatives of the optimal systems properly; as a result, the coefficients entering the expression of the optimal system are labeled with latin letters if they can assume arbitrary nonvanishing values, with greek letters if they can assume the values ± 1 only. In particular, this method uses the `InvariantAutomorphisms[]` method (Algorithm 7) which returns a list of inner automorphisms that leave the p–family invariant.

Algorithm 5: `FindClasses(pfam,adj)`
Returns the classes of the p-families in `pfam` according to the adjacency matrix `adj`

Input: list of p–families `pfam`, adjacency matrix `adj`
Output: list of representatives `reps`, list of classes `classes`
begin
  connected ← `Adj2Classes(pfam,adj)`;
  foreach component *comp* in *connected* **do**
    tmp ← {};
    foreach p-family X in *comp* **do**
      Y ← the first family in *comp* with maximum indegree;
      ind_X ← index of X in `pfam`;
      ind_Y ← index of Y in `pfam`;
      if `adj`$[ind_X, ind_Y] = 1$ **then**
        $tmp \leftarrow tmp \cup \{Y\}$;
      end
    end
    `classes` ← `classes` $\cup$ {*tmp*};
    `reps` ← `reps` $\cup$ {{ family in *tmp* with maximum indegree}};
  end
  return {`reps, classes`};
end

Algorithm 6: `RescaleAlgebra(X,cs,dim)`
Computation of the appropriate constants for a representative

Input: p-family `X`, structure constants `cs`, dimension `dim`
Output: appropriate representative
begin
  $r \leftarrow$ dimension of the Lie algebra;
  $auto \leftarrow$ `InnnerAutomorphisms(cs)`;
  $A \leftarrow$ `InvariantAutomorphisms(X,` $auto$`)`;
  $Y \leftarrow$ `RREF(`$A \cdot$ `X)`;
  $\mathcal{S} \leftarrow$ scaling parameters involved in Y;
  foreach entry $Y_{ij} \notin \{0, 1\}$ of Y **do**
    $P \leftarrow$ parameters in $\mathcal{S}$ involved in the (i, j)-th entry of Y;
    if $P \neq \emptyset$ **then**
      $\mathtt{X}_{ij} \leftarrow$ a suitable greek letter;
      $\mathcal{S} \leftarrow \mathcal{S} \setminus \{P[1]\}$;
    else
      $\mathtt{X}_{ij} \leftarrow$ a suitable latin letter;
    end
  end
  return $\left\{\sum_{j=1}^{r} \mathtt{X}_{ij}\, \Xi_j : i = 1, \ldots, \mathtt{dim}\right\}$;
end

Algorithm 7: `InvariantAutomorphisms(X,auto)`
Find A in `auto` leaving invariant the p–family X, up to RREF

Input: p–family `X`, list of automorphisms `auto`
Output: list of automorphisms `invauto`
begin
  foreach A in `auto` **do**
    $Y \leftarrow A \cdot X$;
    if $c_Y = c_X$ or $c_{RREF(Y)} = c_X$ **then**
      `invauto` $\leftarrow$ `invauto` $\cup \{A\}$;
    end
  end
  return `invauto`;
end

The interested reader can give also a look at the source of the package (available at the url https://mat521.unime.it/oliveri).

12.3.4 *Case Studies with* SymboLie

In this subsection, we present some relevant examples of Lie algebras corresponding to point symmetries of differential equations, and show how the use of the SymboLie package [1, 2] allows us to determine optimal systems of Lie subalgebras.

Some notebooks illustrating the use of the program can be found at the url https://mat521.unime.it/oliveri.

To use SymboLie, open a Mathematica notebook and load the package by issuing

```
<< "SymboLie.wl"
```

The following examples show the results we can get with the package.

Example 12.21 In [38], the authors reported the classification of all real three- and four-dimensional Lie algebras, and listed their complete systems of optimal Lie subalgebras. In particular, there are eleven non-isomorphic three-dimensional real Lie algebras. The complete optimal systems of all non-isomorphic three-dimensional real Lie subalgebas can be computed with SymboLie in less than one minute (the complete systems of optimal Lie algebras of the thirty four-dimensional Lie algebras in less than 20 min). We report in Table 12.1 the results relative to three-dimensional real Lie algebras.

SymboLie gives the classification of the optimal Lie subalgebras in a few seconds, and the results agree with those reported in [38, Table I] but are slightly different due to the choice of some representatives of the equivalence classes.

The Lie algebra $A_{3,8}$, spanned by $\{\Xi_1, \Xi_2, \Xi_3\}$, has the following nonzero Lie brackets:

$$[\Xi_1, \Xi_2] = \Xi_1, \quad [\Xi_2, \Xi_3] = \Xi_3, \quad [\Xi_3, \Xi_1] = 2\Xi_2.$$

Let us write the matrices associated to the inner automorphisms $\exp(t\,\mathrm{ad}_{\Xi_1})$, $\exp(t\,\mathrm{ad}_{\Xi_2})$ and $\exp(t\,\mathrm{ad}_{\Xi_3})$:

$$A_1 = \begin{pmatrix} 1 & -t_1 & -t_1^2 \\ 0 & 1 & 2t_1 \\ 0 & 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} \exp(t_2) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \exp(-t_2) \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & 0 & 0 \\ -2t_3 & 1 & 0 \\ -t_3^2 & t_3 & 1 \end{pmatrix},$$

and denote with A the group generated by $\{A_1, A_2, A_3\}$.

Patera and Winternitz have obtained the following optimal systems:

$$\Theta_A^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_1 + \Xi_3\}\}, \quad \Theta_A^2 \equiv \{\{\Xi_1, \Xi_2\}\}.$$

Table 12.1 Optimal systems of indecomposable real Lie algebras of dimension 3. The constants $a_1, a_2, a_3, a_4 \in \mathbb{R}$ are nonvanishing, and $\alpha_1 = \pm 1$

Algebra	Nonzero Lie brackets	1D optimal system	2D optimal system
$A_{3,1}$	$[\Xi_2, \Xi_3] = \Xi_1$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_2 + a_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_2 + a_1\Xi_3\}$
$A_{3,2}$	$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = \Xi_1 + \Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}$
$A_{3,3}$	$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = \Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_1 + a_1\Xi_2\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_2, \Xi_3\}, \{\Xi_1 + a_1\Xi_2, \Xi_3\}$
$A_{3,4}$	$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = -\Xi_2$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_2\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_2, \Xi_3\}$
$A_{3,5}$	$[\Xi_1, \Xi_3] = \Xi_1$, $[\Xi_2, \Xi_3] = a\Xi_2$ $(0 < \lvert a\rvert < 1)$	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_1 + \alpha_1\Xi_2\}$	$\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_2, \Xi_3\}$
$A_{3,6}$	$[\Xi_1, \Xi_3] = -\Xi_2$, $[\Xi_2, \Xi_3] = \Xi_1$	$\{\Xi_1\}, \{\Xi_3\}$	$\{\Xi_1, \Xi_2\}$
$A_{3,7}$	$[\Xi_1, \Xi_3] = a\Xi_1 - \Xi_2$, $[\Xi_2, \Xi_3] = \Xi_1 + a\Xi_2$ $(a > 0)$	$\{\Xi_1\}, \{\Xi_3\}$	$\{\Xi_1, \Xi_2\}$
$A_{3,8}$	$[\Xi_1, \Xi_2] = \Xi_1$, $[\Xi_1, \Xi_3] = -2\Xi_2$, $[\Xi_2, \Xi_3] = \Xi_3$,	$\{\Xi_1\}, \{\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_3\}$	$\{\Xi_1, \Xi_2\}$
$A_{3,9}$	$[\Xi_1, \Xi_2] = \Xi_3$, $[\Xi_1, \Xi_3] = -\Xi_2$, $[\Xi_2, \Xi_3] = \Xi_1$	$\{\Xi_1\}$	

On the contrary, using SymboLie, we get the following results:

$$\Psi_A^1 \equiv \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_1 + \alpha_1\Xi_3\}\}, \quad \Psi_A^2 \equiv \{\{\Xi_1, \Xi_2\}\}.$$

As far as the two-dimensional optimal system is concerned, both results coincide. On the other hand, comparing Θ_A^1 and Ψ_A^1, the only difference lies in the third representative. Recalling Definition 12.2, the subalgebra $\{\Xi_1 + \Xi_3\}$ is not considered in SymboLie because it does not represent a 2–family, so that we must refer to the corresponding 2–family $X = \{a_1\Xi_1 + a_2\Xi_3\}$. Furthermore, as the coefficients in this linear combination can be rescaled, then we have $X = \Xi_1 + \alpha_1\Xi_2$, where $\alpha_1 = \pm 1$.

Example 12.22 Consider the four-dimensional Lie algebra of point symmetries (see Example 12.19). The required Mathematica code is:

```
vars = {t, x, u};
gens = {{1, 0, 0}, {0, 1, 0}, {0, t, 1}, {3 t, x, -2 u}};
pars = {{},{}};
```

```
cs = StructureConstants[gens,vars];
alg1 = SubAlgebra[cs,pars,1];
alg2 = SubAlgebra[cs,pars,2];
alg3 = SubAlgebra[cs,pars,3];
```

Using `PrintOptimal[]` method it is possible to see the optimal systems of families of Lie subalgebras. Therefore, `PrintOptimal[alg1]` displays the five optimal families of one-dimensional Lie subalgebras:

$$\{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_3\}\}.$$

Similarly, the calls `PrintOptimal[alg2]` and `PrintOptimal[alg3]` show the six optimal families of two-dimensional Lie subalgebras,

$$\{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}\},$$

and the three optimal families of three-dimensional Lie subalgebras,

$$\{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}\},$$

respectively.

Finally, the method `PrintGraph[]` (with argument `alg1`, `alg2` or `alg3`), displays the graph representing the relation $\mathcal{R}$ between the p–families (see Figs. 12.1 and 12.2, where the gray nodes denote the representatives of each class of equivalent families). The legend showing the correspondence between the labels with the representatives families is also displayed. In addition, by setting the second optional argument of the `PrintGraph[]` method equal to 1, it is possible to display the labels of all the families of Lie subalgebras analyzed by SymboLie.

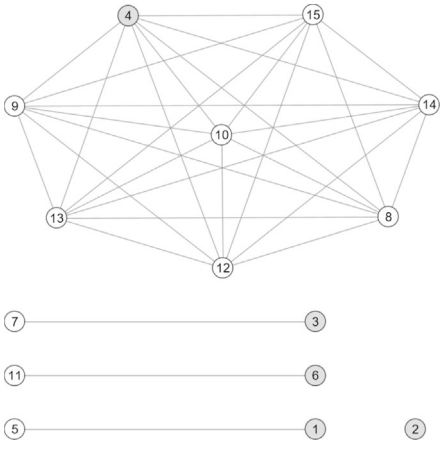


Fig. 12.1 Algebra of Lie symmetries of KdV equation: graph of families of 1D subalgebras, where $1 \to \{\Xi_1\}$, $2 \to \{\Xi_2\}$, $3 \to \{\Xi_3\}$, $4 \to \{\Xi_4\}$, $6 \to \{\Xi_1 + \alpha_1 \Xi_3\}$

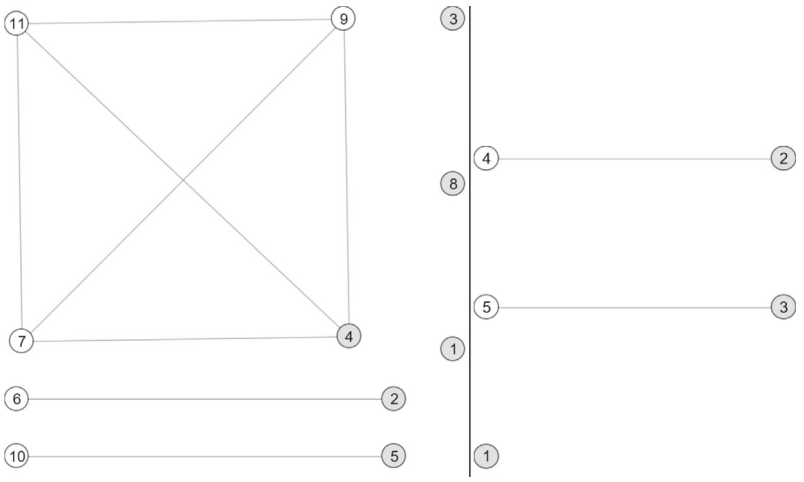


Fig. 12.2 Lie algebra of symmetries of KdV equation: graph of families of 2D subalgebras (left), where $1 \to \{\Xi_1, \Xi_2\}, 2 \to \{\Xi_1, \Xi_4\}, 3 \to \{\Xi_2, \Xi_3\}, 4 \to \{\Xi_2, \Xi_4\}, 5 \to \{\Xi_3, \Xi_4\}, 8 \to \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2\}$, and graph of families of 3D subalgebras (right), where $\{1 \to \{\Xi_1, \Xi_2, \Xi_3\}, 2 \to \{\Xi_1, \Xi_2, \Xi_4\}, 3 \to \{\Xi_2, \Xi_3, \Xi_4\}$

Example 12.23 Let $\mathbb{L}_6$ be a Lie algebra spanned by the vector fields

$$\left\{\Xi_1 = \frac{\partial}{\partial x},\ \Xi_2 = \frac{\partial}{\partial y},\ \Xi_3 = t\frac{\partial}{\partial x},\ \Xi_4 = t\frac{\partial}{\partial y},\ \Xi_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ \Xi_6 = \frac{\partial}{\partial t}\right\}.$$

The complete optimal system of Lie subalgebras has been studied in [36]. Using SymboLie, we obtain the following optimal system of families of Lie subalgebras:

$$\begin{aligned}
\Theta_A^1 \equiv\ & \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_5\}, \{\Xi_6\}, \{\Xi_2 + a_1\Xi_3\}, \{\Xi_3 + a_1\Xi_6\}, \{\Xi_5 + a_1\Xi_6\}\},\\
\Theta_A^2 \equiv\ & \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_6\}, \{\Xi_3, \Xi_4\}, \{\Xi_5, \Xi_6\},\\
& \{\Xi_1, \Xi_2 + a_1\Xi_3\}, \{\Xi_1, \Xi_3 + a_1\Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_6\}, \{\Xi_1, \Xi_4 + a_1\Xi_6\},\\
& \{\Xi_1 + a_1\Xi_3, \Xi_4\}, \{\Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3 + a_1\Xi_4 + a_2\Xi_6\},\\
& \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}\},\\
\Theta_A^3 \equiv\ & \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_6\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_6\},\\
& \{\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_6\}, \{\Xi_1, \Xi_2, \Xi_5 + a_1\Xi_6\},\\
& \{\Xi_1, \Xi_3, \Xi_4 + a_1\Xi_6\}, \{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_6\},\\
& \{\Xi_1, \Xi_2 + a_1\Xi_3, \Xi_4 + a_2\Xi_6\}\},\\
\Theta_A^4 \equiv\ & \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_6\}, \{\Xi_1, \Xi_2, \Xi_5, \Xi_6\},\\
& \{\Xi_1, \Xi_2, \Xi_3 + \alpha_1\Xi_6, \Xi_4 + a_1\Xi_6\}\},\\
\Theta_A^5 \equiv\ & \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_6\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5 + a_1\Xi_6\}\},
\end{aligned}$$

with $a_1, a_2 \neq 0$. This result is in agreement with the one given by Ovsiannikov [36], except for the choice of some representatives of the equivalence classes.

The graphs of the optimal systems computed by SymboLie are shown in Figs. 12.3, 12.4, and 12.5.

As far as the graph of 5D families of optimal Lie subalgebras are concerned, we observe that it is the trivial graph made of three isolated vertices.

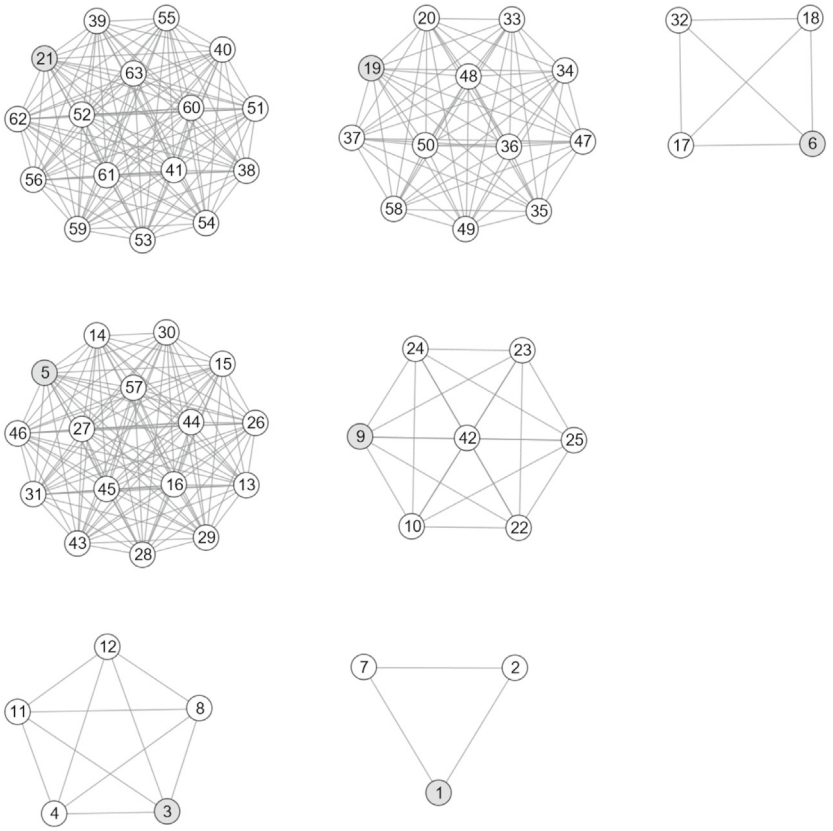


Fig. 12.3 Graph of families of 1D Lie subalgebras of the algebra of Example 12.23, where $1 \to \{\Xi_1\}$, $3 \to \{\Xi_3\}$, $5 \to \{\Xi_5\}$, $6 \to \{\Xi_6\}$, $9 \to \{\Xi_2 + a_1\Xi_3\}$, $19 \to \{\Xi_3 + a_1\Xi_6\}$, $21 \to \{\Xi_5 + a_1\Xi_6\}$.

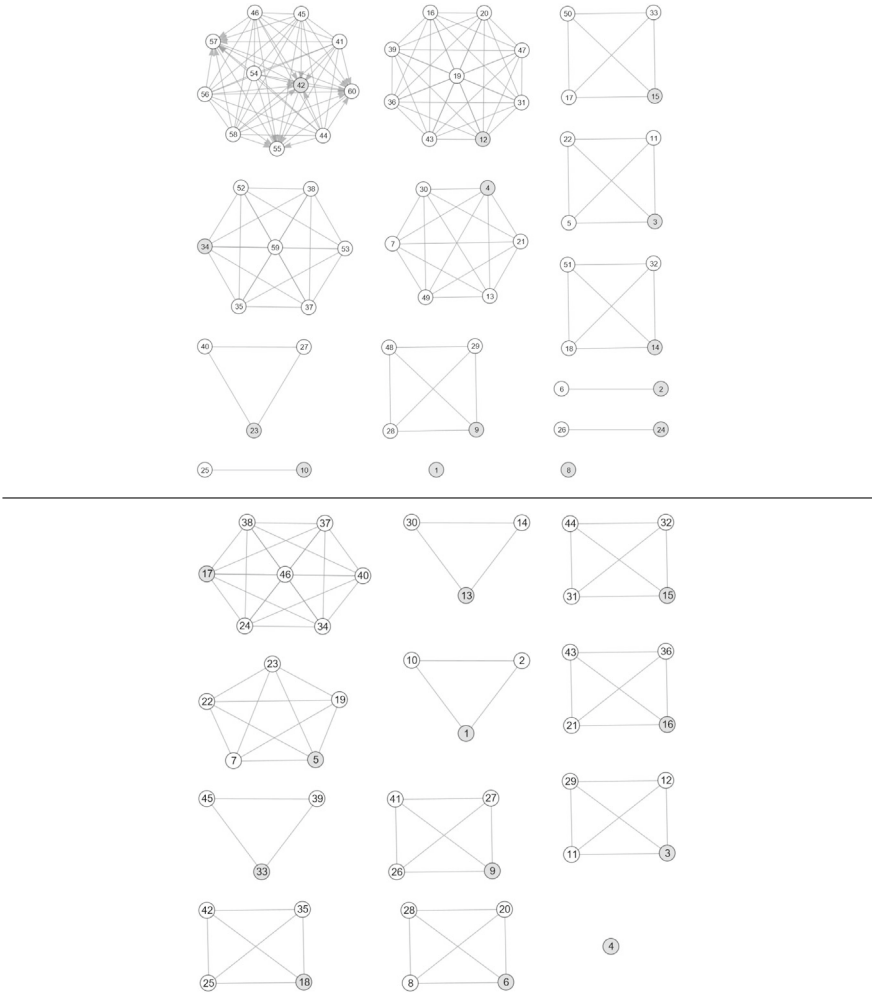


Fig. 12.4 Algebra of Example 12.23: graph of families of 2D Lie subalgebras (top), where $1 \to \{\Xi_1, \Xi_2\}$, $2 \to \{\Xi_1, \Xi_3\}$, $3 \to \{\Xi_1, \Xi_4\}$, $4 \to \{\Xi_1, \Xi_6\}$, $10 \to \{\Xi_1, \Xi_2 + a_1\Xi_3\}$, $12 \to \{\Xi_1, \Xi_3 + a_1\Xi_4\}$, $14 \to \{\Xi_1, \Xi_3 + a_1\Xi_6\}$, $15 \to \{\Xi_1, \Xi_4 + a_1\Xi_6\}$, $23 \to \{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $24 \to \{\Xi_2 + a_1\Xi_3, \Xi_4\}$, $34 \to \{\Xi_1, \Xi_3 + a_1\Xi_4 + a_2\Xi_6\}$, $42 \to \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}$, and of 3D Lie subalgebras (bottom), where $1 \to \{\Xi_1, \Xi_2\}$, $2 \to \{\Xi_1, \Xi_3\}$, $3 \to \{\Xi_1, \Xi_4\}$, $4 \to \{\Xi_1, \Xi_6\}$, $10 \to \{\Xi_1, \Xi_2 + a_1\Xi_3\}$, $12 \to \{\Xi_1, \Xi_3 + a_1\Xi_4\}$, $14 \to \{\Xi_1, \Xi_3 + a_1\Xi_6\}$, $15 \to \{\Xi_1, \Xi_4 + a_1\Xi_6\}$, $23 \to \{\Xi_1 + a_1\Xi_3, \Xi_4\}$, $24 \to \{\Xi_2 + a_1\Xi_3, \Xi_4\}$, $34 \to \{\Xi_1, \Xi_3 + a_1\Xi_4 + a_2\Xi_6\}$, $42 \to \{\Xi_1 + a_1\Xi_4, \Xi_2 + a_2\Xi_3\}$

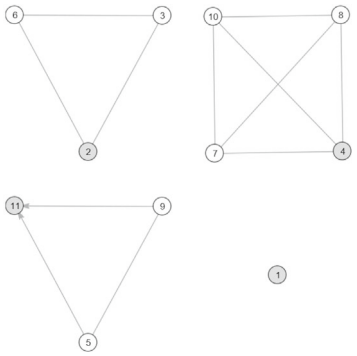


Fig. 12.5 Graph of families of 4D Lie subalgebras of the algebra of Example 12.23, where $1 \to \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$, $2 \to \{\Xi_1, \Xi_2, \Xi_3, \Xi_6\}$, $4 \to \{\Xi_1, \Xi_2, \Xi_5, \Xi_6\}$, $11 \to \{\Xi_1, \Xi_2, \Xi_3 + \alpha_1 \Xi_6, \Xi_4 + a_1 \Xi_6\}$

Example 12.24 Let $\mathbb{L}_5$ be the Lie algebra of continuous symmetries of $(2+1)$-dimensional ZK-BBM equation [44] spanned by the vector fields:

$$\Xi_1 = \frac{\partial}{\partial x}, \quad \Xi_2 = \frac{\partial}{\partial y}, \quad \Xi_3 = \frac{\partial}{\partial t},$$
$$\Xi_4 = \left(-x + \frac{2b}{k} y\right) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \left(\frac{1}{2a} - u\right) \frac{\partial}{\partial u},$$
$$\Xi_5 = t \frac{\partial}{\partial t} + \left(\frac{1}{2a} - u\right) \frac{\partial}{\partial u},$$

where a, b, and k are real constants.

Recently, Tanwar [44] computed the optimal one-dimensional Lie subalgebras of the Lie algebra of symmetries of $(2+1)$-dimensional ZK-BBM equation with the aim of determining group-invariant solutions. The use of `SymboLie` gives the classification of one-dimensional Lie subalgebras in few seconds, and the results, though slightly different in the choice of some representatives of the equivalence classes from those reported in [44], coincide. Moreover, SYMBOLIE is able to compute the complete system of optimal Lie subalgebras in about two minutes. The results are listed below:

$$\begin{aligned}\Theta^1_A \equiv\ & \{\{\Xi_1\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_5\}, \{\Xi_1 + \alpha_1 \Xi_2\}, \{\Xi_1 + \alpha_1 \Xi_3\},\\ & \{\Xi_1 + \alpha_1 \Xi_2 + \alpha_2 \Xi_3\}, \{\Xi_3 + \alpha_1 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_5\},\\ & \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_5\}, \{\Xi_4 + a_1 \Xi_5\}\},\end{aligned}$$

$$
\begin{aligned}
\Theta_A^2 \equiv\ & \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_1, \Xi_5\}, \{\Xi_1 + \alpha_1 \Xi_2, \Xi_3\},\\
& \{\Xi_1 + \alpha_1 \Xi_2, \Xi_5\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\}, \{\Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1 \Xi_3\},\\
& \{\Xi_1, \Xi_3 + \alpha_1 \Xi_4\}, \{\Xi_1, \Xi_2 + \alpha_1 \Xi_5\}, \{\Xi_1, \Xi_4 + a_1 \Xi_5\}, \{\Xi_3, \Xi_4 + a_1 \Xi_5\},\\
& \{\Xi_1 + \alpha_1 \Xi_3, \Xi_2 + \alpha_2 \Xi_3\}, \{\Xi_1 + \alpha_1 \Xi_5, \Xi_2 + a_1 \Xi_5\}, \{\Xi_1 + \alpha_1 \Xi_5, \Xi_3\},\\
& \{\Xi_1 + \alpha_1 \Xi_2 + a_1 \Xi_5, \Xi_3\}\},\\
\Theta_A^3 \equiv\ & \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_5\},\\
& \{\Xi_1, \Xi_4, \Xi_5\}, \{\Xi_1 + \alpha_1 \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_3 + \alpha_1 \Xi_4\},\\
& \{\Xi_1, \Xi_2, \Xi_4 + a_1 \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4 + a_1 \Xi_5\}, \{\Xi_1, \Xi_2 + \alpha_1 \Xi_5, \Xi_3\},\\
& \{\Xi_1 + \alpha_1 \Xi_5, \Xi_2 + a_1 \Xi_5, \Xi_3\}\},\\
\Theta_A^4 \equiv\ & \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_3, \Xi_5\}, \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_5\},\\
& \{\Xi_1, \Xi_2, \Xi_3, \Xi_4 + a_1 \Xi_5\}\},
\end{aligned}
$$

where $a_1 \neq 0$, and $\alpha_1, \alpha_2 \in \{-1, 1\}$.

Example 12.25 Consider the finite-dimensional Lie algebra of symmetries of $u_{,1} - u_{,22} = 0$ (linear heat equation) spanned by (we use the same basis as the one considered in [14])

$$
\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \quad \Xi_2 = \frac{\partial}{\partial x_2}, \quad \Xi_3 = 2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \quad \Xi_4 = u \frac{\partial}{\partial u},\\
&\Xi_5 = 2x_1 \frac{\partial}{\partial x_2} - x_2 u \frac{\partial}{\partial u}, \quad \Xi_6 = 4x_1^2 \frac{\partial}{\partial x_1} + 4x_1 x_2 \frac{\partial}{\partial x_2} - (2x_1 + x_2^2) u \frac{\partial}{\partial u}.
\end{aligned}
$$

SymboLie recovers a system of 9 optimal 1D families of Lie subalgebras, 13 families in the 2D case, 8 families in the 3D case, 5 families in the 4D case and only 1 5D family:

$$
\begin{aligned}
\Theta_A^1 \equiv\ & \{\{\Xi_1\}, \{\Xi_2\}, \{\Xi_3\}, \{\Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_4\}, \{\Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_5\},\\
& \{\Xi_1 + \alpha_1 \Xi_6\}, \{\Xi_1 + \alpha_1 \Xi_3 + a_1 \Xi_6\}\},\\
\Theta_A^2 \equiv\ & \{\{\Xi_1, \Xi_2\}, \{\Xi_1, \Xi_3\}, \{\Xi_1, \Xi_4\}, \{\Xi_2, \Xi_3\}, \{\Xi_2, \Xi_4\}, \{\Xi_3, \Xi_4\}, \{\Xi_3, \Xi_5\},\\
& \{\Xi_3, \Xi_6\}, \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2\}, \{\Xi_1, \Xi_3 + a_1 \Xi_4\}, \{\Xi_2, \Xi_3 + a_1 \Xi_4\},\\
& \{\Xi_1 + \alpha_1 \Xi_5, \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_6, \Xi_4\}\},\\
\Theta_A^3 \equiv\ & \{\{\Xi_1, \Xi_2, \Xi_3\}, \{\Xi_1, \Xi_2, \Xi_4\}, \{\Xi_1, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_3, \Xi_4\}, \{\Xi_2, \Xi_4, \Xi_5\},\\
& \{\Xi_3, \Xi_5, \Xi_6\}, \{\Xi_1, \Xi_2, \Xi_3 + a_1 \Xi_4\}, \{\Xi_1 + \alpha_1 \Xi_5, \Xi_2, \Xi_4\}\},\\
\Theta_A^4 \equiv\ & \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}, \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}, \{\Xi_1, \Xi_3, \Xi_4, \Xi_6\},\\
& \{\Xi_2, \Xi_3, \Xi_4, \Xi_5\}, \{\Xi_1 + \alpha_1 \Xi_6, \Xi_2, \Xi_4, \Xi_5\}\},\\
\Theta_A^5 \equiv\ & \{\{\Xi_1, \Xi_2, \Xi_3, \Xi_4, \Xi_5\}\}
\end{aligned}
$$

with $a_1 \neq 0$ and $\alpha_1 = \pm 1$. The corresponding nontrivial graphs are displayed in Figs. 12.6 and 12.7.

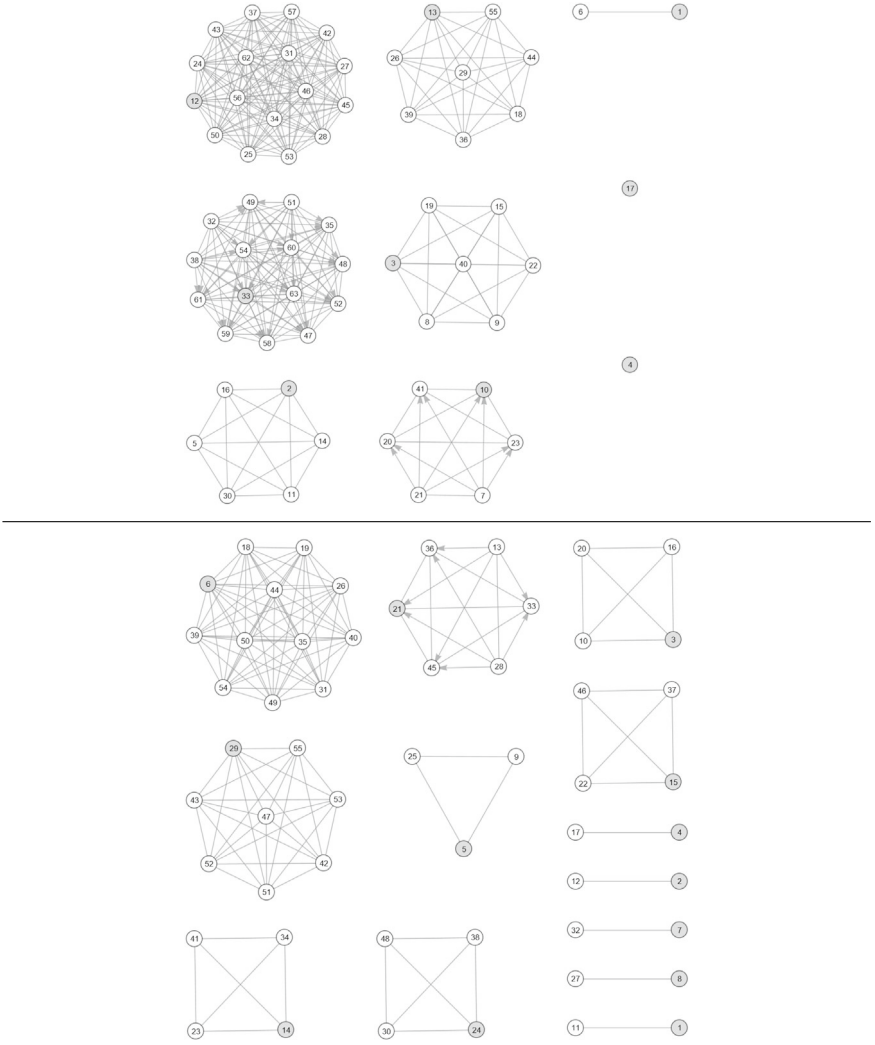


Fig. 12.6 Algebra of symmetries of heat equation (Example 12.25): graph of families of 1D Lie subalgebras (top), where $1 \to \{\Xi_1\}$, $2 \to \{\Xi_2\}$, $3 \to \{\Xi_3\}$, $4 \to \{\Xi_4\}$, $10 \to \{\Xi_1 + \alpha_1 \Xi_4\}$, $12 \to \{\Xi_3 + a_1 \Xi_4\}$, $13 \to \{\Xi_1 + \alpha_1 \Xi_5\}$, $17 \to \{\Xi_1 + \alpha_1 \Xi_6\}$, $33 \to \{\Xi_1 + \alpha_1 \Xi_3 + a_1 \Xi_6\}$, and of 2D Lie subalgebras (bottom), where $1 \to \{\Xi_1, \Xi_2\}$, $2 \to \{\Xi_1, \Xi_3\}$, $3 \to \{\Xi_1, \Xi_4\}$, $4 \to \{\Xi_2, \Xi_3\}$, $5 \to \{\Xi_2, \Xi_4\}$, $6 \to \{\Xi_3, \Xi_4\}$, $7 \to \{\Xi_3, \Xi_5\}$, $8 \to \{\Xi_3, \Xi_6\}$, $21 \to \{\Xi_1 + \alpha_1 \Xi_4, \Xi_2\}$, $14 \to \{\Xi_1, \Xi_3 + a_1 \Xi_4\}$, $15 \to \{\Xi_2, \Xi_3 + a_1 \Xi_4\}$, $24 \to \{\Xi_1 + \alpha_1 \Xi_5, \Xi_4\}$, $29 \to \{\Xi_1 + \alpha_1 \Xi_6, \Xi_4\}$

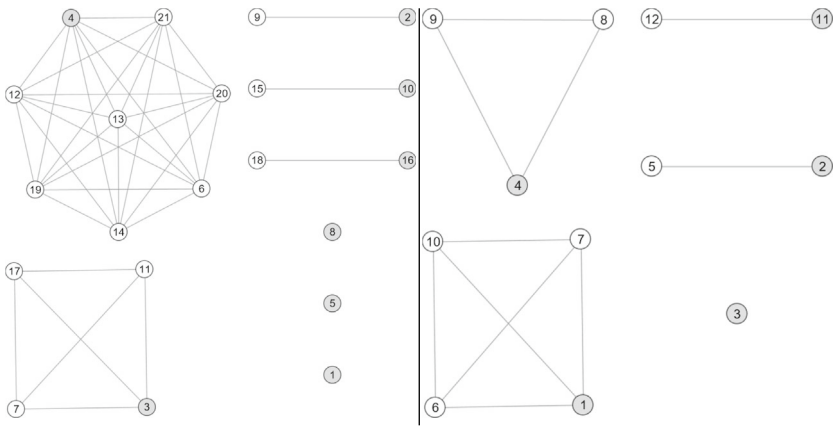


Fig. 12.7 Algebra of symmetries of heat equation (Example 12.25): graph of families of 3D Lie subalgebras (left), where $1 \to \{\Xi_1, \Xi_2, \Xi_3\}$, $2 \to \{\Xi_1, \Xi_2, \Xi_4\}$, $3 \to \{\Xi_1, \Xi_3, \Xi_4\}$, $4 \to \{\Xi_2, \Xi_3, \Xi_4\}$, $5 \to \{\Xi_2, \Xi_4, \Xi_5\}$, $8 \to \{\Xi_3, \Xi_5, \Xi_6\}$, $10 \to \{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$, $16 \to \{\Xi_1 + \alpha_1\Xi_5, \Xi_2, \Xi_4\}$, and of 4D Lie subalgebras (right), where $1 \to \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$, $2 \to \{\Xi_1, \Xi_2, \Xi_4, \Xi_5\}$, $3 \to \{\Xi_1, \Xi_3, \Xi_4, \Xi_6\}$, $4 \to \{\Xi_2, \Xi_3, \Xi_4, \Xi_5\}$, $11 \to \{\Xi_1 + \alpha_1\Xi_6, \Xi_2, \Xi_4, \Xi_5\}$

Comparing the results recovered by SymboLie with the ones found in the literature, some comments are in order:

- in [14, 21, 34, 48], there is exhibited a system of optimal 1D subalgebras with the same cardinality, but with different representatives;
- in the graphs representing the relation $\mathcal{R}$ for families of 1D and 2D subalgebras, there are unidirectional edges: this occurs because some families cannot be *entirely* mapped to others. For example, the 1D 2–family $X = (f_1, f_2, 0, 0, 0, 0)$ (label 7 in the graph) is mapped via $A = \exp(t\, ad_{\Xi_5})$ into $(f_1, 0, 0, \frac{f_2^2}{4f_1}, 0, 0)$, which satisfies Definition 12.2, so that we identify it with the 2–family $Y = (f_1, 0, 0, f_4, 0, 0)$ (label 10). Vice versa, acting on Y with the inner automorphisms A gives $(f_1, 2f_1t, 0, f_4 - f_1t^2, 0, 0)$, that cannot be reduced to the family X for all choices of f_1 and f_4;
- compared to the two-dimensional optimal system found in [14], SymboLie also returns the 2-families $\{\Xi_3, \Xi_5\}$ and $\{\Xi_3, \Xi_6\}$. It can be seen that they can be mapped into a subfamily of $\{\Xi_2, \Xi_3 + a_1\Xi_4\}$ and $\{\Xi_1, \Xi_3 + a_1\Xi_4\}$, respectively, so that it does not satisfy Definition 12.5;
- the 3D optimal system discussed in [14] includes the Lie subalgebra $\{\Xi_1, \Xi_3 - \frac{1}{2}\Xi_4, \Xi_6\}$. Since SymboLie deals with p–familes of subalgebras, it does not include $\{\Xi_1, \Xi_3 + a_1\Xi_4, \Xi_6\}$, with $a_1 \neq 0$, among the candidates, since for a_1 arbitrary the closure with respect to the Lie bracket is not guaranteed. Furthermore,

the family $\{\Xi_3, \Xi_5, \Xi_6\}$ does not appear in the optimal system of [14]. In fact, it is mapped into a subfamily of $\{\Xi_1, \Xi_2, \Xi_3 + a_1\Xi_4\}$, and so SYMBOLIE does not consider them belonging to the relation;
- the optimal systems of 4D and 5D subalgebras computed by SYMBOLIE coincide with those recovered in [14].

12.3.5 Concluding Remarks

The package SYMBOLIE can be extended by introducing additional features. In particular, it would be desirable to allow the program to suitably use special properties of the Lie algebra to be analyzed (for instance, especially for high-dimensional Lie algebras, its decomposition as direct sum of an ideal and a subalgebra), as well as to implement some routines for constructing the submodels [37] once the optimal system of Lie subalgebras of a Lie algebra of symmetries of a differential equation has been obtained [31]. Also, an effort should be devoted to speed up the algorithms.

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