Differential Equations

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March 25, 2026

Contents

Part I

Fundamentals

Chapter 1

Introduction to ODES

At the beginning of one's mathematical journey, they learn the 4 arithmetic operations and how to do calculations with them. Later on, the 4 operations (and possibly more functions) connect variables in algebraic equations, where one needs to find a number (or set of numbers) that satisfies the equation, possibly restricted to some set. When first solving algebraic equations, students are unfamiliar with complex numbers and hence implicitly restrict themselves to real number solutions.

$$x^3 - 6x^2 + 11x - 6 = 0, x \in \mathbb{R}$$

In solving this cubic equation, one finds the following set of solutions on $\mathbb{R}$.

$$x \in \{1, 2, 3\}$$

Similarly when one learns mathematical analysis, they learn about the derivative and integral of functions and the techniques to calculate them. Similarly, we can consider *differential equations* which relate a variable function with its derivatives.

Our goal is to find such a function that satisfies the equation. In this part, we will only consider differential equations with solutions that are real, complex, and $\mathbb{R}^n$ functions.

$$y' = y, y \in C^1(\mathbb{R})$$

It turns out that this differential equation has the following solution set on $C^1(\mathbb{R})$.

$$y \in \{f \in C^1(\mathbb{R}) : f(x) = ke^x, k \in \mathbb{R}\}$$

Such equations prove useful in areas of mathematical analysis, but are also heavily used in modelling phenomenon for applied mathematics.

Algebraic equations can be represented as $P(x) = 0$, where P is some polynomial. In a similar guise we can form a definition for ODEs; *ordinary differential equations.*

Definition 1.1 (Algebraic equation)**.** An *algebraic equation* is an equation of the following form, defined by some polynomial P.

$$P(x) = 0$$

Solutions for x are in $\mathbb{C}$

Definition 1.2 (Ordinary differential equation (ODE))**.** An *ordinary differential equation (ODE)* is an equation of the following form, defined by some $f : D \subseteq \mathbb{R} \times \mathbb{R}^n\mathbb{R}$

$$f(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0$$

Solutions for $y : I \to \mathbb{R}$ are in $C^n(I)$, for some interval I.

If those are ordinary differential equations, does that mean we have... extraordinary differential equations? The 'ordinary' means that we consider differential equations to be in terms of and solved by univariate functions rather than multivariate functions. Differential equations stated and solved in terms of multivariate functions are known as *partial differential equations*, whose theory becomes immensely more difficult.

As if this wasn't enough, there are still many other types of equations relating a univariate function to its derivatives that aren't covered in our definition of an ODE.

- Systems of ODES
- ODEs with multiple function variables (like $y_1 y_2' = y_1 - y_2$)
- Differential equations of univariate functions that employ the idea of fractional derivatives

That said, the power of solving a regular ODE can go quite far in tackling such types of differential equations.

Definition 1.3 (General solution of an ODE)**.** Given an ODE $y' = f(x, y(x))$, its *general solution* is a unique function that satisfies the DE up to a set of constants (if such a function exists).

Not all ODEs can be reduced to a single general solution, however many that we will discuss in this book can.

If one constructs and solves an ODE to model some physical phenomenon, they would typically like to ascribe meaning to the otherwise meaningless free constants in the general solution. Suppose that physical restraints of a problem imply that a useful solution must satisfy $y(0) = 1$, or $y(2) = 0$ or some related condition; we call this an IVP.

Definition 1.4 (Initial value problem (IVP))**.** An *initial value problem (IVP)* is an ODE with a specified $(x_0, y_0) \in \text{dom}(f)$, so $y(x_0) = y_0$

IVPs are vital in applications to boil an ODE down to a specific function that work in the context of application, and it is interesting to consider from a pure mathematical perspective for the uniqueness of solutions that this permits (more on this in the advanced chapter).

On that note, it would be beneficial to us to have at least a vague inclination on what ODEs will have solutions and better yet, unique solutions. We will tackle this result in the advanced part of the book, but for now we offer a mantra that you'll just have to take on faith for a little while.

1.1 Types of differential equations

1.1.1 Order and degree

Definition 1.5 (Order of a differential equation)**.** Highest order derivative

Definition 1.6 (Degree of a differential equation)**.** Highest power applied to a derivative

1.1.2 Forms of equations

Definition 1.7 (Degree of a differential equation)**.**

There exist some more exotic types of differential equatiosn such as fractional differential equations

1.2 First order separable ODE

Separable ODEs are those that can be written in such a way that it is the , that is, the y and x can be 'separated' as variables of 2 different functions.

Separable ODEs of order 1 turn out to be the simplest type of ODEs, having solution method that is elementary and elegant.

By doing some algebraic manipulation and using the reverse chain rule, we can make the problem into that of solving an algebraic equation for y.

$$\begin{aligned} y' &= f(y)g(x) \\ \frac{y'}{f(y)} &= g(x) \\ \int \frac{y'}{f(y)} dx &= \int g(x)dx \\ \int \frac{1}{f(y)} dy &= \int g(x)dx \end{aligned}$$

Theorem 1.1 (First order separable ODE)**.** Given a first order separable ODE $y' = f(y)g(x)$, its general solution satisfies the following algebraic equation.

$$\int \frac{1}{f(y)} dy = \int g(x)dx$$

1.3 Integrating factor

Integrating factor proves vital in the solutions to first order linear ODEs.

Recall what a first order linear ODE looks like

$$y' + q_1(x)y = q_2(x)$$

One may notice that the $y' + q_1(x)y$ part looks almost like the result of applying the product rule. If we had some $I(x)$ satisfying $I' = q_1(x)I$, then $I(x)y' + q_1(x)I(x)y$ would be reversible by the product rule.

$I' = q_1(x)I$ is separable so we can solve it and discover that $I(x) = e^{\int q_1(x)dx}$; here is how multiplying by this factor is useful.

$$y' + q_1(x)y = q_2(x)$$

$$I(x) = e^{\int q_1(x)dx}$$
$$I(x)y' + q_1(x)I(x)y = I(x)q_2(x)$$
$$[I(x)y]' = I(x)q_2(x)$$
$$I(x)y = \int I(x)q_2(x)dx$$
$$y(x) = \frac{\int I(x)q_2(x)dx}{I(x)}$$

The moral of the story is that multiplying by I allows us to reverse the product rule and integrate both sides of the equation, we call this the *integrating factor*, since multiplying our (first order linear) ODE allows us to integrate towards a solution.

The general idea of an integrating factor is that it is some 'factor' such that multiplying an ODE by it allows us to use some rule (like the product rule) to integrate to the solution. Integrating factors may be used for other types of ODEs, however they depend on the context of the equation.

Theorem 1.2 (First order linear ODE)**.** Given a first order linear ODE $y'+q_1(x)y = q_2(x)$, its general solution is the following where $I(x) = e^{\int q_1(x)dx}$.

$$y(x) = \frac{\int I(x)q_2(x)dx}{I(x)}$$

1.4 Second order linear, homogeneous, constant coefficient ODE

We'll now take a look at the simplest possible 2nd order ODE. It arises often in applications and has a simple general solution.

$$y'' + by' + cy = 0$$

Definition 1.8 (Characteristic equation)**.** Given an ODE $\sum_{k=0}^{n} a_k y^{(k)} = 0$, its *characteristic equation* is the following algebraic equation.

$$\sum_{k=0}^{n} a_k \lambda^k = 0$$

Theorem 1.3. The ODE $y'' + by' + cy = 0$ with $a, b \in \mathbb{C}$ has the following general solution if its characteristic equation has 2 distinct zeros.

$$y(x) = K_1 e^{\lambda_1 x} + K_2 e^{\lambda_2 x}$$

If the characteristic equation has a zero of multiplicity 2, the ODE has the following general solution.

$$y(x) = (Ax + b)e^{\lambda_1 x}$$

Corollary 1.1. The ODE $y'' + by' + cy = 0$ with $a, b \in \mathbb{C}$ has the following general solution if its characteristic equation has zeros with nonzero imaginary part ($b^2 - 4c < 0$).

$$y(x) = e^{\alpha x}[K_1 \cos(\omega x) + K_2 \sin(\omega x)]$$

There are a few ways

A basic solution method is to transform second order linear ODES with constant coefficients into a first order linear ODE. There are a few ways we can do this, but using the theory available to us, we chose the method of using an integrating factor.

Letting $I(x) = e^{bx+k}$

$$y'' + by' + cy = 0$$

$$Iy'' + bIy' + cIy = 0$$

$$(Iy')' = -cIy$$

$$Iy = I\frac{m}{m+n}y'$$

$$y = \frac{m}{m+n}y'$$

$$y = \frac{m}{m+n}y' + K_1$$

With this f

1.5 Exact equations

We will now look at an example of a class of 1st order ODEs that isn't necessarily linear. Though it may be non-linear, its form can still be relatively easy to work to. This is due to the idea of a *conservative function*.

$$I(x,y) + J(x,y)y' = 0$$
$$y' = \frac{I(x,y)}{J(x,y)}$$
$$I(x,y)dx + J(x,y)dy = 0$$

The last is not 'exactly' rigorous (haha), but offers a hint about one way in whichwe can integrate towards the solution.

1.5.1 Solving by conservative functions

Imagine there exists some underlying function V, satisfying the following.

$$V_x(x,y) = I(x,y)$$

$$V_y(x,y) = J(x,y)$$

Then we are left with the following.

$$V_x(x,y) + y'V_y(x,y) = 0$$
$$\int [V_y(x,y) + y'V_x(x,y)]dx = \int 0dx$$
$$2V(x,y) + C_1 = C_2$$
$$V(x,y) = K$$

Wouldn't that be nice? The only problem is knowing when such a potential function exists, that is when do we have the following equations?

$$V_x(x,y) = J(x,y)$$
$$V_y(x,y) = I(x,y)$$

By differentiating both equations with respect to the 'other' variable not yet differentiated,we get 2 different forms of V_{xy}, and immediate see that we require $J_y = I_x$.

Theorem 1.4 (Solution to exact equations with conservative function)**.** Let IJ be functions satisfying $J_y = I_x = V_{xy}$

$$I(x,y) + J(x,y)y' = 0$$

Then the following algebraic equation holds. $V(x,y) = K$

1.5.2 Solving by homogeneity

Say that we can't find a conservative function, to solve exact equations analytically we'll need to take a new approach, using the alternative form.

Theorem 1.5 (Solving exact equations with homogeneous functions)**.** We will now study when both I, J are homogeneous of the same degree.

$$y' = \frac{I(x,y)}{J(x,y)}$$

Then the ansatz $y(x) = xv(x)$ makes the ODE separable.

1.6 Bernoulli equations

$$y' + q_1(x)y = q_2(x)y^n$$

A common technique in differential equations (and even algebraic equations!) is to transform a difficult equation into an equation that can be solved. The Bernoulli equation can be transformed into a first order linear equation in the following way.

- Make the substitution $u = y^{1-n}$
- Make the substitution $u' = (1-n)y'y^{-n}$
- Solve the first order linear equation

Fortunately, we are able to calculate analytic solutions for first order linear differential equations, so we can therefore analytically solve Bernoulli equations! However, sometimes we cannot transform our DE into something so nice.

1.7 Riccati equations

$$q_1(x)y' + q_2(x)y + q_3(x)y^2 = q_4(x)$$

Notice that when $q_4(x) = 0$ we have a Bernoulli equation, Riccati equations are perhaps useful to view as possibly non-homogeneous Bernoulli equations.

- Make the substitution $y(x) = \frac{A(x)u'(x)}{u(x)}$
- Determine the $A(x)$ required to cancel nonlinear terms
- Solve the second order linear ODE

Second order linear ODEs are generally much harder to solve; we have studied the case of constant coefficients but a general solution is not yet available to us. We will soon study various techniques to solve such problems.

An interesting sidenote is the following formula for generating a new solution for the Ricatti equation using 3 knowl solutions.

Proposition 1.1 (Nonlinear superposition of Riccati solutions)**.**

$$y_4 = \frac{y_1(y_3 - y_2) + ay_2(y_1 - y_3)}{y_3 - y_2 + a(y_1 - y_3)}$$

Proving such a proposition requires the machinery of Lie theory; this will be covered in the advanced part of this book.

1.8 Basic tools for solving ODEs

1.9 Method of undetermined coefficients

Method of forming an ansatz to reduce differential equations into algebraic equations

$$ke^{ax} \iff Ce^{ax}$$

$$k\sin(ax) \iff K\cos(ax) + M\sin(ax)$$

$$kx^n \iff \sum_{i=0}^{n} K_i x^i$$

1.10 Differential operators

1.11 Annihilator method

Similar to the method of undetermined coefficients, however the first goal is to find some differential operator called the *annihilator* which when composed with the ODE from the left makes the RHS equal to 0.

1.12 Picard iteration

It is used as a tool in the Picard-Lindelöf theorem (proven and explained in the advanced part), however like in many constructive proofs, the tool itself is actually of considerable value in determining an analytic solution. The algorithm is a specific example of the Banach fixed-point theorem in action.

Chapter 2

Linear ODEs

These constitute a class of ODEs that are particularly well behaved, and there exist a myriad of tools that make use of the linearity of their solutions. Perhaps the most fundamental property of such ODEs is the following.

Proposition 2.1. The general solution of a linear ODE of degree n is the linear combination of n distinct functions.

Corollary 2.1. The set $F = \{f \in C^n(I) : \forall x \in I[\sum_{k=1}^{n} q_k(x) f^{(k)}(x) = 0]\}$ forms a linear space over $\mathbb{R}$ (where addition and scalar multiplication are as usual)

Knowing that solutions of Linear ODEs form a linear space suggests the philosophy that solving our ODE amounts to finding a basis for this space.

2.1 Wronskian

Definition 2.1 (Wronskian)**.** Given a vector of real or complex functions $\mathbf{y}$, their *Wronskian* is the function W which is the determinant of the matrix where each column is a vector with a solution to a linear DE and its $n-1$ derivatives.

$$W(\mathbf{y})(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is a vector of solutions to a linear DE

2.1.1 Properties of the Wronskian

$$\mathbf{y} \text{ are linearly dependent on } I \implies W(I) = \{0\}$$

Where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is a vector of solutions to a linear DE

$$\mathbf{y} \text{ are analytic on } I \implies [\mathbf{y} \text{ are linearly dependent on } I \iff W(I) = \{0\}]$$

$$\mathbf{y} \text{ satisfy } y''+q_1(x)y'+q_2(x) = 0 \text{ on } I \implies [\mathbf{y} \text{ are linearly dependent on } I \iff W(I) = \{0\}]$$

2.1.2 Abel's formula

Theorem 2.1 (Abel's formula)**.**

$$y_1, y_2 \text{ satisfy } y'' + q_1(x)y' + q_2(x)y = 0 \implies W(y_1, y_2) = K_{12}e^{-\int q_1(x)dx}$$

Corollary 2.2.

$$y_1 \text{ satisfies } y''+q_1(x)y'+q_2(x)y = 0 \wedge \neg(\forall x[y_1(x) = 0]) \implies y_2 = y_1 \int \frac{e^{-\int q_1(x)dx}}{y_1(x)^2}dx \text{ satisfies } y''$$

As wonderful as it is that we can abuse the linear space formed by the linear ODE to generate novel solutions using known solutions, this still requires that we know some solutions for the ODE.

To get an initial foothold on the problem, the Annihilator method or Cauchy-Euler method can be used to obtain a first set of solutions before expanding our solutions using Abel's formula. Unfortunately such methods are not always possible to apply due to assumptions not being met, but there exists some more potent methods for evaluating solutions to linear ODEs.

2.2 Variation of Parameters (VOP)

VOP is essentially just assuming the ansatz $y(x) = u(x)y_1(x) + v(x)y_2(x)$ and $u'y_1 + v'y_2 = 0$

2.3 Series methods

One powerful idea is to assume that the solution to the linear ODE is an analytic function and simply apply the ansatz $y(x) = \sum_{n=0}^{\infty} a_n x^n$. It's quite simple, yet incredibly effetive. Often the solutions to an ODE are nonelementary, so this method comes in clutch where others fail.

For differential equations such that $y^{(n)} + \sum q_k(x)y^{(k)} = 0 \wedge q_k$ are analytic on $I \wedge 0 \in I$, solutions are always analytic and hence has a Taylor series representation and hence a power series representation. This is due to considering the DE on a complex neighborhood and by noting that holomorphic functions are analytic, see Complex Analysis.

- Assume the ansatz $y(x) = \sum_{n=0}^{\infty} a_n x^n$
- Substitute the ansatz into the DE and manipulate into one single sum $[\sum_{n=0}^{\infty} a_n x^n]^{(k)} = \sum_{n=k}^{\infty} a_n \frac{n!}{(n-k)!} x^{n-k}$ ¡li¿Since the RHS equals zero, equate the coefficient of the new sum to 0¡ ¡li¿Solve recursive formula for a_n ¡/ul¿

Definition 2.2 (Ordinary point)**.**

$$x_0 \text{ is an ordinary point of } \sum q_n(x)y^n(x) = 0 \iff \forall k \in \{0, 1, .., n\} a_k \text{ is analytic at } x_0)$$

Translating from a recurrence relation into a more recognizable notation can be a bit laborious, however a few elementary identities could help out.

$$\prod_{k=1}^{n} 2k = 2^n n!$$

$$\prod_{k=1}^{n} (2k-1) = \frac{(2n-1)!}{2^(n-1)(n-1)!}$$

Definition 2.3 (Singular point)**.**

x_0 is a singular point of $\sum q_n(x)y^n(x) = 0 \iff \neg[x_0$ is a singular point of $\sum q_n(x)y$

All coefficient function analytic at a point means all DEs solutions taylor series equal function itself

Series method Method of Frobenius

Chapter 3

Laplace transform

A specific integral transform that serves as a powerful method for transforming differential equations into algebraic equations and real numbered equations.

Because the Laplace transform maps derivatives of a function to expressions in terms of the Laplace transform of the original function.

Definition 3.1 (Heaviside step function)**.** The characteristic function of real numbers greater than or equal to 0, $\chi_{\mathbb{R}_+}$, it serves useful for the Laplace inversion of function with a factor e^{-sa}

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Definition 3.2 (Laplace transform)**.** Integral transform on $(0, \infty)$ with kernel e^{-st}

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt$$

- $f : [0, \infty) \to \mathbb{R}$ is a real valued function dominated by an exponential function
- $\mathcal{L}\{f\} : S \to \mathbb{C}$ is the Laplace transform of f

$$\mathcal{L}\{f\} \in L^\infty([0, \infty])$$

$$\mathcal{L}\{af+bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$$

$$\mathcal{L}\{f\} \text{ is analytic on } \mathrm{dom}(\mathcal{L}\{f\})$$

$$\mathcal{L}\{t^n f\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}(s)$$

$$\mathcal{L}\{e^{-at}f\}(s) = \mathcal{L}\{f\}(s+a)$$

$$\sum_{n=0}^{\infty} \frac{a_n}{n!}t^n \text{ is absolutely convergent } \implies \mathcal{L}\{\sum_{n=0}^{\infty} \frac{a_n}{n!}t^n\}(s) = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}$$

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{a}\mathcal{L}\{f\}(\frac{s}{a})$$

$$\exists \mathcal{L}\{f'\}, \mathcal{L}\{f\} \implies \mathcal{L}\{f'\}(s) = -f(0) + s\mathcal{L}\{f\}(s)$$

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - \sum_{i=0}^{n-1} s^{n-i-1}f^{(i)}(0)$$

$$\mathcal{L}\{f\} = 0 \implies f = 0 \text{ almost everywhere}$$

$$\mathcal{L}\{f\} = \mathcal{L}\{g\} \implies f = g \text{ almost everywhere}$$

$$\mathcal{L}\{H(t-a)f(t-a)\}(s) = e^{-sa}\mathcal{L}\{f\}(s)$$

Here we list the Laplace transforms associated with common functions

- $\mathcal{L}\{t^n\}(s) = \frac{\Gamma(n+1)}{s^{n+1}}$
- $\mathcal{L}\{e^{-at}\}(s) = \frac{1}{s+a}$
- $\mathcal{L}\{\ln(at)\}(s) = \frac{\ln(a)-\gamma-\ln(s)}{s}$
- $\mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2+a^2}$
- $\mathcal{L}\{\cos(at)\}(s) = \frac{s}{s^2+a^2}$
- $\mathcal{L}\{J_0(t)\}(s) = \frac{1}{\sqrt{1+s^2}}$

Definition 3.3 (Inverse Laplace transform)**.** Since a property of Laplace transforms is that two functions with the same transform must be equal almost everywhere, Laplace transforms are unique and hence invertible.

$$\mathcal{L}^{-1}\{\mathcal{L}\{f\}\} = f$$

$$\mathcal{L}^{-1}\{af+bg\} = a\mathcal{L}^{-1}\{f\} + b\mathcal{L}^{-1}\{g\}$$

Theorem 3.1 (Convolution theorem (Laplace transform))**.**

$$\mathcal{L}\{f * g\}(s) = \mathcal{L}\{f\}(s)\mathcal{L}\{g\}(s)$$

The Laplace transform can be used to transform ODEs into algebraic equations! Here is the outline of the method.

- Compute the Laplace transform for both sides of the equation and equate them
- Solve algebraically for y_1
- Compute $\mathcal{L}^{-1}\{y_1\}(x)$

Definition 3.4 (Confluent hypergeometric functions)**.** Linearly independent solutions to the ODE $zw'' + (b - z)w' - aw = 0$ (Kummer's equation) First kind

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!}$$

- $x^{(n)} = \prod_{k=0}^{n-1}(x + k)$ is the rising factorial
- $(x)_n = \prod_{k=0}^{n-1}(x - k)$ is the falling factorial

Second kind

$$U(a, b, z)$$

Chapter 4

PDEs

Definition 4.1 (Partial differential equation (PDE))**.** A *partial differential equation (PDE)* is an equation of the following form, defined by some $f : S \subseteq \mathbb{R}^n \times \mathbb{R}^m$ and $D^k\mathbf{y}$ representing the matrix of possible partial derivatives of order k.

$$f(\mathbf{x}, y(\mathbf{x}), Dy(\mathbf{x}), \ldots, D^m y(\mathbf{x})) = 0$$

Solutions for $y : U \to \mathbb{R}$ are (usually, depending on whether all variables can be differentiated m times) in $C^m(U)$, for some set U open in (Euclidean topology) $\mathbb{R}^m$.

The introduction of a multivariable function already seriously complicates the formal definition from an ODE, and complication is generally the name of the game with PDEs.

This chapter aims to look at solution methods on the more elementary side that one can apply in the study of PDEs, generally mapping a PDE to multiple ODEs.

4.1 Separation of variables

The most elementary solution method I know of is *separation of variables*, which basically makes a general yet bold ansatz that is an attempt to split our PDE into ODEs with respect to one variable each, then prays to the math gods that things work out.

This solution method has a chance at being succesful when the RHS (or any 1 of the sides) of the PDE reduces to a constant after we make our ansatz because this allows us to split the PDE into multiple ODEs. Here is the general method.

- Assume the ansatz $f(x, y) = X(x)Y(y)$ and substitute into equation
- Separate variables to both sides and equate both sides to a constant $\pm\lambda$
- Solve each problem using SL theory or otherwise and then substitute the solutions into the original ansatz; the general solution is a linear combination of these eigenfunctions
- Equate the general solution to the final boundary condition and find the projection.

4.2 Fourier transform method

Using the ansatz (guess) that a solution can be represented as the Fourier transform of some function u is quite effective.

Fourier studied a PDE now hailed as the heat equation, and he tried to find new solutions to it based on the trivial , which were sinudoids. His idea was basically to consider convolutions of some unknown function with these elementary solutions to find further solutions.

Fourier conjectured that (sufficiently nice) periodic functions could always be represented as something called a Fourier series, and that non-periodic function (again, sufficiently nice ones, whatever that means) could still be represented by a 'Fourier integral'. Though such statements could not be verified until much later, Fourier's steadfast beliefs ultimately paid off, producing legitimate solutions to the heat equation.

Fourier analysis warrant much attention due to its usefulness in PDEs as well as signal processing, and though we do not discuss the theory here, you can read about it in Fourier Analysis. To those already familiar with the theory, I offer you a neat way to solve PDEs with Fourier analysis.

4.3 Qualitative theorems

4.3.1 Maximum principle

4.3.2 Harnack's theorem

Part II

Advanced

Chapter 5

Existence and uniqueness of differential equations

We know that the solutions of a differential equation can be represented as a (possibly empty, possibly infinite, etc.) solution set.

For algebraic equations, we understand the nature of these sets quite well due to the fundamental theorem of algebra; algebraic equations of degree n always have n solutions in $\mathbb{C}$ up to multiplicity. We also know other neat facts like that algebraic equations of odd degree always have at least 1 real solution, and so forth.

The nature of differential equations and their solutions is much more difficult to pin down, however ideas from topology and analysis can be used to make some remarks about certain classes of differential equations. Mostly, these theorems reveal classes of differential equations for which at least 1 solution exists, or even better, when a unique solution exists.

5.1 Peano existence theorem

Among the simplest existence theorems is the Peano existence theorem.

Theorem 5.1 (Peano existence theorem)**.** Let $D \subseteq \mathbb{R} \times \mathbb{R}$ be a set open in $\mathbb{R}^2$ (Euclidean topology) with $(t_0, y_0) \in (D)$. If $f : D \to \mathbb{R}$

is continuous then there exists some interval $t_0 \in I$ on which the IVP $y'(t) = f(t, y(t)), y(t_0) = \mathbf{y}_0$ has a solution.

5.2 Carathéodory existence theorem

To those familiar with measure theory, there is a generalization of the theorem that weakens the conditions.

Theorem 5.2 (Carathéodory existence theorem)**.**

5.3 Picard–Lindelöf theorem

As nice as these theorems are, they don't say anything about the uniqueness of a solution, indeed there are examples of such ODEs having multiple solutions.

The following theorem employs further conditions that show uniqueness for certain well-behaved IVPs on a neighborhoood of the initial value.

Theorem 5.3 (Picard–Lindelöf theorem)**.** Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be a closed rectangle $(t_0, \mathbf{y}_0) \in \text{int}(D)$. If $f : D \to \mathbb{R}^n$ is t-continuous and $\mathbf{y}$-Lipschitz continuous then there exists some closed interval $t_0 \in I$ on which the IVP $y'(t) = f(t, y(t)), y(t_0) = \mathbf{y}_0$ has a unique solution.

Proof is based on applying the Banach fixed-point theorem to the integral equation form of f.

Due to the constructive nature of the Banach fixed-point theorem, the proof of the Picard-Lindelöf theorem inspires a solution method called *Picard iteration*.

5.3.1 Picard iteration

5.4 Cauchy-Kovalevskaya theorem

Theorem 5.4 (Cauchy-Kovalevskaya theorem)**.**

5.5 Okamura's theorem

Theorem 5.5 (Okamura's theorem)**.**

Chapter 6

Sturm-Liouville theory

Operator are an object from functional analysis; familiar with differential operators is required for this chapter.

Sturm-Liouville studies differential equations of specific type, appropriately called Sturm-Liouville equations.

Definition 6.1 (Sturm-Liouville equation)**.** A *Sturm-Liouville equation* is a second order ODE in terms of $p, q, w \in C^1(\mathbb{R})$ and $\lambda \in \mathbb{R}$

$$[p(x)y']'q(x)y = -\lambda w(x)y$$

In modern times, it is studied with a 'functional analysis' flavour, where the Sturm-Liouville equation is constructed by means of an operator, called (what a surprise) Sturm-Liouville operators, so function solutions of the Sturm-Liouville equation are considered as eigenfunctions of some Sturm-Liouville operator.

Definition 6.2 (Sturm-Liouville operator)**.** A *Sturm-Liouville operator* is an operator in terms of $p, q, w \in C^1(\mathbb{R})$

$$\mathcal{L}(y) = -\frac{1}{w(x)}[p(x)y']'q(x)y$$

From this perspective, the Sturm-Liouville equation is the eigenequation of the Sturm-liouville operator $\mathcal{L}y = \lambda y$.

Chapter 7

Fuchsian theory

Perhaps the most powerful method familiar to us so far is the power series method and method of Frobenius. The Fuchsian theory studies in depth the extent to which these methods can be used.

Chapter 8

Lie theory

Lie's theory employs a structure called Lie groups to study the algebraic sy. His ultimate goal was that Lie theory for differential equations would compare with Galois theory for algebraic equations. A -Lie point symmetry

- Holonomic function - D-module