Differential Topology

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Chapter 1

Manifolds

Real analysis and vector analysis gives us notions of differentiability on $\mathbb{R}^n$, however what is the most general topologial space that differentiability can be defined on? How "coordinate invariant" can our space be

Topological spaces arise in many analytical contexts, and therefore it is useful to explore differential structure on them.

The answer is a *smooth manifold*; the idea of a manifold is a special type of topological space that looks "locally" like Euclidean space (done by a homeomorphism from the manifold to $\mathbb{R}^n$, also acting as a local coordinate system) , as well as "globally" having some nice topological properties. The adjective "differentiable" refers to the fact that when viewing the intersection of 2 open sets through different local coordinates, the local coordinate systems can be by a differentiable function; this is important to ensure that one can define a differentiable function over several local patches, and differentiable properties are not dependent on the type of local coordinates.

We commence by developing the notions of *charts*; they will be used to give a formal way of descibing what "locally Euclidean" means.

Definition 1.1 (n-chart)**.** An n-chart is an ordered pair (U, u), where $U \subseteq M$ is open in M and $u : U \to \phi(U) \subseteq \mathbb{R}^n$ is a homeomorphism such that $\phi(U)$ is open in $\mathbb{R}^n$.

Definition 1.2 (n-atlas)**.** An *n-atlas on M* is a set $A$of n-charts on M that cover M

$$\bigcup_{(U_a,u_a)\in A} U_a = M$$

Definition 1.3 (Locally Euclidean space)**.** A locally Euclidean space is a topological space equipped with an atlas

The presence of an atlas can easily be used to show that every point in the manifold had an open neighboerhood homeomorphic to a connected open set of $\mathbb{R}^n$ (another way of defining a locally Euclidean space); the charts themselves represent homeomorphisms to such sets, and since the charts cover the manifold, an arbitrary point must belong to some chart.

The "locally Euclidean" property is an important one for considering a space with coordinates, but also interpreting such spaces without a sense of coordinates.

Although the "locally Euclidean" property does imply some "global" properties, we often consider a locally Euclidean space with some extra "nice" properties; arriving at a *manifold*.

Definition 1.4 (n-dimensional Manifold)**.** An *n-dimensional manifold* is a second-countable, Hausdorff, locally Euclidean space

A sphere or circle are basic examples of manifolds; all points have some open neighborhood homeomorphic to open sets of $\mathbb{R}^2$ and $\mathbb{R}$ respectively, however these spaces

Informally, they capture the idea that some topological spaces 'look' Euclidean locally, but are possibly not Euclidean. For example, Earth's surface 'looks' like $\mathbb{R}^2$ from a human's point of view (ignoring bumps and mountains on the ground), but we know that the Earth is topologically a 2-sphere; if you walk in a straight line for long enough, you eventually reach your starting point again. Alternatively, if one shines a light parallel with the ground, it will traject above a far enough observer due to the curvature of the earth; this is how we know the Earth isn't flat.

Manifolds are useful for studying spaces that could be equipped with a coordinate system, though may benefit from studying it outside that context.

1.1 Submanifolds

There exist 2 variants of a notion of a submanifold in mathematical discourse.

1.1.1 Immersed submanifold

an *n-immersed submanifold of M* is a subset $N \subseteq M$ such that there exists an n-manifold (parameter space) P and bijective map $\psi : P \to N$.

1.1.2 Embedded (regular) submanifold

Immersed submanifold whose inclusion map is an embedding.

1.1.3 Implicit submanifold

1.2 Smooth manifolds

The motivation of this subject is to study topological spaces on which one can perform mathematical analysis, and study the properties that are conserved under diffeomorphisms (to be covered soon). The manifold is "almost" the perfect tool for this; the only caveat is that by itself, the "locally Euclidean" condition eventually prevents the possibility of a coordinate-free notion of a smooth function; the extra requirement is that all transition maps be smooth functions.

This is the sitation that explains why; if we let $f : M \to \mathbb{R}$ be a function such that for any 2 charts ψ, η both $f \circ \psi^{-1}$ and $f\eta^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \eta^{-1})$ are smooth, then we would like $\psi \circ \eta^{-1}$ to always be smooth too, since this allows the chain rule to apply and hence strongly link the smoothness properties of any chart padding of f; this leaves us with a *smooth manifold*.

We will be studying using smooth (C^∞) manifolds, however much of the theory can be applied to (continuously) differentiable (C^1) manifolds. There are a few reasons why I focus on smooth manifolds:

- Smooth manifolds are nicer mathematically and hence may permit more results
- The theory for differentiable manifolds is still similar to that of smooth manifolds
- I can't be bothered to type "differentiable"

Definition 1.5 (Smooth manifold)**.** A *smooth manifold* is a manifold such that for any 2 charts $(U, \phi), (W, \psi)$, the transition maps $\phi \circ \psi^{-1}$ are smooth in the sense of $\mathbb{R}^n$.

Definition 1.6 (Differentiable manifold)**.** Manifold with a maximum differentiable atlas (consider a base atltas and generate the 'maximal atlas' by taking the union with all other compatibel atlases to base atlas).

Definition 1.7 (Compatible atlases)**.** 2 atlases for M are *compatible atlases* if their union is a smooth atlas for M

1.2.1 Smooth functions

Now that we have smooth manifolds, we can extend our notions of smoothness from $\mathbb{R}^n$ to smooth manifolds by padding functions with charts.

Courtesy of all the transition maps being smooth, one can prove that if there exists one "chart padding" on f that makes it a smooth function, then any chart padding will make f smooth.

We can now introduce the notion of a smooth function on a smooth manifold knowing that our definition is independent of any chart chosen.

Definition 1.8 (Smooth function between smooth manifolds)**.** Let M, N be differentiable manifolds, $f : M \to N$ is *differentiable at* p iff for any differentiable chart (U, ϕ) of p and (W, η) of $f(p)$ such that $\eta \circ f \circ \phi^{-1}$ is differentiable at $\phi(p)$

Notice that for a function $f : M \to \mathbb{R}$ (where $\mathbb{R}$ has the Euclidean topology) we only require showing that some chart ψ exists such that $f \circ \psi^{-1}$ is smooth; this is because the "left" chart can trivially be taken as the identity map on $\mathbb{R}$.

It is great that we can now consider smooth functions, however can we similarly consider derivatives? The obvious notion of "chart padding" and then finding the partial derivative is unfortunately chart-dependent, however is adequate when our functions of interest happen to be contained all in the same chart.

Definition 1.9 (Local partial derivative)**.** The derivative of the following with respect to some variable of the domain chart

$$\phi \circ F \circ \eta^{-1}$$

If the choice of chart is obvious (or if one chart alone is being used), we may write $\frac{\partial F}{\partial x^i}$

We will eventually introduce the *differential*, which offers a notion of a derivative that considers all charts simultaneously.

Diffeomorphism

The notion of a homeomorphism essentialy defines the whole spirit of topology, and a diffeomorphism acts analogously for differential topology; differential topology is the study about properties of that are conserved between diffeomorphic smooth manifolds. The definition of a smooth functions places this directly in our grasp.

Definition 1.10 (Diffeomorphism)**.** A *diffeomorphism* between two differential manifolds N and M is a bijective function $f : N \to M$ such that both f and f^{-1} are smooth.

1.2.2 Smooth curves

Smooth curves turn out to be quite interesting to study geometrically, and differential geometry uses them to prove the isoperimetric inequality. At the same time, we'll require smooth curves for various topological constructions.

Definition 1.11 (Smooth curve)**.** A *smooth curve* is a C^∞ function $\gamma : I \to M$ mapping from a real interval to a manifold. Sometimes the image of this function is also reffered to as the differentiable curve.

Chapter 2

Tangent spaces

We've seen that by padding a curve by local charts, we can calculate its derivative respective to the local charts used.

It is undesirable, however, that this calculation is dependent on the local chart used. Nevertheless, the fact that local charts exist and are all compatible with eachother is enough to construct a *tangent space*; a linear space associated with every point on a manifold, containing the possible *tangent vectors* to that point on the manifold.

Instead of calculating the derivative of a curve at a point with respect to local charts (where results depend on the local charts used), differential topology prefer to consider the tangent vector of a curve at a point.

When working in $\mathbb{R}^n$, a differentiable curve at some point could evolve in any direction; for any vector you can imagine, there is a differentiable curve that has that vector as its tangent at some point. When working generally on differentiable manifolds, we may be subject to a more limited amount of tangents; for instance when considering differentiable curves on a sphere at a certain point, we can't have tangents can have a component that would make the particle 'pop off' the sphere. The tangents are limited to the tangent plane of the sphere at that point, rather than any vector.

So given a point in some differentiable manifold, we need to consider that not all tangents are possible and we should enumerate a set that contains the possible tangents at that point.

How will we generate all the tangents of a point on the manifold? By considering all possible differentiable curves on that manifold passing through that point, and putting all the derivatives into a set.

To define tangent spaces, we'll first formalize and discuss tangent vectors at p; all possible derivatives (i.e tangents) at p of smooth curves in the manifold.

There are 2 main tricks one can use to initially define tangent vectors of p of a smooth manifold:

- As equivalence classes of smooth curves through p with same derivative at p (direction of a curve's tangent)
- As derivations on functions representing the directional derivative at p (direction of a function's increase?)

Formally these are completely different mathematical objects, however since there is a perfect correspondence between curve equivalence classes and derivations (as we will later prove), both constructions are called tangent vectors and used interchangeably.

Definition 2.1 (Tangent vector)**.** Given a differentiable manifold M, point $p \in M$ and chart around p $\varphi : U \subseteq M \to \mathbb{R}^n$, then a *tangent vector of M at p* is the equivalence class of some curve $\gamma : [-1, 1] \to M$ with $\gamma(0) = p$, defined in the following way.

$$\mathbf{v} = \gamma'(0) = \{\eta \in C^\infty(I, M) : \gamma(0) = \eta(0) = p \wedge \exists\phi[(\phi \circ \eta)' = (\phi \circ \gamma)']\}$$

Assume we decide on one coordinate chart ϕ, the derivative of any smooth curve in $[\gamma]_p$ evaluated at p is $(\phi \circ \gamma)'(0) = \sum_{k=1}^n c_k \mathbf{e}_{x_k}$, so we can write the $\mathbf{v}_p = [\gamma]_p = \sum_{k=1}^n c_k \partial_{x_k}$

Note that instead of using the basis vectors $\mathbf{e}_{x_k}$, we use the notation ∂_{x_k} or $\frac{\partial}{\partial x_k}$ for the basis vectors in the local chart (I prefer the former); this stresses that the vectors are tangent vectors.

The idea is motivated by a geometric perspective; we are essentially defining the tangent vectors at p are captured as all possible tangents of smooth curves at p.

We can indeed check that this set is a well-defined equivalence relation; notice that if one has $(\psi \circ \eta)' = (\psi \circ \gamma)'$, then a second chart ϕ, then we have the following

$$\begin{aligned}(\phi \circ \eta)' &= (\phi \circ \psi^{-1} \circ \psi \circ \eta)' \\ &= (\phi \circ \psi^{-1})' \circ (\psi \circ \eta)' \\ &= (\phi \circ \psi^{-1})' \circ (\psi \circ \gamma)' \\ &= (\phi \circ \gamma)'\end{aligned}$$

Definition 2.2 (Tangent derivation)**.** Let $\mathbf{v}_p = [\gamma]_p$ be a tangent vector of M at p, then ???? is defined as such for any smooth $f : M \to \mathbb{R}$ we have the following. $\mathbf{v}_p(f) = (f \circ \gamma)'(0)$

Notice that any $\eta \in [\gamma]_p$ returns the same value by definition of a tangent vector

$= \sum_{k=1}^{n} c_k \frac{\partial(f \circ \phi^{-1})}{\partial x_k}(\phi(p))$

Those familiar with differential algebra (which is not assumed) may note that this is a *derivation*. In mortal language, this means it is linear and obeys the Leibniz law (product rule); in some contexts it is prefered to define tangent vectors as derivations and then show its relation to the curve-based definition.

The main idea is that equivalence classes of curves with the same derivative at p generate a derivation by considering its tangent vector derivative, and conversely, a derivation can be shown to correspond uniquely to said equivalence class of curves; since the notions are just as powerful as one another, there are two different perspectives of tangent vectors.

It's often advantageous to consider both perspectives, taking whichever is needed in the moment.

If we consider the set of all possible tangent vectors at p, we have the *tangent space of* p.

Definition 2.3 (Tangent space)**.** Given a differentiable manifold M, point $p \in M$ and chart around p $\varphi : U \subseteq M \to \mathbb{R}^n$, then the *tangent space of* M *at* p is the set T_pM of all tangent vectors of M at p.

$$T_pM = C^\infty(I, M)/\sim$$

Tangent spaces are extremely important due to the fact that they naturally form a linear space;

Proposition 2.1. Let M be an n-manifold, define $[\gamma]_p + [\eta]_p = [\psi^{-1}((\psi \circ \gamma)'(0) + (\psi \circ \eta)'(0))]_p$ Define $c \cdot [\gamma]_p[\psi^{-1}(c(\psi \circ \gamma)'(0))]_p$ Then $(\mathbb{R}, T_pM, +, \cdot)$ is a $\mathbb{R}$-linear space of dimension n

Since tangent spaces form a linear space, we would like to identify a basis to represent tangent vectors. We will employ ∂_x as the notation for a tangent space basis element associated with an infinitesimal increase in the x parameter.

We now introduce tangent bundles.

Definition 2.4 (Tangent bundle)**.**

$$TM = \bigsqcup_{p \in M} T_p M$$

The tangent bundle, well, 'bundles up' all the tangent spaces (double counting elements in multiple tangent spaces, hence why disjoint product is used).

2.1 Vector fields

In vector analysis we see $\mathbb{R}^n$-vector fields, however the fact that the domain and codomain of $\mathbb{R}^n$-vector fields is the same is a consequence of a more general definition for smooth manifolds.

Definition 2.5 (Vector field)**.** A *vector field* is a function $\mathbf{F} \in C^\infty(M, TM)$ such that $\pi(\mathbf{F}(x)) = x$

$\mathbb{R}^n$-vector fields are such because of the fact that $T\mathbb{R}^n = \mathbb{R}^n$.

2.1.1 Vector flows

Definition 2.6 (Flow)**.** Consider the differentiable manifold M, a *flow on* M is a (left) group action of the group $(\mathbb{R}, +)$ acting on a manifold M

We will now take a look at a type of flow that naturaly arises in the study of vector spaces, the *vector flow* (how creative).

Imagine a particle placed at point p in a vector field; the vector field pushes the particle around, making it "flow" around the field. The "flow" of this particle is the integral curve that p lies on.

Definition 2.7 ($\mathbf{F}$-integral curve at x)**.** Given a manifold M with a smooth vector field $\mathbf{F}$, the $\mathbf{F}$*-integral curve at* x is a smooth curve $\mathbf{r} : I \to M$ with defined by the following system of IVPs in local coordinates

$$\mathbf{r}'(t) = \mathbf{F}(\mathbf{r}(t)), \mathbf{r}(0) = x$$

Essentially, the vector field governs the tangent (i.e derivative) of the integral curve. Now imagine we would like a function that can conveniently represent all integral curves at once, where one parameter of this function is a point on the manifold, that when fixed at x returns the $\mathbf{F}$-integral curve at x; this is exactly the idea of a *vector flow*; the flow of all $\mathbf{F}$-integral curves.

This motivated the definition of a *flow*; a group action representing all integral curves on a vector field at once.

The situation we have described is the use of a flow to represent integral curves; this is known as a *vector flow*.

Definition 2.8 (Vector flow)**.** Given a manifold M with smooth vector field $\mathbf{F}$ the *vector flow of* $\mathbf{F}$ ψ is a flow on M such that $\psi(t, x)$ represents the $\mathbf{F}$-integral cuve through x

Calculating a vector flow is tantamount to solving the related system of IVP ODEs, however thanks to the Peano existence theorem solutions of this specific set of equations always exist and and are unique.

Since the Peano existence theorem creates a bijective correspondence between vector flows and vector fields, we are led to a method to recover a vector field given a vector flow (which is much, much easier than calculating a vector flow from a vector field).

Proposition 2.2.

$$\mathbf{F}(x) = \psi_t(0, x)$$

Vector fields and vector flows have evident applications in physics, but they are also useful in the study of continuous transformation groups!

Theorem 2.1. Let p be a point on smooth manifold where vector field doesn't vanish, then there is a local coordinate chart such that the vector field is 1 in one direction and zero in the others in this chart.

2.1.2 Vector field derivation

Vector fields map

Recalling the tangent vector, tangent derivation equivalence, one can analogously define "derivation fields" by considering tangent derivations associated with the tangent vectors of a vector field.

We have seen that a vector field is associated uniquely with a vector flow, however they are also uniquely associated with a differential operator!

This differential operator is

$$\mathbf{v}(f)(x) = \mathbf{v}(\mathbf{x}) \cdot \nabla f$$

One can prove the following representation of this differential operator with respect to the vector flow rather than vector field

Definition 2.9 (Derivation field)**.**

$$\mathbf{v}(f)(x) = (f \circ \psi_{\mathbf{v}})_t(0, x)$$

Where psi is the vector flow of $\mathbf{v}$.

Proposition 2.3. Let $\mathbf{v}(x) = \sum v^i(x)\partial_{x^i}$

$$\mathbf{v}(f)(x) = \sum v^i(x)(f \circ \psi^{-1})_{x^i}(\psi(x))$$

Where psi is a local chart.

Naturally, one can prove that $\mathbf{v}$ is a derivation.

Lie series

This differential operator gives a way of extending the methods of generating a Taylor series to generate a series for functions composed with a vector flow (with the spacial point fixed).

2.2 Differential

The tangent space also allows for the notion of the *differential* or *pushforward*; which is the generalization of the total derivative to any differentiable manifold. Indeed, in $\mathbb{R}$ and $\mathbb{R}^n$ it becomes the derivative and the total derivative respectively.

the main idea is that if you have a smooth function f m to n if you move in some "direction" (tangent vector) from p in m, the differential tells you what "direction" your corresponding image element will change from f(p).

Definition 2.10 (Differential (pushforward))**.** Let M, N be differentiable manifolds and $f : M \to N$ a differentiable function, The *differential of* f *at* p is a function $\mathrm{d}f_p : T_pM \to T_{f(p)}N$ defined as such, where $\gamma'(0)$ is a tangent vector of T_pM

$$df_p(\gamma'(0)) = (f \circ \gamma)'(0)$$

Note that one should check that such a function is well defined. More explicitly, any choice of smooth curve from the same tangent vector equivalence class produces the same result.

This definition, while intuitive in a qualitative sense, is initially not very useful for calculating the differential. By expressing the differential in terms of local coordinates through the chain rule, we discover properties of the differential that aliken it to the total derivative and reveals more concrete properties.

Theorem 2.2.

$$dF_p(\gamma'(0)) = \mathbf{J}_{\phi\circ F\circ\psi^{-1}}(\psi(p))\gamma'(0)$$

Corollary 2.1. The differential is a linear map

$$dF_p([\gamma]_p)$$

Differentials act as a way to communicate a linear approximation of differential change from one manifold to the other.

2.2.1 Related vector fields

Now that the differential is at out disposal, we can use it to demonstrate a special relationship between vector spaces.

Definition 2.11 (F-related pair of vector fields)**.** Let $\mathbf{F}, \mathbf{G}$ be vector fields, then they are F-related when the following holds.

$$dF_x(\mathbf{F}(x)) = \mathbf{G}(F(x))$$

One can prove that for F-related vector fields, the integral paths of $\mathbf{V}$ are mapped to integral paths of $\mathbf{W}$ through the differential.

Proposition 2.4. Let $\mathbf{F}$ and $dF_x \circ \mathbf{F}$ be F-related, then

$$F(\psi_{\mathbf{F}}(t, x)) = \psi_{dF_x\circ\mathbf{F}}(t, F(x))$$

If $\mathbf{F}$ and $dF_x \circ \mathbf{F}$ are F-related then we have

$$dF_x(\mathbf{F}(x)) = dF_x(\mathbf{F}(F(x))$$

2.2.2 Critical points

Definition 2.12 (Critical point)**.** For a function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $p \in U$ is critical point iff its pushforward df_p is not surjective

2.3 Lie bracket

Given vector fields, there is a special operation defined upon it that resembles a commutator (from ring theory), called the *Lie bracket*.

Definition 2.13 (Lie bracket)**.** Let $\mathbf{v}, \mathbf{u}$ be derivation fields, then the *Lie bracket* is the derivation field with the associated vector field operator.

$$[\mathbf{v}, \mathbf{u}](f) = \mathbf{v}(\mathbf{u}(f)) - \mathbf{u}(\mathbf{v}(f))$$

Due to the correspondence between vector fields and derivation fields, one can think of the Lie bracket as an operation on vector fields. The Lie bracket is meant to be a differential operator on some manifold, and hence it is easiest to define using derivation fields, though one may want to work using vector fields.

Though this describes the unique differential operator associated with $[\mathbf{v}, \mathbf{u}]$, one requires applications of the chain rule to describe $[\mathbf{v}, \mathbf{u}]$ in local coordinates.

Proposition 2.5 (Lie bracket in local coordinates)**.**

$$[\mathbf{v}, \mathbf{u}] = \sum_{i=1}^{m} \sum_{j=1}^{m} (v^j u^i_{x^j} - u^j v^i_{x^j}) \partial_{x^i}$$

Needless to say, this form is concrete but potentially messy to work with.

Proposition 2.6.

$$[\mathbf{v}, \mathbf{u}] = -[\mathbf{u}, \mathbf{v}]$$
$$[c\mathbf{v}, d\mathbf{u}] = cd[\mathbf{v}, \mathbf{u}]$$
$$[\mathbf{v} + \mathbf{w}, \mathbf{u}] = [\mathbf{v}, \mathbf{u}] + [\mathbf{w}, \mathbf{u}]$$
$$[\mathbf{v}, \mathbf{u} + \mathbf{w}] = [\mathbf{v}, \mathbf{u}] + [\mathbf{v}, \mathbf{w}]$$

Somewhat surprisingly, the Lie bracket can be represented as the derivative of 4 compositions of the vector flow, evaluated at 0.

$$f(t, \mathbf{x}) = \varphi(-\sqrt{t}, \psi(-\sqrt{t}, \varphi(\sqrt{t}, \psi(\sqrt{t}, \mathbf{x}))))$$
$$[\mathbf{v}, \mathbf{u}](\mathbf{x}) = f_t(0, \mathbf{x})$$

2.4 Tangent spaces of submanifolds

Implicitly defined submanifolds as the zeroset of F are the zeroset of the differential dF

Theorem 2.3 (Frobenius' theorem)**.** A system of vector fields is integrable iff it is in involution.

Chapter 3

Jet spaces

Tangent spaces are a useful tool for setting up a notion of a derivative on a manifold independently of chosen coordinates (the pushforward). It would be useful to generalize this idea to support the notion of higher order derivatives; this is particularly important in modelling differential varieties (submanifolds representing space of solutions to some differential equation).

We do this by *jet spaces*; these are spaces that keep track of functions up to their Taylor series truncated to n terms, or more essentially, a space that captures the first n derivatives.

Definition 3.1 (Jet space).

$$J^n$$

Similar to tangent spaces, we often want to represent all Jet spaces as sets for the image of some functions, hence we analogously consider objects called Jet bundles.

Definition 3.2 (Jet bundle).

Chapter 4

Differential forms

The differential offers a notion of a derivative independent of local charts thanks to tangent spaces. Given that the bread and butter of mathematical analysis is differentiation and integration, we're naturally lead to the following question; can a notion of an integral be developed independent of local charts?

To do this, we mix the concept of an exterior algebra with tangent spaces.

Definition 4.1 (Differential k-form)**.** $\bigwedge_k T_x^* M$

We can think of differential k-forms as alternating covectors of the tangent space with k parameters.

Definition 4.2 (Smooth ifferential k-form)**.** $\bigwedge_k T_x^* M$

Definition 4.3 (1-form)**.** Function between

4.1 Exterior derivative

$$d\omega = \sum d$$

Chapter 5

Orientability

Orientable atlas is an atlas where all transitions of charts have positive Jacobian determinant.

Chapter 6

Fiber bundles

Chapter 7

De Rham cohomology

7.1 De Rham complex

7.2 Stokes-Cartan theorem

The Stokes theorem of vector analysis generalizes to differential topology by means of differential forms.

Theorem 7.1 (Stokes-Cartan theorem)**.**

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

Chapter 8

Grassmanian

The Grassmanian allows us to understand and give meaning to the following question; What does it mean to change subspaces ”smoothly”

Definition 8.1 (Grassmanian)**.** smooth manifold formed from k-dimensional linear subspaces of V $\mathbf{Gr}(k, V)$

Proposition 8.1. $\mathbf{Gr}(k, \mathbb{R}^n)$ is a compact smooth manifold of dimension $k(n-k)$