Enumerative Combinatorics

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Chapter 1

Basic combinatorial methods

Put simply, combinatorics is the branch of mathematics concerned with cardinalities of finite sets. Enumerative combinatorics aims to develop techniques to directly enumerate (or at least approximate) the cardinality of finite sets; given X, what is $|X|$?

Extremal combinatorics aims to identify and enumerate the "largest" (or "smallest") set within a family of sets; it studies techniques that can be use when direct enumeration of the whole family is not possible.

Combinatorics can be studied for its own sake, however it is also useful for problems in applied mathematics, and combinatorial problems even often arise in the midst studying other branches of pure mathematics! Conversely (and like all of all of mathematics), techniques from many other branches of pure mathematics arise in combinatorics; this is demonstrated by the studies of analytic combinatorics and algebraic combinatorics, which apply methods of their respective fields to enrich enumerative combinatorics.

Although set theory suffices to describe basic combinatorial objects, many mathematical objects that have become central belong to other fields of mathematics; among some of these fields an the related mathematical object are the following.

- Graph theory (graphs)
- Formal language theory (strings)
- Number theory (integer partitions)
- Group theory (words)

Since combinatorics revolves around cardinalities of finite sets, set theory is foundational for combinatorics, and indeed many set theoretic constructs are among the most important combinatorial objects.

First consider

Lemma 1.1. Let X, Y be finite sets There exists an injective function $f \in \mathrm{Map}(X, Y)$ iff $|X| \leq |Y|$ There exists a surjective function $f \in \mathrm{Map}(X, Y)$ iff $|X| \geq |Y|$ There exists a bijective function $f \in \mathrm{Map}(X, Y)$ iff $|X| = |Y|$

This lemma instantly gives 2 fundamental combinatorial techniques that will be employed incessantly in this book.

1.1 Bijection principle

The result relating to bijective functions will be an extremely important lemma for rigorously building combinatorics, however it is also useful for comparing combinatorial problems; problems with a bijection have the same solution.

Theorem 1.1 (Bijection principle)**.** Let $f : X \to Y$ be bijective, then $|X| = |Y|$.

For those familiar with the terminology of abstract algebra, a bijection is a "combinatorial isomorphism", meaning that combinatorial problems whose resulting sets have a bijection between them are in some way "essentially the same" problem.

One important application of this is that the perspective offered by a bijection different formulae for "essentially the same" problem, meaning that the different formulae can be equated; this is the notion of a combinatorial proof and it has serious repercussions on number theory in particular.

1.2 Pidgeonhole principle

The part of the lemma relating to injective functions is fundamental in extremal combinatorics and is called the *pidgeonhole principle.*

To get to the heart of mathematics, both formalism and intuition should be given adequate attention; a point that was motivated by the famous cominatorialist Gian-Carlo Rota at a time when mathematicians became obsessively concentrated on creating networks of theorems and maniulating mathematical notation. In this spirit, we suggest a "way of thinking" that will motivate the pidgeonhole principle.

The pidgeonhole principle gets its name from the adage "if there are more pidgeons than pidgeonholes (and every pidgeon is in a pidgeonhole), then some pidgeonhole has multiple pidgeons". In many languages it is translated literally as the *drawer principle*, and an analogous analogy supports this nomenclature.

More abstractly, imagine the process of creating a function between finite sets with the hope of making it an injective function, considering each $x_i \in X$ and deciding on some $f(x_i) \in Y$. If $|X| > |Y|$, then when deciding a mapping for $x_{|Y|+1}$ we notice that every $y \in Y$ already has a mapping, and we will be required to "reuse" some $y \in Y$ for $f(x_{|Y|+1}$, violating injectivity.

Theorem 1.2 (Pidgeonhole principle)**.** Let X, Y be finite sets, then there exists an injective function $f \in \text{Map}(X, Y)$ iff $|X| \leq |Y|$

In a group of 8 people, there is a pair of people born on the same day of the week.

This can be generalized even further.

Theorem 1.3 (Generalized pidgeonhole principle)**.** Let X, Y be finite sets, $f : X \to Y$ be a function and $|X| > n|Y|$ for some $n \in \mathbb{N}$, then there exists some $y \in Y$ such that $|f^{-1}(\{y\})| > n$

In a group of 29 people, there are at least 5 people born on the same day of the week.

Here is an even cooler combinatorial problem.

Example 1.1. A university is conducting 2006 seminars, each with 40 attendees. For each pair of seminars, there is exactly one person attending both. Is it true that there must be someone attending all 2006 seminars?

By the extended pidgeon hole principle we can see that there exist some person attending at least $\lceil \frac{2005}{40} \rceil + 1 = 52$ seminars. Since in each class he has no peers from another one of his classes, eminars, however one can prove that someone attending strictly more than 40 seminars must attend them all.

Let p attend $t > 40$ seminars $S_1, S_2, ..., S_t$, and assume by hypothesis that p doesn't attend U. Futhermore assume we have $U \cap S_i = U \cap S_j = \{x_i\}$, then $|S_i \cap S_j| = 2$ since they contain both p and x_i, creating a contradiction, implying that $U \cap S_i$ is disjoint to any $U \cap S_j$.

The inequality $|U| \geq |\bigcup_{i=1}^{t} U \cap S_i|$ should hold, but this yields $40 \geq t$, which is a contradiction that disproves our hypothesis.

1.3 Sum and product rules

There are various techniques by which one can split a problem into (hopefully easier) subproblems which are connected to the original problem by means of arithmetic operations on cardinal numbers.

The principles introduced in this section are based on set theoretic notions alone, and are uncreatively named after the operation that links the subproblems to the original problem.

1.3.1 Sum rule

Recalling how addition of cardinalities is defined in set theory, we can prove the following result by demonstrating a bijection between disjoint union and a union of disjoint sets. As intuition suggests, one can count a set by breaking it ito parts, counting each part, and then summing these results. As elementary as this sounds, finding a clever way to partition a set is is an extremely powerful technique.

To prove the sum rule, we first examine the following lemma.

Lemma 1.2. Let X, Y be disjoint finite sets, then $|X \cup Y| = |X| + |Y|$.

Naturally, this lemma extends to any finite collection of pairwise disjoint sets.

Since partitions are disjoint finite sets, we end up with the following result.

Proposition 1.1 (Sum rule)**.** Let $(X_k)_{k=1}^{n}$ form a partition on X, then the following holds.

$$|X| = \sum_{k=1}^{n} |X_k|$$

The sum rule says that knowing a partition on a combinatorial problem allows the problem to become "broken up" into several (hopefully easier) subproblems using addition.

Example 1.2. If one can only wear a necklace or a ring, and there are 7 necklaces and 5 rings, how many possible outfits are there? $7 + 5$ or 12 different fits.

1.3.2 Product rule

Enumerating cartesian products is simply done by taking the product of the cardinalities of the base sets; this is the *product rule.* This rule is useful since many problems can be modelled by cartesian products, or are otherwise bijective to such a representation.

Lemma 1.3.
$$|X \times \{0\}| = |X|$$

we prove inductively that the following holds where multiplication is interpreted as repeated addition

Theorem 1.4 (Product rule)**.** Let $(X_k)_{k=1}^n$ be a sequence of sets, then the following holds.
$$|\prod_{k=1}^{n} X_k| = \prod_{k=1}^{n} |X_k|$$
Let X, Y be finite sets, then $|X \times Y| = |X||Y|$.

Cartesian products contain pairs (or tuples) where each index is chosen from a certain set; this makes the product rule useful for combinatorial problems where multiple choices hold simultaneously

Example 1.3. A pizzeria makes pizzas with a 'protein' (cut of meat), a sauce, and a choice of condiments. If there are 5 protien choices, 3 sauces, and 10 condiment choices, how many pizza combinations are there? $5 \cdot 3 \cdot 10$ or 150

1.3.3 Quotient rule

When considering the cardinality of sets reduced to equivalence classes, the set may be enumerated by the quotient rule when all equivalence clases have identical cardinality; it is a special application of the sum rule.

Proposition 1.2 (Quotient rule)**.** Let $\sim$ be an equivalence class on X, where each resulting equivlence class has cardinality n_0 (so $|[x]| = n_0$), then the following holds.

$$|X/\sim| = \frac{|X|}{n_0}$$

Sometimes we have solved a combinatorial problem, however we notice that some different elements are "essentially the same" in some way and we want to consider them as the same element rather than distinct ones (we use equivalence classes to describe "essentially the same" elements); the quotient rule shows when division can be used to forward the original solution to the new problem.

Later on, we will employ the quotient rule to use the permutation problem to solve the combination problem (we make different permutations of the same string apart of the same equivalence class).

Example 1.4. 2 red, 1 green, and a 1 blue marble are placed in a line on a desk. How many ways can the marbles be ordered if they are all used in the process and the red marbles are indistinguishable? $\frac{4 \cdot 3 \cdot 2 \cdot 1}{2}$ or 12 First consider 3 disjoint cases

How many strings of length 5, no repeating characters, from only alphabet symbols, and ignoring the order of symbols can be created? there are 120 ways a set of 5 distinct characters can be arranged, and 7893600 distinct strings that could be made; the quotient rule implies that $\frac{7893600}{120}$ or 65780 possibilities

1.4 Factorial function

Recall how in our last example we took the product of 26,25, and 24; numbers adjacent to eachother. We'll introduce a powerful notation to compactly represent the product of the

Definition 1.1 (Factorial function)**.** The *factorial function* is the function $! : \mathbb{N} \to \mathbb{N}$ that represents the product of the first n natural numbers, where $0! = 1$ by convention.

$$0! = 1$$
$$n! = \prod_{k=1}^{n} k$$

Alternatively, one can define the factorial function recursively in the dollowing manner.

Proposition 1.3. The factorial function obeys the following recurrence relation

$$0! = 1$$

$$n! = (n-1)!n$$

The factorial function appears abundantly in the study of enumerative combinatorics, however it also arises in many other fields of mathematics.

There

Definition 1.2.

$$(n)_m = \prod_{k=0}^{m-1}(n-k) = \frac{n!}{(n-m)!}$$

Definition 1.3.

$$n^{(m)} = \prod_{k=0}^{m-1}(n+k) = \frac{(n+m-1)!}{(n-1)!}$$

1.5 Recursive formulae

When finding a general formula for the cardinalities of an ordered family of sets, one may find a relationship between a set's cardinality and its predecessors in the family, leading to a *recursive formula* Such formulae are usually proven by mathematical induction.

1.5.1 Fibonacci numbers

One of the most famous sequences in mathematics arises from a recursive formula for enumerating the cardinalities in a particular family of sets; the *Fibonacci sequence*. The Fibonacci sequence is also studied as an object of interest in number theory, and appears unexcpectedly throughout many branches of mathematics.

Chapter 2

Arrangements, permutations, and combinations

Now that we are familiar with the basic principles of combinatorics, we will be able to look at enumerating sets of functions. These combinatorial problems are so frequently encountered in applications as well as in more complex problems, and are hence of particular interest to solve.

How many functions between a specified domain and codomain are there?

Note that to beautify the results of combinatorics (cardinality bars in complex formulae look horrendous) we usually consider functions such that the domains and codomains as integer intervals; replacing these sets with any other set with the same cardinality will yield the same combinatorial results.

Lemma 2.1.

$$|\mathrm{Map}(X_1 \cup X_2, Y)| = |\mathrm{Map}(X_1, Y)||\mathrm{Map}(X_2, Y)|$$

Proposition 2.1.

$$|\mathrm{Map}(X, Y)| = |Y|^{|X|}$$

Apply the lemma by considering the partition of singletons; we have $|\mathrm{Map}(\{x\}, Y)| = |Y|$ by the bijection $f_y \mapsto y$ for the $f_y : \{x\} \to Y$ such that $\mathrm{Im}(f_y) = \{y\}$, so the result follows.

Proposition 2.2.

$$|\mathcal{P}(X)| = 2^{|X|}$$

This is proven by the product rule by considering for each $x \in X$, the sets with x and without x; this forms a base set $\{x, \emptyset\}$ in the cartesian product with cardinality 2.

2.1 Arrangements

$$[a..b] = \mathbb{Z} \cap [a, b]$$

arrangement is a function

$$f \in \mathrm{Map}(X, Y)$$

arrangement without repetition is an injective function

$$f \in \mathrm{Inj}(X, Y)$$

Proposition 2.3. There are n^k k-arrangements of n

$$|\mathrm{Map}(k, n)| = n^k$$

$$|\mathrm{Inj}(k, n)| = \frac{n!}{(n-k)!} = (n)_k$$

This is proven by noting that $\mathrm{Map}(k, n)$ is bijective to $\prod_{i=1}^{k}[1..n]$, which has cardinality n^k by the product rule.

The latter is similarly proven by product rule, however one notes that each subsequent base set in the cartesian product has one less value in $[1..n]$ it can map to to ensure injectivity.

2.1.1 String notation

$f : [1..n] \to \Sigma$

`ZAC` is formally an arrangement mapping 1 to Z, 2 to A, and 3 to C. This string notation can essentially be thought of as "notational sugar" for arrangements.

Example 2.1. Given the set $\Sigma = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots, \mathrm{Z}\}$, we want to find how many possible strings of length 3 exists, given that the set has replacement. There are 26 possibilities for the first letter, and since we have replacement, ther are 26 possibilities for the second letter since the previously chosen letter 'regenerated' back into Σ. Similarly, there are 26 possibilities for that third letter. So there are 26^3 or 17576 possible strings.

Example 2.2. Given the set $\Sigma = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \dots, \mathrm{Z}\}$, we want to find how many possible strings of length 3 exists, given that the set doesn't have replacement. There are 26 possibilities for the first letter, and since we don't have replacement, ther are 25 possibilities for the second letter since the previously chosen letter was 'deleted' from Σ when we chose it. Similarly, there are 24 possibilities for that third letter. So there are $26 \cdot 25 \cdot 24$ or 15600 possible strings.

Many interesting problems in combinatorics can be phrased in terms of strings,

In this setting we merely use strings as a notation, however abstract algebra can create an intrinsic mathematical strucure for them called the *free monoid of order n with no rules*; this is covered in algebraic combinatorics (advanced part of this book) and in the advanced section of formal language theory.

Consider a multiset (a set that allows for multiple instances of elements); how many k-tuples (ordered sequences of k instances) can be made from the multiset? These are called permutations; . However there are 2 different 'rules' we can place on ourselves when counting permutations or doing any other activity based on drawing instances from multisets.

Now that the basic techniques of combinatorics are understood, we'll use them to solve more complex problems. The first of which is the problem of *permutations*; consider a set of n elements; how many k-tuples (ordered sequence with k terms) can be formed by elements of this set?

Here's another perspective; given a set of n symbols Σ, how many strings of length k can be formed?

One may note there are 2 different 'rules' we can place on ourselves when counting permutations.

2.2 Permutations

We're now able to count arrangements with and without replacement, which is quite useful. Now imagine that we want to consider strings "up to a scrambling of symbols", so `EARTH` and `HEART` should be considered the same, or $LISTEN$ and $SILENT$, or LIFE and FILE, or $HXEUI$ and $UHIXE$. Essentially, the "order" in which the symbols appear is unimportant for distinguishing strings.

One trick to formalize this idea is to group arrangements into equivalence classes, which contains all the arrangemets tha are "the same up to some scrambling". The way we represent a possible "scrambling" is by a special function called a *permutation* (to permute means to swap around, or to scramble up).

A *permutation* is a bijective operation $\pi : X \to X$

$$\pi \in \mathrm{Map}(X, X)$$

We represent the set of permutations on X as $\mathrm{Sym}(X)$, and we allow writing $\mathrm{Sym}([1..n])$ as $\mathrm{Sym}(n)$

Permutations happen to be extremely important in group theory as well as combinatorics (indeed, the 2 branches of mathematics have much cross over), and a group theoretic study of permutations can lead to some deep combinatorics. Alas, for now we'll study and apply permutations using our basic principles.

Lemma 2.2.

$$|\mathrm{Sym}(n)| = n!$$

2.3 Combinations

We can now form the equivalence classes that classify groups of strings "equal up to a permutation"; these classes are called *combinations* (i.e all "combinations" of the same symbols are the in the same class).

Definition 2.1 (Combination)**.**

$$f \sim g \iff \exists \pi \in \mathrm{Sym}(g(X))[f = \pi \circ g]$$

combination is an equivalence class of arrangements equal to eachother up to a permutation

$$\mathrm{Map}(k, n)/\sim$$

A *combination without repetition* is an equivalence class of arrangement without repetition equal to eachother up to a permutation

$$\mathrm{Inj}(k, n)/\sim|$$

Sets do not impose an order on its elements nor do elements appear more than once; this suggests a correspondence between combinations without repetition and subsets

Lemma 2.3.

$$|\mathrm{Inj}(k,n)/\sim| = |\{K \subseteq [1..n] : |K| = k\}|$$

This problem is extremely similar to the problem of permutation without repetition, however for combinations without repetition we use the quotient rule to ignore order.

Proposition 2.4. The following formula counts k-combinations in n with repetition

$$|\mathrm{Inj}(k,n)/\sim| = \frac{n!}{(n-k)!k!}$$

When repetitions are allows, one cannot similarly use the quotient rule to calculate combinations from arrangements since not every combination equivalence class has the same cardinality.

Among elementary methods, there is a very particular use of the bijection principle to calculate the amount of combinations with repetition is commonly known as the *stars & bars approach*.

To appeal to the intuition, we will equivalently view the problem from the perspective of forming combinations of k-string from an alphabet of n symbols, with repetition allowed.

Since we only care about combinations, we consider taking strings in some "canonical form" repetitions of symbols are adjacent

`ALBUQUERQUE` goes to `ABEELQQRUUU` `WATERFALL` goes to `AAEFLLRTW` `MARIO` goes to `AIMOR`

By considering stars correspond to symbols bars correspond to "symbol collections"

Considering the distinct strings using $n-1$ bars and k stars is a combinatorial problem is bijective to $C((n,k))$, and is much easier to calculate.

Proposition 2.5. The following formula counts k-combinations in n without repetition

$$|\mathrm{Map}(k,n)/\sim| = \frac{(n+k-1)!}{k!(n-1)!} = \binom{n+k-1}{k}$$

2.4 Binomial coefficient

The expression $\frac{n!}{(n-k)!k!}$ is so common that it warrants an alternate notation called the *binomial coefficient*. It gets its name from its role as coefficients of the resulting polynomial in the binomial theorem of elementary algebra.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Since the combination function is quite frequent, this notation is more convenient to employ. It gets its name for being related to the coefficients of a binomial when expanded.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{0} = 1$$

$$\binom{n}{1} = n$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n-m}{m-j} = \binom{n}{k}$$

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Theorem 2.1 (Vandermonde's convolution)**.**

$$\binom{r+s}{n} = \sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k}$$

Chapter 3

Partitions

2 combinatorial objects that can be ”partitioned”; sets and integers.

3.1 Integer partitions

Definition 3.1 (Integer partition symbol)**.** λ is a multiset of integers whose sum is n

$$\lambda \vdash n$$

3.1.1 Integer partition function

We introduce a function to represent this combinatorial problem.

Definition 3.2 (Parititon function)**.** The *partition function* is the function $p : \mathbb{N} \to \mathbb{N}$ that returns the amount of multisets whose sum of elements equal the input element

$$|\{K \subseteq [1..n] : K \text{ is a multiset } \wedge \sum_{k \in K} k = n\}|$$

The partition function is an arithmetic function studied in number theory, and without resorting to algebraic combinatorics its study is rather difficult.

3.1.2 Young diagram

Also known as Ferrer diagrams to the French,

3.2 Set partitions

The idea of partitions is so important in combinatorics (we have already made great use of them) that we have specific terminlology and notation for them.

Definition 3.3 (Set partition symbol)**.** ρ is a set of disjoint sets whose union is X

$$\rho \vdash X$$

Definition 3.4 (Blocks of a partition)**.** Let $\rho \vdash X$ then blocks of ρ are the elements of ρ

Partitions are useful for applying combinatorial methods (notably the sum rule and quotient rule), however partitions also serve as interesting combinatorial objects to study themselves; namely, how many possible partitions on X are there?

3.2.1 Bell numbers

The Bell numbers are defined precisely as the solution to this problem.

Definition 3.5 (Bell numbers)**.**

$$B_n = |\{\rho \subseteq \mathcal{P}([1..n]) : \rho \vdash [1..n]\}|$$

Our combinatorial methods are sufficient to derive the following formula for Bell numbers.

Theorem 3.1.

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-k-1}$$

The history of Bell numbers traces back to medieval Japan in a parlor game called "Genjikō" described in the novel "tale of Genji". The game goes as such; you are blindfolded and sniff 5 sticks of incense, can you identify which sticks of incense were of the same type? Thinking of this as a set of 5 elements (5 distinct physical sticks) where one creates equivalence classes containing the sticks of incense of the same type, there are B_5 or 52 possible answers one could propose.

3.2.2 Stirling numbers

If we perhaps only

Definition 3.6 (Stirling numbers of the second kind)**.**

$$\left\{ {n \atop k} \right\} = |\{\rho \subseteq \mathcal{P}([1..n]) : X \vdash [1..n] \wedge |\rho| = k\}|$$

What are the first kind of Stirling numbers? These are deferred for the advanced part of the book, specifically the study of algebraic combinatorics.

Since Bell numbers count all partitions, Bell numbers are therefore the sum of all related Stirling numbers (of the second kind).

$$B_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}$$

Chapter 4

Further elementary methods

We are coming near the end of elementary counting methods; beyond this chapter we will be required to draw from other branches of mathematics to obtain deeper results. This chapter explains the remaining combinatorial methods that are accessible with the prerequisite of this part.

4.1 Twelvefold way

4.2 Generating polynomials

4.3 Inclusion-exclusion principle

We understand through the sum rule that the cardinality of a finite union of finite disjoint sets is merely the sum of said set's cardinalities (i.e to find the total among of options between 2 groups of distinct options, simply sum the groups).

If we are forced to deal with non-disjoint sets (i.e groups that contain overlap), we must consider a more general principle that tells how to calculate the finite union of arbitrary finite sets.

We will consider the case of 2 sets.

Proposition 4.1 (Inclusion-exclusion principle (2 sets))**.**

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$

Noticing using set theory that one has $X \cup Y = X \cup (Y \setminus X)$ and that $Y = (X \cap Y) \cup (Y \setminus X)$

Proposition 4.2 (Inclusion-exclusion principle (3 sets))**.**

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|$$

Theorem 4.1 (Inclusion-exclusion principle)**.** Let X_k be finite sets, then the following holds.

$$|\bigcup_{k=1}^{n} X_k| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{J \subseteq [1..n], |J|=i} |\bigcup_{j \in J} X_j|$$

Notice that if we assume that the X_k are pairwise disjoint, then we have a partition rather than a covering and we obtain the sum rule.

4.3.1 Derangement numbers

A particularly interesting formula for derangements that follows from this is one including e! One may realise the similarity of our previous series to the definition of the exponential function; it is essentially a partial sum of the series for $\frac{1}{e}$!

Definition 4.1 (n-derangements)**.** surjective operations of n elements with no fixed points.

$$D_n = |\{f \in \mathrm{Map}(\mathbb{N} \cap [1, n], \mathbb{N} \cap [1, n]) : f \text{ is bijective } \wedge \nexists n \in \mathbb{N} \cap [1, n][f(n) = n]\}|$$

$$D_0 = 1$$

A closed form for n-derangements is found using the inclusion-exclusion principle to count the amount of permutations with each k being mapped to k, then minusing this from $n!$.

Proposition 4.3.

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

There exists a particularly concise expression for derangements that can be obtained by a neat trick with the previous proposition; a well known fact of real analysis is the following power series for e^x (that if we haven't seen before, we will just take for granted).

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Noting the similarity of this formula to the partial sums of the series for e, we can prove the following.

$$D_n = \frac{n!}{e} - \sum_{k=n+1}^{\infty} \frac{(-1)^k n!}{k!}$$

Knowing that derangements must be integers, we obtain the following.

$$D_n = [\frac{n!}{e}]$$

Alternatively, a recursive formula can also be found using the sum rule; bijections (of which there are $n!$) are partitioned into derangements (of which there are D_n) and non-derangements. Non-derangements are calculated again by applying the sum rule with a very particular partition as described below.

$$\bigcap_{k=1}^{n} \{f \in \mathrm{Map}(\mathbb{N} \cap [1, n], \mathbb{N} \cap [1, n]) : f(k) = k \wedge \forall i > k[f(i) \neq i]\}$$

With this partition, we can apply the sum rule to obtain the following result.

Proposition 4.4.

$$D_n = n! - [\sum_{k=2}^{n-1} \binom{n}{n-k} D_k] - 1$$

$$D_n = n! - \sum_{k=1}^{n} \binom{n}{k} D_k$$

Derangements are a great testiment to the power of the inclusion-exclusion principle; often finding a partition for a combinatorial problem requires more care and may only lead to a recursive formula, while the inclusion-exclusion principle can lead to more concrete formulae and requires weaker conditions than the sum rule.

4.4 Catalan numbers

Chapter 5

Combinatorial numbers

Many integer sequences arise from problems in combinatorics. There are also integer sequences studied in number theory that have applications in combinatorics. To complicate matters, number theorists occasionally study combinatorial numbers from a number theory perspective. It has been a challenge to decipher which types of numbers should be ascribed to Enumerative Combinatorics or Elementary Number Theory; the philosophy I've taken is that any sequence that has historical recognition as combinatorial or is conventionally studied in a combinatorial perspective rather than a number theoretic one shall be listed under this chapter. There are a few points of contention on what I have included, however if the reader should be interested in a holistic view of integer sequences, please do read Elementary Number Theory; the two books go hand in hand in many respects.

Before defining and studying the main combinatorial numbers, we shall look at combinatorial applications from particular sequences from Elementary Number Theory. Familiarity with such sequences will be assumed, so their definitions will not be listed here.

5.1 Fibonacci sequence

The Fibonacci sequence is the familiar recurrence relation below.

Definition 5.1 (Fibonacci sequence).

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1$$

It is studied as an integer sequence in its own right in number theory, however we will focus on the combinatorial aspects of this sequence. Using the definition of Fibonacci numbers

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$$

5.1.1 Sums of 1 and 2

Example 5.1.

$$\mathbf{F_n} = |\{(a_k)_{k\in\mathbb{N}} : a_k \in \{1,2\} \wedge \sum_{k=1}^{m} a_k = n\}|$$

1 1 2 11,2 3 111,12,21 5 1111,112,121,211,22 8 11111,1112,1121,1211,2111,122,212,221

This combinatorial problem is solved by noting that one can calculate the terms recursively by considering sequences starting with '2' next to any sequence adding up to $n-2$, or '1' next to any sequence adding up to $n-1$. The combinatorial problem can be solved in another way by

$$\mathbf{F_n} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-k-1}{k}$$

$$\mathbf{F_n} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-k-1}{k}$$

Deriving new closed forms in this way is an example of the power of double counting in combinatorics. It would be possible (but messy) to derive this using real analysis to calculate the Taylor series for the Fibonacci number's generating function, although the method of double counting generally appeals better to the intuition and may be considered more elegant.

5.2 Catalan numbers

The Catalan numbers are numbers that relate to a wide range of bijective combinatorial problems; we will take the following as the definition of the Catalan numbers, this is known as Senger's recurrence relation.

Definition 5.2 (Catalan numbers)**.**

$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$$

The Catalan numbers have a closed form which can be found by a special application of the bijection principle (it is alternatively derived using the theory of generating functions from analytic combinatorics). To see this, however, we require knowing some combinatorial problems that the Catalan numbers enumerate.

5.2.1 Dyck words

Consider *Dyck words*, words consisting of balanced brackets (or more generally, words formed from a binary alphabet such that any initial segment has more instances of (than) , and the whole word has an equal amount of both symbols).

We will prove that, and then employ a double counting technique to derive a closed form for the Catalan numbers.

The problem of Catalan numbers arises in the counting of balancing brackets, or Dyck words. 1 () 2 (()),()() 5 ((())),(()()),()(()),(())(),()()() 14 (((()))),((()())),(()(())),((())()),(()()()),()((())),()(()()),(())(()),()()(()),((()))(),(()())(),()(())(),(())()(),()

$$C_n = \binom{2n}{n} - \binom{2n}{n+1}$$

5.2.2 Mountain ranges

5.2.3 Lattice paths

5.3 Eulerian numbers

Part II

Advanced

Chapter 6

Generating functions

Generating functions have been described as a "clothesline" to "hang" a sequence on; it is a single analytic object that represents the data of the whole sequence, all at once.

Though the nature of its uses are rather abstract, generating functions are an object that arises commonly in both combinatorics and number theory, particularly as a means to involve methods from mathematical analysis in the settings of combinatorics and number theory.

In more elementary combinatorics, they can also be used as an enumeration; the way that coefficients of polynomials multiply together can be leveraged as a technique to enumerate certain combinatorial problems.

Generating functions are interesting curiosities to beginners, however their use is often not understood; we'll require some motivating examples and observations to lay strong foundations for this chapter. Recall the binomial theorem taught in Elementary Algebra.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Even though this result is taught as a method in algebra to quickly calculate binomial expansions, the common proof for this result relies on basic combinatorics.

The idea is that we think of as strings.

$$(x+y)^5$$

$$xxxyy, xxyxy, xyxxy, yxxxy, xxyyx, xyxyx, yxxyx, xyyxx, yxyxx, yyxxx$$

Example 6.1. You want to buy a Gameboy that costs \$80, but the seller will only accept notes (5,10,20,50,100). How many possible ways are there to pay the seller with no change by using distinct note combinations?

It is the coefficient of x^{80} in the following polynomial.

$$(\sum_{k=1}^{16} x^{5k})(\sum_{k=1}^{8} x^{10k})(\sum_{k=1}^{4} x^{20k})(\sum_{k=1}^{1} x^{50k})$$

To calculate this coefficient in a more efficient manner than a complete direct expansion, we can consider the closed form for partial sums of a geometric series.

Since we can reduce coefficients of polynomials. This is really the bijection principle in action; calculation coefficients of multiplied polynomials can be put in bijection to many combinatorial problems, in the case of the binomial theorem, coefficients of exponentiated binomials have a bijective relationship to permutations of k amount of $x$es and $n-k$ amount of ys.

If we can encode our problem by means of polynomials, we can invoke results from mathematical analysis relating to polynomials to solve combinatorial problems. The moral is that the bijection principle is truly the heart of combinatorics; translating our problem into equivalent structures that are easier to calculate!

6.1 Ordinary generating functions

Since polynomial encodings are so important, it is useful to tabulate some frequent encodings for common combinatorial problems. We standardize this by the notion of a *generating funtion*, the most fundamental being the ordinary generating function.

Definition 6.1 (Ordinary generating function)**.**

$$\sum_{k=0}^{\infty} S_k x^k$$

6.2 Exponential generating functions

Often generating functions have a trivial factor known to occur in every coefficient.

$$\sum_{k=0}^{\infty} S_k \frac{x^k}{k!}$$

6.3 Lambert series

6.4 Bell series

6.5 Dirichlet series

By substituting this definition into the generating function series and keeping in mind that we want $C_0 = 1$, we can use our definition to deduce the generating function of the Catalan numbers.

Proposition 6.1.

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

One can use methods from real analysis to find a Taylor series of $\frac{1-\sqrt{1-4x}}{2x}$ and in doing so, find a closed form for the Catalan numbers.

Proposition 6.2.

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

This is a rather modern, mechanical approach combinatorics permits to evaluate a closed form of an integer sequence, however combinatorial beauty is found by studying the nature of specific counting problems in which this sequence appears.

Chapter 7

Analytical combinatorics

7.1 Generating functions

7.2 Circle method

7.3 Saddle-point method

7.3.1 Dobiński's formula