

Fourier Analysis

Zac Zerafa

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Part I
Fundamentals

Chapter 1

Fourier series

Fourier analysis is the fundamental (perhaps described as most elementary) field of harmonic analysis, studying functions between Euclidean spaces and their trigonometric series representations.

Abstract harmonic analysis generalizes the ideas and tools of Fourier analysis by considering more complex spaces that may require a different type of sum (or integral) decomposition to best suit the space's symmetry.

Trigonometric series were used notably by Euler, Lagrange, and Gauss in various applications, however Fourier and Bessel were the first mathematicians to think about them as a way of representing an arbitrary function; it was his boldness that brought this field to life!

Taylor series are based on a premise that a Taylor series should have the same derivatives as the function it models. Fourier series work by means of a different train of thought, we're projecting a function onto a space spanned by infinite 'sinusoids'.

1.1 Fourier series

Definition 1.1 (Trigonometric series).

$$a_0 + \sum_{n=1}^{\infty} [a_n \cos(n \frac{2\pi}{P} x) + b_n \sin(n \frac{2\pi}{P} x)]$$

$$\sum_{n=-\infty}^{\infty} c_n e^{in \frac{2\pi}{P} x}$$

Due to convenience, we will study Fourier analysis by means of the its complex form.

We also have the following formulae to convert the coefficients between the 2 representations.

Proposition 1.1.

$$\begin{aligned}c_0 &= a_0 \\c_n &= \frac{a_n - ib_n}{2} \\c_{-n} &= \frac{a_n + ib_n}{2}\end{aligned}$$

We are familiar with the Taylor series decomposition for analytic functions by matching derivatives at some reference point, can we also model some rich class of functions by means of a trigonometric series? This was the exact motivation of Fourier and Bessel.

If we restrict our attention to $L^1([0, 2\pi])$ functions (complex functions integrable on $[0, 2\pi]$), we can devise a general method by using the *orthogonality of trigonometric functions on $[0, 2\pi]$* , in other words, we can abuse the following identities.

$$\int_{[0, 2\pi]} e^{inx} e^{imx} dx = 0$$

Let's say we have some function $f : [0, 2\pi] \rightarrow \mathbb{C}$ and we're hoping to model a trigonometric series converging to f on $[0, 2\pi]$.

$$\begin{aligned}& \int_{[0, 2\pi]} f(x) e^{inx} dx \\&= \int_{[0, 2\pi]} \left[\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx} \right] e^{imx} dx \\&= \sum_{n=-\infty}^{\infty} \widehat{f}(n) \int_{[0, 2\pi]} e^{inx} e^{imx} dx \\&= \widehat{f}(n) \int_{[0, 2\pi]} e^{imx} e^{imx} dx + \widehat{f}(-n) \int_{[0, 2\pi]} e^{-imx} e^{imx} dx \\&= 2\pi \widehat{f}(-n)\end{aligned}$$

If we rearrange for $\widehat{f}(-n)$ and then flip $-n \rightarrow n$, we have a method to calculate the coefficients we want so that the trigonometric series equals f (well, at least for sufficiently nice functions)!

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{[0, 2\pi]} f(x) e^{-inx} dx$$

There's one slightly subtle but mathematically crucial caveat to this method; the 3rd equality swaps the infinite sum and the integral. This cannot be done in general, so we must keep in mind that this trigonometric series converges to f iff this sum-integral swap is permitted (we will discuss this in further detail later).

We expanded on the interval $[-\pi, \pi]$, though one can generalize this to any interval $[-\frac{T}{2}, \frac{T}{2}]$ by considering some $2\pi/T$. So far our theory had been about developing Fourier series on $[-\pi, \pi]$, however this can be generalized to $[-\frac{T}{2}, \frac{T}{2}]$.

$$\widehat{f}(n) = \frac{\langle f, e^{-inx} \rangle}{\langle e^{-inx}, e^{-inx} \rangle}$$

Definition 1.2 (Fourier series). For a real function f defined on $[-\frac{T}{2}, \frac{T}{2}]$, the *Fourier series of f on $[-\frac{T}{2}, \frac{T}{2}]$* (discrete spectrum) is the following related trigonometric series (if it exists).

$$f \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{in\omega x}$$

- $\omega = \frac{2\pi}{T}$
- $\widehat{f}(n) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-in\omega x} dx$ are the *Fourier coefficients*

We remind the reader that a Fourier series of a function does not necessarily equal the function, even if this is the main motivation for its definition (just like Taylor series). Indeed, it is quite complicated to determine when and in what sense (pointwise or uniformly, everywhere or almost everywhere etc.) a Fourier series converges to its function. We will come back and answer this question later.

The Fourier coefficient function is worthy of much study, and it obeys many elementary, yet interesting properties.

Proposition 1.2. • $f \in L^2 \implies \|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$

- $\widehat{af + bg}(n) = a\hat{f}(n) + b\hat{g}(n)$
- $\widehat{f(x - a)}(n) = e^{-in\omega a} \hat{f}(n)$
- $\widehat{f'}(n) = in\omega \hat{f}(n)$
- $\widehat{f^*}(n) = [\hat{f}(-n)]^*$

The fourier coefficients together are called the (discrete) spectrum of a function. Their modulus and argument are called the amplitude and phase spectrum respectively.

1.2 Forced series development

Furthermore, one can also consider a Fourier sine series or Fourier cosine series on $[0, 2\pi]$ (or generally $[0, T]$), that only expands the function in terms of one type of sinusoid (sine or cosine)

1.3 Bessel's inequality

Lemma 1.1 (Bessel's inequality (Fourier coefficients)). Let f be a piecewise continuous periodic function whose Fourier series exists, then the following holds.

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$$

Considering partial sums of the Fourier series as $s_n(t)$, the Bessel inequality hinges off the fact that $\int_{-\frac{T}{2}}^{\frac{T}{2}} (f(t) - s_n(t))(f(t) - s_n(t))^* dt$ must be positive and real since it is the integral of the square of a modulus $zz^* = |z|^2$

1.4 Riemann-Lebesgue lemma

To prove the fundamental theorem of Fourier series, we will eventually need to employ the *Riemann-Lebesgue lemma*, which states that Fourier coefficients tend to 0 for larger frequencies.

To make mathematical proofs less complicated, chunks of the proof that amount to a result in and of itself will be set aside as its own lemma so that it can be applied to the proof and any other proofs.

Lemma 1.2 (Riemann-Lebesgue lemma (Fourier coefficients)). Let f be a piecewise continuous periodic function whose Fourier series exists, then the following holds.

$$\lim_{n \rightarrow \pm\infty} \hat{f}(n) = 0$$

The intuition is that because continuous functions have close image elements associated for close domain elements, the more oscillatory trigonometric functions bounce at a higher frequency and are therefore required less and less in the series.

This is one of the lemmas we will require to propound the fundamental theorem of Fourier series; a theorem that assures that for any piecewise smooth periodic function f , the Fourier series converge to $\frac{f(t+) + f(t-)}{2}$ pointwise! Our analysis in this theorem will be based on analyzing the partial sums of a Fourier series, and this lemma will be the final touch to prove the theorem.

1.5 Fundamental theorem of Fourier series

This theorem allows us to understand when and how Fourier series of a function converge to the function itself.

Theorem 1.1 (Fundamental theorem of Fourier series). Let f be a piecewise smooth periodic function, then the Fourier series of f converges pointwise to the following.

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\omega x} = \frac{f(x+) + f(x-)}{2}$$

This theorem doesn't give a necessary condition; it is a possibility that some non-piecewise smooth functions converges to its Fourier series, and there are examples of this (such as the Weierstrass function)

Proposition 1.3.

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$$

$$\widehat{fg}(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m)\hat{g}(n - m)$$

Furthermore, this theorem operates within the realm of elementary real analysis, playing with a Riemann integral until our result is derived. It is indeed possible to find a wider class of functions that converges in some weaker sense; this however requires working with the Lebesgue integral and spaces of functions called L^p spaces, covered in the advanced section of this book.

1.6 Parseval's theorem

Definition 1.3 (Parseval's theorem). Let f, g be piecewise smooth periodic functions, then the following holds.

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)g^*(x)dx = \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)^*$$

Parseval's theorem can be used to derive many interesting series. Though the fundamental theorem of Fourier series can do this too, Parseval's theorem gives more of an edge to create such series.

1.7 Convolution kernels

Taylor's theorem considers partial Taylor series as apart of its result; it is a good start to study partial Fourier series too. We know that since it is a finite sum that it indeed converges, so things are much nicer.

Note that this section requires familiarity with the convolution operator, discussed in the advanced section of Real Analysis. The idea of this section is to show a way of using convolutions with a function to form their partial Fourier series. This streamlines some of our previous observations that we required in the fundamental theorem of Fourier series.

1.7.1 Dirichlet kernel

We can actually represent partial Fourier series by means of a special integral transform on the related function f ; partial Fourier series have an integral representation!

For an informal derivation, let's operate under the ansatz that such a integral transform exists, then we have the following.

$$\sum_{n=-N}^N \widehat{f}(n)e^{inx} = \int_0^{2\pi} f(x)g(x,t)dt$$

Since we know how Fourier series coefficients are calculated, we replace the Fourier coefficient symbols with what they calculate to.

$$\sum_{n=-N}^N \widehat{f}(n)e^{inx} = \int_0^{2\pi} f(x)g(x,t)dt$$

Applying the linearity of the integral (which is possible in general since we are considering finite sums), it becomes clear what kind of kernel our integral transform must take!

Our kernel is $\frac{\sum_{n=-N}^N e^{-inx}}{2\pi}$. For the sake of a clean notation, we can consider partial Fourier series as the convolution of f and $\frac{\sum_{n=-N}^N e^{inx}}{2\pi}$.

Let's make some definitions.

Definition 1.4 (Dirichlet kernel).

$$D_n(x) = \sum_{k=-n}^n e^{ikx} = \frac{\sin(\frac{n+1}{2}x)}{\sin(\frac{x}{2})}$$

Proposition 1.4.

$$(D_n * f)(x) = 2\pi \sum_{k=-n}^n \widehat{f}(k)e^{ikx}$$

$$\int_{-\pi}^{\pi} D_n(x)dx = 2\pi D_n(2\pi k) = 2n+1 \quad D_n \text{ is even} \quad (D_n * f)(x) = 2\pi \sum_{k=-n}^n \widehat{f}(k)e^{ikx}$$

$$\exists f'(x) \implies \lim_{n \rightarrow \infty} (D_n * f)(x) = f(x) \quad \exists f'(x) \implies \lim_{n \rightarrow \infty} (D_n * f)(x) =$$

$$\frac{1}{2}[f(x^+) + f(x^-)] \quad \exists f'(x) \text{ p.w} \implies \lim_{n \rightarrow \infty} (D_n * f)(x) = f(x)$$

The Dirichlet kernel can be used to make the proof of the fundamental theorem of Fourier series much more mechanical since we now have prepackaged results that could be applied to hasten the proof, rather than toying around with integrals.

1.7.2 Fejér kernel

The Cesàro summation of the first n partial Fourier series can also be represented as an integral transform. If one were to reverse engineer under this assumption like we did to derive the Dirichlet kernel, we would end up discovering the *Fejér kernel*.

Chapter 2

Fourier transform

Fourier series are a representation for periodic functions, however what about extending these techniques to non-periodic functions? This can be done by upgrading from a Fourier series to a 'Fourier integral'.

We will require different tools; hence the Fourier transform will become our analogue for the Fourier coefficients, acting as a kernel for the Fourier integral.

If f equals some Fourier series, we want to find some sequence $\widehat{f}(n)$ such that

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx}$$

if f equals some 'Fourier integral', we want to find some function $\widehat{f}(\xi)$ such that

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)e^{ix\xi}d\xi$$

The basic idea is that instead of considering a countably infinite, discrete set of frequency functions (in our case, e^{inx} with $n \in \mathbb{Z}$), we now consider an uncountably infinite, continuous set of frequency functions ($e^{ix\xi}$ with $\xi \in \mathbb{R}$). Due to this continuous nature we end up studying an integral representation instead of a sum representation like in the Fourier series.

In our study of Fourier series, we started with the question of finding a trigonometric series representation if a function and finished with our solution; a method to compute the appropriate Fourier series coefficients.

For 'Fourier integrals', we will start the other way around; we will define the Fourier transform and then show that it is the appropriate weight for the 'Fourier integral' as a result called the Fourier inversion theorem.

2.1 Fourier transform

As alluded to, the Fourier transform will be the kernel that weighs how much each $e^{i\xi x}$ wave should be included in the Fourier integral to formulate the function.

Definition 2.1 (Fourier transform). For a real function f defined on \mathbb{R} , the *Fourier transform* (continuous spectrum) is the integral transform \mathcal{F} operating on real functions that describes how trigonometric functions of frequency ξ are present in a function (if the Fourier transform exists).

$$\mathcal{F}\{f\}(\xi) = \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x} dx$$

- f is a real function
- $\mathcal{F}\{f\}$ is the Fourier transform

Proposition 2.1.

f, g are real functions

- $\mathcal{F}\{af + bg\} = a\mathcal{F}\{f\} + b\mathcal{F}\{g\}$
- $\mathcal{F}\{f'\} = i\xi\mathcal{F}\{f\}$
- $\mathcal{F}\{f^*\}(\xi) = [\mathcal{F}\{f\}(-\xi)]^*$
- $\mathcal{F}\{f\}$ is uniformly continuous
- $\mathcal{F}\{f(x - a)\} = e^{-ia\xi}\mathcal{F}\{f\}$
- $\int_{-\infty}^{\infty} \widehat{f}(x)g(x)dx = \int_{-\infty}^{\infty} f(x)\widehat{g}(x)dx$
- $\mathcal{F}\{f\}(\xi - a) = \mathcal{F}\{e^{iax}f\}(\xi)$
- $\mathcal{F}\{f(ax)\}(\xi) = \frac{1}{a}\mathcal{F}\{f\}(\frac{\xi}{a})$

These elementary properties may seem similar to those for Fourier coefficients. Indeed (no book on mathematics is complete without copious proliferation of the word 'indeed'), we will discuss many properties of the Fourier transform that are also applicable for Fourier series coefficients!

Both being integral transforms with similar kernels, the Fourier coefficients are related to the Fourier transform in the following way!

$$\hat{f}(n) = \frac{1}{T} \mathcal{F}\{f\}(n\omega)$$

The selfduality of the Fourier transform $\int_{-\infty}^{\infty} \mathcal{F}\{f\}(\xi)g(\xi)d\xi = \int_{-\infty}^{\infty} f(\xi)\mathcal{F}\{g\}(\xi)d\xi$

If f is a real function that is absolutely integrable on \mathbb{R} then its Fourier transform exists. Furthermore, the Fourier transform of such a function would be continuous.

If f is a continuously differentiable real function whose continuous spectrum (Fourier transform) exists then the Fourier transform of f' exists and follows $\mathcal{F}\{f'\} = i\xi\mathcal{F}\{f\}$

2.2

The Schwartz space is discussed in the advanced section of Real Analysis; it is a space that happens to find very nice interactions with the Fourier transform.

2.3 Convolution theorem (Fourier transform)

Now we will lead to the big result relating to the convolution operator in Fourier analysis. We know that $\mathcal{F}\{f\} + \mathcal{F}\{g\} = \mathcal{F}\{f + g\}$ due to the linearity of the Fourier transform, but is there a similar result for compactifying $\mathcal{F}\{f\}\mathcal{F}\{g\}$ into 1 Fourier transform?

The answer is 'yes', and our friend convolution comes in clutch.

Theorem 2.1 (Convolution theorem (Fourier transform)). Let f, g be absolutely integrable on \mathbb{R} , then the following holds

$$\mathcal{F}\{f * g\}(\xi) = \mathcal{F}\{f\}(\xi)\mathcal{F}\{g\}(\xi)$$

This is good news for signal processing engineers, since they now have an easier way to calculate the Fourier transform for their superimposed signals. It also facilitates problems in PDEs.

2.4 Riemann-Lebesgue lemma

Similar to our study on Fourier series, we will require an analogue of the Riemann-Lebesgue lemma.

Lemma 2.1 (Riemann-Lebesgue lemma (Fourier transform)). Let f be an absolutely integrable, piecewise continuous function on \mathbb{R} , then the following holds.

$$\lim_{\xi \rightarrow \pm\infty} \mathcal{F}\{f\}(\xi) = 0$$

2.5 Fundamental theorem of Fourier integrals

When studying Fourier series of periodic functions, the ultimate question was when the Fourier series would converge to its own function. We will ask ourselves the same thing with the Fourier integral, when (and how, recalling that there exist different notions of convergence) does the Fourier integral converge to its own function?

The Fourier transform must exist in the first place, but even if it does, it does not guarantee that the Fourier integral will converge, yet alone back to the original function!

Definition 2.2 (Inverse Fourier transform (Fourier integral)).

$$\mathcal{F}^{-1}\{f\}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{i\xi x} d\xi$$

Theorem 2.2 (Fundamental theorem of Fourier integrals). If f is absolutely integrable and piecewise smooth function on \mathbb{R} , then the following CPV holds.

$$\mathcal{F}^{-1}\{\mathcal{F}\{f\}\}(x) = \frac{f(x+) + f(x-)}{2}$$

Proposition 2.2. Let f, g be absolutely integrable piecewise smooth functions such that $\mathcal{F}\{f\} = \mathcal{F}\{g\}$, then $f = g$ whenever they are both continuous.

This proposition justifies us as talking about the Fourier integral as the inverse Fourier transform.

Proposition 2.3.

$$\mathcal{F}\{f(-x)\}(\xi) = 2\pi f(\xi)$$

2.6 Plancherel's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{F}\{f\}(\xi)|^2 d\xi$$

2.7 Fourier transform for distributions

Like with the derivative of distributions, we would like the Fourier transform for distributions and real functions to coincide for real functions that are also distributions.

Definition 2.3 (Fourier transform for (Schwartz) distributions).

$$\langle \mathcal{F}\{T\}, \phi \rangle = \langle T, \mathcal{F}\{\phi\} \rangle$$

2.7.1 Fourier transform on Schwartz functions

Since we are dealing with Schwartz distributions, it is useful to understand how the Fourier transform works on Schwartz functions. We first note that the fundamental theorem of Fourier integrals applies to Schwartz functions, and that the following fact holds.

Proposition 2.4. The Fourier transform as an operator on Schwartz space is injective.

$$\mathcal{F}(\mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R})$$

$$\mathcal{F}\{1\}(\xi) = 2\pi\delta(\xi)$$

$$\mathcal{F}\{e^{iax}\}(\xi) = 2\pi\delta(\xi - a)$$

Definition 2.4 (Dirac comb).

$$\text{comb}_T(x) = \sum_{n=-\infty}^{\infty} \delta(x - nT)$$

We now define what it means to shift a distribution.

it comes from the idea of again thinking of distributions as idealized integral transform kernels; if a kernel was translated, the reverse chain rule would relate it such that the following would hold

$$\langle T(x - c), \phi \rangle = \langle T, \phi(x + c) \rangle$$

$$\langle \mathcal{F}\{T'\}, \phi \rangle = \langle \mathcal{F}\{T\}, ix\phi(x) \rangle$$

$$\langle \mathcal{F}\{T'\}, \phi \rangle = \langle \mathcal{F}\{T\}, ix\phi(x) \rangle$$

$$\langle \mathcal{F}\{T\}(-\xi), \phi \rangle = \langle 2\pi T, \phi x \rangle$$

$$\langle \mathcal{F}\{\mathcal{F}\{T(-x)\}\}, \phi \rangle = \langle 2\pi T, \phi x \rangle$$

Part II
Advanced

Chapter 3

L^p analysis on Fourier series

The fundamentals part of this book develops the theory of Fourier analysis with reference only to fundamental real analysis and basic complex analysis, however the introduction of more advanced tools such as Lebesgue integration, L^p spaces and ideas from functional analysis allows for a much more advanced study of Fourier series and integrals.

On a less important note, the domain of periodic functions on $[-\frac{T}{2}, \frac{T}{2}]$ can be considered as the circle group when adding domain elements; Fourier series essentially exist on the circle group.

In this chapter, we will employ the Lebesgue integral rather than the Riemann integral so that we can study an alternatives to the fundamental theorem of Fourier series, notably the fact that the Fourier series of L^2 functions converges pointwise almost everywhere.

This is a slightly weaker type of convergence being considered, however it applies to a larger class of functions as well as being a Hilbert space, meaning that we can apply functional analysis to study Fourier series!

Theorem 3.1 (Carleson's theorem). Let f be a function in L^2 , then the Fourier series of f converges to f pointwise almost everywhere.

Chapter 4

L^p analysis on Fourier integral

Functional analysis have since become a major tool in the study of Fourier analysis, since (Lebesgue) integrable functions form their own space (called a Banach space) on which the Fourier transform is a functor.

In the same vein as the previous chapter, we apply functional analysis and measure theory to commence a deeper study on the Fourier integral, and use them to generalize the Fourier integral for other spaces.

$$\mathcal{F}\{f\} \in L^\infty \quad \|\widehat{f}\|_\infty \leq \|f\|_1$$

Theorem 4.1 (Plancherel theorem).

$$f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \implies \|\mathcal{F}\{f\}\|_2 = \|f\|_2$$

4.1 Fourier integral on distributions

We define distributions in Functional Analysis, and now we generalize the Fourier integral to them.

4.1.1 Poisson's summation formula

4.2 Fourier integral on \mathbb{R}^n

Lemma 4.1 (Riemann-Lebesgue lemma for \mathbb{R}^n). Let f be a $L^1(\mathbb{R}^n)$ function, then the following holds (i.e its Fourier transform's tails tend to 0).

$$\lim_{\xi \rightarrow \pm\infty} \mathcal{F}\{f\}(\xi) = 0$$

$$f \in L^1(\mathbb{R}^n) \implies \lim_{\xi \rightarrow \pm\infty} \mathcal{F}\{f\}(\xi) = 0$$