General Topology

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Part I

Fundamentals

Chapter 1

Topological spaces

1.1 Topology

Geometry studies spaces and objects within them, particularly the notions of distance, length, angles, curvature, area, volume; features that can be quantified concretely.

Eventually, a problem in mathematics was posed that challenged the perspective that mathematicians would have towards geometry forever. In the Prussian town of Königsberg (today the Russian town of Kalingrad), there were 7 bridges connecting 4 distinct sides of the town. Is it possible to walk a path between these 'islands' that traverses each bridge exactly once?

In one sense the problem appeared vaguely geometric due to its relation to traversing through a space, however it is clearly not related to ; it relied soley on how the 'islands' were connected between themselves by bridges. This lead to the creation of a field called graph theory which was subsequently used to prove that such a traversal is impossible.

This problem spurred much thought of how spaces could be abstractified, particularly on how some properties of 'closeness' could stand alone without the need for distance.

A space with distance defined is called a metric space; it is the space at the heart of geometry. Mathematicians experimented with such spaces and discovered that *open sets* of the space (sets not containing their boundary, that is, the surface covering them) could always be expressed in terms of set algebraic laws. Therefore one could create an abstract 'system of open sets' so long as they follow. This is precisely the idea of a *topology*.

For example, the open sets of the metric space $(\mathbb{R}, |\cdot|)$ (real line with distance defined by the absolue difference) are (possibly countably infinite) unions of open intervals $\bigcup_{i=1}^{\infty}(a_i, b_i)$, since the boundary points would be those a_i, b_i, which are excluded from the set. When we learn the definition of a topology, we will see this is indeed a topology.

Topology is therefore called the study of spaces, and the properties retained when continuously deformed (i.e homeomorphic, this is when the properties related to open sets are unchanged). The reason it is described like this is because continuous deformations on spaces are the types of functions that don't change the properties of open sets. Therefore this is just saying that topology is the study of open sets.

Topology is useful for abstractifying ideas from geometric spaces, but in mathematics it is sometimes easier to conduct proofs in more general settings. Such proofs are also beneficial in that it allows the result to be applicable to more situations; this is where much of the power of topology lies!

Overtime, we will see how open sets are sufficient to handle many concepts without the need to resort to a metric space. When we study metric spaces, we will see how this definition of a topology was derived and all will be clear, however for now we will take for granted this abstract formalization of a topology by the following set algebra.

Definition 1.1 (Topology)**.** A *topology on a set* X is a set $\mathcal{T}$ of subsets of X such that:

- X and $\emptyset$ are in $\mathcal{T}$
- $\mathcal{T}$ is closed under finite intersections
- $\mathcal{T}$ is closed under countable unions

$$\mathcal{T} \subseteq \mathcal{P}(X) \text{ is a topology on } X \iff X, \emptyset \in \mathcal{T} \wedge [\bigcap_{i=0}^{n} U_i \in \mathcal{T}] \wedge [\bigcup_{i=0}^{\infty} U_i \in \mathcal{T}]$$

Definition 1.2 (Topological space)**.** A *topological space* is an ordered pair $(X, \mathcal{T})$ of a set X and a topology $\mathcal{T}$ on X denoted as . Elements of X are referred to as *points*.

- X is a set
- $\mathcal{T}$ is a topology over X

The sets in $\mathcal{T}$ are called the *open sets*, and they are said to be *open in X*

$$(X, \mathcal{T}) \text{ is a topological space } \iff \mathcal{T} \text{ is a topology on } X$$

$$U \text{ is open in } X \iff U \in \mathcal{T}$$

When the topology is apparent, one may denote the topological space $(X, \mathcal{T})$ as just X.

We won't define boundaries for topological spaces just yet since we'll require the notion of 'neighborhoods and limit points', but we know from our study in metric spaces that we'll eventually prove that closed sets contain their whole boundary, and open sets contain none of it.

For any given set X, there are two 'obvious' topologies that could be made.

$$\mathcal{T} \text{ is the discrete topology on } X \iff \mathcal{T} = \mathcal{P}(X)$$

$$\mathcal{T} \text{ is the indiscrete topology on } X \iff \mathcal{T} = \{X, \emptyset\}$$

We introduce a topology that can be put upon infinite sets.

$$\mathcal{T} \text{ is the cocountable topology on } X \iff \mathcal{T} = \{U \subseteq X : U = \emptyset \vee |X \backslash U| \leq \aleph_0\}$$

$$\mathcal{T} \text{ is the cofinite topology on } X \iff \mathcal{T} = \{U \subseteq X : U = \emptyset \vee |X \setminus U| < \aleph_0\}$$

$$X \text{ is finite } \wedge \mathcal{T} \text{ is the cofinite topology on } X \implies \mathcal{T} \text{ is the discrete topology on } X$$

1.1.1 Comparing topologies

Definition 1.3. Given 2 topologies $\mathcal{T}$,$\mathcal{U}$, we say that $\mathcal{U}$ *is finer than* $\mathcal{T}$ *and* $\mathcal{T}$ *is coarser than* $\mathcal{T}$ iff $\mathcal{T} \subset \mathcal{U}$.

Similar to how $\leq$ orders the real numbers, we may use $\subseteq$ to form an order of topologies. The main difference is that not all topologies can be compared to eachother like real numbers. if $a \leq b$ is false, then $b \leq a$ is true, however it is posible that we can have $\mathcal{T} \subseteq \mathcal{U}$ and $\mathcal{U} \subseteq \mathcal{T}$ both false.

In order theory, we say that this is a partial order (or poset); an order where uncomparable pairs of elements exist.

1.1.2 Examples of topological spaces

To build some intuition for topological spaces, we offer some basic examples of what topological spaces look like (and what they don't look like).

Basic examaples

Example 1.1.

$$(\{1,2,3\},\{\emptyset,\{1,2,3\},\{2,3\},\{1,2\},\{2\}\})$$

$$(\{1,2,3,4\},\{\emptyset,\{1,2,3,4\},\{2,3\},\{1,2,3\},\{1,4\},\{1\}\})$$

Examples that aren't topological spaces

Example 1.2.

$$(\{1,2,3,4\},\{\emptyset,\{1,2,3,4\},\{2,3,4\},\{1,2,4\}\})$$

$$(\{1,2,3\},\{\emptyset,\{2,3\},\{1,2\},\{2\}\})$$

Example 1.3. When one applies the cofinite topology on the set $\mathbb{N}$, the result is the following topological space.

$$(\mathbb{N},\{U : |\mathbb{N}\setminus U| < \aleph_0\})$$

Example 1.4. The *Sierpiński space* is the topological space defined as such

$$(\{1,0\},\{\emptyset,\{1\},\{1,0\}\})$$

Though that last example was kind of cool, it's perhaps not entirely clear why we're doing topology in the first place. We now discuss a topological space that is quite familiar to us.

Example 1.5 (Euclidean topology on $\mathbb{R}$)**.**

$$(\mathbb{R},\{U\subseteq\mathbb{R} : U=\bigcup_{n\in\mathbb{N}}(a_n,b_n)\})$$

It is interesting to note the nature of how Euclidean topologies are defined; and close them up under countable unions. This is the idea of a *topological basis.*

1.1.3 Basis of a topological space

Those familiar with linear algebra are familiar with the idea of a *basis*; a set of elements that under some operation can generate an entire space. Topological spaces follow the same principle; often we can find some basis that can generate the topological space that helps our analysis of the space. Better yet, perhaps we want to define a topological space by means of a basis!

Definition 1.4. Let $(X, \mathcal{T})$ be a topological space. A *basis* of $\mathcal{T}$ is a set $\mathcal{B} \subseteq \mathcal{T}$ such that any set open in X is a union of sets in $\mathcal{B}$. Elements of $\mathcal{B}$ are called *basic sets*.

Every topological space can be represented by a basis since the topology itself forms a trivial basis for itself.

Proposition 1.1. Let $(X, \mathcal{T})$ be a topological space. $\mathcal{T}$ is a basis for $\mathcal{T}$

A topological basis can be though of as providing the 'ingredients' and the three set algebra laws generate these ingredients into a topology. But unfortunately not every set of sets can generate a topology under countable unions; when does a topological basis acually generate some topology?

By ensuring our set of sets agrees with the 3 conditions of the algebra of sets, we can be sure that our set is basis for some topology on the space.

Definition 1.5. A set $\mathcal{B} \subset \mathcal{P}(X)$ generate some topology on X iff both of the following hold

- $X = \bigcup_{i \in \mathbb{N}} \mathcal{B}_i$
- For any sets $\mathcal{B}_i \mathcal{B}_j \in \mathcal{B}$, we have $\mathcal{B}_i \cap \mathcal{B}_j \in \mathcal{B}$

Proposition 1.2. Let $\mathcal{P}$ be a partition on X. Then $\mathcal{P}$ is a basis for some topology on X.

We can now check whether a set of sets can actually form a topology, but what if we want to check if our basis forms a *particular* topology in question?

We can refine our definition of a basis to be more 'constructive' in this sense.

Proposition 1.3. Let $(X, \mathcal{T})$ be a topological space. A set of sets $\mathcal{B}$ is a basis for $\mathcal{T}$ iff any of the following hold.

- If U is an open set, it is a union of sets in $\mathcal{B}$
- If U is an open set, for any $u \in U$ there is a set in $\mathcal{B}$ containing u that is completely contained in U

Example 1.6. $\{U \subset X : U = \{x\}, x \in X\}$ is a topological basis for the discrete topology on X

A basis is a nice way to define a topology, however it is possible that different basis' can actually generate the same topology!

Proposition 1.4. Two basis $\mathcal{B}_1, \mathcal{B}_2$ generate the same topology iff all the following hold.

- For any $B_1 \in \mathcal{B}_1$ each $b_1 \in B_1$ has a set $B_2 \in \mathcal{B}_2$ such that $b_{\in} B_2$
- For any $B_2 \in \mathcal{B}_2$ each $b_2 \in B_2$ has a set $B_1 \in \mathcal{B}_1$ such that $b_2 \in B_1$

When we have two basis' for a topology, we can compare their *refinement*.

Definition 1.6. Let $(X, \mathcal{T})$ be a topological space. If $\mathcal{B}, \mathcal{C}$ are two basis' for $\mathcal{T}$, we say that $\mathcal{B}$ is a refinement of $\mathcal{C}$ iff $\mathcal{B} \subseteq \mathcal{C}$.

1.1.4 Euclidean topology

We have seen some rather rudimentary ways of constructing topologies on a general space X, however one of the most fundamental topologies for our intuition is the *Euclidean topology*, which characterizes open sets in a Euclidean space.

$$\mathcal{B}_{\mathbb{R}^n} = \{U \subseteq X : U = \{x \in X : \sum_{i=1}^{n} (p_i - x_i)^2 < r^2\}\}$$

Note that we are implicitly using a metric to define the basis elements, this alludes to a way to create a topology from any metric space; more on this later.

In $\mathbb{R}$, the basis elements are the open intervals.

Since we define open sets as unions of open intervals, we can see that the open intervals form a basis for the Euclidean topology.

$$\mathcal{B}_{\mathbb{R}} = \{(a, b) : a, b \in \mathbb{R} \wedge a < b\}$$

We'll briefly use the notation $B(\mathbf{p}, r)$

$$B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{p} - \mathbf{x}\| < r\}$$

$$\mathcal{B}_{\mathbb{R}^n} = \{B(\mathbf{p}, r) : \mathbf{p} \in \mathbb{R}^n \wedge r \in \mathbb{R}^+\}$$

Noting that $(a, b) = (\frac{a+b}{2} - \frac{b-a}{2}, \frac{a+b}{2} + \frac{b-a}{2}) = B_{\mathbb{R}}(\frac{a+b}{2}, \frac{b-a}{2})$ we see that the open intervals are actually open balls of the single dimension Euclidean metric space. This alludes to the fact that this is actually a metric space; indeed this is true for any dimensional Euclidean topology, however we'll savour the details for later.

Due to the general familiarity that readers tend to have with the Euclidean topology, much of the theory developed for general topology will be applied to the Euclidean topology as a mode of demonstration.

Chapter 2

Limit points

We've alluded to the fact that we're trying to model open sets as sets that don't contain any of their 'boundary', so we'd ultimately like to define what a boundary is on a formal level.

2.1 Neighborhoods

Now we will start developing more specific theory of topology that culminates to generalizing limits to topological spaces. We do this by means of beighborhoods.

Definition 2.1 (Neighborhood)**.** A *neighborhood of* p is a set V containing some open set U containing p. An *open neighborhood of* p is a neighborhood of p that is an open set.

$$V \text{ is a neighborhood of } p \iff \exists U \subseteq V [U \text{ is an open set } \wedge p \in U]$$

$$V \text{ is an open neighborhood of } p \iff V \text{ is an open set } \wedge p \in V$$

Some authors define neighborhoods as open neighborhoods, however this book does not make that assumption. It might be interesting to note that neighborhoods of p can play a similar role to open balls of p in the sense that they both contain open sets containing p. Though there are some notable differences like open balls relying on a metric and neighborhoods being much more general, they can be used with similar functions in some circumstances.

Proposition 2.1. If p has a neighborhood V, then p has an open neighborhood $U \subseteq V$.

$$V \text{ is a neighborhood of } p \implies \exists U \subseteq V[U \text{ is an open neighborhood of } p]$$

2.2 Limit points

We're familiar with the idea that open intervals don't include their endpoints, however can we define this phenomena given any topology? This is the idea of a *limit point.*

Imagine a set S in $\mathbb{R}^2$ and examine the neighborhoods of various types of points. Points well outside the open set have neighborhoods disjoint to S, however neighborhoods of points $p \in S$ always have a nontrivial intersection with S, because they both contain p as a bare minimum. What's interesting is that there are points outside of S whose neighborhoods also always have a nontrivial intersection. It's like these points are 'nearly' in S; it turns out that having nontrivial intersection of all neighborhoods is the right way to define limit points.

Definition 2.2 (Limit point of a set)**.** A *limit point of a set* S is a point p such that all neighborhoods of p include another point in S that isn't p.

$$p \text{ is a limit point of } S \iff \forall V[V \text{ is a neighborhood of } p \implies (V \cap S) \setminus \{p\} \neq \emptyset]$$

2.2.1 Constructions from limit points

Limit points allow us to make various properties surrounding them.

Definition 2.3 (Boundary of a set)**.** Let $(X, \mathcal{T})$ be a topological space. The *boundary of* S is the set of all points $p \in X$ such that all their neighborhoods have intersections with S and $X \setminus S$. We denote the boundary of S as ∂S, and elements of ∂S are called *boundary points* of S.

Definition 2.4 (Closure of a set)**.** The *closure of a set* S is the union of S and the set of all limit points of S. Given that the topological space is $(X, \mathcal{T})$, the closure of S is denoted as $\text{cl}_X(S)$, and its elements are called points of closure of S.

$$\text{cl}_X(S) := S \cup \{p : p \text{ is a limit point of } S\}$$

Definition 2.5 (Interior of a set)**.** The *interior of a set* S is the union of all subsets $U \subseteq S$ that are open. Given that the topological space is $(X, \mathcal{T})$, the interior of S is denoted as $\text{int}_X(S)$.

Definition 2.6 (Closed set)**.**

$$(X, \mathcal{T})$$

A *closed set in* X is a set F that completely contains its boundary ∂F.

In order to avoid this loss of sanity, it's important to note that closed sets and open sets are not opposites in the sense meant by the English language. The definition of open and closed sets make this clear, however sometimes linguistic intuition can muddle the facts.

A testament to this is the fact that 'clopen' sets (sets that are closed and open simultaneously) exist.

Definition 2.7 (Clopen set)**.**

$$U \text{ is clopen in } (X, \mathcal{T}) \iff U \text{ is closed and open in } (\mathcal{T}, X)$$

2.3 Properties of closure

Closures are very useful constructs in topology with many important properties.

Proposition 2.2. Let $(X, \mathcal{T})$ be a topological space. For any set S, $\text{cl}_X(S)$ is closed.

$$(X, \mathcal{T})$$

$$\forall S \in X[\text{cl}_X(S) \text{ is closed in } (X, \mathcal{T})]$$

Proposition 2.3. Let $(X, \mathcal{T})$ be a topological space. $\text{cl}_X(S)$ is the smallest possible closed set containing S.

$$\forall T[T \text{ is closed in } (X, \mathcal{T}) \land S \subseteq T \implies \text{cl}_X(S) \subseteq T]$$

Here's one intuitive way to think about that proposition; think of a swarm of all the open sets disjoint to S, all making a union around S. This swarm of open sets is trying to engulf everything around S, so any points that 'survive' are in the smallest possible closed superset of S.

Proposition 2.4. Let $(X, \mathcal{T})$ be a topological space. For any set S, $\text{cl}_X(S)$ is closed.

$$(X, \mathcal{T})$$
$$\forall S \in X[\text{cl}_X(S) \text{ is closed in } (X, \mathcal{T})]$$

Limit points follow a 'transitive property', that is, if the limit points of S have limit points themselves, they are also limit points of S; we've had them the entire time. This leads to the following proposition.

Proposition 2.5. Let $(X, \mathcal{T})$ be a topological space. For any S, $\text{cl}_X(S) \setminus S$ is closed.

Proposition 2.6.

$$\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$$
$$\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$$

- closure of S is intersection of all closed sets containing S

2.4 Properties of interior

Proposition 2.7. Let $(X, \mathcal{T})$ be a topological space. For any set S, $\text{int}_X(S)$ is open.

$$(X, \mathcal{T})$$
$$\forall S \in X[\text{int}_X(S) \text{ is open in } (X, \mathcal{T})]$$

Proposition 2.8.

$$\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$$
$$\text{int}(A \cup B) \supseteq \text{int}(A) \cup \text{int}(B)$$

2.5 Equivalent definitions

Using the language of closures and interiors, we can prove the following equivalent definitions. This will give us a bigger picture about the behaviour of topologies.

Theorem 2.1 (Equivalent definitions of a boundary)**.** Let $(X, \mathcal{T})$ be a topological space. ∂S is the boundary of S iff any of the following hold.

- For each point in ∂S, all the points neighborhoods have intersections with S and $X \setminus S$.
- $\partial S = \text{cl}_X(S) \setminus \text{int}_X(S)$
- $\partial S = \text{cl}_X(S) \cap \text{cl}_X(X \setminus S)$

Theorem 2.2 (Equivalent definitions of a closure)**.** Let $(X, \mathcal{T})$ be a topological space. cl(S) is the closure of S iff any of the following hold.

- cl(S) is the union of S and its limit points
- $\text{cl}(S) = S \cup \partial S$
- cl(S) is the intersection of all closed sets containing S

Theorem 2.3 (Equivalent definitions of a closed set)**.** Let $(X, \mathcal{T})$ be a topological space. A set F is closed in X iff any of the following hold.

- $X \setminus F$ is open.
- $\text{cl}_X(F) = F$ (F contains all its limit points)
- $\partial F \subseteq F$

$$F \text{ is closed in } X \iff [X \setminus F \text{ is open in } X] \vee [\text{cl}_X(F) = F] \vee [\partial F \subseteq F]$$

Theorem 2.4 (Equivalent definitions of an open set)**.** Let $(X, \mathcal{T})$ be a topological space. A set U is open in X iff any of the following hold.

- $X \setminus U$ is closed.
- $\text{int}_X(U) = U$
- $\partial U \cap U = \emptyset$

$$F \text{ is closed in } X \iff [X \setminus F \text{ is open in } X] \vee [\text{cl}_X(F) = F] \vee [\partial F \subseteq F]$$

Corollary 2.1. Closure is idempotent.

$$\text{cl}_X(\text{cl}_X(S)) = \text{cl}_X(S)$$

Corollary 2.2. Let $(X, \mathcal{T})$ be a topological space. For any S, the set of its limit points is closed.

$$\forall S[\{p : p \text{ is a limit point of } S\} \text{ is closed in } (X, \mathcal{T})]$$

Definition 2.8. Let $(X, \mathcal{T})$ be a topological space, a set S is *dense in* X iff its closure equals X.

Here's an example demonstrating this idea in the Euclidean topological space.

Proposition 2.9. Let $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ be the Euclidean topological space, then $\mathbb{Q}$ is dense in $\mathbb{R}$.

The topologies on X form a poset with $\subseteq$ as the order. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ and $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$.

Chapter 3

Continuous functions

3.1 Continuous function (Topological space)

Students of real analysis typically get their first taste of rigorously defining continuous function by means of limits, and consequently, an epsilon-delta condition. This does the trick for real functions,

Closeness in the image implies closeness in the domain. We start by defining continuity at a point.

Definition 3.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ A function $f : X \to Y$ is *continuous at* $p \in X$ iff for any neighborhood $V \subseteq Y$ of $f(p)$, there exists a neighborhood $U \subseteq X$ of p such that $f(U) \subseteq V$.

Definition 3.2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ A *continuous function* is a function that is continuous at all points of its domain.

We can form the following definition for entire domains.

Definition 3.3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ A function $f : X \to Y$ is *continuous* iff for any V open in Y, $f^{-1}(V)$ is open in X.

As we start to study metric spaces more intensely, readers familiar with real analysis will come to understand how this definition transforms into the familiar epsilon-delta definition of a continuous function. For now, just trust me bro.

Notably, we can recylce our fact that neighborhoods are always within open balls to reverse engineer the definition of a continuous function from real analysis to obrain the topological definition.

In real analysis, one proves that continuity of real functions is preserved under the 4 arithmetic operations and composition; the only operator that can be well defined for continuous functions on generic topological spaces is the composition operation and we will prove that continuity is closed under it.

Proposition 3.1. Let f and g be continuous functions, then $f \circ g$ is continuous.

- Equivalent definitions of continuous function - clousre based, - point based - point based

3.2 Homeomorphisms

Like how group homomorphishms preserve a groups structure, continuous functions preserve neighborhood structure of a topology. This leads to the question; like how group isomorphisms show that two groups are 'algebraically equivalent', is there some class of function to show that two topological spaces are 'topologically equivalent' (in the sense that they have the 'isomorphic neighborhoods')?

Definition 3.4 (Homeomorphism)**.** A *homeomorphism* between two topological spaces T and U is a bijective function $f : T \to U$ such that both f and f^{-1} are continuous.

Since topological spaces are defined by their open sets, homeomrophic topological spaces have indistinguishable topological properties.

$$f(\partial_X U) = \partial_Y f(U)$$
$$\mathrm{cl}_Y(f(U)) = f(\mathrm{cl}_X(U))$$

This leads to the common mathematical joke that topologists can't tell the difference between a coffee mug and a donut; since a torus (donut) can be continuously deformed to a coffee mug, they are homeomorphic and hence have the same properties insofar as topology is concerned.

Here are some examples of types of properties conserved under homeomorphisms.

- Connectedness properties

- Compactness properties
- Separation properties
- Countablility properties
- Metrizability properties

3.3 Embeddings

Sometimes mathematicians realize that the topological space they're working with is just a subspace, or at least homeomorphic to one.

Think of Pac-man's world, where the top of the screen warps to the bottom and the left warps to the right. This is homeomorphic to a torus, so Pac-man is living within a torus-shaped 'planet' in $\mathbb{R}^3$; it is a subspace that is 'embedded' within $\mathbb{R}^3$.

Definition 3.5 (Embedding)**.** a continuous function between topological spaces is an embedding iff it is a homeomorphism from X to a topological subspace of Y

-topological properties

Chapter 4

Constructions of topological spaces

Now that we are familiar with the notion of a topological space and what they represent, it is useful to look at some general methods that can be used to construct new topological spaces from existing ones.

4.1 Topological subspaces

To extend a topology to a subset of a topological space, we introduce topological subspaces.

Definition 4.1. A topological subspace of $(X, \mathcal{T})$ is a topological space $(Y, \mathcal{T}_Y)$.

- $Y \subseteq X$ is a subset of X
- $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$ is the induced topology for the topological subspace

Indeed, topological subspaces always form a topological space, however its topological properties aren't necessarily (and often aren't) the same as the original space. That said, much can be said regarding a topological subspaces' relationship with the original space.

Proposition 4.1. Let Y be a topological subspace of X. A set if closed in Y iff it is of the form $Y \cap F$ where F is closed in X.

Proposition 4.2. Let Y be a topological subspace of X. If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$

Proposition 4.3. Let Y be a topological subspace of X. If Y is open in X, then open sets in Y are open sets in X. Let Y be a topological subspace of X. If Y is closed in X, then closed sets in Y are closed sets in X.

Proposition 4.4. Let Y be a topological subspace of X and $A \subseteq Y/$

$$\mathrm{cl}_X(A) \cap Y = \mathrm{cl}_Y(A)$$

4.2 Product topological spaces

Definition 4.2 (Box topological space)**.** Let $(X_i, \mathcal{T}_i)_{i \in I}$ be a family of topological spaces. The *box topological space* $\prod_{j \in I} X_j$ is the topological space generated by the following basis.

$$\mathcal{B} = \{\prod j \in IU_j : U_j \in \mathcal{T}_j\}$$

Essentially, the cartesian product of open sets from each X_j is a basis element.

finite products mean product topology equals box topology

Proposition 4.5. Let $(\prod_{i=1}^n X_i, \mathcal{T})$ be a box topological space, then all $\mathcal{T}$ is the coarsest topology such that all projection functions are continuous.

This result doesn't generalize for infinite box topological spaces.

Definition 4.3 (Product topological space)**.** For each $i \in \mathbb{N} \cap [1, n]$, let $(X_i, \mathcal{T}_i)$ be topological spaces. The *box topological space* $\prod_{j \in I} X_j$ is the topological space generated by the following basis. Let $(\prod_{i=1}^n X_i, \mathcal{T})$ be a box topological space, then all $\mathcal{T}$ is the coarsest topology such that all projection functions are continuous.

When we consider infinite products, the projection functions may not be continuous. We define product topological spaces to be more robust.

Definition 4.4 (Box topological space)**.** projection functions are all continuous.

4.3 Quotient topological spaces

How can we formally define the idea of constructing a new topological space 'gluing' points of the original space?

The idea is to use equivalence relations as our 'glue'; we want points within the same equivalence class to be seen as the same point (i.e glued together)

Quotient spaces do exactly this; the term 'quotient' appears because by using equivalence clases, we partition (i.e 'divide') our original set.

Definition 4.5 (Quotient space)**.** Given a topological space $(X, \mathcal{T})$ and equivalence relation $\sim$ on X, let $X/\sim$ be the set of equivalence classes and $q : x \to X/\sim$ be a surjective function defined as $q(x) = [x]$, called the *quotient map*.

The *quotient topology* $\mathcal{T}_{X/\sim}$ is the topology $\mathcal{T}_{X/\sim} = \{V \in X/\sim : q^{-1}(V) \in \mathcal{T}\}$. The topological space $(X/\sim, \mathcal{T}_{X/\sim})$ is the *quotient space induced by* $\sim$.

4.4 Final topology

We can generalize the idea of our quotient space

Definition 4.6 (Final topology)**.** Given a set X and topological spaces $(Y_i, \mathcal{U}_i)$ with functions $f_i : Y_i \to X$, the *final topology on* X is the finest topology $\mathcal{T}$ on X such that all f_i are continuous.

Chapter 5

Metric spaces

5.1 Metric space

The notion of distance is extremely powerful and intuitive; it is the basis of many ideas in analysis such as limits and continuous functions. However, it is possible to generalize these interesting properties to topological spaces; spaces where distance may not be defined, but closeness is represented by neighborhoods. To begin our journey in topology, we first study the basics of *metric spaces*; spaces where we can take distance for granted.

5.2 Metric space

Definition 5.1. A *metric space*is an ordered pair (X, d).

- X is a set
- $d : X \times X \to \mathbb{R}^+$ is a *distance function* or *metric* that defines the notion of distance.

The metric must satisfy the following properties.

- $d(x, y) \in [0, \infty)$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

When the metric is apparent, one may denote the metric space (X, d) as just X.

This last condition for the distance function is known as *the triangle inequality*; it is an incredibly useful tool in proofs.

Here are some examples to familiarize us with metrics; the first of which is already known to us.

Example 5.1. The *Euclidean metric* is the following metric defined on $\mathbb{R}^n$.

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (\mathbf{x}_i - \mathbf{y}_i)^2}$$

Notice that for $\mathbb{R}$, this is just $d(x, y) = |x - y|$, and higher dimensions are variations of the Pythagorean theorem.

Example 5.2. The *Chebyshev metric* is the following metric defined on $\mathbb{R}^n$.

$$d(\mathbf{x}, \mathbf{y}) = \max_i(|\mathbf{x}_i - \mathbf{y}_i|)$$

This metric represents the largest difference between two points on an axis.

Example 5.3. The *taxicab metric* is the following metric defined on $\mathbb{R}^n$.

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |\mathbf{x}_i - \mathbf{y}_i|$$

This metric represents the distance of the smallest path between the points if one could only 'walk' along the axises.

Many definitions of topology will make much more sense once we see how they generalize and relate to metric spaces. Admittedly, topology seems kind of strange without the context of how it was derived.

Furthermore, metric spaces generally follow nicer conditions than a generic topological space.

5.3 Topological space induced by a metric space

We now introduce a useful tool in the analysis of metric spaces; the open ball. Since we can measure distance, we can construct sets with everything within a certain distance of some element. One decision we will have to make is whether or not to include elements that are exactly the specified distance away; this decision will indeed change many properties of these special sets. Open balls are defined not to include elements with exactly the specified distance.

5.3.1 Open balls

Definition 5.2 (Open ball)**.** Let (X, d) be a metric space, $p$be an element in X and $r \in (0, \infty)$ a nonnegative real number. An *open ball centered at* p *with radius* r is a set $B(p, r)$ defined as such.

$$B(p, r) = \{x \in X : d(x, p) < r\}$$

It is called a 'ball' because open balls made with the $\mathbb{R}^3$ Euclidean metric happens to look like a ball. As mentioned, the idea is that these sets cover any points strictly closer than r units away. The 'open' part of the name corresponds to the strict inequality $<$ rather than $\leq$, so that the ball doesn't contain the boundary points of the ball.

It is only because open balls exclude these boundary points that we can prove the following fundamental result.

Proposition 5.1. If $x \in B(p, r)$, then there exists some $B(x, s)$ such that $B(x, s)$ is contained completely in $B(p, r)$. Elements in open balls of p have their own open balls completely contained in that open ball of p.

$$x \in B(p, r) \implies \exists s \in (0, \infty)[B(x, s) \subseteq B(p, r)]$$

Theorem 5.1. A metric space (X, d) naturally induces a topological space by the basis $\mathcal{B}_d = \{B(p, r) \subseteq X : p \in X \wedge r \in \mathbb{R}^+\}$.

So all metric spaces induce a topological space, however for which topological spaces does there exist a metric space that generates said topology? This is the class of *metrizable topological spaces*; topological spaces that can be reformulated as metric spaces.

Proposition 5.2. Let (X, d) be a metric space. The open balls of (X, d) are a basis for a topological space $(X, \mathcal{T})$. The resulting topological space is called the *topological space induced by* (X, d).

Sometimes I may refer as an abuse of notation as the metric space and the topological space that it induces as the same thing. Though this is technically incorrect, it's well understood what is meant.

Definition 5.3. A *metrizable topological space* is a topological space $(X, \mathcal{T})$ such that there exists a metric $d : X \times X \to [0, \infty)$ such thatthe topology induced by d is $\mathcal{T}$.

Different metrics on the same set can possibly induce the same topology; they are said to be equivalent metrics if they both form the same topology. -equivalent metrics

5.4 Continuous function (Metric space)

Definition 5.4. Let (X, d_X) and (Y, d_Y) A function $f : X \to Y$ is *continuous at* $p \in X$ iff for any ε, there is a δ such that if $d_X(p, q) < \delta$ then $d_Y(f(p), f(q)) < \varepsilon$.

As always, we have to strip away that metric to generalize our definition to topological spaces. We can start by reformulating this definition in terms of open balls, and then generalize to neighborhoods to get the final result.

Proposition 5.3. Let (X, d_X) and (Y, d_Y) A function $f : X \to Y$ is *continuous at* $p \in X$ iff for any $B_Y(f(p), \varepsilon)$, there is a $B_X(p, \delta)$ such that if $q \in B_X(p, \delta)$ then $f(q) \in B_Y(f(p), \varepsilon)$.

Limits can be used to express continuous functions between metric spaces.

Theorem 5.2 (Continuous function)**.** A function $f : X \to Y$ between 2 metric spaces $(X, d_X), (Y, d_Y)$ is continuous iff for any sequence x_n in (X, d) converging to x, the following holds.

$$\lim_{n \to \infty} f(x_n) = f(x)$$

Corollary 5.1 (Continuous function)**.** A function $f : X \to Y$ between 2 metric spaces $(X, d_X), (Y, d_Y)$ is continuous iff for each $x_0 \in X$ and $\varepsilon > 0$, there is some δ where the following holds for all $x \in X$.

$$d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$$

Corollary 5.2 (Continuous function)**.** A function $f : X \to Y$ between 2 metric spaces $(X, d_X), (Y, d_Y)$ is continuous iff the following holds for any $x_0 \in X$.

$$\lim_{x \to x_0} f(x) = f(x_0)$$

- same sequence converging to x and y means x=y
-cauchy sequence -subsequence -bolzano weierstrass therem topology

5.5 Complete metric space

Though convergence of sequences may be established without a metric space, metric spaces provide a more intuitive definition for convergence.

Definition 5.5 (Convergent sequence)**.** In a metric space (X, d), a *limit* of a sequence is a point $p \in X$ that is arbitrarily close all remaining terms of a sequence. A *convergent sequence* is a sequence with a limit.

$$(X, d)$$

$$\lim_{n \to \infty} x_n = p \iff \forall \varepsilon \in (0, \infty)[\exists N \in \mathbb{N}[n > N \implies d(x_n, p) < \varepsilon]]$$

$$\lim_{n \to \infty} x_n = p \iff \forall \varepsilon \in (0, \infty)[\exists N \in \mathbb{N}[n > N \implies x_n \in B_X(p, \varepsilon)]]$$

Proposition 5.4.

$$\lim_{n \to \infty} x_n = x \land \lim_{n \to \infty} x_n = y \implies x = y$$

Definition 5.6 (Cauchy sequence)**.** In a metric space (X, d), a *Cauchy sequence* is a sequence where the absolute difference of terms are eventually bounded by any arbitrary positive number.

$$(X, d)$$

$$(x_n)_{n \in \mathbb{N}} \text{ is Cauchy} \iff \forall \varepsilon \in (0, \infty)[\exists N \in \mathbb{N}[n, m > N \implies d(x_n, x_m) < \varepsilon]]$$

$$(x_n)_{n \in \mathbb{N}} \text{ is Cauchy} \iff \forall \varepsilon \in (0, \infty)[\exists N \in \mathbb{N}[n, m > N \implies x_n \in B_X(x_m, \varepsilon)]]$$

Definition 5.7 (Complete space)**.** A *complete space* is a metric space (X, d) where all Cauchy sequences are convergent.

Students of real analysis know that real convergent sequences and real Cauchy sequences are the exact same thing, however in general topology they are not! Consider the Euclidean topology on $\mathbb{Q}$; the sequence $x_n = \sum_{k=0}^{n} \frac{(-1)^k 4}{2k+1}$ is Cauchy but not convergent since its limit π is not in the space $\mathbb{Q}$.

Proposition 5.5. A set S is closed in the induced topology of a metric spaces iff any sequence in S converges to some $p \in S$

5.6 Isometries

Topological properties are conserved under homeomorphisms; metric properties are conserved under isometries.

Definition 5.8 (Isometry)**.** Let X, Y be two metric spaces. An *isometry* is a function $f : X \to Y$ such that distance is preserved in the following way.

$$d_X(x, y) = d_Y(f(x), f(y))$$

If there exists an isometry between two metric spaces, then they are said to be *isometric*.

Proposition 5.6. Isometries under composition from a group. Isometries of $\mathbb{R}^n$ are rotations and reflections

5.7 Banach fixed-point theorem

One of my favourite objects in mathematics as a youth were convergent recursive sequences. I had derived the sequence $x_{n+1} = x_n + \frac{a - x_n^2}{a}$ to calculate $\sqrt{a}$ by using the fact that $\lim_{n\to\infty} x_n = \sqrt{a}$.

I relied on heuristic arguments, however the theory of metric spaces allows us to prove the correctness of this claim.

Definition 5.9 (Contraction mapping)**.** Given a metric space (X, d), a *contraction mapping* is a function $f : X \to X$ such that there exists some $r \in (0, 1)$ where the following holds for any $x, y \in X$.

$$d(f(x), f(y)) \leq r d(x, y)$$

It is easily seen that contraction mappings are a very special case of Lipschitz continuous functions that map to the same metric space and with a factor in $(0, 1)$. Since Lipschitz functions are continuous, we have the following lemma.

Lemma 5.1. Contraction mappings are continuous functions.

Theorem 5.3 (Banach fixed-point theorem)**.** Let (X, d) be a complete metric space and $f : X \to X$ a contraction mapping then f has a unique fixed point. Furthermore, $x_{n+1} = f(x_n)$ converges to this fixed point.

The Banach fixed-point theorem was precisely the theory of why my sequence converged, specifically, the contraction mapping was $f(x) = x + \frac{a-x^2}{a} = x + 1 - \frac{x^2}{a}$ and it indeed had a unique fixed point $\sqrt{a}$ and convergent sequence as I had derived from philosophically concerning techniques.

5.8 Baire spaces

Theorem 5.4 (Baire category theorem)**.** For a complete metric space (X, d), let X_n be a sequence of open dense subsets of X, then $\bigcap_{n=1}^{\infty} X_n$ is dense in X.

5.9 Metrizations

Metric spaces are much nicer to work with than topological spaces, so given a topolgical space, mathematicians often want to see if it is metrizable.

We search for a set of minimal properties that a topological space can obey that can be used to construct a metric. Such results are called *metrization theorems*; sufficient conditions for a topological space to be metrizable.

Note that this study makes use of the separation properties.

Theorem 5.5 (Urysohn Metrization theorem)**.**

5.10 Open and closed sets

We may be acutely aware that open balls exclude the 'endpoints' and closed balls keep them, this is essentially the notion of a *boundary*.

Definition 5.10. Let (X, d) be a metric space. The *boundary of* S is the set of all elements $p \in X$ such that all their open and closed balls have intersections with S and $X \setminus S$. We denote the boundary of S as ∂S, and elements of ∂S are called boundary points of S.

Due to those 2 findamental resulds, we can translate what they mean using the language of boundaries.

Proposition 5.7. Open balls are disjoint from their boundary. Closed balls contain their boundary.

There a more types of sets that don't contain their boundary rather than just open balls, so let's consider them.

Definition 5.11. Let (X, d) be a metric space. A *open set of* X is a set U that is disjoint to ∂U. We say that U *is open in* X.

The same can be said about sets completely containing their boundary.

Definition 5.12. Let (X, d) be a metric space. A *closed set of* X is a set F that completely contains ∂F. We say that F *is closed in* X.

Noting that $\partial S = \partial(X \setminus S)$, we can prove the following curious proposition.

Proposition 5.8. F is closed iff $X \setminus F$ is open.

This gives us a nice set theoretic representation for closed sets that doesn't include a metric; we'll make this the prime definition once we've defined what topologies are!

Theorem 5.6. Let (X, d) be a metric space. If U is open in X then any element in U has an open ball completely contained in U.

This brings us 2 intuitive corollaries.

Corollary 5.3. Let (X, d) be a metric space. Open balls are open sets.

Corollary 5.4. Let (X, d) be a metric space. open sets are unions of open balls.

5.11 Origins of topologies

We'll now use our theory of metric spaces to understand how the set algebra laws for a topology were derived.

Proposition 5.9. Let (X, d) be a metric space.

- X and $\emptyset$ are open sets
- Open sets are closed under countable unions
- Open sets are closed under finite intersections

If we can't define a metric on a space, we can at least ensure it has a notion of open sets defined in this way. These set algebraic laws give a neat way to define open sets without any resort to a metric function; this is what we base the definition of a topology on.

Since a family of open sets must obey these 3 properties, if we define open sets by a family of sets that satisfy these 3 properties rather than the open balls, we generalize the idea of a metric space to a topological space.

Chapter 6

Connected spaces

6.1 Connected space

Homeomorphisms can be used to show 2 topological spaces are the same, however it is interesting to note what properties distinguish topological spaces.

Consider the topological subspaces of the Euclidean topological space $A = [0, 1]$ and $B = [0, 1] \cup [2, 3]$. They have a clear distinction; the former is a uniform interval while the latter has a gap in between.

We see that B is comprised of 2 closed intervals 1 unit of distance away from another, but topologies cannot work in terms of distance. However the topology can detect that there are 2 disjoint open sets that split B up.

Formally, the idea of connectedness captures the property of a topological space where it cannot be broken down into disjoint open sets.

We can consider the connectedness of entire spaces, and then extend this notion to general sets by considering the topological subspaces they induce (analogous to the above example).

Definition 6.1. A *connected space* $(X, \mathcal{T})$ is a topological space such that the only union of a pair of disjoint open sets is the space with the emptyset. There are no disjoint open sets U, V (other than $, X$) where $U \cup V = X$.

The following theorem provides an alternative but equivalent definition of connected spaces that is often easier to work with.

Theorem 6.1 (Equivalent definitions of a connected space)**.** Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is connected space iff any of the following hold.

- There are no disjoint open sets U, V (other than $, X$) where $U \cup V = X$.

- There are no disjoint closed sets U, V (other than , X) where $U \cup V = X$.
- $\emptyset, X$ are the only clopen sets in $(X, \mathcal{T})$

Using the least upper bound property, the following many be proven.

Proposition 6.1. Euclidean topological space is connected.

6.1.1 Connected sets

Definition 6.2 (Connected set)**.** Let $(X, \mathcal{T})$ be a topological space, then S is a *connected set of* X iff its subspace topology induced by X is a connected space.

Our knowlegde of topological subspaces comes in clutch here.

Proposition 6.2. Let $(X, \mathcal{T})$ be a topological space and $X = U \cup V$ a disjoint union of open sets. All connected sets are a subset of either U or V.

Given a connected set S, extending S by including limit points preserves its connectedness since the neighborhoods of limit points are never disjoint from S ()

Proposition 6.3. Let $(X, \mathcal{T})$ be a topological space and S a connected set of X, then any set T satisfying $S \subseteq T \subseteq \text{cl}(S)$ is a connected set.

Proposition 6.4. Let $(X, \mathcal{T})$ be a topological space and $(S_i)_{i \in I}$ a family of connected sets such that $\bigcap_{i \in I} S_i \neq \emptyset$, then $\bigcup_{i \in I} S_i$ is a connected set

Proposition 6.5. finite box topology of connected spaces is connected

In real analysis, there are many theorems that only apply to connected sets, namely the intermediate value theorem and the fact that $f'(x) = 0$ for all x on a connected set (i.e an interval) implies f is constant on said set.

Proposition 6.6 (A fixed point theorem)**.** Let $f : [0, 1] \to [0, 1]$ be a continuous function, then f has a fixed point.

6.2 Path connected space

An even more natural idea of connectedness is if any pair of points in the topological space can be, well, connected!

Definition 6.3 (Curve)**.** A *curve* is a continuous function $f : I \to X$, where I is a non-degenerate interval of $\mathbb{R}$.

Definition 6.4 (Path)**.** A *path* is a curve $f : [0,1] \to X$ with its domain as $[0,1]$

We could have defined a path with any compact interval domain, but all compact intervals are homeomorphic, so $[0,1]$ was chosen for simplicity; a change of variables can make any curve with compact onterval domain a path.

Definition 6.5. A *path connected space* $(X, \mathcal{T})$ is a topological space such that for any pair of distinct points $x, y \in X$, there exists a continuous function $f : [0,1] \to X$ where $f(0) = x$ and $f(1) = y$. Essentially, any two points in a space has a continuous path from one another.

Proposition 6.7. Path connected spaces are connected spaces.

It's notable that connected spaces aren't always path connected; there is a famous counterexample called the *Topologist's sine curve* that demonstrates this by being a discontinuous function that can still pass as connected. Consider the following topological space as this graph with the induced Euclidean topology $\mathbb{R}^2$.

$$T = \{ \begin{bmatrix} x \\ \sin(\frac{1}{x}) \end{bmatrix} : x \in (0,1]\} \cup \{\mathbf{0}\}$$

Note that no path can go through $\mathbf{0}$, since this would not be a continuous function and hence not a path (or even a curve). $\sin(\frac{1}{x})$ just oscillates like crazy near the origin; $\sin(\frac{1}{x})$ doesn't get closer to 0 as x approaches 0, therefore we looe continuity here and hence it is not a path connected space. That said, T is still connected!

There is also the notion of a simply connected space; this is useful in the study of complex analysis, however to give a propert topological treatment of this property we require the methods of algebraic topology.

6.3 Locally connected spaces

Chapter 7

Compact spaces

7.1 Compact space

Let's consider sequences with elements drawn from some set U. Real analysis tells us that closed and bounded intervals

As with most of this book, we introduce the metric space version first and generalize to topological spaces. However, as we've seen, limit points of a set and limits of a topological sequence are related, but distinct in general topological spaces (they are the same in metric spaces, however).

This leads to 2 philosophies of what 'compact' means; in the sense of limit points of a set, or in the sense of sequential limits.

Another property that can be attributed to spaces and be used to distinguish them is whether they are compact.

In the Euclidean topology, the sets $(0,1)$ and $[0,1]$ contain none and all of their limit points respectively, but consider the subspaces they induce; they aren't homeomorphic. The same goes for comparing any closed ball and open ball of $\mathbb{R}^n$

The notion of compactness with its formal definition is not as natural as connectedness, so we merely present its definion and build intuition later.

Definition 7.1 (Compact space)**.** Let $(X,\mathcal{T})$ be a topological space, then $(X,\mathcal{T})$ is a *compact space* iff every open cover of X has a finite subcover of X.

$$X = \cup_{i\in I} U_i$$

$$X = \cup_{i=1}^{k} U_{n_i}$$

Since X is bein interpreted as an entire space (no elements outside of X), we use equality. Similarly to how we extended connected spaces to generic connected sets, we can do the same with compactness.

Definition 7.2 (Compact set)**.** Let $(X, \mathcal{T})$ be a topological space, then S is a *compact set of* X iff the topological subspace induced by S is a compact space.

Definition 7.3. Let $(X, \mathcal{T})$ be a compact space, a closed set F is a compact set.

Theorem 7.1. Let $(X, \mathcal{T})$ be a topological space, then S is a compact set of X iff every open cover of S has a finite subcover of S.

$$X \subseteq \cup_{i \in I} U_i$$

$$X \subseteq \cup_{i=1}^{k} U_{n_i}$$

compactness preserved under homoemoprhism compactness preserved under continuous functions closed intervals are compact in Euclidean space compactness need only open cover by basic sets

Theorem 7.2 (Tychonoff's theorem)**.** $\prod_{i \in I} S_i$ is compact in product topological space iff each S_i is compact.

Theorem 7.3 (Bounded set)**.** Let (X, d) be a metric space, a *bounded set of* X is some set S such that for any points $s, t \in S$ we have $d(s, t) < M$.

Theorem 7.4 (Heine-Borel theorem)**.** S is compact in $\mathbb{R}^n$ with Euclidean topology iff S is closed and bounded.

7.2 Sequentially Compact space

Vague introduction to Sequential compactness, shortcomings This definition is much more intuitive; no sequence

Definition 7.4 (Sequentially compact space)**.**

Definition 7.5 (Sequentially compact set)**.**

Proposition 7.1. Let X be a metric space, then X is sequentially compact iff it is compact.

Recall the Heine-Borel theorem, since we know that in metric spaces sequential compactness and compactness are equivalent, the Heine-Borel theorem implies the Bolzano-Weierstrass theorem!

Theorem 7.5 (Bolzano-Weierstrass theorem)**.** *S is sequentially compact in $\mathbb{R}^n$ with Euclidean topology iff S is closed and bounded.*

7.3 Other?

Lemma 7.1 (Lebesgue's number lemma)**.**

7.4 Compactifications

- One-point compactification

Chapter 8

Separation properties

The properties we're going to explore arise from a deeper study of topology rather than from inspiration of mathematical analysis.

It's interesting to classify topological spaces by how their points can be separated by open sets. Given that we know how neighborhoods of points should relate to one another, we can prove many interesting results.

8.1 Kolmogorov space

Definition 8.1 (Kolmogorov space)**.** A T_0 *space (Kolmogorov space)* is a topological space such that for every distinct pair of points, at least 1 point in the pair has a neighborhood not containing the other point.

8.2 Fréchet spaces

Definition 8.2 (Fréchet space)**.** A T_1 *space (Fréchet space)* is a topological space such that for every distinct pair of points, both points in the pair has a neighborhood not containing the other point.

Proposition 8.1. Let $(X, \mathcal{T})$ be a Fréchet space, then any singleton of X is closed.

From this one can prove a modest amount of corollaries, such as that finite frechet spaces are discrete topological spaces.

However, more interesting properties follow a stronger separation property; Hausdorff spaces.

8.3 Hausdorff spaces

Metric spaces have many wonderful properties over any old topological space; we've used our theory our theory of metric spaces to inspire topological spaces, however do metric spaces have any inrinsic separation properties?

Notice that for any pair of points in a metric space, we can find open balls for them that are small enough to be disjoint; every pair of distinct points in a metric space have disjoint neighborhoods then.

We'll now consider the class topological spaces that also have this property, they are known as *Hausdorff spaces*.

Definition 8.3. A T_2 *space (Hausdorff space)* is a topological space such that for every distinct pair of points, there exists a pair of neighborhoods of both points which are disjoint.

Felix Hausdorff originally included this separation property within his own definition of a topological space! It's a nice property indeed, however for the sake of making a more minimalistic and general theory of topology, topological spaces are no longer defined to be Hausdorff spaces.

As discussed, the topologies induced by metric spaces are all Hausdorff spaces, however the converse isn't necessarily true.

Proposition 8.2. Topological spaces induced by metric space are Hausdorff spaces.

This hints us to the fact that some of the nice properties that metric spaces offer can be invoked by this separation property rather than the explicit need for distance. If this is true, we'd be able to strengthen many of our theorems on metric spaces to just Hausdorff spaces, let's see what we can do with this goal in mind!

Proposition 8.3. Let $(X, \mathcal{T})$ be a Hausdorff space, a compact set C is a closed set.

8.4 Regular spaces

regular space normal space - Urysohn's lemma

Chapter 9

Countability properties

9.1 First countable spaces

- first countable space - metric spaces are first countable spaces

9.2 Second countable spaces

-second countable space - Euclidean spaces are second countable spaces

Chapter 10

Topological sequences

10.1 Topological sequences

Definition 10.1 (Limit point of a sequence)**.** In a topological space $(X, \mathcal{T})$, a *limit point of a sequence* is a point p where all its neighborhoods contain all remaining terms of a sequence. A *convergent sequence* is a sequence with a limit.

$$(X, \mathcal{T})$$

$$p \text{ is a limit point of } (x_n)_{n\in\mathbb{N}} \iff \forall V \subseteq X[V \text{ is a neighborhood of } p \implies \exists N \in \mathbb{N}[n > N \implies a_n \in V$$

Some readers may know that convergent sequences of real numbers converge to a single real number, however for general topological spaces, limit points are not necessarily unique.

If one restricts the terms of a sequence to some set, it may still be possible that the limit of the sequence lies *outside* this set. Consider the Euclidean topology on $\mathbb{R}$, the open set $(0, 1)$ and the sequence $a_n = \frac{1}{n+1}, n \geq 1$. Though we have $a_n \in (0, 1)$, we also have $\lim_{n\to\infty} a_n = 0$, which is out of the set!

The phenomenon where limits can exceed the set their terms are chosen from is interesting indeed; any point that is the limit of some sequence of terms within a set is called a *limit point* of that set.

10.2 Nets

10.3 Filters