Harmonic Analysis

Zac Zerafa

February 2, 2026

Contents

I Fourier analysis **1**

1 Fourier series **3**
1.1 Linear algebra and functional analysis 3
1.1.1 Inner product spaces 4
1.1.2 L^p spaces 4
1.2 Fourier series 4

2 Fourier transform **7**
2.1 Fourier transform 8
2.2 Fourier inversion theorem 9
2.3 Convolution theorem (Fourier transform) 9
2.4 Riemann-Lebesgue lemma 10
2.5 Plancherel's theorem 11
2.6 Parseval's theorem 11

3 More Fourier series **13**
3.1 Fourier convergence theorem 13
3.2 Kernels 14
3.2.1 Dirichlet kernel 14
3.2.2 Fejér kernel 15

4 Hankel transform **17**
4.1 Hankel transform 17

II Abstract harmonic analysis **19**

Part I

Fourier analysis

Chapter 1

Fourier series

Fourier analysis is the fundamental field of harmonic analysis, studying functions between Euclidean spaces and focusing on their trigonometric sum (or integral) decomposition.

Abstract harmonic analysis generalizes the ideas and tools of Fourier analysis by considering more complex spaces that may require a different type of sum (or integral) decomposition to best suit the space's symmetry.

Trigonometric series were used notably by Euler, Lagrange, and Gauss in various applications, however Fourier and Bessel were the first mathematicians to think about them as a way of representing

1.1 Linear algebra and functional analysis

Taylor series are based on a premise that a Taylor series should have the same derivatives as the function it models. Fourier series work by means of a different train of thought, we're projecting a function onto a space spanned by infinite 'sinusoids'.

Considering this interpretation, to truly understand Fourier analysis in all its mathematical rigour, we require some assitance from linear algebra and functional analysis.

REFRESHER/BRIEF INTRODUCTION TO LEBESGUE INTEGRATION

1.1.1 Inner product spaces

1.1.2 L^p spaces

Originally we learn integration as the Riemann integral; integration by partitioning the domain by basic set theory. Here, we will instead use the Lebesgue integral; integration by partitioning the range by measure theory.

Lebesgue integration offers many theoretical advantages. For Fourier analysis, the use of Lebesgue integration is particularly important as sets of Lebesgue integrable functions form a linear space, something Riemann integrable functions cannot do.

1.2 Fourier series

Definition 1.1 (Trigonometric series)**.**

$$b_0 + \sum_{n=1}^{N} [a_n \sin(nx) + b_n \cos(nx)]$$

$$\sum_{n=-N}^{N} c_n e^{inx}$$

We will study Fourier analysis by means of the latter form.

We are familiar with the Taylor series decomposition for analytic functions by matching derivatives at some reference point, can we also model some rich class of functions by means of a trigonometric series? This was the exact motivation of Fourier and Bessel.

If we restrict our attention to $L^1([0, 2\pi])$ functions (complex functions integrable on $[0, 2\pi]$), we can devise a general method by using the *orthogonality of trigonometric functions on* $[0, 2\pi]$, in other words, we can abuse the following identities.

$$\int_{[0,2\pi]} e^{inx} e^{imx} dx = 0$$

Let's say we have some function $f : [0, 2\pi] \to \mathbb{C}$ and we're hoping to model a trigonometric series converging to f on $[0, 2\pi]$.

$$\int_{[0,2\pi]} f(x) e^{inx} dx$$

$$= \int_{[0,2\pi]} [\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}] e^{imx} dx$$

$$= \sum_{n=-\infty}^{\infty} \widehat{f}(n) \int_{[0,2\pi]} e^{inx} e^{imx} dx$$

$$= \widehat{f}(n) \int_{[0,2\pi]} e^{imx} e^{imx} dx + \widehat{f}(-n) \int_{[0,2\pi]} e^{-imx} e^{imx} dx$$

$$= 2\pi \widehat{f}(-n)$$

If we rearrange for $\widehat{f}(-n)$ and then flip $-n \to n$, we have a method to calculate the coefficients we want so that the trigonometic series equals f (well, at least for sufficiently nice functions)!

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{[0,2\pi]} f(x) e^{-inx} dx$$

There's one slightly subtle but mathematically crucial caveat to this method; the 3rd equality swaps the infinite sum and the integral. This cannot be done in general, so we must keep in mind that this trigonometric series converges to f iff this sum-integral swap is permitted (we will disuss this in further detail later).

$$\widehat{f}(n) = \frac{\langle f, e^{-inx} \rangle}{\langle e^{-inx}, e^{-inx} \rangle}$$

Definition 1.2 (Fourier series)**.** Given a (Lebesgue) integrable complex function f, the *Fourier series of* f is the following related trigonometric series.

$$f \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}$$

- $\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(n) e^{-inx} dx$ are the *Fourier coefficients*

We remind the reader that a Fourier series of a function does not necessarily equal the function, even if this is the main motivation for its definition (just like Taylor series). Indeed, it is quite complicated to determine when and in what sense (pointwise or uniformly, everywhere or almost everywhere etc.) a Fourier series converges to its function. We will come back and answer this question later.

The Fourier coefficient function is worthy of much study, and it obeys many elementary, yet interesting properties.

Proposition 1.1.

- $f \in L^2 \implies \|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$
- $\widehat{af + bg}_n = a\widehat{f}(n) + b\widehat{g}(n)$
- $\widehat{f(x-a)}_n = e^{-ina}\widehat{f}(n)$
- $\widehat{f}'(n) = in\widehat{f}(n)$

Chapter 2

Fourier transform

The Fourier transform is the continous analogy of the idea of the Fourier series coefficients.

If f equals some Fourier series, we want to find some sequence $\widehat{f}(n)$ such that

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx}$$

if f equals some 'Fourier integral', we want to find some function $\widehat{f}(\xi)$ such that

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{ix\xi} d\xi$$

The basic idea is that instead of considering a countably infinite, discrete set of frequency functions (in our case, e^{inx} with $n \in \mathbb{Z}$), we now consider an uncountably infinite, continuous set of frequency functions ($e^{ix\xi}$ with $\xi \in \mathbb{R}$). Due to this continuous nature we end up studying an integral representation instead of a sum representation like in the Fourier series.

In our study of Fourier series, we started with the question of finding a trigonometric series representation if a function and finished with our solution; a method to compute the appropriate Fourier series coefficients.

For 'Fourier integrals', we will start the other way around; we will define the Fourier transform and then show that it is the appropriate weight for the 'Fourier integral' as a result called the Fourier inversion theorem.

2.1 Fourier transform

Is there a way of determining how much of a wave is in the function? We use a specific integral transform called the Fourier transform to recover this information.

An integral transform is a transform (self-map) of a function space of the following form

$$\mathcal{T}\{f\}(t) = \int_{\Omega} I(x,t)f(x)dx$$

- $\mathcal{T} : \mathbb{L}^1 \to \mathbb{R}$ is the integral transform
- $f : X \to Y$ is the input function
- $I : X \times T \to Y$ is the *integral transform kernel*

Now let's introduce the integral transform we need to decompose a function.

Definition 2.1 (Fourier transform)**.** The *Fourier transform* is the integral transform $\mathcal{F}$ on functions in $L^1(\mathbb{R})$ that describes how periodic functions of frequency $2\pi\xi$ are present in a function.

$$\mathcal{F}\{f\}(\xi) = \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi x}dx$$

- $f \in L^1$ is a Lebesgue intergable function
- $\mathcal{F}\{f\}$ is the Fourier transform

Proposition 2.1.

$$f, g \in L^1(\mathbb{R})$$

- $\mathcal{F}\{f\} \in L^\infty$
- $\mathcal{F}\{af + bg\} = a\mathcal{F}\{f\} + b\mathcal{F}\{g\}$
- $\|\widehat{f}\|_\infty \leq \|f\|_1$
- $\mathcal{F}\{f'\} = 2\pi i\xi\mathcal{F}\{f\}$
- $\mathcal{F}\{f\}$ is uniformly continuous

- $\mathcal{F}\{f(x-a)\} = e^{2\pi i a\xi}\mathcal{F}\{f\}$
- $\int_{-\infty}^{\infty} \widehat{f}(x)g(x)dx = \int_{-\infty}^{\infty} f(x)\widehat{g}(x)dx$
- $\mathcal{F}\{f\}(\xi - a) = \mathcal{F}\{e^{2\pi i a} f\}(\xi)$
- $\mathcal{F}\{f(ax)\}(\xi) = \frac{1}{a}\mathcal{F}\{f\}(\frac{\xi}{a})$

These elementary properties may seem similar to those for Fourier coefficients. Indeed (no book on mathematics is complete without copious proliferation of the word 'indeed'), we will discuss many properties of the Fourier transform that are also applicable for Fourier series coefficients!

2.2 Fourier inversion theorem

Functional analysis have since become a major tool in the study of Fourier analysis, since (Lebesgue) integrable functions form their own space (called a Banach space) on which the Fourier transform is a functor.

Definition 2.2 (Inverse Fourier transform)**.**

$$\mathcal{F}^{-1}\{f\}(x) = \int_{-\infty}^{\infty} f(\xi)e^{i2\pi\xi x}d\xi$$

2.3 Convolution theorem (Fourier transform)

There exists a that manages to show up in many places within mathematics and its applications, the *convolution operator*. Originally it found applications in the following ways.

- PDF for sum of random variables
- Solutions to the heat equation
- Superposition of signals in signal processing

When studied within harmonic analysis, mathematicians realized that this operator has some notable, mathematically interesting properties. We first define such an operator.

Definition 2.3 (Convolution operator)**.** Operator on $L^1(\mathbb{R})$ functions such that the integral of f is evaluated with g as a weight function that is translated by t

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

Proposition 2.2.

$$(f * g) = (g * f)$$

$$\|f * g\|_1 \leq \|f\|_1\|g\|_1$$

Now we will lead to the big result relating to the convolution operator in Fourier analysis. We know that $\mathcal{F}\{f\} + \mathcal{F}\{g\} = \mathcal{F}\{f+g\}$ due to the linearity of the Fourier transform, but is there a similar result for compactifying $\mathcal{F}\{f\}\mathcal{F}\{g\}$ into 1 Fourier transform?

The answer is 'yes', and our friend convolution comes in clutch.

Theorem 2.1 (Convolution theorem (Fourier transform))**.**

$$f, g \in L^1(\mathbb{R}) \implies \mathcal{F}\{f * g\}(\xi) = \mathcal{F}\{f\}(\xi)\mathcal{F}\{g\}(\xi)$$

This is good news for signal processing engineers, since they now have an easier way to calculate the Fourier transform for their superimposed signals.It also facilitates problems in PDEs.

2.4 Riemann-Lebesgue lemma

Intuition tells us that when a function makes heavy use of more.

If we consider our relatively nice class of $L^1(\mathbb{R}^n)$ functions, can we say something about how the Fourier transform must behav

Lemma 2.1 (Riemann-Lebesgue lemma)**.** Let f be a $L^1(\mathbb{R}^n)$ function, then the following holds (i.e its Fourier transform's tails tend to 0).

$$\lim_{\xi \to \pm\infty} \mathcal{F}\{f\}(\xi) = 0$$

$$f \in L^1(\mathbb{R}^n) \implies \lim_{\xi \to \pm\infty} \mathcal{F}\{f\}(\xi) = 0$$

2.5 Plancherel's theorem

Theorem 2.2 (Plancherel theorem)**.**

$$f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \implies \|\mathcal{F}\{f\}\|_2 = \|f\|_2$$

2.6 Parseval's theorem

Definition 2.4 (Parseval's theorem)**.** The Fourier transform is a unitary operator.

Chapter 3

More Fourier series

Although we're familiar with the motive, definition, and derivation of Fourier series, we haven't talked about the interesting results relating to it like we've done with the Fourier transform.

We'll start by porting all the Fourier transform results that apply to Fourier series coefficients. Recall that we calculate Fourier series for functions on compact intervals, so the general idea to port such results over is to consider the function on $\mathbb{R}$, however mapping all domain elements outside this compact interval to 0 (this is called giving the function 'support'). This essentially makes the Fourier transform the Fourier series coefficient function!

Theorem 3.1 (Plancherel theorem (Fourier coefficients))**.**

Theorem 3.2 (Parseval's theorem (Fourier coefficients))**.**

$$f \in L^2 \implies \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$$

One can actually solve the Basel problem by applying Parseval's theorem to $|x|$ on $[-\pi, \pi]$.

Lemma 3.1 (Riemann-Lebesgue lemma (Fourier coefficients))**.**

$$f \text{ is continuous on } I \land f \text{ is } 2\pi\text{periodic on } \mathbb{R} \implies \lim_{n\to\infty} |\widehat{f}(n)| = 0$$

3.1 Fourier convergence theorem

When does a Fourier series converge to its function?

Taylor's theorem is the main result used to ensure that a Taylor series converges to its function, it would be nice to have such an analogue for Fourier series.

3.2 Kernels

Taylor's theorem considers partial Taylor series as apart of its result; it is a good start to study partial Fourier series too. We know that since it is a finite sum that it indeed converges, so things are much nicer.

3.2.1 Dirichlet kernel

We can actually represent partial Fourier series by means of a special integral transform on the related function f; partial Fourier series have an integral representation!

For an informal derivation, let's operate under the ansatz that such a integral transform exists, then we have the following.

$$\sum_{n=-N}^{N} \widehat{f}(n)e^{inx} = \int_{0}^{2\pi} f(x)g(x,t)dt$$

Since we know how Fourier series coefficients are calculated, we replace the Fourier coefficient symbols with what they calculate to.

$$\sum_{n=-N}^{N} \widehat{f}(n)e^{inx} = \int_{0}^{2\pi} f(x)g(x,t)dt$$

Applying the linearity of the integral (which is possible in general since we are considering finite sums), it becomes clear what kind of kernel our integral transform must take!

Our kernel is $\frac{\sum_{n=-N}^{N} e^{-inx}}{2\pi}$. FOr the sake of a clean notation, we can consider partial Fourier series as the convolution of f and $\frac{\sum_{n=-N}^{N} e^{inx}}{2\pi}$.

Let's make some definitions.

Definition 3.1 (Dirichlet kernel)**.**

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \frac{\sin(\frac{n+1}{2}x)}{\sin(\frac{x}{2})}$$

Proposition 3.1.

$$(D_n * f)(x) = 2\pi \sum_{k=-n}^{n} \widehat{f}(n)e^{inx}$$

$\int_{-\pi}^{\pi} D_n(x)dx = 2\pi$ $(D_n*f)(x) = 2\pi \sum_{k=-n}^{n} \widehat{f}(n)e^{inx}$ $\exists f'(x) \implies \lim_{n\to\infty}(D_n*f)(x) = f(x)$ $\exists f'(x) \implies \lim_{n\to\infty}(D_n*f)(x) = \frac{1}{2}[f(x^+)+f(x^-)]$ $\exists f'(x)$p.w $\implies \lim_{n\to\infty}(D_n * f)(x) = f(x)$

3.2.2 Fejér kernel

The Cesàro summation of the first n partial Fourier series can also be represented as an integral transform. If one were to reverse engineer under this assumption like we did to derive the Dirichlet kernel, we would end up discovering the *Fejér kernel*.

Definition 3.2 (Fejér kernel)**.**

$$F_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(x) = \frac{\sin^2(\frac{n}{2}x)}{n \sin^2(\frac{x}{2})}$$

Proposition 3.2.

$$(F_n * f)(x) = \frac{2\pi}{n} \sum_{k=0}^{n-1} \sum_{k=-n}^{n} \widehat{f}(n)e^{inx}$$

$F_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(x)$ $\forall x, F_n(x) \geq 0$ $\int_{-\pi}^{\pi} F_n(x)dx = 2\pi$ $\forall \delta \in (0,\pi)$ $\lim_{n\to\infty} \int_{\delta<|x|<\pi} F_n(x)dx = 0$ $\lim_{n\to\infty} T_n(x) \to S_n(x)$uniformly

Chapter 4

Hankel transform

$$\mathcal{F}\{f\}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} d\mathbf{x}$$

4.1 Hankel transform

For radial functions,there exists a different set of orthogonal functions that better suits radial symmetry.

Definition 4.1 (Hankel transform)**.** Integral transform that extends the notion of Fourier transform for radial functions

$$\mathrm{H}_n\{f\}(\rho) = \int_0^\infty f(r) J_n(\rho r) r dr$$

- $f \in L^1$ is a Lebesgue intergable radial function
- $\mathrm{H}_n\{f\}$ is the radial Hankel transform of order n
- J_n is the Bessel function of the first kind of order n

Part II

Abstract harmonic analysis