Lie Theory

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Part I

Fundamentals

Mathematical logic Set theory Group theory Order theory Real analysis Ordinary differetial equations Linear algebra General topology Vector analysis Fourier analysis Partial differential equations Differential topology

Representation theory?

Chapter 1

Lie groups

Lie theory is often explained as being an analogue of Galois theory for differential equations.

Galois theory was motivated by the idea of using transformation groups to study polynomial zeros. More concretely, given a polynomial P, we want some transfomation group G (with an action) such that with some r satisfying $P(r) = 0$ and $g \in G$, one has $P(\alpha(g, r)) = 0$; the transformation group tells us how to map zeros to other zeros. This perspective gives interesting methods to recover zeros of polynomials, but it also permits proofs of the Abel-Ruffini theorem which states that there is no general formula in radicals for polynomials of degree 5 and above.

Lie theory attempts to create a similar model for solutions of differential equations rather than solutions of polynomial equations. We seek a transfomation group G (with an action) such that with some y satisfying $f(x, y, y', \ldots) = 0$ and some $g \in G$, we have $f(\alpha(g, (x, u, u', \ldots))) = 0, u = \alpha(g, y)$.

Though the spirit of both theories are extremely close, the differing nature of the equations and the nature of the symmetries permit theories of very different flavors. One of the main differences is that Lie theory describe the ways in which a solution may smoothly flow toward other solutions, while Galois theory describes

the technique was to describe transformation groups that map any zero offor polynomials (i.e groups that when acting on a polynomial's domain map zeros of the polynomial to other zeros). Lie theory similarly tries to study differential equations by means of symmetry groups, however due to the differing nature of differential equations, one requires the use of "continuous" (technically "smooth") symmetries to develop a successful theory.

Unfortunately this definition of a Lie group offers practically no insight whatsoever as to why we want to consider them

1.1 Continuous transformation groups

Since solutions of differential equations are functions on a differentiable manifold, the theory is developed in the context of differential topology, where the field is applied and enriched by the fields of differential geometry and differential equations.

Given a "structure" X, an *automorphism group* is a group G of the automorphisms of X; these automorphisms naturally give a way for G to act on X. Most of the time there are so many different automorphisms with varying structure, so we often consider subgroups of this group; subgroups of automorphism groups are *transformation groups*.

Advanced group theory is used to model symmetries on finite spaces, Transformation groups appear and are studied in various fields of mathematics, depending on what types of spaces the transformation group is modelling (polynomial zeros, linear transforms on shapes that leave them invariant, solutions of a differential equation) and "how" the transformation group functions (continuous, discrete, or smooth symmetries)

Definition 1.1 (Topological group)**.**

To model continuous symmetries we require not only a group, but one that can "act smoothly". This desire is satisfied by employing *Lie groups* as continuous transform groups; they are groups that are also smooth manifolds with smooth operations.

1.2 Lie groups

The titular object that Lie theory concerns itself with is *Lie groups*; smooth manifolds that are also a group. When used as continuous transformation groups, they are particularly nice to work with since their topology is very nice.

Definition 1.2 (Lie group)**.** Lie group is a group $(G, \cdot)$ such that G is also a smooth manifold and multiplication and the inverse operation are smooth functions.

Say we have an n-order PDE, we consider some submanifold of $X \times U^{(n)}$ representing solutions of said PDE; by studying the vector fields on this submanifold (with the help of a tool called a "prolongation") we can determine a basis to represent any vector field on this submanifold, then finally "exponentiate" it to calculate the orbits of the symmetry group.

Definition 1.3 (Lie subgroups)**.**

Closed-subgroup theorem Let G be a Lie group, then any closed subgroup H is an embedded Lie group.

1.3 Transformation groups

Transformation groups are groups of "transformations" (or in the worst cases, just elements that correspond to a transformation) that act on some space.

Most Lie groups we will see are groups formed by special automorphisms that act on the manifold; these are known as *transformation groups*.

Transformation groups are associated with a Lie group action

semiregular actions are those whose orbits are embedded smooth manifolds that all have the same dimension.

Alternatively, they are such that the stabilizer of each point on manifold is trivial.

1.4 Local Lie group

One often only wants to deal with an open neighborhood of the identity, and therefore end up formally working with a *local Lie group* instead of a (global) Lie group. While our definition for a local Lie groups removes the guarante of the closure of the group operation, it does allow working directly with some local coordinates of a chart covering said this neighborhood.

Definition 1.4 (Local Lie group)**.** Essentially a Lie Pregroup

Local lie groups are always a neighborhood of some global Lie group

Proposition 1.1. Let $(G, \cdot)$ be a connected Lie group and U be an open neighborhood of 1_G, then any element $g \in G$ is a finite product of elements in U

1.4.1 Local transformation groups

There is a local analogue of transformation groups too.

1.5 Classical Lie groups

The matrix groups are perhaps the most frequenly occuring examples of Lie groups. In many contexts they are used as transformation groups for some

As a manifold, $\mathrm{GL}(n)$ is interpreted as the subspace topology of the Euclidean topology $\mathbb{R}^{n^2}$; this topology immediately makes matrix groups Lie groups.

Chapter 2

Lie algebrae

Finding continuous transformation groups on some space X is usually tantamount to solving a Lie point symmetry group.

In practice, one often starts with very limited information regarding the Lie group. Fortunately there is an extremely useful intermediate structure that proves vital in deducing data of the Lie group; the *Lie algebra*.

As will be discussed, Lie algebrae and Lie groups have a very close correspondence with eachother; this can be summarized by the following:

- Every Lie group gives rise to a Lie algebra
- Every Lie algebra gives rise to a Lie group
- Isomorphic Lie groups correspond to isomorphic Lie algebrae
- The Exponential map is the major tool used for obtaining data of the Lie group from the Lie algebra

This chapter will develop the motivation, definition, properties, and basic applications of Lie algebrae.

vector fields that "infinitesimally nudge"

Irregardless of what we know about a specific Lie group G, left (or right) multiplication maps are a natural class of autodiffeomorphisms on a Lie group. Informally speaking, this represents a class of smooth symmetries for the Lie group itself; these maps are ways that the Lie group's elements can be smoothy permuted!

In an attempt to gain insight to the Lie group, we take a look at the local behaviour of these maps on the identity element.

In what direction does each multiplication map L_g "nudge" its image as it works on elements from 1_G along $\mathbf{v}_{1_G}$? This is essentially the evaluation of $(dL_g)_{1_G}(\mathbf{v}_{1_G})$; it is useful to encode all these "nudges" by different L_g associated with a single direction from the identity into one single vector field as such.

$$\mathbf{V}(g) = (dL_g)_{1_G}(\mathbf{v}_{1_G})$$

Definition 2.1 (Infinitesimal generator)**.** Let $\mathbf{v}_{1_G}$ be a tangent vector in $T_{1_G}G$, then an *infinitesimal generator of* G is a vector field $\mathbf{V}$ of the following form.

$$\mathbf{V}(g) = (dL_g)_{1_G}(\mathbf{v}_{1_G})$$

Denote the set of all IGs of G as $\mathfrak{g}$

Essentially, each IG captures data related to how all the multiplication maps behave as they drift away from the identity in a certain direction. The reason we define infinitesimal generators as such is because it relies only on the tangent space at the identity, and because left (or right) multiplication maps being *bijective* autodiffeomorphisms are most likely to capture the most group elements.

As it turns out, knowledge of the IGs alone is extremely powerful at recovering the structure of a Lie group; in special cases one can even determine the whole Lie group back!

Proposition 2.1. $\mathfrak{g}$ is a linear space

We have seen that the IGs form a linear space, but what's more is that they are also closed under the Lie bracket, meaning they form not only an F-linear space, but more generally an *F-algebra*!

Proposition 2.2. Let $\mathbf{V}, \mathbf{U} \in \mathfrak{g}$, then $[\mathbf{U}, \mathbf{U}] \in \mathfrak{g}$

Because the infinitesimal generators of a Lie group form an algebra over a field we call this space a *Lie algebra* of that Lie group.

Definition 2.2 (Lie algebra of a Lie group)**.** Let G be a Lie group, then the *Lie algebra of* G is the linear space $\mathfrak{g}$ of all *infinitesimal generators*

At the heart of an IG is some tangent vector at the identity that generates it, and one can show that these tangent vectors do more than just generate IGs, but characterize their structure completely!

Every tangent vector at 1_G produces its own linear space, and by the linearity of the differential, the addition of these tangent vectors is compatible with the addition of IGs and the zero vector of T_{1_G} is associated with the zero vector of the Lie algebra.

Moreover, there is a bijective correspondence between IGs and these tangent vectors;

$$(dL_g)_{1_G}(\mathbf{u}_{1_G}) = (dL_g)_{1_G}(\mathbf{v}_{1_G})$$

Since L_{1_G} is the identity map on G, letting $g = 1_G$ shows the following

$$\mathbf{u}_{1_G} = \mathbf{v}_{1_G}$$

Therefore each tangent vector at the identity corresponds uniquely to some IG.

All these conditions come together to mean that one can treat the Lie algebra as the Tangent space at the identity!

Proposition 2.3. Let G be a Lie group, then the tangent vectors at 1_G and Lie algebra $\mathfrak{g}$ are isomorphic as linear spaces.

$$\mathfrak{g} \cong T_{1_G}G$$

2.0.1 Generic Lie algebrae

As we have demonstrated, Lie algebrae naturally arise out of Lie groups, however the motivation for Lie algebrae is to construct a Lie group back. We will introduce a definition of a Lie algebra independent of a Lie group and eventually show how a generic Lie algebra has sufficient data to determine its own Lie group.

Definition 2.3 (Lie algebra)**.** A *Lie algebra* $\mathfrak{g}$ is an F-linear space with an alternating bilinear operation satisfying the Jacobi identity.

2.0.2 Invariant vector fields

There is an alternative way of arriving that the construction of an IG, This brings useful properties of IGs to light.

Definition 2.4 (Left-invariant vector field)**.** Let G be a Lie group. $df_h(\mathbf{V}(g)) = \mathbf{V}(hg)$

Proposition 2.4. Infinitesimal generators are precisely the left invariant vector field on Lie group.

2.0.3 Vector flows on infinitesimal generators

Theorem 2.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$, the vector flow on an IG of $\mathfrak{g}$ is Lie isomorphic to $(\mathbb{R}, +)$ or $\mathrm{SO}(2)$

In the spirit of seeking symmetries, we seek for a natural class of autodiffeomorphisms on a Lie group to form an algebraic structure with. In group theory, the maps $f_h(g) = hg$ are general group automorphisms and autodiffeomorphisms

Definition 2.5 (Lie algebra of a Lie group)**.** Let G be a Lie group, then its Lie algebra $\mathfrak{g}$ is the linear space of all right-invariant vector fields on G (also called *infinitesimal generators*) Let G be a Lie group, then its Lie algebra $\mathfrak{g}$ is the linear space of all right-invariant vector fields on G (also called *infinitesimal generators*)

Consider "right-invariant" vector fields on a lie group. note that they are entirely characterized by $\mathbf{F}(1_G)$, so any tangent vector of $T_{1_G}G$ corresponds to an infinitesimal generator; this means we can colloquially think of a Lie algebra as $T_{1_G}G$

Since all Lie algebrae have a basis of vector fields, and any Lie brakcet of vector fields lies within the Lie algebra, then the Lie bracket of basis vector fields (which can be used to calculate any Lie bracket) is representable by the Hamel basis; we can find *structure constants* for the Lie bracket to assist with computing Lie brackets and conveniently represent them using *Commutator tables.*

Since the Lie bracket is bound by certain algebraic relations, these have a direct impact on the relationship between the structure constants.

2.0.4 Exponential map

As preluded to, the Lie algebra can be used to reconstruct (at least partially) the related Lie group. Interpreting the Lie algebra as the tangent space to the identity,

Given that the elements of $\mathfrak{g}$ are smooth vector fields on G, one can recover group elements by considering vector flows from the identity; choosing an IG and its vector flow for 1 unit of time is how we define the *exponential map.*

Definition 2.6 (Exponential map)**.** Let G be a Lie group with Lie algebra of IGs $\mathfrak{g}$, the *exponential map* is the map from the Lie algebra to its Lie group defined as such, where $\psi_{\mathbf{V}}$ is the vector flow of $\mathbf{V}$ $\exp : \mathfrak{g} \to G$

$$\exp(\mathbf{V}) = \psi_{\mathbf{V}}(1, 1_G)$$

Scaling an IG $\mathbf{V}$ by t can be used to find other times for the vector flow on $\mathbf{V}$; indeed one has the following expression

$$\exp(t\mathbf{V}) = \psi_{\mathbf{V}}(t, 1_G)$$

This map serves as a direct tool to recover data of the Lie group using its corresponding Lie algebra.

What makes this map "exponential" is that for the Lie group $(\mathbb{R}_+, \cdot)$ the exponential map is exactly the exponential function.

The Lie algebra of $(\mathbb{R}_+, \cdot)$ is spanned only by the infinitesimal generators is $x\partial_x$, and so the vector flow for $c_1 x\partial_x$ is solved to be $\psi(\varepsilon, x) = e^{c_1\varepsilon}x$, and so the exponential function is $\exp(c_1 x\partial_x) = e^{c_1}$.

Please consider the flow through the identity of a Lie group for some vector field, $\psi_{\mathbf{v}}(t, 1_G)$; this actually forms a 1-dimensional Lie subgroup (isomorphic to either $\mathbb{R}$ or SO(2)).

Proposition 2.5. Given a right-invariant vector field $\mathbf{F}$ on Lie group G, its integral curve at the identity is a one-parameter Lie subgroup. Given vector flow ψ, elements of the form $\psi(t, 1_G)$ for any t form this Lie subgroup.

Essentially just taking the integral curve through the identi

Notice that any space of dimension 1 has trivial Lie bracket.

2.1 Lie subalgebrae

Linear subspace of a Lie algebra closed under the Lie bracket.

Theorem 2.2 (Lie subalgebra-Lie subgroup correspondence)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$, then any Lie subgroup $H \leq G$ is associated with some Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$. Conversely, any Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is of some Lie subgroup $H \leq G$.

Theorem 2.3 (Ado's theorem)**.** Let $\mathfrak{g}$ be a n-dimensional Lie algebra, $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n)$

This allows a proof of a more general theorem

Theorem 2.4 (Lie algebra-Lie group correspondence)**.**

2.1.1 Lie algebrae of local Lie groups

There is a proposition that easily allows the calculation of a basis for such Lie algebrae.

Proposition 2.6.

$$\mathbf{F}_k(x) = \sum \xi_k^i(x)\partial_{x^i}$$

Notice that the right-invariant maps can be used to represent the group operation. Using the properties of right-invariant vector fields gives the following.

Lie derivative

2.2 Classical Lie algebrae

An important class of Lie algebrae are that of the classical Lie groups. These examples are widespread, concrete examples of Lie algebrae, and working with them will help strengthen one's understanding of Lie algebrae in general.

We will first examine $\mathfrak{gl}(n)$; the Lie algebra of $\mathrm{GL}(n)$. Interpreting Lie algebrae from the perspective of a tangent space to the identity permits the best insight for its calculation; since the identity is $\mathbf{I}$, one may consider the Lie group as the tangent vectors from this element.

To calculate the permissible tangent vectors at the identity matrix, one considers arbitrary "smooth curves of invertible matrixes", say $\mathbf{A}(t)$, where $\mathbf{A}(0) = \mathbf{I}$. Since we take matrixes with the Euclidean topology on $\mathbb{R}^{n^2}$, one can directly differentiate this curve and evaluate at 0 to obtain matrixes as the tangent vector at the identity matrix.

Futhermore, since we are working in $\mathrm{GL}(n)$, every matrix on the curve must be invertible; matrixes on the curve are bound by the expression $\det(\mathbf{A}(t)) = c, c \neq 0$. Differentiating this expression gives a condition that elements of the Lie algebrae must obey, so the matrixes in the Lie algebrae are determined by $\det'(\mathbf{A}(t)) = 0$.

Results pertaining to Lie subalgebrae will allow the results of $\mathfrak{gl}(n)$ to extend to other classical Lie algebrae.

$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$

The matrix commutator is the Lie bracket on any such Lie subalgebra.

To calculate, we use the fact that $\mathfrak{gl}(n)$ is isomorphic to $T_{\mathbf{I}}\mathrm{GL}(n)$

Chapter 3

Smooth transformation groups

3.1 Invariant functions

Given a manifold M, we may want to conside the Lie group G with action α such that the group action "smoothly preserves some property". We can represent a "property" by means of a function, hence our Lie group is a smooth transformation group when the group action on some manifold point does not change the image mapping of this *invariant function*.

Definition 3.1 (Invariant function)**.** Smooth function f obeying $f(\alpha(g, x)) = f(x)$

Invariant functions are used generally across mathematics to describe the sense of symmetry one is interested in.

Given a group action, how can we find an invariant function such that the Lie group is a smooth tangsformation group in the sense of said function? Conversely, if we know the type of smooth symmetry we are interested in, we have an invariant function to represent this; however how can we find the appropriate Lie group associated to this notion of smooth symmetry?

The group action isdetermined by the vector flows on the infinitesimal generators, so consider $f(\psi_{\mathbf{V}}(\varepsilon, x))$, but if f is G-invariant then the derivative of this function respecting ε is 0.

This leads to the following theorem

Proposition 3.1. Let G be a connected transformation group acting on M, then $f : M \to \mathbb{R}$ is invariant function iff for every infinitesimal generator $\mathbf{F}$ of G, $\mathbf{F}(f) = 0$

Consider submanifolds defined by zero sets of some function $f : M \to \mathbb{R}^n$; they essentially represent the solutions to a system of n equations on the manifold of the form $f_i(x) = 0$ (if $M = \mathbb{R}^m$, these could potentially be algebraic equations, which is a familiar system of interest).

It may be useful to find a smooth transformation group and group action for which f is an invariant function, since such a smooth transformation group would help in finding more zeros of f in M.

3.1.1 Invariant subsets

The notion of invariant subsets is often useful in specific applications of group theory, notably that of Lie theory.

α-invariant subset is a subset on which any group action on an element of that set is also in the set. More clearly put, it i a subset $X \subseteq S$ such that any group element $g \in G$ acting on $x \in X$ obeys $\alpha(g, x) \in X$

We can see that the properties of invariant functions are essentially bound to invariant subsets by this simple proposition; this might be convenient in some situations.

Let G be a transformation group acting on M then f is an invariant function iff all its level sets are invariant subsets.

3.2 Further results?

Submanifold is locally G-invariant set iff the infinitesimal generators of $\mathfrak{g}$ are in T_xN

Chapter 4

Applications to differential equations

Of course, the main propsect of Lie theory is its use in determining smooth transformation groups for differential equations!

4.1 Prolongations

To construct a framework in which we can apply Lie theory, we consider the space solutions of a differential equation as a subvariety of a jet space; this is our manifold. Given a differential equation $f(x, u^{(n)}) = 0$, we desire to find a smooth transformation group G; a Lie group with a group action on the jet space that is G-invariant over f.

4.1.1 Prolongation of IGs

Since the Lie group is the object to be calculated, we must consider a way to prolong infinitesimal generators, corresponding to the prolongation of the group action.

Theorem 4.1. Let G be a Lie group and $f(x, u^{(n)})$ be a differential equation of maximal rank, and for any element $\mathbf{V} \in \mathfrak{g}$

$$\mathrm{Pr}\mathbf{V}[f(x, u^{(n)})] = 0$$

Then G is a smooth transformation group for the differential equation

4.1.2 Prolongation formula

This is the key to determining the Lie algebra, from which the exponential map can recover the Lie group.

4.1.3 Properties of prolonged IGs

is linear, can pass within Lie bracket

Chapter 5

Solvable Lie groups

Related to a condition on the lie bracket for the lie algebra