Measure Theory

Zac Zerafa

November 22, 2025

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Part I

Measure functions

Chapter 1

σ-algebrae

1.1 σ-algebra

If you've studied general topology, you may recall that a topological space can be defined as a particular family of sets called the topology. It was inspired by the properties of open sets generated by metric spaces.

In the context of measure theory, we'll eventually do a similar thing; we'll use a family of sets known as the σ-algebra to form *measurable spaces.* We define what a σ-algebra is so that we have familiarity once we encounter it.

Definition 1.1. A *σ-algebra* is a family of sets Σ

$$\emptyset, X \in \Sigma$$

$$\sigma \in \Sigma \implies X \setminus \sigma \in \Sigma$$

$$\bigcup_{n=1}^{\infty} \sigma_n \in \Sigma$$

Chapter 2

Topology

2.1 Borel σ-algebra

It is best to come back to this section after having read the rest of this part, since the motivation for Borel σ-algebrae comes from a desire to combine measure theory with topology, but thus far we have covered no measure theory.

Imagine you have a topological space and you want to find a way to measure your open sets. You want all your open sets to be measurable, but since measures are defined on σ-algebra, such a measure may also be well defined on sets that aren't open. A topology is closed under finite intersections and uncountable unions, while a σ-algebra is closed under countable intersection and unions. Since σ-algebrae have a weaker condition for closure of intersections, when a σ-algebrae is generated it also includes countable intersections of open sets; which aren't necessarily in the topology!

Definition 2.1 (Borel σ-algebra)**.** The *Borel σ-algebra of* $(X, \mathcal{T}\ \mathcal{B}_{\mathcal{T}}$ is the σ-algebra generated by the sets open in $(X, \mathcal{T})$. Sets of the Borel σ-algebra are called *Borel sets of* $(X, \mathcal{T})$.

Definition 2.2 (Borel space)**.** Let $(X, \mathcal{T})$ be a topological space, then the *Borel space of* $(X, \mathcal{T})$ is the measurable space $(X, \mathcal{B}_{\mathcal{T}}$.

Proposition 2.1. Sets closed in $(X, \mathcal{T})$ are Borel sets of $(X, \mathcal{T})$.

Since the Borel σ-algebra is a σ-algebra generated by open sets (which are already Lebesgue measurable for the Euclidean topology on $\mathbb{R}$), the following proposition naturally follows.

Proposition 2.2. Let $(\mathbb{R}, \mathcal{T})$ be the Euclidean topology on $\mathbb{R}$. The Borel sets of $(\mathbb{R}, \mathcal{T})$ are measurable.

What's more interesting is the following.

Proposition 2.3. Open sets in the Euclidean topology on $\mathbb{R}$ are Lebesgue measurable. In other words, $\mathcal{B}_{\mathcal{T}} \subseteq \operatorname{dom}(\lambda)$.

Chapter 3

Lebesgue measure

THe concept of cardinality gives us a way of measuring the 'size' of sets, but it doesn't always capture the .

Although the notion of cardinality is extremely useful in some contexts, it fails to describe the notion of 'size' in others. Take the intervals $[0, 1]$ and $[3, 9]$. Their cardinalities are both 'infinite', but in another sense, $[0, 1]$ has a 'length' of 1 and $[3, 9]$ has a 'length' of 6. This notion of size is called the *Lebesgue measure*, and is useful in mathematical analysis, specifically integration.

The Lebesgue measure as we will define it will be able to assign a 'measure' to more complicated sets on $\mathbb{R}$ rather than just intervals (we can generalize the Lebesgue measure to $\mathbb{R}^n$, but this part will only consider the Lebesgue measure on sets of $\mathbb{R}$). However we will eventually ask if all sets of $\mathbb{R}$ can be assigned a measure according to the Lebesgue measure. Spoiler alert; no.

The ultimate goal we have is to define a general framework for ways of 'measuring' the size of sets in different ways, including our Lebesgue measure. However like much of mathematics, we need to understand concrete examples before we abstractify them.

Those who have read my book on general topology recall that I had to begin with metric spaces before I could generalize them to topological spaces. This book will similarly start with the study of the Lebesgue measure and later generalize to generic measure functions. Though we're ambitious, we still need to crawl before we learn to climb.

3.1 Vitali sets

We have an idea of some properties that our Lebesgue measure should obey, so let's idealize some essential properties for the Lebesgue measure λ.

We now introduce a rather pathological type of set.

Definition 3.1 (Vitali set)**.**

When we officially define our Lebesgue measure, it will obey these properties; We'll take this on faith for now. Assuming that our Lebesgue measure obeys these properties, we're disillusioned with the following discovery.

Proposition 3.1 (Vitali's theorem)**.** Let V be a Vitali set and λ be the Lebesgue measure, then $\lambda(V)$ undefined.

The Vitali sets are therefore an example of sets that we won't be able to measure with the Lebesgue measure. If we surrender the i, j or kth of these idealized properties (or more radically, the axiom of choice) we can have every set of $\mathbb{R}$ to be measurable, however dropping any of those properties would make the Lebesgue measure much less of a powerful tool. Under this premise, Lebesgue decided to accept the reality of non-measurable sets and continue his theory.

Aware of the limitation of our Lebesgue measure, we're prompted to give a constructive definition of the Lebesgue measure and classify what sets the Lebesgue measure is well defined on.

3.2 Constructing the Lebesgue measure

We'll start our journey to costruct such a function; we've got the intuition of how we want to measure intervals, so we can define the *interval length function* as a precursor to the Lebesgue measure.

Definition 3.2 (Interval length function)**.** The *interval length function* is the function ℓ defined

$$\ell([a,b]) = \ell((a,b)) = b - a$$

To handle sets that aren't necessarily intervals, we consider collections of open intervals that cover the set in question and take the cover with the smallest measure to be the *Lebesgue outer measure* of that set. We're essentially 'shrinking in' on the set from the outside by means of open intervals.

Definition 3.3 (Lebesgue outer measure)**.** The *Lebesgue outer measure* is the function $\lambda^* : \mathcal{P}(\mathbb{R}) \to [0, \infty) \cup \{+\infty\}$ defined as such.

$$\lambda^*(E) = \inf\{\sum_{n=1}^{\infty} \ell(I_n) : (I_n)_{n\in\mathbb{N}} \text{ is a sequence of open intervals } \wedge E \subseteq \bigcup_{n=1}^{\infty} I_n\}$$

Note that the sequence of I_n may be finite or countable (as like any sequence).

It's worth mentioning that we have permitted the domain of the Lebesgue outer measure to be the nonnegative extended real numbers; we will allow sets to be given the measure $+\infty$. This is because the sum in this definition may possibly diverge, and it will allow arithmetic of the extended real numbers to be conducted on the Lebesgue measure.

This is almost the Lebesgue measure, however λ^* is well defined for any real set and in terms of countable additivity, the following proposition is the best that can be done.

Proposition 3.2. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{R}$, then the following holds.

$$\lambda^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \lambda^*(E_n)$$

With the Lebesgue outer measure, this proposition cannot be promoted to an equality, since that would make λ^* exactly obey those 'idealized' Lebesgue measure properties, but then there would be sets undefined for the Lebesgue outer measure, which is a contradiction.

We still want countable additivity though, so this means that we want to restrict the domain of λ^* from $\mathcal{P}(\mathbb{R})$ to $\{\bigcap_{n=1}^{\infty} E_n \in \mathcal{P}(\mathbb{R}) : (E_n)_{n\in\mathbb{N}} \text{ are pairwise disjoint } \implies \lambda^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \lambda^*(E_n)\}$.

An alternative but equivalent way to characterize this is through *Carathéodory criterion*.

Theorem 3.1 (Carathéodory criterion)**.**

So the Lebesgue outer measure restricted to sets obeying the Carathéodory criterion is exactly what we want out Lebesgue measure to be; we can now define the Lebesgue measure.

Definition 3.4 (Lebesgue measure)**.** The *Lebesgue measure* is the function $\lambda : \Sigma \subset \mathcal{P}(\mathbb{R}) \to [0, \infty]$ defined as below, where elements $E \in \Sigma$ obey Carathéodory criterion.

$$\lambda(E) = \lambda^*(E)$$

Definition 3.5 (Lebesgue measurable set)**.** A subset E of $\mathbb{R}$ is a *Lebesgue measurable set* iff $E \in \operatorname{dom}(\lambda)$

It's interesting to note that we could also define the Lebesgue measure by considering a *Lebesgue inner measure*, and taking the sets where the inner and outer measures are equal; this was Lebesgue's original idea.

Proposition 3.3. $E \subseteq \mathbb{R}$ is Lebesgue measurable iff the following holds.

$$\lambda^*(E) = \lambda_*(E)$$

Proposition 3.4. Let $I \subseteq \mathbb{R}$ be an interval, then I is Lebesgue measurable.

Proposition 3.5. For any $y \in \mathbb{R}$ we have the following.

$$\lambda(E) = \lambda(\{x + e : e \in E\})$$

Proposition 3.6. $\operatorname{dom}(\lambda)$ is a σ-algebra.

Lebesgue null set countable sets are lebesgue null sets Cantor set is a lebesgue null set

Chapter 4

Measures

Now that we have an concrete understanding of one type of measure function, we propose a definition for a generic measure function; this will allow us to apply measure theory to diverse and abstract settings.

Definition 4.1 (Measurable space)**.** A *measurable space* is an ordered pair (X, Σ). It represents a set with potential for a well defined measure defined on Σ.

- X is a space (set)
- Σ is a σ-algebra on X

Measurable spaces have potential for a measure to be endowned onto it, while a measure space is a measurable space with a specific measure attached.

Definition 4.2 (Measure space)**.** A *measure space* is a 3-tuple (X, Σ, μ). It represents a measurable space equipped with a specific measure on its Σ.

- X is a space (set)
- Σ is a σ-algebra on X
- μ is a measure on Σ

We define measurable spaces as their own object since there are things that we can say about a sets measurability without the need to involve specific measures; so long as our chosen σ-algebra includes the set, any measure that could be applied to that measurable space will give that set a measure.

Definition 4.3 (Measure)**.** A *measure on* (X, Σ) is a function $\mu : \Sigma \to [0, \infty) \cup \{+\infty\}$ that satisfies the following.

- $\mu(\emptyset) = 0$
- $\mu(\bigcup_{i \in \mathbb{R}} E_i) = \sum i \in \mathbb{R} \mu(E_i)$

Definition 4.4 (measurable set)**.** E is a *measurable set of* (X, Σ) iff $E \in \Sigma$

Note that this doesn't depend on any measure, but rather the σ-algebra which determines the 'measurability structure' chosen for X

The idea of a null set however is dependent on the chosen measure.

Definition 4.5 (null set)**.** E is a *null set of* (X, Σ, μ) iff $\mu(E) = 0$. When the measurable space is known but there may be multiple measures, this may be called a μ-null set.

$$E \text{ is a null set of } (X, \Sigma, \mu) \iff \mu(E) = 0$$

It's obvious that all null sets are measurable (since by definition they have a measure, specifically 0).

Definition 4.6 (Almost everywhere property)**.** A property P holds *almost everywhere on* (X, Σ, μ) iff the set of elements on which it doens't hold is a null set.

- complete measure space

Definition 4.7 (Absolute continuity)**.** Let (X, σ) be a measurable space compatible with measures μ, ν. Then μ *is absolutely continuous with respect to* ν iff every ν-null set is a μ-null set.

Chapter 5

Classes of measure functions

- finite measure - atomic measure

Chapter 6

Measure-like functions

In our construction of the Lebesgue measure, we have already encountered various functions that were used as stepping stones to obtain an actual measure (Lebesgue inner measure and length function); many measure-like functions are made with this intention.

Some measure-like functions are made to make functions similar to measures but with more sophisticated codomains. For example, signed measures allow sets with negative measure and vector measures allow the measure of sets to have a direction. Such measures arise in the contextx of mathematical analysis and probability theory, hence it is worth defining them.

6.1 Inner and outer measures

- inner measure - outer measure

6.2 Premeasures

Chapter 7

Signed measures

7.1 Inner and outer measures

Part II

Measurable functions

Chapter 8

Measurable functions

This part will set the foundations of applications of measures to functions and vice versa; it will be necessary for when we establish Lebesgue integration later on.

Definition 8.1 (Lebesgue measurable function)**.** A *Lebesgue measurable function* is a function $f : X \subseteq \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ such that $\{x : f(x) > y\}$ is Lebesgue measurable for any $y \in \mathbb{R}$.

One can prove equivalence to the following definition.

Proposition 8.1. $f : X \subseteq \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is a Lebesgue measurable function iff for any set U open in $\mathbb{R}$, $f^{-1}(U)$ is measurable.

Definition 8.2 (Measurable function)**.** Let $(X, \Sigma_X), (Y, \Sigma_X)$ be measurable spaces. A *measurable function* is a function $f : X \to Y$ such that for any $Ein\Sigma_Y$, $f^{-1}(E) \in \Sigma_X$.

This is similar to the topological definition of a continuous function; continuous functions maps open sets of one topology from open sets of another. Measurable functions map measurable sets of one measurable space from measurable sets of another.

One step in Lebesgue integration is partitioning the range into intervals (which are Borel sets of the Euclidean topology) and use the preimage of the function to determine what subset of the domain is mapped to each interval. Lebesgue integration will require these domain subsets to be measurable to calculate our integral, so our Lebesgue integral will naturally only work on measurable functions.

That said, I've hinted that the Lebesgue integral will only partition its range by means of intervals (which are open sets, assuming we're talking about open intervals for now), which means that we can restrict our attention to what are called *Borel functions* for simplicity. Sure, the Lebesgue integral looses a tiny bit of power by doing this, but the functions that suffer from this downsizing are so pathological that we kind of don't care.

Definition 8.3 (Borel function)**.** Let $(X, \Sigma), (Y, \mathcal{B}(\mathcal{T}_Y))$ be a measurable space and Borel space respectively. A *Borel function* is a measurable function $f : X \to Y$.

This means that Borel functions are defined such that the preimage of any Borel set is a measurable set. Recall that all open sets are Borel sets, so with Borel functions, preimages of open sets are measurable sets.

Though the converse doesn't hold (Borel sets aren't necessarily open sets), since $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B)$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$ for countable unions and intersections respectively, and Borel sets are indeed countable unions and intersections of open sets, letting open sets map from measurable sets is enough to create an equivalent definition!

Proposition 8.2. Let $(X, \Sigma), (Y, \mathcal{B}(\mathcal{T}_Y))$ be a measurable space and Borel space respectively. $f : X \to Y$ is a Borel function iff for any U open in $(Y, \mathcal{T}_Y)$, $f^{-1}(U)$ is measurable.

The Lebesgue integral will consider functions with a codomain of $\mathbb{R}$, so we'll now prove the following proposition that will give us some good intuition for when we study Lebesgue integration.

Proposition 8.3. Let $(X, \Sigma), (\mathbb{R}, \mathcal{B}(\mathcal{T}_\mathbb{R}))$ be a measurable space and Borel space on the Euclidean topology of $\mathbb{R}$ respectively. $f : X \to \mathbb{R}$ is a Borel function iff for any $a \in \mathbb{R}$, $f^{-1}((-\infty, a))$ is measurable.

However as it turns out, we don't even require the intervals to be strictly open. Borel spaces contain not only all open sets, but also closed sets and therefore any type of interval! Alternatively, using countable unions and intersections on open sets as a Borel σ-algebra permits, we can indeed create any type of interval.

Proposition 8.4. Let $(X, \Sigma), (\mathbb{R}, \mathcal{B}(\mathcal{T}_\mathbb{R}))$ be a measurable space and Borel space on the Euclidean topology of $\mathbb{R}$ respectively. $f : X \to \mathbb{R}$ is a Borel function iff for any $a \in \mathbb{R}$, $f^{-1}((-\infty, a))$ is measurable.

So in the end, it doesn't even matter if we're using closed, open, or even semi-closed and semi-open intervals; any of these will do!

8.1 Constructing Borel functions

When we finish constructing our Lebesgue integral, we'd like to know what class of functions it can work its magic on; I've spoiled that one condition it will require is that the function be Borel. However we currently haven't seen concrete examples of Borel functions; we hope that this class of functions is rich so that our Lebesgue integral isn't a downgrade from the Riemann integral.

Besides, with an intuitive notion of what this new class of function looks like, we could form work arounds. We'll start by examining characteristic functions and simple functions (these will be building blocks in the study of Lebesue integration).

Note that the codomain is taken as the Euclidean topology with its Borel σ-algebra.

Proposition 8.5. Let (X, Σ) be a measurable space. $\chi_E : X \to \{0, 1\}$ is a Borel function iff E is measurable.

Proposition 8.6. Real continuous functions are Borel functions

The property of being measurable is closed under many arithmetic operations.

Proposition 8.7. f+g is measurable f-g is measurable fg is measurable cf is measurable —f— is measurable min(f,g) is measurable max(f,g) is measurable

continuous functions are Borel functions

Chapter 9

Sequences of measurable functions

pointwise convergent sequence of measurable function converges to measurable function

$$\sup\{f_1, \ldots, f_n\}$$
$$\inf\{f_1, \ldots, f_n\}$$
$$\sup_n (f_n)$$
$$\inf_n (f_n)$$

Chapter 10

Theorems on measurable functions

10.1 Littlewood's 3 principles

10.2 Egorov's theorem

Theorem 10.1 (Egorov's theorem)**.** Measurable sequences of functions that converge pointwise converge uniformly almost everywhere.

There exists a (much less exciting) partial converse to Egorov's theorem

Measurable sequences of functions that converge uniformly almost everywhere converge pointwise almost everywhere.

I suppose this is interesting in that it means that almost everywhere pointwise and uniform convergence are the same for sequnces of measurable functions.

10.3 Luzin's theorem

Theorem 10.2 (Luzin's theorem)**.** An almost everywhere finite function whose domain is a measurable set is continuous almost everywhere.

Chapter 11

Convergence in measure

11.1 Convergence in measure

11.2 Cauchy in measure

Part III

Lebesgue integration

Chapter 12

Lebesgue integral

After much anticipation, we now develop the Lebesgue integral. We have much to look forward to; our new integral will have many desirable features, notably the following. - Ability to integrate larger class of functions - Ability to integrate over larger class of domains - Backwards compatibility with Riemann integral - Easier proof framework for proving theorems about swapping limits and integrals - Ability to define a Hilbert space for integrable functions (this is an idea of functional analysis)

12.1 Lebesgue integral for simple functions

We will do this by means of simple functions, since as we are about to prove, we can use them to

Definition 12.1 (Lebesgue integral of a simple function)**.** If f is a a simple function $f : X \to \mathbb{R}$ of the form $f = \sum_{k=1}^{n} c_k \chi_{E_k}$ where the E_k are pairwise disjoint, the *Lebesgue integral of* f is the following sum.

$$\int f d\mu = \sum_{k=1}^{n} c_k \mu(E_k)$$

If E is a measurable set, then the following sum is called the *Lebesgue integral of* f *on* E.

$$\int_E f d\mu = \sum_{k=1}^{n} c_k \mu(E_k \cap E)$$

Definition 12.2 (Integrable simple function (ISF))**.** An *Lebesgue integrable on* E is a simple function $f : X \to \mathbb{R}$ of the form $f = \sum_{k=1}^{n} c_k \chi_{E_k}$ where the E_k are pairwise disjoint such that its Lebesgue integral on E is finite. An *integrable simple function* is a simple function $f : X \to \mathbb{R}$ of the form $f = \sum_{k=1}^{n} c_k \chi_{E_k}$ where the E_k are pairwise disjoint such that its Lebesgue integral is finite.

f+g fg cf

$$|\int f d\mu| = \int |f| d\mu$$

$$\int |f+g| d\mu \leq \int |f| d\mu + \int |g| d\mu$$

12.2 Lebesgue integral

p

Definition 12.3 (Lebesgue integral of a nonnegative function)**.** For a function $f : X \to \mathbb{R}$ with a sequence of ISFs that converge to f in measure, the *Lebesgue integral of* f is the following limit.

$$\int f d\mu = \lim_{n\to\infty} f_n d\mu$$

If E is a measurable set, then the following limit is called the *Lebesgue integral of* f *on* E.

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$$

Definition 12.4 (Lebesgue integrable function)**.** An *Lebesgue integrable function* is a function $f : X \to \mathbb{R}$ such that there exists a sequence of ISFs that converge to f in measure and its Lebesgue intergal is finite. A function is *Lebesgue integrable on* E is a function $f : X \to \mathbb{R}$ such that there exists a sequence of ISFs that converge to f restricted to E in measure and its Lebesgue intergal on E is finite.

Proposition 12.1. *For a function* $f : X \to \mathbb{R}$ *Lebesgue integral of* f *is the following limit.*

$$\int f d\mu = \lim_{n\to\infty} f_n d\mu$$

If E is a measurable set, then the following limit is called the *Lebesgue integral of f on E*.

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$$

If it isn't finite almost everywhere, it's not Lebesgue integrable.

Definition 12.5 (Lebesgue integrable function)**.** A function $f : X \to \mathbb{R}$ is a *Lebesgue integrable function* iff there exists a sequence of ISFs

$$\int f d\mu = \lim_{n\to\infty} f_n d\mu$$

We'll use the following proposition to link our definition of the Lebesgue integral for simple functions o that of nonnegative Borel functions defined on a measurable set.

Proposition 12.2. for nonnegative Borel function on measurable set, exists sequence of simple functions that converges pointwise. if function bounded, converges uniformly.

12.3 Relation to improper Riemann integral

Chapter 13

Properties of Lebesgue integral

f=g almost everywhere means same integral

Chapter 14

L^p spaces?

Chapter 15

Measure decompositions

Hahn Decomposition theorem Jordan decomposition Radon-Nikodym theorem Radon-Nikodym derivative

Chapter 16

Convergence theorems

A major limitation of the Riemann integral is that it does not provide sufficient tools to discuss when a limit and integral commute. The Lebesgue integral offers.

Bounded convergence theorem Dominated convergence theorem Monotoneconvergence theorem Fatou's lemma

Chapter 17

Improper Lebesgue integral

Part IV

Product measures

product measure product measurable space product measure space Caratheodory extension theorem

Chapter 18

Fubini-Tonelli theorem

Fubini's theorem Tonelli's theorem