# Real Analysis

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# Contents


**I Fundamentals** **1**

**1 Real numbers** **5**
1.1 Algebraic and order properties of $\mathbb{R}$ . . . . . 6
1.2 Euclidean metric of $\mathbb{R}$ . . . . . 7
1.2.1 Open intervals . . . . . 7
1.3 Notable properties of $\mathbb{R}$ . . . . . 8
1.3.1 Density of $\mathbb{Q}$ in $\mathbb{R}$ . . . . . 8
1.3.2 Archimedean property of $\mathbb{R}$ . . . . . 9

**2 Real Sequences and series** **11**
2.1 Real Sequences . . . . . 11
2.1.1 Bolzano-Weierstrass theorem . . . . . 13
2.2 Real Cauchy sequences . . . . . 14
2.2.1 Rationals are not Cauchy complete . . . . . 15
2.3 Squeeze theorem . . . . . 15
2.4 Limit supremum and infimum . . . . . 16
2.5 Series . . . . . 16
2.6 Harmonic series . . . . . 17
2.7 Geometric series . . . . . 17
2.8 Convergent series tests . . . . . 18

**3 Bachmann-Landau family of notations** **21**
3.1 Original Bachmann-Landau notation . . . . . 21
3.2 Knuth's notation . . . . . 22
3.3 Vinogradov-Hardy-Littlewood notation . . . . . 23

**4 Continuous function** **25**
4.1 Limits . . . . . 25

4.2 Continuous function . . . . . 26
4.3 Main results . . . . . 27
4.4 Stronger variants of continuity . . . . . 28

# Part I

# Fundamentals

Mathematical logic Set theory Order theory

# Chapter 1

# Real numbers

Mathematical analysis refers to the study of functions (particularly continuous, differentiable and integrable ones) on various spaces. We will start our journey through analyis by studying analysis on $\mathbb{R}$, so functions that map to real numbers. Readers are already be familiar with such functions, and they are the basic building blocks of more advanced functions (for instance, complex or vector function), so this is the ideal topic for the beginner of analysis.

Along the way we will make a rigorous construction of calculus, and learn how to use the results of calculus to prove deeper theorems.

Many ideas that will be discussed in this book can be generalized by studying topology, and indeed this book serves as a strong precursor to many ideas arising in topology.

Before even looking at functions, we will develop a deeper insight of the properties that characterize the real numbers. We are already familiar with how algebra and how ordering works on the real numbers, but we will also need to appeal to more properties that although kind of obvious, we may not be used to applying.

**Definition 1.1** (Real numbers)**.** The *real numbers* is the set $\mathbb{R}$ that is the (conditional) Dedekind completion of $\mathbb{Q}$ with the standard order $\leq$

In other words, the real numbers are an extension of the rational numbers such that any bounded above (or below) set of real numbers has a supremum (or infimum); a property that does not hold for the rationals or integers.

There are indeed other equivalent ways one can define the real numbers, however they require the notion of convergent sequences, a more "topologi-

cal" concept which we haven't covered yet. The idea of a supremum is much simpler to introduce and build only on our understanding of ordering of the real numbers, hence why Dedekind completeness is a neat way to define real numbers swiftly.

- Axiomatically assuming Dedekind completeness
- Axiomatically assuming the Bolzano-Weierstrass theorem
- Axiomatically assuming the Intermediate value theorem
- Axiomatically assuming Cauchy completeness and Archimedian property

All these approaches require some notion of convergence, except for the first, which draws from order theory rather than topology.

## 1.1 Algebraic and order properties of $\mathbb{R}$

We will explore the algebraic and ordering properties of $\mathbb{R}$, much of which is simply inherited from $\mathbb{Q}$.

To extend the algebraic operations from $\mathbb{Q}$ to $\mathbb{R}$ using our definition of real numbers, we use the following definition.

**Definition 1.2** (Addition on $\mathbb{R}$)**.** $x + y = \sup\{a + b \in \mathbb{Q} : a < x \wedge y < b\}$

**Definition 1.3** (Multiplication on $\mathbb{R}$)**.** $xy = \sup\{ab \in \mathbb{Q} : a < x \wedge y < b\}$

We should be well familiar with algebra on the real numbers; the following proposition captures familiar facts about real algebra that we will depend on.

**Proposition 1.1.** $(\mathbb{R}, +, \cdot, \leq)$ is a Dedekind-complete totally ordered field

$0 \leq x \wedge 0 \leq y \implies 0 \leq xy \, \forall U \subset \mathbb{R}[U$ is bounded above $\implies \sup_{\leq}(U) \in \mathbb{R}] \, \forall U \subset \mathbb{R}[U$ is bounded below $\implies \inf_{\leq}(U) \in \mathbb{R}]$

We'll now consider some bounded sets as exampls of how some familiar irrational numbers can be constructed as the supremum of certain subsets of $\mathbb{Q}$.

$$X_\pi = \{3, 3.1, 3.14, 3.141, \ldots\}$$

$$X_e = \{2, 2.7, 2.71, 2.718, \ldots\}$$

$$X_{\sqrt{2}} = \{x \in \mathbb{Q} : x^2 \leq 2\}$$

By carefully crafting an appropriate subset of $\mathbb{Q}$, we can realize any real number as a supremum!

Once we have considered all supremums of each bounded rational set, we end up with the rational and irrational numbers; now the supremum of any real set will be some number in $\mathbb{R}$. This property is known as *dedekind completeness* or *the least upper bound property*; the algebraic laws of real numbers, ordering laws of real numbers, and dedekind completness of real numbers is enough to completely define real numbers.

## 1.2 Euclidean metric of $\mathbb{R}$

Perhaps the most powerful tool of mathematical analysis is the *Euclidean metric*; it essentially characterizes distance between real numbers. Time and time again, we will rely on this tool to create meaningful definitions related to "convergence and limits", and its properties are essential when conducting low-level proofs related to "convergence and limits".

Distance between numbers on $\mathbb{R}$ is naturally interpreted as the absolute difference of the numbers (this represents how "far apart" they are additively, without sign), so $d(x, y) = |x - y|$ is the *Euclidean metric on* $\mathbb{R}$, which just means that the following properties hold (it is good routine to actually check these yourself!).

$$d(x, y) = 0 \iff x = y$$

$$d(x, y) = d(y, x)$$

$$d(x, z) \leq d(x, y) + d(y, z)$$

(Triangle inequality)

More generally, one can consider sets with a function satisfying these 3 conditions to form *metric spaces*; spaces with a suitable notion of distance.

### 1.2.1 Open intervals

A 'topological space' deals all about what sets we call 'open'. In the case of this book, we deal with *Euclidean topology* $(\mathbb{R}, \mathcal{T})$; the open sets of $\mathbb{R}$ are

the open intervals, and unions of open intervals (even infinite unions of open intervals). The reason for this definition requires a bit more understanding of general topology, but for now, it's intuitive enough that we can accept it.

Neighborhoods of $p$ are open sets containing $p$; this idea will return to us along our journey. When restricted to our Euclidean topology on $\mathbb{R}$, we can think of neighborhoods of $p$ as open sets which include at least one open interval containing $p$. The term 'neighborhood' can be thought of here as elements that are 'neighbors' of $p$ since they 'live' in the same open set; they're 'close' in the sense that they're next door neighbors in this open set.

More noteworthy for our purposes however, this topological space is metrizable (i.e we can define what 'distance' means) in the following way.

**Definition 1.4** (Open interval)**.**

$$(a, b)$$

**Definition 1.5** (Open ball)**.**

$$B(p, r) = (p - r, p + r)$$

## 1.3 Notable properties of $\mathbb{R}$

We are yet to study real sequences, however with our definition of the real numbers and their structure we can prove various properties that give us a deeper understanding of the real numbers.

### 1.3.1 Density of $\mathbb{Q}$ in $\mathbb{R}$

Often one finds it useful to approximate irrational numbers with rational approximants, for instance the approximation $\pi \approx \frac{22}{7}$ has been used since antiquity for applications of mathematics that do not require high precision. the; indeed one can always improve their rational appximation.

**Proposition 1.2** ($\mathbb{Q}$ is dense in $\mathbb{R}$)**.** For any $x \in \mathbb{R}$ and any $\varepsilon \in \mathbb{R}_+$, there is some $y \in \mathbb{Q}$ such that $|x - y| < \varepsilon$

The core idea is that this proposition becomes interesting when we consider really small values of $\varepsilon$; it says that given any real number, we can choose a rational number that approximates it with arbitrarily small error $\varepsilon$.

This is important for the basis of numerical analysis, however it will also be a useful trick throughout this book, and we will now use it to demonstrate the Archimedian property.

### 1.3.2 Archimedean property of $\mathbb{R}$

The Dedekind completeness of the real numbers is powerful indeed, and it will be a vital tool when studying convergent real sequences and limits of real functions. Before we study such concepts, we can realize more ordering properties of the reals.

**Proposition 1.3** (The Archimedian property for of real numbers)**.**

$$\forall \varepsilon \in \mathbb{R}_+ [\exists n \in \mathbb{N}(\frac{1}{n} < \varepsilon)]$$

Choose our real number $\varepsilon$ and consider the set $[\frac{\varepsilon}{2}, \varepsilon] \cap \mathbb{Q}$; it cannot be empty since $\mathbb{Q}$ is dense in $\mathbb{R}$, and since any element of this set must be less than or equal to $\varepsilon$, we can consider some element $\frac{p}{q}$ such that $\frac{p}{q} \leq \varepsilon$ so therefore $\frac{1}{q} < \overset{p,q}{\leq} \varepsilon$, proving the Archimedian property.

This essentially means that there are no 'infinitesimals' in the real numbers, or equivalently (by considering $n > \frac{1}{\varepsilon}$), there are no infinite values. The Archimedean property is obeyed by the rational numbers as well.

The Archimedian property can be used to prove that $\mathbb{R}_+$ has no minimum or maximum value.

# Chapter 2

# Real Sequences and series

Real analysis is essentially the study of real functions and real sequences, especially in relation to limits and convergence. We will begin by looking at a *real sequences*; they are somewhat interesting on their own, but also supplement the study of real functions.

## 2.1 Real Sequences

Sequences are first covered in Set Theory as an ordered list of (possibly infinite) values, where repetition is allowed. It is then formally defined as a function mapping integers to elements of an arbitrary set. Set theory being as fundamental as it is, it didn't have much more to say about sequences.

Elementary Number Theory goes further and delegates a chapter to 'integer sequences'; sequences of integers and rational numbers (so much for the 'integer part)'. This book discusses properties of any real sequence and particularly focuses on their behaviour as the amount of terms approaches infinity; elementary number theory does not concern itself with such concepts.

This chapter introduces sequences in the space of real numbers and aims to build a correct rigorous understand of precisely what it means for a sequence to 'converge' to a value.

This chapter additionally aims to act as a stepping stone for dealing with real functions. Real sequences are still functions, and despite being simpler, we will see that they form the fundamentals for rigorously defining ideas of analysis such as limits. Even if our sequences terms are all happen to be

rational, it is still interesting to note that the sequence might be approaching an irrational number. These are the kinds of fun things that happen when you get 'real' about mathematics (okay, that was pretty lame).

**Definition 2.1** (Real sequence)**.** A *real sequence* is a sequence $(x_n)_{n=1}^{\infty}$ contained in $\mathbb{R}$, so each $x_n \in \mathbb{R}$

In the context of mathematical analysis, we are mostly concerned about *convergence and limits* of sequences.

Some of the more interesting sequences are those that follow special properties

**Definition 2.2** (Real bounded sequence)**.** A real sequence is *bounded above* iff there is some $M$ such that for all $n$ we have $x_n \leq M$ . A real sequence is *bounded below* iff there is some $M$ such that for all $n$ we have $x_n \geq M$ A *real bounded sequence* is a real sequence bounded above and below, equivalently, there is some $M$ such that for all $n$ we have $|x_n| \leq M$.

We can think of bounded sequences as installing a set of walls that the sequence cannot pass.

**Definition 2.3** (Real monotone sequence)**.** A real sequence is *monotone increasing* iff $x_n > x_{n-1}$. A real sequence is *monotone decreasing* iff $x_n < x_{n-1}$.

Monotone sequences get strictly larger (or smaller); they can never "backtrack" with respect to the order. Simultaneously, bounded sequences are always "capped" at some maximum (or minimum) possible value.

But what happens when an unstoppable force meets an immovable object? What happens when our monotone sequence is forced to increase (or decrease) but it is bounded, so our sequence faces the wall? Such a sequence would have to converge! It gets squeezed against the least upper bound of the sequence since by being monotone it cannot oscillate backwards.

**Theorem 2.1** (Monotone convergence theorem)**.** Bounded, monotone sequences are convergent

Note that there is another theorem called the monotone convergence theorem; it's completely different and it refers to a sufficient condition for swapping the order of taking a limit and an integral.

Another tool that will become indispensable to us is the notion of a subsequence.

**Definition 2.4** (Real subsequence)**.** Let $(x_n)$ be a real sequence, then a *real subsequence of* $(x_n)$ is a real sequence of the form $(x_{n_k})$, where $(n_k)$ is some natural numbered sequence used index certain term from the original sequence.

The idea is that subsequences of a sequence can 'skip' whatever terms of the original sequence as desired, so the multiples of 4 are a subsequence of the even numbers since it skips every odd index.

**Definition 2.5** (Real convergent sequence)**.** For a real sequence $(a_n)$, a *limit of* $(a_n)$ is a real number $L$ such that for any positive $\varepsilon$, we can find an integer $N$ so that whenever $n > N$ we have $|a_n - L| < \varepsilon$. Basically, $L$ is arbitrarily close all remaining terms of a sequence. A *convergent real sequence* is a real sequence with a limit.

$$a_n \to L \iff \forall \varepsilon \in (0, \infty)(\exists N \in \mathbb{N}[n > N \implies |a_n - L| < \varepsilon]])$$

We'll also call sequences that aren't convergent to be divergent sequences.

We'll now establish some of the more elementary properties of convergent sequences; later on we will

**Proposition 2.1.** The limit of a convergent real sequence is unique.

$$x_n \to L \wedge x_n \to M \implies L = M$$

Note that there is an $n$ such that for any $\varepsilon$, $|x_n - L - x_n + M| < |x_n - L| + |x_n - M| = 2\frac{\varepsilon}{2} = \varepsilon$. Simultaneously, $|x_n - L - x_n + M| = |M - L| < \varepsilon$, but since $\varepsilon$ is any positive real, we have $M - L = 0$, proving our point.

Due to the uniqueness of limits of convergent real sequences, we introduce the following notation for limits.

$$x_n \to x \iff \lim_{n\to\infty} x_n = x$$

### 2.1.1 Bolzano-Weierstrass theorem

**Theorem 2.2** (Bolzano-Weierstrass theorem (BWT))**.** Let $(x_n)$ be a real, bounded (above and below) sequence, then $(x_n)$ has a convergent subsequence.

Obviously if $(x_n)$ is convergent, so are its subsequences. When $(x_n)$ is not convergent, we This follows from the fact that for any non convergence sequence we can choose

## 2.2 Real Cauchy sequences

From our definition of a convergent sequence, proving that a sequence is convergent is tantamount to proving that it has a limit. There are, however, cases where we may not know the limit, and so resort to proving that a limit at least exists.

The idea of Cauchy sequences is to define a class of sequences with "convergent behaviour" (i.e the sequence eventually confines itself to an arbitrarily small interval); Eventually we will see that in real analysis, Cauchy sequences are precisely the convergent sequences.

Now that we know about convergent sequences, it's time to abstractify this conect a little bit with Cauchy sequences. These sequences are one that eventually 'settle down' (i.e they have 'convergent behaviour').

**Definition 2.6** (Real Cauchy sequence)**.** A *real Cauchy sequence* is a real sequence $(a_n)_{n\in\mathbb{N}}$, where for any positive $\varepsilon$, we can find an integer $N$ so that whenever $n, m > N$ we have $|a_n - a_m| < \varepsilon$. Basically, the absolute difference between all remaining terms can be bouned arbitrarily close to 0 given enough terms of the sequence have passed.

$$(x_n)_{n\in\mathbb{N}} \text{ is Cauchy } \iff \forall\varepsilon \in (0,\infty)(\exists N \in \mathbb{N}[n, m > N \implies |a_n - a_m| < \varepsilon]])$$

Isn't this the same as a convergent sequence? Due to the completeness of the real numbers, real cauchy sequences and real convergent sequences are the same, however Cauchy sequences are more general in that they don't require the limit to exist within the space itself; just the convergent behaviour. In spaces without completeness, convergent sequences are merely a special type of Cauchy sequence. The notion of convergent sequences and Cauchy sequences being the same is *Cauchy completeness*.

**Proposition 2.2.** A real sequence is convergent iff it is Cauchy. In other words, the real numbers are Cauchy complete.

Proving that real sequences are Cauchy is not so bad.For real convergent sequence $x_n \to L$, we can let $|x_n - L| < \frac{\varepsilon}{2}$ for any $\varepsilon$ and proceed as such.

$$|x_n - x_m| = |x_n - x_m + L - L| \leq |x_n - L| + |x_m - L| < \frac{2\varepsilon}{2} = \varepsilon$$

To prove that Cauchy sequences are convergent sequences is more difficult; we need to prove that Cauchy sequences are bounded, and then the Bolzano Weierstrass theorem will immediately imply that it is convergent.

Since the real numbers are Cauchy complete, Cauchy sequences are a useful tool for proving convergence of a sequence without the need to conjecture its limit. Let's say we have a real sequence and we want to see if it is convergent, but we don't know its limit. We can't prove it directly using the definition of a convergent sequence since it requires a limit, however one can resort to proving that the sequence is Cauchy to imply that the sequence is convergent.

### 2.2.1 Rationals are not Cauchy complete

**Definition 2.7** (Rational sequence)**.** A *rational sequence* is a sequence $(x_n)_{n=1}^{\infty}$ of rational numbers, so each $x_n \in \mathbb{Q}$. Futhermore, we say that $x_n$ converges iff $|x_n - x_m|$ can be made as small as we like by finding some $N \leq n, m$.

Let's consider the rational sequence $x_n = \frac{4}{x_{n-1}+\frac{2}{x_{n-1}}}, x_1 = 1$. This is a rational Cauchy sequence, however as it turns out, this sequence does not approach any rational number! This means it is not convergent in $\mathbb{Q}$. This leads us to the following; instead of defining the real numbers as the Dedekind completeness of $\mathbb{Q}$, we can define the real numbers as the Cauchy completion of $\mathbb{Q}$ (with the Archimedian property, since some of the ordering properties are not implied by Cauchy completion).

Similar to how Dedekind completion refers to creating equivalence classes of bounded sets with the same upper bounds, the Cauchy completion refers to creating equivalence classes of Cauchy sequences whose difference converges to 0.

In either case, the whole point of $\mathbb{R}$ is to 'patch up' $\mathbb{Q}$ so that Cauchy sequences and convergent sequences are equivalent notions and supremums always exist.

## 2.3 Squeeze theorem

Yet another intuitive result that we would like to prove.

**Theorem 2.3** (Squeeze theorem)**.** Let $a_n \to L$ and $c_n \to L$ with $a_n \leq b_n \leq c_n$, then $b_n \to L$.

## 2.4 Limit supremum and infimum

What does the supremum of a sequence become eventually?

**Definition 2.8** (Limit supremum)**.**

$$\limsup x_n = \lim_{N\to\infty} \sup(\{x_n : n \geq N\})$$

**Definition 2.9** (Limit infimum)**.**

$$\liminf x_n = \lim_{N\to\infty} \inf(\{x_n : n \geq N\})$$

BWT can be used to tell us that bounded real sets always have limit supremums and infimums.

$x_n$ is a convergent real sequence iff $\limsup x_n = \liminf x_n$.

## 2.5 Series

Series are merely the limit of a type of infinite sequence where terms are added to the previous result, however the study of a particular sequences that converge to a series is particularly colorful.

- partial sum

**Definition 2.10** (Series)**.** Given a sequence $\{x_n\}_{n=1}^{\infty}$, its *series* is the limit of the sequence $s_1 = x_1, s_n = s_{n-1} + x_n$, or otherwise $s_n = \sum_{k=1}^{n} x_k$; the partial sums made from $x_n$ (if the limit exists). If the limit exists, it is a convergent series, otherwise it is a divergent series.

$$\sum_{n=1}^{\infty} x_n = S$$

One can note that $x_n \to 0$ to have a chance of being a convergent series. However this condition alone will not suffice to prove that a series is convergent; we will look at the harmonic series as a counterexample to this claim.

- absolutely convergent series - conditionally convergent series

## 2.6 Harmonic series

**Definition 2.11** (Harmonic series)**.**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Even though the series progresses by quantities that get closer to 0, the series still doesn't converge.

**Proposition 2.3.** The harmonic series is divergent

The idea is that at any point in the series, we can still group enough terms to make an arbitrarily large number; this is essentially the *Cauchy condensation test*, which we will cover soon.

If we consider the behaviour of the Harmonic sequence rather than its divergent series, we note that it grows in a similar fashion to the natural logarithm, in fact their difference converges! It converges to a rather interesting number, the *Euler-Mascheroni constant*.

**Definition 2.12** (Euler-Mascheroni constant)**.**

$$\gamma := \lim_{n\to\infty} [H_n - \ln(n)]$$

## 2.7 Geometric series

The geometric sequence is typically introduced in high school.

**Definition 2.13** (Geometric series)**.** A *geometric series* is a series of the following form.

$$\sum_{n=1}^{\infty} ar^n$$

**Proposition 2.4** (Closed form for geometric sequence)**.**

$$\sum_{k=1}^{n} ar^k = a\frac{1-r^{n+1}}{1-r}$$

**Corollary 2.1** (Closed form of a geometric series)**.**

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$$

## 2.8 Convergent series tests

Series have a bit more structure than any ordinary limit, so they permit a wide range of 'tests' we can derive as general tools for checking convergence.

**Theorem 2.4** (Absolute convergence test)**.** Any absolutely convergent series is convergent.

**Theorem 2.5** (Comparison test)**.** If $\sum_{n=1}^{\infty} y_n$ converges and $x_n \leq y_n$ then $\sum_{n=1}^{\infty} x_n$ converges.

**Theorem 2.6** (Limit comparison test)**.** Consider the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ with strictly positive terms, if $\lim_{n\to\infty} \frac{x_n}{y_n} \neq 0$ then the series either both converge or both diverge.

**Theorem 2.7** (Ratio test)**.** Consider the series $\sum_{n=1}^{\infty} x_n$ with strictly positive terms

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$$

The series converges when $L < 1$, diverges when $L > 1$. When $L = 1$ the test is inconclusive.

**Theorem 2.8** (Leibniz test)**.** If $x_n$ is monotone sequence and $x_n \to 0$ then $\sum_{n=1}^{\infty} (-1)^n x_n$ converges.

**Theorem 2.9** ($n$th root test)**.** Consider the series $\sum_{n=1}^{\infty} x_n$

$$\text{limsup}\sqrt{n}|x_n| = L$$

The series converges when $L < 1$, diverges when $L > 1$. When $L = 1$ the test is inconclusive.

**Theorem 2.10** ($P$ test)**.** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$

**Theorem 2.11** (Cauchy condensation test)**.** $x_n$ nonnegative, monotone decreasing sequence $\sum_{n=1}^{\infty} x_n$ converges iff $\sum_{n=1}^{\infty} 2^n x_{2^n}$ converges.

**Theorem 2.12** (Integral test)**.** $\sum_{n=N}^{\infty} f(n)$ iff $\int_N^{\infty} f(x)dx$ converges

These tests are useful to verify that a series converges, however they don't offer a method to calculate the series itself. In the case of the geometric series we deduced its value by considering a closed form for partial sums and finding

the limit of said closed form; we got lucky here, we won't always be able to do that.

It's much more difficult to determine what a convergent series equals, however when we formally study differentiation we will unlock a tool called *Taylor series* which are extremely useful for calculating series.

That said, we should be happy; we can calculate geometric series and check convergence for a wide range of series.

# Chapter 3

# Bachmann-Landau family of notations

It is a tool to describe the growth of functions and is used extensively in mathematical analysis, analytical number theory, numerical analysis.

In computer science, one may way to consider a function that eats a number representing an input to some model of computation and spits out the amount of atomic steps that algorithm takes to terminate. Algorithms may have extremely complex behaviour, so usually such functions are unable to be explicitly documented by its mappings, but rather the asymptotic class it belongs to.

Consider $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$

## 3.1 Original Bachmann-Landau notation

They notation were introduced in analytic number theory by Paul Bachmann and popularized by Edmund Landau. Hardy and Littlewood also made use of these notations and extended upon them.

Unfortunately in modern times, computer science and analytic number theory have their own notations in use, however the use of big-$O$ and little-$o$ is understood universally in mathematics.

**Definition 3.1** (Big-$O$ function class)**.**

$$f \in O(g) \iff \exists M \in \mathbb{R}_+[\forall n \in X[|f(n)| \leq Mg(n)]]$$

$$f \in O(g) \iff \sup_{n \in X} \frac{|f(n)|}{g(n)} < \infty$$

**Definition 3.2** (Little-$o$ function class)**.**

$$f \in o(g)$$

There are a following conventional abuse of notations associated with $O$, $o$, and related notations. The primary abuse of notation is using the equality sign instead of set inclusion sign, so some write $f = O(g)$ instead of $f \in O(g)$.

The second prevalent abuse of notation is doing arithmetic with asymptotic classes, so some write $f + O(g)$ instead of $f + h, h \in O(g)$.

## 3.2 Knuth's notation

In the 1970s, Donald Knuth devised a range of standard function classes alongside $O$ and $o$ for use in computational complexity theory. Specifically, he introduced $\Omega$ ($\Omega$ existed before Knuth, however he created his own distict definition of the symbol to better suit the nature of his field), $\Theta$, and $\omega$.

Computational complexity theory adheres to Knuth and the original Bachmann-Landau notations to describe the complexity of algorithms (where the function generally maps some notion of 'input size' to the amount of irreducible steps the algorithm takes).

**Definition 3.3** (Big-$\Omega$ function class (Knuth))**.**

$$f \in \Omega(g) \iff \exists M \in \mathbb{R}_+ [\forall n \in X [f(n) \geq M g(n)]$$

$$f \in \Omega(g) \iff \inf_{n \in X} \frac{f(n)}{g(n)} > 0$$

**Definition 3.4** (Big-$\Theta$ function class (Knuth))**.**

$$f \in \Theta(g) \iff \exists M_1, M_2 \in \mathbb{R}_+ [\forall n \in X [M_1 g(n) \leq f(n) \leq M_2 g(n)]]$$

$$f \in \Theta(g) \iff f \in O(g) \wedge g \in O(f)$$

**Definition 3.5** (Little-$\omega$ function class)**.**

$$f \in \omega(g)$$

## 3.3 Vinogradov-Hardy-Littlewood notation

Before the time of Knuth and little after the creation of $O$ and $o$, various mathematicians working in analytic number theory extended upon Bachmann-Landau notation to describe a wider range of asymptotic behaviour of functions.

Unlike Knuth whose notation only introduces more function classes, VHL notation introduced function relations. Though they could have been introduced as function classes, they are conventionally treated as relations.

**Definition 3.6** ($\ll$ function relation (Vinogradov))**.**

$$f \ll g \iff \sup_{n \in X} \frac{|f(n)|}{g(n)} < \infty$$

**Definition 3.7** ($\gg$ function relation (Vinogradov))**.**

$$f \gg g \iff \inf_{n \in X} \frac{f(n)}{g(n)} > 0$$

**Definition 3.8** ($\asymp$ function relation (Hardy-Littlewood))**.**

$$f \asymp g \iff f \ll g \wedge g \ll f$$

We can see that we have $f \ll g$ iff $f \in O(g)$, and $f \gg g$ iff $f \in \Omega(g)$ in the sense of Knuth, and hence $f \asymp g$ iff $f \in \Theta(g)$; so far these have been equivalent notations to Knuth.

There are 2 more notations in analytic number theory that are not widespread in computational complextiy theory: $\Omega$ in the sense of Hardy-Littlewood and $\sim$.

**Definition 3.9** (Big-$\Omega$ function class (Hardy-Littlewood))**.**

$$f \in \Omega(g) \iff \limsup_{n \to \infty} \frac{|f(n)|}{g(n)} > 0$$

**Definition 3.10** ($\sim$ function relation (Hardy-Littlewood))**.**

$$f \sim g \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

# Chapter 4

# Continuous function

## 4.1 Limits

**Definition 4.1** (Real function)**.** A *real function* is a function $f : X \to \mathbb{R}$

Many (maybe even all) functions you'd be familiar with are real functions; polynomial functions, trigonometric functions, exponential functions, logarithms, rational functions, anything that sends a real number to another real number.

We are familiar with limits of real sequences, however there is a distinct but related notion of a limit of a real function.

**Definition 4.2** (Limit at a point of a real function)**.** For a real function $f$, its *limit at* $p$ is a number $L$ suuch that for any positive $\varepsilon$, we can find a positive $\delta$ so that whenever $|x - p|$ we have $|f(x) - L\| < \varepsilon$. Basically, as $x$ converges to $p$, $f(x)$ converges to $L$.

$$\lim_{x\to c} f(x) = L \iff \forall \varepsilon \in (0,\infty)(\exists \delta \in (0,\infty)[|x-c| < \delta \implies |f(x)-L| < \varepsilon])$$

$$\lim_{x\to c} f(x) = L \iff \forall (x_n)_{n\in\mathbb{N}}[\lim_{n\to\infty} x_n = c \implies \lim_{n\to\infty} f(x_n) = L]$$

**Definition 4.3** (Limit at a point of a real function)**.** Let $f : X \to \mathbb{R}$ be a real function, then the *limit of* $f$ *at* $p$ is a number $L$ such that any sequence $(x_n)$ contained in $X$ converging to $p$, $(f(x_n))$ converges to $L$

$$\lim_{x\to p} f(x) = L \iff \forall (x_n)_{n\in\mathbb{N}} \subseteq X[x_n \to p \implies \lim_{n\to\infty} f(x_n) = L]$$

Although our intuition already suggests this, it is important for the sake of mathematical rigor to check that a real function's limit at a point is a unique real number.

Since we are working in a 1-dimensional space, we can consider limits taken from the left (or right) side of the real line. This is known as a *one-sided limit.*

-Left sided limit -Right sided limit

Since these are the only 2 directions a limit may take on the real line, we have the following.

**Proposition 4.1.**

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = L \wedge \lim_{x\to c^-} f(x) = L$$

## 4.2 Continuous function

Although we may have a naiive notion of what a continuous function is, the lack of a formal definition prevents us from leveraging its properties or realizing moments when they arise in mathematics. We will motivate a definition by recalling how continuous functions

If we think of a continuous $\mathbb{R}$-function graphically as some plotted curve that is "unbroken" and look at a certain point $(x_0, f(x_0))$ on the plot, we have the expectation that $f(x_0 + \varepsilon) \approx f(x_0)$ when $\varepsilon$ is "small enough". One also expects that when $\varepsilon$ is made smaller and smaller, the approximation improves (if one is somehow certain the function is continuous but doesn't observe this, the original $\varepsilon$ was probably not sufficiently small).

The approximation relationship $\approx$ is not formally defined on $\mathbb{R}$, however it is a useful symbol for communicating heuristic arguments such as the one previously given. Fortunately, limits are a formal notion that capture the core ideas of this argument, specifically the following limit

$$\lim_{\varepsilon\to 0} f(x_0 + \varepsilon) = f(x_0)$$

.

Instead of considering the limit in terms of sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \to 0$, one can equivalently construct sequences of the form $(x_0 + a_n)_{n\in\mathbb{N}}$, and one can easily show that $x_0 + a_n \to x_0$. We now have the formal definition of a continuous real function.

**Definition 4.4** (Continuous real function)**.** A real function is *continuous at* $x_0$ iff the following limit exists.

$$\lim_{x \to x_0} f(x) = f(x_0)$$

A function is said to be continuous on $U$ iff it is continuous for every $u \in U$. A *continuous function* is a function continuous on it's domain.

**Proposition 4.2.** A real function is continuous at $x_0$ iff for any open interval $I$ with $x_0 \in I$, $f^{-1}(I)$ is a union of open intervals.

Subsequently, this translates to an alternative definition for a continuous function.

**Corollary 4.1.** A real function is continuous on the open interval $U$ iff for any open interval $I \subseteq U$, $f^{-1}(I)$ is a union of open intervals.

Where working with the Euclidean metric is inconventient, this definition of a continuous function may prove effective to work with.

We will now introduce some powerful results for identifying continuous functions. Due to the properties of limits previously discussed, continuous functions obey analogous closure properties to that of limits.

**Proposition 4.3.** closed under addition, subtraction, multiplication and division (when not zero)

continuity is conserved even through composition.

**Proposition 4.4.** Let $f, g$ be continuous real functions, then $f \circ g$ is a continuous real function.

## 4.3 Main results

In mathematical analysis, continuity is among the most basic properties that one desires their function to have.

**Proposition 4.5.** Let $f$ be a real continuous function, then $f([a, b]) = [c, d]$

**Theorem 4.1** (Extreme value theorem)**.** Let $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ be a real function continuous on some compact interval $[a, b]$, then there exists some maximum and minimum of $f$ on $[a, b]$.

**Theorem 4.2** (Intermediate value theorem)**.** Let $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ be a real function continuous on some compact interval $[a, b]$, then for any $\eta \in [f(a), f(b)]$, there is some $\xi \in [a, b]$ such that $f(\xi) = \eta$.

## 4.4 Stronger variants of continuity

Eventually, we'll require certain properties of functions similar to continuity but stronger. A real function is continuous if the limit in the definition of a continuous real function always exists,

Unpacking the limit within the definition of a real continuous function,

**Definition 4.5** (Uniformly continuous real function)**.** A real function is *uniformly continuous on $U$* iff

$f$ is uniformly continuous on $U \iff \forall\varepsilon\exists\delta\forall x_1, x_2 \in U[|x_1-x_2| < \delta \implies |f(x_1)-f(x_2)| < \varepsilon]$

**Definition 4.6** (Lipschitz continuous real function)**.** A real function $f$ is *Lipschitz continuous on $U$* iff the following logical expression holds for $f$

$$\exists M[\forall x_1, x_2 \in U[|f(x_1) - f(x_2)| \leq M|x_1 - x_2|]]$$

$f$ is Lipschitz continuous on $U \iff \exists M[\forall x_1, x_2 \in U[|f(x_1)-f(x_2)| \leq M|x_1-x_2|]]$

Uniformly continuous functions are continuous Lipschitz functions are uniformly continuous

**Definition 4.7** ($\alpha$-Hölder continuous real function)**.** A real function $f$ is *$\alpha$-Hölder continuous on $U$* iff the following logical expression holds for $f$

$$\exists M[\forall x_1, x_2 \in U[|f(x_1) - f(x_2)| \leq M|x_1 - x_2|^\alpha]]$$

$\alpha$-Hölder continuous real functions are uniformly continuous