# Set Theory

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December 17, 2025

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# Part I

# Naive Set Theory

# Chapter 1

# Sets

At the beggining of the 19th century, mathematicians sought to create a robust mathematical framework that ensured consistent proofs, propositions and algorithms. To do this, they had to create general, unambiguous, and well-formed definitions and axioms throughout mathematics. Formalizing mathematical logic would be the first step, however there would need to be some 'intermediate' field of mathematics that defines objects on which mathematical logic can state propositions on.

Georg Cantor has created a paradise which connects almost all the rest of mathematics between eachother and to mathematical logic (the notable exception being type theory; an alternative to set theory). Philosophically, set theory is extremely fundamental and powerful, so having a strong command of it is an essential tool for every mathematician.

Indeed, there are many ways to go about establishing set theory depending on one's philosophy on how mathematics 'should' be (this is a topic in its own), however the earlierst step forward was the development of *naive set theory*.

Georg Cantor's development of set theory contained some paradoxes, leading it to be called naive set theory. Axiomatic set theories such as *ZFC* (Zermelo's Fried Chicken?!?) fix these paradoxes, however naive set theory gives a mostly correct intuition as well as the proper language to make branches of mathematics rigorous. As we study naive set theory, we'll allude to its limitations due to some noteworthy paradoxes.

We now commence our journey through Cantor's paradise.

## 1.1 Sets

A mathematical object is anything that can be well defined in mathematics. They include numbers, functions, geometric objects, spaces (linear spaces, topological spaces, measure spaces etc.), algebraic strucutres (groups, rings etc.) and even (or perhaps especially) sets themselves.

Then Cantor said, "Let there be sets," and there were sets. And Cantor saw that the sets were good.

**Definition 1.1.** A *set* is an unordered collection of unique objects (one cannot have two instances of the same object in the same set). Mathematical objects within a set are called *elements of the set*.

Typically capital case letters are used to symbolically represent sets. One may also specify a set using curly brackets around the objects, for example, $\{1, 2, 3, 4, 5\}$ is the set containing integers 1 through to 5.

**Example 1.1.**

$$Z = \{4, 6502, 64, \frac{\pi^2}{6}\}$$

**Example 1.2.**

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

**Example 1.3.**

$$V = \{\mathrm{A}, \mathrm{E}, \mathrm{I}, \mathrm{O}, \mathrm{U}\}$$

Although the idea of a set is simple, extremely complicated sets with unintuitive properties can be formed. We have seen that sets can be specified by exaustively listing each of its elements, however one can use a logical statement to determine which elements lie within a set; this is known as *set builder notation*.

$$X = \{x : P(x)\}$$

**Example 1.4.**

$$P = \{n \in \mathbb{N} : n \text{ is prime}\}$$

**Definition 1.2.** A *disjoint pair of sets* are a pair of sets $A, B$ that share no elements in common. We say that $A$ and $B$ are *disjoint*.

## 1.2 Examples of sets

The most basic set is the emptyset; a set which contains nothing.

**Definition 1.3** (Empty set)**.** The *emptyset* is the set containing no elements. It is denoted as $\emptyset$

**Definition 1.4** (Singleton)**.** A *singleton* is a set containing 1 element. If this element is $x$, the singleton is often denoted as $\{x\}$.

## 1.3 Subsets

**Definition 1.5.** Let $X$ be a set, then $Y$ is a *subset* of $X$ if all the elements of $Y$ are in $X$, or in other words, the elements of $Y$ form a part of $X$.

$$Y \subseteq X \iff \forall y \in Y[y \in X]$$

**Definition 1.6.** The *powerset* of a set $X$ is the set of all subsets of $X$. It is denoted as $\mathcal{P}(X)$.

$$\mathcal{P}(X) = \{Y : Y \subseteq X\}$$

## 1.4 Cardinality

**Definition 1.7** (Cardinality of a set)**.** The *cardinality of a set* $X$ is the number of elements within $X$. It is denoted as $|X|$.

One interesting question to ask is what cardinality sets of infinite elements have. The truth is that there are different types of 'infinities', and therefore to do proper analysis on these sets we require more theory; this will be covered in part 3.

Some observations can be made using all this theory that we have developed

$$Y \subseteq X \implies |Y| \leq |X|$$

$$\mathcal{P}(X) = 2^{|X|}$$

- combina - universe of discourse

## 1.5 Set operations

### 1.5.1 Intersection

$$A \cap B = \{x : x \in A \land x \in B\}$$
$$A \cap B = B \cap A$$

$$|A \cap B| \leq \min\{|A|, |B|\}$$

Where 'min' refers to the smallest element of this set, see Order Theory.

$$B \subseteq A \implies A \cap B = B$$

Intersections allow us to formally define the disjoint property between sets.

**Definition 1.8.** A *disjoint pair of sets* are a pair of sets $A, B$ that satisfy $A \cap B = \emptyset$. In other words, they share no elements in common.

### 1.5.2 Union

$$A \cup B = \{x : x \in A \lor x \in B\}$$
$$A \cup B = B \cup A$$

$$|A \cap B| \geq \max\{|A|, |B|\}$$
$$B \subseteq A \implies A \cup B = A$$

### 1.5.3 Complement

$$A^{\complement} = \{x : x \notin A\}$$
$$(A^{\complement})^{\complement} = A$$

### 1.5.4 Set difference

$$A \setminus B = \{x : x \in A \land x \notin B\}$$
$$A \setminus B = A \cap B^{\complement}$$

$$(A \setminus B) \subseteq A$$
$$|A \setminus B| \leq |A|$$

### 1.5.5 Cartesian product

$$A \times B = \{(a, b) : a \in A \land b \in B\}$$

$$|A \times B| = |A||B|$$
$$|A \times B| = |B \times A|$$

- symmetric difference

### 1.5.6 De morgan's laws (set theory)

## 1.6 Closure operators

$$X \subseteq \mathrm{cl}(X)$$
$$\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$$
$$X \subseteq Y \implies \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$$

Related to topological closure of a set. Also have use in universal algebra.

## 1.7 Covers

**Definition 1.9** (Cover)**.** A *cover of* $X$ is a family of sets $\mathcal{C} = (S_i)_{i \in I}$ whose union is a superset of $X$.

$$X \subseteq \bigcup_{i \in I} S_i$$

Covers appear in branches of mathematics such as topology and measure theory.

## 1.8 Partitions

**Definition 1.10** (Partition)**.** A *partition on* $X$ is a family of sets $\mathcal{P}$ of subsets of $X$ such that all the following hold.

$\emptyset \notin \mathcal{P}$

$\bigcup_{P \in \mathcal{P}} P = X$

Any two distinct sets in the partition are disjoint.

**Definition 1.11.** A partition $A$ is a *refinement of* $B$ iff every $a \in$ A is a subset of some $b \in B$. We say that A is finer than B and B is coarser than A.

### 1.8.1 Russel's paradox

# Chapter 2

# Other collection objects

## 2.1 Multiset

Here is a useful set-like object which is crucial to combinatorics.

**Definition 2.1.** A *multiset* is an unordered collection of objects, where the same object may have multiple instances. Unique objects within the multiset are called *elements of the multiset* The amount of instances of an element is called its *multiplicity*.

# Chapter 3

# Relation

## 3.1 Relation

Sometimes we wish to describe that 2 objects are 'related' in some way; maybe they share a specific property, and we want a set to keep track of all pairs of 'similar elements'. This leads to the idea of a relation.

**Definition 3.1.** A *relation* is a set $R$ of ordered pairs from two sets. We use the notation $xRy$ to say that $(x, y)$ is in $R$, meaning that $x$ is $R$-related to $y$.

$$R \text{ is a relation of } X \times Y \iff R \subseteq X \times Y$$

$$xRy \iff (x, y) \in R$$

### 3.1.1 Examples of relations

We are familiar with a few relations already.

**Example 3.1.** The following is a relation that relates real numbers that are equal to eachother.

$$E = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y\}$$

In this relation, each $x$ is related only to the $y$ equal to it; it's kind of a trivial relation.

**Example 3.2.** The following is a relation that relates real numbers with real number less that or equal to it.

$$L = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq y\}$$

In this relation, each $x$ is related to an infinite amount of $y$.

**Example 3.3.** The following is a relation that characterizes cartesian coordinates on the unit circle

$$L = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$$

In this relation, each $x$ is related to at most 2 $y$.

Relations don't even need to have deep mathematical properties; they simply need to be subsets of the cartesian product of two sets!

**Example 3.4.** The following is an example relating my sister an I to our favourite numbers.

$$F = \{(\mathbf{Zac}, 64), (\mathbf{Zac}, 6502), (\mathbf{Zac}, 4), (\mathbf{Alyssa}, 6), (\mathbf{9})\}$$

This is a subset of $\{\mathbf{Zac}, \mathbf{Alyssa}\} \times \mathbb{N}$, hence it is a relation.

## 3.2 Function

We now look to study a special type of relation that is still quite general, but as we will later see is particularly interesting regarding the things we can say about them.

These kind of relations give every element in the first set strictly 1 mapping, and are called *functions*.

**Definition 3.2.** A *function* is a relation $f$ where no two ordered pairs have the same first element. Since functions relate every element from the first entry of the ordered pair only once, we can denote the $y$ such tht $xfy$ as $f(x)$.

$$f \text{ is a function } \iff f \text{ is a relation } \land \forall (x, y), (x, z) \in f[y = z]$$

$$f(x) = y \iff xfy$$

Functions are perhaps the most interesting objects in mathematics; just ask Thomas Garrity `https://youtu.be/zHU1xH6Ogs4?si=i3G81IifMUrq8sP9`. Because of their special status and properties, they are often described using different notations to relations; more on this later.

## 3.3 Equivalence relation

Readers familiar with some number theory know that $7 = 2(\mod 5)$, although obviously $7 \neq 2$. Readers familiar with Star Wars know that Anakin Skywalker is Darth Vader, but Darth Vader often exclaims that he isn't Anakin and that "Anakin Skywalker is dead".

The point is that in certain contexts, distinct (unequal) objects may exhibit identical behaviour. $7, 2$ are 'equivalent' in $\mathbb{Z} \setminus 5\mathbb{Z}$, and 'from a certain point of view' Darth Vader is Anakin.

Relations that seek to relate entities that are 'the same from a certain point of view' are called *equivalence relations*, and are defined as relations that obey the 3 properties of reflexivity, symmetry, and transitivity.

**Definition 3.3.** An *equivalence relation* $\sim$ is a relation that is reflexive, symmetric, and transitive. It generalizes the properties that equality has.

$$\sim \text{ is an equivalence relation } \iff \sim \text{ is symmetric, reflexive and transitive}$$

$$\sim \text{ is an equivalence relation } \iff \forall x \in X[x \sim x] \land \forall x, y \in X[x \sim y \implies \sim x] \land forall x, y, z \in X[x \sim y$$

$$R \text{ is reflexive } \iff \forall x \in X, xRx$$
$$R \text{ is symmetric } \iff (\forall x, y \in X, xRy \implies yRx)$$
$$R \text{ is antisymmetric } \iff (\forall x, y \in X, xRy \land xRy \implies x = y)$$
$$R \text{ is transitive } \iff (\forall x, y, z \in X, xRy \land yRz \implies xRz)$$

**Definition 3.4.** An *equivalence class* of an element $a$ is the set of all elements that an equivalence relation deems 'equal' to $a$.

$$[a] = \{x \in X : a \sim x\}$$

**Proposition 3.1.**

$$b \in [a] \implies [b] = [a]$$

**Proposition 3.2.** Equivalence classes form partitions.

# Chapter 4

# Functions and Maps

## 4.1 Functions

Authors may differentiate between a 'function' and a 'map', but they are really the same thing; 'map' tends to be used for special types of functions but in this text the definitions are synonymous. Generally the term 'function' will be used, unless the function in question is conventionally referred to as a map.

Functions were defined earlier as a special type of relation. We further study the different range of functions that exist

**Definition 4.1.** *map* is a synonym for function; this word is favoured when the is associated with a 'special structure' (like linear spaces, topological spaces, groups, rings, and so; don't worry if you don't know what these are).

**Definition 4.2.** A *homomorphism* is a map between 'special structures' that preserves the behaviour of

**Definition 4.3.** The *domain of a function* is the set of all well defined inputs for the function.

$$f : X \to Y \implies \mathrm{Dom}(f) = \{x \in X : f(x) \text{ is well defined}\}$$

**Definition 4.4.** The *predomain of a function* is the set representing the 'space' of which the domain is a subset of.

$$f : X \to Y \implies X \text{ is the predomain of } f$$

**Definition 4.5.** The *codomain of a function*

$$f : X \to Y \implies \text{Codom}(f) = Y$$

**Definition 4.6.** The *image of a subset* is the set of all outputs the function has mapped to some subset of its domain.

$$f(U) = \{f(u) : u \in U\}$$

The *image of a function* is the set of all outputs the function has mapped to its domain elements.

$$\text{Im}(f) = \{f(x) : x \in \text{Dom}(f)\} = f[\text{dom}(f)]$$

**Definition 4.7.** The *preimage of a subset* is the set of all inputs the function has mapped to some subset of its image.

$$f^{-1}(U) = \{x \in \text{dom}(f) : f(x) \in U\}$$

*Fibers* are preimages of a singleton.

$$f^{-1}(\{u\}) = \{x \in \text{dom}(f) : f(x) = u\}$$

The term 'range' is used ambiguously to denote either the codomain or image; from my experience it is used informally in contexts where the image is equal to the codomain. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 3x$, indeed $\text{Im}(f) = \text{Codom}(f) = \mathbb{R}$, hence no confusion can occur on the function's 'range'.

## 4.2 Types of functions

Functions can differ significantly regarding what type of objects it maps in its domain and image. Functions might also obey certain specific properties on how domain elements are mapped, however we can discuss some general properties of functions that rely on no other mathematics other than set theory.

**Definition 4.8** (Injective function)**.** An *injective function*

$$f \text{ is injective} \iff f(x) = f(y) \iff x = y$$

**Definition 4.9** (Surjective function)**.** A *surjective function*

$$f \text{ is surjective } \iff \forall y \in \text{Codom}(f) \exists x \in \text{Dom}(f)[f(x) = y]$$

**Theorem 4.1.** If a function is surjective then its image has the same cardinality as its codomain.

$$f \text{ is surjective } \implies |\text{Im}(f)| = |\text{Codom}(f)|$$

**Definition 4.10** (Bijective function)**.** A *bijective function* is a function that is both injective and surjective.

$$f \text{ is bijective } \iff f \text{ is injective } \wedge f \text{ is surjective}$$

**Theorem 4.2.** If a function is surjective then its domain the same cardinality as its range.

$$f \text{ is bijective } \implies |\text{Im}(f)| = |\text{Dom}(f)|$$

**Definition 4.11.** The *composition operator* $\circ$ is a $(g \circ f) = g(f(x))$

**Definition 4.12.** An *inverse function* of $f$

**Theorem 4.3.** A function is invertible iff it is bijective.

**Definition 4.13.** An *operation* is a function whose domain is the same set as its image.

**Definition 4.14.** An *$n$-ary function* is a function that takes $n$ arguments. $n$ is called the *arity* of this function.

**Definition 4.15.** Given a function $f$, a *fixed point* $x_0 \in \text{dom}(f)$

$$x_0 \text{ is a fixed point of } f \iff f(x_0) = x_0$$

## 4.3 Indexed families

Sets are powerful objects; we've been able to develop much of mathematics through their grace. However, sets are limited in that they are unordered and can only hold one 'instance' of some mathematical object (i.e sets don't allow repetition of elements). The theory of functions can build upon the notion of a set to allow 'ordered collections' structures.

**Definition 4.16.** An *indexed family* is a function $f : I \to X$ $(x_i)_{i \in I}$ or $(x_i)$

**Definition 4.17** ($n$-tuple (function definition))**.** An *$n$-tuple* (also called a *sequence*) is an indexed family indexed by a set of $n$ elements.

**Definition 4.18** (Ordered pair)**.** An *ordered pair* is a 2-tuple.

## 4.4 Tuples and sequences

**Definition 4.19** ($n$-tuple (collection definition))**.** An *$n$-tuple* is an ordered collection of objects, where the same object may appear multiple times. The $k$th object in the tuple is called the *$k$th term of the $n$-tuple*.

**Definition 4.20.** The *characteristic function of $S$ on $U$* is the function $\chi_S : U \to \{0,1\}$ with $S \subseteq U$ defined in the following manner.

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

# Chapter 5

# Cardinality

The fundamentals of cardinality have been stated previously, however the deeper ideas associated with cardinality were inaccessible to us without first learning more about functions. Sets of infinite elements may not necessarily be of the same 'size'; some infinities are 'larger' than others. This intriguing fact is the first item on our agenda.

## 5.1 Countable sets

**Definition 5.1.** A *countable set* is a set that either has a bijection to $\mathbb{N}$ or has finite cardinality.

**Definition 5.2.** A *countably infinite set* is a set that has a bijection to $\mathbb{N}$.

### 5.1.1 Propositions on countable sets

### 5.1.2 Examples of countable sets

$\{0, 2, 4, 5, 6\}$ is countable. $\mathbb{N}$ is countable.

These 2 examples follow directly from the definition of a countable set, however our propositions allow us to deduce more complicated sets as countable. $\mathbb{Z}$ is countable. $\mathbb{Q}$ is countable.

## 5.2 A notable uncountable set

$\mathbb{R}$ is uncountable.

Because the nature of a sets cardinality is not adequately described by natural numbers when dealing with infinite sets, a set of ordered symbols calld *cardinal numbers* are introduced.

## 5.3 Cardinal numbers

We call the 'set' of ordered symbols used to compare cardinalities as *cardinal numbers*. Their definition is designed to compare the sizes of sets. So long as there is a way of deciding which cardinal number is 'bigger' in a pair of cardinal numbers, we are happy.

So far the natural numbers have been our cardinal numbers, however this contains no symbol to adequately describe the cardinality of $\mathbb{N}$; a countable but infinite set. Better yet, we also want to describe the cardinality of $\mathbb{R}$; an infinite set that isn't even countable.

We want a 'set' of well ordered cardinal numbers that allows to compare finite and infinite cardinalities.

### 5.3.1 Aleph numbers

We want to contain the natural numbers in this set of cardinal numbers to handle the finite sets, however we want to augment a cardinal to represent the infinite, but countable sets. These type of sets are the most basic and smallest type of infinite set, and so this cardinal number should be greater than all natural numbers, but less than any future infinite cardinal numebrs we append.

For now, let's denote the cardinality of countably infinite sets with the cardinal number $\aleph_0$.

**Proposition 5.1.** Countably infinite sets are the sets of cardinality $\aleph_0$.

$$A \text{ is countably infinite} \iff |A| = \aleph_0$$

We can inductively define the rest of the cardinal numbers by introducing new cardinal number as the next smallest cardinal.

The details are complicated, however we are assured that this is permissible due to the *well-ordering theorem*, which states that for any set we can create some order upon it that makes it well-ordered (every subset contains a smallest element). Hence if one considers the set of all cardinal largers number than $\aleph_0$, our theorem states this contains a smallest cardinal number, which we may denote as $\aleph_1$.

This can be generalized

**Definition 5.3** (Aleph numbers)**.** The *aleph numbers* are a sequence of ordered cardinal numbers $(\aleph)_{n\in\mathbb{N}}$ representing the $n$th smallest infinite cardinal.

$$\forall n \in \mathbb{N}(\aleph_0 > n)$$

$$n > m \implies \aleph_n > \aleph_m$$

Recall that for finite sets, a set with cardinality $n$ has its powerset with cardinality $2^n$; mathematicians allow a similar notation to describe the cardinalities of infinite powersets. This allows the representation $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$.

### 5.3.2 Cardinality of the continuum

The *cardinality of the continuum*

$$\mathfrak{c} = |\mathbf{R}|$$

$$\mathfrak{c} = |\mathcal{P}(\mathbf{N})|$$

### 5.3.3 Beth numbers

**Theorem 5.1.** *Cantor's theorem*

$$|\mathcal{P}(X)| > |X|$$

Cantor's theorem is obvious for finite cardinalities since $2^n > n$ for any natural number $n$. The real kicker is the fact that it holds for the infinite cardinalities too; meaning you can create arbitrarily larger infinities as desired.

One interesting question is whether taking the powerset of the countable set yields a set with the very next cardinal $\aleph_1$.

- Mention continuum hypothesis

$$\aleph_1 = 2^{\aleph_0}$$

In words, this means that there is no infinite cardinal between the cardinalities $\aleph_0$ and $\mathfrak{c}$, so $\mathfrak{c}$ is the very next cardinal and hence $\mathfrak{c} = \aleph_1$. The truth of this hypothesis is kind of complicated, we will return to this when we study axiomatic set theory.

There is also a generalization of this hypothesis that extends to all the aleph numbers.

- Mention generalized continuum hypothesis

$$\aleph_{\alpha+1} = 2^{\aleph_\alpha}$$

This generalization means that taking powersers of any infinite set always yields the very next cardinal number; Cantor's theorem instantly implies that $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$, but there is no apparent reason to believe that $\aleph_{\alpha+1} \geq 2^{\aleph_\alpha}$.

We now introduce the *beth numbers* to offer a clean notation to describe constructing cardinal numbers by repeatedly taking powersets. We will later tie these in with the generalized continuum hypothesis.

**Definition 5.4** (Beth numbers)**.** The *beth numbers* are a sequence of ordered cardinal numbers $(\beth)_{n\in\mathbb{N}}$ representing the cardinality of the powerset of a set with cardinality of the previous beth number.

$$\beth_0 = \aleph_0$$

$$\beth_{n+1} = 2^{\beth_n}$$

**Proposition 5.2.**

$$\mathfrak{c} = \beth_1$$

This gives us a nice representation for the continuum hypothesis.

$$\aleph_1 = \beth_1$$

And the generalized continuum hypothesis is as such.

$$\aleph_\alpha = \beth_\alpha$$

### 5.3.4 Arithmetic on cardinal numbers

$$|A| = a \wedge |B| = b \wedge A \cap B = \emptyset$$

$$a + b = |A \cup B|$$

$$|A| = a \wedge |B| = b \wedge A \cap B = \emptyset$$
$$ab = |A \times B|$$

Addition of cardinal numbers is a commutative monoid.

$$a + b = b + a$$

$$a + (b + c) = (a + b) + c$$
$$a + 0 = a$$

Multiplication of cardinal numbers is a commutative monoid.

$$a + b = b + a$$

$$a + (b + c) = (a + b) + c$$
$$a + 0 = a$$

## 5.4 Ordinal numbers

**Definition 5.5** (First infinite ordinal). $\omega$

**Definition 5.6** (Epsilon numbes).

## 5.5 Large countable numbers

# Part II

# Formalized Set Theory

# Chapter 6

# Zermelo-Fraenkel Set Theory (ZFC)

The standard formal set theory construction used by mathematicians is ZFC. Due to its popularity, this will be the first formalized set theory that we will develop.

## 6.1 Axiom of choice

Historically the most contoversial axiom in mathematics has been the axiom of choice and its equivalent forms (which we will discuss later). Even today, some mathematicians are hesitant to accept the axiom of choice in thei set theories.

### 6.1.1 Zermelo's theorem

### 6.1.2 Zorn's lemma