Vector Analysis

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Contents

Part I

Real Multivariate and vector functions

Chapter 1

Multivariate functions

The goal of this chapter is to translate knowledge from real analysis to higher dimensions of Euclidean spaces. Real analysis studied functions of the form $f : \mathbb{R} \to \mathbb{R}$, however we now want to consider $f : \mathbb{R}^m \to \mathbb{R}^n$; we'll be looking at functions between real Euclidean spaces, possibly with different dimensions. One consequence is that the absolute value as a metric generalizes to the norm.

It's worth mentioning that many of the ideas addressed in this book are standards for any multidimensional generalization of analysis. We first look at three types of functions that are used across all types of multidimenional analysis.

Definition 1.1. A *vector valued function* is a function $\mathbf{r} : X^n \to \mathbb{R}^m$ that maps elements of a space to a Euclidean space.

Definition 1.2. A *vector field* is a vector valued function $\mathbf{f} : X^n \to \mathbb{R}^n$ that assigns each vector in some space X^n to a vector in a real space of the same dimension. It is a vector valued function where the domain and image spaces have the same dimension.

Definition 1.3. A *scalar field* is a vector valued function $f : X^n \to \mathbb{R}$ that assigns a real number to each pont in some space X^n.

Though these definitions apply to functions whose domains are arbitrary spaces X, this book will only *vector valued functions with Euclidean domains* $f : \mathbb{R}^m \to \mathbb{R}^n$, so the following variants of these 3 definitions will be frequent objects of interest within this book.

- Real multivariate functions $f : \mathbb{R}^n \to \mathbb{R}, n > 1$; scalar fields with Euclidean domain.
- Real vector functions $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n, m, n > 1$; vector valued functions with Euclidean domain.
- Real vector fields $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n, n > 1$; vector fields with Euclidean domain.

There are special classes within these functions, such as *differentiable curves and surfaces*; we'll delve into this theory soon.

After having learned the theory of this book, functional analysis can be used to explore these functions on more general spaces called *manifolds*.

Example 1.1. The following function $f : \mathbb{R}^2 \to \mathbb{R}$ is an example of real multivariate function.

$$f(x, y) = x^2 - y^2$$

Example 1.2. The following function $\mathbf{r} : [0, 6\pi] \to \mathbb{R}^3$ is an example of real vector function (it's also a 'differentiable curve') that maps a single variable to a $\mathbb{R}^2$ vector. This is a helix that makes 3 revolutions.

$$\mathbf{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

Example 1.3. The following function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is an example of a real vector field. It returns a vector that is orthogonal to the input vector.

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \mathbf{x}_2 \\ -\mathbf{x}_1 \end{bmatrix} = \mathbf{x}_2\mathbf{i} - \mathbf{x}_1\mathbf{j}$$

Chapter 2

Multivariate limits

In terms of theory and rigor, limits are defined in an extremely similar way to real analysis. In terms of reality and intuition, the primary difference is that in real analysis, it suffices to check that both the left and right limits exist and are equal. In higher dimensions, there are an uncountably infinite directions in which a limit may be approached.

limit in $\mathbb{R}^n$

Definition 2.1. For a real vector function $\mathbf{f}$, its *limit at* $\mathbf{p}$ is a vector $\mathbf{L}$ such that for any positive ε, we can find a positive δ so that whenever $\|\mathbf{x}-\mathbf{p}\| < \delta$ we have $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon$. Basically, as $\mathbf{x}$ converges to $\mathbf{p}$, $\mathbf{f}(\mathbf{x})$ converges to $\mathbf{L}$.

$$\lim_{\mathbf{x}\to\mathbf{p}} \mathbf{f}(\mathbf{x}) = \mathbf{L} \iff \forall\varepsilon \in (0,\infty)(\exists\delta \in (0,\infty)[\|\mathbf{x}-\mathbf{p}\| < \delta \implies \|\mathbf{f}(\mathbf{x})-\mathbf{L}\| < \varepsilon])$$

Proposition 2.1. for real vector functions limits areclosed under addition subtraction scalar multiplication dot product cross product norm

Chapter 3

Multivariate continuity

Proposition 3.1.

$$\mathbf{f} \text{ is continuous at } \mathbf{x}_0 \iff lim_{\mathbf{x}\to\mathbf{x}_0}\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0)$$

Proposition 3.2. Let $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$. $\mathbf{f}$ is continuous at $\mathbf{x}_0$ iff each component function $\mathbf{f}_i$ continuous at $\mathbf{x}_0$.

Proposition 3.3. for real vector functions continuity is closed under addition subtraction multiplication composition

Chapter 4

Multivariate differentiation

In real analysis, the derivative is a tool that describes the 'tangent' or 'best linear approximator' at a point of a function.

For multivariate Functions, there are several directions to approach domain elements by and the function may increase at different rates for different directions taken; tangents in one direction may not be tangents in another.

One idea is to only consider the derivative along a specific direction to a domain element; this is what *directional derivatives and partial derivatives* strive to do, and it will prove an adequate stating point for now.

4.1 Directional derivative

Partial derivatives can take the derivative with respect to any variable, however does there exist a way of taking the derivative of a multivariate function with respect to any arbitrary 'direction', rather than the directions dictated by the variables (usually axis')? If one represents their direction by $\mathbf{u}$, the *directional derivative* acts as a framework to take the derivative along $\mathbf{u}$.

We start by considering the derviatives at a point with respect to a specific direction specified by a unit vector. These are the *directional derivatives.*

Definition 4.1. Let $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ be a real vector function and $\hat{\mathbf{u}}$ be a unit vector, the *directional derivative of* $\mathbf{f}$ *at* $\mathbf{x}_0$ is defined as the following. It represents the derivative in the sense of real analysis along $\mathbf{u}$.

$$\nabla_{\mathbf{u}}\mathbf{f}(\mathbf{x}_0) = \lim_{h\to 0} \frac{\mathbf{f}(\mathbf{x}_0 + h\mathbf{u}) - \mathbf{f}(\mathbf{x}_0)}{h\|\mathbf{u}\|}$$

4.2 Partial derivative

A special case of directional derivatives is when we take the directional derivative by purely along a single variable. These are the *partial derivatives*; the derivative of a multivariate function with respect to a single variable.

Definition 4.2. The *partial derivative of* $\mathbf{f}$ *at* $\mathbf{x}_0$ *with respect to* $\mathbf{x}_k$ is defined as the following.

$$\mathbf{f}_{\mathbf{x}_k}(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_k \\ \vdots \\ \mathbf{x}_m \end{bmatrix}) = \lim_{h\to 0} \frac{\mathbf{f}(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_k + h \\ \vdots \\ \mathbf{x}_m \end{bmatrix}) - f(\begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_k \\ \vdots \\ \mathbf{x}_m \end{bmatrix})}{h}$$

This is essentially differentiation in the sense of real analysis where other variables behave as constants.

We've used the notation f_x to describe the partial derivative of f with respect to x, however most notations of analysis support for partial derivatives. *Leibniz notation* expresses partial derivatives in the following way.

$$\frac{\partial f}{\partial x}$$

$$\frac{\partial^2 f}{\partial x \partial y}$$

Lagrange notation expresses partial derivatives in the following way.

$$f_x$$

$$f_{xy}$$

Euler-Arbogast notation expresses partial derivatives in the following way.

$$\partial_x f$$

$$\partial_{xy} f$$

Newton's notation does not support partial derivatives.

4.3 The Jacobian matrix and Hessian matrix

Now that the notions of partial derivatives are accessible, we discuss 2 interesting matrixes; they will be useful tools later on.

Definition 4.3. The *Hessian matrix of* f is the matrix of second-order partial derivatives of a real multivariate function.

$$\mathbf{H}_f$$

$$(\mathbf{H}_f)_{ij} = \frac{\partial^2 f}{\partial \mathbf{x}_i \partial \mathbf{x}_j}$$

Definition 4.4. The *Jacobian matrix of* $\mathbf{f}$ is the matrix of first-order partial derivatives for each function within a real vector function.

$$\mathbf{J}_\mathbf{f}$$

$$(\mathbf{J}_\mathbf{f})_{ij} = \frac{\partial f_i}{\partial \mathbf{x}_j}$$

Chapter 5

Total derivative

Though a point may have directional derivatives defined for all directions, it is still possible for the function to be discontinuous at that point! Just because a function is differentiable and continuous through at a point on a certain path,

A more holistic approach that captures the geometry of multidimensional space is needed for deeper insight of differentiability of such functions.

In an informal sense, differentiable functions express the fact that a function can be approximated as a linear function on a small enough scale; if we 'zoom in' infinitely on the function, it is linear. Let's try and formulate this in the sense of real analysis.

$$f(x_0 + h) \approx f(x_0) + f'(x_0)h$$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) = 0$$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - f'(x_0)h}{h} = 0$$

$$\lim_{h \to 0} \frac{f(x_0 + h) - [f(x_0) + f'(x_0)h]}{h} = 0$$

This proves that differentiable functions are such that $f(x_0 + h)$ can be represented by a linear approximate $f(x_0) + f'(x_0)h + r(h)$, where $r(h)$ is some remainder function that is dominated by h in the sense that $\lim_{h\to 0} \frac{r(h)}{h} = 0$; it is 'negligible' in the limit.

Differentiation from this perspective is easier to generalize; we don't necessarily need to define differentiability by expressing a direct Newton quotient, but by specifying that some linear function *exists* that can approximate the function in this specific way. All we need to do now is generalize the idea of a real linear function from real analysis to real vector linear function!

Linear transforms are the generalization for higher dimension spaces, and these can be represented my matrixes; the analogue of out $f'(x_0)h$ will be $\mathbf{Jh}$, where $\mathbf{J}$ is some matrix that makes such a linear transform. In this vain, we can let our linear approximator be $\mathbf{f}(\mathbf{x}_0) + \mathbf{Jh}$.

Definition 5.1. A real vector function $\mathbf{f}$ is *(totally) differentiable* iff there exists some matrix $\mathbf{J}$ where the following holds. The matrix $\mathbf{J}$ is the *total derivative of* $\mathbf{f}$

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - (\mathbf{f}(\mathbf{x}_0) + \mathbf{Jh})\|}{\|\mathbf{h}\|} = 0$$

$$\mathbf{f} \text{ is differentiable at } \mathbf{x}_0 \iff \exists \mathbf{J}[\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - (\mathbf{f}(\mathbf{x}_0) + \mathbf{Jh})\|}{\|\mathbf{h}\|} = 0]$$

This is the ultimate definition for differentiability for any real vector function (I'll often drop the 'totally' part because I'm lazy). This condition is stronger than the notions of directional and partial derivatives; it entirely encapsulates the idea that the function can locally be approximated by a single linear transform.

We introduce a second equivalent totally differentiable function that mentions the use of a remainder function; this is extremely useful in proofs.

Proposition 5.1. A real vector function $\mathbf{f}$ is *(totally) differentiable* iff there exists some matrix $\mathbf{J}$ and real vector function $\mathbf{r}$ where the following holds.

$$\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) = \mathbf{f}(\mathbf{x}_0) + \mathbf{Jh} + \mathbf{r}(\mathbf{h})$$

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{r}(\mathbf{h})\|}{\|\mathbf{h}\|} = 0$$

We now delve into the properties of totally differentiable functions.

Proposition 5.2. Let $\mathbf{f}$ be differentiable at $\mathbf{x}_0$, then $\mathbf{f}$ is continuous at $\mathbf{x}_0$.

Proposition 5.3. Differentiable implies directional derivative eists in all directions.

The following lemma will be used prove a theorem that will tell us exactly how to calculate the total derivative, however the lemma also has some standalone value since it links the total derivative to the directional derivative.

Lemma 5.1. Let $\mathbf{f}$ be a function differentiable at $\mathbf{x}_0$ with total derivative $\mathbf{J}$, then $\nabla_{\mathbf{u}}\mathbf{f}(\mathbf{x}_0) = \mathbf{J}^i(\mathbf{x}_0)\mathbf{u}$, where $\mathbf{J}^i$ is the ith column of $\mathbf{J}$.

We now prove the following revelation.

Theorem 5.1. Let $\mathbf{f}$ be differentiable at $\mathbf{x}_0$, then the total derivative at $\mathbf{x}_0$ is the Jacobian matrix evaluated at $\mathbf{x}_0$ $\mathbf{J}_{\mathbf{f}}(\mathbf{x}_0)$

It was our Jacobain matrix all along; the Jacobian matrix represent this best linear transform at a point! If for a real function $f(x_0 + h) \approx f(x_0) + f'(x_0)h$, then for a real vector function we have $\mathbf{f}(\mathbf{x}_0)$!

Using this fact, we can prove the following.

Corollary 5.1.

$$\nabla_{\mathbf{u}}\mathbf{f} = \sum_{i=1}^{n} \mathbf{u}_i f_{\mathbf{x}_i}$$

We will have an even more elegant representation for this when we study differential operators.

5.0.1 Properties of Jacobian matrix

Now that we've discovered that the Jacobian matrix embodies the total derivative in a Euclidean space, it is fitting that we study this structure more closely.

Proposition 5.4.

$$\mathbf{J}_{\mathbf{f}+\mathbf{g}} = \mathbf{J}_{\mathbf{f}} + \mathbf{J}_{\mathbf{g}}$$

$$\mathbf{J}_{w\mathbf{f}} = w\mathbf{J}_{\mathbf{f}} + \mathbf{f}\mathbf{J}_{w}$$

5.1 Multivariate chain rule

We've already seen that composition of continuous functions are continuous in any space; however are compositions of differentiable functions differentiable? We've seen this to be true in real analysis, and the chain rule gives us a neat formula on how to compute derivatives with respect to the composed functions and their derivatives.

In higher dimensions the chain rule holds similarly in terms of the Jacobian matrix.

Theorem 5.2 (Multivariate chain rule)**.**

$$\mathbf{J}_{\mathbf{f}\circ\mathbf{g}} = \mathbf{J}_{\mathbf{f}}(\mathbf{g})\mathbf{J}_{\mathbf{g}}$$

We require a generalization of the chain rule for such compositions.

Proposition 5.5.

$$\frac{\partial \mathbf{f}}{\partial t} \sum \frac{\partial \mathbf{f}}{\partial \mathbf{x}_j} \frac{\partial \mathbf{x}_j}{\partial t}$$

5.2 Clairaut's theorem

Theorem 5.3 (Clairaut's theorem)**.**

This theorem has many names *Clairaut's theorem, Young's theorem, Schwarz' theorem,* or simply the *symmetry of second derivatives.*

5.3 Multivariate inverse function theorem

Theorem 5.4 (Multivariate inverse function theorem)**.**

$$\mathbf{J}_{\mathbf{f}^{-1}}(\mathbf{x}_0) = [\mathbf{J}_{\mathbf{f}}(\mathbf{x}_0)]^{-1}$$

5.3.1 Multivariate inverse function theorem

5.3.2 Multivariate implicit function theorem

5.4 Multivariate Taylor's theorem

5.4.1 Multivariate Taylor's theorem

5.4.2 Multivariate Taylor series

5.5 Derivative tests

5.5.1 Properties of the Hessian matrix

Proposition 5.6 (Multivariate derivative test)**.** If a real multivariate function f with continuous second derivatives has $\nabla f(\mathbf{x}_0) = \mathbf{0}$ and $\mathbf{H}_f(\mathbf{x}_0)$ is positive definite, then x_0 is a local minimum of f.

Proposition 5.7. Let f have continuous second-order partial derivatives on U. Then f is convex on U iff its Hessian matrix $\mathbf{H}_f$ is positive semi-definite.

Proposition 5.8. Let f have continuous second-order partial derivatives on U. If the Hessian matrix $\mathbf{H}_f$ is positive definite on U, then f is convex on U.

5.5.2 Lagrangian multipliers

Chapter 6

Multivariate integration

6.1 Multivariate integration

6.2 Fubini's theorem and Tonelli's theorem

6.2.1 Nested integration

The theorems of Fubini and Tonelli assist immensely in the calculation and interpretation of multivariate integrals. They provide two different sufficient conditions for when one can swap the order of integral nesting. It turns out we can do this for quite a rich class of functions, however proving these theorems requires us to interpret integration as Lebesgue integration rather than Riemann integration. If you are unfamiliar with the Lebesgue integral, this is covered at the end of Real Analysis.

6.2.2 Fubini's theorem

Theorem 6.1. Let f be Lebesgue integrable on $X \times Y$, then we have the following.

$$\iint_{X\times Y} f(x,y)d(x,y) = \int_X (\int_Y f(x,y)dy)dx = \int_Y (\int_X f(x,y)dx)dy$$

6.2.3 Tonelli's theorem

Though similar to Fubini's condition, Tonelli introduces a similar yet distinct condition that implies that nested integration can be swapped.

Theorem 6.2. Let f be a nonnegative measurable function on $X \times Y$, then we have the following.

$$\iint_{X\times Y} f(x,y)d(x,y) = \int_X (\int_Y f(x,y)dy)dx = \int_Y (\int_X f(x,y)dx)dy$$

6.3 Fubini's theorem and Tonelli's theorem

¡h3 class=cyan¿Vector-valued function¡/h3¿ ¡p¿Function $\mathbf{r} : X^n \to \mathbb{R}^m, m \geq 2$ that takes parameter vector or scalar as input and outputs a cartesian vector¡/p¿ ¡h4 class=cyan¿Vector field Campo vettoriale¡/h4¿ ¡p¿Vector-valued function $\mathbf{F} : X^n \to \mathbb{R}^n$ that assigns a cartesian vector with same dimension as the space, to each point in said space X^n¡/p¿ ¡p¿$\mathbf{F}(\mathbf{x})$¡/p¿ ¡h4 class=cyan¿Position function¡/h4¿ ¡p¿Vector valued function $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ that takes a scalar parameter t as input and outputs a cartesian vector, essentially representing a path in a space¡/p¿

¡h3 class=cyan¿Vector operators¡/h3¿ ¡h4 class=cyan¿Nabla symbol¡/h4¿ ¡p¿Differential operator ∇ used as a notation for vector operators and hints towards their methods of calculation¡/p¿ ¡p¿$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$¡/p¿ ¡h4 class=cyan¿Gradient Gradiente ¡/h4¿ ¡p¿Vector operator that ¡b¿returns the vector of maximum change of a point in a scalar field¡/b¿ f¡/p¿ ¡p¿$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$¡/p¿ ¡ul¿ ¡li¿$f$ is the scalar field¡/li¿ ¡/ul¿ ¡p¿See ¡a href='/daigaku/$2022_sum/m2.php' > Mathematics2 < /a >< /p >$

¡h3 class=cyan¿Divergence Divergenza ¡/h3¿ ¡p¿Vector operator that ¡b¿returns the scalar quantity of flow in and out of a point in a vector field¡/b¿ $\mathbf{F}$¡/p¿ ¡p¿To calculate based on intuition, look to the top, bottom, left and right of the point and note how regarding these adjacent vectors the point absorbs and emits¡/p¿ ¡p¿$\nabla \cdot \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$¡/p¿ ¡ul¿ ¡li¿$\mathbf{F}$ is the vector field¡/li¿ ¡li¿F_n is the vector field's n argument¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Sink¡/h4¿ ¡p¿Points in vector fields with more ¡b¿inward¡/b¿ flow¡/p¿ ¡p¿(x,y) is a sink $\iff \nabla \cdot f(x,y)0$¡/p¿ ¡h4 class=cyan¿Source¡/h4¿ ¡p¿Points in vector fields with more ¡b¿outward¡/b¿ flow¡/p¿ ¡p¿(x,y) is a source $\iff \nabla \cdot f(x,y)0$¡/p¿

¡h3 class=cyan¿Curl Rotore ¡/h3¿ ¡p¿Vector operator that ¡b¿returns the vector normal to the direction of counterclockwise rotation with its magnitude representing the intensity of the rotation at a point in vector field¡/b¿ $\mathbf{F}$¡/p¿ ¡p¿$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix}$¡/p¿ ¡ul¿ ¡li¿$\mathbf{F}$ is the vector field¡/li¿ ¡li¿F_n is the vector field's n argument¡/li¿ ¡/ul¿

¡h3 class=cyan¿Laplacian Laplaciano ¡/h3¿ ¡p¿Vector operator that ¡b¿returns the scalar quantity of 'curvature' at a point in the scalar field¡/b¿ f¡/p¿ ¡p¿This works by capturing the gradient of the scalar field and finding the divergence of this gradient at some point¡/p¿ ¡p¿$\nabla^2 f = \nabla \cdot (\nabla f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$¡/p¿

¡h3 class=cyan¿Coordinate system¡/h3¿ ¡p¿System of variables used to define a set of points in a space¡/p¿ ¡h4 class=cyan¿Point representation¡/h4¿ ¡p¿Each coordinate system has a set of equations to translate points in another system to said coordinate system¡/p¿ ¡h4 class=cyan¿Vector representation¡/h4¿ ¡p¿Each coordinate system has an orthonormal basis (frame) relative to a point in space that represents any vector from that point, e see ¡a href=/daigaku/2024$_aut/la.php > LinearAlgebra < /a >< /p >< p > Notethatsomeorthonormalbasisesmaybedependentonsome\theta$ or ϕ relational to the vector's base from the origin¡/p¿ ¡h4 class=cyan¿$\mathbb{R}^2$¡/h4¿ ¡ul¿ ¡li¿Cartesian¡/li¿ ¡li¿Polar¡/li¿ ¡li¿Parabolic¡/li¿ ¡li¿Bipolar¡/li¿ ¡li¿Elliptic¡/li¿ ¡/ul¿ ¡h4 class=cyan¿$\mathbb{R}^3$¡/h4¿ ¡ul¿ ¡li¿Cartesian¡/li¿ ¡li¿Cylindrical¡/li¿ ¡li¿Spherical¡/li¿ ¡/ul¿

¡h3 class=cyan¿Cartesian coordinates¡/h3¿ ¡p¿Ordered 3-tuple (x, y, z) that represents a point in a 3D space¡/p¿ ¡ul¿ ¡li¿$x \in \mathbb{R}$ is the translation along the x-axis¡/li¿ ¡li¿$y \in \mathbb{R}$ is the horizontal translation perpendicular to the x-axis¡/li¿ ¡li¿$z \in \mathbb{R}$ is the vertical translation¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Vector transform¡/h4¿ ¡p¿span$\{\hat{i}, \hat{j}, \hat{k}\} = \mathbb{R}^3$¡/p¿ ¡ul¿ ¡li¿$\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$¡/li¿ ¡li¿$\hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$¡/li¿ ¡li¿$\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Volume element¡/h4¿ ¡p¿$dV = dxdydz$¡/p¿

¡h3 class=cyan¿Polar coordinates¡/h3¿ ¡p¿Ordered pair (r, θ) that represents any point and vector in a 2D space¡/p¿ ¡ul¿ ¡li¿$r \in \mathbb{R}_+$ represents the modulus¡/li¿ ¡li¿$\theta \in [0, 2\pi]$ represents the azimuthal angle (angle from x to y)¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Point transfom¡/h4¿ ¡ul¿ ¡li¿$r = \sqrt{x^2 + y^2}$¡/li¿ ¡li¿$\theta = \arctan(\frac{y}{x})$¡/li¿ ¡/ul¿ ¡ul¿ ¡li¿$x = r\cos(\theta)$¡/li¿ ¡li¿$y = r\sin(\theta)$¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Vector transform¡/h4¿ ¡p¿For some chosen angle θ¡/p¿ ¡p¿span$\{\hat{r}, \hat{\theta}\} = \mathbb{R}^2$¡/p¿ ¡ul¿ ¡li¿$\hat{r} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$¡/li¿ ¡li¿$\hat{\theta} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$¡/li¿ ¡/ul¿ ¡p¿$P_{(r,\theta)\to(x,y)} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$¡/p¿ ¡h4 class=cyan¿Vector operators¡/h4¿ ¡ul¿ ¡li¿$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$¡/li¿ ¡li¿$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$¡/li¿ ¡li¿$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Area element¡/h4¿ ¡p¿$dA = rdrd\theta$¡/p¿

¡h3 class=cyan¿Cylindrical coordinates¡/h3¿ ¡p¿Ordered 3-tuple (r, θ, z) that represents any point and vector in a 3D space¡/p¿ ¡ul¿ ¡li¿$r \in \mathbb{R}_+$ represents the modulus projected along the x-y plane¡/li¿ ¡li¿$\theta \in [0, 2\pi]$ represents the azimuthal angle (horizontally from +x-axis to +y-axis)¡/li¿ ¡li¿$z \in \mathbb{R}$ represents vertical translation¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Point transform¡/h4¿ ¡ul¿ ¡li¿$r = \sqrt{x^2 + y^2}$¡/li¿ ¡li¿$\theta = \arctan(\frac{y}{x})$¡/li¿ ¡li¿$z = z$¡/li¿ ¡/ul¿ ¡ul¿ ¡li¿$x = r\cos(\theta)$¡/li¿ ¡li¿$y = r\sin(\theta)$¡/li¿ ¡li¿$z = z$¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Vector transform¡/h4¿ ¡p¿For some chosen angle θ¡/p¿ ¡p¿span$\{\hat{r}, \hat{\theta}, \hat{z}\} = \mathbb{R}^3$¡/p¿ ¡ul¿ ¡li¿$\hat{r} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix}$¡/li¿ ¡li¿$\hat{\theta} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix}$¡/li¿ ¡li¿$\hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$¡/li¿ ¡/ul¿ ¡p¿$P_{(r,\theta,z)\to(x,y,z)} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$¡/p¿ ¡h4 class=cyan¿Vector operators¡/h4¿ ¡ul¿ ¡li¿$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$¡/li¿ ¡li¿$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$¡/li¿ ¡li¿$\nabla \times \mathbf{F} = (\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z})\hat{r} + (\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r})\hat{\theta} + \frac{1}{r}(\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta})\hat{z}$¡/li¿ ¡li¿$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Volume element¡/h4¿ ¡p¿$dV = rdrd\theta dz$¡/p¿

¡h3 class=cyan¿Spherical coordinates¡/h3¿ ¡p¿Ordered 3-tuple (ρ, θ, ϕ) that represents any point and vector in a 3D space¡/p¿ ¡ul¿ ¡li¿$\rho \in \mathbb{R}_+$ represents the modulus¡/li¿ ¡li¿$\theta \in [0, \pi]$ represents the inclination angle (vertically from +z-axis to x-y-plane)¡/li¿ ¡li¿$\phi \in [0, 2\pi]$ represents the azimuthal angle (horizontally from +x-axis to +y-axis)¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Point transform¡/h4¿ ¡ul¿ ¡li¿$\rho = \sqrt{x^2 + y^2 + z^2}$¡/li¿ ¡li¿$\theta = \arccos(\frac{z}{\rho})$¡/li¿ ¡li¿$\phi =$

$\arctan(\frac{y}{x})$¡/li¿ ¡/ul¿ ¡ul¿ ¡li¿$x = \rho\sin(\theta)\cos(\phi)$¡/li¿ ¡li¿$y = \rho\sin(\theta)\sin(\phi)$¡/li¿ ¡li¿$z = \rho\cos(\theta)$¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Vector transform¡/h4¿ ¡p¿$\forall\theta,\phi \in [0,2\pi], \operatorname{span}\{\hat{\rho},\hat{\theta},\hat{\varphi}\} = \mathbb{R}^3$¡/p¿ ¡ul¿ ¡li¿$\hat{\rho} = \begin{pmatrix}\sin(\theta)\cos(\phi)\\ \sin(\theta)\sin(\phi)\\ \cos(\theta)\end{pmatrix}$¡/li¿ ¡li¿$\hat{\theta} = \begin{pmatrix}\cos(\theta)\cos(\varphi)\\ \cos(\theta)\sin(\varphi)\\ -\sin(\theta)\end{pmatrix}$¡/li¿ ¡li¿$\hat{\varphi} = \begin{pmatrix}-\sin(\varphi)\\ \cos(\varphi)\\ 0\end{pmatrix}$¡/li¿ ¡/ul¿ ¡p¿$P_{(\rho,\theta,\phi)\to(x,y,z)} = \begin{bmatrix}\sin(\theta)\cos(\phi) & \cos(\theta)\cos(\phi) & -\sin(\phi)\\ \sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & \cos(\phi)\\ \cos(\theta) & -\sin(\theta) & 0\end{bmatrix}$¡/p¿ ¡h4 class=cyan¿Vector operators¡/h4¿ ¡li¿$\nabla f = \frac{\partial f}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{\rho\sin(\theta)}\frac{\partial f}{\partial\varphi}\hat{\varphi}$¡/li¿ ¡li¿$\nabla\cdot\mathbf{F} = \frac{1}{\rho^2}\frac{\partial(\rho^2 F_\rho)}{\partial\rho} + \frac{1}{\rho\sin(\theta)}\frac{\partial(\sin(\theta)F_\theta)}{\partial\theta} + \frac{1}{\rho\sin(\theta)}\frac{\partial F_\varphi}{\partial\varphi}$¡/li¿ ¡li¿$\nabla\times\mathbf{F} = \frac{1}{\rho\sin(\theta)}(\frac{\partial(\sin(\theta)F_\varphi)}{\partial\theta} - \frac{\partial F_\theta}{\partial\varphi})\hat{\rho} + \frac{1}{\rho}(\frac{1}{\sin(\theta)}\frac{\partial F_\rho}{\partial\varphi} - \frac{\partial(\rho F_\varphi)}{\partial\rho})\hat{\theta} + \frac{1}{\rho}(\frac{\partial(\rho F_\theta)}{\partial\rho} - \frac{\partial F_\rho}{\partial\theta})\hat{\varphi}$¡/li¿ ¡li¿$\nabla^2 = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2\frac{\partial f}{\partial\rho}) + \frac{1}{\rho^2\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)\frac{\partial f}{\partial\theta}) + \frac{1}{\rho^2\sin^2(\theta)}\frac{\partial^2 f}{\partial\phi}$¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Volume element¡/h4¿ ¡p¿$dV = \rho^2\sin(\theta)d\rho d\theta d\phi$¡/p¿

Part II

Differentiable curves and surfaces

Chapter 7

Differentiable curves

7.1 Differentiable curve

We will define differentiable curves by means of *position functions*; functions that represent one's position in a space.

Definition 7.1. A *position function* is a vector valued function $\mathbf{r} : I \to \mathbb{X}^n$ that maps a real interval I to some space X^n, essentially representing a path in a space. The argument of a position function will be referred to as *time* since the parameter often represents time.

As one could imagine, position functions are crucial in physics as physicists are often concerned with how forces modify the 'position' of objects.

As intuition may suggest, this function is 'nice' when it is continuous (the position doesn't magically 'teleport') and differentiable (we can calculate how much the direction of this position is changing with respect to time). Thus we arrive at the idea of a *differentiable curve.*

Definition 7.2. A *differentiable curve* is a vector valued function $\boldsymbol{\gamma} : I \to \mathbb{R}^n$ that maps a real interval I to some space X^n, essentially representing a path in a space.

$$\boldsymbol{\gamma}'(t_0) = \lim_{t \to t_0} \frac{\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(t_0)}{t - t_0}$$

Definition 7.3. A *differentiable curve* is a position function $\mathbf{r} : I \to X$ that is totally differentiable. The image of this function is also called the differentiable curve. X is some space, technically a differentiable manifold.

$$\mathrm{r} : [t_0, t_1] \to X \text{ is closed} \iff \mathbf{r}(t_0) = \mathbf{r}(t_1)$$

A curve that is continuous but not differentiable is known as a *topological curve*; this is a notion studied further in general topology.

7.2 Closed differentiable curve

7.3 Line integral (scalar field)

Definition 7.4. Let $\mathbf{r} : [t_0, t_1] \to \mathcal{C}$ be a differentiable curve. The *arc length function of* $\mathbf{r}$ is the function $s : [t_0, t_1] \to [0, \infty)$ that returns the length of the differentiable curve at a given time.

$$s_{\mathbf{r}}(t) = \int_{t_0}^{t} \|\mathbf{r}'(t')\| dt'$$

Definition 7.5.

Definition 7.6. A *line integral (scalar field)* is an integral of a scalar field taken along a differentiable curve. One can think about it as curve's arc length the scaled by the scalar field's intensity along it.

$$\int_{\mathcal{C}} f(\mathbf{x}) ds = \int_{t_0}^{t_1} f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt$$

- $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar field
- $\mathbf{r} : [t_0, t_1] \to \mathcal{C}$ is a differentiable curve parametrization

The reparametrization from s to t is from the reverse chain rule.

When representing a line integral along a closed curve, we will use the integral sign $\oint$ instead of $\int$

Definition 7.7. A *line integral (vector field)* is an integral of a vector field taken along a differentiable curve. One can think about it as curve's arc length the scaled by the curve's dot product with the vector field's vectors along it. It is the 'work' of a vector field along that curve.

$$\int_{\mathcal{C}} \mathbf{f}(\mathbf{x}) \cdot d\mathbf{r} = \int_{\mathbf{r}(t_0)}^{\mathbf{r}(t_1)} \mathbf{f}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

- $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}$ is a vector field
- $\mathbf{r} : [t_0, t_1] \to \mathcal{C}$ is a differentiable curve parametrization

Chapter 8

Differentiable surfaces

8.1 Differentiable surface

8.2 Various parametrization

8.3 Surface integral

8.4 Flux integral

Part III

Frames

Chapter 9

Frames of coordinate systems

Readers will be familiar with coordinate systems.

Coordinate frame is a Basis that formed based on some arbitrary 'origin' coordinate. It encapsulates the idea For instance, at some coordinate p, one may want a frame representing the directions of the increase of each coordinate (cartesian, cylindrical, spherical frames), or perhaps one wants a frame with directions along a curve, orthogonal to the bend of a curve, and another direction orthogonal to both of those (Frenet-Serret frame).

We first study the former; frames of coordinate systems.

Note that although many frames we use are orthonormal basis', but this is not a requirement; we'll see examples of these much later on.

coordinate frame

Definition 9.1. Let x be a coordinate system and $\mathbf{r}$ map the coordinate system to S, then the *orthonormal coordinate frame with respect to* p is the orthonormal basis of vectors defined by the set of $\mathbf{e}_x(p) = \frac{\mathbf{r}_x(p)}{\|\mathbf{r}_x(p)\|}$ for each coordinate.

scale factor

divide by scale factor to make normalized coordinate frame

Cartesian frame

$$(x, y, z)$$

$$\mathbf{r}(x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathcal{F} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \} = \{\mathbf{ij}, \mathbf{k}\}$$

This is a *fixed frame*; the frame is the same relative to any coordinate point. Often students are confused when learning other coordinate frames because the cartesian coordinate frame is fixes, and they expect all others to be fixed too, but the fact that we have nonfixed frames is precisely why we like to deal with them.

Cylindrical frame

$$(r, \theta, z)$$

$$\mathbf{r}(r, \theta, z) = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \\ z \end{bmatrix}$$

$$\mathcal{F} = \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\} = \{ \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}, \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$$

Spherical frame

$$(\rho, \theta, \phi)$$

$$\mathbf{r}(\rho, \theta, \phi) = \begin{bmatrix} \rho\sin(\phi)\cos(\theta) \\ \rho\sin(\phi)\sin(\theta) \\ \rho\cos(\phi) \end{bmatrix}$$

$$\mathcal{F} = \{\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\phi\} = \{ \begin{bmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\phi) \end{bmatrix}, \begin{bmatrix} \cos(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta) \\ -\sin(\phi) \end{bmatrix}, \begin{bmatrix} \cos(\phi) \\ -\sin(\phi) \\ 0 \end{bmatrix} \}$$

9.1 Applications for line integrals

9.2 Applications for surface and flux integrals

Chapter 10

Frenet-Serret frame

Definition 10.1. Let $\mathbf{r}$ be a differentiable curve, the *Frenet-Serret frame of* $\mathbf{r}$ is an orthonormal basis spanning $\mathbb{R}^3$ relative to a a point on the differentiable curve $\mathbf{r}(t)$ with arclength s.

$$\mathbf{T}(s) = \mathbf{r}'(s)$$

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}$$

$$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$$

$\mathbf{T}$ (tangent) direction tangent to the differentiable curve $\mathbf{N}$ (normal) direction the differentiable curve is turning into against its tangent $\mathbf{B}$ (binormal) direction orthogonal to $\mathbf{T}, \mathbf{N}$

The chain rule can be used to express the Frenet-Serret frame with respect to 'time' (it's argument).

Definition 10.2. The *curvature of a differentiable curve* is the measure of directional change on the tangential plane.

$$\kappa(s) = \|\mathbf{T}'(s)\|$$

Definition 10.3. The *torsion of a differentiable curve* is the measure of directional change out of the tangential plane.

$$\tau(s) = \|\mathbf{B}'(s)\|$$

Theorem 10.1 (Frenet-Serret formulae)**.**

$$\mathbf{T}' = \kappa\mathbf{N}$$

$$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$$

$$\mathbf{B}' = -\tau\mathbf{N}$$

¡h3 class=cyan¿Frenet-Serret frame¡/h3¿ ¡p¿Orthonormal basis spanning $\mathbb{R}^3$ derived from a point in a path function $\mathbf{r}(t)$ with arclength s.¡/p¿ ¡ul¿ ¡li¿¡b¿Tangent¡/b¿, unit vector with direction tangent to the path at the point¡/li¿ ¡li¿¡b¿Normal¡/b¿, unit vector with direction normal to the path at the point (differentiation of tangent with respect to arclength)¡/li¿ ¡li¿¡b¿Binormal¡/b¿, unit vectot with direction normal to the two other vectors¡/li¿ ¡/ul¿ ¡ul¿ ¡li¿$\mathbf{T}(s) = \mathbf{r}'(s)$¡/li¿ ¡li¿$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}$¡/li¿ ¡li¿$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$¡/li¿ ¡/ul¿ ¡p¿The chain rule can be used to represent these basis functions with respect to time¡/p¿ ¡!– ¡h4 class=cyan¿Position-dependent definition¡/h4¿ ¡ul¿ ¡li¿$\widehat{\mathbf{T}}(t) = \frac{\frac{d\mathbf{r}}{dt}}{\|\frac{d\mathbf{r}}{dt}\|}$¡/li¿ ¡li¿$\widehat{\mathbf{N}}(t) = \frac{\frac{d\widehat{\mathbf{T}}}{dt}}{\|\frac{d\widehat{\mathbf{T}}}{dt}\|}$¡/li¿ ¡li¿$\widehat{\mathbf{B}}(t) = \widehat{\mathbf{T}} \times \widehat{\mathbf{N}}$¡/li¿ ¡/ul¿

¡h3 class=cyan¿Curvature¡/h3¿ ¡p¿Measure of directional change on the tangential plane¡/p¿ ¡p¿$\kappa = \|\mathbf{T}'\|$¡/p¿ ¡p¿$\kappa = \mathbf{T}'\cdot\mathbf{N}$¡/p¿ ¡h4 class=cyan¿Propositions¡/h4¿ ¡p¿$\kappa = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3}$¡/p¿

¡h3 class=cyan¿Torsion¡/h3¿ ¡p¿Mesure of directional change out of the tangential plane¡/p¿ ¡p¿$\tau = \|\mathbf{B}'\|$¡/p¿ ¡p¿$\tau = -\mathbf{B}'\cdot\mathbf{N}$¡/p¿ ¡h4 class=cyan¿Propositions¡/h4¿ ¡p¿$\kappa = \frac{\mathbf{r}'\cdot(\mathbf{r}''\times\mathbf{r}''')}{\|\mathbf{r}'\times\mathbf{r}''\|^2}$¡/p¿

¡h3 class=cyan¿Frenet-Serret formulae¡/h3¿ ¡ul¿ ¡li¿$\mathbf{T}' = \kappa\mathbf{N}$¡/li¿ ¡li¿$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$¡/li¿ ¡li¿$\mathbf{B}' = -\tau\mathbf{N}$¡/li¿ ¡/ul¿

¡h3 class=cyan¿Fundamental planes¡/h3¿ ¡ul¿ ¡li¿Osculating plane span$\{\mathbf{T},\mathbf{N}\}$¡/li¿ ¡li¿Rectifying plane span$\{\mathbf{T},\mathbf{B}\}$¡/li¿ ¡li¿Normal plane span$\{\mathbf{N},\mathbf{B}\}$¡/li¿ ¡/ul¿

Part IV

Multivariate and vector differential operators

Chapter 11

Nabla symbol

∇ is pronounced 'nabla'; it is a symbol used in denoting many multivariate and vector differential operators. It can be given the following interpretation. This differential operator can be paired with notations of linear algebra (done to hint towards the similar calculations of differential operators and some linear maps of linear algebra) to create new differential operators.

Chapter 12

Multivariate differential operators

Differential operators on scalar fields

12.1 Gradient operator

Fundamentally, the gradient operator on f

Definition 12.1. The *Gradient operator* is a multivariate differential operator ∇ such that ∇f is the unique vector whose dot product with $\mathbf{u}$ gives the directional derivative along $\mathbf{u}$.

Proposition 12.1 (Gradient operator of $\mathbb{R}^n$)**.**

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial \mathbf{x}_1} \\ \frac{\partial f}{\partial \mathbf{x}_2} \\ \vdots \\ \frac{\partial f}{\partial \mathbf{x}_n} \end{bmatrix}$$

$$\nabla f_i = \frac{\partial f}{\partial \mathbf{x}_i}$$

The idea behind the gradient operator is that it direction is towards the largest change in the function. Since partial derivatives calculate the amount of change along each axis, it is the linear combination of the increases along each axis; forming the direction of total change.

It also allows for an elegeant representation of the directional derivative.

Proposition 12.2.

$$f'_{\mathbf{u}(\mathbf{x})=\frac{\mathbf{u}}{\|\mathbf{u}\|}\cdot\nabla f(\mathbf{x})}$$

The gradient operator is perhaps one of the most fundamental in vector analysis, and it is present in the *gradient theorem*; a fundamental theorem of calculus for vector fields.

12.2 Laplace operator

Definition 12.2. The *Laplace operator* or *Laplacian* is a multivariate differential operator ∇^2 (or sometimes written Δ) defined as the following.

$$\nabla^2 f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial \mathbf{x}_i^2}$$

The vector differential operator called the *divergence operator* allows for a particularly neat definition for the Laplacian.

$$\nabla^2 f = \nabla \cdot (\nabla f)$$

The $\nabla\cdot$ is this divergence operator; we will now commence our studies of such vector differential operators.

Chapter 13

Vector differential operators

Differential operators on vector fields

13.1 Divergence operator

- general def - integral def

$$\nabla \cdot \mathbf{F} = \sum_{i=1}^{n} \frac{\partial f}{\partial \mathbf{x}_i}$$

$$\nabla \cdot \mathbf{F} = \lim_{\Delta V \to 0} \iint_S \mathbf{F} \cdot d\mathbf{S}$$

- Gauss' theorem ¡h3 class=cyan¿Gauss' theorem Teorema di Gauss ¡/h3¿ ¡p¿Theorem asserting that the divergence of all points in a volume equals the flux integral of the volume's closed surface.¡/p¿ ¡p¿ Intuitively, this is because by thinking of the divergence of each infinitesimal volume element bounded by the surface, each infinitesimal volume elements has its flux 'cancelled out' by adjacent volume elements¡/p¿ ¡p¿$\iiint_V (\nabla \cdot \mathbf{F}) dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$¡/p¿ ¡ul¿ ¡li¿$S$ is the closed differentiable surface¡/li¿ ¡li¿$\mathbf{F}$ is the vector field¡/li¿ ¡/ul¿

13.2 Curl operator

- general def

$$\nabla \times \mathbf{F} = \hat{n} \lim_{\Delta s \to 0} \frac{1}{\Delta s} \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

¡li¿$\hat{n}$ is the unit vector normal to the direction of counterclockwise rotation¡/li¿ ¡li¿$\mathcal{C}$ is a piecewise smooth (differentiable) closed (end point is the same as start point) curve¡/li¿ ¡li¿Δs is area bound by $\mathcal{C}$ (this equation is happens when the curve is infinitely tight)¡/li¿

- basic properties

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{f}) = 0$$

Chapter 14

Differential operators in different frames

¡h4 class=cyan¿Vector operators¡/h4¿ ¡ul¿ ¡li¿$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta}$¡/li¿ ¡li¿$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$¡/li¿ ¡li¿$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$¡/li¿ ¡h4 class=cyan¿Vector operators¡/h4¿ ¡ul¿ ¡li¿$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$¡/li¿ ¡li¿$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$¡/li¿ ¡li¿$\nabla \times \mathbf{F} = (\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z})\hat{r} + (\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r})\hat{\theta} + \frac{1}{r}(\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta})\hat{z}$¡/li¿ ¡li¿$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$¡/li¿ ¡/ul¿

¡h4 class=cyan¿Vector operators¡/h4¿

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{\rho \sin(\theta)}\frac{\partial f}{\partial \varphi}\hat{\varphi}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial (\rho^2 F_\rho)}{\partial \rho} + \frac{1}{\rho \sin(\theta)}\frac{\partial (\sin(\theta) F_\theta)}{\partial \theta} + \frac{1}{\rho \sin(\theta)}\frac{\partial F_\varphi}{\partial \varphi}$$

$$\nabla \times \mathbf{F} = \frac{1}{\rho \sin(\theta)}(\frac{\partial (\sin(\theta) F_\varphi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \varphi})\hat{\rho} + \frac{1}{\rho}(\frac{1}{\sin(\theta)}\frac{\partial F_\rho}{\partial \varphi} - \frac{\partial (\rho F_\varphi)}{\partial \rho})\hat{\theta} + \frac{1}{\rho}(\frac{\partial (\rho F_\theta)}{\partial \rho} - \frac{\partial F_\rho}{\partial \theta})\hat{\varphi}$$

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial f}{\partial \rho}) + \frac{1}{\rho^2 \sin(\theta)}\frac{\partial}{\partial \theta}(\sin(\theta)\frac{\partial f}{\partial \theta}) + \frac{1}{\rho^2 \sin^2(\theta)}\frac{\partial^2 f}{\partial \phi}$$

¡/ul¿ ¡h4 class=cyan¿Volume element¡/h4¿ ¡p¿$dV = \rho^2 \sin(\theta) d\rho d\theta d\phi$¡/p¿

Chapter 15

Integral representation

Chapter 16

Introduction to the integral theorems

Theorem 16.1 (Gradient theorem).

$$\int_{\mathbf{r}_0}^{\mathbf{r}_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_1) - f(\mathbf{r}_0)$$

This essentialy means that vector fields expressible as the gradient of some scalar field have path-independent integrals; all paths between the same two points return the same integral! This leads us to the idea of a *conservative field*; vector fields where the integral between two points is 'conserved' by any path between said points.

¡h3 class=cyan¿Conservative field ¡/h3¿ ¡p¿Vector field such that the line integral result is ¡b¿ purely dependent on the endpoints¡/b¿ of the line ¡/p¿ ¡p¿$\mathbf{F}$ is conservative $\iff$ ¡/p¿ ¡ul¿ ¡li¿$\exists f : \nabla f = \mathbf{F}$¡/li¿ ¡li¿$\nabla \times \mathbf{F} = \mathbf{0}$¡/li¿ ¡li¿$\mathcal{C}$ is closed $\implies \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Potential function¡/h4¿ ¡p¿f is the potential function of $\mathbf{F} \iff \nabla f = \mathbf{F}$¡/p¿

- Stoke's theorem ¡h3 class=cyan¿Stoke's theorem¡/h3¿ ¡p¿Theorem asserting that the curl of all points on an open surface equals the line integral of the open surface's edge.¡/p¿ ¡p¿ Intuitively, this is because by thinking of the curl of each infinitesimal surface element, each infinitesimal volume elements has its curl 'cancelled out' by adjacent surface elements¡/p¿ ¡p¿$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$¡/p¿ ¡ul¿ ¡li¿$S$ is the open differentiable surface¡/li¿ ¡li¿$\mathcal{C}$ is the closed differentiable curve representing the edge of S, oriented counterclockwise the surface's normal¡/li¿ ¡li¿$\mathbf{r}(t) : [t_0, t_1] \to \mathcal{C}$ is the position function along $\mathcal{C}$¡/li¿ ¡li¿$\mathbf{F}$ is the vector field¡/li¿ ¡/ul¿

- Green's theorem

¡h3 class=cyan¿Green's theorem¡/h3¿ ¡p¿Corrolary of Stoke's theorem, form of Stoke's theorem of a function projected in the xy-plane.¡/p¿ ¡p¿$\iint_S (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy = \oint_{\partial \mathcal{S}} P(x,y)dx + Q(x,y)dy$¡/p¿

¡h3 class=cyan¿Vector operators¡/h3¿ ¡h4 class=cyan¿Nabla symbol¡/h4¿ ¡p¿Differential operator ∇ used as a notation for vector operators and hints towards their methods of calculation¡/p¿ ¡p¿$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$ ¡/p¿ ¡h4 class=cyan¿Gradient Gradiente ¡/h4¿ ¡p¿Vector operator that ¡b¿returns the vector of maximum change of a point in a scalar field¡/b¿ f¡/p¿ ¡p¿$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$¡/p¿ ¡ul¿ ¡li¿$f$ is the scalar field¡/li¿ ¡/ul¿ ¡p¿See ¡a href='/daigaku/2022$_{s}um/m2.php' > Mathematics2 < /a >< /p >$

¡h3 class=cyan¿Divergence Divergenza ¡/h3¿ ¡p¿Vector operator that ¡b¿returns the scalar quantity of flow in and out of a point in a vector field¡/b¿ $\mathbf{F}$¡/p¿ ¡p¿To calculate based on intuition, look to the top, bottom, left and right of the point and note how regarding these adjacent vectors the point absorbs and emits¡/p¿ ¡p¿$\nabla \cdot \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$¡/p¿ ¡ul¿ ¡li¿$\mathbf{F}$ is the vector field¡/li¿ ¡li¿F_n is the vector field's n argument¡/li¿ ¡/ul¿ ¡h4 class=cyan¿Sink¡/h4¿ ¡p¿Points in vector fields with more ¡b¿inward¡/b¿ flow¡/p¿ ¡p¿(x,y) is a sink $\iff \nabla\cdot f(x,y)0$¡/p¿ ¡h4 class=cyan¿Source¡/h4¿ ¡p¿Points in vector fields with more ¡b¿outward¡/b¿ flow¡/p¿ ¡p¿(x,y) is a source $\iff \nabla \cdot f(x,y)0$¡/p¿

¡h3 class=cyan¿Curl Rotore ¡/h3¿ ¡p¿Vector operator that ¡b¿returns the vector normal to the direction of counterclockwise rotation with its magnitude representing the intensity of the rotation at a point in vector field¡/b¿ $\mathbf{F}$¡/p¿ ¡p¿$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix}$¡/p¿ ¡ul¿ ¡li¿$\mathbf{F}$ is the vector field¡/li¿ ¡li¿F_n is the vector field's n argument¡/li¿ ¡/ul¿

¡h3 class=cyan¿Laplacian Laplaciano ¡/h3¿ ¡p¿Vector operator that ¡b¿returns the scalar quantity of 'curvature' at a point in the scalar field¡/b¿ f¡/p¿ ¡p¿This works by capturing the gradient of the scalar field and finding the divergence of this gradient at some point¡/p¿ ¡p¿$\nabla^2 f = \nabla \cdot (\nabla f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$¡/p¿

Chapter 17

That's all folks

This book has discussed ways of how ideas in real analysis are extended into higher Euclidean spaces and how this is assisted by the means of linear algebra.