Classical Mechanics

1 Introduction

Classical mechanics is important as it gives the foundation for most of physics. The theory, based on **Newton's laws of motion**, provides essentially an **exact** description of almost all **macroscopic** phenomena. The theory requires modification for

1. *microscopic systems*, e.g. atoms, molecules, nuclei - use **quantum mechanics**
2. *particles travelling at speeds close to the speed of light* - use **relativistic mechanics**

These other theories must reduce to classical mechanics in the limit of large bodies travelling at speed much less than the speed of light.

The subject is usually divided into

1. **statics** - systems at rest and in *equilibrium*,
2. **kinematics** - systems in motion, often accelerating. Concerned here with general relationships, e.g. $\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$, (Newton's second law, without specifying the details of the force.)
3. **dynamics** - details of the force law are specified, e.g. gravitational force, force due to a stretched spring.

2 Newton's laws of motion

These were formulated in his book *Principia Mathematica* in 1687. They are the basis of all classical mechanics.

1. A body remains at rest or in a state of uniform motion (non-accelerating) unless acted on by an **external** force.
2. Force = time rate of change of momentum, i.e.

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \tag{1}$$

where $\mathbf{p} = m\mathbf{v}$ = momentum of body of mass m moving with velocity $\mathbf{v}$. If m is constant then

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{a}, \tag{2}$$

with acceleration, $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$.

3. To every **force** (**action**) there is an equal but opposite **reaction**.

These laws are only true in an **inertial** (non-accelerating) **frame of reference**. We shall discuss later how we can treat motion relative to an accelerating frame of reference.

From the second law, if $\mathbf{F} = 0$, then the acceleration $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0$, so the velocity, $\mathbf{v}$, is constant. Thus first law is special case of the second law.

We can also derive the third law from the second law as follows. Apply a force $\mathbf{F}$ to body 1. Body 1 pushes on body 2 with force $\mathbf{F}_2$ and body 2 pushes back on body 1 with force $\mathbf{F}_1$ as shown in fig 1.

Applying Newton's second law, for the combined system,

$$\mathbf{F} = (m_1 + m_2)\,\mathbf{a}. \tag{3}$$

For body 1,

$$\mathbf{F} + \mathbf{F}_1 = m_1\mathbf{a}, \tag{4}$$

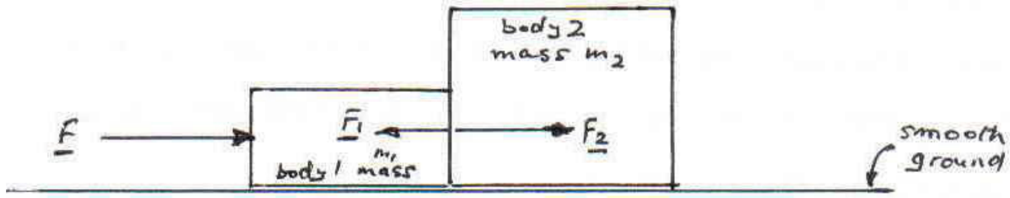

Figure 1: Demonstration of Newton's Third Law

for body 2,

$$\mathbf{F}_2 = m_2\mathbf{a}. \tag{5}$$

Adding

$$\mathbf{F} + \mathbf{F}_1 + \mathbf{F}_2 = (m_1 + m_2)\,\mathbf{a} = \mathbf{F} \tag{6}$$

$$\mathbf{F}_1 + \mathbf{F}_2 = 0 \tag{7}$$

and hence

$$\mathbf{F}_1 = -\mathbf{F}_2. \tag{8}$$

So that forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal in magnitude but opposite in direction.

3 Scalars and Vectors

There are two main types of variables in mechanics.

Scalar – has only magnitude, e.g. mass, energy, speed.

Vector – has magnitude and direction, e.g. position, velocity, acceleration, force.

A vector may be represented graphically by a directed line segment. The length of the line represents the magnitude of the vector, the direction of the line shows the direction of the vector. In printed text vectors are often written in **bold** type, and will be underlined in hand written text. A certain knowledge of vectors is assumed, in particular their Cartesian form.

Consider Newton's second law,

$$\mathbf{F} = m\mathbf{a}. \tag{9}$$

This is a vector equation which means that not only is the magnitudes on both sides of the equation equal, but the direction of the acceleration $\mathbf{a}$ is the same as the force, $\mathbf{F}$. If we express $\mathbf{F}$ and $\mathbf{a}$ in Cartesian form,

$$\mathbf{F} = F_x + F_y + F_z \tag{10}$$

$$\mathbf{a} = a_x + a_y + a_z, \tag{11}$$

then

$$F_x = ma_x = m\frac{\mathrm{d}v_x}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \tag{12}$$

$$F_y = ma_y = m\frac{\mathrm{d}v_y}{\mathrm{d}t} = m\frac{\mathrm{d}^2y}{\mathrm{d}t^2}, \tag{13}$$

$$F_z = ma_z = m\frac{\mathrm{d}v_z}{\mathrm{d}t} = m\frac{\mathrm{d}^2z}{\mathrm{d}t^2}. \tag{14}$$

4 Units and Dimensions

In order to discuss concepts such as velocity, force energy, etc. we must introduce a standard set of **units** of the fundamental variables or dimensions. The fundamental variables are mass $[M]$, length $[L]$ and time $[T]$. The most widely used set of units for these variables is the Système International (S.I.).

The base units of the S.I. system are

variable	**unit name**	**abbreviation**
mass $[M]$	kilogram	kg
length $[L]$	metre	m
time $[T]$	second	s

From these base units we can obtain derived units for other variables, e.g. speed = distance/time, with dimensions of speed $[v] = [L] / [T]$ and the S.I. unit is m/s, or ms^{-1}. For force = mass $\times$ acceleration the dimensions are $[F] = [m]\,[L]\,[T]^{-2}$ and the unit is $\mathrm{kg\,m\,s}^{-2}$ = Newton (N). Units have dimensions and **all** equations in physics must be **dimensionally homogeneous**; i.e. both sides and each term of the equation must have the same dimensions. As an example,

1. distance travelled in time t at constant speed v is $s = vt$. The dimension are

$$[L] = [L]\,[T]^{-1}\ [T] = [L]\,. \tag{15}$$

2. distance travelled in time t under a constant acceleration a from an initial speed u is

$$s = ut + \frac{1}{2}at^2. \tag{16}$$

The dimensions are

$$\begin{aligned} [L] &= [L]\,[T]^{-1}\ [T] + [L]\,[T]^{-2}\ [T]^2 && (17)\\ [L] &= [L] + [L]\,. && (18) \end{aligned}$$

Dimensional analysis of equations provides a very useful check on the correctness of algebraic expressions. However it does not give information about dimensionless constants, such as the $\frac{1}{2}$ in equation (16) above.

5 Time rate of change of vectors

Newton's second law, $\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$, involves the time rate of change of the velocity vector, $\mathbf{v}$. Consider

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}. \tag{19}$$

Change of position of particle in time interval Δt is $\Delta\mathbf{r}$, so velocity (see fig 2

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}. \tag{20}$$

In Cartesian coordinates

$$\begin{aligned} \mathbf{r} &= x\widehat{\mathbf{i}} + y\widehat{\mathbf{j}} + z\widehat{\mathbf{k}}, && (21)\\ \mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\widehat{\mathbf{i}} + \frac{\mathrm{d}y}{\mathrm{d}t}\widehat{\mathbf{j}} + \frac{\mathrm{d}z}{\mathrm{d}t}\widehat{\mathbf{k}}, && (22)\\ &= v_x\widehat{\mathbf{i}} + v_y\widehat{\mathbf{j}} + v_z\widehat{\mathbf{k}}. && (23) \end{aligned}$$

The speed of the particle is

$$v = |\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_x^2 + v_y^2 + v_z^2}. \tag{24}$$

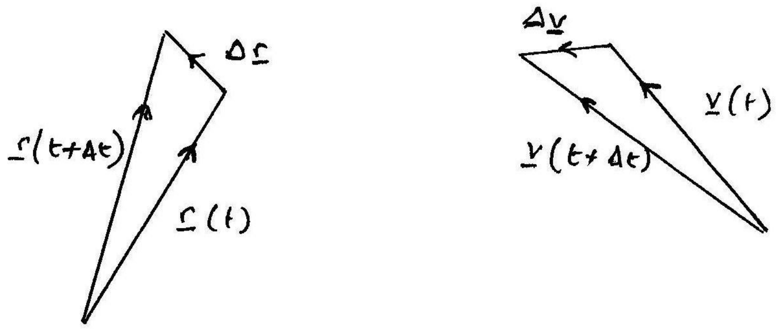

Figure 2: Time-dependent vectors

For acceleration, replace $\mathbf{r}$ by $\mathbf{v}$ in the above expressions,

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}. \tag{25}$$

In Cartesian coordinates

$$\mathbf{a} = v_x \widehat{\mathbf{i}} + v_y \widehat{\mathbf{j}} + v_z \widehat{\mathbf{k}}, \tag{26}$$

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}v_x}{\mathrm{d}t}\widehat{\mathbf{i}} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\widehat{\mathbf{j}} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\widehat{\mathbf{k}}, \tag{27}$$

$$= \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\widehat{\mathbf{i}} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\widehat{\mathbf{j}} + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\widehat{\mathbf{k}}. \tag{28}$$

6 Motion in one dimension

Consider a particle of mass, m, moving along the positive x-axis as in fig 3.

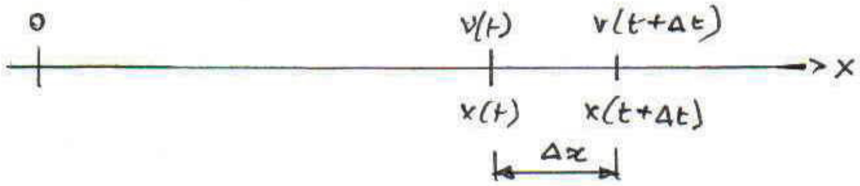

Figure 3: Motion along x-axis

The velocity is positive for motion in sense of x increasing and negative for x decreasing.

In time $\mathrm{d}t$ distance travelled by particle is $\mathrm{d}x = v\mathrm{d}t$. In the finite time interval between t_1, when position of particle is x_1, and time t_2 when position is x_2, distance travelled is

$$s = (x_2 - x_1) = \int_{t_1}^{t_2} v\,\mathrm{d}t. \tag{29}$$

This is represented by the shaded area in fig 4(a). We need to know how v varies with t in order to calculate s.

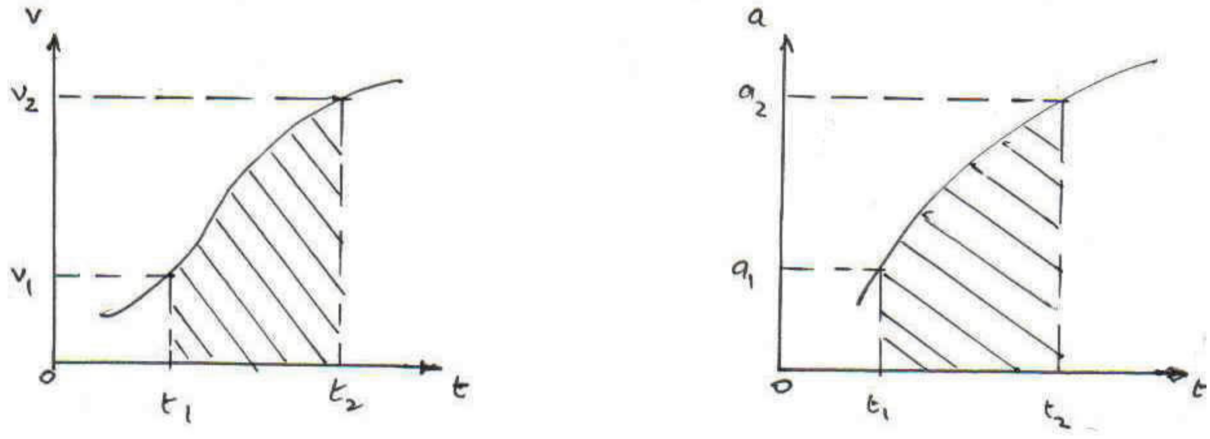

Figure 4: (a) Left: Velocity - time graph; (b) Right: Acceleration - time graph

Similarly for acceleration, a, the change in velocity in time interval between time t_1 and t_2 is

$$(v_2 - v_1) = \int_{t_1}^{t_2} a\,\mathrm{d}t, \tag{30}$$

and is represented by the shaded area in fig 4(b). As before we need to know how a varies as a function of time.

7 Motion under a constant force

If the force is constant, then by Newton's second law, $\mathbf{F} = m\mathbf{a}$, then so is the acceleration. Hence from fig 3

$$\begin{aligned} v_2 &= v_1 + a\,(t_2 - t_1)\,, && (31)\\ v &= at + K && (32) \end{aligned}$$

where K is an arbitrary constant of integration. Suppose $v = u$ at $t = 0$, this is a **boundary** or **initial condition**, so

$$\begin{aligned} v &= u + at && (33)\\ \frac{\mathrm{d}x}{\mathrm{d}t} &= u + at && (34) \end{aligned}$$

and integrating gives

$$x = ut + \frac{1}{2}at^2 + K_2, \tag{35}$$

where K_2 is another constant of integration determined by another boundary (initial) condition. Suppose $x = x_0$ at $t = 0$, then

$$x = x_0 + ut + \frac{1}{2}at^2 \tag{36}$$

and distance travelled in time t, is

$$s = (x - x_0) = ut + \frac{1}{2}at^2. \tag{37}$$

Since $t = (v - u)\,/a$, then

$$s = u\frac{(v-u)}{a} + \frac{1}{2}a\left[\frac{(v-u)}{a}\right]^2 \tag{38}$$

and re-arranging,

$$v^2 = u^2 + 2as \tag{39}$$

Collecting together the equations of motion we have for linear motion under a constant acceleration (or force)

$$\begin{aligned} v &= u + at, & (40)\\ s &= ut + \frac{1}{2}at^2, & (41)\\ v^2 &= u^2 + 2as, & (42)\\ s &= \frac{1}{2}(u+v)\,t. & (43) \end{aligned}$$

7.1 Free fall under gravity

In this case the constant acceleration is downwards, with $a = -g$, as x, v, and a are positive when measured upwards. Consider a body thrown upwards with an initial speed u. Forces are as shown in

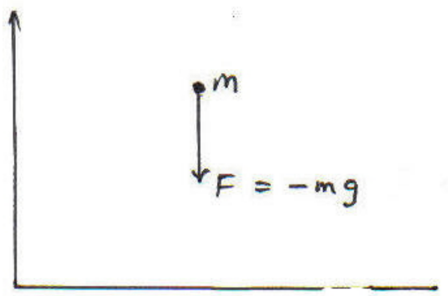

Figure 5: Mass falling freely under influence of gravity

fig 5. The initial conditions are, $x = 0$, $v = u$ at $t = 0$. To find the maximum height reached, we find s where $v = 0$. Thus from eq(42), $0 = u^2 - 2gs$ giving $s = u^2/(2g)$. The time to the maximum height is found using eq(40) giving $t = u/g$. To find the time to reach some specified height, x_1, then use eq(41),

$$\begin{aligned} x_1 &= ut - \frac{1}{2}gt^2 & (44)\\ & gt^2 - 2ut + 2x_1 = 0. & (45) \end{aligned}$$

This is a quadratic equation in t, with two roots,

$$t = \frac{u \pm \sqrt{u^2 - 2gx_1}}{g}. \tag{46}$$

If $u^2 > 2gx_1$, the body can reach a height greater than x_1 and the two roots are real, with

$$t_1 = \frac{u - \sqrt{u^2 - 2gx_1}}{g} \tag{47}$$

giving the time to reach x_1 on the way up and

$$t_2 = \frac{u + \sqrt{u^2 - 2gx_1}}{g} \tag{48}$$

giving the time to reach height x_1 going down after reaching the maximum height. Note that t_1 and t_2 are symmetric about the time to maximum height $t = u/g$. If $x_1 > u^2/g$ no particle can reach this height (the two roots of the quadratic are complex quantities).

8 Force of friction

Consider a block of mass m sliding on a **rough** surface. The frictional force, $\mathbf{F}_f$, **as it acts on the block**, is in a direction **opposite** to the motion of the block. It acts at the surface of the block in contact with the rough surface. The **normal reaction**, **N**, of the surface **on the block** is perpendicular to the surface. As there is no resultant vertical force on the block, (see fig 6) then the magnitude

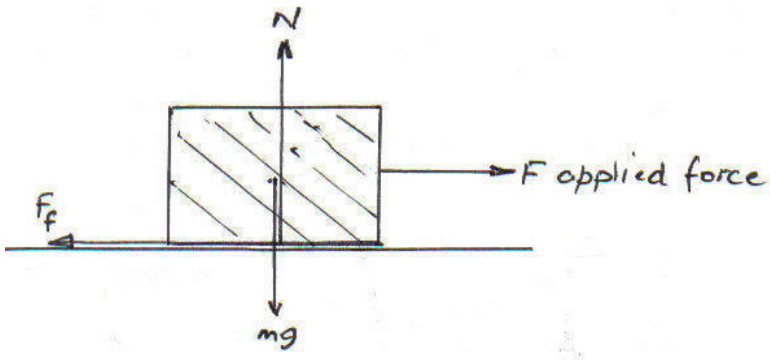

Figure 6: Block sliding over rough horzontal surface

$$N = mg. \tag{49}$$

If the block is sliding the frictional force is proportional to the normal reaction. The constant of proportionality is called the coefficient of friction, often denoted by μ. Hence we write $F_f = \mu N$ for the magnitude.

Some typical values of μ are

surfaces	$\boldsymbol{\mu}$
steel on steel	0.4
teflon on teflon	0.04
lead on steel	1

If applied force **F** is not sufficiently large to cause the block to slide, there is still a frictional force $\mathbf{F}_f$ acting in the direction **opposite** to that in which the body **would move** if there were no friction, but $F_f < \mu N$.

If the body is **just about to slide** we shall assume that $F_f = \mu N$. In practice the coefficient of **sliding friction** is slightly less than the coefficient of **static friction**.

Consider the motion of a block sliding on a rough horizontal surface. Resolving the forces vertically,

$$mg - N = 0 \tag{50}$$

so

$$F_f = \mu N = \mu mg \tag{51}$$

and the resultant horizontal force on the body is

$$(F - F_f) = ma = m\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{52}$$

This is the **equation of motion** of the body.

Example 1 *Suppose body is initially sliding to the right and the applied force $F = 0$. The body slides to rest under the influence of the frictional force. Equation of motion is*

$$-F_f = -\mu mg = ma, \tag{53}$$

so the acceleration

$$a = -\mu g \tag{54}$$

is constant. Hence previous results for motion under a constant acceleration can be used. In particular the time to come to rest if initial velocity is **u** *is given by*

$$\begin{aligned} 0 &= u - \mu g t, & (55) \\ t &= u/(\mu g). & (56) \end{aligned}$$

8.1 Motion of body on rough inclined plane

Consider the body being pulled/pushed **up** the plane at a constant speed. The forces are as in the diagram, fig 7. The body is sliding so we take frictional force $F_f = \mu N$. There is no resultant force

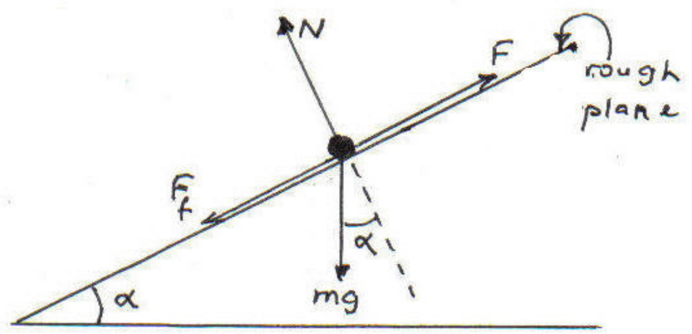

Figure 7: Mass pulled up an inclined plane

perpendicular to the plane (it remains in contact) so

$$N = mg\cos\alpha. \tag{57}$$

Resolving forces parallel to plane surface,

$$\begin{aligned} F - F_f - mg\sin\alpha &= ma = m\frac{\mathrm{d}v}{\mathrm{d}t} & (58) \\ F - \mu mg\cos\alpha - mg\sin\alpha &= m\frac{\mathrm{d}v}{\mathrm{d}t}. & (59) \end{aligned}$$

Example 2 *What is largest value for the angle of inclination of the plane for which body remains at rest on the plane* ***without sliding down****. In this case the body wants to slide down, so the frictional force F_f acts up the plane, as in fig 8. For equilibrium there is no resultant force on the particle in any direction.*

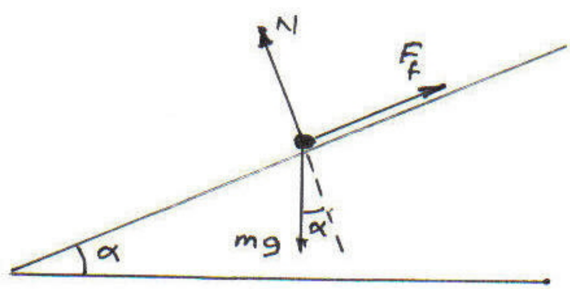

Figure 8: Body sliding down an inclined plane

Resolving forces perpendicular to the plane

$$N = mg\cos\alpha \tag{60}$$

and parallel to the plane

$$F_f = mg \sin \alpha. \tag{61}$$

The maximum value of F_f is μN, so

$$F_f = \mu N = \mu mg \cos \alpha \tag{62}$$

so the maximum value of α is when

$$\mu mg \cos \alpha = mg \sin \alpha \tag{63}$$
$$\tan \alpha = \mu. \tag{64}$$

8.2 Role of friction in accelerating a car

Forces are as shown for a rear-wheel drive car in fig 9. The frictional force between the rear wheels and

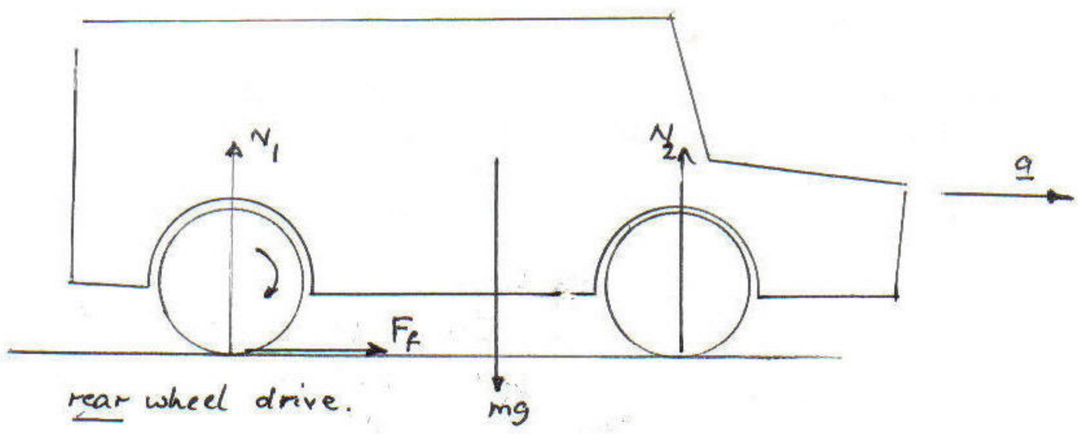

Figure 9: Rear-wheel driven vehicle

the ground is F_f, and this force provides the force needed to accelerate the car, $F_f = ma$.

9 Kinematical relations

Before considering other one-dimensional problems we need to develop some other general aspects of mechanics.

9.1 Work

Force $\mathbf{F}$ is applied to a body. If point of application of the force, i.e. body moves a distance d$\mathbf{r}$ as in fig 10, the **work done by the force** is defined to be

$$\mathrm{d}W = \mathbf{F} \cdot \mathrm{d}\mathbf{r} = (F \cos \alpha)\ \mathrm{d}r \tag{65}$$

where $F \cos \alpha$ is component of force in direction of motion of the body. Hence **work is a scalar quantity**.
For finite displacements of a particle from position $\mathbf{r}_1$ to $\mathbf{r}_2$ as in fig 10 the work done **by** the force is

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{66}$$

Note that $\mathbf{F}$ may vary with position.

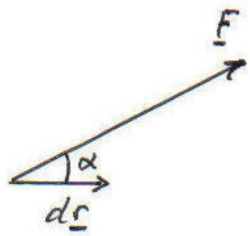

Figure 10: Force displaced

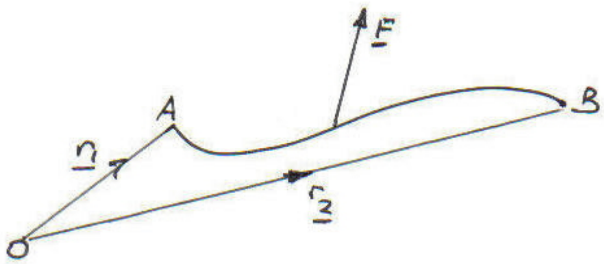

Figure 11: Body moving over curved path

For a **constant force** (in both **magnitude** and **direction**), work done is

$$W = \mathbf{F} \cdot (\mathbf{r}_2 - \mathbf{r}_1)\,. \tag{67}$$

Work is **energy** and the S.I. unit is the newton metre (N m).

If force acts in the x-direction and the particle can only move in the x-direction, then we can drop the scalar product in eq(66) and write

$$W = \int_{x_1}^{x_2} F\,\mathrm{d}x \tag{68}$$

as the work done by force F in moving particle from position x_1 to x_2.

Example 3 *Work that must be done to stretch a spring, as illustrated in fig 12: Tension in the spring*

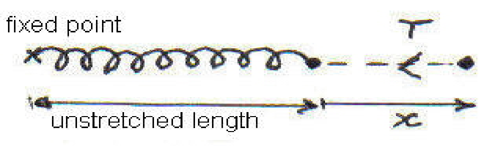

Figure 12: Stretched spring

when extended is proportional to the (small) extension, (Hooke's law), so $T = -kx$ *where* k *is the stiffness constant. To stretch the spring we must apply a force* $F = +kx$. *The work done by force* F *(i.e. by us) to stretch spring to an extension* x_1 *is*

$$W = \int_0^{x_1} F\,\mathrm{d}x = \int_0^{x_1} kx\,\mathrm{d}x = \frac{1}{2}kx_1^2. \tag{69}$$

*We assume there are no **dissipative** forces (e.g. friction) so this energy is stored in the stretched spring as **potential energy**. Potential energy will be discussed in more detail later.*

Example 4 *Work done **by** gravity when particle falls from position x_1 to x_2 where $(x_1 - x_2) = h$. The*

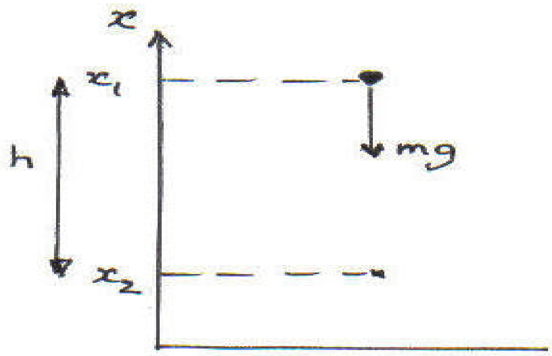

Figure 13: Body falling through distance h under gravity

force $F = -mg$ and the work

$$W = \int_{x_1}^{x_2} F \,\mathrm{d}x = \int_{x_1}^{x_2} (-mg) \,\mathrm{d}x = (-mgx)\big|_{x_1}^{x_2} = -mg\,(x_2 - x_1) = mgh. \tag{70}$$

The work done manifests itself as kinetic energy of the particle.

Example 5 *Suppose particle is constrained to fall under gravity inside a **smooth** tube, as in the fig 14. Force of tube **on** the particle is N and is normal to the tube. (The tube being smooth produces no*

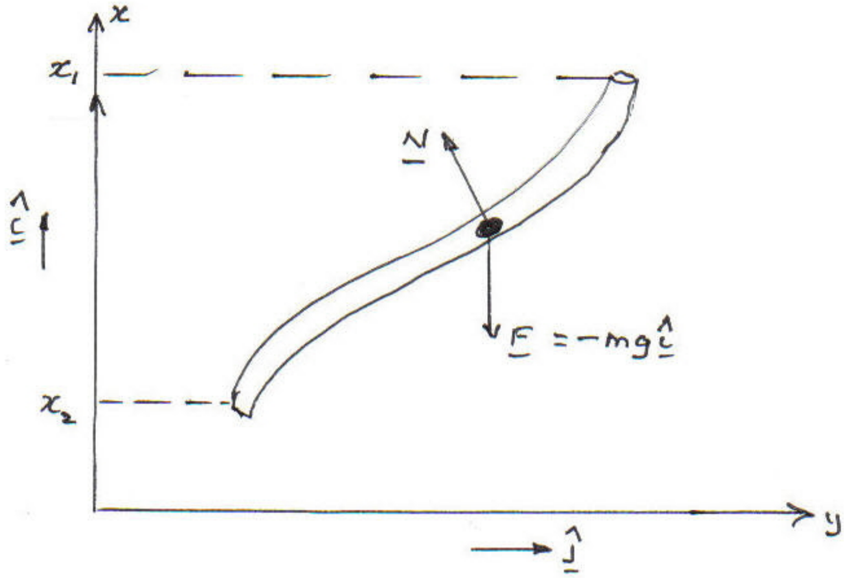

Figure 14: Contrained motion of particle under gravity

frictional force - an idealized situation!) So resultant force on particle is $\mathbf{N} + \mathbf{F}$. Work done by these forces as particle moves a distance $\mathrm{d}\mathbf{r}$ along the tube is

$$\mathrm{d}W = (\mathbf{N} + \mathbf{F}) \cdot \mathrm{d}\mathbf{r} = \mathbf{N} \cdot \mathrm{d}\mathbf{r} + \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \tag{71}$$

But $\mathbf{N}$ *is orthogonal (perpendicular) to* $\mathrm{d}\mathbf{r}$ *at every point along tube, so* $\mathbf{N}\cdot\mathrm{d}\mathbf{r}=0$. *Hence*

$$\mathrm{d}W=\mathbf{F}\cdot\mathrm{d}\mathbf{r} \tag{72}$$

and constraining forces which are not dissipative (i.e. no friction) do no work. For this problem, $\mathbf{F}=-mg\widehat{\mathbf{i}}$ *and* $\mathrm{d}\mathbf{r}=\mathrm{d}x\widehat{\mathbf{i}}+\mathrm{d}y\widehat{\mathbf{j}}$, *so*

$$\mathrm{d}W=\mathbf{F}\cdot\mathrm{d}\mathbf{r}=-\,mg\widehat{\mathbf{i}}\cdot\left(\mathrm{d}x\widehat{\mathbf{i}}+\mathrm{d}y\widehat{\mathbf{j}}\right)=-mg\mathrm{d}x \tag{73}$$

and hence the work done by gravity when particle falls from height x_1 *to* x_2 *is*

$$W=\int_{x_1}^{x_2}(-mg)\,\mathrm{d}x=mgh \tag{74}$$

as before.

9.2 Power

Power, P, is defined as the rate of doing work,

$$P=\frac{\mathrm{d}W}{\mathrm{d}t}. \tag{75}$$

The S.I. unit is the joule/second $\left(\mathrm{J\,s^{-1}}\right)$ or watt (W). Since $\mathrm{d}W=\mathbf{F}\cdot\mathrm{d}\mathbf{r}$ then

$$P=\frac{\mathrm{d}W}{\mathrm{d}t}=\mathbf{F}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}=\mathbf{F}\cdot\mathbf{v}=F_xv_x+F_yv_y+F_zv_z. \tag{76}$$

Average power over extended time interval t is $P=W/t$, where W is work done in time t.

9.3 Impulse

Consider a force $\mathbf{F}$ which acts on a body of mass m during a time interval from t_1 to t_2. The impulse (a vector quantity) of $\mathbf{F}$ during this time interval is defined as

$$\mathbf{I}=\int_{t_1}^{t_2}\mathbf{F}\,\mathrm{d}t \tag{77}$$

But from Newton's second law

$$\mathbf{F}=\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{78}$$

so

$$\mathbf{I}=\int_{t_1}^{t_2}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\,\mathrm{d}t=\int_{\mathbf{p}_1}^{\mathbf{p}_2}\mathrm{d}\mathbf{p}=(\mathbf{p}_2-\mathbf{p}_1) \tag{79}$$

where $\mathbf{p}_1=m\mathbf{v}_1$, the momentum of particle at time t, and similarly $\mathbf{p}_2=m\mathbf{v}_2$ at time t_2. Hence Newton's second law can also be stated as Impulse = change in momentum,

$$\mathbf{I}=\Delta\mathbf{p}=(\mathbf{p}_2-\mathbf{p}_1)\,. \tag{80}$$

If force, $\mathbf{F}$, is constant (in direction and magnitude) throughout the time interval $\Delta t=(t_2-t_1)$, then

$$\mathbf{I}=\mathbf{F}\,(t_2-t_1)=\mathbf{F}\Delta t. \tag{81}$$

An important special case is when force, $\mathbf{F}$, is very large and Δt is very small, so that $\mathbf{I}=\Delta\mathbf{p}$ is finite. Note as $\mathbf{I}=\int_{t_1}^{t_2}\mathbf{F}\,\mathrm{d}t$ then

$$\mathbf{F}=\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}=\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}t} \tag{82}$$

Example 6 *Particle of mass* m *bouncing off a wall elastically (no loss of kinetic energy) as in fig 15. Change in momentum of particle is* $\Delta\mathbf{p}=(-m\mathbf{v})-m\mathbf{v}=-2m\mathbf{v}$ *is the impulse on the particle. Therefore by Newton's third law the impulse on the wall is* $+2m\mathbf{v}$.

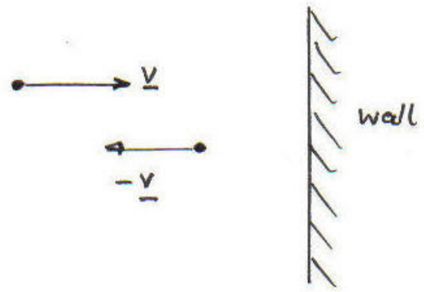

Figure 15: Particle bouncing elastically from a wall (very large mass)

9.4 Kinetic energy and potential energy

Consider a one-dimensional problem with the force $\mathbf{F}$ in the x-direction acting on a particle of mass m. The equation of motion is

$$F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = mv\frac{\mathrm{d}v}{\mathrm{d}x}. \tag{83}$$

The work done by the force in moving the particle from position x_1 to position x_2 in fig 16 is

Figure 16: Work done between two points

$$W = \int_{x_1}^{x_2} F\,\mathrm{d}x = \int_{x_1}^{x_2} mv\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int_{v_1}^{v_2} mv\,\mathrm{d}v = \frac{1}{2}mv^2|_{v_1}^{v_2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2, \tag{84}$$

where v_1 is velocity of particle at position x_1 and similarly at position 2. We **define kinetic energy** (K.E.) as $K_E = \frac{1}{2}mv^2$. Hence the work done by the force is equal to the change (increase) in kinetic energy.

We can also consider the work done by the force as the **difference** in **potential energy** (P.E.) between positions x_1 and x_2, thus we define (mechanical) potential $V(x)$ by

$$W = \int_{x_1}^{x_2} F\,\mathrm{d}x = V(x_1) - V(x_2) = -[V(x_2) - V(x_1)] = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \tag{85}$$

and

$$V(x_1) + \frac{1}{2}mv_1^2 = V(x_2) + \frac{1}{2}mv_2^2. \tag{86}$$

The above relationship only defines the **difference** in potential energy. To define an **absolute potential energy** of a particle at x_1 we need an arbitrary but conveniently chosen position, x_0, at which the potential $V = 0$ by definition. Then

$$V(x_1) = \int_{x_1}^{x_0} F\,\mathrm{d}x \tag{87}$$

and

$$V(x_1) - V(x_2) = \int_{x_1}^{x_0} F\,\mathrm{d}x - \int_{x_2}^{x_0} F\,\mathrm{d}x = \int_{x_1}^{x_0} F\,\mathrm{d}x + \int_{x_0}^{x_2} F\,\mathrm{d}x = \int_{x_1}^{x_2} F\,\mathrm{d}x = W. \tag{88}$$

Since

$$V(x_1) + \frac{1}{2}mv_1^2 = V(x_2) + \frac{1}{2}mv_2^2 \tag{89}$$

then potential energy + kinetic energy is constant, i.e. total energy is constant. Hence **total energy**, $E = V + K_E$, **is conserved** throughout the motion provided force $\mathbf{F}$ is non-dissipative (i.e. no friction). If we know the potential energy as a function of position, $V(x)$ we can determine the force F from

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{90}$$

Note F is in the direction of decreasing V.

Example 7 *Gravitational force $F = -mg$. Then*

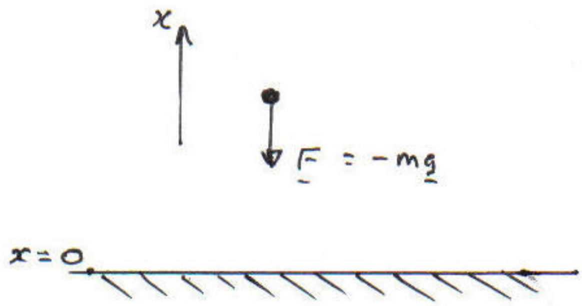

Figure 17: Particle falling under gravity

$$V(x_1) = \int_{x_1}^{x_0} F\,\mathrm{d}x = \int_{x_1}^{x_0} (-mg)\,\mathrm{d}x = (-mgx)\big|_{x_1}^{x_0} = mgx_1 - mgx_0. \tag{91}$$

We can choose $x_0 = 0$ as the ground, giving

$$V(x) = mgx \tag{92}$$

and

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x} = -mg \tag{93}$$

as above.

Example 8 *Equation for motion under constant acceleration of gravity given earlier,*

$$v^2 = u^2 - 2g(x - x_0) \tag{94}$$

can be re-written by multiplying by $\frac{1}{2}m$ as

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{1}{2}m2g(x - x_0) \tag{95}$$

$$\frac{1}{2}mv^2 + mgx = \frac{1}{2}mu^2 + mgx_0 \tag{96}$$

i.e. conservation of total energy.

In three-dimensions we have

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2. \tag{97}$$

also

$$W = V(\mathbf{r}_1) - V(\mathbf{r}_2) = \text{change of P.E.} \tag{98}$$

and

$$V(\mathbf{r}_1) = \int_{\mathbf{r}_1}^{\mathbf{r}_0} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \tag{99}$$

$$V(\mathbf{r}_0) = 0 \tag{100}$$

$$\mathbf{F} = -\boldsymbol{\nabla} V \tag{101}$$

where the gradient operator

$$\boldsymbol{\nabla} \equiv \widehat{\mathbf{i}}\frac{\partial}{\partial x} + \widehat{\mathbf{j}}\frac{\partial}{\partial y} + \widehat{\mathbf{k}}\frac{\partial}{\partial z}. \tag{102}$$

We can only write $\mathbf{F} = -\boldsymbol{\nabla} V$ and define a potential energy function $V(\mathbf{r})$ for a **non-dissipative** or **conservative** force. A conservative force is one such that the work done by the force in moving the particle from position $\mathbf{r}_1$ to $\mathbf{r}_2$ is **independent of the path** between $\mathbf{r}_1$ and $\mathbf{r}_2$, see fig 18 The work

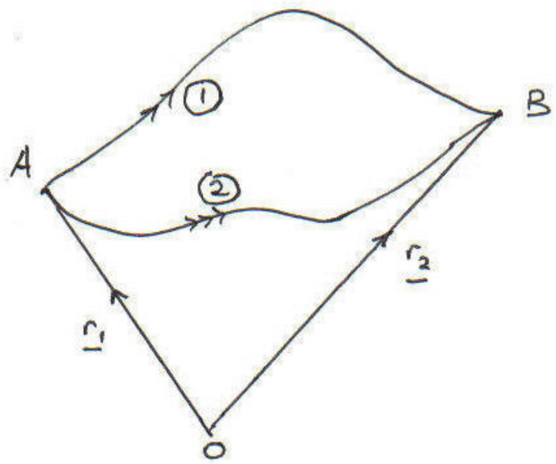

Figure 18: Work done over different paths by a conservative force

done just depends on the initial and end points,

$$W_{\mathrm{AB}} = W = \int_{\mathbf{r}_\mathbf{A}}^{\mathbf{r}_\mathbf{B}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = V(\mathbf{r}_\mathrm{A}) - V(\mathbf{r}_\mathrm{B}). \tag{103}$$

Examples of conservative forces are: electrostatics, gravitation, force due to a stretched spring.

10 Simple Harmonic motion

Consider a particle of mass m attached to a light (massless) spring with a displacement x from the equilibrium (unstretched) position $x = 0$ as shown in fig 19. Positive x means extension, negative x means compression of the spring. The restoring force is given by Hooke's law, $F = -kx$. The equation of motion is

$$F = -kx = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \tag{104}$$

or, writing $\omega = \sqrt{k/m}$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0. \tag{105}$$

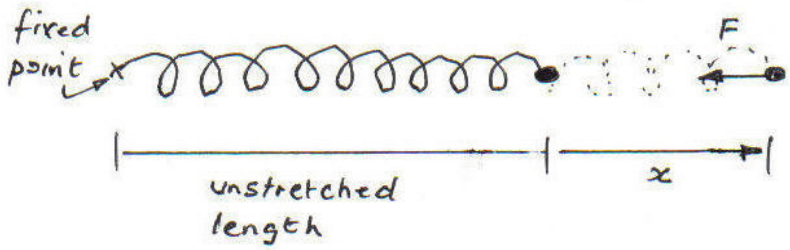

Figure 19: Stretched spring

This equation, in one form or another, is one of the most common equations in physics.

The general solution is

$$x(t) = A\cos\omega t + B\sin\omega t, \tag{106}$$

where A and B are two arbitrary constants of integration of the second-order differential equation eq(105). The functions $\sin\omega t$ and $\cos\omega t$ are shown in figs 20 and 21. This solution can also be written in the form

$$x(t) = C\mathrm{e}^{i\omega t} + D\mathrm{e}^{-i\omega t}, \tag{107}$$

where C and D are two arbitrary constants of integration, or as

$$x(t) = r\cos(\omega t + \phi) \tag{108}$$

with distance r and angle ϕ arbitrary constants. The values of these arbitrary constants are determined by the **initial** or **boundary conditions**.

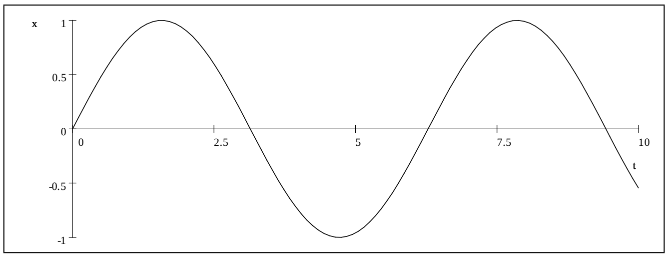

Figure 20: sinωt

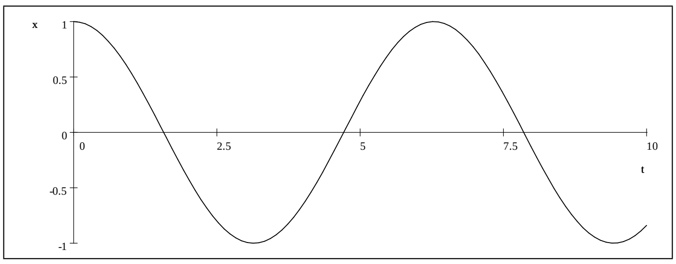

Figure 21: cosωt

Any solution to the equation of motion is **periodic**, the time for one complete period/oscillation being

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}. \tag{109}$$

The frequency of oscillation is

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{Hz.} \tag{110}$$

and the angular frequency $\omega = \sqrt{k/m}$ rad/s.

Example 9 *Suppose $x = a$ and $\mathrm{d}x/\mathrm{d}t = 0$ at $t = 0$ are the initial conditions. Then from*

$$\begin{aligned} x(t) &= A\cos\omega t + B\sin\omega t, && (111)\\ \frac{\mathrm{d}x}{\mathrm{d}t} &= -\omega A\sin\omega t + \omega B\cos\omega t && (112) \end{aligned}$$

substituting the initial values gives,

$$\begin{aligned} a &= A, && (113)\\ 0 &= \omega B. && (114) \end{aligned}$$

Thus a particular solution satisfying the initial conditions is

$$\begin{aligned} x(t) &= a\cos\omega t && (115)\\ v(t) &= -a\omega\sin\omega t. && (116) \end{aligned}$$

The largest displacement a is called the amplitude.

10.1 Potential and kinetic energy in simple harmonic motion

We now consider the variations of the potential energy and kinetic energy of the particle as it undergoes simple harmonic motion. The force $F = -kx$ is a conservative force so the potential energy

$$V(x) = \frac{1}{2}kx^2 + V_0. \tag{117}$$

We can choose $V(x = 0) = 0$ giving $V_0 = 0$ and the potential energy is

$$V(x) = \frac{1}{2}ka^2\cos^2(\omega t) \tag{118}$$

The kinetic energy K_E is

$$K_E = \frac{1}{2}mv^2 = \frac{1}{2}ma^2\omega^2\sin^2(\omega t). \tag{119}$$

The total energy $E = K_E + V$ is

$$E = \frac{1}{2}ma^2\omega^2\sin^2(\omega t) + \frac{1}{2}ka^2\cos^2(\omega t). \tag{120}$$

But $\omega^2 = k/m$ so

$$\begin{aligned} E &= \frac{1}{2}ma^2\frac{k}{m}\sin^2(\omega t) + \frac{1}{2}ka^2\cos^2(\omega t), && (121)\\ &= \frac{1}{2}ka^2\left[\sin^2(\omega t) + \cos^2(\omega t)\right], && (122)\\ E &= \frac{1}{2}ka^2 = K_E + V. && (123) \end{aligned}$$

Fig 22 shows the potential energy curve for a stretched spring. If total energy is E, then the motion is restricted to $|x| < a$. At $x = a$, $V = \frac{1}{2}ka^2 = E$, and $K_E = 0$; at $x = 0$, $V = 0$, and $K_E = \frac{1}{2}ka^2 = E$.

Whenever the potential energy function of a system has a rounded minimum, the vibrational motion about the equilibrium position of minimum potential energy approximates to simple harmonic motion. An example is two atoms bound together to form a molecule, such as in fig 23,. If r_0 is the separation for minimum potential energy, (the equilibrium separation), the two atoms vibrate about this value r_0.

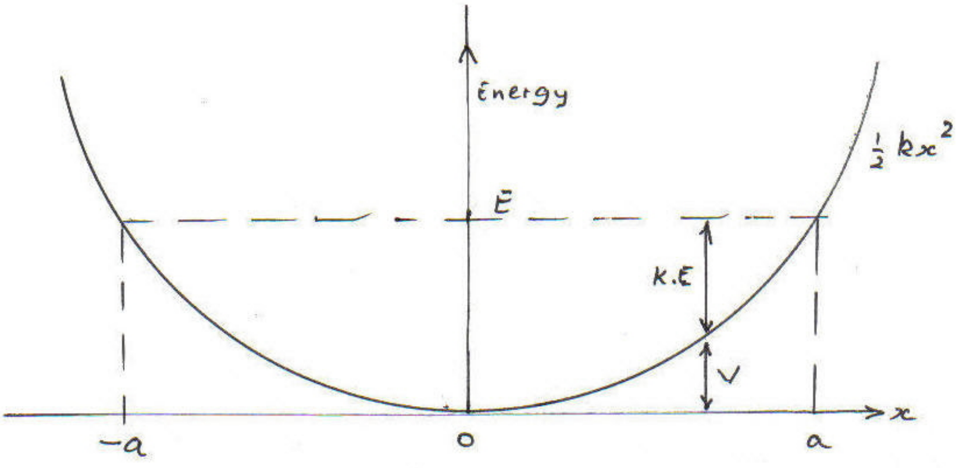

Figure 22: Potential energy curve for simple harmonic motion

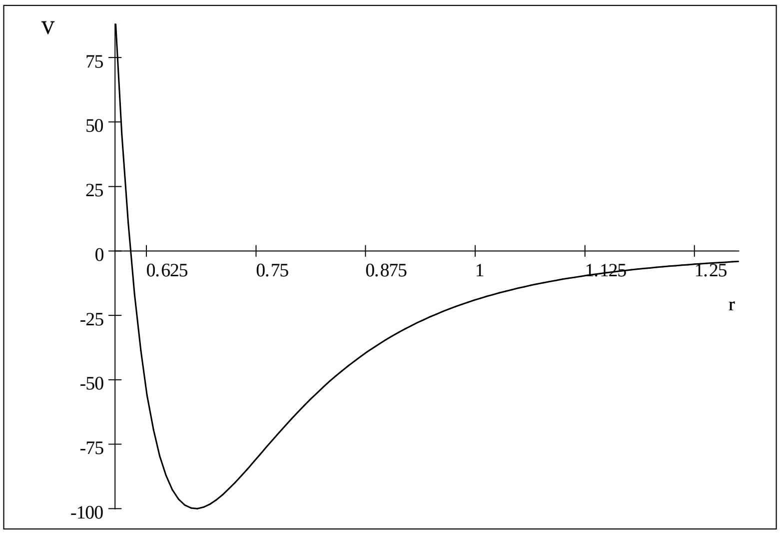

Figure 23: Potential energy curve as a function of the separation of two atoms

This can be shown by taking a Taylor's series expansion about the equilibrium position. If x_0 is the exquilibrium separation, then

$$V(x) - V(x_0) = \left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)_{x=x_0} (x - x_0) + \frac{1}{2}\left(\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right)_{x=x_0} (x - x_0)^2 + \ldots . \tag{124}$$

But at $x = x_0$ is a minimum, so $\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)_{x=x_0} = 0$ and $\left(\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right)_{x=x_0} > 0$. Since the force for small displacements $(x - x_0)$ is

$$F(x) = -\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right), \tag{125}$$

$$= -\left(\frac{\mathrm{d}^2V}{\mathrm{d}x^2}\right)_{x=x_0} (x - x_0). \tag{126}$$

Hence force is of the form $F = -kx$ where x is displacement from equilibrium, with $k > 0$, i.e.S.H.M.

10.2 Damped oscillations

In all **real** mechanical oscillators there is some **damping** (or friction). The damping force opposes the motion of the particle. Consider a damping force that is proportional to the velocity of the particle, i.e. $F_f = -\lambda \frac{\mathrm{d}x}{\mathrm{d}t}$ with λ a positive constant. Suppose the velocity is positive, i.e. motion in direction of increasing x, then the damping force is in the opposite direction (see fig 24). The equation of motion of

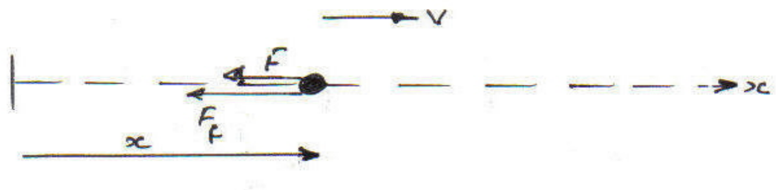

Figure 24: Displacement and forces in a damped harmonic oscillator

the particle is

$$F + F_f = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \tag{127}$$

$$-kx - \lambda\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \tag{128}$$

or

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \lambda\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0. \tag{129}$$

This is a second-order, linear, homogeneous differential equation with constant coefficients, m, λ and k. The general solution is

$$x(t) = A\mathrm{e}^{q_1 t} + B\mathrm{e}^{q_2 t}, \tag{130}$$

where q_1 and q_2 are the roots of the quadratic equation (the auxiliary equation)

$$mq^2 + \lambda q + k = 0, \tag{131}$$

$$q = \frac{-\lambda \pm \sqrt{\lambda^2 - 4mk}}{2m}. \tag{132}$$

In the absence of damping, $\lambda = 0$, then the natural angular frequency of oscillation, denoted by $\omega_0 = \sqrt{k/m}$. Then

$$\begin{aligned} q &= -\frac{\lambda}{2m} \pm \sqrt{\frac{\lambda^2}{4m^2} - \frac{k}{m}} = -\frac{\lambda}{2m} \pm i\sqrt{\frac{k}{m} - \frac{\lambda^2}{4m^2}} && (133) \\ &= -\frac{\lambda}{2m} \pm i\sqrt{\omega_0^2 - \frac{\lambda^2}{4m^2}} && (134) \\ &= -\frac{\lambda}{2m} \pm i\omega, && (135) \end{aligned}$$

where the angular frequency

$$\omega = \sqrt{\omega_0^2 - \frac{\lambda^2}{4m^2}}. \tag{136}$$

There are three possibilities;
(a) $\lambda^2 < 4mk$, so $\lambda^2/4m < k/m = \omega_0$, and ω is real and positive and $0 < \omega < \omega_0$
(b) $\lambda^2 = 4mk$, so $\lambda^2/4m = k/m = \omega_0$ and $\omega = 0$,
(c) $\lambda^2 > 4mk$, so$\lambda^2/4m > k/m = \omega_0$ so ω is complex and q is wholly real, and there is no oscillation.
We will consider these cases in turn.

(a) $\lambda^2 < 4mk$, known as light damping ($0 < \omega < \omega_0$).
The roots are complex quantities, $q_1 = -\frac{\lambda}{2m} + i\omega$ and $q_2 = -\frac{\lambda}{2m} - i\omega$, so

$$\begin{aligned} x(t) &= A\mathrm{e}^{q_1 t} + B\mathrm{e}^{q_2 t}, &(137)\\ &= \mathrm{e}^{-(\lambda/2m)t}\left[A\mathrm{e}^{i\omega t} + B\mathrm{e}^{-i\omega t}\right]. &(138)\end{aligned}$$

From de Moivre's theorem $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$, $x(t)$ can be re-written as

$$x(t) = \mathrm{e}^{-(\lambda/2m)t}\left[C\cos\omega t + D\sin\omega t\right] \tag{139}$$

which is a product of a function $\mathrm{e}^{-(\lambda/2m)t}$ exponentially decaying with time and an oscillating function $[C\cos\omega t + D\sin\omega t]$ of time with angular frequency ω as shown in fig 25.

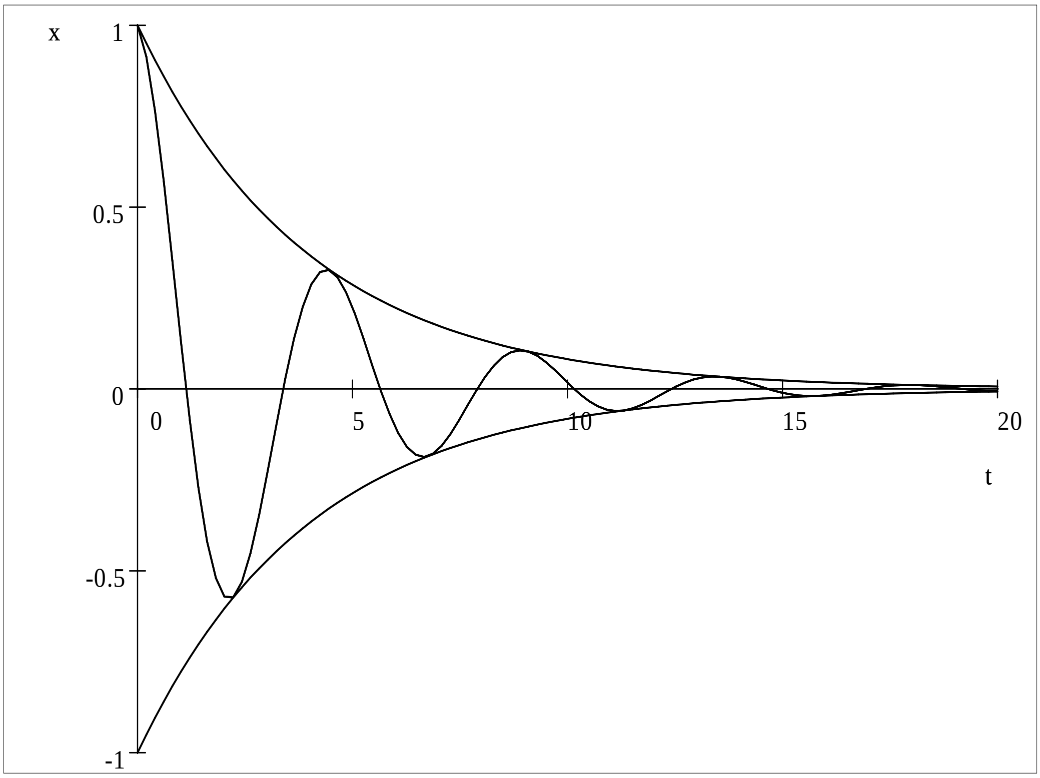

Figure 25: Displacement as function of time in an under damped oscillator

Energy is continually being lost due to the damping force but at any given time the energy of the oscillator is

$$E = K_E + V = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \tag{140}$$

Using the above expressions for $x(t)$ it can be shown that for very light damping

$$E(t) \simeq E_0\mathrm{e}^{-\lambda t/m}, \tag{141}$$

where E_0 is the total energy at $t = 0$. Therefore $E_0 = \frac{1}{2}ka^2$ for an initial displacement a, and

$$\frac{\mathrm{d}E}{\mathrm{d}t} \simeq -\frac{\lambda}{m}E_0\mathrm{e}^{-\lambda t/m} \simeq -\frac{\lambda}{m}E. \tag{142}$$

Hence in time interval $\mathrm{d}t$ the fractional energy loss

$$\frac{\mathrm{d}E}{E} = -\frac{\lambda}{m}\mathrm{d}t. \tag{143}$$

In a time interval $\mathrm{d}t = 1/\omega$, the fractional energy loss is $\lambda/(m\omega)$. The reciprocal of this quantity

$$\frac{E}{\mathrm{d}E} = Q = \frac{m\omega}{\lambda} \simeq \frac{m\omega_0}{\lambda} \tag{144}$$

is called the **Q-factor** or **quality factor**. For a lightly damped oscillator the Q-factor is large.

(b) $\lambda^2 = 4mk$, ($\omega = 0$)known as critical damping
The two roots are equal to $-\lambda/(2m)$. In this case the general solution to the equation of motion is

$$x(t) = (A + Bt)\, e^{-(\lambda/2m)t}. \tag{145}$$

If $x = a$ and $\mathrm{d}x/\mathrm{d}t = 0$ at $t = 0$, $x(t)$ is never negative - there is no oscillation and the system returns to the equilibrium position is the shortest time (see fig 26).

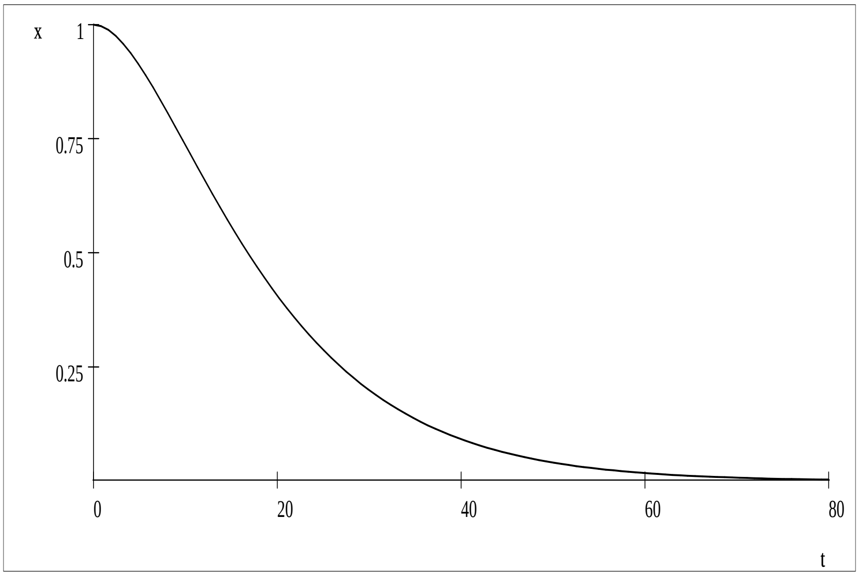

Figure 26: Displacement as function of time for critical damping

Critical damping is required for swing-doors and car suspensions.

(c) $\lambda^2 > 4mk$, known as heavy damping, ω is complex and q is wholly real. The two roots are

$$q_1 = -\frac{\lambda}{2m} + \sqrt{\frac{\lambda^2}{4m^2} - \omega_0^2} < 0, \tag{146}$$

$$q_2 = -\frac{\lambda}{2m} - \sqrt{\frac{\lambda^2}{4m^2} - \omega_0^2} < 0. \tag{147}$$

Both roots are negative but q_2 is more negative then q_1, i.e. $|q_2| > |q_1|$. The displacement

$$x(t) = A\mathrm{e}^{-|q_1|t} + B\mathrm{e}^{-|q_2|t} \tag{148}$$

is the addition of two term, the first $A\mathrm{e}^{-|q_1|t}$ dies away slowly (middle curve in fig 27), the second $B\mathrm{e}^{-|q_2|t}$ dies away more quickly (lower curve in fig 27). As damping increases $|q_1|$ becomes smaller and the displacement decreases more slowly with time.

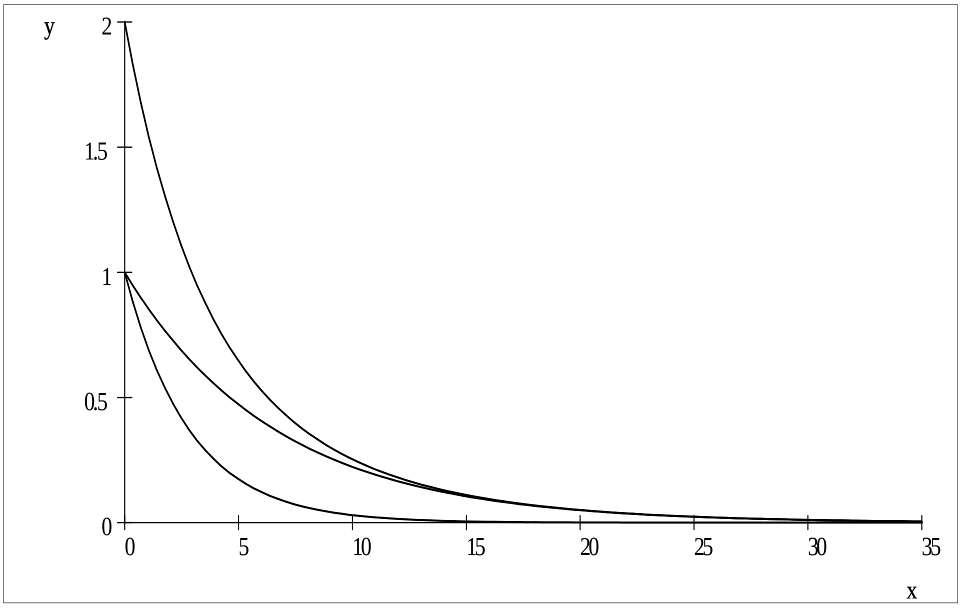

Figure 27: Displacement as function of time for overdamped (heavy) oscillator

10.3 Forced damped oscillator

Consider applying a periodic force $F = F_0 \cos \omega_f t$ to a damped oscillator in order to keep it oscillating. The equation of motion of the particle is

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \lambda\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0 \cos \omega_f t \tag{149}$$

where the term on the R.H.S. is the driving or forcing term. The general solution to this equation is the sum of the general solution to the homogeneous equation (zero on R.H.S., eq(129)) plus a **particular solution** to the full equation. It was shown above that the solution to the homogeneous equation eq(129) dies away exponentially (this is called the transient solution) so that after a sufficiently long time the motion of the particle is described by the particular solution alone.

Let's take as an ansatz solution a particular solution of the form

$$x(t) = A\cos(\omega_f t + \phi). \tag{150}$$

That is one having the same frequency ω_f as the driving frequency but with an unknown phase difference ϕ and amplitude A. To determine A and ϕ substitute the assumed solution into the differential equation of motion. As

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\omega_f A \sin(\omega_f + \phi), \tag{151}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega_f^2 A \cos(\omega_f + \phi), \tag{152}$$

we have

$$-m\omega_f^2 A\cos(\omega_f + \phi) - \lambda\omega_f A\sin(\omega_f + \phi) + kA\cos(\omega_f t + \phi) = F_0\cos\omega_f t \tag{153}$$

$$A\left(k - m\omega_f^2\right)\cos(\omega_f + \phi) - \lambda\omega_f A\sin(\omega_f + \phi) = F_0\cos\omega_f t \tag{154}$$

Expanding the cosine and sine terms,

$$A\left(k - m\omega_f^2\right)[\cos\omega_f t\cos\phi - \sin\omega_f t\sin\phi] - \lambda\omega_f A[\sin\omega_f t\cos\phi + \cos\omega_f t\sin\phi] = F_0\cos\omega_f t \tag{155}$$

This equation must be true for all values of t. (Note cosine and sine are orthogonal functions.) Choose t such that $\cos\omega_f t = 1$, then $\sin\omega_f t = 0$, and

$$A\left(k - m\omega_f^2\right)\cos\phi - \lambda\omega_f A\sin\phi = F_0. \tag{156}$$

Now choose t such that $\sin\omega_f t = 1$, then $\cos\omega_f t = 0$, and

$$-A\left(k - m\omega_f^2\right)\sin\phi - \lambda\omega_f A\cos\phi = 0. \tag{157}$$

Squaring eq(156) and eq(157) and adding gives,

$$A^2\left(k - m\omega_f^2\right)^2 + \lambda^2\omega_f^2 A^2 = F_0^2, \tag{158}$$

and

$$A = \frac{F_0}{\left[\left(k - m\omega_f^2\right)^2 + \lambda^2\omega_f^2\right]^{1/2}}. \tag{159}$$

For an undamped, unforced oscillator the natural angular frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{160}$$

so $k = m\omega_0^2$, which gives

$$A = \frac{F_0}{\left[m^2\left(\omega_0^2 - \omega_f^2\right)^2 + \lambda^2\omega_f^2\right]^{1/2}}. \tag{161}$$

We can now investigate the variation of A with ω_f. If there were no damping, $\lambda = 0$, then

$$A = \frac{F_0}{m\left|\left(\omega_0^2 - \omega_f^2\right)\right|}. \tag{162}$$

Hence $A \to \infty$ as $\omega_f \to \omega_0$. Even with damping, A has its maximum value when $\omega_f \simeq \omega_0$. This is the phenomena of **resonance**.Amplitude of the oscillation becomes much larger when the angular frequency

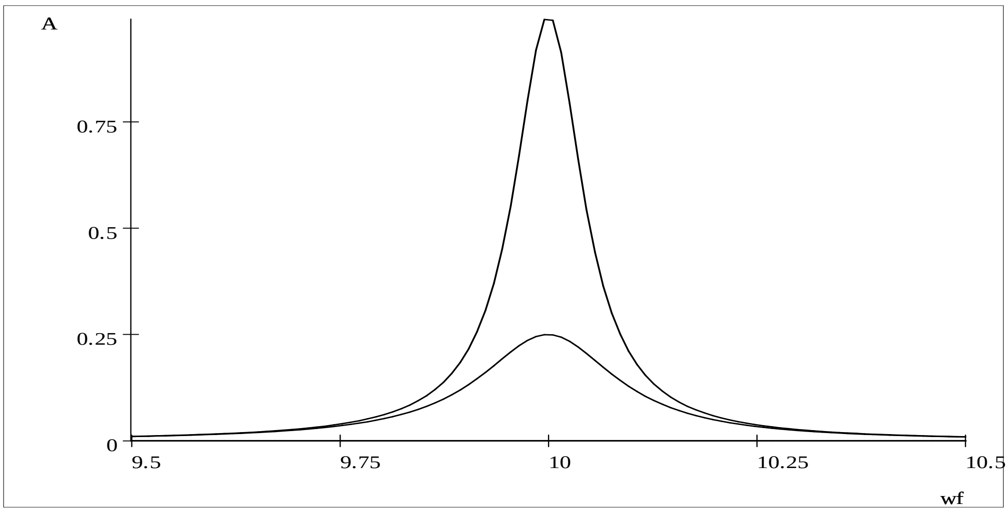

Figure 28: Amplitude of forced oscillator; Lower curve - heavy damping; Upper curve - light damping

of the driving force is equal to the natural frequency of the undamped oscillator as shown in fig 28. Dramatic example - Facoma Narrows bridge.

From eq(157)

$$\tan\phi = \frac{\lambda\omega_f}{\left(m\omega_f^2 - k\right)} = \frac{\lambda\omega_f}{m\left(\omega_f^2 - \omega_0^2\right)}. \tag{163}$$

Below the resonance frequency, $\omega_f < \omega_0$ the phase angle ϕ is fairly small, but negative as shown in fig

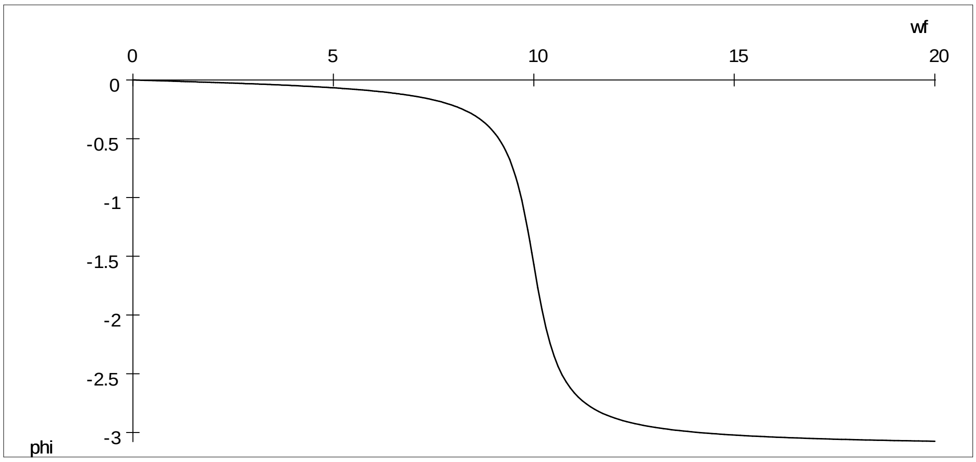

Figure 29: Phase in forced oscillator as function of driving frequency

29. Therefore the displacement x is approximately **in phase** with the driving force. As ω_f increases through ω_0 the phase angle increases rapidly and drops through $-\pi/2$ to $-\pi$, at which stage x is then out of phase with the driving force for $\omega_f > \omega_0$.

11 Motion in a plane (two-dimensions)

Particle at point P as in fig 30. Its position, velocity and acceleration are

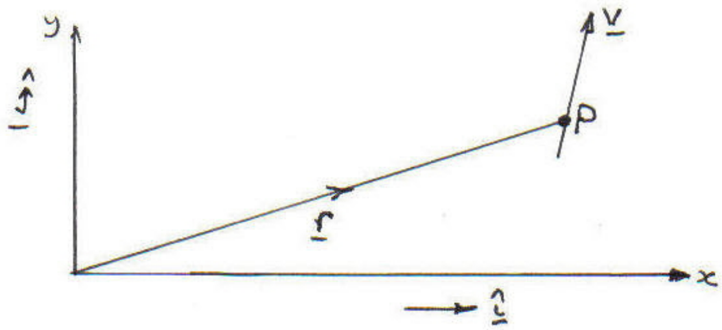

Figure 30: Vectors for motion in two dimensions

$$\begin{aligned}
\mathbf{r} &= x\widehat{\mathbf{i}} + y\widehat{\mathbf{j}}, && (164)\\
\mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\widehat{\mathbf{i}} + \frac{\mathrm{d}y}{\mathrm{d}t}\widehat{\mathbf{j}}, && (165)\\
\mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\widehat{\mathbf{i}} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\widehat{\mathbf{j}}. && (166)
\end{aligned}$$

We will consider various examples of motion in a plane

11.1 Trajectories of particle in a uniform gravitational field (ballistic trajectories)

Initial velocity $\mathbf{u}_0$ makes an angle α with the horizontal (sometimes called the angle of elevation) as in fig 31. Equation of motion is

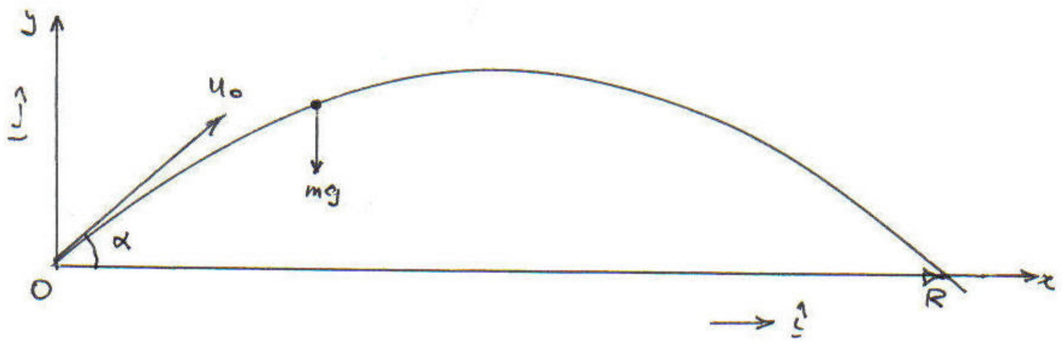

Figure 31: Ballistic trajectory

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\left(\frac{\mathrm{d}v_x}{\mathrm{d}t}\widehat{\mathbf{i}} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\widehat{\mathbf{j}}\right) = -mg\widehat{\mathbf{j}}. \tag{167}$$

Hence immediately we see that

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = 0 \tag{168}$$

and so v_x is constant, equal to its initial value

$$v_x(t) = u_0 \cos\alpha, \tag{169}$$
$$x(t) = u_0 t \cos\alpha. \tag{170}$$

Also

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -g, \tag{171}$$
$$v_y = u_0 \sin\alpha - gt, \tag{172}$$
$$y = u_0 t \sin\alpha - \frac{1}{2}gt^2, \tag{173}$$

and (from $v^2 = u^2 + 2as$)

$$v_y^2 = u_0^2 \sin^2\alpha - 2gy. \tag{174}$$

11.1.1 Shape of trajectory

Since $x(t) = u_0 t \cos\alpha$ then $t = x/(u_0\cos\alpha)$. Hence using the y position, eq(173)

$$y = u_0\left(\frac{x}{u_0\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{u_0\cos\alpha}\right)^2 \tag{175}$$
$$= x\tan\alpha - \left(\frac{gx^2}{2u_0^2\cos^2\alpha}\right). \tag{176}$$

This is the equation of a parabola.

The range R on the horizontal plane is found from setting $y = 0$. Then

$$0 = x \tan\alpha - \frac{1}{2} g \left(\frac{x^2}{u_0^2 \cos^2 \alpha} \right) \tag{177}$$

which has one solution, $x = 0$, and the other

$$x = R = \frac{2u_0^2}{g} \sin\alpha \cos\alpha = \frac{u_0^2}{g} \sin 2\alpha. \tag{178}$$

Time to reach $y = 0$ is found from

$$0 = u_0 t \sin\alpha - \frac{1}{2} g t^2 \tag{179}$$

giving either $t = 0$ (the initial position) or for the whole range

$$T = \frac{2u_0 \sin\alpha}{g}. \tag{180}$$

Maximum range occurs when $\sin 2\alpha = 1$, so when $\alpha = \pi/4 = 45°$, and $R_{\max} = u_0^2/g$.

For a range $R < R_{\max}$ there are two values of α which give rise to the same range, as illustrated in fig 32 and fig 33.

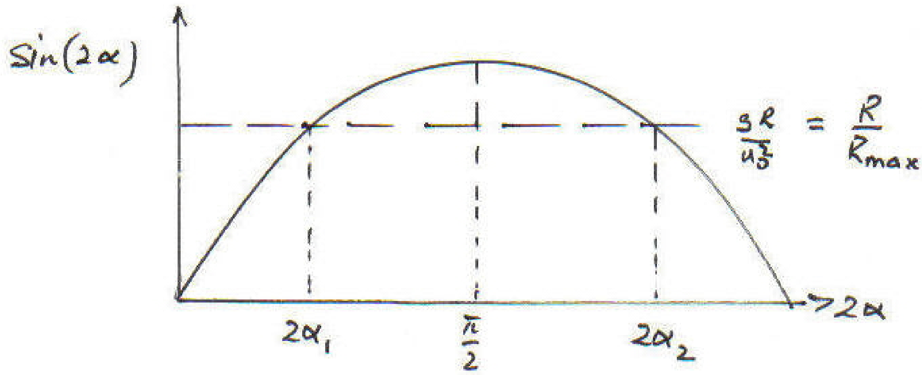

Figure 32: Angle for different ranges

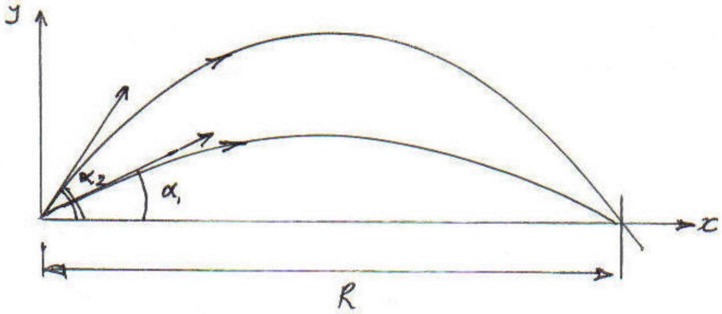

Figure 33: Two trajectories giving the same range

Since $R = R_{\max} \sin 2\alpha$, and $\sin 2\alpha = \sin(\pi - 2\alpha)$ then

$$2\alpha_1 = \sin^{-1}\left(\frac{R}{R_{\max}}\right) = \sin^{-1}\left(\frac{gR}{u_0^2}\right), \tag{181}$$

$$(\pi - 2\alpha_2) = \sin^{-1}\left(\frac{R}{R_{\mathrm{max}}}\right) = 2\alpha_1. \tag{182}$$

Hence

$$\alpha_2 = \frac{\pi}{2} - \alpha_1 \tag{183}$$

and then

$$\alpha_2 - \frac{\pi}{4} = \frac{\pi}{4} - \alpha_1 \tag{184}$$

so we see that the two angles for the same range are symmetric about $\pi/4$, see fig 33. Maximum range occurs when $\alpha_1 = \alpha_2 = \pi/4$.

12 Conservation of momentum for isolated systems of particles

From Newton's second law, $\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$, if $\mathbf{F} = 0$, then $\mathbf{p}$ is constant. Consider a body of mass M moving with velocity $\mathbf{V}$ and not subject to any forces. Its momentum remains constant, i.e. $\mathbf{p} = M\mathbf{V} =$ constant. The body then explodes into several pieces (say at least three) with masses m_i, $i = 1, 2, 3\ldots$ travelling with velocities $\mathbf{v}_i$, $i = 1, 2, 3\ldots$ as illustrated in fig 34. The total momentum of the pieces is

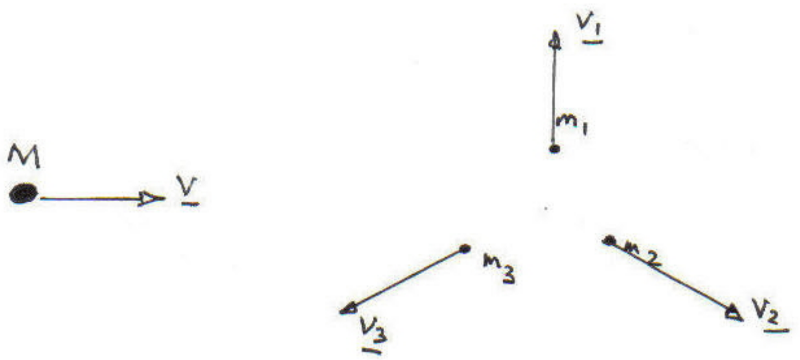

Figure 34: Exploding mass

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + \ldots. \tag{185}$$

The explosion gives impulses $\mathbf{I}_i$ to each of the three pieces. Because of Newton's third law (action and reaction are equal an opposite). If impulse on particle 1 due to particle 2 is $\mathbf{I}_{12}$ and that due to particle 2 on particle 1 is $\mathbf{I}_{21}$, then

$$\mathbf{I}_{12} = \mathbf{I}_{21}. \tag{186}$$

For particle 1 the total impulse on it due to the other particles is

$$\mathbf{I}_1 = \mathbf{I}_{12} + \mathbf{I}_{13} \tag{187}$$

and for the whole system we have

$$\begin{aligned} \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 &= (\mathbf{I}_{12} + \mathbf{I}_{13}) + (\mathbf{I}_{21} + \mathbf{I}_{23}) + (\mathbf{I}_{31} + \mathbf{I}_{32}) && (188)\\ &= (\mathbf{I}_{12} + \mathbf{I}_{21}) + (\mathbf{I}_{23} + \mathbf{I}_{21}) + (\mathbf{I}_{31} + \mathbf{I}_{32}) && (189)\\ &= 0 + 0 + 0 && (190) \end{aligned}$$

Thus total impulse is zero and so the total momentum is the **same before and after** the explosion,

$$\begin{aligned} \mathbf{p} &= \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots && (191)\\ M\mathbf{V} &= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + \ldots && (192) \end{aligned}$$

that is, **total momentum is conserved**. In this case (non-relativistic) mass is conserved,

$$M = m_1 + m_2 + m_3 + \ldots. \tag{193}$$

On the other hand **kinetic energy is not conserved**,

$$\frac{1}{2}MV^2 \neq \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 + \dots \tag{194}$$

but the extra kinetic energy comes from chemical potential energy or energy stored in springs.

Example 10 *Two particles, one initially at rest, collide and stick together (opposite of the explosion). Initial momentum is*

$$\mathbf{p}_i = m_1 \mathbf{v}_1 + 0. \tag{195}$$

Final momentum is

$$\mathbf{p}_f = M\mathbf{V} = (m_1 + m_2)\,\mathbf{V}. \tag{196}$$

Since total momentum is conserved, $\mathbf{p}_i = \mathbf{p}_f$*, so* $m_1\mathbf{v}_1 = (m_1 + m_2)\,\mathbf{V}$ *and*

$$\mathbf{V} = \frac{m_1}{m_1 + m_2}\mathbf{v}_1 \tag{197}$$

and combined particle moves in the same direction as the incoming one, but at a reduced speed. The initial kinetic energy is

$$K_{\mathrm{Ei}} = \frac{1}{2}m_1 v_1^2 + 0. \tag{198}$$

The final kinetic energy

$$T_f = \frac{1}{2}(m_1 + m_2)\,V^2 = \frac{1}{2}(m_1 + m_2)\left(\frac{m_1}{m_1 + m_2}v_1\right)^2 = \frac{1}{2}\frac{m_1^2}{(m_1 + m_2)}v_1^2 < \frac{1}{2}m_1 v_1^2. \tag{199}$$

*Thus there is a **loss** of kinetic energy in a 'sticking' collision. The energy is converted into heat and/or sound.*

13 Collisions between bodies

These are very important in many branches of physics and chemistry.

13.1 With a rigid wall

If wall is smooth, i.e. no friction, then impulse given to the particle by the wall is perpendicular to the wall, there is no component parallel to the wall's surface. as in fig 35 Hence tangential component of velocity is unaltered so

$$v_i \sin\alpha = v_f \sin\beta. \tag{200}$$

If collision is perfectly elastic, i.e. no loss of kinetic energy,

$$\frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 \tag{201}$$

and so

$$v_i = v_f, \tag{202}$$
$$\alpha = \beta. \tag{203}$$

Thus the normal component of velocity is simply reversed in the collision.

In practice the normal component of velocity is reduced by a factor $e < 1$ (called coefficient of restitution), such that

$$v_f \cos\beta = e v_i \cos\alpha. \tag{204}$$

Since

$$v_f \sin\beta - v_i \sin\alpha \tag{205}$$

and

$$\tan\beta = \frac{\tan\alpha}{e}, \tag{206}$$

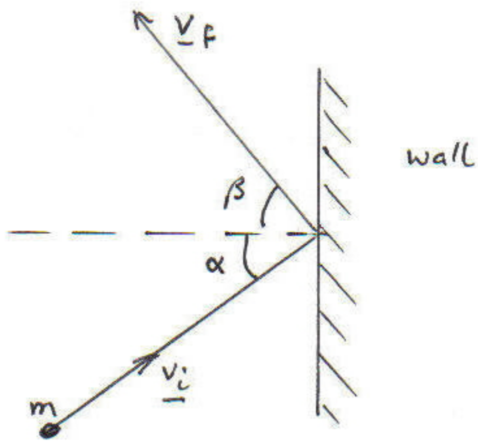

Figure 35: Collision of small mass with a rigid wall

so $v_f < v_i$ and $\beta > \alpha$. There is a loss of kinetic energy since

$$v_f^2 = v_i^2(\sin^2\alpha + e^2\cos^2\alpha) = v_i^2 + v_i^2\left(e^2 - 1\right)\cos^2\alpha \quad (207)$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \frac{1}{2}mv_i^2\left(e^2 - 1\right)\cos^2\alpha < 0. \quad (208)$$

For an elastic collision, $e = 1$, and kinetic energy is conserved

13.2 Between two bodies of finite mass, one initially at rest

No external forces act during the collision so the total momentum is conserved (always true).

13.2.1 Head-on elastic collision ($e = 1$)

Figure 36: Head-on elastic collision

Conservation of momentum gives from fig 36

$$m_1u_1 + 0 = m_1v_1 + m_2v_2, \quad (209)$$

$$v_2 = \frac{m_1}{m_2}\left(u_1 - v_1\right) \quad (210)$$

Conservation of kinetic energy gives

$$\begin{aligned}\frac{1}{2}m_1u_1^2 + 0 &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 && (211)\\
m_1u_1^2 &= m_1v_1^2 + m_2\left[\frac{m_1}{m_2}(u_1 - v_1)\right]^2 && (212)\\
u_1^2 &= v_1^2 + \frac{m_1}{m_2}(u_1 - v_1)^2 && (213)\\
u_1^2 - v_1^2 &= \frac{m_1}{m_2}(u_1 - v_1)^2 && (214)\\
(u_1 - v_1)(u_1 + v_1) &= \frac{m_1}{m_2}(u_1 - v_1)^2. && (215)\end{aligned}$$

Thus one solution is $v_1 = u_1$. The other is found from

$$\begin{aligned}(u_1 + v_1) &= \frac{m_1}{m_2}(u_1 - v_1), && (216)\\
v_1 &= \frac{(m_1 - m_2)}{(m_1 + m_2)}u_1. && (217)\end{aligned}$$

The corresponding velocities for particle 2 are

$$\begin{aligned}v_2 &= 0, && (218)\\
v_2 &= \frac{m_1}{m_2}\left(u_1 - \frac{(m_1 - m_2)}{(m_1 + m_2)}u_1\right) && (219)\\
v_2 &= \frac{2m_1u_1}{(m_1 + m_2)}. && (220)\end{aligned}$$

The solution $v_1 = u_1$, $v_2 = 0$ corresponds to no collision. The non-trivial solution is

$$v_1 = \frac{(m_1 - m_2)}{(m_1 + m_2)}u_1 < u_1 \qquad v_2 = \frac{2m_1u_1}{(m_1 + m_2)} > 0. \tag{221}$$

If $m_1 > m_2$ then v_1 is positive; if $m_1 < m_2$ then v_1 is negative. If $m_1 = m_2$ then $v_1 = 0$ and $v_2 = u_1$, the incident particle is brought to rest and the struck one moves of with the initial velocity. If $m_1 \gg m_2$ then $v_1 \simeq u_1$ and $v_2 \simeq 2u_1$. If $m_2 \gg m_1$ then $v_1 \simeq -u_1$ and $v_2 \simeq 0$, i.e. collision with a "fixed" wall.

13.2.2 Glancing collision of two balls

We shall assume balls are smooth, so that the impulse on each ball can only be along line of their centres. Suppose ball 2 is initially at rest as shown in fig 37. The impulse on it, $\mathbf{I}_2 = m_2\mathbf{v}_2$, is along the line of centres, so the direction of $\mathbf{v}_2$ is along line of ball centres at impact. Impulse on ball 1 is

$$\mathbf{I}_1 = -\mathbf{I}_2 = m\mathbf{v}_1 - m_1\mathbf{u}_1. \tag{222}$$

The direction of $\mathbf{v}_2$ is determined by angle ϕ, the angle between line of centres and the initial direction of $\mathbf{u}_1$.

Momentum is conserved in the collision, so

$$m_1\mathbf{u}_1 + 0 = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 \tag{223}$$

If collision is elastic then kinetic energy is conserved and

$$\frac{1}{2}m_1u_1^2 + 0 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2. \tag{224}$$

Expressing momentum in the x and y directions

$$m_1u_1 = m_1v_1\cos\theta + m_2v_2\cos\phi, \tag{225}$$

$$m_1v_1\sin\theta - m_2v_2\sin\phi = 0. \tag{226}$$

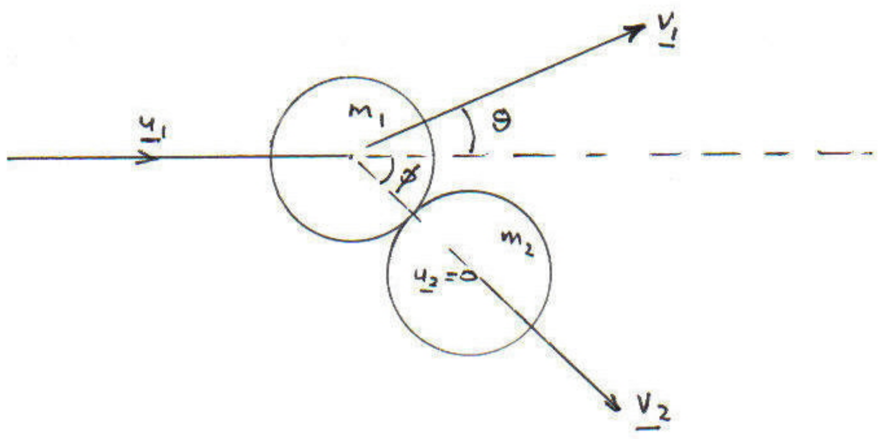

Figure 37: Glancing collision

The angle ϕ is determined by the geometry of the collision. Thus we have three equations to solve for three unknowns v_1, v_2 and θ. The solution is rather complicated for general masses m_1 and m_2 but is quite simple for the special case of $m_1 = m_2$. Then conservation of momentum gives

$$m\mathbf{u}_1 = m\mathbf{v}_1 + m\mathbf{v}_2, \tag{227}$$
$$\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2. \tag{228}$$

Kinetic energy conservation gives

$$\frac{1}{2}mu_1^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \tag{229}$$
$$u_1^2 = v_1^2 + v_2^2 \tag{230}$$

But from the magnitude of $\mathbf{u}_1$

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = (\mathbf{v}_1 + \mathbf{v}_2) \cdot (\mathbf{v}_1 + \mathbf{v}_2) \tag{231}$$
$$u_1^2 = v_1^2 + v_2^2 + 2\mathbf{v}_1 \cdot \mathbf{v}_2 \tag{232}$$

Comparing equations eq(230) and eq(232) we see that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \tag{233}$$

and so $\mathbf{v}_1$ is perpendicular to $\mathbf{v}_2$, and $\theta + \phi = \pi/2$.

14 Motion in a plane expressed in plane polar coordinates

Particle of mass m is at position P defined by Cartesian coordinates (x, y) or polar coordinates (r, θ) or vector $\mathbf{r}$ as shown in fig 38. Let us introduce a unit radial vector $\widehat{\mathbf{r}}$ and unit transverse vector $\widehat{\boldsymbol{\theta}}$; $\widehat{\mathbf{r}}$ and $\widehat{\boldsymbol{\theta}}$ are orthogonal (perpendicular) to each other. The position P is $\mathbf{r} = r\widehat{\mathbf{r}}$. In terms of unit vectors along the fixed Cartesian axes

$$\widehat{\mathbf{r}} = \cos\theta\widehat{\mathbf{i}} + \sin\theta\widehat{\mathbf{j}} \tag{234}$$
$$\widehat{\boldsymbol{\theta}} = -\sin\theta\widehat{\mathbf{i}} + \cos\theta\widehat{\mathbf{j}}. \tag{235}$$

As the particle moves the **directions** of $\widehat{\mathbf{r}}$ and $\widehat{\boldsymbol{\theta}}$ **vary**.

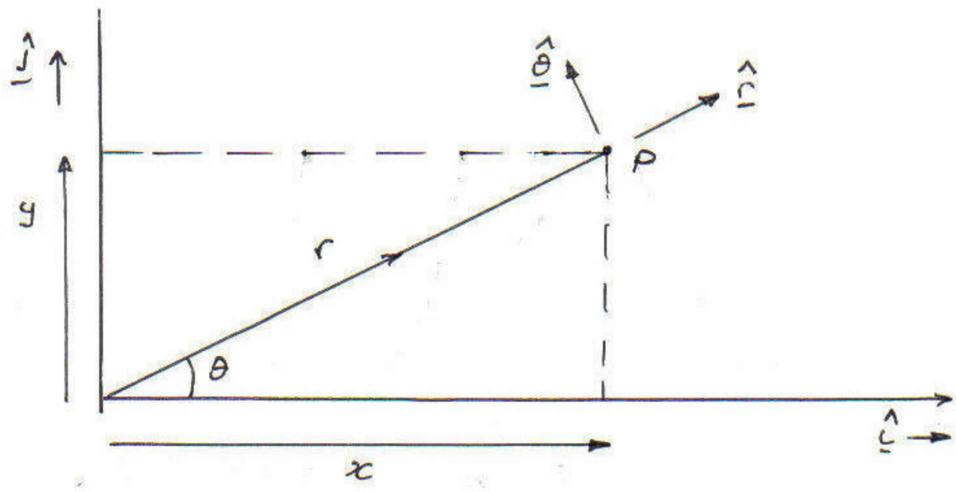

Figure 38: Diagram for unit vectors for two-dimensional motion

Consider velocity of the particle,

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(r\widehat{\mathbf{r}}\right) = \frac{\mathrm{d}r}{\mathrm{d}t}\widehat{\mathbf{r}} + r\frac{\mathrm{d}\widehat{\mathbf{r}}}{\mathrm{d}t}. \tag{236}$$

An expression for $\mathrm{d}\widehat{\mathbf{r}}$ can be found algebraically or graphically as shown in fig 39. Algebraically we have

$$\begin{aligned}
\widehat{\mathbf{r}} &= \cos\theta\widehat{\mathbf{i}} + \sin\theta\widehat{\mathbf{j}}, && (237)\\
\frac{\mathrm{d}\widehat{\mathbf{r}}}{\mathrm{d}t} &= -\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\mathbf{i}} + \cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\mathbf{j}}, && (238)\\
&= \frac{\mathrm{d}\theta}{\mathrm{d}t}\left(-\sin\theta\widehat{\mathbf{i}} + \cos\theta\widehat{\mathbf{j}}\right), && (239)\\
\frac{\mathrm{d}\widehat{\mathbf{r}}}{\mathrm{d}t} &= \frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\boldsymbol{\theta}}. && (240)
\end{aligned}$$

So

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\widehat{\mathbf{r}} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\boldsymbol{\theta}}. \tag{241}$$

The radial component of $\mathbf{v}$ is $\frac{\mathrm{d}r}{\mathrm{d}t}$; transverse component is $r\frac{\mathrm{d}\theta}{\mathrm{d}t}$.

An often used abbreviation for differentiation with respect to time is to put a dot above the quantity, so the above expression for $\mathbf{v}$ is written as

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\widehat{\mathbf{r}} + r\dot{\theta}\widehat{\boldsymbol{\theta}} \tag{242}$$

Speed of the particle is

$$v = |\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}. \tag{243}$$

Consider acceleration of the particle

$$\begin{aligned}
\mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\widehat{\mathbf{r}} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\boldsymbol{\theta}}\right) && (244)\\
&= \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\widehat{\mathbf{r}} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\widehat{\mathbf{r}}}{\mathrm{d}t} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\boldsymbol{\theta}} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\widehat{\boldsymbol{\theta}} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\widehat{\boldsymbol{\theta}}}{\mathrm{d}t}. && (245)
\end{aligned}$$

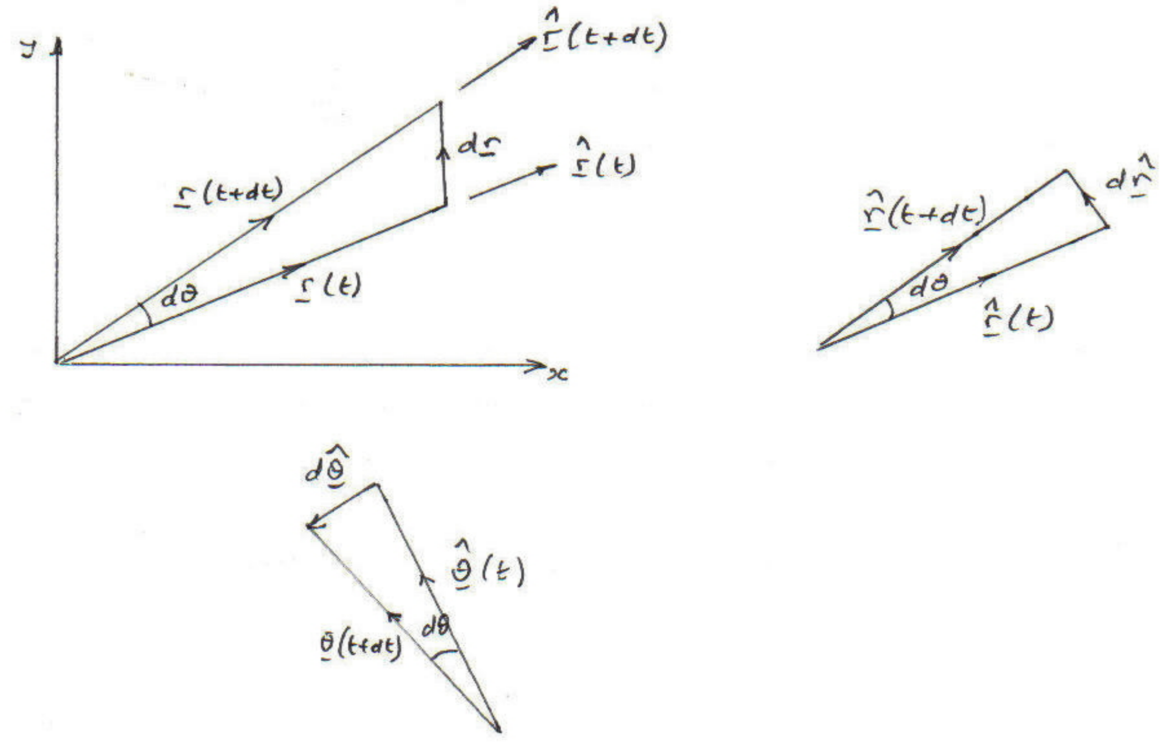

Figure 39: Time dependence of polar vectors

The time derivative $\frac{\mathrm{d}\widehat{\boldsymbol{\theta}}}{\mathrm{d}t}$ can be found also algebraically or graphically, as illustrated in fig 39.

$$\widehat{\boldsymbol{\theta}} = -\sin\theta\widehat{\mathbf{i}} + \cos\theta\widehat{\mathbf{j}}, \tag{246}$$

$$\frac{\mathrm{d}\widehat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\mathbf{i}} - \sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\mathbf{j}}, \tag{247}$$

$$\frac{\mathrm{d}\widehat{\boldsymbol{\theta}}}{\mathrm{d}t} = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\left(\cos\theta\widehat{\mathbf{i}} + \sin\theta\widehat{\mathbf{j}}\right) = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\mathbf{r}}. \tag{248}$$

Hence the acceleration

$$\mathbf{a} = \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\widehat{\mathbf{r}} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\boldsymbol{\theta}} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\boldsymbol{\theta}} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\widehat{\boldsymbol{\theta}} - r\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\widehat{\mathbf{r}}, \tag{249}$$

$$= \left[\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]\widehat{\mathbf{r}} + \left[2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right]\widehat{\boldsymbol{\theta}}, \tag{250}$$

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\widehat{\boldsymbol{\theta}}. \tag{251}$$

The radial component of the acceleration is $\left(\ddot{r} - r\dot{\theta}^2\right)$, the transverse component is $\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)$.

14.1 Circular motion

Consider a particle moving in a circle of radius r as shown in fig 40. Then r is constant, so $\dot{r} = 0$, $\ddot{r} = 0$. Hence

$$\mathbf{v} = r\dot{\theta}\widehat{\boldsymbol{\theta}} \tag{252}$$

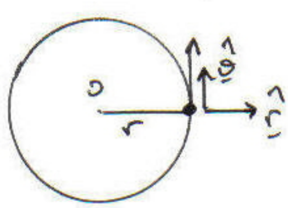

Figure 40: Motion in a circle

and the velocity is purely transverse.

We sometimes write $\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}$, in which case

$$\mathbf{v} = r\omega\widehat{\boldsymbol{\theta}}. \tag{253}$$

The acceleration

$$\mathbf{a} = \left(-r\dot{\theta}^2\right)\widehat{\mathbf{r}} + r\ddot{\theta}\widehat{\boldsymbol{\theta}}. \tag{254}$$

If the angular velocity is constant, $\dot{\omega} = \ddot{\theta} = 0$, and

$$\mathbf{a} = -r\dot{\theta}^2\widehat{\mathbf{r}} = -r\omega^2\widehat{\mathbf{r}}. \tag{255}$$

Since $\mathbf{v} = r\omega\widehat{\boldsymbol{\theta}}$ then for the magnitudes $v = \omega r$ and so

$$\mathbf{a} = -\frac{v^2}{r}\widehat{\mathbf{r}} \tag{256}$$

and the acceleration is directed towards the centre of the circle. This acceleration is called the **centripetal acceleration**. It follows that there **must** be a force $\mathbf{F} = m\mathbf{a} = -\, m\omega^2 r\widehat{\mathbf{r}} = -m\frac{v^2}{r}\widehat{\mathbf{r}}$ acting radially.

15 Central force

A central force is one which is purely directed along a radius, either towards or away from the centre, so that

$$\mathbf{F}(\mathbf{r}) = F(r)\,\widehat{\mathbf{r}}. \tag{257}$$

If $F(r)$ is **positive** the force is **repulsive** or **centrifugal**; if $F(r)$ is **negative** the force is **attractive** or **centripetal**. Examples of central forces are

(a) electrostatic force between point or spherically symmetric charges; like charges repel and unlike charges attract,

(b) gravitational force

$$\mathbf{F}(\mathbf{r}) = -G\frac{M_1 M_2}{r^2}\widehat{\mathbf{r}} \tag{258}$$

between point or spherically symmetric masses; this is always attractive.

(c) tension in a string or spring.

The expressions above for velocity (see fig 41) and acceleration in plane polar coordinates are particularly important in the case of central forces. The equation of motion is

$$m\mathbf{a} = \mathbf{F} = F(r)\,\widehat{\mathbf{r}}, \tag{259}$$

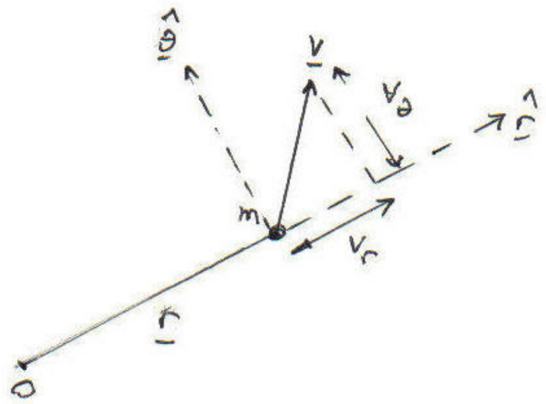

Figure 41: Vectors for rotational motion

$$m\left[\left(\ddot{r}-r\dot{\theta}^{2}\right)\widehat{\mathbf{r}}+\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)\widehat{\boldsymbol{\theta}}\right]=F\left(r\right)\widehat{\mathbf{r}} \tag{260}$$

We see that that the force **only** enters into the radial component,

$$m\left(\ddot{r}-r\dot{\theta}^{2}\right)=F\left(r\right) \tag{261}$$

and the transverse component

$$m\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)=0. \tag{262}$$

is independent of the nature of the force. Hence one may therefore make significant deductions about the motion regardless of the force. The radial component of velocity is $v_r=\dot{r}$ and the transverse component is $v_\theta=r\dot{\theta}$. We **define the magnitude of the angular momentum**, L, of the particle about point O as

$$L=\left(mv_\theta\right)r=mr^2\dot{\theta}. \tag{263}$$

[In general, to be discussed later, angular momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}=m\mathbf{r}\times\mathbf{v}$. For $\mathbf{r}$ and $\mathbf{v}$ in the plane of the paper, then $\mathbf{L}$ is out of the paper and perpendicular to it.] Thus

$$\begin{aligned} L &= mr^2\dot{\theta} && (264)\\ \frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right)=m\left(2r\dot{r}\dot{\theta}+r^2\ddot{\theta}\right) && (265)\\ &= mr\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)=mra_\theta && (266)\end{aligned}$$

where we have used the expression for the transverse acceleration $a_\theta=2\dot{r}\dot{\theta}+r\ddot{\theta}$. But for a central force the transverse acceleration must be zero, so

$$\frac{\mathrm{d}L}{\mathrm{d}t}=0, \tag{267}$$

and hence the angular momentum L is **constant** during the motion for **any central force**.

15.1 Examples of central force motion

15.1.1 Motion in a circle

(1) Consider a planet, mass m, moving in a circular orbit about a star of mass M as in fig 42 The

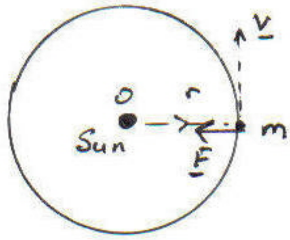

Figure 42: Planet in circular motion about Sun

centripetal force is due to gravity,

$$\begin{aligned}
\mathbf{F}(\mathbf{r}) &= -G\frac{Mm}{r^2}\widehat{\mathbf{r}} && (268)\\
-G\frac{Mm}{r^2}\widehat{\mathbf{r}} &= m\left(-r\dot{\theta}^2\right)\widehat{\mathbf{r}} && (269)\\
r\dot{\theta}^2 &= \frac{GM}{r^2} && (270)\\
\dot{\theta}^2 &= \omega^2 = \frac{GM}{r^3}. && (271)
\end{aligned}$$

The period of rotation $T = 2\pi/\omega$, so

$$\begin{aligned}
\left(\frac{2\pi}{T}\right)^2 &= \frac{GM}{r^3} && (272)\\
T^2 &= \frac{4\pi^2}{GM}r^3 && (273)\\
T^2 &\propto r^3, && (274)
\end{aligned}$$

as expressed in Kepler's third law of planetary motion.

Speed of the planet is $v = r\omega$, so

$$v = \sqrt{\frac{GM}{r}}. \qquad (275)$$

(2) Consider a car of mass m moving in a circular path of radius r on a horizontal surface with coefficient of friction μ as in fig 43. What is the maximum speed without skidding sideways (ignore toppling over for the moment)? The centripetal force necessary for it to travel round the circular path is

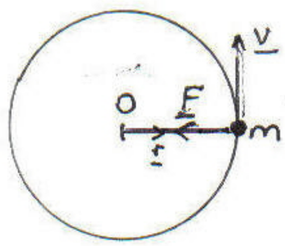

Figure 43: Car in circular motion on horizontal track

$$\mathbf{F} = m\left(-\frac{v^2}{r}\right)\widehat{\mathbf{r}} \qquad (276)$$

and this must be provided by the frictional force between the wheels and the ground. So as the normal reaction $N = mg$, then

$$m\left(-\frac{v^2}{r}\right)\widehat{\mathbf{r}} = -\mathbf{F}_{friction} \tag{277}$$

$$m\left(\frac{v^2}{r}\right) = \mu mg \tag{278}$$

$$v_{\max} = \sqrt{\mu g r}. \tag{279}$$

If $\mu = 1$, $g \simeq 10\,\mathrm{m\,s^{-2}}$ then $v_{\max} \simeq \sqrt{10r}\,\mathrm{m\,s^{-1}}$. Thus for $r = 20\,\mathrm{m}$, $v_{\max} \simeq 14\,\mathrm{m\,s^{-1}} \simeq 31$ mph.

Example 11 *(From 2000 exam paper, question 11.)*

Particle of mass m moves on a smooth horizontal table subjected to a force

$$\mathbf{F} = -\frac{K}{r^3}\widehat{\mathbf{r}}. \tag{280}$$

If particle is initially moving in a circle of radius r as in fig 44we will determine the speed of the particle and its angular momentum. For this motion,

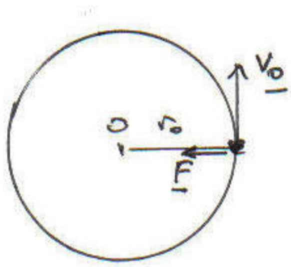

Figure 44: Motion in a circle

$$\mathbf{F} = -\frac{K}{r^3}\widehat{\mathbf{r}} = m\left(-r\dot{\theta}^2\right)\widehat{\mathbf{r}}, \tag{281}$$

so

$$\dot{\theta} = \frac{1}{r^2}\sqrt{\frac{K}{m}}. \tag{282}$$

The initial value of the angular momentum at $r = r_0$ is

$$L = mr^2\dot{\theta} = \sqrt{mK}, \tag{283}$$

is independent of r_0 and so is the only possible value for the angular momentum for motion in a circle under an inverse cube force law.

Suppose particle is given a radially outward impulse. The impulse does not change the angular momentum so it remains at a value $\sqrt{mK}$ for any subsequent motion. Radial equation of motion after the impulse is

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{K}{r^3}. \tag{284}$$

But $\dot{\theta} = \frac{1}{r^2}\sqrt{\frac{K}{m}}$ so

$$m\left[\ddot{r} - r\left(\frac{1}{r^2}\sqrt{\frac{K}{m}}\right)^2\right] = -\frac{K}{r^3}, \tag{285}$$

$$m\ddot{r} = 0. \tag{286}$$

Hence $\dot{r}$ is constant. So particle moves in a spiral with constant radial component of velocity as illustrated in fig 45.

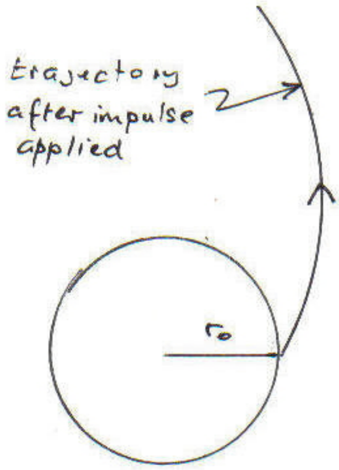

Figure 45: Trajectory of particle

Example 12 *Motion of a particle attached to a string and moving in a vertical circle, radius R, under gravity, see fig 46. Assume string remains taut throughout the motion. At highest point, A, force is*

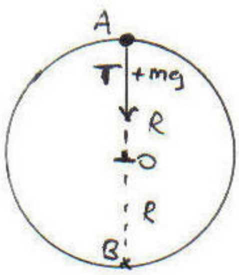

Figure 46: Motion of mass on string swung in circle in vertical plane

$$\mathbf{F} = -\left(T_A + mg\right)\widehat{\mathbf{r}} = m\left(-R\dot{\theta}_A^2\right)\widehat{\mathbf{r}} \tag{287}$$

giving$_A$

$$\dot{\theta}_A = \sqrt{\frac{(T_A + mg)}{mR}}. \tag{288}$$

The velocity at A

$$v_A = R\dot{\theta}_A = \sqrt{\frac{(T_A + mg)\,R}{m}}. \tag{289}$$

The minimum speed at A is when the tension $T_A = 0$*, i.e.*

$$v_{A\,\min} = \sqrt{gR}. \tag{290}$$

By conservation of energy at lowest point B

$$\frac{1}{2}mv_B^2 = \frac{1}{2}mv_A^2 + mg\,(2R) \tag{291}$$

$$v_B^2 = v_A^2 + 4mgR. \tag{292}$$

At lowest point B the tension in the string T_B, is given by

$$(-T_B + mg)\,\widehat{\mathbf{r}} = m\left(-R\dot{\theta}_B^2\right)\widehat{\mathbf{r}} \tag{293}$$

$$T_B - mg = m\left(R\dot{\theta}_B^2\right) = m\frac{v_B^2}{R} \tag{294}$$

$$= \frac{m}{R}\left(v_A^2 + 4mgR\right). \tag{295}$$

If $v_A = v_{A\,\min} = \sqrt{gR}$ then

$$T_B = mg + \frac{m}{R}(gR + 4mgR) = 6mg. \tag{296}$$

16 Motion under inverse square law of force

Motion under an inverse square law of force has been extensively analyzed because gravitation is such a force. This is a central force with

$$\mathbf{F} = \frac{K}{r^2}\widehat{\mathbf{r}}. \tag{297}$$

Thus the potential energy function is

$$V = \frac{K}{r} \tag{298}$$

since

$$\mathbf{F} = -\boldsymbol{\nabla} V = -\widehat{\mathbf{r}}\frac{\partial V}{\partial r} = \frac{K}{r^2}\widehat{\mathbf{r}}. \tag{299}$$

Note $V \to 0$ as $r \to \infty$. If K is positive the force is repulsive; if K is negative the force is attractive. For the gravitational force $K = -GMm$ and is always attractive. The gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$. For the electrostatic force $K = Q_1 Q_2/(4\pi\varepsilon_0)$ where Q_1 and Q_2 are the charges which can be of either sign. The equation of motion of the particle is (see eq(260) and fig 47)

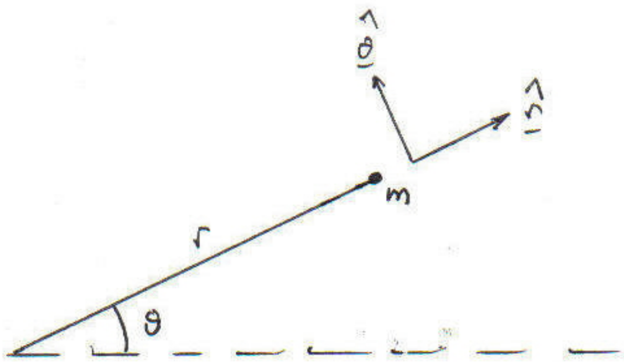

Figure 47: Rotational motion in a plane

$$m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\widehat{\boldsymbol{\theta}}\right] = \frac{K}{r^2}\widehat{\mathbf{r}}. \tag{300}$$

Thus for the radial component

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = \frac{K}{r^2} \tag{301}$$

and for the transverse component

$$\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0. \tag{302}$$

It was noted earlier that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0. \tag{303}$$

Thus as $r \neq 0$, then $r^2\dot{\theta}$ is constant, and so the angular momentum $L = mr^2\dot{\theta}$ is also constant. Hence

$$\dot{\theta} = \frac{L}{mr^2} \tag{304}$$

and the radial equation can therefore be written in the form

$$m\left(\ddot{r} - \frac{L^2}{m^2r^3}\right) = \frac{K}{r^2}. \tag{305}$$

Instead of solving for r and θ as functions of time we will consider the shape of the orbit, i.e. determine r as a function of θ. To do this we define $u = 1/r$. We then have

$$\begin{aligned}
\dot{r} &= \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}, && (306)\\
\dot{r} &= -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\left(\frac{L}{mr^2}\right) = -\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}. && (307)
\end{aligned}$$

Differentiating again we get

$$\begin{aligned}
\ddot{r} &= \frac{\mathrm{d}\dot{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{L}{m}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\left(\frac{L}{mr^2}\right) && (308)\\
\ddot{r} &= -\frac{L^2}{m^2}u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}. && (309)
\end{aligned}$$

In terms of u the radial equation (305)is

$$m\left(-\frac{L^2}{m^2}u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - u^3\frac{L^2}{m^2}\right) = Ku^2. \tag{310}$$

Multiplying through by $-m/\left(L^2u^2\right)$ gives

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{mK}{L^2}. \tag{311}$$

In terms of the variable $y = u + mK/L^2$ this is the equation of simple harmonic motion,

$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} + y = 0 \tag{312}$$

with solution $y = A\cos(\theta - \theta_0)$. Thus the general solution of eq(311) is

$$u = A\cos(\theta - \theta_0) - \frac{mK}{L^2} = \frac{1}{r}. \tag{313}$$

This is of the same form as the general equation of a **conic section**, (see eq(317))

$$u = \frac{1}{r} = \frac{1}{h}\left(1 + e\cos\theta\right). \tag{314}$$

We can choose $\theta_0 = 0$ as this defines the orientation of the trajectory.

If K is negative, i.e. attractive force, the trajectory is of the form shown in fig 48.

If K is positive, i.e. repulsive force, the trajectory is of this form in fig 49

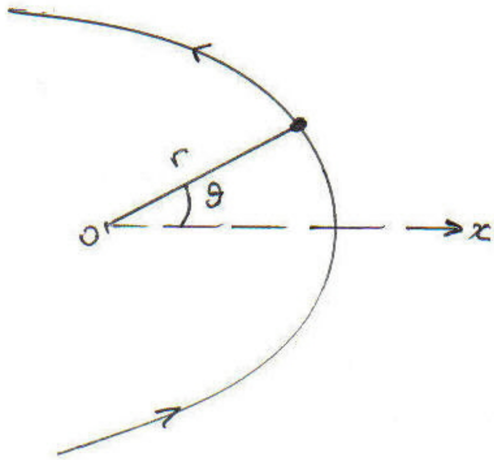

Figure 48: Trajectory for attractive inverse square force law, focus at O

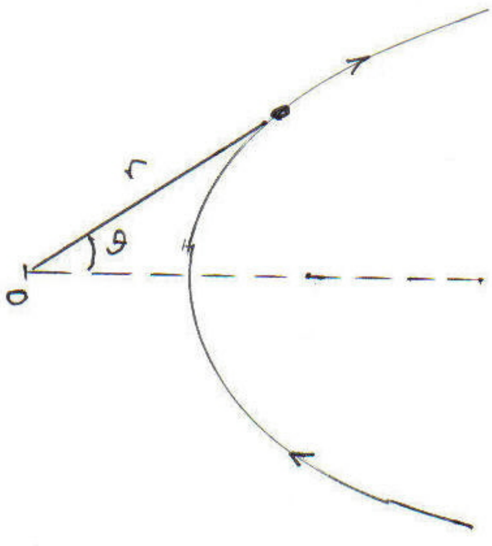

Figure 49: Trajectory for repulsive inverse square law, focus at O

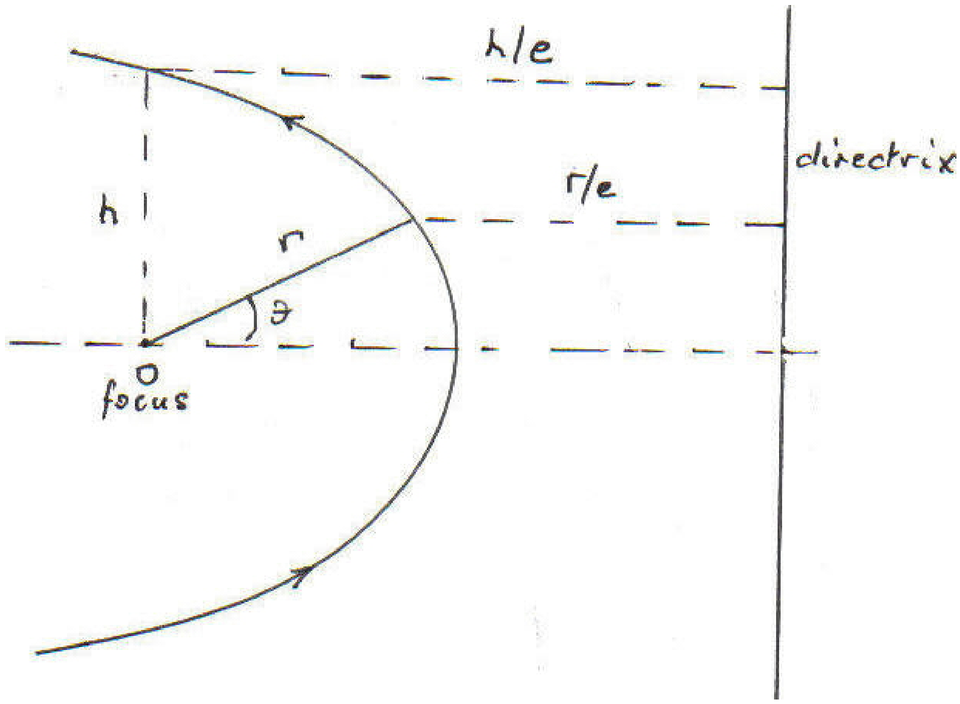

Figure 50: Definition of terms for conic section

16.1 Conic sections

A conic section is a locus of a point which moves in a plane such that its distance from a fixed point, the focus, is a constant ratio e , the eccentricity, to the distance from a fixed straight line, the directrix, see fig 50.

Consider an attractive force so that the relevant conic section is as in the diagram, fig50. The quantity h is the value of r when $\theta = \pi/2$. From the geometry and definitions of e and h, we have

$$r \cos\theta + \frac{r}{e} = \frac{h}{e}, \tag{315}$$

and

$$r\,(1 + e\cos\theta) = h, \tag{316}$$

or in terms of $u = 1/r$,

$$u = \frac{1}{r} = \frac{1}{h}\,(1 + e\cos\theta)\,. \tag{317}$$

We shall see that possible forms of the trajectory depend on whether the force is repulsive or attractive. For the attractive force it matters whether the total energy is positive, zero or negative.

16.1.1 Trajectories - Attractive force (K is negative)

For an attractive central force , such as gravity there are three types of orbit:

1. If total energy $E > 0$, eccentricity $e > 1$ the trajectory is a hyperbola, as in fig 51.
2. If $E = 0$, $e = 0$, the trajectory is a parabola, as in fig 52.
3. If $E < 0$, $e < 1$, the trajectory is an ellipse, as in fig 53.

A circle is a special case of an ellipse with zero eccentricity.

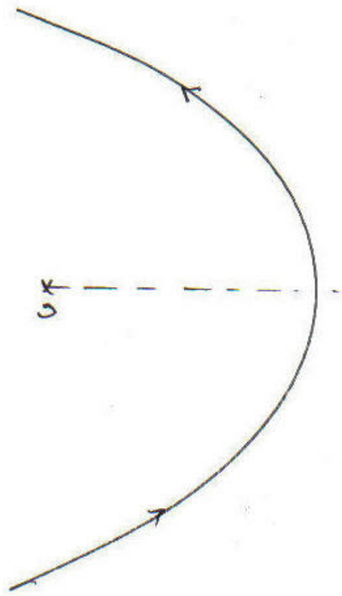

Figure 51: Hyperbola trajectory, $E > 0$, $e > 1$

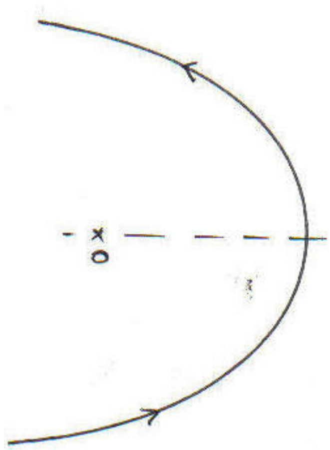

Figure 52: Parabola trajectory, $E = 0$, $e = 0$

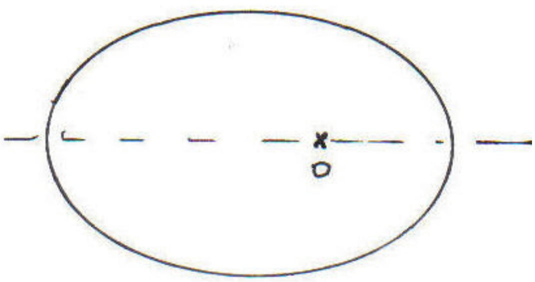

Figure 53: Ellipse trajectory, $E < 0$, $e < 1$

16.1.2 Determination of eccentricity

We shall determine the eccentricity of an orbit for given total energy E and angular momentum L. The total energy of the particle is

$$E = \frac{1}{2}mv^2 + V \tag{318}$$

and constant throughout the motion (gravitation is a conservative force). So

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{K}{r}. \tag{319}$$

But $\dot{r} = -\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}$, and $\dot{\theta} = \frac{L}{mr^2} = \frac{L}{m}u^2$ so

$$\begin{aligned} E &= \frac{1}{2}m\left[\left(\frac{L}{m}\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{1}{u^2}\left(\frac{L}{m}u^2\right)^2\right] + Ku && (320)\\ &= \frac{1}{2}m\left[\frac{L^2}{m^2}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + \frac{L^2}{m^2}u^2\right] + Ku && (321)\\ &= [\text{positive kinetic energy}] + \text{potential energy}. && (322)\end{aligned}$$

Hence we have that

(i) if K is positive then E must also be positive.

(ii) if K is negative then E may be positive, zero or negative.

The equation of the orbit is (see eq(313))

$$u = A\cos\theta - \frac{mK}{L^2}, \tag{323}$$

so

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = -A\sin\theta \tag{324}$$

and the expression for the total energy becomes

$$E = \frac{1}{2}m\left[\frac{L^2}{m^2}(-A\sin\theta)^2 + \frac{L^2}{m^2}\left(A\cos\theta - \frac{mK}{L^2}\right)^2\right] + K\left(A\cos\theta - \frac{mK}{L^2}\right). \tag{325}$$

Since the energy is constant throughout the motion we can evaluate the expression for E at any convenient point. Choose $\theta = \pi/2$. The energy is

$$\begin{aligned} E &= \frac{1}{2}m\left[\frac{L^2A^2}{m^2} + \frac{L^2}{m^2}\frac{m^2K^2}{L^4}\right] - \frac{mK^2}{L^2} && (326)\\ &= \frac{A^2L^2}{2m} - \frac{mK^2}{2L^2}. && (327)\end{aligned}$$

We can use this to determine the arbitrary constant A, since re-arranging

$$A^2 = \frac{2m}{L^2}\left(E + \frac{mK^2}{2L^2}\right) = \frac{m^2K^2}{L^4}\left(1 + \frac{2EL^2}{mK^2}\right), \tag{328}$$

and

$$A = \frac{m\,|K|}{L^2}\sqrt{\left(1 + \frac{2EL^2}{mK^2}\right)}. \tag{329}$$

Thus the orbit is determined completely if the values of E and L are known for a given K and m.

The solution for the motion is

$$u = A\cos\theta - \frac{mK}{L^2} \tag{330}$$

and the general form of a conic section is

$$\begin{aligned} u &= \frac{1}{h}(1 + e\cos\theta) && (331)\\ &= \frac{e}{h}\cos\theta + \frac{1}{h}. && (332)\end{aligned}$$

Hence we have

$$\frac{e}{h} = A, \quad (333)$$

$$\frac{1}{h} = -\frac{mK}{L^2}, \quad (334)$$

$$e = -\frac{L^2 A}{mK} = -\frac{|K|}{K}\sqrt{\left(1+\frac{2EL^2}{mK^2}\right)}. \quad (335)$$

For an attractive inverse square law (such as gravitation) K is negative and so

$$e = \sqrt{\left(1+\frac{2EL^2}{mK^2}\right)}. \quad (336)$$

Clearly
(a) if $E > 0$ then $e > 1$ and the motion is a hyperbola,
(b) if $E = 0$ then $e = 1$ and motion is a parabola,
(c) if $E < 0$ then $e < 1$ and motion is an ellipse. In this case the particle cannot escape to infinity because at infinity $V = 0$ and the kinetic energy ≥ 0. Therefore to escape to infinity requires $E \geq 0$.
Consider an elliptic orbit as in the diagram, If we know speed of particle at position of closest approach,

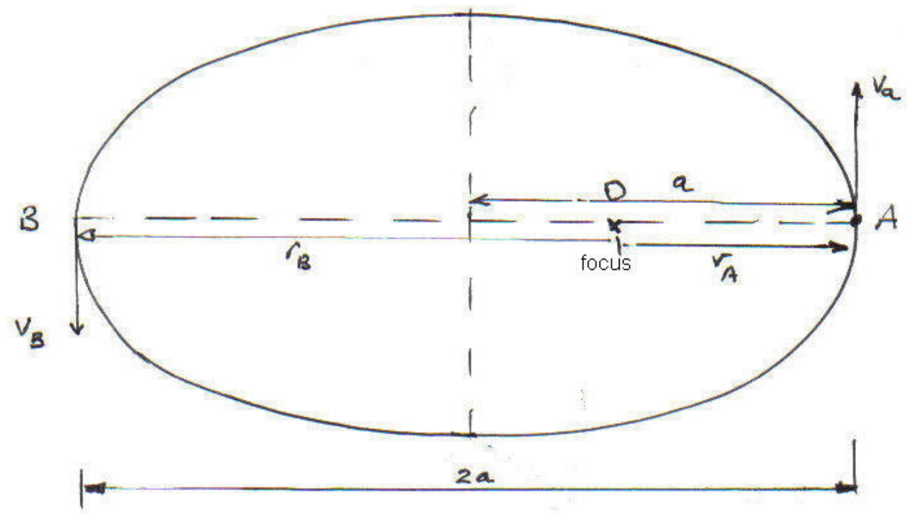

Figure 54: Parameters for an elliptical orbit

we know everything about the orbit. The angular momentum is

$$L = mr_A v_A \quad (337)$$

and energy is

$$E = \frac{1}{2}mv_A^2 - \frac{|K|}{r_A}. \quad (338)$$

We can determine the eccentricity e and the length of the semi-major axis, a. From general equation

$$r\,(1 + e\cos\theta) = h \quad (339)$$

we have for r_A $(\theta = 0)$ and r_B $(\theta = \pi)$,

$$r_A\,(1+e) = h; \qquad r_B\,(1-e) = h, \quad (340)$$

$$2a = r_A + r_B = \frac{h}{(1+e)} + \frac{h}{(1-e)} = \frac{2h}{(1-e^2)}, \quad (341)$$

$$a = \frac{h}{(1-e^2)}. \quad (342)$$

From the expressions for h and e above in terms of E and L,

$$a = \frac{-\frac{L^2}{mK}}{-\frac{2EL^2}{mK^2}} = \frac{K}{2E} = \left|\frac{K}{2E}\right|. \tag{343}$$

For the gravitational force, $K = -GMm$ with G the gravitational constant, $6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$. The point A is the **perihelion** if the Sun is at the focus, and **perigee** if the Earth is at the focus. Similarly point B is the **aphelion** (Sun at the focus) or **apogee** (Earth at the focus).

If origin of coordinates is taken at mid-point of the major axis, the focus is at $(ae, 0)$ and the Cartesian equation of the orbit is

$$\frac{x^2}{\left[\frac{h^2}{(1-e^2)^2}\right]} + \frac{y^2}{\left[\frac{h^2}{1-e^2}\right]} = 1, \tag{344}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{345}$$

with

$$a = \frac{h}{1-e^2}, \qquad b = \frac{h}{\sqrt{1-e^2}}. \tag{346}$$

16.2 Kepler's laws of planetary motion

1. Planets move in elliptic orbits with Sun at a focus,
2. The radius vector sweeps out equal areas in equal times as illustrated in fig 55.

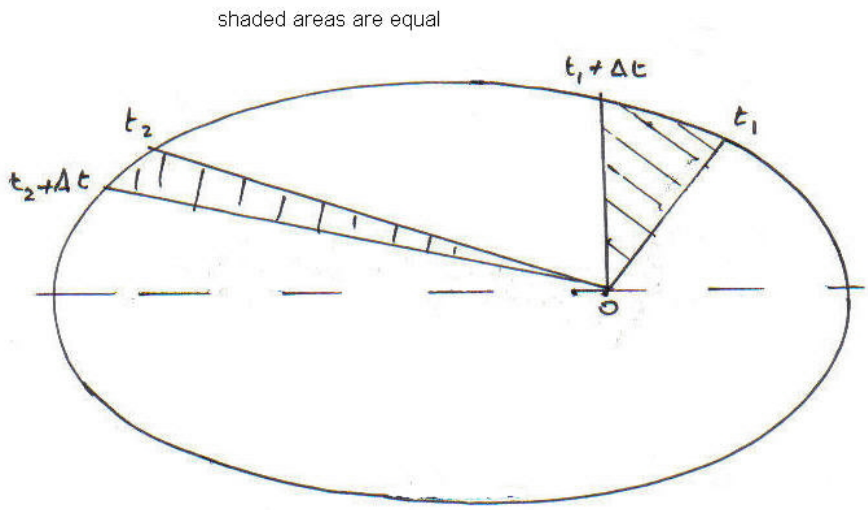

Figure 55: Areas for Kepler's law

3. (Period of rotation)$^2 \propto$ (semi-major axis)^3.

From fig 56, the area of triangle is

$$\mathrm{d}A = \frac{1}{2} r v_\theta \mathrm{d}t = \frac{1}{2} r r \dot{\theta} \mathrm{d}t \tag{347}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} r^2 \dot{\theta}. \tag{348}$$

But the angular momentum $L = mr^2\dot{\theta}$ is a constant for any central force so

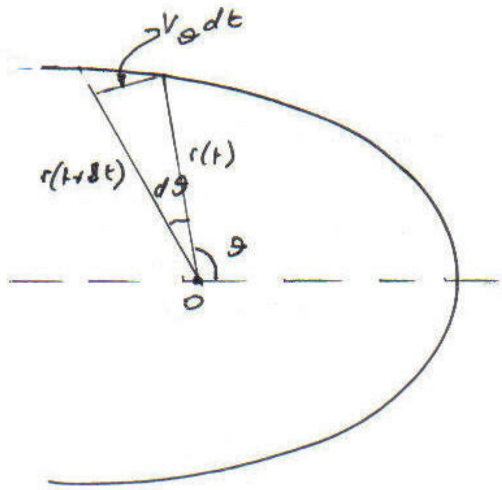

Figure 56: Kepler's law diagram

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}\frac{L}{m} = \texttt{const}. \tag{349}$$

Hence we have Kepler's second law.

The third law was derived earlier for a circular orbit where $a = r$ and will not be done again for the general elliptical motion.

17 Equivalent one-dimensional equation of motion

The radial equation of motion for a central force is

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r) \tag{350}$$

and

$$L = mr^2\dot{\theta} = \text{cons}\tan\text{t}. \tag{351}$$

Thus

$$\dot{\theta} = \frac{L}{mr^2} \tag{352}$$

and

$$m\left(\ddot{r} - r\left(\frac{L}{mr^2}\right)^2\right) = F(r) \tag{353}$$

$$m\ddot{r} - \frac{L^2}{mr^3} = F(r) \tag{354}$$

$$m\ddot{r} = F(r) + \frac{L^2}{mr^3}. \tag{355}$$

This is the same equation of motion for a particle moving in one-dimension under an effective force

$$F_{eff} = F(r) + \frac{L^2}{mr^3}. \tag{356}$$

The second term $\frac{L^2}{mr^3}$ is called a centrifugal force

$$F_C = \frac{L^2}{mr^3}. \tag{357}$$

If $F(r)$ is a conservative force we can introduce a potential function $V(r)$ such that

$$F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r}. \tag{358}$$

We can also introduce the centrifugal potential

$$V_C = \frac{1}{2}\frac{L^2}{mr^2} \tag{359}$$

so that

$$F_C(r) = -\frac{\mathrm{d}V_C}{\mathrm{d}r}. \tag{360}$$

consider an attractive inverse square law, $F(r) = -|K|/r^2$, then $V = -|K|/r$ and the total effective potential function (see fig 57) is

$$V_{eff} = -\frac{|K|}{r} + \frac{L^2}{2mr^2}. \tag{361}$$

If total energy $E < 0$ then the energy line crosses V_{eff} at $r = r_{\min}$ and $r = r_{\max}$, with $r_{\min} + r_{\max} = 2a$,

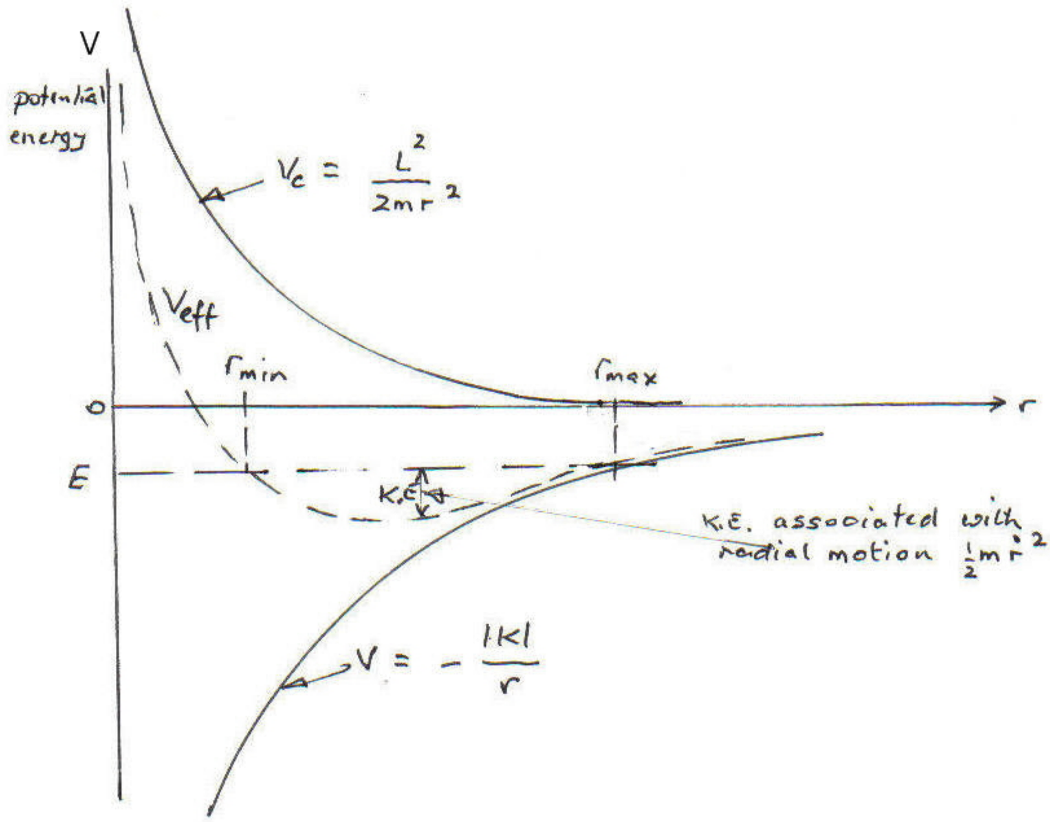

Figure 57: Effective potential

as in fig 57. If total energy $E > 0$ then $r_{\max}$ is infinite and orbit is a hyperbola. If $E = 0$, $r_{\max}$ is infinite but particle has zero kinetic energy at infinity.

18 Reduced mass

When considering a planet orbiting the Sun we have assumed that the Sun is fixed at the focus. However the mass of the Sun, M, is finite and therefore the Sun and planet **both** move with respect to their overall centre of mass. The centre of mass is at point O in fig 58 such that

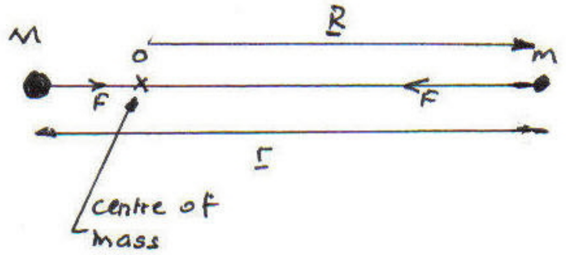

Figure 58: Rotation about centre of mass of two bodies

$$\mathbf{R} = \frac{M}{(M+m)}\mathbf{r}. \tag{362}$$

The acceleration of m relative to O is

$$\mathbf{a} = \frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} = \frac{M}{(M+m)}\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}. \tag{363}$$

Thus the equation of motion of the planet is

$$\mathbf{F} = m\mathbf{a} = \frac{mM}{(M+m)}\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \mu\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}. \tag{364}$$

That is the same as a particle of mass

$$\mu = \frac{mM}{(M+m)} < m \tag{365}$$

at a position $\mathbf{r}$ relative to the **fixed** point O (centre of mass). The real system in which both bodies orbit about a common centre of mass is equivalent to a body of the reduced mass μ orbiting at a distance r from the (centre of mass) **fixed** point, see fig 59.

Figure 59: Equivalent system: reduced mass

Note that $\mu = \frac{mM}{(M+m)} < m < M$. If $m \ll M$, then $\mu \simeq m$; if $m = M$ then $\mu = m/2$.

Now let's look at the kinetic energy of the system. If velocity of mass M is $\mathbf{V}$, that of mass m is $\mathbf{v}$, then the total kinetic energy is

$$E = \frac{1}{2}MV^2 + \frac{1}{2}mv^2. \tag{366}$$

Since from fig 58

$$\mathbf{R} = \frac{M}{(M+m)}\mathbf{r} \tag{367}$$

and both masses must have the same angular velocity about the centre of mass, then

$$v = \omega R = \omega\left(\frac{M}{m+M}\right)r; \qquad V = \omega\left(\frac{m}{m+M}\right)r. \tag{368}$$

The kinetic energy of the system

$$\begin{aligned} E &= \frac{1}{2}M\omega^2 r^2 \left(\frac{m}{m+M}\right)^2 + \frac{1}{2}m\omega^2 r^2 \left(\frac{M}{m+M}\right)^2 && (369)\\ &= \frac{1}{2}\omega^2 r^2 \left(\frac{1}{m+M}\right)^2 \left(Mm^2 + mM^2\right) = \frac{1}{2}\omega^2 r^2 \left(\frac{1}{m+M}\right)^2 mM\left(m+M\right) && (370)\\ &= \left(\frac{mM}{m+M}\right)\omega^2 r^2 = \frac{1}{2}\mu\omega^2 r^2 && (371)\end{aligned}$$

is the same as that of a mass μ orbiting the centre of mass at a radius r at angular velocity ω. Thus the reduced mass and the combined masses have the same kinetic energy.

Similarly if we look at the angular momentum of the system

$$L = m\left(\omega R\right)R + M\omega\left(r-R\right)\left(r-R\right). \tag{372}$$

Since

$$R = \frac{M}{(M+m)}r \tag{373}$$

then

$$(r-R) = \frac{m}{(m+M)}r \tag{374}$$

and

$$\begin{aligned} L &= m\omega\left[\frac{M}{(M+m)}r\right]^2 + M\omega\left[\frac{m}{(m+M)}r\right]^2 && (375)\\ &= \frac{mM}{(m+M)^2}\left(M+m\right) = \frac{mM}{(m+M)}\omega r^2 = \mu\omega r^2. && (376)\end{aligned}$$

The angular,omentum of the reduced mass particle is the same as that of the two particles.

19 Frames of reference

A frame of reference is defined by a set of **coordinate axes** (x, y, z) at rest relative to a particular observer. The observer can express **position**, **velocity** and **acceleration** of a particle **relative to this frame of reference**. We want to consider transformations of these quantities between different frames of reference. This is particularly important in the **theory of relativity**.

Consider one frame of reference S, fixed relative to the Earth and another frame of reference S' fixed relative to a car moving with constant velocity v along the x-axis, as in fig 60. Suppose origins O and O' coincide at time $t = 0$. A point P has coordinates x at time t in S and x' in S', so that

$$x' = x - vt. \tag{377}$$

If the point P is not on the x-axis the y and z coordinates are the same in both frames, i.e.

$$\begin{aligned} x' &= x - vt, && (378)\\ y' &= y, && (379)\\ z' &= z. && (380)\end{aligned}$$

Now suppose the point P is a bird flying at a constant velocity $\mathbf{u}$ along the x-axis relative to the Earth, then

$$u = \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{381}$$

Velocity of the bird relative to the car is

$$u' = \frac{\mathrm{d}x'}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - v = u - v. \tag{382}$$

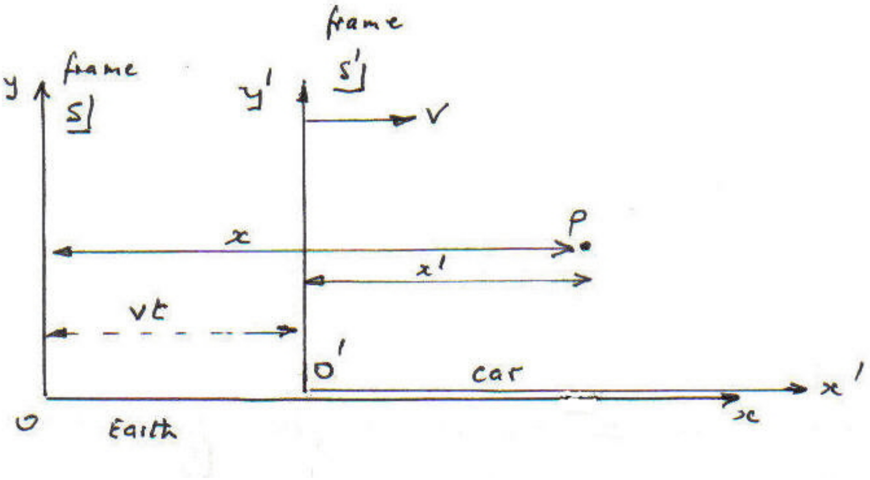

Figure 60: Two Cartesian inertial frames of reference

Generally if the bird has velocity $\mathbf{u}$ and the car has velocity $\mathbf{v}$ relative to the Earth, the velocity of the bird relative to the car is

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}. \tag{383}$$

In terms of components

$$u'_x = u_x - v_x, \tag{384}$$
$$u'_y = u_y - v_y, \tag{385}$$
$$u'_z = u_z - v_z. \tag{386}$$

The velocity diagram is (see fig 61)

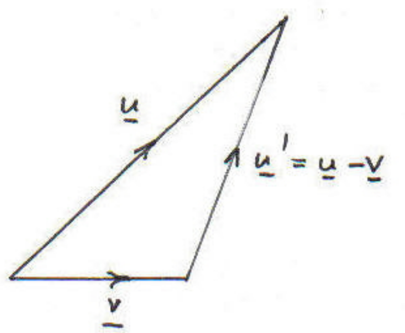

Figure 61: Velocity diagram

Example 13 *Rain falling vertically down with velocity* $\mathbf{u}$ *relative to the Earth; pedestrian is walking horizontally with velocity* $\mathbf{v}$ *relative to the Earth. Velocity of rain relative to pedestrian is* $\mathbf{u}' = \mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$*, see fig 62.*

Example 14 *Ship sailing north-east at velocity* $\mathbf{u}'$ *relative to the water. Current of water flowing with velocity* $\mathbf{v}$ *from east to west. What is velocity of ship relative to the Earth? Then* $\mathbf{u}' = \mathbf{u} - \mathbf{v}$*, so* $\mathbf{u} = \mathbf{u}' + \mathbf{v}$ *is velocity of ship relative to the Earth, as in fig 63. Note ship is pointing in the direction of velocity* $\mathbf{u}'$ *but is travelling along direction of velocity* $\mathbf{u}$*.*

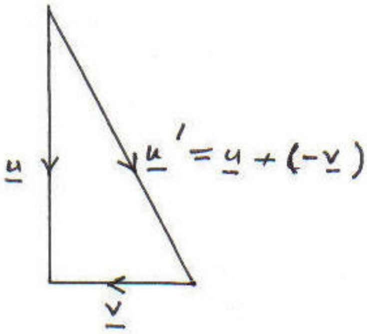

Figure 62: Velocity diagram fro rain and pedestrian

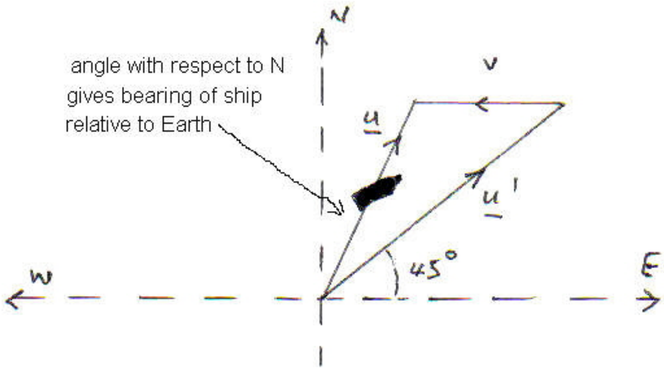

Figure 63: Velocity diagram for boat and water

19.1 Appropriate direction for interception

Consider two ships that want to intercept. Ship 1 is travelling with constant velocity $\mathbf{u}_1$ on a given bearing. Ship 2 is travelling at a constant speed $\mathbf{u}_2$. We need to determine the bearing (direction) of ship 2 such that the two ships will intercept, given that the ships are at known positions A and B respectively at time t. What must angle β be for interception in fig 64? Consider motion of ship 2 relative to ship 1, i.e. $\mathbf{u}_2' = \mathbf{u}_2 - \mathbf{u}_1$. For interception ship 2 must appear relative to ship 1 to be travelling along the direction B to A. Thus $\mathbf{u}_2' = \mathbf{u}_2 - \mathbf{u}_1$ is in direction $\overrightarrow{BA}$. Time to intercept is

$$t_1 = \frac{D}{|u_2'|}. \tag{387}$$

We can draw the velocity diagram with u_1, u_2, α given and use sine rule to determine β.

$$\frac{u_1}{\sin\beta} = \frac{u_2}{\sin\alpha} = \frac{u_2'}{\sin\gamma}. \tag{388}$$

The problem could be solved in the Earth frame, since the positions of ship 1 and 2 are given by

$$\begin{aligned} \mathbf{r}_A(t) &= \mathbf{r}_A + \mathbf{u}_1 t, && (389)\\ \mathbf{r}_B(t) &= \mathbf{r}_B + \mathbf{u}_2 t. && (390) \end{aligned}$$

Interception occurs when $\mathbf{r}_B(t) = \mathbf{r}_A(t)$ i.e.

$$\begin{aligned} \mathbf{r}_B + \mathbf{u}_2 t &= \mathbf{r}_A + \mathbf{u}_1 t, && (391)\\ \mathbf{r}_A - \mathbf{r}_B &= \mathbf{u}_2 t - \mathbf{u}_1 t, && (392)\\ t &= \frac{|\mathbf{u}_2 - \mathbf{u}_1|}{|\mathbf{r}_A - \mathbf{r}_B|}. && (393) \end{aligned}$$

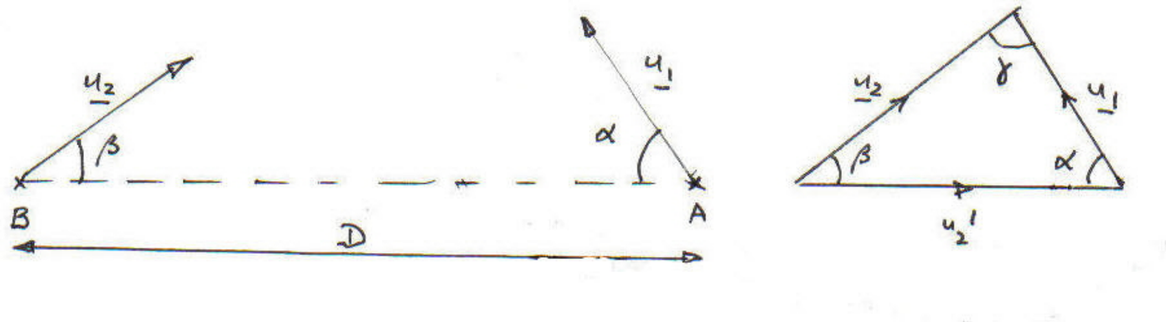

Figure 64: (a) Spatial and (b) velocity diagrams for interception

Thus $\mathbf{u}_2'$ is along $\mathbf{r}_A - \mathbf{r}_B$ and t is the time to interception.

Example 15 *Q3 1B27 Exam 2003.* $u_1 = 25\,\mathrm{km/h}$, $u_2 = 35\,\mathrm{km/h}$, *distance* $AB = 10\,\mathrm{km}$. *Velocity diagram fig 65.* $\mathbf{u}_2'$ *must lie in direction of line* BA*, i.e. due N, and* $\alpha + \beta = 45\,°$*. From sine rule*

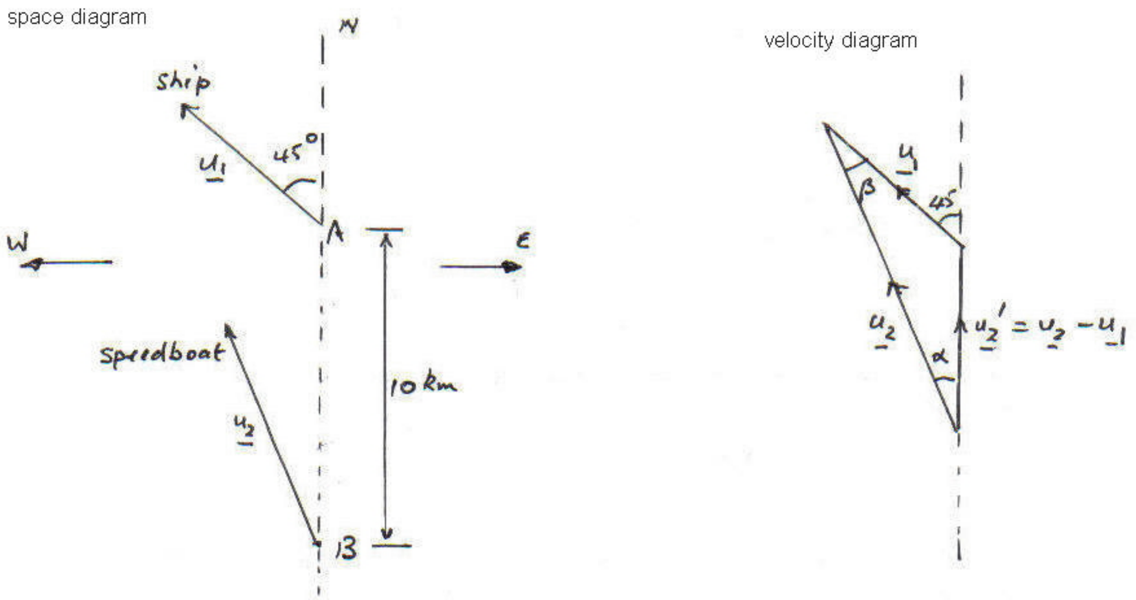

Figure 65: Space and velocity diagram for exam problem

$$\frac{u_1}{\sin\alpha} = \frac{u_2}{\sin 135\,°} = \frac{u_2}{1/\sqrt{2}}, \tag{394}$$

$$\sin\alpha = \frac{u_1}{u_2\sqrt{2}} = \frac{25}{35\sqrt{2}} = 0.505, \tag{395}$$

$$\alpha = 30.34\,°. \tag{396}$$

Therefore $\beta = 45 - \alpha = 14.66\,°$. *Also*

$$\frac{u_2'}{\sin\beta} = \frac{u_1}{\sin\alpha} \tag{397}$$

$$u_2' = u_1\frac{\sin\beta}{\sin\alpha} = \frac{25 \times 0.253}{0.505} = 12.52\,\mathrm{km/h}. \tag{398}$$

Time to interception is

$$t = \frac{10\,\mathrm{km}}{12.52\,\mathrm{km/\,h}} = 0.80\,\mathrm{h}. \tag{399}$$

19.2 Transformation of velocity and acceleration

Consider again a car travelling along the x-axis with velocity $\mathbf{v}$ and a bird flying along the x-axis with velocity $\mathbf{u}$ relative to the ground as in fig 66.

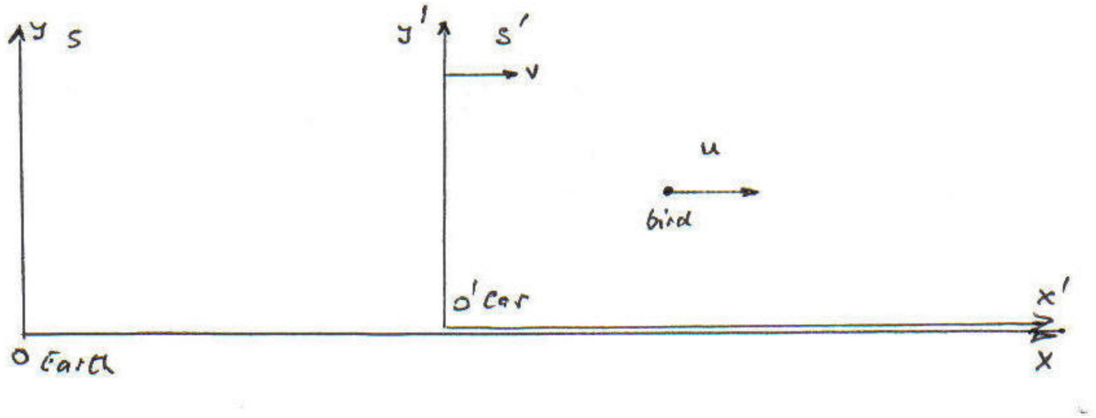

Figure 66: Diagram for acceleration as seen in two frames

Velocity of bird relative to car is (from $x' = x - vt$),

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}. \tag{400}$$

Suppose $\mathbf{v}$ is constant but bird is accelerating relative to the ground, then

$$\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \tag{401}$$

and the **acceleration** of the bird is the **same** in both frames of reference. Suppose now that the car is also accelerating relative to the Earth, then

$$\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{402}$$

with $\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}t}$ being the acceleration of the bird relative to the car, $\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$ being the acceleration of the bird relative to the Earth, and $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ being the acceleration of the car relative to the Earth. The force on the bird necessary to accelerate it in frame S (Earth) is

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}. \tag{403}$$

The equation of motion of the bird in the reference frame of the car, S', is **NOT**

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}t}. \tag{404}$$

It is correctly

$$\mathbf{F}' = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{405}$$

i.e. the effective force on the bird in the car's frame of reference, S', is

$$\mathbf{F}' = \mathbf{F} - m\mathbf{a}, \tag{406}$$

where $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ is the acceleration of the car (in frame S). Thus if we want to apply Newton's laws of motion in an **accelerating frame** we must **add** to the **real** force $\mathbf{F}$ the **fictitious** force ($-m\mathbf{a}$). An accelerating frame of reference is also called a **non-inertial** frame. An **inertial** (or non-accelerating) frame of reference is one in which Newton's laws of motion apply **without** needing to introduce fictitious forces.

20 Rotating frame of reference

Consider a person standing on a rotating roundabout as in fig 67. Real force of friction at the feet

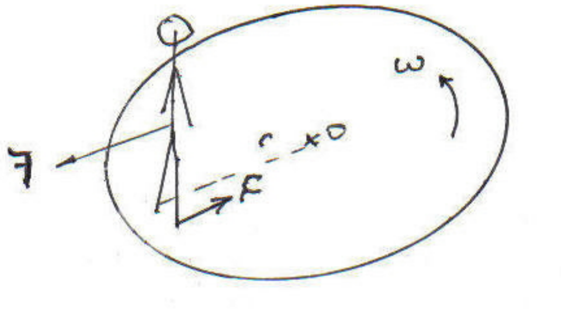

Figure 67: Observer in a rotating frame

provides the centripetal acceleration $r\omega^2$, therefore $F = mr\omega^2$. **Relative to the roundabout** the person is **at rest** under the influence of a real centripetal force $F = mr\omega^2$ and a **fictitious centrifugal force** $\mathfrak{F} = mr\omega^2$ acting through the centre of mass. If ω is large enough, the person may fall over outwards under the influence of these two forces, which constitute a couple acting on the person (see section on rotational motion and couples later).

20.1 Geostationary satellite

Satellite appears stationary relative to an observer on a rotating Earth as shown in fig 67. Real force on

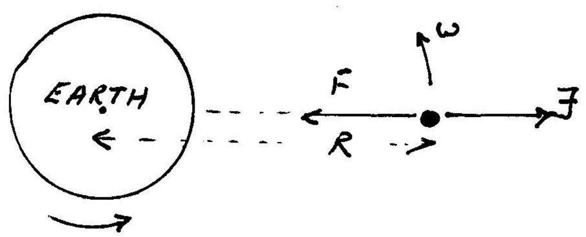

Figure 68: Geostationary satellite

the satellite if the gravitational force (centripetal force)

$$F = -G\frac{Mm}{R^2} = -mR\omega^2. \tag{407}$$

The observer on Earth thinks there is also a fictitious centrifugal force

$$F = mR\omega^2 = G\frac{Mm}{R^2} \tag{408}$$

so there is no net force, and so the satellite is at rest and in equilibrium in this frame.

20.2 Observing a moving body in a rotating frame

Consider a person B at rest on the roundabout observing a moving body, e.g. a flying bird, as in fig 69 Consider the same bird being observed by a second person A at rest on the ground and not on the roundabout. To observer A the path of the bird is a straight line. Relative to observer B on the

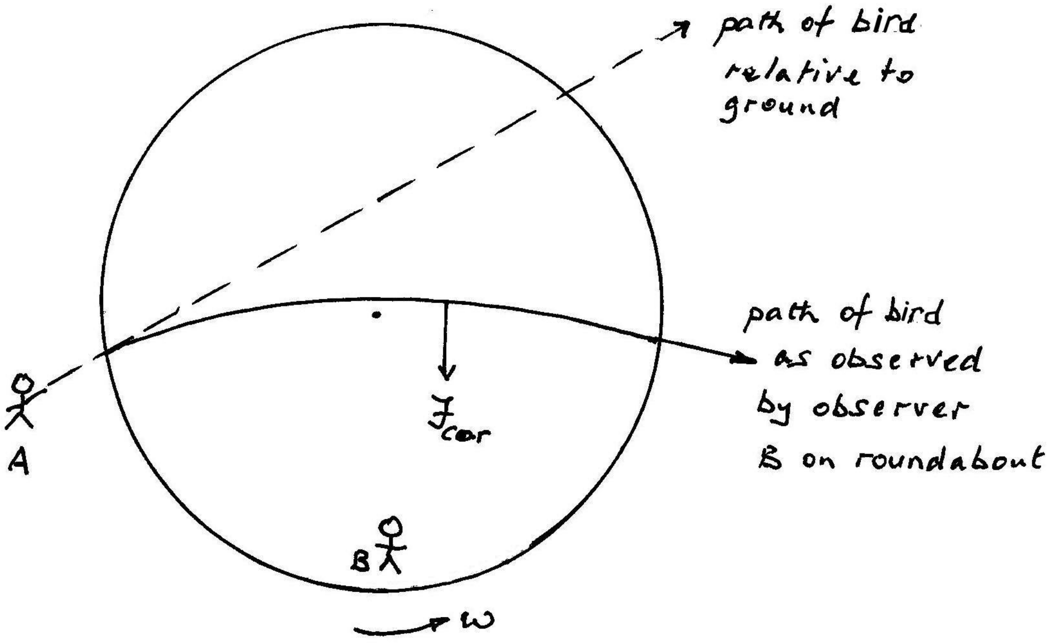

Figure 69: Observers in rotaing and inertial frames

roundabout the path of the bird is curved. Hence observer B assumes there is a transverse horizontal force $\mathfrak{F}_{cor}$ acting on the bird deflecting the bird to the right for the given sense of rotation. This is the **Coriolis force**.

The general expression for the Coriolis force is

$$\mathfrak{F}_{cor} = -2m\boldsymbol{\omega} \times \mathbf{v}, \tag{409}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of rotating frame, $\mathbf{v}$ is velocity of particle of mass m relative to the rotating frame of reference. Note $\mathfrak{F}_{cor}$ is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{v}$ and so always transverse to the direction of motion. There are several interesting consequences of the Coriolis force:

Consider the effect of the rotating Earth:

Throw a ball horizontally due East at three different places on the Earth's surface, A, B, and C as in fig 70 - dotted line is the local horizontal at each place on the Earth. In the northern hemisphere (point A) the Coriolis force deflects the ball to the right as it has a component in this direction. On the equator (point B) the Coriolis force has no horizontal component so the ball is not deflected. (Coriolis force is vertical so affects the local effective force due to gravity.) In the southern hemisphere the ball is deflected to the left.

Weather systems are determined by the Coriolis force. Wind blows anticlockwise (clockwise) around region of low pressure in the northern (southern) hemispheres. Winds blow approximately along isobars instead of down the pressure gradient.

One should set the size of the Coriolis force on the rotating Earth into context. The maximum value of the Coriolis acceleration is $a = -2\omega v$. Since $\omega = (2\pi) / (24 \times 60 \times 60)\,\mathrm{rad\,s^{-1}} = 7.272\,2 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$ then for a speed of $15\,\mathrm{m\,s^{-1}}$ ($\simeq 33\,\mathrm{mph}$), $a = 2.2 \times 10^{-3}\,\mathrm{m\,s^{-2}}$, (quite small compared to the vertical component of acceleration due to gravity on the Earth's surface).

20.3 Simple derivation of expression for Coriolis force

Consider a massless smooth rod on which there is a small ring of mass m. The rod is rotating with constant angular velocity ω in a horizontal plane. As the rod is smooth there can be no radial force (from friction) on the ring. The only force on the ring is a transverse normal reaction force N as shown in fig 71.

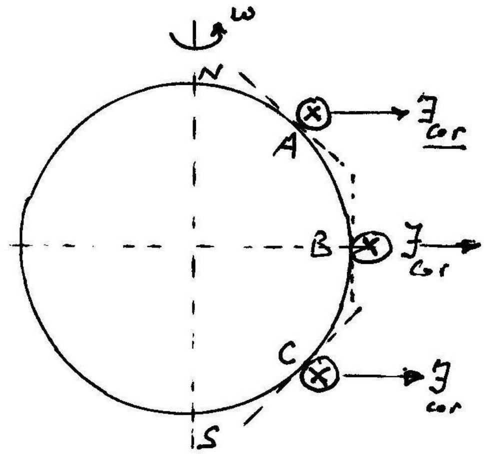

Figure 70: Effect of Coriolis force at Earth's surface

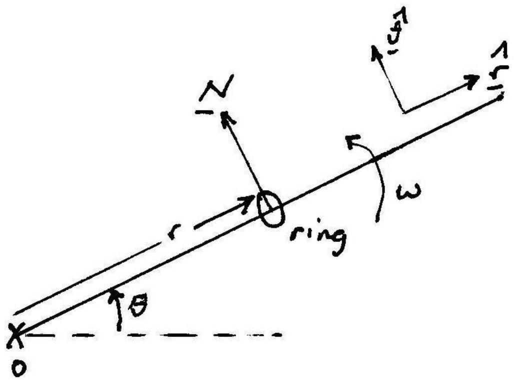

Figure 71: Diagram for simple derivation of expression for Coriolis force

The equation of motion of the ring is, relative to the inertial reference frame,

$$m\mathbf{a} = \mathbf{N} \tag{410}$$

$$m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\widehat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\widehat{\boldsymbol{\theta}}\right] = N\widehat{\boldsymbol{\theta}}. \tag{411}$$

Therefore the radial equation of motion is

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = 0 \tag{412}$$

and the transverse one is

$$m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = N. \tag{413}$$

But $\dot{\theta} = \omega =$ constant, so

$$m\left(\ddot{r} - r\omega^2\right) = 0, \tag{414}$$

$$2m\dot{r}\omega = N. \tag{415}$$

Now consider the equation of motion of the ring relative to the rotating rod. In this frame the ring can only move in one dimension along the rod. If we work in this frame we must add to the real forces the fictitious centrifugal force $\mathfrak{F}_c$ and, because the ring is moving relative to the rotating frame, the Coriolis force $\mathfrak{F}_{cor}$ as in fig 72 In the rotating frame the equations of motion is radially

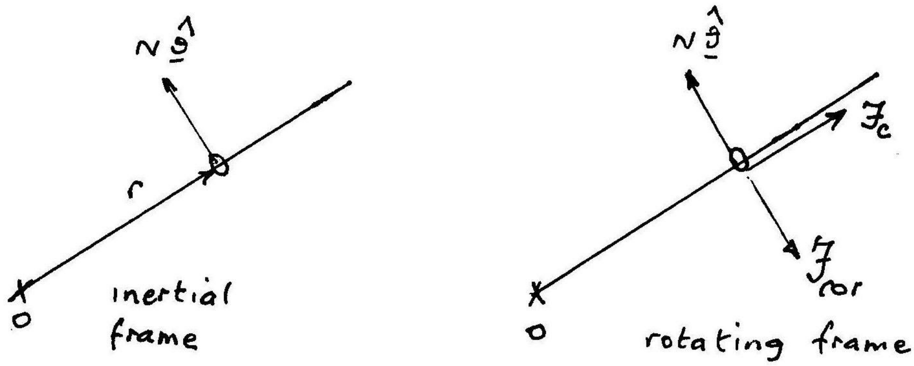

Figure 72: Forces on ring in inertial and rotating frames

$$m\ddot{r} = \mathfrak{F}_c \tag{416}$$

and transversely,

$$N - \mathfrak{F}_{cor} = 0 \tag{417}$$

as there is no transverse motion in the rotating frame. Comparing eq(414) with eq(416) gives a centrifugal force

$$\mathfrak{F}_c = mr\omega^2 \tag{418}$$

and comparing eq(415) and eq(417) gives a Coriolis force

$$\mathfrak{F}_{cor} = 2m\dot{r}\omega. \tag{419}$$

In the rotating frame the ring has only radial velocity $v = \dot{r}$. Thus the magnitude of the Coriolis force

$$\mathfrak{F}_{cor} = 2mv\omega. \tag{420}$$

The direction of $\mathfrak{F}_{cor}$ and its magnitude is consistent with the general expression quoted previously, i.e.

$$\mathfrak{F}_{cor} = -2m\boldsymbol{\omega} \times \mathbf{v} \tag{421}$$

as $\boldsymbol{\omega}$ is out of the page and $\mathbf{v}$ is along the rod.

21 Angular momentum and torques

Particle of mass m has velocity $\mathbf{v}$ at position vector $\mathbf{r}$ as in fig 73. The **angular momentum is defined** by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} \tag{422}$$

about point O. The direction of $\mathbf{L}$ is given by the usual right-hand rule for a vector product.

Consider a force $\mathbf{F}$ acting on the particle. The **torque** or **moment of force** about point O is **defined** by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \tag{423}$$

But

$$\mathbf{F} = \mathbf{m}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{424}$$

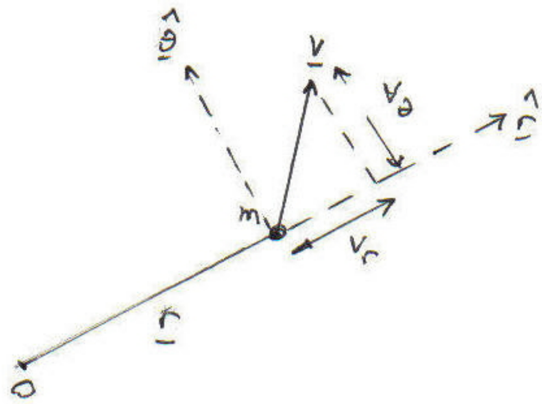

Figure 73: Position and velocity vectors for rotational motion

so

$$\boldsymbol{\tau} = m\mathbf{r} \times \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}. \tag{425}$$

Consider

$$\begin{aligned}
\frac{\mathrm{d}\left(\mathbf{r}\times\mathbf{v}\right)}{\mathrm{d}t} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\mathbf{v} + \mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} && (426)\\
&= \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 0 + \mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} && (427)\\
\frac{\mathrm{d}\left(\mathbf{r}\times\mathbf{v}\right)}{\mathrm{d}t} &= \mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} && (428)
\end{aligned}$$

and therefore

$$\boldsymbol{\tau} = m\mathbf{r}\times\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\frac{\mathrm{d}\left(\mathbf{r}\times\mathbf{v}\right)}{\mathrm{d}t} = \frac{\mathrm{d}\left(m\mathbf{r}\times\mathbf{v}\right)}{\mathrm{d}t} \tag{429}$$

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}. \tag{430}$$

Hence **torque is equal to the rate of change of angular momentum**. This is analogous to the linear case of force equal to rate of change of linear momentum, i.e. Newton's second law. If $\boldsymbol{\tau}$ =0 then $\mathbf{L}$ is constant in magnitude and direction. If $\boldsymbol{\tau}$ =0 then either force $\mathbf{F} = 0$ or $\mathbf{r}\times\mathbf{F}$ in which case force $\mathbf{F}$ and $\mathbf{r}$ are parallel and the force is a central force. Thus $\mathbf{L}$ is constant for any central force, $\mathbf{F}\left(\mathbf{r}\right) = F\left(r\right)\widehat{\mathbf{r}}$.

For a system in equilibrium, $\mathbf{L} = 0$ and constant so there is no net torque about **any** point. This is the basis of taking moments to determine the magnitude of forces.

Example 16 *A massless beam is supported at points A and B and masses m_1 and m_2 are attached as in the fig 74. Resolving vertically,*

$$N_1 + N_2 = m_1 g + m_2 g = \left(m_1 + m_2\right) g. \tag{431}$$

Take moments about A, (assume clockwise moment is positive, anti-clockwise moment is negative)

$$m_1 g x_1 - N_2 x_2 + m_2 g x_3 = 0 \tag{432}$$

$$\begin{aligned}
N_2 &= \frac{\left(m_1 x_1 + m_2 x_3\right) g}{x_2}, && (433)\\
N_1 &= \left(m_1 + m_2\right) g - N_2. && (434)
\end{aligned}$$

Example 17 *Motorcyclist with uniform acceleration such that front wheels lift off the ground as in fig 75!The real forces are, weight mg, friction $F = ma$ and normal reaction $N = mg$. In motorcyclists' own frame of reference (which is accelerating relative to the ground) he is in equilibrium under influences of*

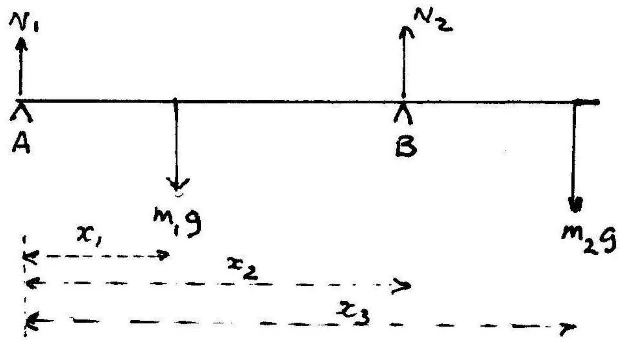

Figure 74: Masses on supported beam

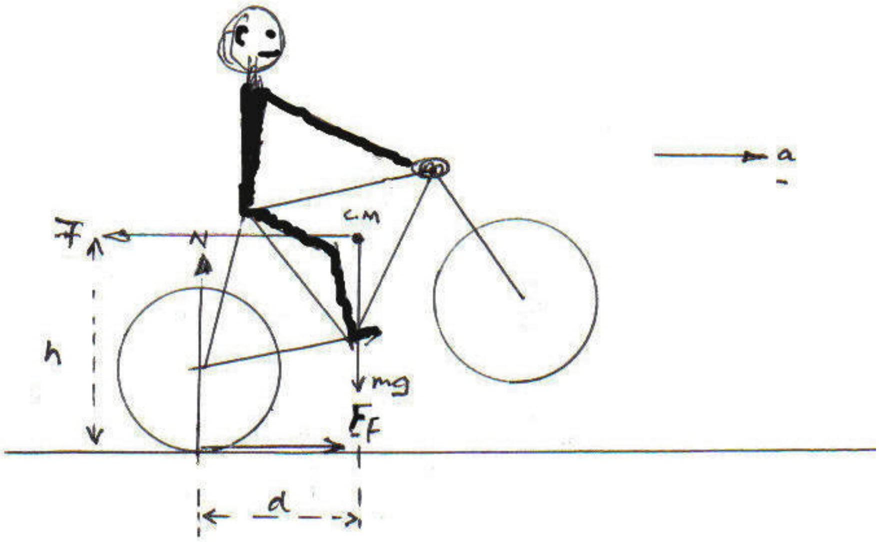

Figure 75: Accelerating cyclist

real ***and*** *fictitious force* $\mathfrak{F} = ma$ *acting through the centre of mass as shown. Taking moments about point of contact of rear wheel with the ground,*

$$mgd - \mathfrak{F}h = 0. \tag{435}$$

Example 18 *An accelerating car as shown in fig 76: We will assume rear-wheel drive; that centre of mass is at height h above the ground; distances d_1 and d_2 from wheels as shown in the diagram. Real forces are: weight mg; friction $F = ma$ at the driven rear wheels contact with the ground (There is also friction between front wheels and the ground acting in the opposite sense causing the front wheels to turn rather than skid.); normal reactions at wheels, N_1 and N_2. We have*

$$N_1 + N_2 = mg. \tag{436}$$

In the car's accelerating frame we add a fictitious force $\mathfrak{F} = ma$ as shown. In this frame the car is now in equilibrium. Taking moments about point A,

$$mgd_1 - N_2(d_1 + d_2) - \mathfrak{F}h = 0, \tag{437}$$

$$N_2 = \frac{1}{(d_1 + d_2)}(mgd_1 - mah) = \frac{m}{(d_1 + d_2)}(gd_1 - ah), \tag{438}$$

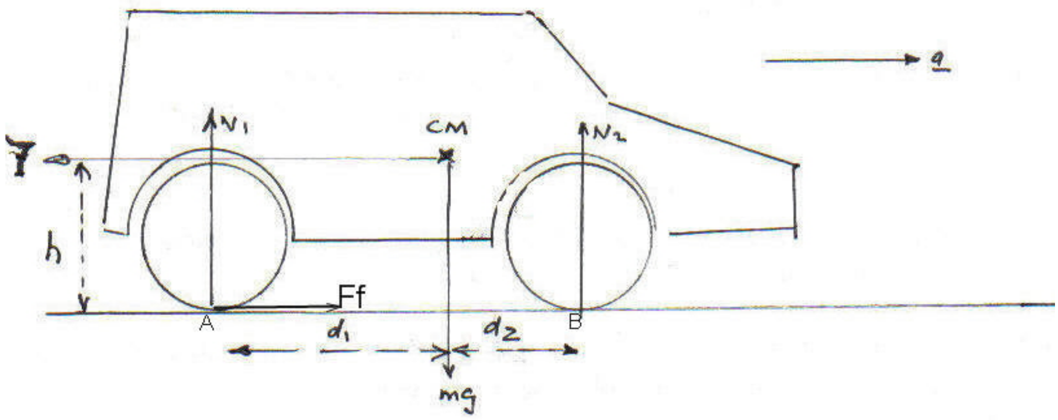

Figure 76: Accelerating vehicle

and

$$N_1 = mg - N_2 = mg - \frac{m}{(d_1 + d_2)}(gd_1 - ah) \quad (439)$$

$$N_1 = \frac{m}{(d_1 + d_2)}(gd_2 + ah). \quad (440)$$

For a rear-wheel drive car the maximum acceleration of the car is either when (a) $F = ma = \mu N_1$ *where* μ *is coefficient of friction, whence*

$$a = \frac{\mu}{(d_1 + d_2)}(gd_2 + ah), \quad (441)$$

or (b) if μ *is large enough when* $N_2 = 0$ *whence*

$$a = g\frac{d_1}{h}. \quad (442)$$

(Hence the need for a low centre of mass in racing cars!) Any higher acceleration than this will cause the car to somersault backwards. For a front-wheel drive car, maximum acceleration occurs for $F = ma = \mu N_2$*. The front wheels can never leave the ground because they will skid first.*

22 Systems of particles - properties of centre of mass

Consider a system of n particles of masses, m_i $(i = 1, 2, \ldots, n)$ at position vectors $\mathbf{r}_i$ $(i = 1, 2, \ldots, n)$ as shown in fig 77. Total mass is

$$M = \sum_{i=1}^{n} m_i. \quad (443)$$

The **centre of mass** is at position

$$\mathbf{R} = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{M}. \quad (444)$$

The position $\mathbf{R}$ is such that if the total mass M were at $\mathbf{R}$ it would have the **same moment** about origin O as the actual system, i.e.

$$M\mathbf{R} = \sum_{i=1}^{n} m_i \mathbf{r}_i. \quad (445)$$

The centre of mass is therefore that point about which a rigid body will balance if in a uniform gravitational field.

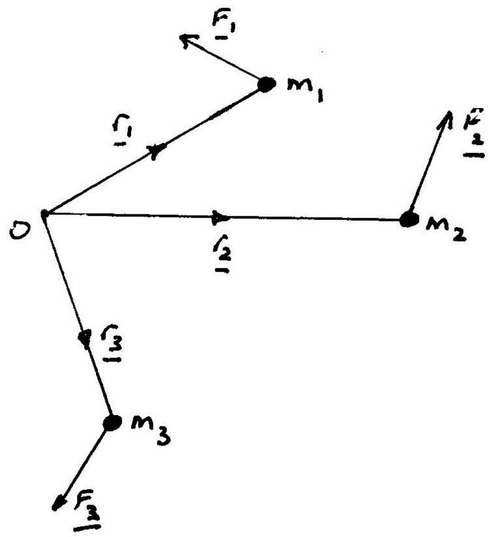

Figure 77: System of particles acted on by forces

If the system of particles is **not subjected** to any **external forces** the centre of mass **remains at rest or in uniform motion**, i.e. no acceleration of the centre of mass. This is true even if the particles in the system are moving relative to each other and interact with each other. Also, if system of particles is subjected to **external forces the centre of mass behaves as if all the forces acted at that position**. In addition, any fictitious force appears to act at through the centre of mass.
Proof. Let $\mathbf{F}_i$ be force on i-th particle, then

$$\mathbf{F}_i = \mathbf{F}_{i\mathrm{ext}} + \mathbf{F}_{i\mathrm{int}}, \tag{446}$$

where $\mathbf{F}_{i\mathrm{ext}}$ is external force on particle i, and $\mathbf{F}_{i\mathrm{int}}$ is the internal force on particle i due to internal interactions with all other particles. The equation of motion of particle i is

$$\mathbf{F}_i = m_i \frac{\mathrm{d}^2 \mathbf{r}_i}{\mathrm{d}t^2} \tag{447}$$

and the resultant force on the system is

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i = \sum_{i=1}^{n} m_i \frac{\mathrm{d}^2 \mathbf{r}_i}{\mathrm{d}t^2} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\sum_{i=1}^{n} m_i \mathbf{r}_i \right) \tag{448}$$

$$\mathbf{F} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} (M\mathbf{R}) = M \frac{\mathrm{d}^2 \mathbf{R}}{\mathrm{d}t^2}. \tag{449}$$

Therefore the centre of mass moves as if all the forces act on a mass M at the position of the centre of mass.

Since

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i = \sum_{i=1}^{n} \mathbf{F}_{i\mathrm{ext}} + \sum_{i=1}^{n} \mathbf{F}_{i\mathrm{int}} \tag{450}$$

and

$$\sum_{i=1}^{n} \mathbf{F}_{i\mathrm{int}} = 0 \tag{451}$$

by Newton's third law, we have

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i = M \frac{\mathrm{d}^2 \mathbf{R}}{\mathrm{d}t^2}. \tag{452}$$

Therefore only external forces influence the motion of the centre of mass. If $\mathbf{F} = 0$ (no resultant external force) then

$$\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2} = 0 \tag{453}$$

and velocity of centre of mass $\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}$ is constant. ■

23 Rigid bodies

23.1 Centre of mass

Rigid bodies can be considered as a system of particles rigidly joined together. An example is a uniform thin rod of length L and mass M as in fig 78. We can determine the position of the centre of mass as

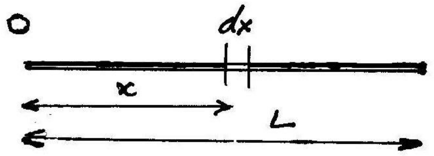

Figure 78: Uniform thin rod

follows:

Mass per unit length is

$$\sigma = \frac{M}{L} \tag{454}$$

so the mass of an element of length $\mathrm{d}x$ is $\mathrm{d}m = \sigma \mathrm{d}x$. Therefore from the definition, the position of the centre of mass is given by

$$X = \frac{\int_0^L x\,\mathrm{d}m}{\int_0^L \mathrm{d}m} = \frac{\int_0^L x\,\sigma \mathrm{d}x}{\int_0^L \sigma \mathrm{d}x} = \frac{\frac{1}{2}\sigma x^2|_0^L}{M} = \frac{\frac{1}{2}\sigma L^2}{M} = \frac{1}{2}\frac{\sigma L^2}{M} = \frac{1}{2}\frac{M}{L}\frac{L^2}{M} = \frac{1}{2}L \tag{455}$$

as might have been expected. Note this procedure works even when σ is not constant.

Example 19 *Position of centre of mass of a thin wedge shaped rod (as shown in fig 79) where the mass per unit length is proportional to distance from one end, i.e. $\sigma = kx$ with k a constant. Then mass of*

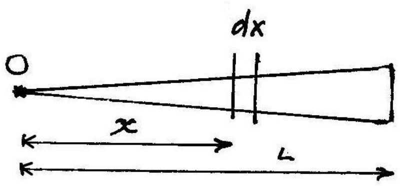

Figure 79: Centre of mass of wedge rod

element of length $\mathrm{d}x$ *is* $\mathrm{d}m = \sigma \mathrm{d}x = kx\mathrm{d}x$ *and total mass is*

$$M = \int_0^L \mathrm{d}m = \int_0^L \sigma \mathrm{d}x = \int_0^L kx\mathrm{d}x = \frac{1}{2}kL^2. \tag{456}$$

The position of the centre of mass is

$$X = \frac{\int_0^L x\,\mathrm{d}m}{\int_0^L \mathrm{d}m} = \frac{\int_0^L x\,\sigma \mathrm{d}x}{M} = \frac{\int_0^L kx^2\,\mathrm{d}x}{M} = \frac{\frac{1}{3}kL^3}{\frac{1}{2}kL^2} = \frac{2}{3}L. \tag{457}$$

23.2 Rotation of a rigid body about a fixed axis

Consider rotation of a rigid body as depicted in fig 80, with the axis of rotation perpendicular to the page. Let ω be the angular velocity. A mass element m_i is at a perpendicular distance r_i from the pivot. The only velocity is transverse to this direction of magnitude $v_i = \omega r_i$. For a rigid body, all elements must have the **same** angular velocity. The kinetic energy of rotation is

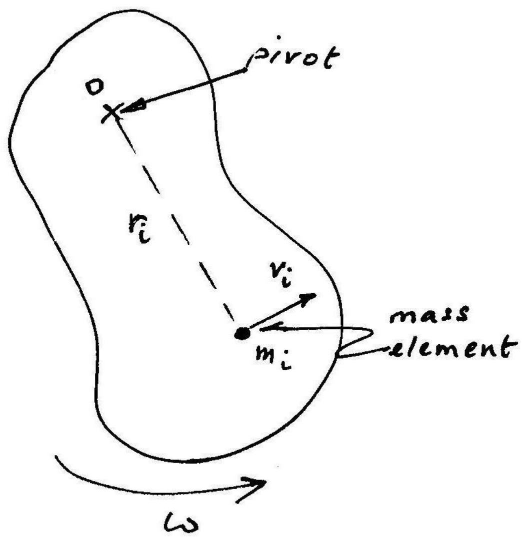

Figure 80: Rotating rigid body

$$K_E = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{n} \frac{1}{2} m_i \left(\omega r_i\right)^2 = \frac{1}{2}\omega^2 \sum_{i=1}^{n} m_i r_i^2. \tag{458}$$

The last summation depends only on the distribution of mass within the body. We can define the **moment of inertia,** I**, of the body about this axis** by

$$I = \sum_{i=1}^{n} m_i r_i^2, \tag{459}$$

and

$$K_E = \frac{1}{2} I \omega^2. \tag{460}$$

For a continuous mass distribution with density ρ which may vary with position, an element of volume $\mathrm{d}v$ has mass $\mathrm{d}m = \rho\,\mathrm{d}v$. If $r_\perp$ is the perpendicular distance from the axis, the moment of inertia is obtained from

$$I = \int_{vol} r_\perp^2 \rho\,\mathrm{d}v. \tag{461}$$

The value of the moment of inertia depends on the mass distribution in the body and the **position of the pivot**. Later we will calculate the moment of inertia of various bodies.

23.3 Angular momentum of a rotating rigid body.

The angular momentum of a point mass m moving in a circle of radius r with angular velocity ω is

$$L = mr^2\omega = mvr. \tag{462}$$

Therefore the angular momentum of an extended rigid body is

$$L = \sum_{i=1}^{n} m_i r_i^2 \omega = \omega \sum_{i=1}^{n} m_i r_i^2 = I\omega. \tag{463}$$

Angular momentum is a vector quantity so we should write

$$\mathbf{L} = I\boldsymbol{\omega} \tag{464}$$

where direction of vector is along the axis of rotation in a sense given by a right-handed screw.

23.4 Correspondence between rotational and translational motion

The previous sections show that we can establish a table of correspondences between quantities related to linear motion and to rotational motion.

Linear			Rotation
mass	m	I	moment of inertia
velocity	$\mathbf{v}$	$\boldsymbol{\omega}$	angular velocity
linear momentum	$\mathbf{p} = m\mathbf{v}$	$\mathbf{L} = I\boldsymbol{\omega}$	angular momentum
linear kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$	rotational kinetic energy
force	$F = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$	$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}$	torque
	$F = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$	$F = I\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}$	

23.5 Compound pendulum

The mass of the body is M. The centre of mass is a distance ℓ from the pivot as shown in fig81. It will be assumed that the moment of inertia, I, of this body about an axis through the pivot point A is known. Torque of the weight about A is

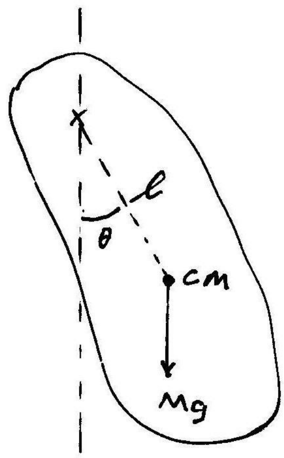

Figure 81: Compound pendulum

$$\tau = -Mg\ell \sin\theta. \tag{465}$$

Note the negative sign as the torque is in the sense of decreasing θ.

The equation of motion for rotation about the axis through A is

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(I\omega) = I\frac{\mathrm{d}\omega}{\mathrm{d}t} = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \tag{466}$$

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -Mg\ell \sin\theta. \tag{467}$$

If the angle θ is always small, then $\sin\theta \simeq \theta$ (in radians) and

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + Mg\ell\theta = 0, \tag{468}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{Mg\ell}{I}\theta = 0, \tag{469}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2\theta = 0, \tag{470}$$

where

$$\omega = \sqrt{\frac{Mg\ell}{I}}. \tag{471}$$

This is the equation for simple harmonic motion (in θ) with a period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Mg\ell}}. \tag{472}$$

If we express the moment of inertia by

$$I = Mk^2 \tag{473}$$

with k called the **radius of gyration** $\left(k = \sqrt{I/M}\right)$ then

$$T = 2\pi\sqrt{\frac{Mk^2}{Mg\ell}} = 2\pi\sqrt{\frac{k^2}{g\ell}}. \tag{474}$$

A **simple pendulum** is a point mass M on the end of a massless inextensible string (!). The moment of inertia of a point mass M at a distance ℓ from a pivot is $I = M\ell^2$, so $k = \ell$. The period becomes

$$T = 2\pi\sqrt{\frac{\ell^2}{g\ell}} = 2\pi\sqrt{\frac{\ell}{g}}. \tag{475}$$

23.6 Determination of moment of inertia

The basic definition is

$$I = \sum_{i=1}^{n} m_i r_i^2 \longrightarrow \int_{vol} r_\perp^2 \rho \,\mathrm{d}v. \tag{476}$$

1. Point mass M at distance ℓ from axis of rotation, $I = M\ell^2$.
2. Ring of mass M and of radius R about an axis through O, the centre of the ring as in fig 82, and perpendicular to the plane of the ring. As all the mass is the same distance from O, then eq(476)

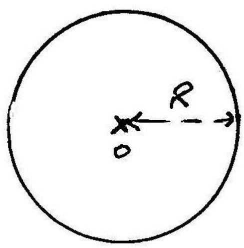

Figure 82: Moment of inertia of a ring

immediately gives $I = MR^2$. It is important to note that for all other axes the moment of inertia is different.

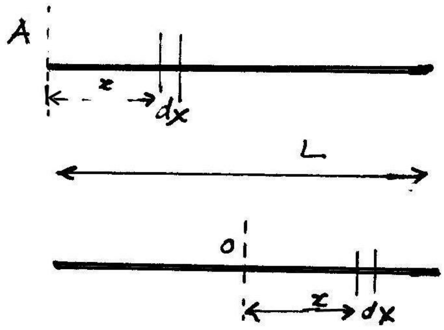

Figure 83: Moment of inertia of a uniform rod

3. Uniform thin rod, mass M, length L about an axis through one end of the rod and perpendicular to the rod, as in fig 83. Mass per unit length is $\rho = \frac{M}{L}$ so mass of length $\mathrm{d}x$ is $\mathrm{d}m = \rho\,\mathrm{d}x$. Thus from eq(476)

$$I_A = \int x^2 \,\mathrm{d}m = \int_0^L x^2 \rho \,\mathrm{d}x = \frac{1}{3}\rho L^3 = \frac{1}{3}\frac{M}{L}L^3 = \frac{1}{3}ML^2. \tag{477}$$

If the axis were through the centre of the rod, point O, and perpendicular to it then

$$I_O = \int_{-L/2}^{L/2} x^2 \rho \,\mathrm{d}x = \frac{1}{3}\rho x^3|_{-L/2}^{L/2} = \frac{1}{24}\rho L^3 + \frac{1}{24}\rho L^3 = \frac{1}{12}\rho L^3 = \frac{1}{12}\frac{M}{L}L^3 = \frac{1}{12}ML^2. \tag{478}$$

This clearly illustrates that the moment of inertia is not a fixed quantity for a rigid body but depends on the choice of axis about which rotation is to occur. Two important theorems help us determine moments of inertia. These theorems are called (a) the theorem of perpendicular axes and (b) the theorem of parallel axes.

23.7 Theorem of perpendicular axes

It is important to note that this theorem **only applies to plane lamina**. Consider a lamina (sheet) lying in the $x - y$ plane, with mass per unit area ρ (not necessarily constant) as shown in fig 84. Mass of an element of area $\mathrm{d}x\mathrm{d}y$ is $\mathrm{d}m = \rho \mathrm{d}x\mathrm{d}y$. Thus moment of inertia of lamina about the z-axis is

$$I_z = \iint_{area} r^2 \,\mathrm{d}m = \iint \left[\rho\left(x^2 + y^2\right)\right] \,\mathrm{d}x\mathrm{d}y, \tag{479}$$

$$= \iint \left[\rho\left(x^2 + y^2\right)\right] \,\mathrm{d}x\mathrm{d}y = \tag{480}$$

$$= \iint \rho x^2 \,\mathrm{d}x\mathrm{d}y + \iint \rho y^2 \,\mathrm{d}x\mathrm{d}y \tag{481}$$

$$= \int_x x^2 \left(\int_y \rho \,\mathrm{d}x\mathrm{d}y\right) + \int_y y^2 \left(\int_x \rho\mathrm{d}x\mathrm{d}y\right). \tag{482}$$

Since

$$\left(\int_y \rho \,\mathrm{d}x\mathrm{d}y\right) = \mathrm{d}M \tag{483}$$

is the mass of a strip of width $\mathrm{d}x$ parallel to the y-axis, the first integral on the right-hand side

$$\int_x x^2 \,\mathrm{d}M = I_y, \tag{484}$$

is the moment of inertia of the lamina about the y-axis as shown in fig 85. Similarly the second integral

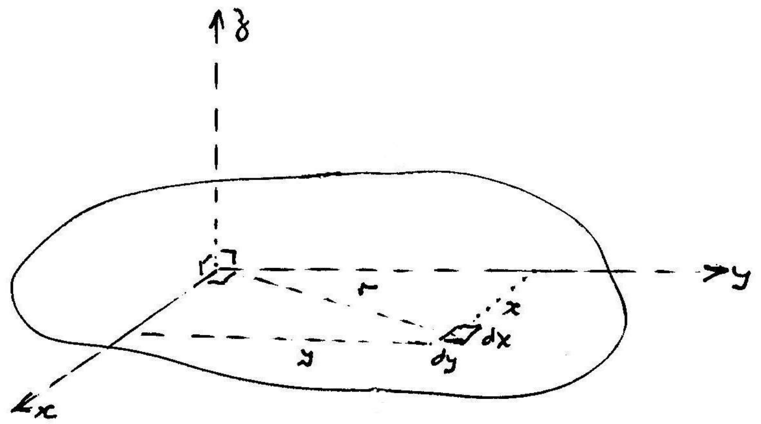

Figure 84: Diagram for perpendicular axes theorem of moments of inertia

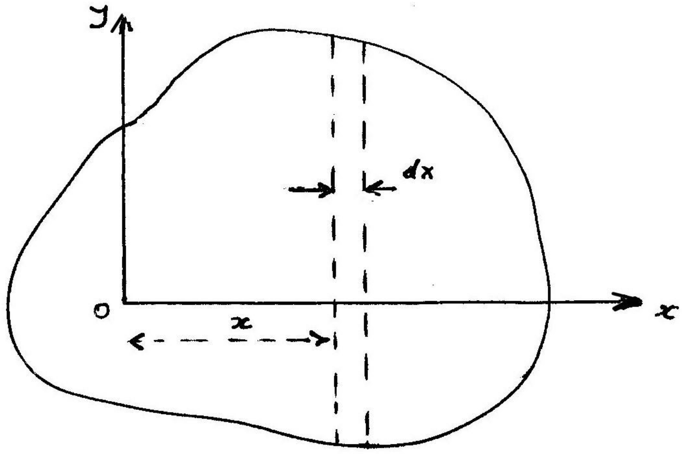

Figure 85: Diagram for identification of integrals

gives I_x the moment of inertia of the lamina about the x-axis. Therefore

$$I_z = I_x + I_y. \tag{485}$$

Note the x and y axes can be chosen anywhere in the lamina, but the z-axis **must** be taken through the point of intersection of the x and y axes.

23.8 Theorem of parallel axes

This theorem applies to any solid body. Consider any axis through the centre of mass. Let I_0 be the moment of inertia of the body about this axis. Now consider a parallel axis at a distance a from the axis through the centre of mass (see fig 86). The moment of inertia about a given axis depends only on the perpendicular distance from the axis of each mass element. Therefore we may squash the three-dimensional body down onto a plane perpendicular to the axis. Suppose this is the $x-y$ plane. Then the distance of an element from the axis through O is r and

$$r^2 = x^2 + y^2, \tag{486}$$

and

$$a^2 = a_x^2 + a_y^2. \tag{487}$$

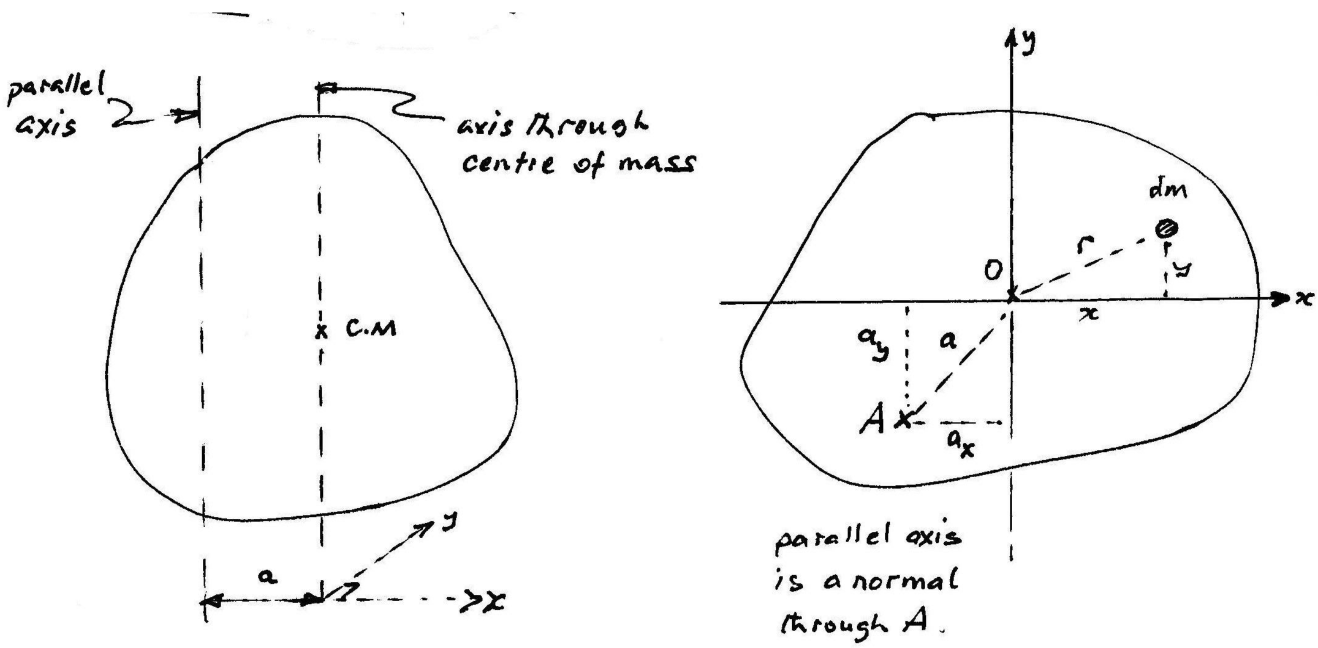

Figure 86: Diagram for parallel axes theorem of moments of inertia

Distance from A to mass element is R such that

$$R^2 = (x + a_x)^2 + (y + a_y)^2 . \tag{488}$$

Moment of inertia of body about axis through O perpendicular to the $x - y$ plane is

$$I_0 = \int_{vol} r^2 \,\mathrm{d}m = \int_{vol} \left(x^2 + y^2\right) \,\mathrm{d}m \tag{489}$$

and the total mass

$$M = \int \mathrm{d}m. \tag{490}$$

Moment of inertia about parallel axis through A is

$$I_A = \int_{vol} R^2 \,\mathrm{d}m \tag{491}$$

$$= \int_{vol} \left[(x + a_x)^2 + (y + a_y)^2\right] \,\mathrm{d}m \tag{492}$$

$$= \int_{vol} \left[\left(x^2 + y^2\right) + \left(a_x^2 + a_y^2\right) + 2xa_x + 2ya_y\right] \,\mathrm{d}m \tag{493}$$

$$I_A = \int_{vol} \left(x^2 + y^2\right) \,\mathrm{d}m + a^2 \int \mathrm{d}m + 2a_x \int x \,\mathrm{d}m + 2a_y \int y \,\mathrm{d}m. \tag{494}$$

But by definition of the centre of mass

$$\int x \,\mathrm{d}m = \int y \,\mathrm{d}m = 0 \tag{495}$$

and so

$$I_A = I_0 + Ma^2. \tag{496}$$

This theorem is true for any body.

Example 20 *Consider previous example of a uniform rod of length L. We showed explicitly that $I_0 = \frac{1}{12}ML^2$. Thus by the parallel axes theorem the moment of inertia about a parallel axis through one end of the rod is $I_A = I_0 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$ as explicitly shown previously.*

23.9 Kinetic energy of rigid body with rotation and translation

We shall find an expression for the kinetic energy of a body which is rotating through its centre of mass and also is in rectilinear motion. Body has mass M and moment of inertia about axis through centre of

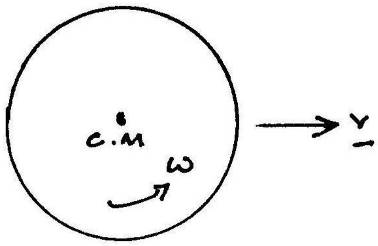

Figure 87: Rotational and translational motion

mass of I_0. The kinetic energy of rotation about axis through centre of mass is $\frac{1}{2}I_0\omega^2$. Kinetic energy of rectilinear motion is $\frac{1}{2}Mv^2$. Thus the total kinetic energy is simply

$$K_E = \frac{1}{2}I_0\omega^2 + \frac{1}{2}Mv^2. \tag{497}$$

Consider a wheel **rolling** on a surface **without skidding**. As the point in contact with the surface is not slipping, it must be momentarily stationary. Thus the speed of the centre of mass $v = \omega R$. The total

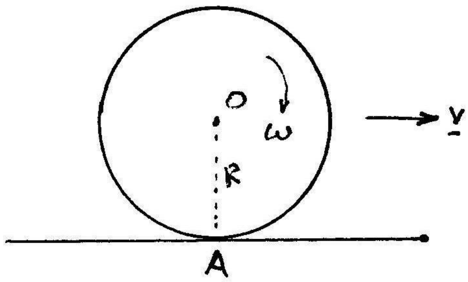

Figure 88: Rolling wheel

kinetic energy is

$$\begin{aligned} K_E &= \frac{1}{2}I_0\omega^2 + \frac{1}{2}Mv^2 && (498)\\ &= \frac{1}{2}I_0\omega^2 + \frac{1}{2}M\omega^2R^2 && (499)\\ &= \frac{1}{2}\left(I_0 + MR^2\right)\omega^2 && (500)\\ &= \frac{1}{2}I_A\omega^2, && (501) \end{aligned}$$

where

$$I_A = I_0 + MR^2 \tag{502}$$

is the moment of inertia of the wheel about an axis through A (the point of contact with the surface) perpendicular to the plane of the wheel. Therefore the wheel can be considered as momentarily rotating about point A (the point of contact with the surface) with angular velocity ω. The moment of inertia about axis through centre of mass of the wheel (ring) perpendicular to the wheel is $I_0 = MR^2$. In this case

$$K_E = \frac{1}{2}\left(I_0 + MR^2\right)\omega^2 = \frac{1}{2}\left(MR^2 + MR^2\right)\omega^2 = M\omega^2 r^2 = Mv^2. \tag{503}$$

23.10 Effect of external force

We wish to see what is the effect of an external force applied to a **free** rigid body if the force does not act through the centre of mass of the body, see fig 89. If we now add forces **F** and **−F** acting through the

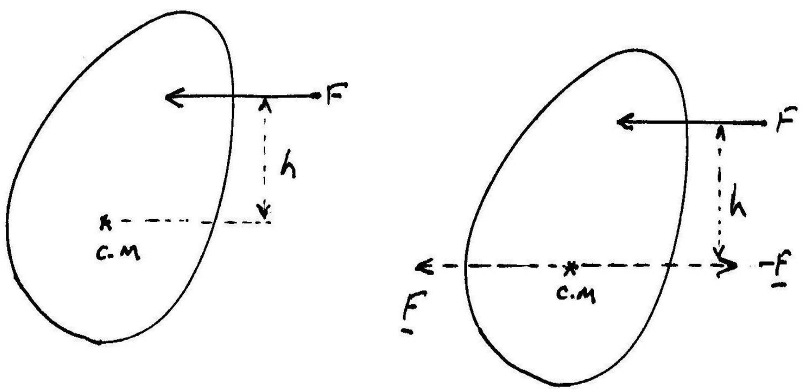

Figure 89: Left: External force not through C.o.M; Right: equivalent force system

centre of mass we now have a system with a force **F** through the centre of mass and a couple made up from the force **−F** through the centre of mass and the original force **F**. Force acting through the centre of mass accelerates the mass according to

$$\mathbf{F} = M\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}t^2}. \tag{504}$$

The couple, with a torque $\tau = Fh$, produces an angular acceleration

$$\tau = \frac{\mathrm{d}L}{\mathrm{d}t} = I_0\frac{\mathrm{d}\omega}{\mathrm{d}t} \tag{505}$$

about an axis through the centre of mass and perpendicular to the plane defined by the direction of the force and the centre of mass.

Suppose force **F** is applied for a very short time dt to a system initially at rest. Then

$$\mathbf{F}\,\mathrm{d}t = M\mathbf{V} \tag{506}$$

where **V** is velocity of centre of mass, and

$$\boldsymbol{\tau}\mathrm{d}t = I_0\boldsymbol{\omega}, \tag{507}$$

where $\boldsymbol{\omega}$ is the angular velocity about axis through the centre of mass. Immediately after the impulse is applied the speeds of the points A and B (see fig 90) are

$$v_A = V + \ell_A\omega \tag{508}$$
$$v_B = V - \ell_B\omega. \tag{509}$$

It may be that $v_B = 0$. The point at which the impulse is applied for which $v_B = 0$ is called the **centre of percussion**. If a body, e.g. a door is hinged at B and is struck at the centre of percussion there is no impulsive reaction at the hinge. Similarly if a cricket bat or tennis racket is held at one end (i.e. B) and the ball strikes the centre of percussion there is no painful jarring sensation at the player's hand.

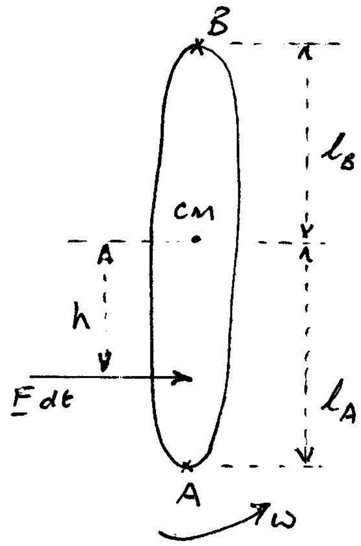

Figure 90: Impulse given to body free to rotate

Determination of position of centre of percussion If $v_B = 0$ then $V = \ell_B\omega$. But

$$MV = F\mathrm{d}t \tag{510}$$

and

$$I_0\omega = \tau\mathrm{d}t = Fh\mathrm{d}t = MVh \tag{511}$$

so

$$I_0\omega = M\ell_B\omega h \tag{512}$$

$$h = \frac{I_0}{M\ell_B}. \tag{513}$$

For a uniform rod of length L and mass M, we have $I_0 = \frac{1}{12}ML^2$ and $\ell_B = \frac{1}{2}L$, so $h = \frac{1}{6}L$, i.e. 2/3 of the length of the rod from B.

23.11 Simple theory of the gyroscope

Consider spinning a disc with moment of inertia I_0 about an axis perpendicular to the disc, as in fig91.The torque of the weight about O is $\tau = Mgx$. The direction of the torque vector $\boldsymbol{\tau}$ is into the paper. Since

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \tag{514}$$

then in time dt the angular momentum of the spinning disc changes by an amount

$$\mathrm{d}\mathbf{L} = \boldsymbol{\tau}\,\mathrm{d}t \tag{515}$$

in the direction of $\boldsymbol{\tau}$. Thus viewed from above we have the vector diagram with $\mathbf{L}\,(t)$ the angular momentum at time t. As can be seen from the diagram, the angular momentum vector $\mathbf{L}$ is rotated through an angle $\mathrm{d}\alpha$ in time $\mathrm{d}t$ but it is still horizontal and of the same magnitude. Hence

$$\mathrm{d}\alpha = \frac{\mathrm{d}L}{L} = \frac{\tau\mathrm{d}t}{L} = \frac{Mgx\mathrm{d}t}{I_0\omega}. \tag{516}$$

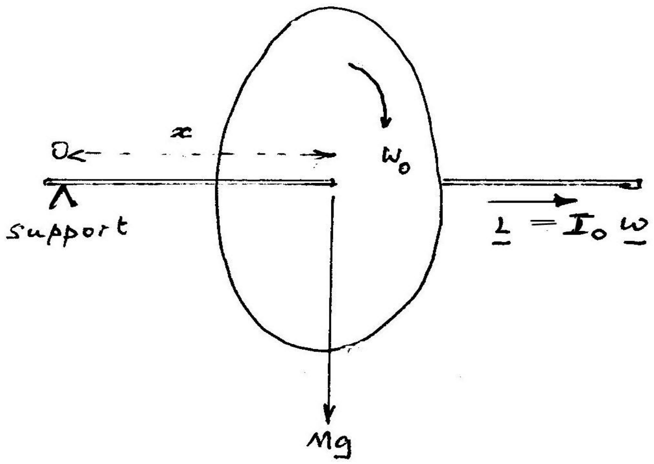

Figure 91: Spinning disc

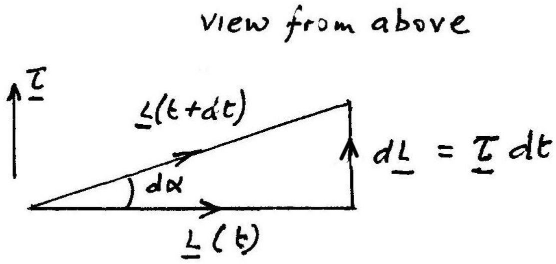

Figure 92: Vector diagram

The rate of rotation of **L**, or **rate of precession**, is

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\tau}{L} = \frac{Mgx}{I_0\omega}. \tag{517}$$

This simple theory is valid if $\frac{\mathrm{d}\alpha}{\mathrm{d}t} \ll \omega$.

There are many applications of gyroscopes in maintaining stability of rotating bodies and in various control systems.

24 Fluid Mechanics

A fluid is a substance that can readily be deformed, that flows when an external force is applied. Obvious examples are gases and liquids.

Hydrostatics - is study of fluids at rest and in equilibrium.

Hydrodynamics - study of fluids in motion.

24.1 Hydrostatics

Consider equilibrium of a small cube of liquid in a tank under gravity as in fig 93. Density of the liquid is ρ. There is no horizontal net force so forces on each pair of opposite parallel vertical faces must cancel. Hence if P_1 is average pressure on face 1, and P_2 is average pressure on face 2, then force on face 1 is

$$\mathrm{d}F_1 = P_1 \mathrm{d}y\, \mathrm{d}z \tag{518}$$

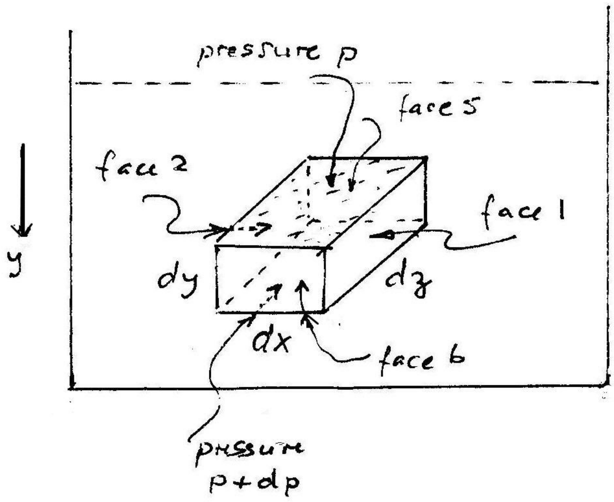

Figure 93: Small element of volume of a fluid

and force of face 2 *is*

$$\mathrm{d}F_2 = P_2 \mathrm{d}y\, \mathrm{d}z \tag{519}$$

but $\mathrm{d}F_1 = \mathrm{d}F_2$ and hence $P_1 = P_2$. Therefore **pressure is same at all points of same depth**.

Consider forces on upper and lower horizontal faces. Downward force on top force is

$$\mathrm{d}F_5 = P\, \mathrm{d}x\, \mathrm{d}z. \tag{520}$$

Upward force on bottom face is

$$\mathrm{d}F_6 = (P + \mathrm{d}P)\ \mathrm{d}x\, \mathrm{d}z. \tag{521}$$

Therefore for equilibrium of mass of liquid $\rho\, (\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z)$,

$$\begin{aligned} \mathrm{d}F_6 - \mathrm{d}F_5 &= \rho\, (\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z)\, g, & (522) \\ \mathrm{d}P\, \mathrm{d}x\, \mathrm{d}z &= \rho\, (\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z)\, g & (523) \\ \mathrm{d}P &= \rho g\, \mathrm{d}y\, . & (524) \end{aligned}$$

On integrating

$$P = \rho g y + P_0, \tag{525}$$

where P_0 is pressure at $y = 0$. Hence pressure in a static liquid under gravity varies only with the vertical height.

Pressure is measured in Pascals (Pa), 1 Pa = 1 Nm^{-2}.

24.1.1 Hydraulic press

Apply force F_2 to piston 2 of area A_2 as in fig 94 The pressure in the liquid $P_2 = F_2/A_2$. Same pressure under piston 1 of area A_1 so upward force on piston 1 is

$$F_1 = P_1 A_1 = \frac{A_1}{A_2} F_2. \tag{526}$$

If $A_1 > A_2$ then $F_1 > F_2$.

Piston 2 moves a distance x_2, piston 1 moves distance x_2 then if liquid is incompressible,

$$A_1 x_1 = A_2 x_2 \tag{527}$$

and

$$x_1 = \frac{A_2}{A_1} x_2. \tag{528}$$

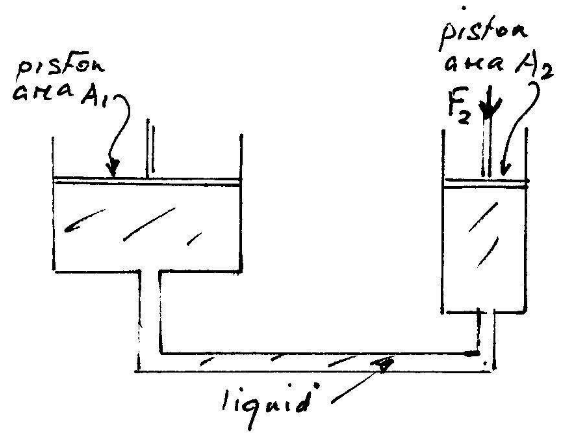

Figure 94: Hydraulic press

Work done by force F_1 is

$$F_1 x_1 = \frac{A_1}{A_2} F_2 \frac{A_2}{A_1} = F_2 x_2 \tag{529}$$

if there are no dissipative forces such as friction at the pistons or viscosity of the liquid.

24.1.2 Buoyancy

Consider a cylinder of mass m with end-face area A floating partially submerged in a liquid of density ρ as in fig 95. Force on top face $F_T = P_0 A$. Force on bottom face is $F_B = P_1 A = (P_0 + \rho g y)\,A$. Thus for

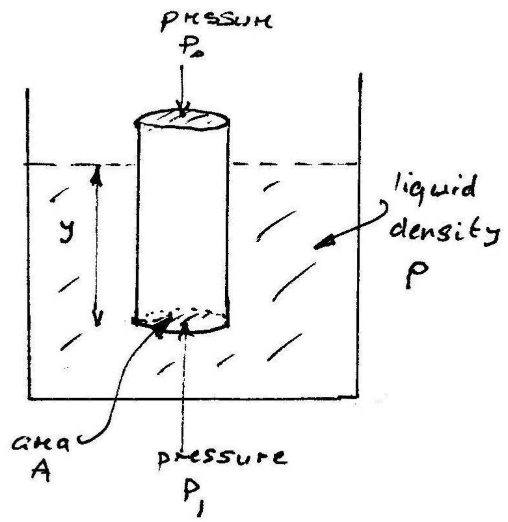

Figure 95: Floating cylinder

equilibrium

$$mg = F_B - F_T = \rho g y A = \left[(yA)\,\rho\right] g, \tag{530}$$

i.e. upthrust $= F_B - F_T$ is equal to the weight of liquid displaced.

Consider a fully immersed body. Let N be reaction of cylinder on legs as in fig 96. Force on top face is

$$F_T = P_1 A. \tag{531}$$

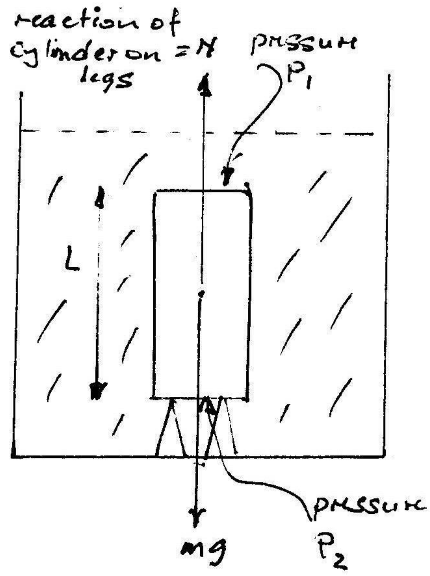

Figure 96: Submerged cylinder

Force on bottom face is

$$F_B = P_2 A + \rho g \, (lA) \, . \tag{532}$$

Thus upthrust $F_B - F_T = \rho \, (lA) \, g$. Again upthrust equals weight of liquid displaced. For equilibrium

$$mg = \text{upthrust} + N. \tag{533}$$

Whether body is partially or fully immersed there is an upthrust equal to weight of fluid displaced. This is **Archimedes' Principle**. The result applies to any shape of body.

24.2 Hydrodynamics - fluids in motion

Consider an **ideal liquid** which is

1. incompressible,
2. non-viscous,
3. constant temperature throughout,
4. flowing steadily, i.e. no time variation in flow pattern,
5. non-turbulent.

At each point with coordinates (x, y, z) the liquid has a velocity vector $\mathbf{v}$ and pressure p (scalar). Consider following a small element of liquid as it travels along, as in fig 97. The velocity may change in magnitude and direction along the path. This path of a fluid element is called a **streamline**. Consider another fluid element. This will follow another streamline. Two streamlines can never cross because it would mean that the velocity vector would have two different directions at the same point which is impossible (see fig 98).

Flow along stream lines, with the velocity vector changing smoothly is called **streamline flow**. Consider a thin bundle of adjacent streamlines as in fig 99 forming a **stream tube**. Because streamlines cannot cross any fluid that enters a stream tube at end A must exit through end B. There is no loss of fluid through the side of the stream tube.

Figure 97: A streamline

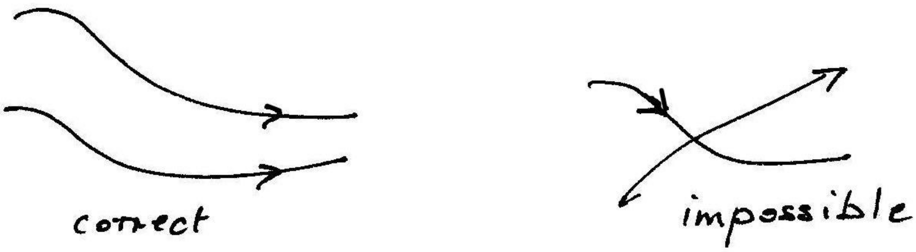

Figure 98: Two streamlines

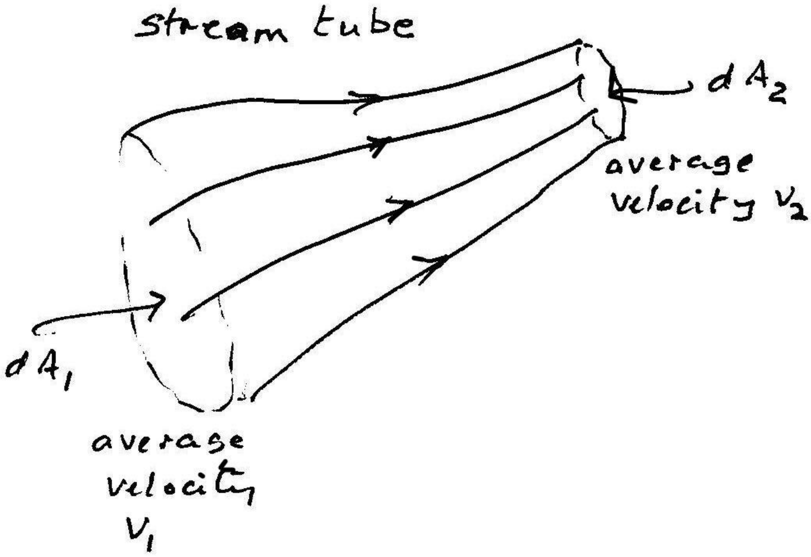

Figure 99: A streamtube

If density of fluid at A is ρ_1 and density at B is ρ_2, mass of fluid entering tube at A in time t is $m_1 = \rho_1 v_1 t \mathrm{d}A_1$ and the mass of fluid leaving through end B is $m_2 = \rho_2 v_2 t \mathrm{d}A_2$. Since mass is conserved, $m_1 = m_2$ and

$$\begin{aligned} \rho_1 v_1 t \mathrm{d}A_1 &= \rho_2 v_2 t \mathrm{d}A_2, \quad (534) \\ \rho_1 v_1 \mathrm{d}A_1 &= \rho_2 v_2 \mathrm{d}A_2. \quad (535) \end{aligned}$$

This is the **equation of continuity**. This is true for liquids and gases. For an ideal liquid which is incompressible, so $\rho_1 = \rho_2$, then

$$v_1 \mathrm{d}A_1 = v_2 \mathrm{d}A_2. \quad (536)$$

Example 21 *Hose-pipe nozzle of area A_2; volume rate of flow of water along the pipe of cross sectional area A_1 is $v_1 A_1 = v_2 A_2$, so $v_2 = v_1 A_1 / A_2$ at the nozzle, and hence $v_2 > v_1$ since $A_2 < A_1$.*

24.2.1 Bernoulli's equation

This is a most important equation relating to fluid flow. Consider an ideal liquid flowing along under gravity as in fig 100. Assume there is no viscosity (i.e. no dissipative forces) so that the total energy of a small volume of liquid as it flows along a stream tube must be conserved.

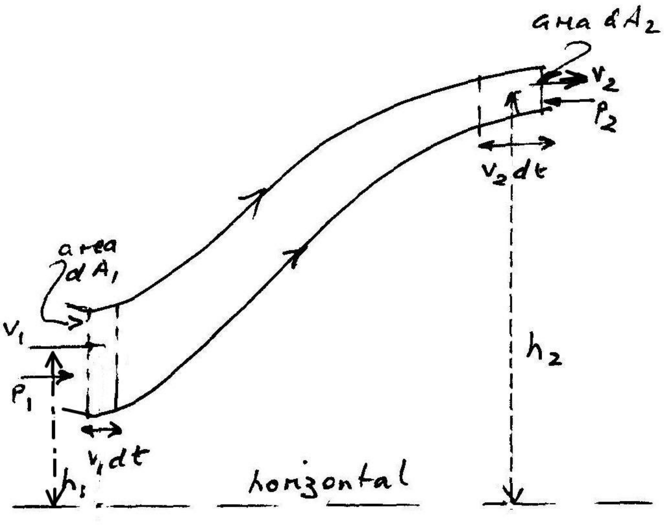

Figure 100: Diagram for Bernoulli's eqaution

Consider the work done by the pressure in time dt. Work done **by** pressure at end A_1 is

$$\mathrm{d}W_1 = (p_1 \mathrm{d}A_1)\, v_1 \mathrm{d}t, \quad (537)$$

and work done **against** pressure at end A_2 is

$$\mathrm{d}W_2 = (p_2 \mathrm{d}A_2)\, v_2 \mathrm{d}t. \quad (538)$$

By the continuity equation, mass of liquid element dm is (note $\rho_1 = \rho_2 = \rho$ as liquid is incompressible)

$$\mathrm{d}m = \rho \mathrm{d}A_1 v_1 \mathrm{d}t = \rho \mathrm{d}A_2 v_2 \mathrm{d}t. \quad (539)$$

Change of kinetic energy of this mass from end A_1 to end A_2 is

$$\begin{aligned} \mathrm{d}W_3 &= \frac{1}{2}\mathrm{d}m \left(v_2^2 - v_1^2\right) \quad (540) \\ &= \frac{1}{2}\rho \mathrm{d}A_1 v_1 \mathrm{d}t \left(v_2^2 - v_1^2\right). \quad (541) \end{aligned}$$

Work done against gravity in raising this mass of liquid from h_1 to h_2 is

$$\begin{aligned} \mathrm{d}W_4 &= \mathrm{d}m\, g\,(h_2 - h_1) && (542)\\ &= \rho \mathrm{d}A_1 v_1 \mathrm{d}t\, g\,(h_2 - h_1)\,. && (543)\end{aligned}$$

Therefore from conservation of energy

$$\begin{aligned} \mathrm{d}W_1 - \mathrm{d}W_2 &= \mathrm{d}W_3 + \mathrm{d}W_4 && (544)\\ (p_1 \mathrm{d}A_1)\, v_1 \mathrm{d}t - (p_2 \mathrm{d}A_2)\, v_2 \mathrm{d}t &= \frac{1}{2}\rho \mathrm{d}A_1 v_1 \mathrm{d}t \left(v_2^2 - v_1^2\right) + \rho \mathrm{d}A_1 v_1 \mathrm{d}t\, g\,(h_2 - h_1)\,. && (545)\end{aligned}$$

The continuity equation gives $\mathrm{d}A_1 v_1 = \mathrm{d}A_2 v_2$, so

$$\begin{aligned} p_1 - p_2 &= \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) + \rho g\,(h_2 - h_1) && (546)\\ p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 &= p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2. && (547)\end{aligned}$$

Thus in general along a stream tube

$$p + \frac{1}{2}\rho v^2 + \rho g h = \mathtt{cons}\tan\mathtt{t}. \qquad (548)$$

This is Bernoulli's equation. This has been derived for an ideal liquid, but is approximately true for gases if there is no great change in density, otherwise the equation becomes

$$\int \frac{\mathrm{d}p}{\rho} + \frac{1}{2}v^2 + gh = \mathtt{cons}\tan\mathtt{t}. \qquad (549)$$

If fluid is not moving, $v = 0$, then Bernoulli's equation becomes

$$p + \rho g h = \mathrm{cons}\tan\mathrm{t} \qquad (550)$$

i.e. pressure increases with depth as derived earlier. as h is measured in upwards sense.

Example 22 *Flow of water through a small hole in a tank, as in fig 101. At the surfaces $p_0 = p_1$ is the atmospheric pressure and $v \simeq 0$ as the water level falls slowly(!) So*

$$p_0 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_0 + 0 + \rho g h_2 \qquad (551)$$

$$v_1^2 = 2g\,(h_2 - h_1)\,. \qquad (552)$$

This is the same speed as free fall under gravity through a height $h = h_2 - h_1$.
Rate of flow (volume per second) through area A is Av.

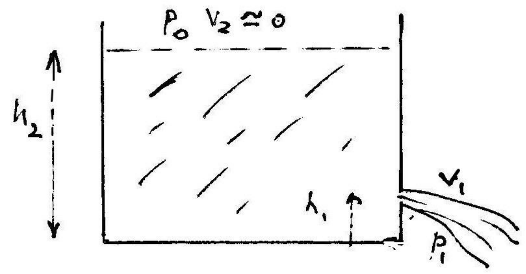

Figure 101: Flow from orifice in tank

24.2.2 Venturi meter

If the flow is horizontal so with no change in height then

$$p + \frac{1}{2}\rho v^2 = \mathtt{cons}\tan\mathtt{t} \tag{553}$$

and pressure is greater where speed of flow is less. This is the basis of the Venturi meter as illustrated in fig 102.

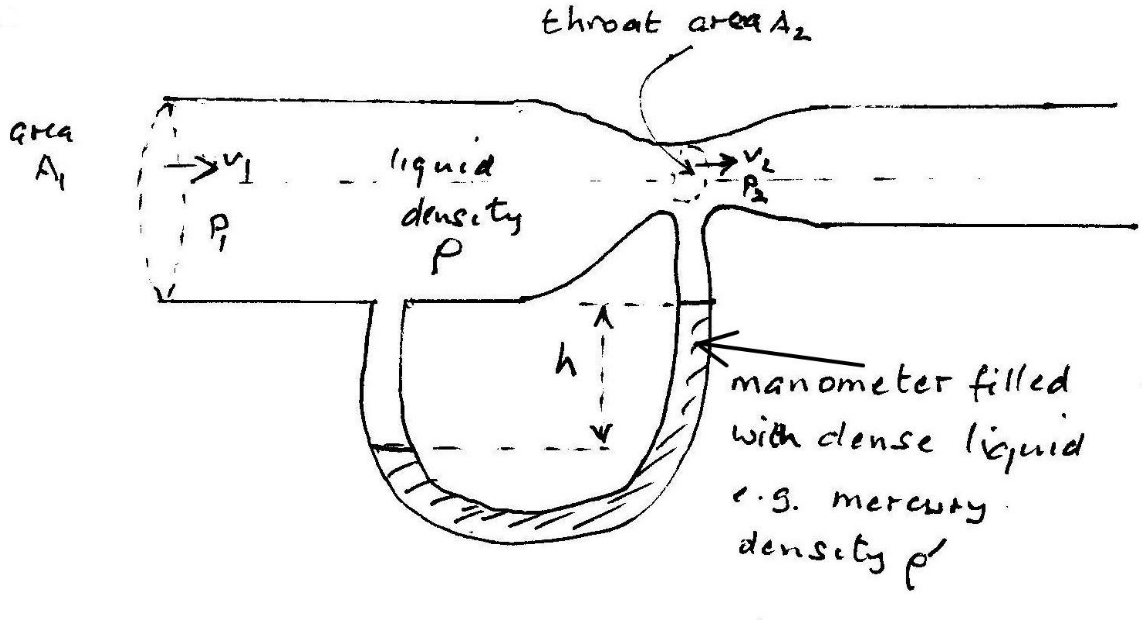

Figure 102: Venturi meter

From the diagram

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2 \tag{554}$$

and

$$p_1 - p_2 = \rho' gh \tag{555}$$

so

$$\rho' gh = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2. \tag{556}$$

But the continuity equation is $v_1 A_1 = v_2 A_2$ so $v_2 = v_1 A_1 / A_2$ so substituting for v_2

$$\rho' gh = \frac{1}{2}\rho v_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right) \tag{557}$$

$$v_1 = \sqrt{\frac{2\rho' gh A_2^2}{\rho\left(A_1^2 - A_2^2\right)}} \tag{558}$$

and volume rate of flow along pipe is

$$A_1 v_1 = \sqrt{\frac{2\rho' gh A_1^2 A_2^2}{\rho\left(A_1^2 - A_2^2\right)}}. \tag{559}$$

Example 23 *Flow of air over an aircraft wing as in fig 103.Air flow over the top of the wing, v_1 is faster than under the bottom, v_2 so pressure above wing, p_1 is less than than under the wing p_2 which gives rise to the upward force (lift) on the wing.*

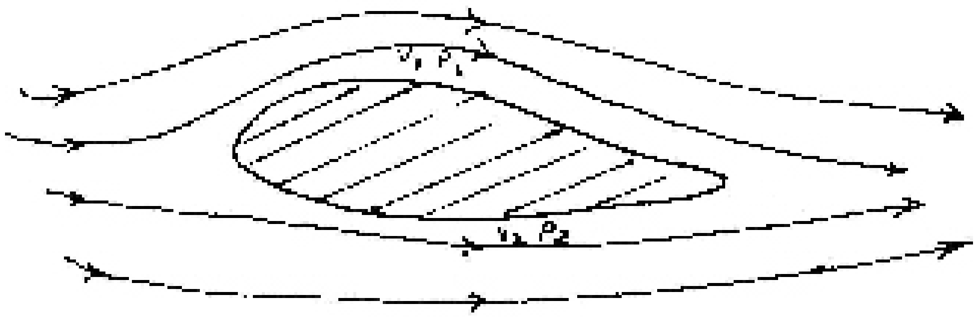

Figure 103: Flow over an aerofoil

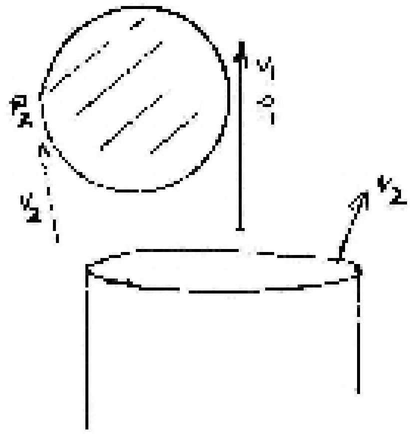

Figure 104: Ball suspended in upward air jet

Example 24 *Stability of table-tennis ball suspended on air jet. The air speed* $v_1 > v_2$ *so* $p_1 < p_2$ *producing a resultant force pushing the ball towards the centre of the air jet (see fig 104).*

Example 25 *Deflection of a spinning rough ball. Ball moving at speed* v *through the air and spinning about a vertical axis as in fig 105.*

As ball is rough it slows down air at A and tend to increase speed of air at B, such that $v_1 < v_2$, so that $p_1 > p_2$ producing a resultant force in direction from A to B. The ball is deflected horizontally. Note the ball is rough, a smooth ball does not drag the air round with it and the effect is not observed.

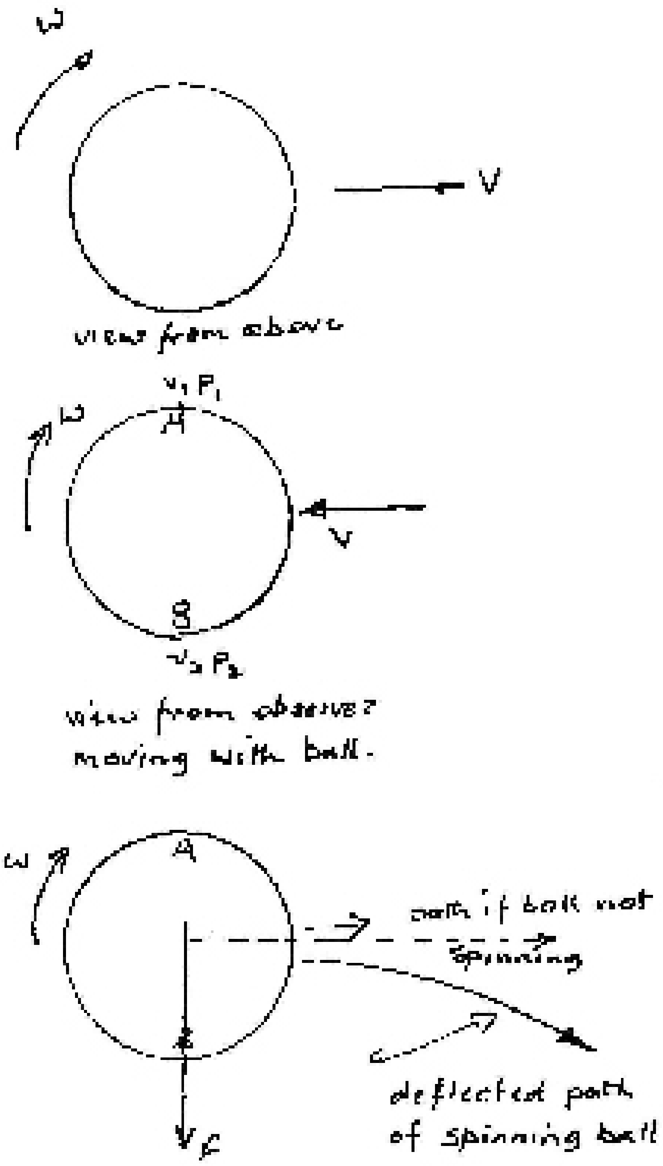

Figure 105: Deflection of spinning ball