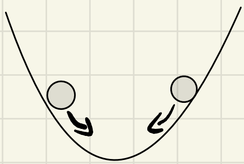
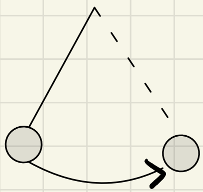
Simple Harmonic Motion

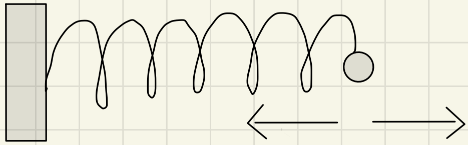
Physical Examples



- Ball in a well

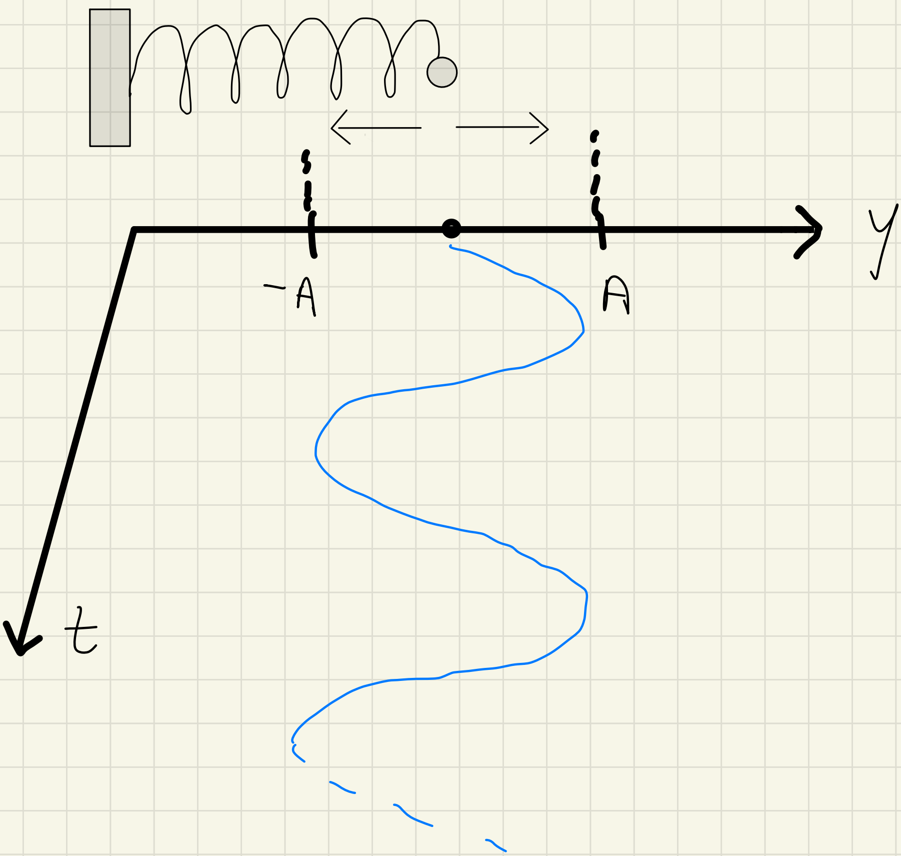


- Pendulum



- Mass on spring

Simple Harmonic Motion (SHM)



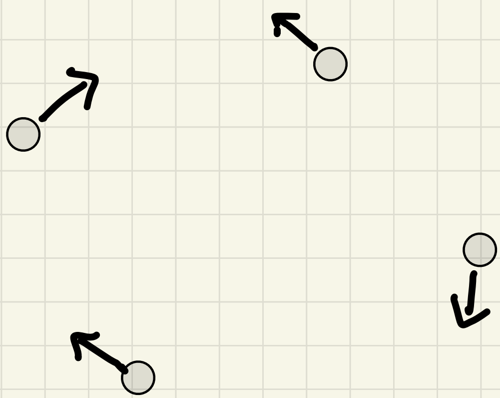
Key object = 'Wave function'

- $y(t) = A\cos(\omega t - \phi)$

Recall: Particle Mechanics

- $$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2$$
  $$v(t) = v_0 + a_0 t$$

- $$\bar{F} = m\bar{a} \qquad \text{(Force)}$$
  $$K = \frac{1}{2} m \bar{v}^2 \qquad \text{(Energy)}$$



Philosophy

Particle Mechanics —

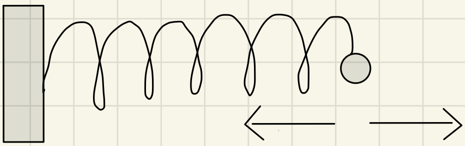
- Evaluate position, velocity, force, energy, ... at given instants in time

Wave Mechanics —

- wave function captures position, velocity, force, ... at all times
- Possible because motion is periodic

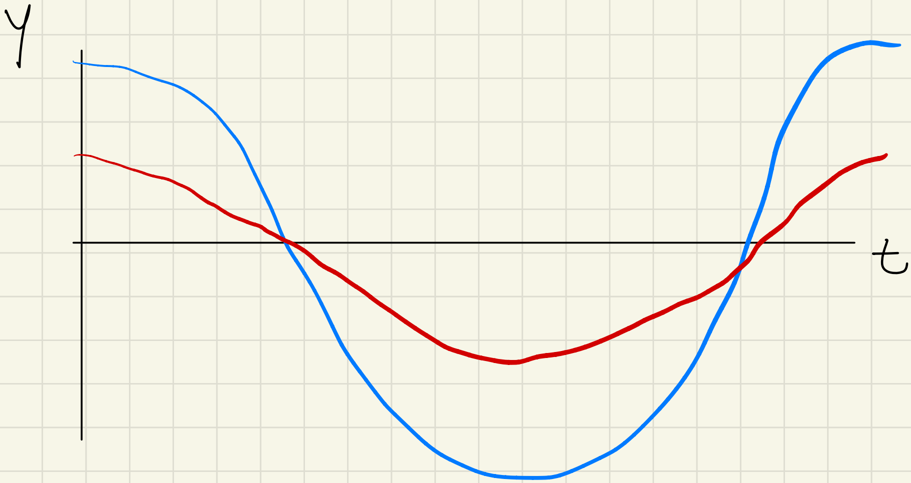
The Wave Function (WF)

Physical Example



Generic WF

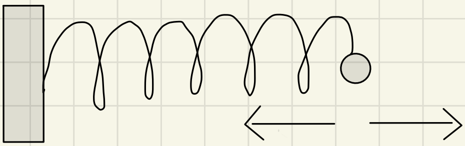
$$y(t) = A\cos(\omega t - \phi)$$



- $A > A$
- $\omega = \omega$
- $\phi = \phi$

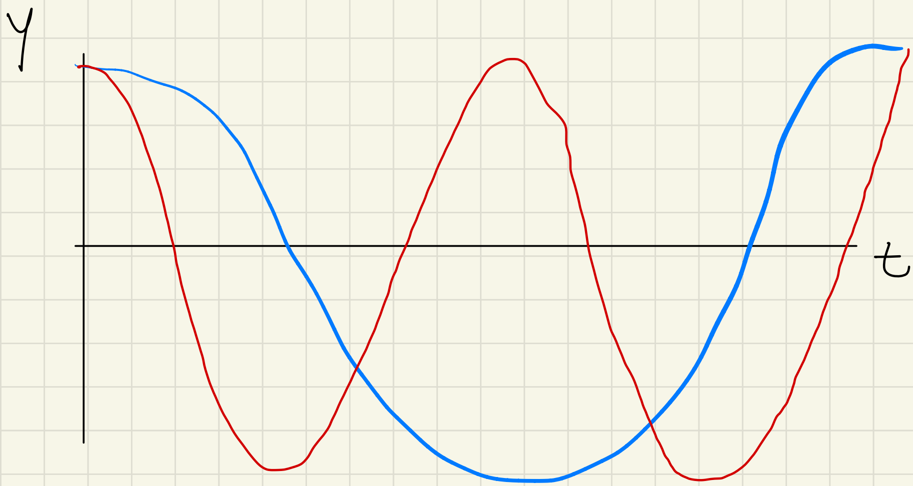
The Wave Function (WF)

Physical Example



Generic WF

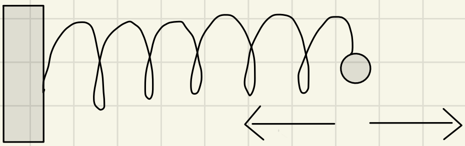
$$y(t) = A\cos(\omega t - \phi)$$



- $A = A$
- $\omega < \omega$
- $\phi = \phi$

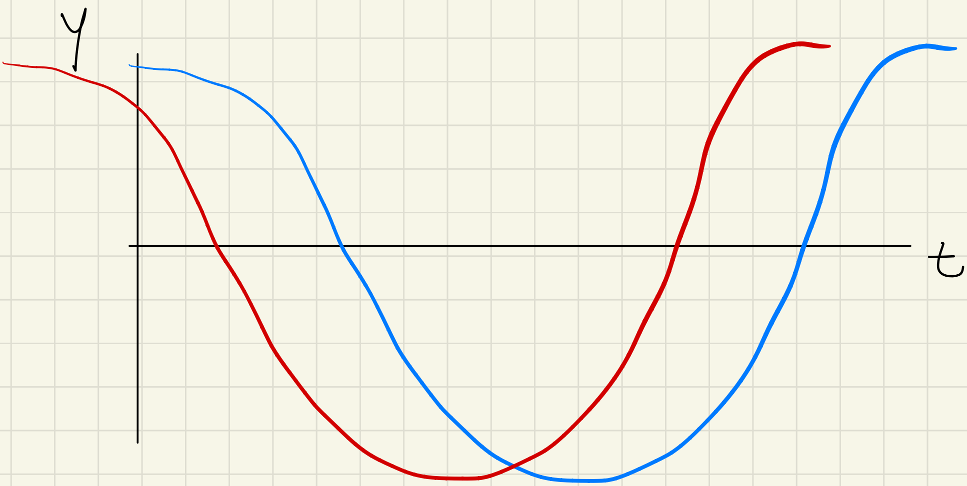
The Wave Function (WF)

Physical Example



Generic WF

$$y(t) = A\cos(\omega t - \phi)$$



- $A = A$
- $\omega = \omega$
- $\phi \neq \phi$

Wavefunction Summary

$$y(t) = A\cos(\omega t - \phi)$$

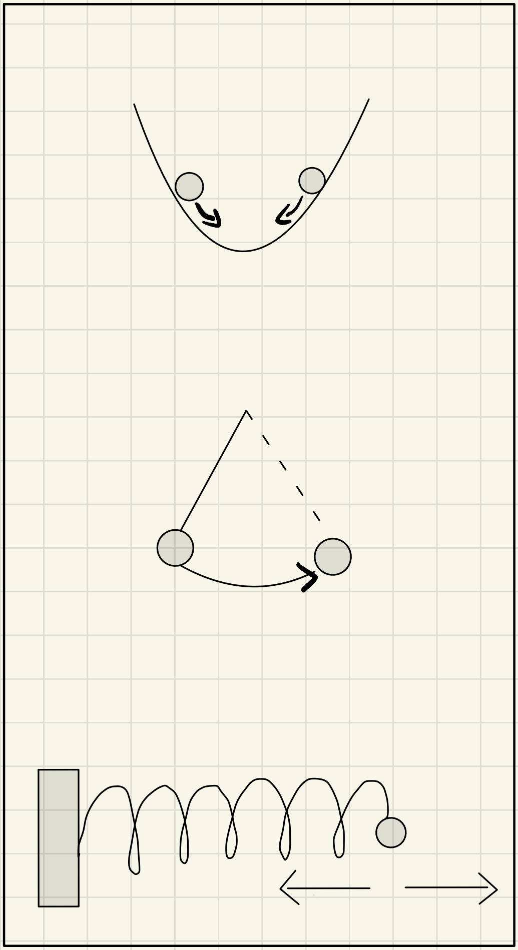
- A = amplitude
- ϕ = "offset" phase
- ω = angular frequency

Also

$f = \frac{\omega}{2\pi}$ = 'frequency'

$T = \frac{2\pi}{\omega} = 1/f$ = period

SHM Wave Mechanics



① Forces

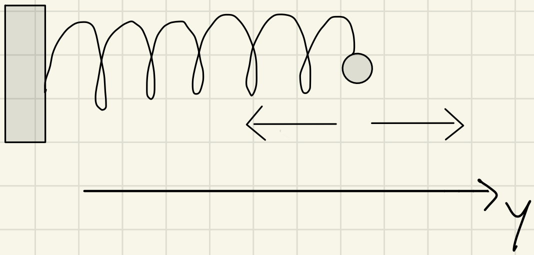
$F = -ky$

= restoring force

② Energy

$K + U = \text{constant}$

SHM Wave Mechanics

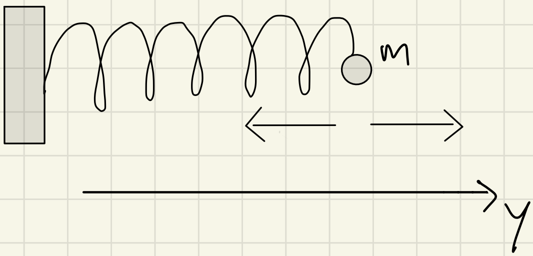


$$y(t) = A\cos(\omega t - \phi)$$

- Question: where did the wavefunction come from?

① Forces (SHM)

$$\Sigma F = ma = -Ky$$
$$m\ddot{y}(t) = -Ky$$
Guess: $y(t) = A\cos(\omega t - \phi)$
$$\ddot{y}(t) = -\omega^2 A\cos(\omega t - \phi)$$
$$-m\omega^2 \cancel{A\cos(\omega t - \phi)} = -K\cancel{A\cos(\omega t - \phi)}$$
$$\omega^2 = \frac{K}{m}$$



- $\Sigma F = ma = -Ky$
  $$a(t) = \ddot{y}(t) \left(= \frac{d^2}{dt^2} y(t) \right)$$
- $$m\ddot{y}(t) = -Ky(t)$$

  $y(t)$ = wave func. = solution
- 'Guess':
  $$y(t) = A\cos(\omega t - \phi)$$

show that $y(t)$ is a solution (Exercise)

Exercise

$$m\ddot{y}(t) = -\kappa y(t)$$

Guess: $y(t) = A\cos(\omega t - \phi)$

$$\text{LHS} = m\frac{d^2}{dt^2}\left(A\cos(\omega t - \phi)\right)$$

$$= m\frac{d}{dt}\left(-\omega A\sin(\omega t - \phi)\right)$$

$$= m\left(-\omega^2 A\cos(\omega t - \phi)\right)$$

$$= -\underline{m\omega^2} A\cos(\omega t - \phi)$$

$$\text{RHS} = -\underline{\kappa} A\cos(\omega t - \phi)$$

If $\kappa = m\omega^2$ then LHS = RHS

$$\omega = +\sqrt{\frac{\kappa}{m}}$$

Summary

$$y(t) = A\cos(\omega t - \phi)$$

is a solution when

$$\omega = \sqrt{\frac{k}{m}}$$

spring constant

mass

$$= \text{angular frequency}$$

2 Energy

$$K + U = \text{constant}$$

Definitions

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{y}^2$$

$$U = \frac{1}{2} k y^2$$

Solution $y(t)$:

$$\frac{1}{2} m \dot{y}(t)^2 + \frac{1}{2} k y(t) = \text{constant}$$

Exercise:

Check $y(t) = A\cos(\omega t - \phi)$ is a solution

- $\dot{y}(t) = -\omega A \sin(\omega t - \phi)$

- $\cos^2\theta + \sin^2\theta = 1$

- $LHS = \frac{1}{2} m \dot{y}^2 + \frac{1}{2} k y^2$

$= \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t - \phi) + \frac{1}{2} k A^2 \cos^2(\omega t - \phi)$

$= \frac{1}{2} k A^2 \left(\sin^2(\omega t - \phi) + \cos^2(\omega t - \phi) \right)$

$= \frac{1}{2} k A^2$

$= RHS$

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

What did we learn?

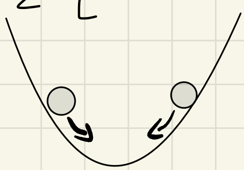
$$y(t) = A\cos(\omega t - \phi)$$

ω = set by spring constant & mass
= independent of initial conditions

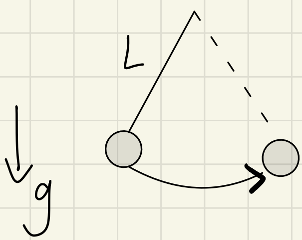
$$\left.\begin{aligned} A &= \text{unrestricted} \\ \phi &= \text{unrestricted} \end{aligned}\right\} \text{set by initial conditions}$$

Angular Frequency

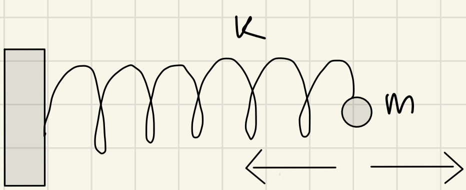
$$h(y) = \frac{1}{2}\frac{y^2}{L}$$



$$\omega = \sqrt{\frac{g}{L}}$$

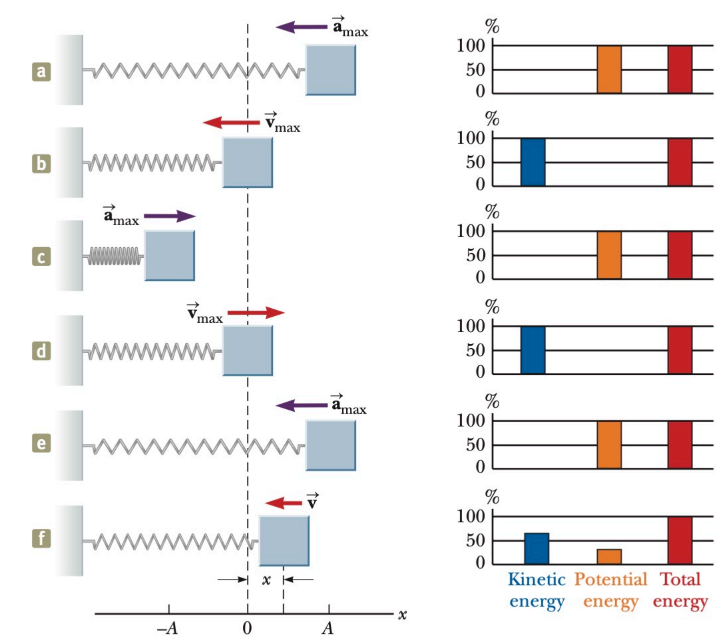


$$\omega = \sqrt{\frac{g}{L}}$$

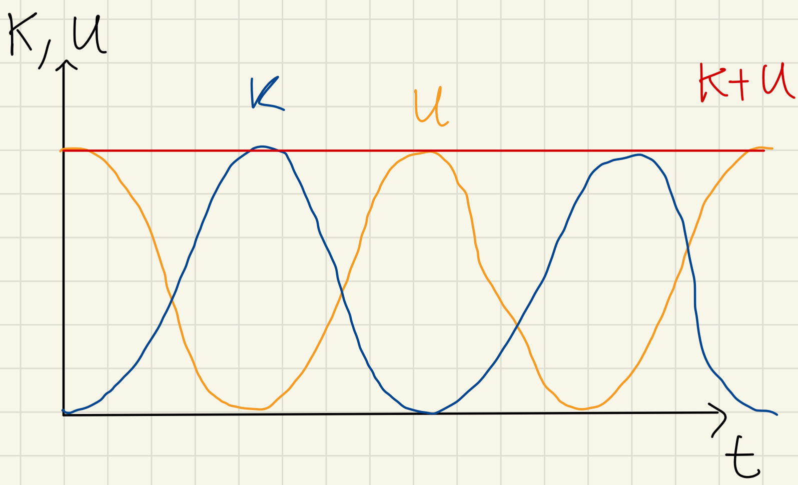


$$\omega = \sqrt{\frac{k}{m}}$$

Energy Summary



t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$\frac{T}{4}$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$\frac{T}{2}$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$\frac{3T}{4}$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
t	x	v	$-\omega^2 x$	$\frac{1}{2}mv^2$	$\frac{1}{2}kx^2$



Exercises

1. Object of mass 'm' is hung from a spring and set into SHM with period 'T.' Now replace mass with '2m.' What is the new period

(a) $2T$ (c) T (e) $T/2$

(b) $\sqrt{2}\,T$ (d) $T/\sqrt{2}$

2. A 200 g block connected to a light spring for which the force constant is 5 N/m is free to oscillate on a frictionless, horizontal surface. The block is displaced 5 cm from equilibrium and released from rest.

(a) Find the period of motion.

(b) Determine the maximum speed.

(c) Determine max acceleration.

3. A car of mass 1,300 kg can be modelled as system of four springs. Each spring has $k = 20{,}000$ N/m. Two people riding in the car have a combined mass of 160 kg. Find the frequency of vibration of the car after it hits a pothole.

4. Less words.

Particle mass 'm'
Oscillates with SHM with period 'T'
Max transverse velocity 'v_{max}'

(a) Find ω
(b) Find max displacement 'y_{max}'
(c) Force required to hold particle at $y = y_{max}$
(d) Kinetic and potential energies at $y = y_{max}$

Express in terms of $\{m, T, v_{max}\}$

5. A grandfather clock depends on the period of a pendulum to keep correct time. (i) Suppose a grandfather clock is calibrated correctly, but then the 'bob' slides down the rod.

Does the grandfather clock run (a) slow, (b) fast, or (c) correctly?

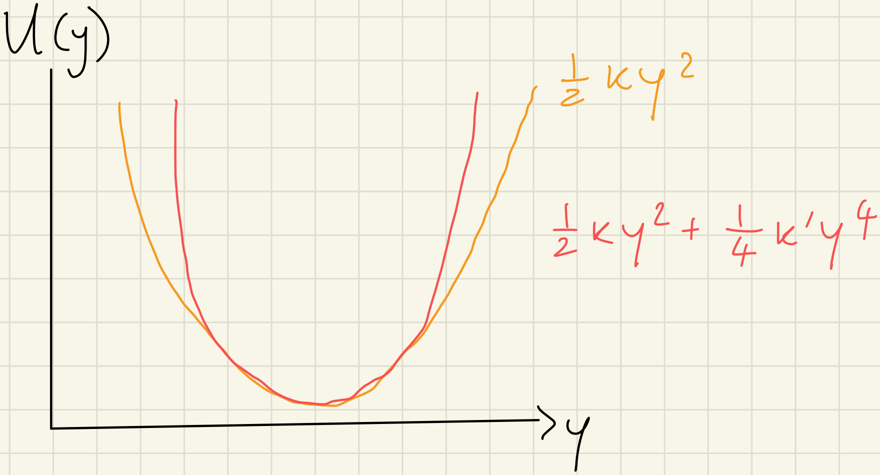
(ii) Suppose it runs correctly at sea level, how does it run atop a large mountain? (a), (b) or (c)

6. Final Exercise.

Huygens (1629–1695), a great clockmaker and physicist, suggested that an international unit of length could be defined as the length required to have $T = 1\,s$ on a simple pendulum. What would that unit of length be?

Non-simple Harmonic Motion (Discussion)

Energy:



Forces:

$$F_{spring} = -ky - k'y^3$$

$$m\ddot{y}(t) = -ky(t) - k'y^3(t)$$

$$y(t) \neq A\cos(\omega t - \phi)$$

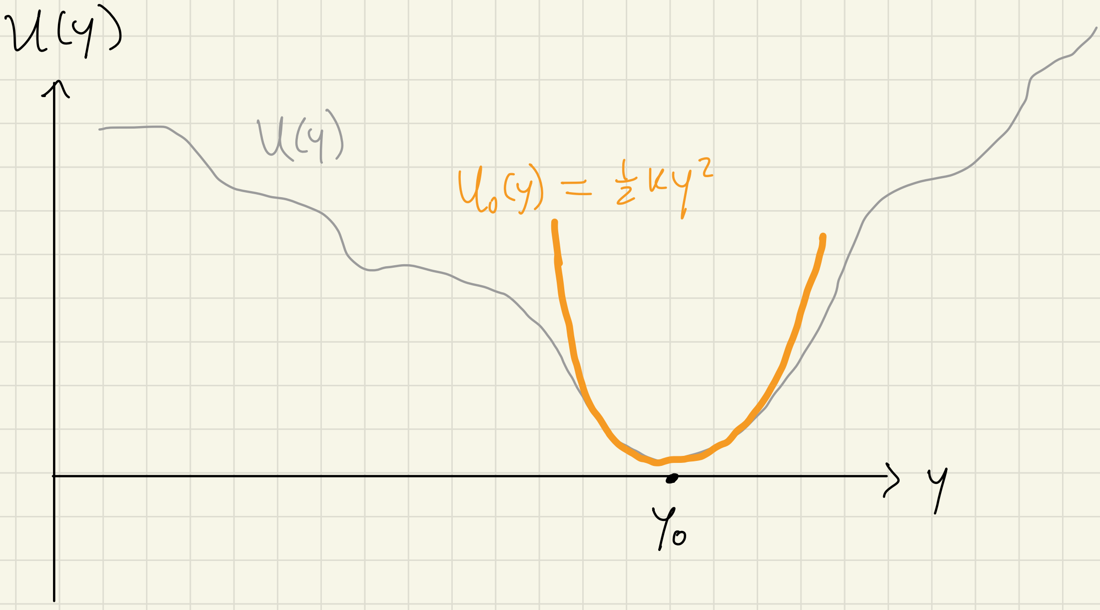
Non-simple Harmonic Motion
(Discussion)

- Is SHM a good approximation of Nature?
- Or is SHM just convenient?

SHM is a great approximation to many many natural systems!

Two ingredients:

- Equilibrium $\sum F = 0$
- Taylor expansion



[Aside]

- Equilibrium

$$\sum F(y) = -\frac{dU}{dy} = 0$$

$U(y)$ has a minimum when $\sum F(y) = 0$ (equilibrium)

- Taylor expansion

$$f(y-y_0)$$

$$= f(y_0) + (y-y_0) f'(y_0) + \frac{1}{2}(y-y_0)^2 f''(y_0)$$

$$+ \frac{1}{6}(y-y_0)^2 f'''(y_0) + \cdots$$

- Taylor expansion

$$f(y-y_0)$$

$$= f(y_0) + (y-y_0) f'(y_0) + \frac{1}{2}(y-y_0)^2 f''(y_0)$$

$$+ \frac{1}{6}(y-y_0)^2 f'''(y_0) + \cdots$$

Simplify: $y_0 = 0$

0 (Equilibrium / minimum)

$$f(y) = f(0) + y f'(0) + \frac{1}{2} y^2 f''(0) + \cdots$$

$$= f(0) + \frac{1}{2} y^2 f''(0) + \cdots$$

$$= \text{const.} + \frac{1}{2} k y^2 + \cdots$$