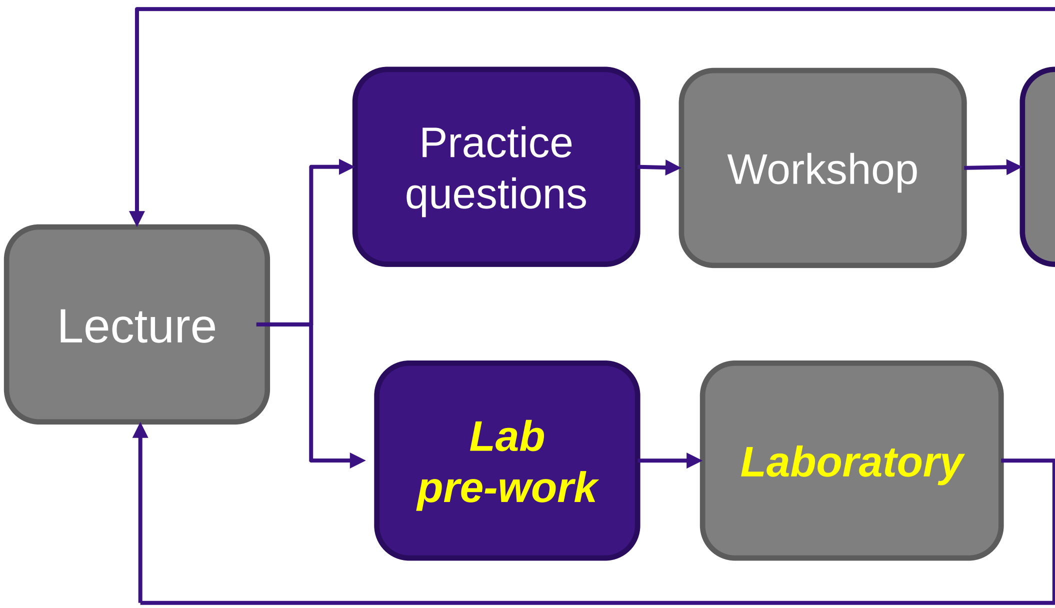
Lecture 2: Kinematics in two and three dimensions; Vectors

How you'll learn in this subject

- You'll be introduced to new concepts and principles with worked examples in lectures
- You'll practise applying these concepts and models in workshops
- You'll test the validity of some models in laboratory



Grey box – timetabled activity
Blue box – done in own time
Yellow font – assessed activity

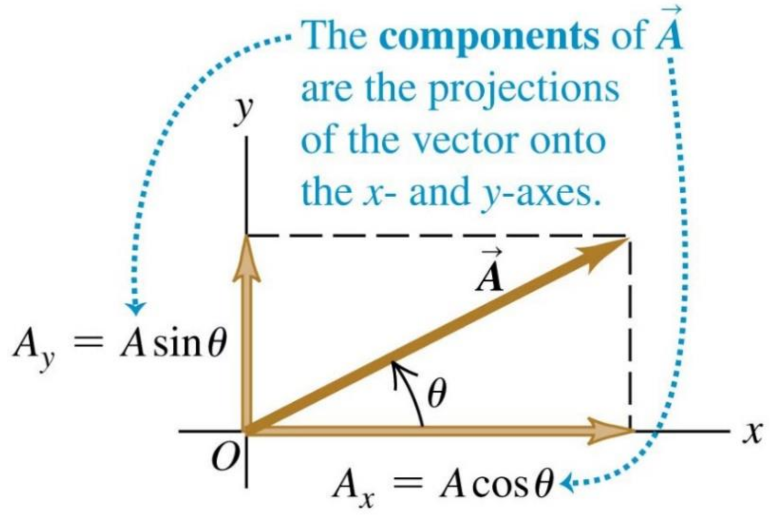
In this lecture

- Position, velocity and acceleration in motion in two and three dimensions
- Vectors
- Motion of a projectile
- Circular motion

(With acknowledgments to Danica Solina)

Kinematics in two dimensions

- We will extend the description of motion to two dimensions. We will use vectors to represent the position and velocity of an object.
- A vector quantity has both a magnitude and a direction in space.
- Draw a vector as a line with an arrowhead at its tip.
- The length of the line shows the vector's magnitude.



In this case, both A_x and A_y are positive.

Scalars and Vectors

Scalars

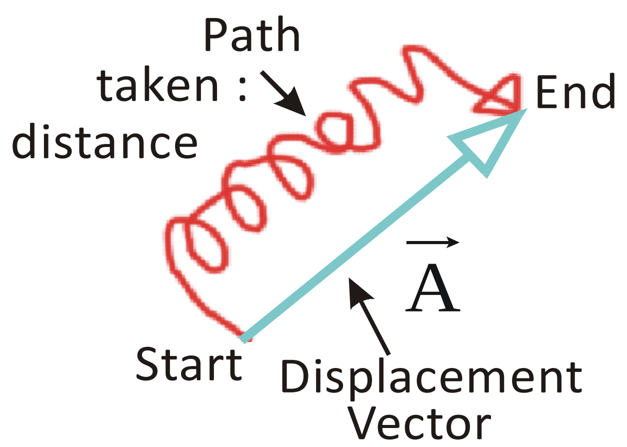
- Have magnitude but not direction

  Example: time, mass, energy, distance
- May be positive or negative

Vectors

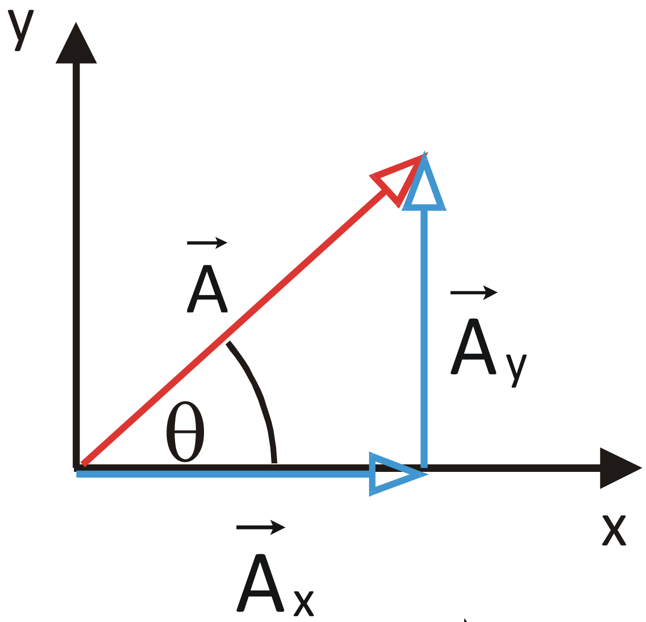
- Have both magnitude and direction

  Examples: velocity, acceleration, force, displacement
- Can be represented with an arrow
- Given relative to something e.g 5 km 30º West of North



Displacement vector $\vec{A}$: change of position of an object relative to the starting position.

Vector components



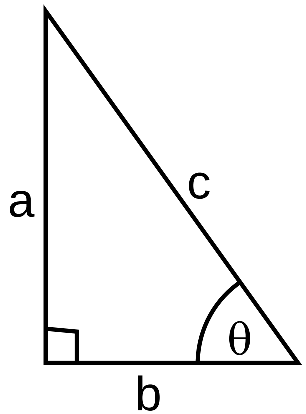
Vector $\vec{A}$ has an x and y component

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

The length/magnitude of each component is

$$A_x = |A| \cos\theta \qquad A_y = |A| \sin\theta$$

where $|A|$ is the length/magnitude of $\vec{A}$



Right triangle

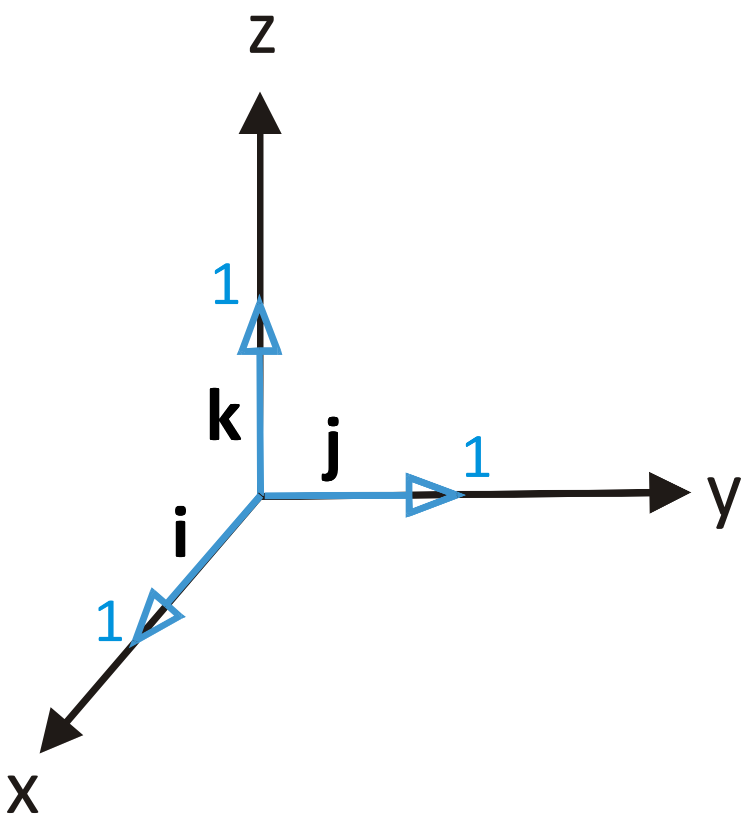
Trigonometry (Maths revision)

$$\sin\theta = \frac{a}{c}, \cos\theta = \frac{b}{c}, \tan\theta = \frac{a}{b}$$

Pythagoras theorem: $c^2 = a^2 + b^2$

Area of triangle: $\frac{1}{2}ab$

Unit vectors



- A unit vector is a vector which has a magnitude of one
- **i**, **j**, **k** are unit vectors in the x, y and z axes, respectively.
- A vector can be represented mathematically using its magnitude in each direction multiplied by the corresponding unit vector:

$$\overrightarrow{A} = \overrightarrow{A_x} + \overrightarrow{A_y} + \overrightarrow{A_z}$$

$$\boxed{\overrightarrow{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}}$$

Position vector

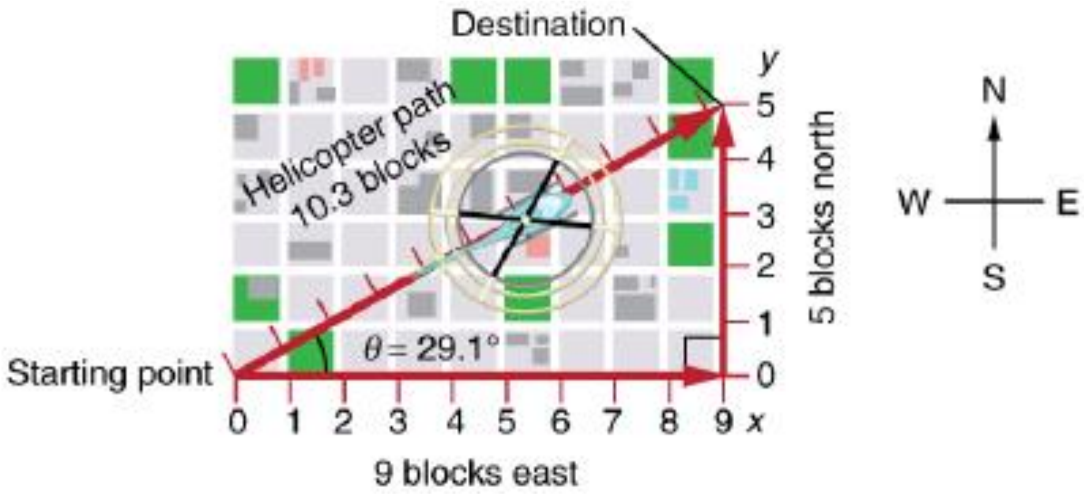
- Choose a suitable x-y reference frame with unit vectors.
- In kinematics in 2D the position of a point can be described by the position vector **r.**

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y$$

The unit vectors ***i*** and ***j*** point in the x- and y-direction, respectively.

Example 1

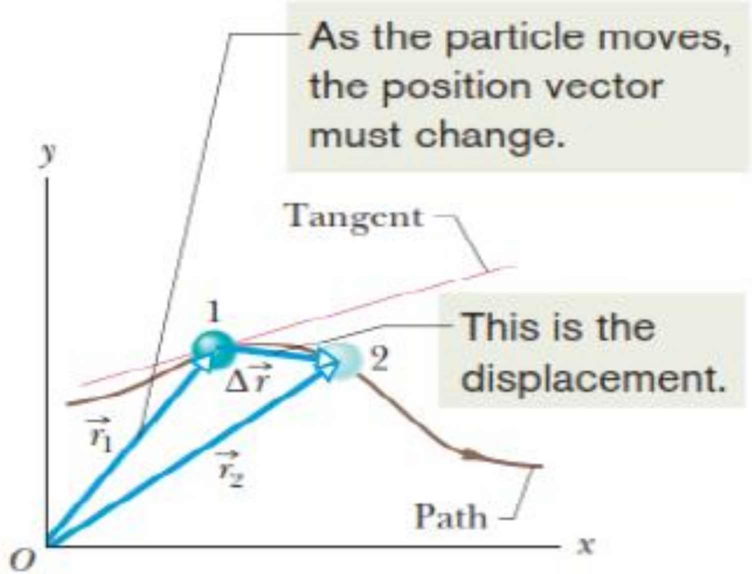
A helicopter flies straight to a point which is 9 blocks East and 5 blocks North of its starting point. Find the position (displacement vector) of final destination.



Displacement

If the position of an object changes from position vector $\mathbf{r}_1$ to $\mathbf{r}_2$ then the displacement of this object is $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

Displacement is a vector which joins the initial position with the final position.



$\mathbf{r}_1 = \mathbf{i}x_1 + \mathbf{j}y_1$ $\mathbf{r}_2 = \mathbf{i}x_2 + \mathbf{j}y_2$ $\Rightarrow$ $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{i}(x_2 - x_1) + \mathbf{j}(y_2 - y_1)$

Velocity

- An object which changes its position from $\mathbf{r}_1$ to $\mathbf{r}_2$ in the time interval t_2 - t_1 has average velocity:

$$\overline{\mathbf{v}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}$$

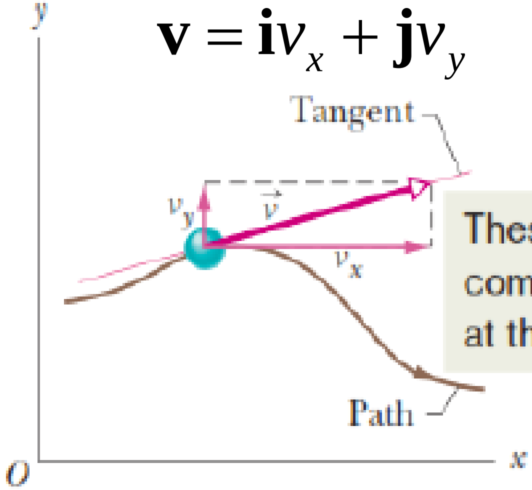
- The instantaneous velocity equals to the limit of the object's average velocity as the time interval approaches zero.

$$\mathbf{v} = \lim_{\Delta t \to 0} \overline{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

- The direction of the instantaneous velocity vector is tangent to the traveling path (red vector in the diagram)

The velocity vector is always tangent to the path.

$$\mathbf{v} = \mathbf{i}v_x + \mathbf{j}v_y$$



These are the x and y components of the vector at this instant.

Acceleration

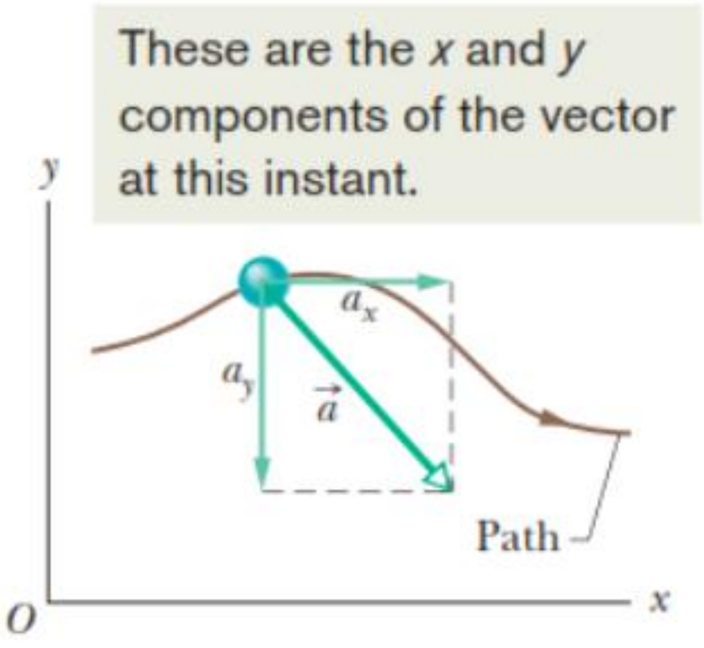
If the velocity of an object changes from $\mathbf{v}_1$ to $\mathbf{v}_2$ during time interval $t_2 - t_1$ then the average acceleration of this object is

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1}$$

The instantaneous acceleration equals to the limit of the object's average acceleration as the time interval approaches zero.

An instantaneous acceleration is the acceleration of an object measured at the given moment of time

$$\mathbf{a} = \lim_{\Delta t \to 0} \bar{\mathbf{a}} = \lim_{\Delta t \to 0} \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$



Example 2

A rocket is launched from the Earth surface with an initial velocity vector $\overrightarrow{v_o}$ of magnitude 3500 m/s at an angle of 60^o above the horizontal. The rocket's engines provide a constant acceleration of 20 m/s^2 in the direction of its velocity vector.

a) Determine the horizontal and vertical components of the initial velocity vector.

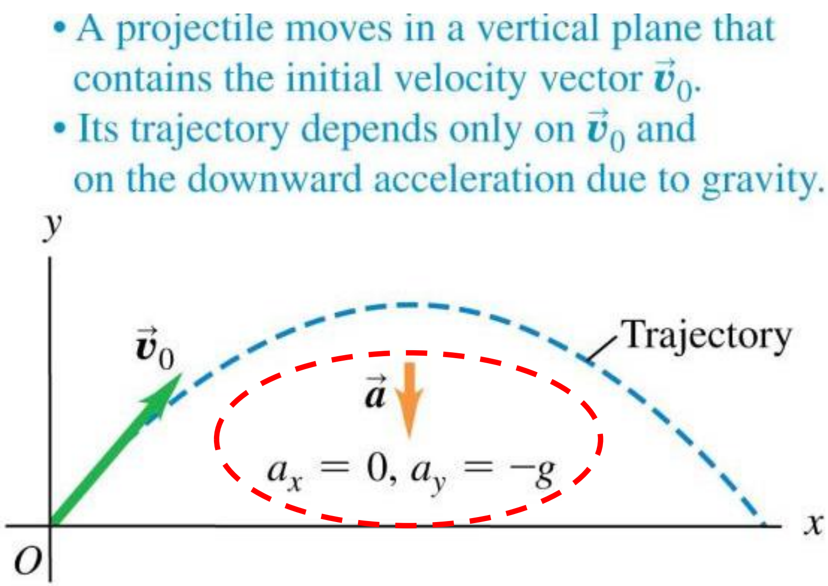
b) Find the velocity vector of the rocket after 15 seconds.

Projectile Motion

- An object thrown (or projected) into air is called a projectile
- If we neglect the air resistance than the only force which acts on this object is the Earth gravitational pull
- The Earth gravitational pull is directed downward and provides a constant acceleration *g* to the projectile.
- The only acceleration it is subject to is due to gravity:

$$a_x = 0 \qquad a_y = -g = -9.8\ \text{m/s}^2.$$

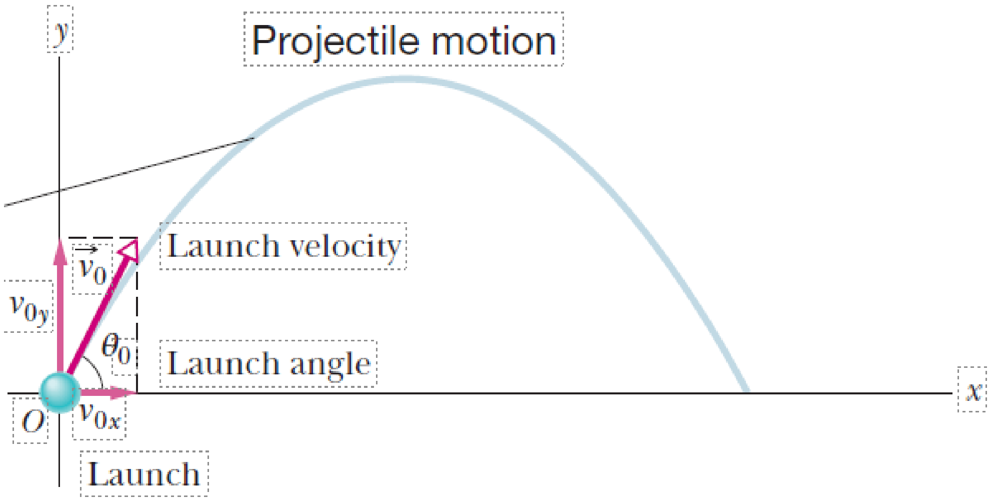
- The trajectory is a parabola.



Projectile Motion

A projectile is launched with an initial velocity **$\mathbf{v_0}$** and angle θ_0 Write down the position vector velocity (and acceleration) of the object.

The initial velocity vector has x and y components: $v_0 = v_{0x}\mathbf{i} + v_{0y}\mathbf{j}$



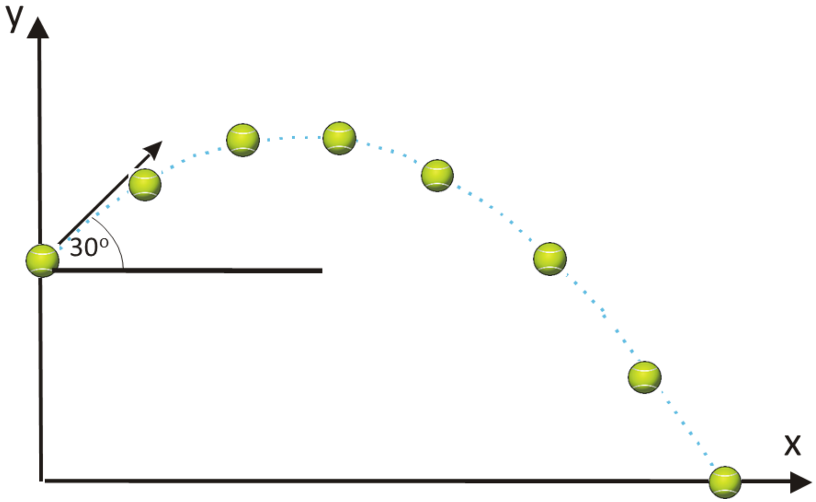
A projectile motion is a complex motion. It can be viewed as composed of two motions: horizontal and vertical.

- In horizontal direction: acceleration is zero
- In vertical direction: acceleration is constant and downwards

$$a_y = -g = -9.81\ \mathrm{m/s^2}$$

Example 3

A soccer ball is kicked towards the end of the field from a height of 1.0 m above the ground. Its initial velocity is 36.5 m/s at an angle of 30^o above the horizontal. Ignore air resistance. What is the maximum height that the ball can reach?



Revision: Equations of Motion

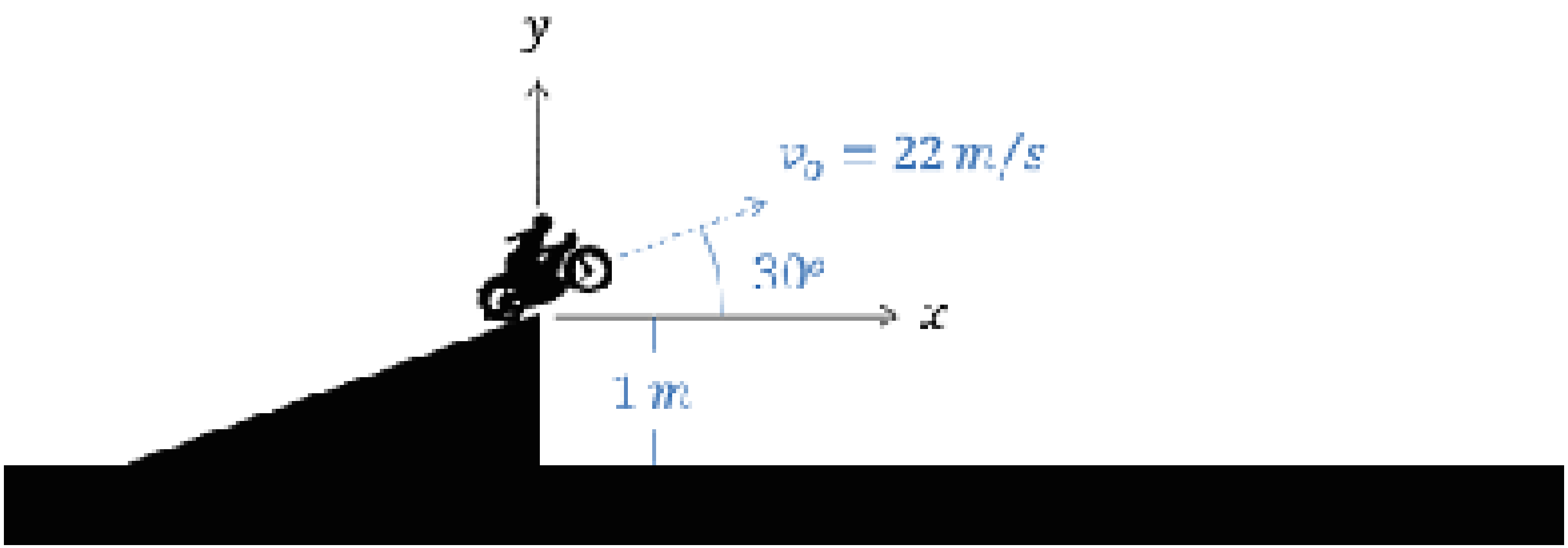
- In the previous lecture, we learned the below equations in situations where acceleration is constant
- If the acceleration, a, is constant these equations are very useful for calculating displacements and velocities

$$v = v_o + at$$
$$s = v_o t + \frac{1}{2}at^2$$
$$v^2 = v_o^2 + 2as$$

where: v_o is the initial velocity
v is the final velocity
t is the time interval for the motion
a is the acceleration
s is the displacement

Example 4

A motorbike launches off a ramp with a height of 1.0 m at a velocity of 22 m/s and an angle of 30°. Determine the equations for the acceleration, velocity and position as functions of time. How far does the motorbike move in the x-direction before hitting the ground?



Circular rotation

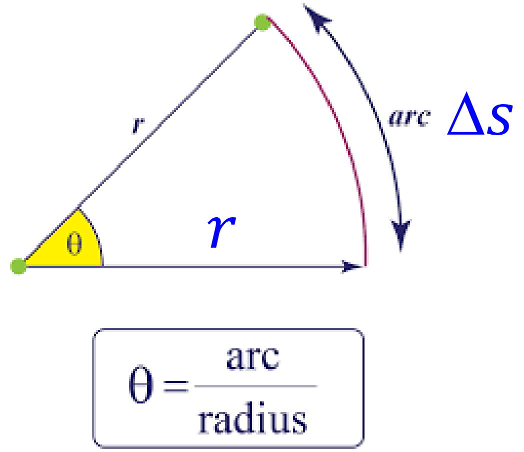
An object rotating on a plane about a fixed axis will have a motion that follows a circular arc

Δs is the arc length

r is the radius of curvature

θ is the rotation angle

$$\theta = \frac{\Delta s}{r}$$

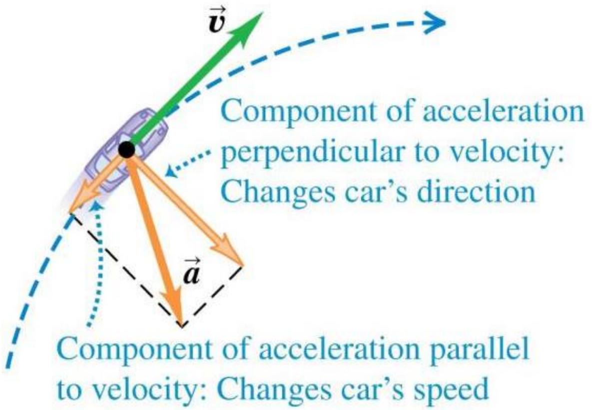


For one complete revolution the arc length is the circle's circumference

$$\theta = \frac{2\pi r}{r} = 2\pi$$

The SI unit for rotation is the radian

$$1\, rad = \frac{360^0}{2\pi} \approx 57.3^0$$



Car slowing down along a circular path

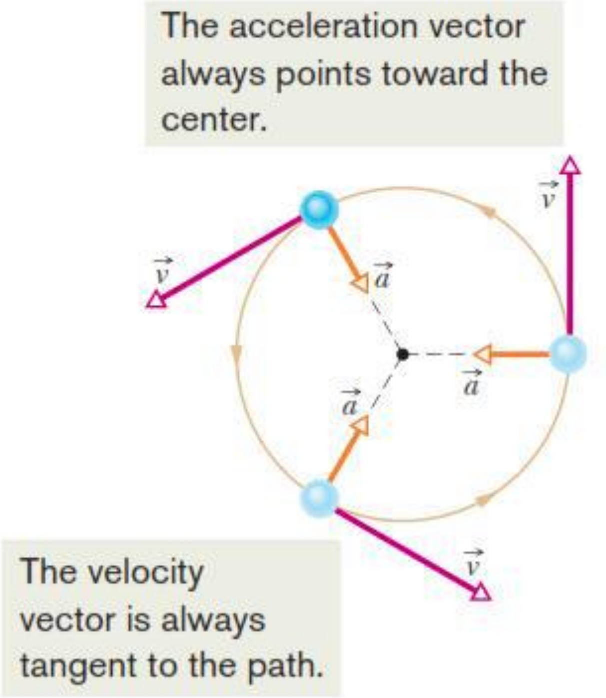
Uniform circular rotation

Uniform circular motion occurs when an object travels around a circle with a constant speed.

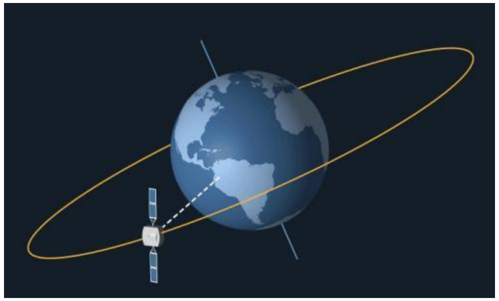
The direction of the object's velocity is constantly changing so the velocity is not constant. Thus it is always subject to an acceleration that we call centripetal acceleration (center seeking).

$$a_c = \frac{v^2}{r}$$

Centripetal acceleration is perpendicular to the velocity, so it changes the direction of the velocity – but not its magnitude (speed)



Example: the Moon or satellite orbiting around the Earth



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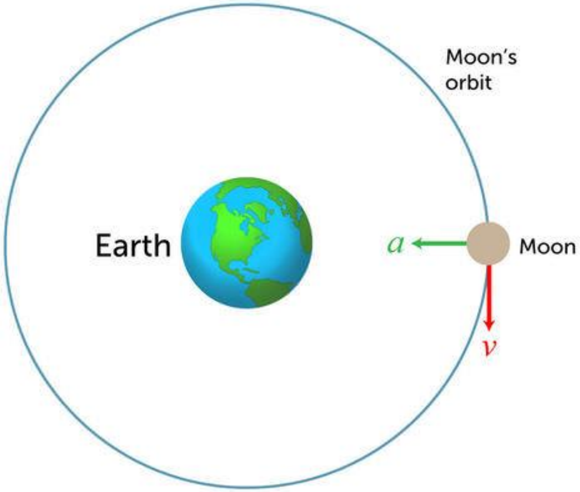
$$a_c = \frac{v^2}{r}$$

r is the radius of the circular orbit

See the derivation on the next slide.



Examples: the Moon going around the Earth

A derivation of centripetal acceleration

We can state that $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ is directed to the centre of the circle

Using:

$$\frac{\Delta v}{v} = \frac{\Delta r}{r}$$

$$\frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta r}{\Delta t}$$

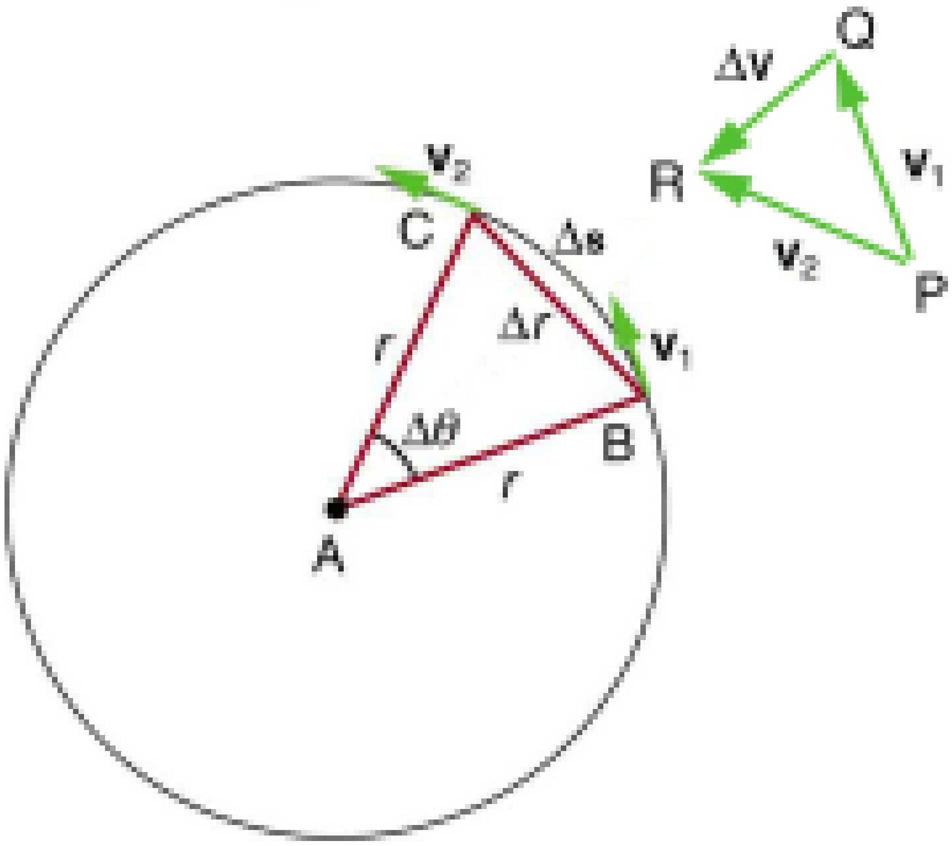
We also have:

$$a_c = \frac{\Delta v}{\Delta t}$$

$$v = \frac{\Delta r}{\Delta t}$$

Therefore

$$a_c = \frac{v^2}{r}$$



Example 5

A particle in a centrifuge a distance of 15 cm from the axis of rotation experiences a force which is 250 times that which it would experience due to gravity alone. What is the speed of the particle in the centrifuge?

Uniform circular rotation

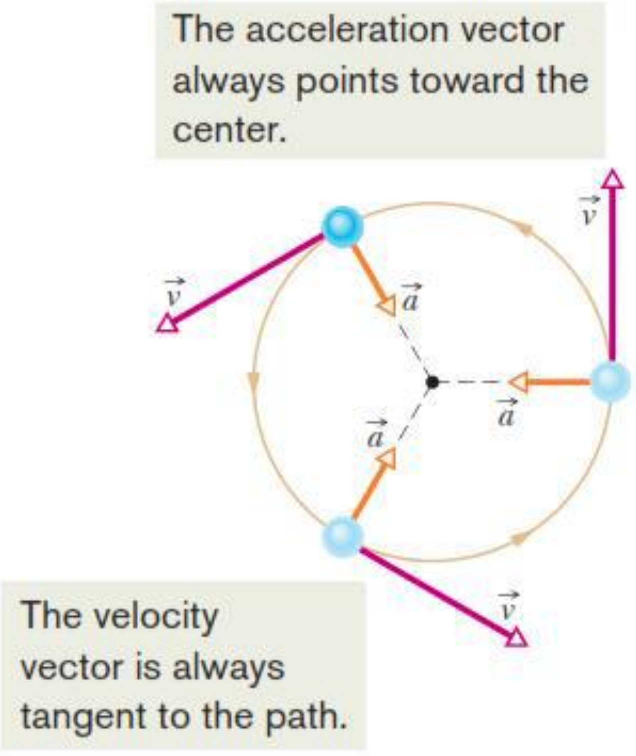
Th time taken for the object to make one complete rotation is called the period of revolution, *T*.

$$T = \frac{2\pi r}{v}$$

Frequency of the rotation *f* is given by

$$f = \frac{1}{T}$$

Frequency of the rotation tells how many rotations are taking place per unit time



Angular and linear velocities

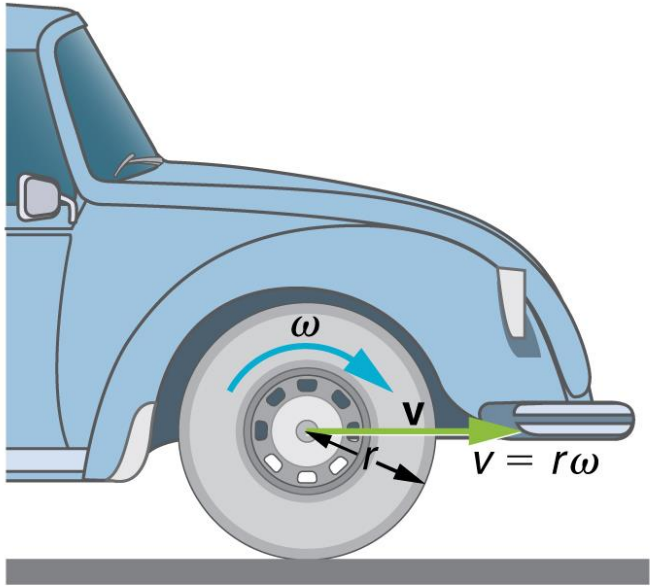
Angular velocity ω measures how fast an object is rotating (in rad/s). It is the rate of change of the rotation angle θ

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T}$$

Linear velocity v measures how far an object moves in a given time.

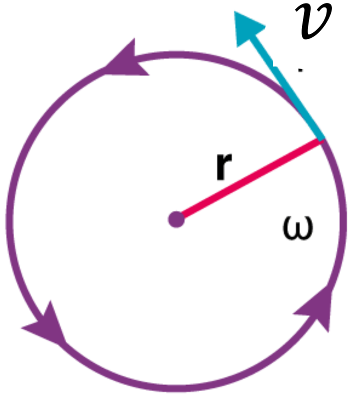
$$v = \frac{\Delta s}{\Delta t} = \frac{r\theta}{\Delta t}$$

$$v = \frac{2\pi r}{T} = \omega r$$

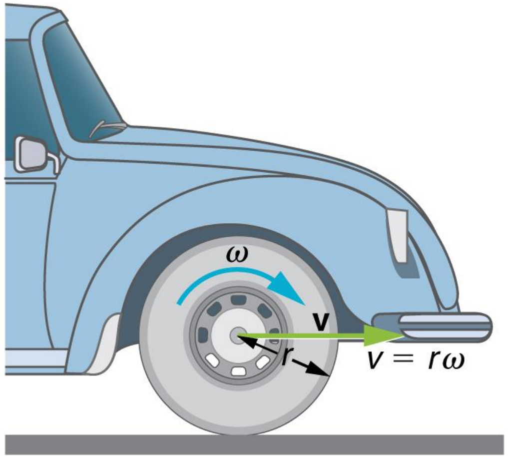


Angular variables

$$v = \frac{2\pi r}{T} = \omega r$$



Linear velocity υ (i.e. the tangential speed) is proportional to the distance from the axis of rotation. Linear velocity is expressed in m/s.

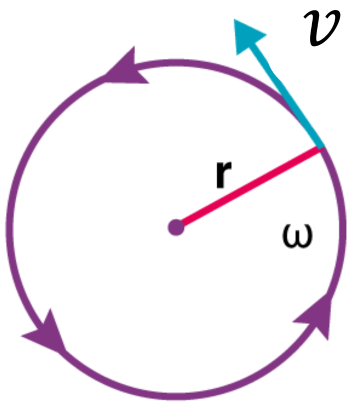


Angular velocity ω of a point is independent of how far it is from the axis of rotation. Angular velocity is expressed in rad/s.

Angular acceleration

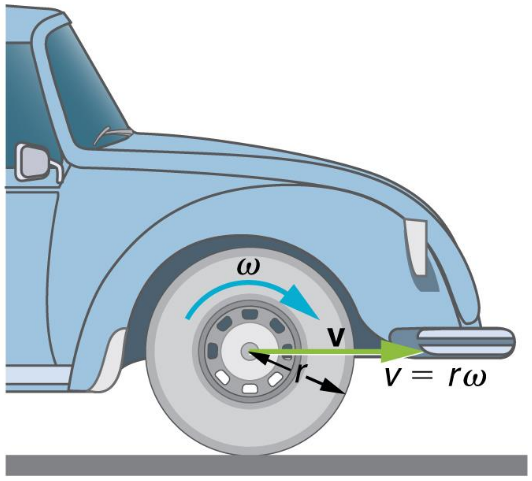
The rate at which the angular velocity change is called the angular acceleration, α.

$$\alpha = \frac{\Delta\omega}{\Delta t}$$



The tangential linear acceleration, *a*, of a point along the circular arc of radius *r* given by

$$a_{tang} = r\alpha$$



Example 6

A centrifuge rotates with an angular velocity of 5320 rad/s. A test tube is placed in this device. Find the acceleration of a particle situated 9.0 cm from the axis of rotation.

Example 7

A radar station is tracking a rocket with a speed of 400 m/s in the direction shown below. The rocket is 3.6 km away at an angle of 20^o with respect to the horizontal. Find:

a) The radial and transverse components of the velocity of the rocket.
b) The transverse angular velocity of the rocket.