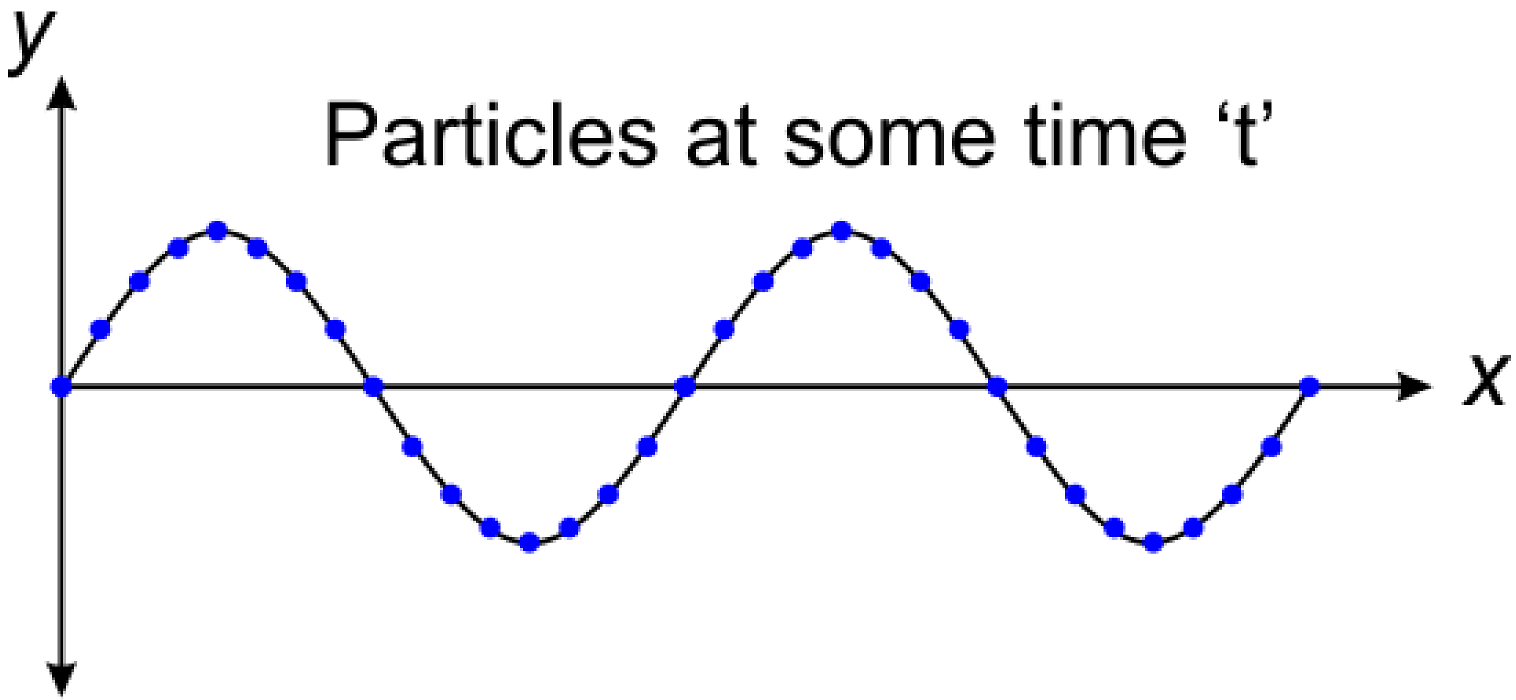
Mathematical Description of Waves

Text: Walker *etal. (2021), Halliday's Fundamentals of Physics – First Australian and New Zealand Edition*
John Wiley & Sons Australia (HW)

Mathematical Description of a Wave

The WAVE FUNCTION, y(*x*,*t*), describes the displacement, *y*, of each particle from its equilibrium position at a distance, *x*, from the starting point as some time, *t*. Think about the particles along a string



It gives a ‘snapshot’ in time of the position of each particle.

Wave Number (k)

The wave number k is defined as $k = \frac{2\pi}{\lambda}$ and has units rad/m.

Substituting $k = \frac{2\pi}{\lambda}$ and $f = \frac{1}{T} = \frac{\omega}{2\pi}$ we get:

$$y(x,t) = A\cos(kx - \omega t + \phi)$$

When our wave/energy is travelling from left to the RIGHT.

Where A : amplitude

k: wave number and

ω: angular frequency

ϕ: initial phase angle in radians

Wave Function for a Travelling Wave

If our energy is travelling from right to the LEFT then our wave function is:

$$y(x,t) = Acos(kx + \omega t + \phi)$$

So we have:

$$y(x,t) = Acos(kx \mp \omega t + \phi)$$

That quantity $kx \mp \omega t$ is called the phase of the wave in radians and is constant.

The wave speed is given by $v = f\lambda$

Example 1

Given at a time $t = 0$, the wave speed of a sinusoidal wave through a string is 12.0 m/s with frequency of 2.00 Hz and amplitude 0.075m. Find the angular frequency, period, wavelength and wave number of the wave. What is the wave function if the initial phase is zero?

Example 2

Write the equation for a progressive wave moving along the negative *x*-axis and having amplitude 0.020 m, frequency 550 Hz, and velocity 330 m/s.

Example 3

When a train of plane waves of wavelength 3.40m traverses a medium, individual particles execute a periodic motion given by the relation, $y = 50.0 \sin \frac{\pi t}{3}$ where *y* is in mm and *t* in seconds.

a. Find the velocity of the wave.

b.Find difference in phase (in degrees), for two positions of the same particle at time intervals 1.00 apart.

c.Find the difference in phase of two particles 2.10m apart?

d.If the displacement of a certain particle at a given time is 30 mm, determine its possible position two seconds later.

Example 3 cont…

When a train of plane waves of wavelength 3.40m traverses a medium, individual particles execute a periodic motion given by the relation, $y = 50.0 \sin \frac{\pi t}{3}$ where *y* is in mm and *t* in seconds.

a. Find the velocity of the wave.

b.Find difference in phase (in degrees), for two positions of the same particle at time intervals 1.00 apart.

c.Find the difference in phase of two particles 2.10m apart?

d.If the displacement of a certain particle at a given time is 30 mm, determine its possible position two seconds later.

Velocity and Acceleration of the PARTICLE at distance '*x*'

If our energy is travelling from left to the RIGHT then our wave function is:

$$y(x,t) = A\cos(kx - \omega t + \phi)$$

Velocity

Note the difference between wave speed and particle speed!

$$v(x,t) = \frac{\partial y(x,t)}{\partial t} = A\omega\sin(kx - \omega t + \phi)$$

Acceleration

$$a(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -A\omega^2\cos(kx - \omega t + \phi) = -\omega^2 y(x,t)$$

These are partial derivatives, treating '*x*' as a constant while differentiating with respect to '*t*'. There will more about this in Mathematics 2.

The Wave Equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

This is important! When it occurs we know a disturbance can propagate as a wave along a medium. This can be any wave, not just a sinusoidal wave. You will see this relationship when you reach quantum.

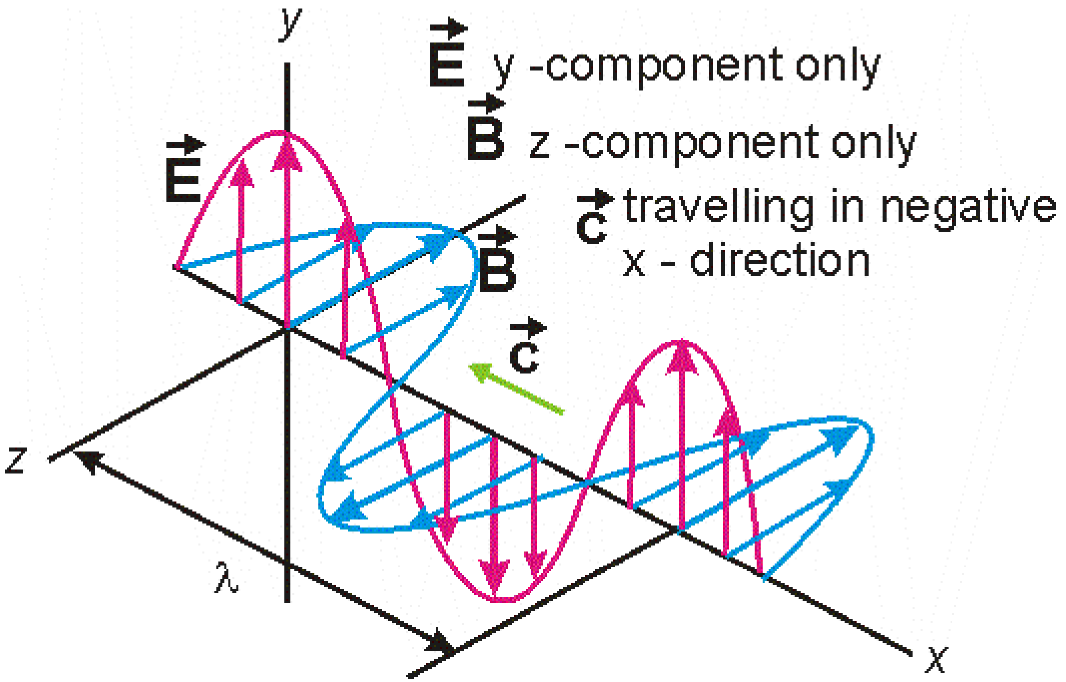
Note: Once again, this mathematics requires partial derivatives which are taught in Mathematics 2. For this subject, I will not examine the higher level calculus presented. Only the resulting algebraic equations.

Electromagnetic Radiation

Text: Walker *etal. (2021), Halliday's Fundamentals of Physics – First Australian and New Zealand Edition*
John Wiley & Sons Australia (HW)

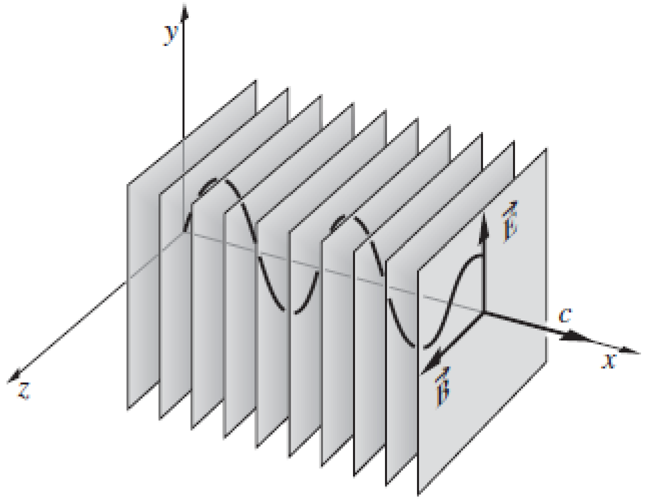
Key Properties of EM Waves

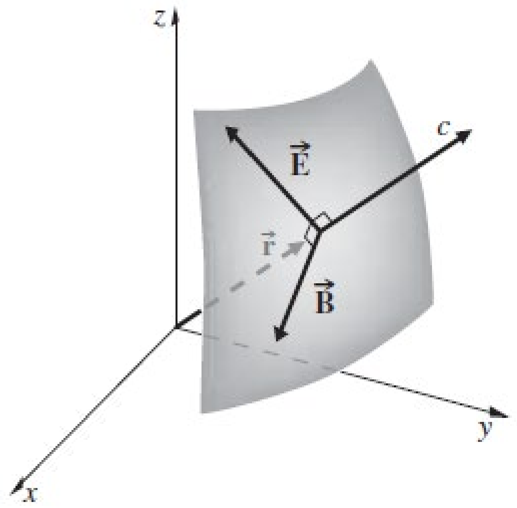
This is a diagram of a ***1-D*** EM wave



1. The wave is transverse.
2. Both $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation, $\vec{c}$, of the wave.
3. $\vec{E}$ and $\vec{B}$ are perpendicular to one another.
4. Direction of $\vec{c}$ is given by $\vec{E} \times \vec{B}$
5. $E_{\max} = cB_{\max}$ where c is the speed of light.
6. The waves travel in vacuum with a definite and unchanging speed.
7. Unlike mechanical waves, EM waves require no medium.
8. What is oscillating is the electric and magnetic field.
9. The fields always vary sinusoidally, have the same frequency, and are in phase with each other.

A slightly more helpful visualisation in 3-D





A plane wave

- The electric field is the same (mag and dir) everywhere on a given plane
- What would the field line representation look like?

A spherical wave

- The electric field is the same (mag and dir relative to **r**) everywhere on a given spherical surface
- What would the field line representation look like?

Time varying $\vec{E}$ and $\vec{B}$

$$E_y(x,t) = E_{\max}\cos(kx-\omega t) \quad \text{and} \quad B_z(x,t) = B_{\max}\cos(kx-\omega t)$$

where E_{max} and B_{max} are the amplitudes of the fields, ω is the angular frequency and $k = 2\pi/\lambda$ is the wave number.

The speed of all EM waves (in vacuum) is given by $c = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}$

where μ_0 is the permeability of free space = $4\pi\times10^{-7}$ N/A^2 and ε_0 is the permittivity of free space = 8.85×10^{-12} C^2 /N.m^2

Also can be shown that: $c = \dfrac{E_{\max}}{B_{\max}}$ (amplitude ratio)

Energy in EM waves

Energy is associated with EM waves: think sunburn, microwave ovens, radio transmitters and lasers which make use of EM energy.

The **rate** of energy transport per unit area is given by the Poynting vector, $\vec{S}$ (after John Poynting, 1852- 1914).

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \quad \mathrm{W/m^2}$$

with the intensity, I, of an EM wave is given by the average value of the Poynting vector,

$$I = S_{ave} = \frac{E_{\max}B_{\max}}{2\mu_0} = \frac{1}{c\mu_0}E^2_{rms} \quad N.B. \quad E_{rms} = \frac{E_{\max}}{\sqrt{2}}$$

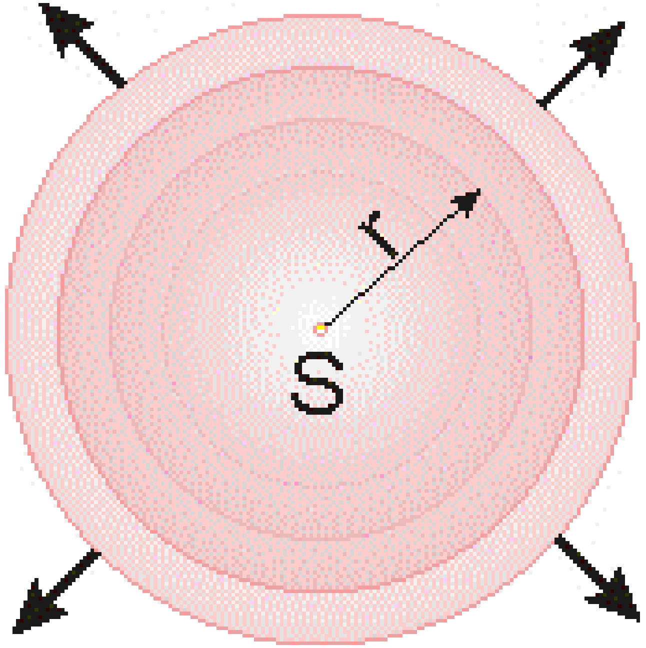
The energy density within an electric field is given by

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{B^2}{2\mu_0}$$

Example 5

Sunlight just outside Earth's atmosphere has an intensity of 1.4 kW/m^2. Assuming the light is a plane-wave, find E_{max} and B_{max}.

Energy radiation from a point source



If we assume a point source, S, emits light isotropically then the light must pass through the sphere of radius, r.

The intensity of the wave at a distance r from the source is:

$$I = \frac{power}{area} = \frac{P_S}{4\pi r^2}$$

Radiation Pressure, p_r

EM waves have linear momentum and can exert a pressure on objects. Extremely small – notice no kickback from camera flash!

During an interval Δt, the object gains an energy ΔU from the radiation. If the object is free to move and the radiation is entirely absorbed by the object, then the momentum change is given by:

$$\Delta p = \frac{\Delta U}{c}$$

If the radiation is entirely reflected back along its original path, the magnitude of the momentum change of the object is twice that given above, or

$$\Delta p = \frac{2\Delta U}{c}$$

Radiation Pressure cont.....

Since $F = \dfrac{\Delta p}{\Delta t}$ and $I = \dfrac{power}{area} = \dfrac{(energy / time)}{area}$, it follows that

$F = \dfrac{IA}{c}$ total absorption

$F = \dfrac{2IA}{c}$ total reflection

Since $pressure = \dfrac{F}{A}$, the radiation pressures, p_r in the two cases become

$p_r = \dfrac{I}{c}$ total absorption

$p_r = \dfrac{2I}{c}$ total reflection

Example 6

The maximum electric field 10 m from an isotropic point source of light is 2.0 V/m. What are

a. The maximum value of the magnetic field?
b. The average intensity of the light there?
c. What is the power of the source?

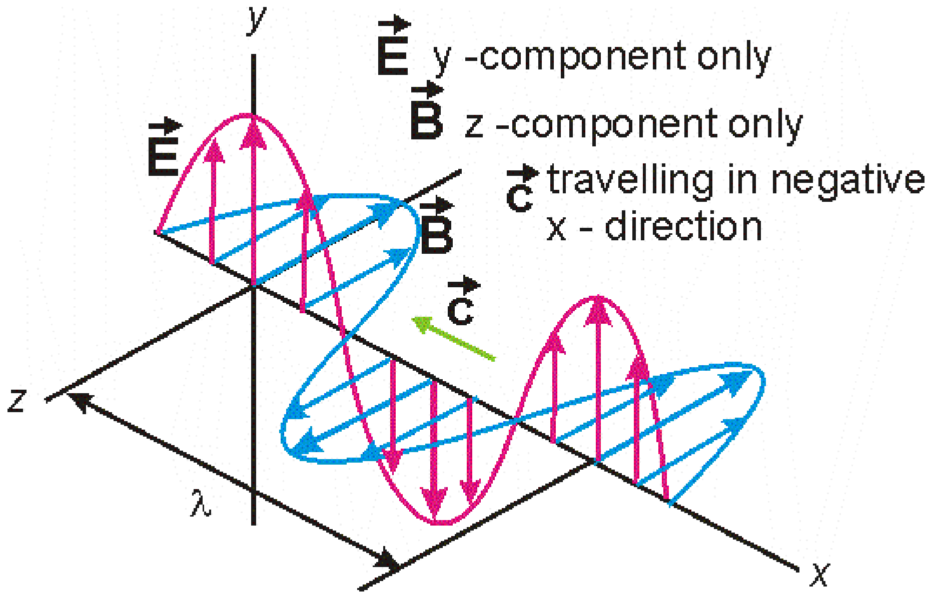
Example 7

High power lasers are used to compress a plasma (a gas of charged particles) by radiation pressure. A laser generating radiation pulses with peak power of 1.5×10^3 MW is focused onto $1.0\ mm^2$ of high-electron density plasma.

Find the pressure exerted on the plasma if the plasma reflects all the light directly back along their paths.

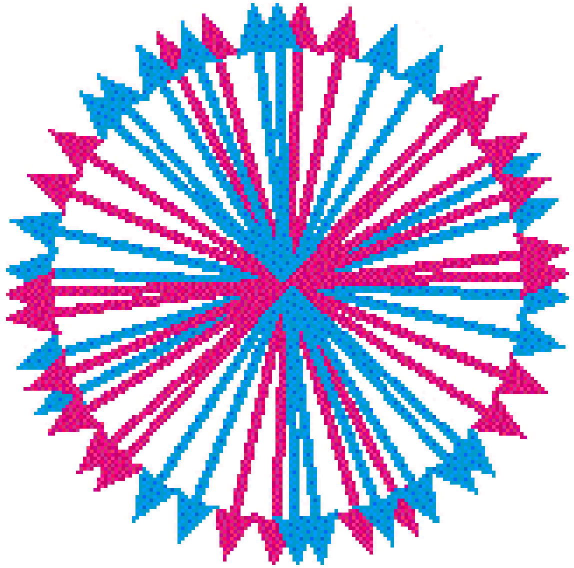
Polarisation

Our little diagram for an EM is a little deceptive as it suggests that EM waves are perfect and the E and B fields are always oriented the same way. This is known as polarised light.



(Randomly polarised over very short periods of time)

If an EM wave is unpolarised, the E & B fields lie in randomly directed (but mutually perpendicular planes). The bottom diagram shows how the E and B fields could point in random directions as the wave travel towards you.



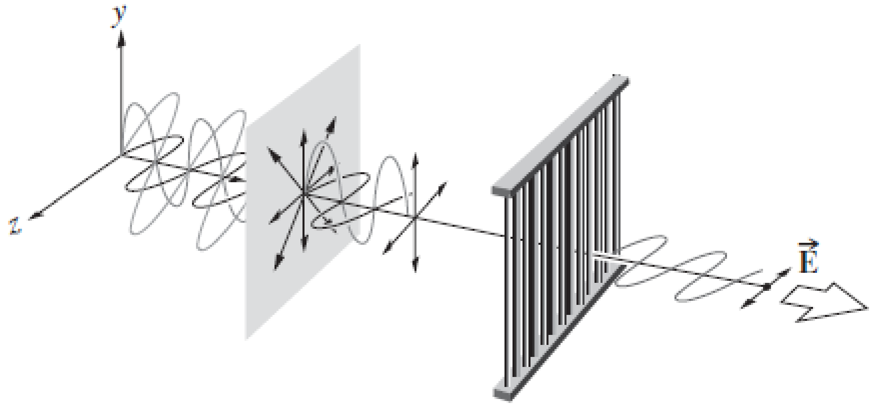
Polarising Materials

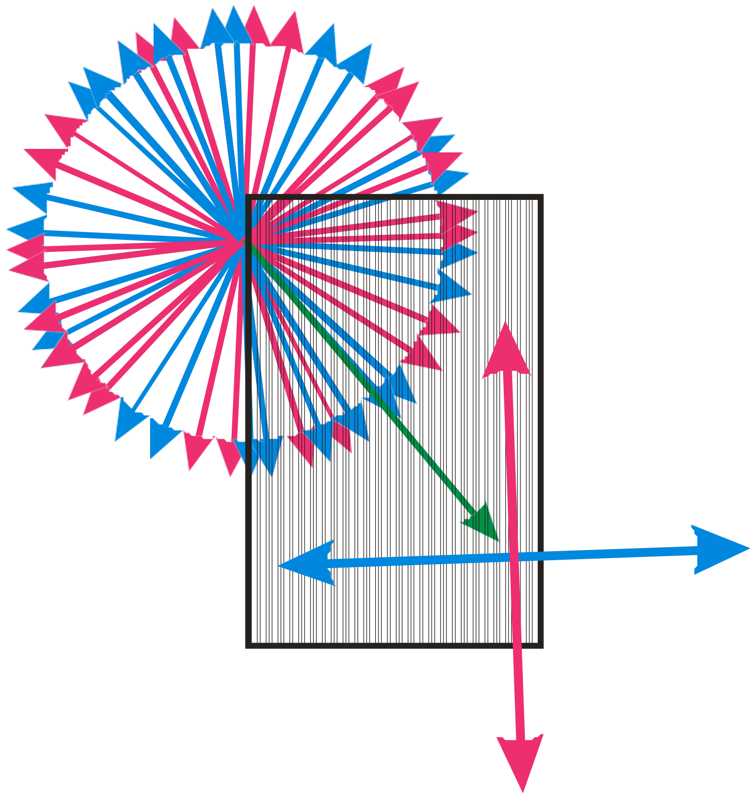
A polariser will allow one E-field orientation through.

If the intensity of the original unpolarised light is I_0, then the intensity of the light emerging through a polariser, I, is half the incident intensity.

Which E field gets through?

$$I = \frac{1}{2} I_0$$



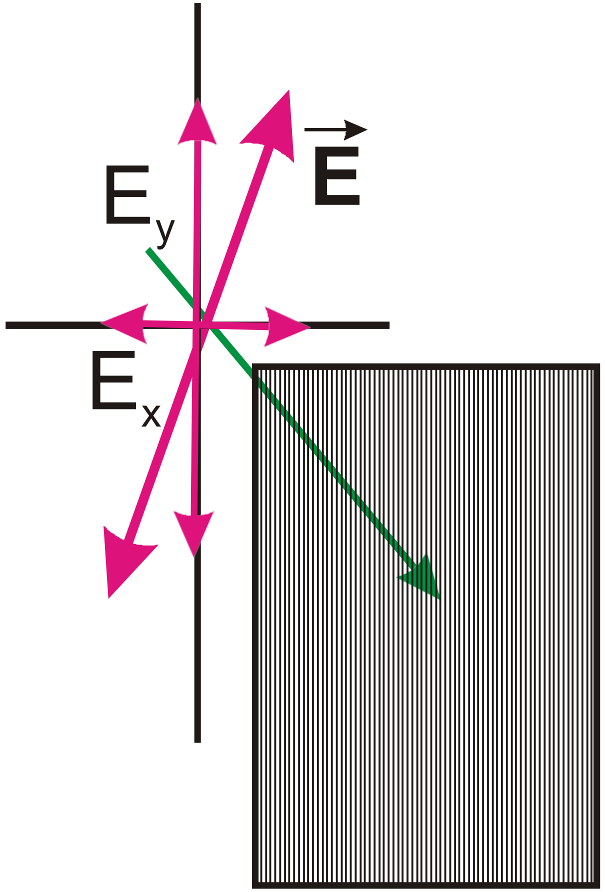


Intensity of the Polarised Wave

If the light incident on a polarising sheet is already polarised with intensity I_0, only those components of the E field that are parallel to the polarising direction are transmitted.

$$E_y = E\cos\theta \qquad \text{and} \qquad I = \frac{E_{rms}^2}{c\mu_0}$$

Thus, $I = I_0 \cos^2\theta$



Crossed Polariser

If unpolarised light is incident on a polariser then this is incident on a second polariser rotated some angle θ to the first the resultant intensity will be

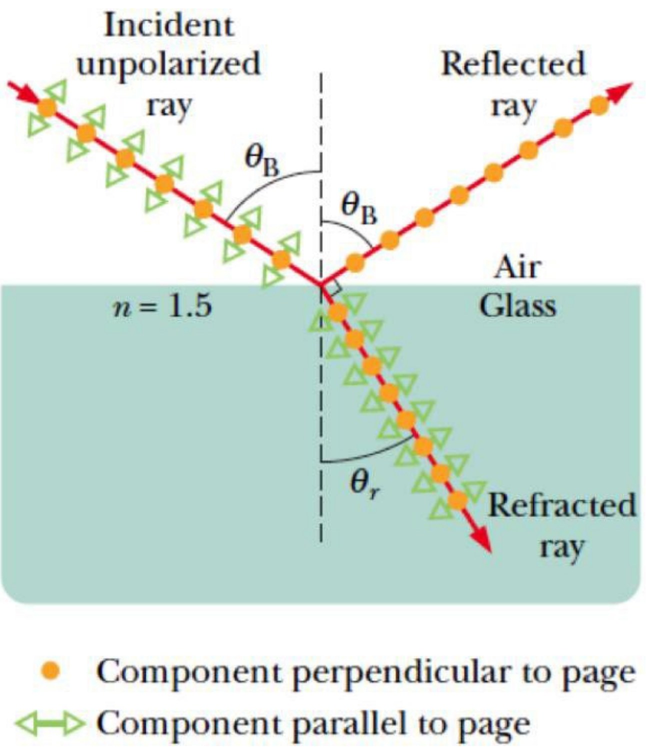
$$I = \frac{1}{2} I_0 \cos^2\theta$$

Example 8

A beam of light with intensity 43 W/m^2 and polarisation parallel to the y-axis is sent into a system of two polarising sheets with polarizing directions at angles of $\theta_1 = 70^o$ and $\theta_2 = 90^o$ to the y – axis. What is the final intensity?

Polarisation by reflection – Brewster's Angle, θ_B

Light reflected from a surface is either fully or partially polarised. For rays incident on a surface at the Brewster angle the reflected and refracted rays are perpendicular to one another. The



HRW. Fig. 33-25, pg. 912.

reflected ray consists only of components perpendicular to the page while the refracted ray consists of the original parallel components and weaker perpendicular components. It can be shown:

$$\theta_B = \tan^{-1}\frac{n_2}{n_1}$$

Brewster's Angle – it comes down to little current oscillations

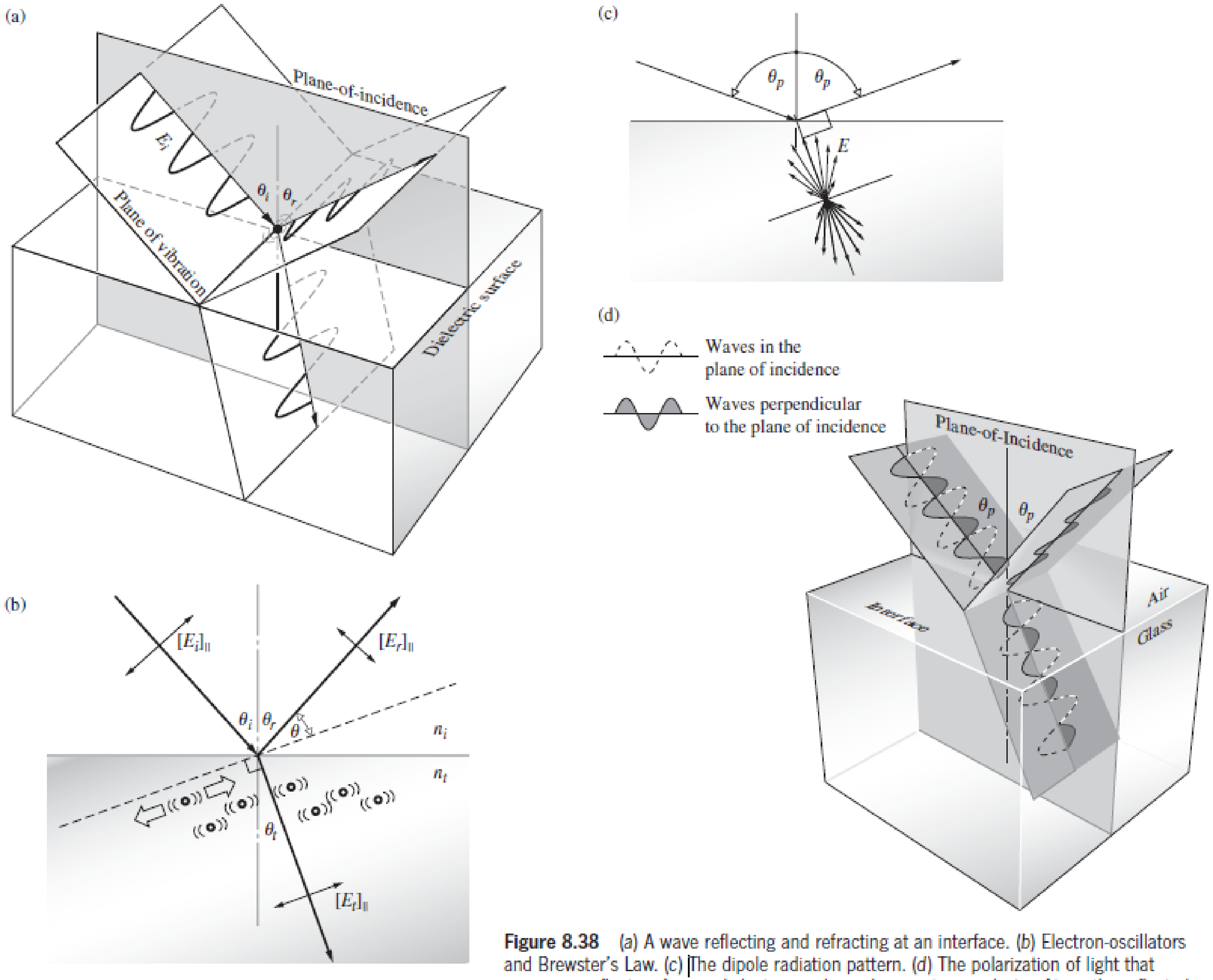


Figure 8.38 (*a*) A wave reflecting and refracting at an interface. (*b*) Electron-oscillators and Brewster's Law. (*c*) The dipole radiation pattern. (*d*) The polarization of light that occurs on reflection from a dielectric, such as glass, water, or plastic. At θ_p, the reflected beam is a $\mathscr{P}$-state perpendicular to the plane-of-incidence. The transmitted beam is strong in $\mathscr{P}$-state light parallel to the plane-of-incidence and weak in $\mathscr{P}$-state light perpendicular to the plane-of-incidence—it's partially polarized.

(Interesting background - Not examinable)

Example 9

Light travelling in water of refractive index 1.33 is incident on a plate of glass with index of refraction 1.53. At what angle of incidence is the reflected light fully polarised?