R-L-C and AC Circuits

These lectures slides are based on ones prepared by Dr. Danica Solina

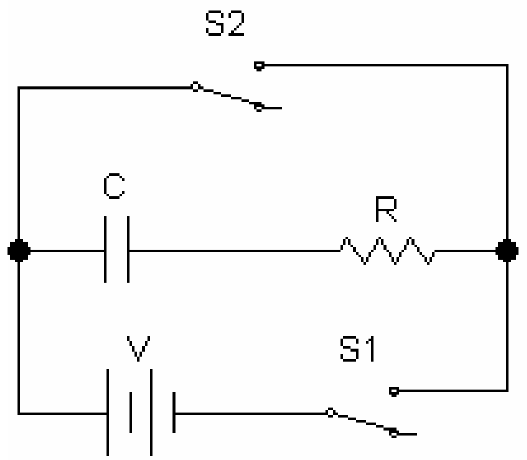
Text: Walker *etal. (2021), Halliday's Fundamentals of Physics – First Australian and New Zealand Edition*
John Wiley & Sons Australia (HW)

Introduction:

Previously we have only dealt with currents that do not vary with time. Now let us extend this to time-varying currents.

- R-C Circuits
- R-L Circuits
- R-L-C Circuits
- AC Circuits

R-C Circuits:

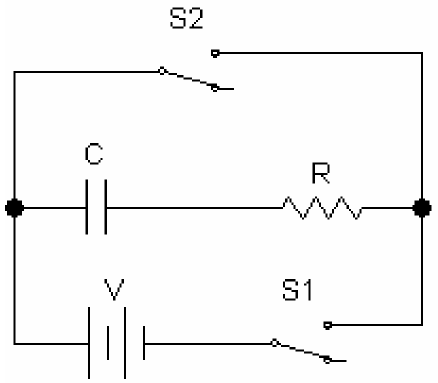


The capacitor is initially uncharged. To charge we close switch S1.

At the instant S1 is closed the potential difference across C is zero and the entire battery voltage appears across R, causing an initial current $I = \frac{V}{R}$ to flow.

- As the capacitor charges, its voltage increases and (by Kirchoff's Voltage Rule) the potential difference across R decreases, corresponding to a decrease in current.
- After a long time, the capacitor becomes fully charged (the entire battery voltage appears across it), there is no potential difference across the resistor, and the current becomes zero.

R-C Circuits – Mathematical model:



Let the charge on C at some time *t* after the switch is closed be *q* and let the circuit current be *i*. Then,

$$V_R = iR \qquad V_C = \frac{q}{C}$$

By Kirchoff's Rule,

$$V = i(t)R + \frac{q(t)}{C}$$

or

$$i(t) = \frac{V}{R} - \frac{q(t)}{RC}$$

The solution of this equation is

$$q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)$$

or

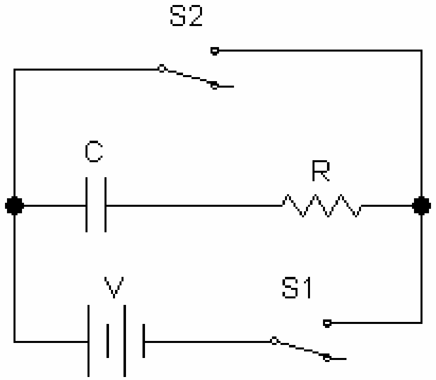
$$i(t) = \frac{dq}{dt} = \frac{V}{R}e^{-\frac{t}{RC}}$$

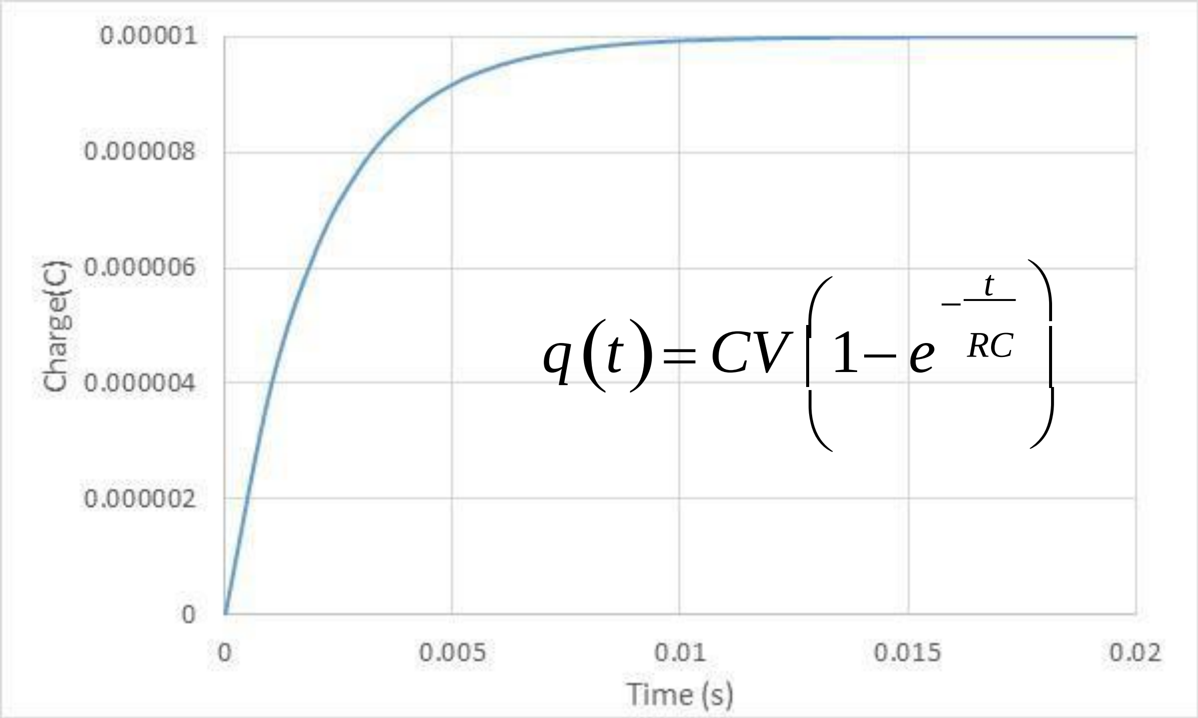
At $\quad t = 0, i = \frac{V}{R} \quad$ and $\quad t \to \infty, i = 0$

t = *RC* is the ***time constant***, τ, of the circuit.

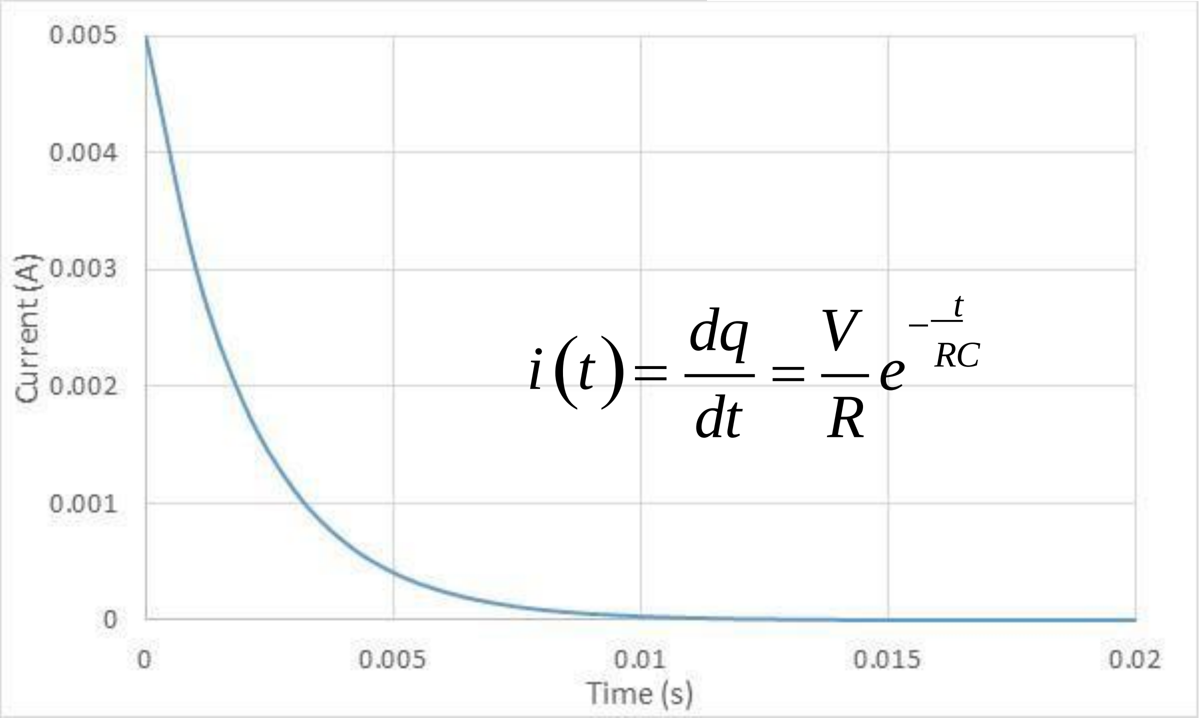
R-C Circuits - Graph:

V = 10 V, R = 2000 Ω, C = 1μF





$$q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)$$



$$i(t) = \frac{dq}{dt} = \frac{V}{R} e^{-\frac{t}{RC}}$$

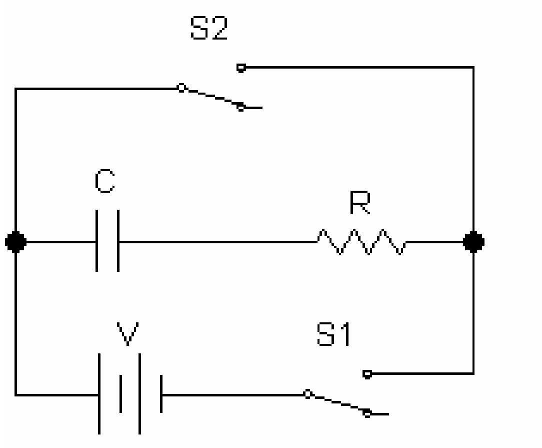
R-C Circuits - Example:

For the circuit below, $R = 1000\ \Omega$, $C = 1\ \mu F$ and $V = 10V$. Find

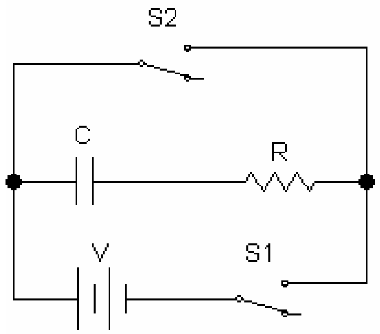
(i) the time constant

(ii) the current after, 4, 5 and 6 time constants once S1 is closed.

(iii) the time at which the capacitor charge is $2\mu C$.



R-C Circuits - Discharging:



Suppose after a very long time (>>τ, when the current is 0), S1 is opened and S2 immediately closed. What happens?

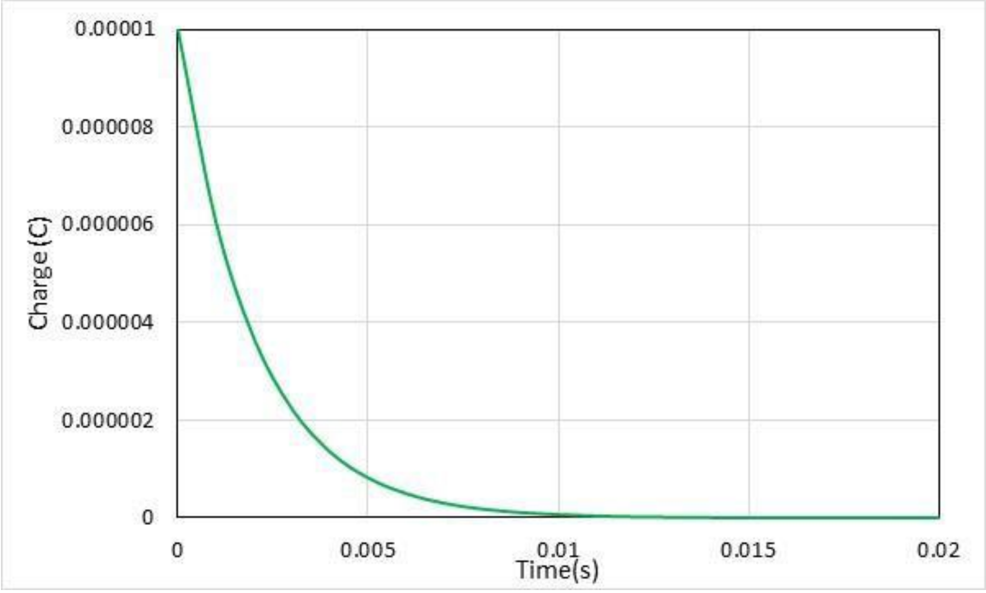
- The capacitor discharges through the resistor and its charge eventually decreases to zero.

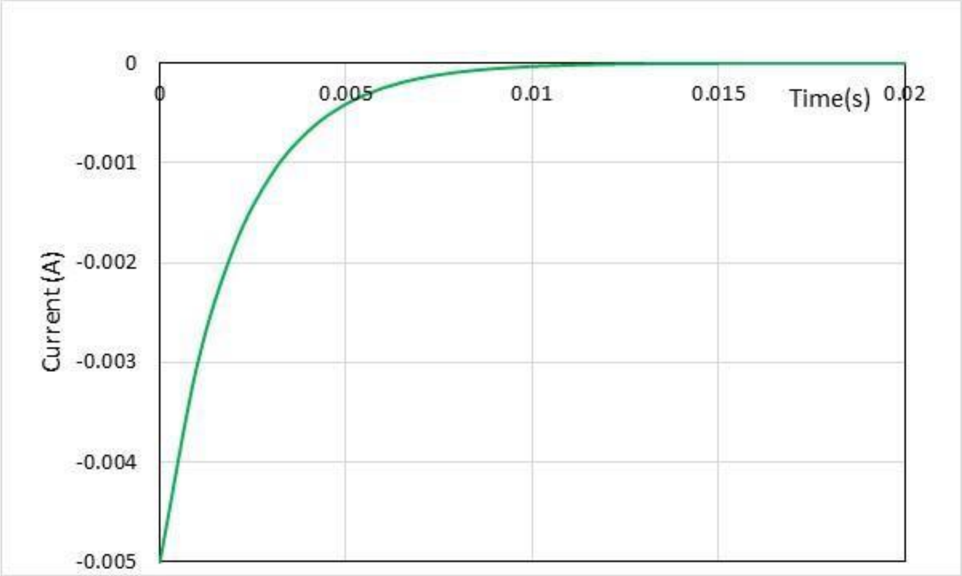
We now have

$$i(t) = -\frac{V}{R} e^{-\frac{t}{RC}}$$

i.e. in the opposite direction to charging case. And

$$q(t) = CVe^{-\frac{t}{RC}}$$

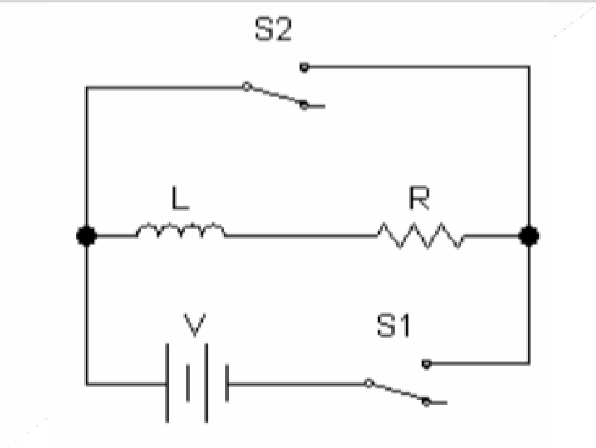




R-L (series) Circuits :

All real inductors can be represented as:

where L = inductance and R = resistance of turns.



Suppose switch S1 is suddenly closed. At this instant,

- there is a max. self-induced emf (why?)
- the current gradually increases at a rate depending on the values of L and R.

Let the current at time *t* after closing the switch be *i* and let its rate of increase at that time be di/dt.

The potential difference across the inductor is then $V_L = L\frac{di}{dt}$ and across the resistor it is $V_R = iR$.

By Kirchoff's rules: $V = V_L(t) + V_R(t)$

$$= L\frac{di}{dt} + i(t)R$$

R-L Circuits - Solution:

The solution of this equation is

$$i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

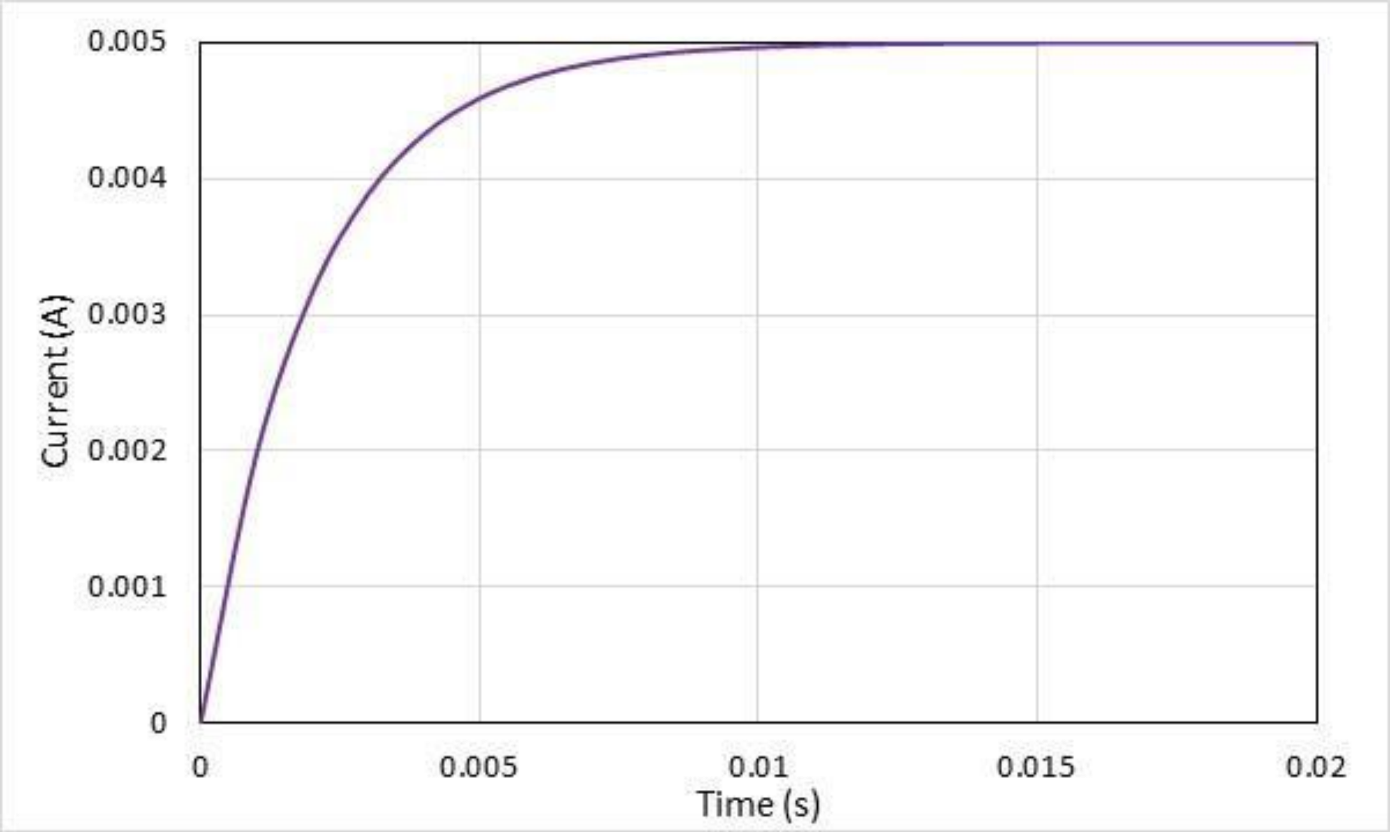
At

$$t = 0, i = 0 \quad \text{and} \quad t \to \infty$$

$$t = \frac{L}{R}, i = \frac{V}{R}\left(1 - \frac{1}{e}\right) = 0.63\frac{V}{R}$$

$t = \frac{L}{R}$ is called the ***time constant***, τ, of the circuit.

V = 10 V, R = 2000 Ω, L = 4.0 H



R-L Circuits - Example:

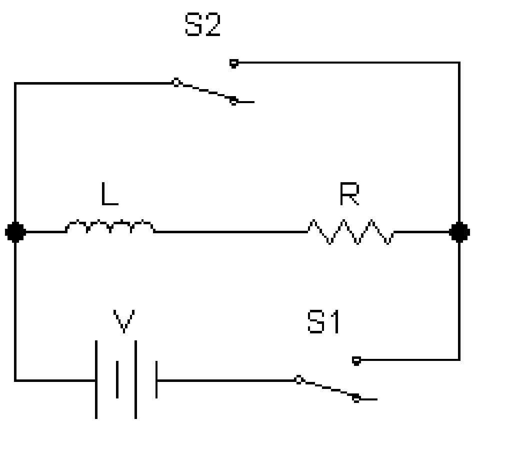
For the circuit below, R = 10 Ω, L= 50 mH and V = 10V. Find

(i) the time constant

(ii) the maximum current flow

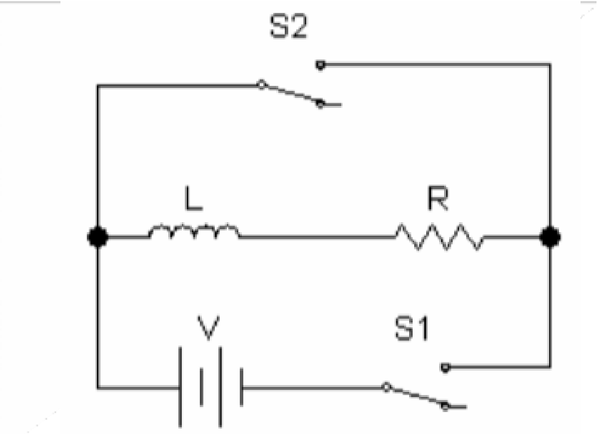
once S1 is closed.

How long does it take the current to reach 50% of its max. value?



R-L Circuits – Current decay:

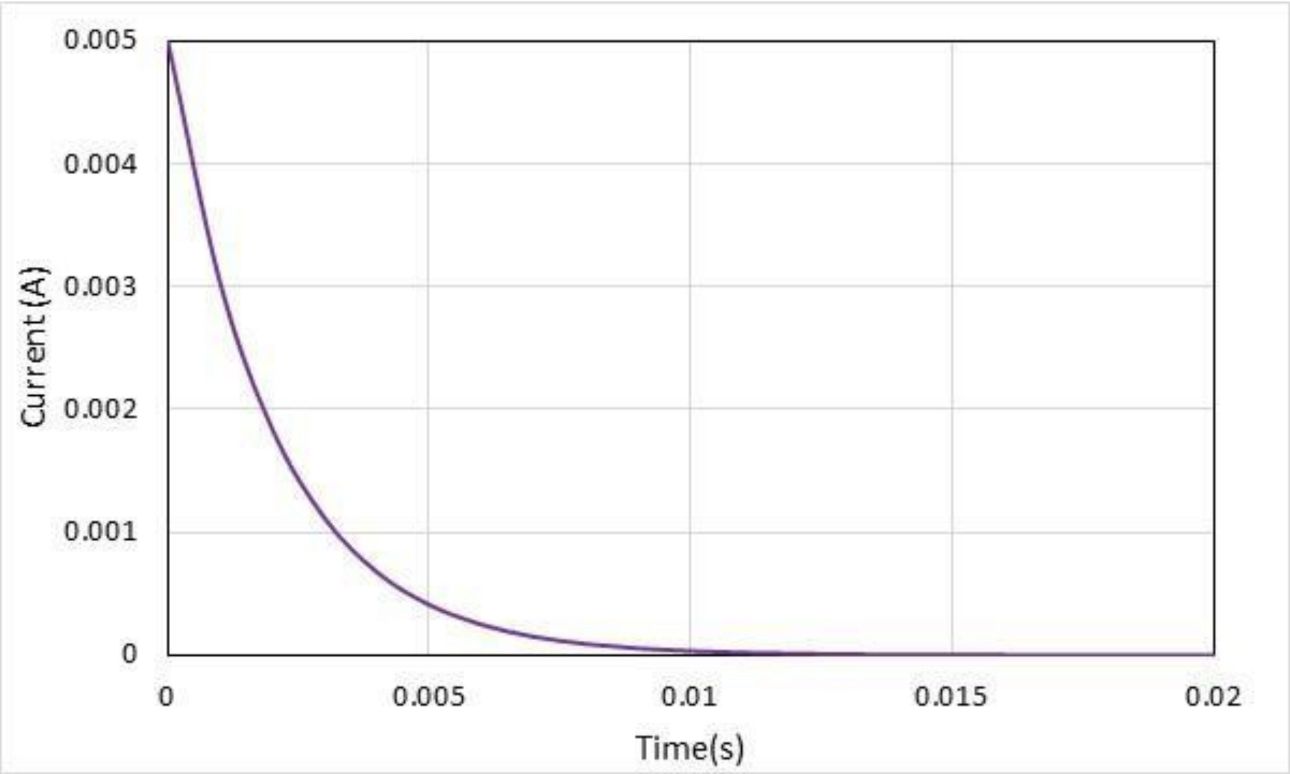
Suppose after a very long time (>>τ, when the current is I=V/R), S1 is opened and S2 immediately closed. What happens?



- The current now "decays" according to the equation

$$i(t) = -\frac{V}{R} e^{-\frac{R}{L}t}$$

The time constant, , is now the time taken for the current to decrease (i.e. exponentially decay) to 1/e (=0.368) of its starting value, I_0.



L-C (series) circuit:

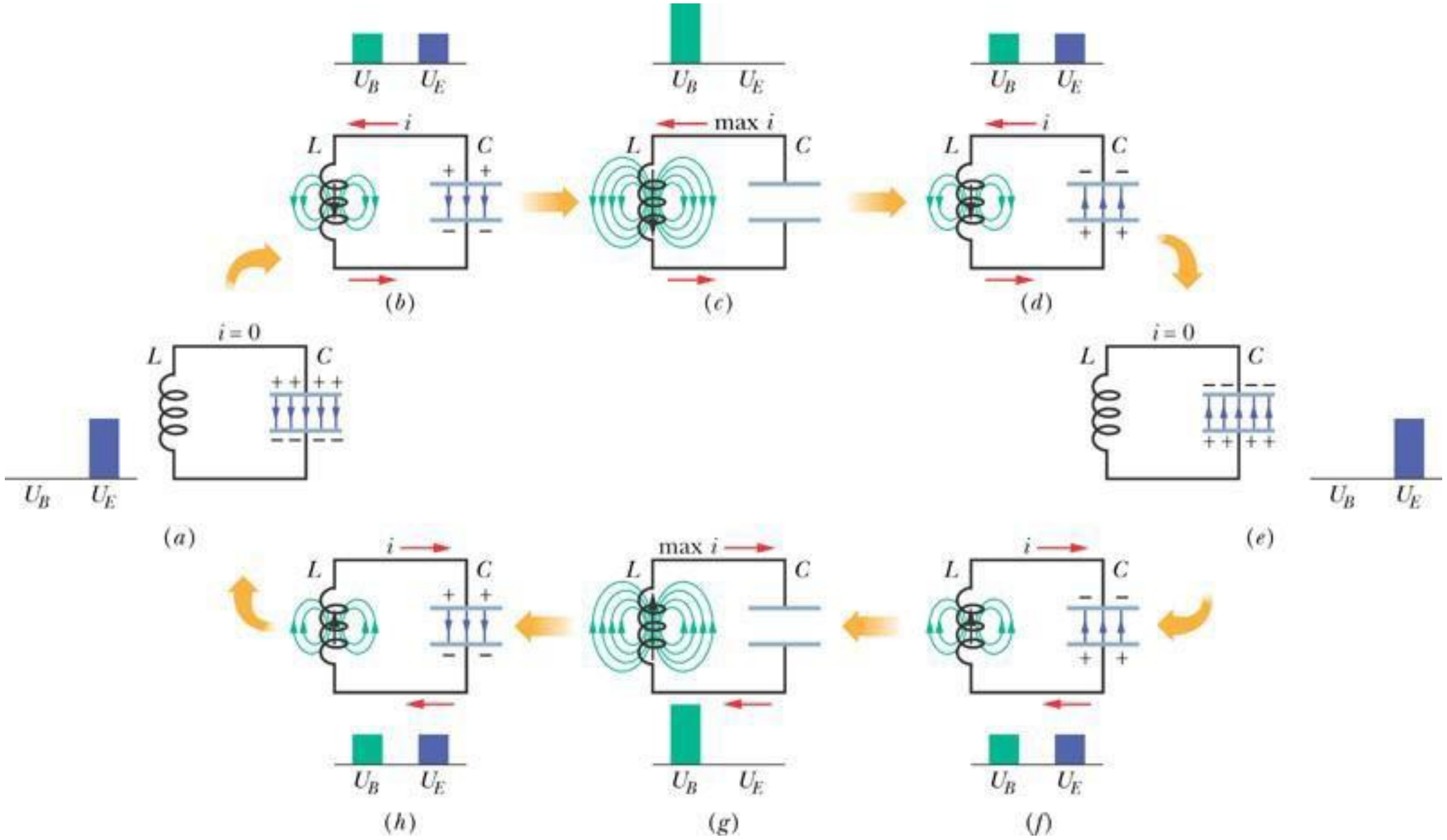
Suppose L is ideal (R=0) and C is initially charged.

This process keeps repeating (how long?), with charges going back and forth.

It is called electrical oscillation and forms the basis of many electronic tuned circuits.

The frequency of oscillation can be shown to be

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$



, Diagram from HW

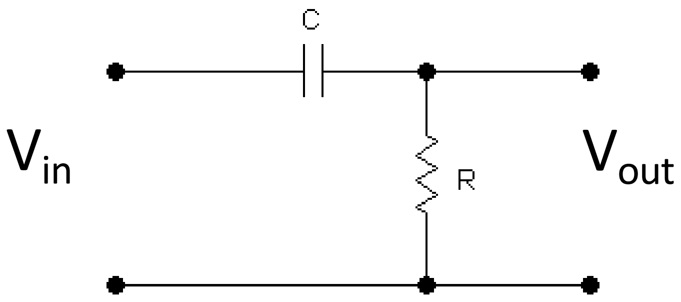
Differentiating circuits:

A differentiating circuit is when the output voltage of a circuit is proportional to the derivative of the input voltage i.e.

$$V_{out} = RC\frac{dV_{in}}{dt}$$

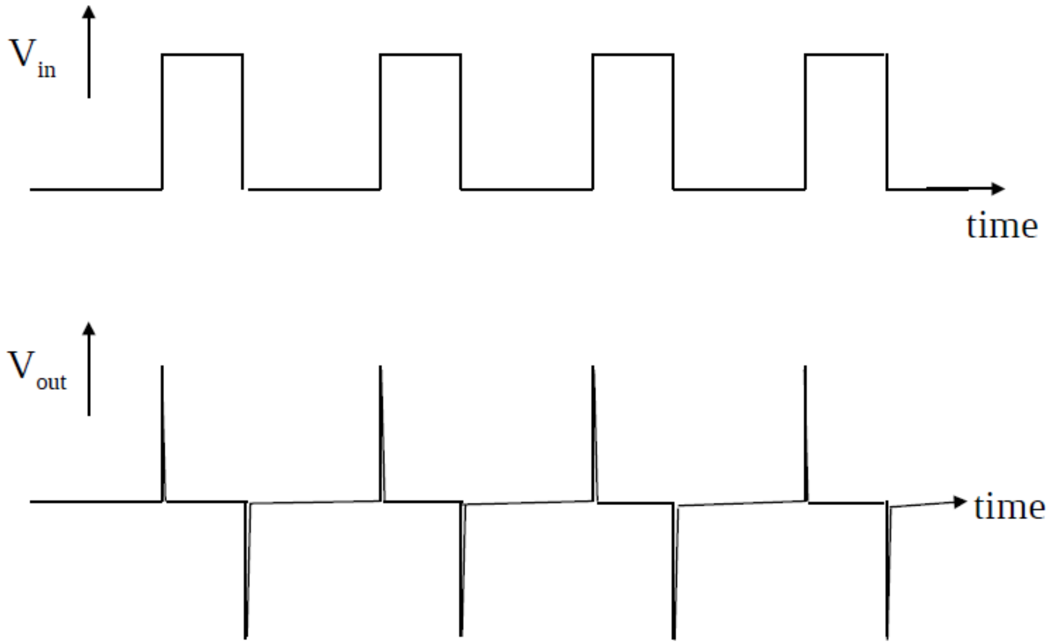
These circuits have important applications in control-circuits for electronics e.g transmitters and receivers for TV.

The RC circuit when the output is taken across the resistor is a differentiating circuit.



Important considerations:

1. Time constant must be much smaller than the period of the input voltage
2. X_C should be > 10R at operating frequency



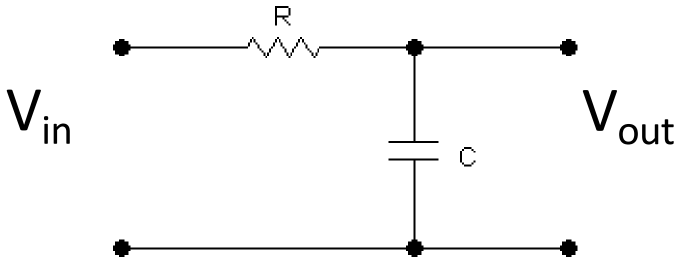
Integrating circuits:

An integrating circuit is when the output voltage of a circuit is proportional to the integral of the input voltage i.e.

$$V_{out} = \frac{1}{RC}\int V_{in}\,dt$$

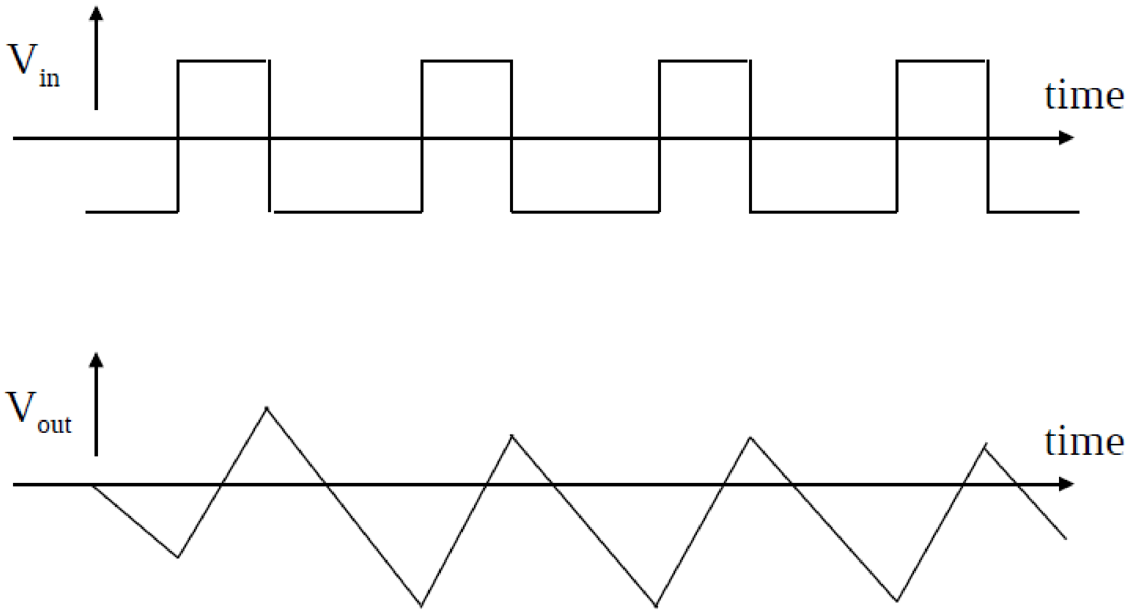
These circuits have important applications in control-circuits for electronics e.g transmitters and receivers for TV.

The RC circuit when the output is taken across the capacitor is an integrating circuit.



Important considerations:

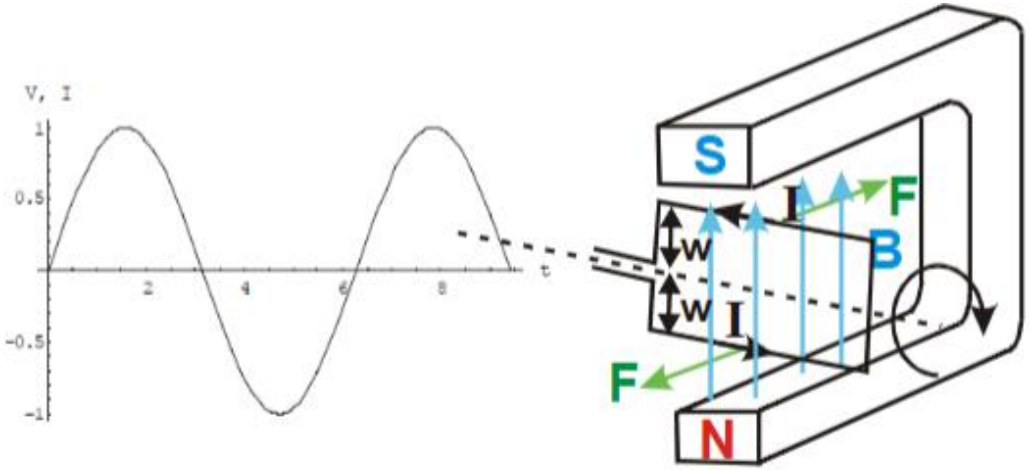
1. Time constant must be much larger than the period of the input voltage.
2. R should be > X_C at operating frequency



For more info: https://electronicspost.com/differentiating-circuit-and-integrating-circuit/

Alternating Current (AC) :

If we consider the coil of wire rotating in the field we can see as the coil rotates the area changes and so too does the current induced. Once the coil has rotated 180 degrees not only does the current reduce but also changes direction in the coil.



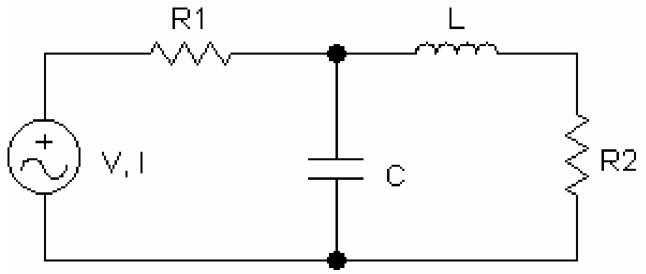
The input voltage oscillates back and forth periodically with the shape of a sine wave.

Current and voltage can be described by:

$$i = I_p \sin \omega t \; or \quad v = V_p \sin(\omega t + \phi)$$

where I_p = Peak Current and V_p = Peak Voltage and $\omega = 2\pi f$ and $f = \frac{1}{T}$. ϕ is included as current may not be in phase with emf.

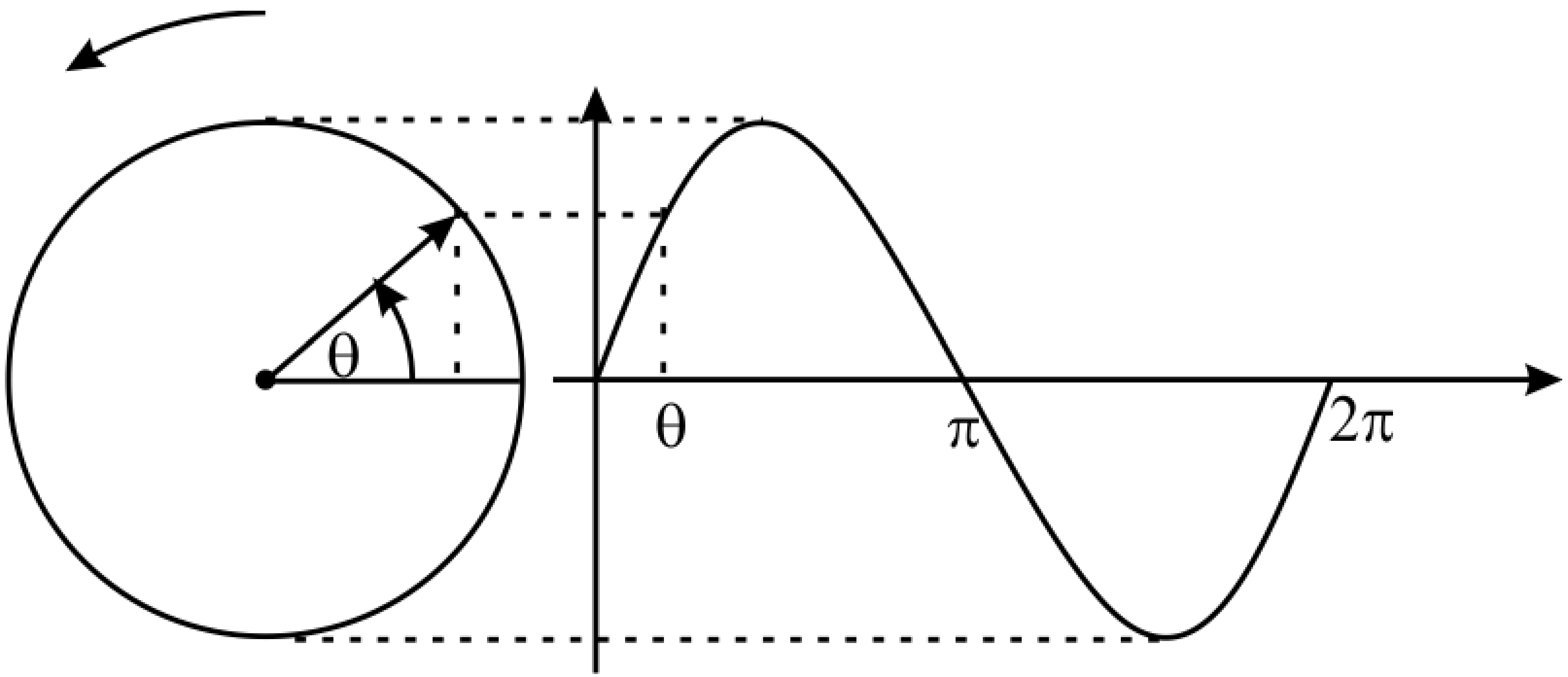
The use of lowercase *v* and *i* refers to the instantaneous voltage and current respectively. Uppercase refers to the maximum value of V or I which we refer to as the ***Voltage*** or ***Current amplitude***.



Phasor:

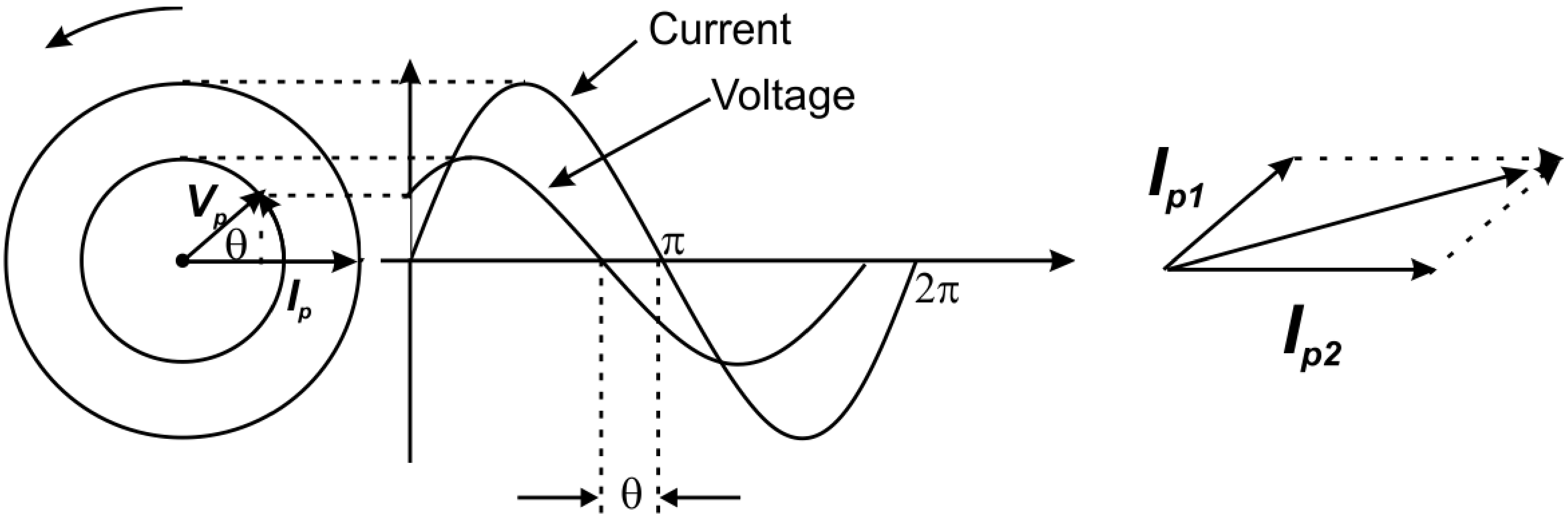
Sine Curve

A sine curve can be constructed by rotating the radius of a circle.



Phasor:

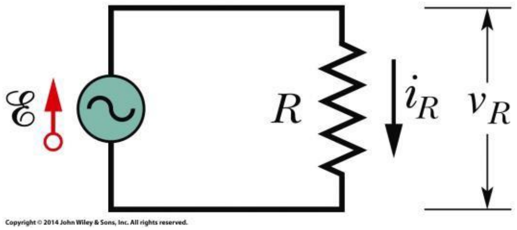
Representing alternating current and voltage (later impedance) using a phasor



Alternating Current (AC) – Purely Resistive load :

Resistors:

$$V = IR$$



Since

$$i = I_p \sin(\omega t - \phi)$$

Voltage is

$$v = I_p R \sin(\omega t - \phi)$$

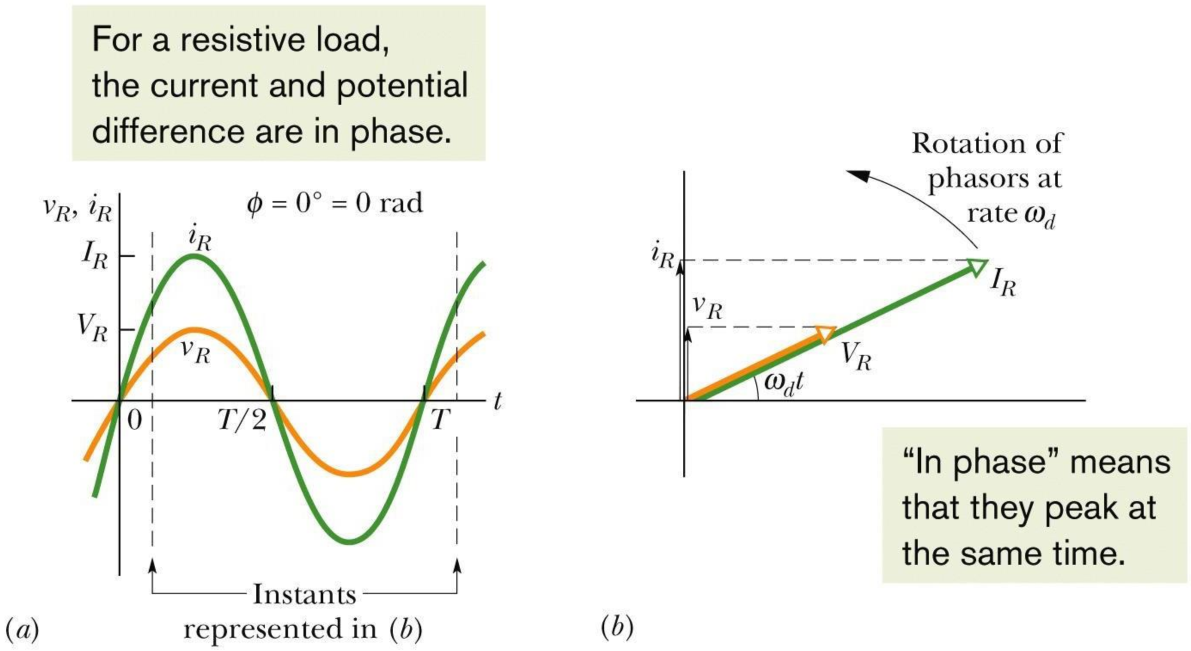
For a purely resistive load ϕ =0 so

$$i = I_p \sin(\omega t)$$

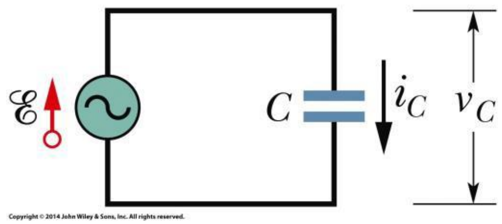
$$v = I_p R \sin(\omega t)$$

Since ϕ =0 therefore the current and voltage are said to be in ***phase***.

The voltage and current can be represented as phasors (rotating vector diagrams). The phasor rotates anticlockwise. Note, a phasor is a geometric entity that help us analyze quantities that vary sinusoidally with time.



Alternating Current (AC) – Capacitor:



$$C = \frac{q}{V} \quad \Rightarrow \quad V$$

Differentiate :

$$\frac{dV_C}{dt} = \frac{1}{C}\frac{dq}{dt} = \frac{1}{C} I(t)$$

Integrate:

$$V_C(t) = \int \frac{1}{C} I(t) dt$$

$$= \int \frac{1}{C} I_p \sin(\omega t) dt$$

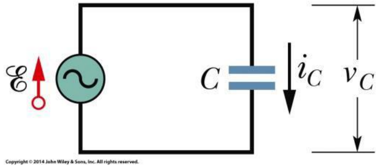
$$= -\frac{I_p}{\omega C}\cos(\omega t)$$

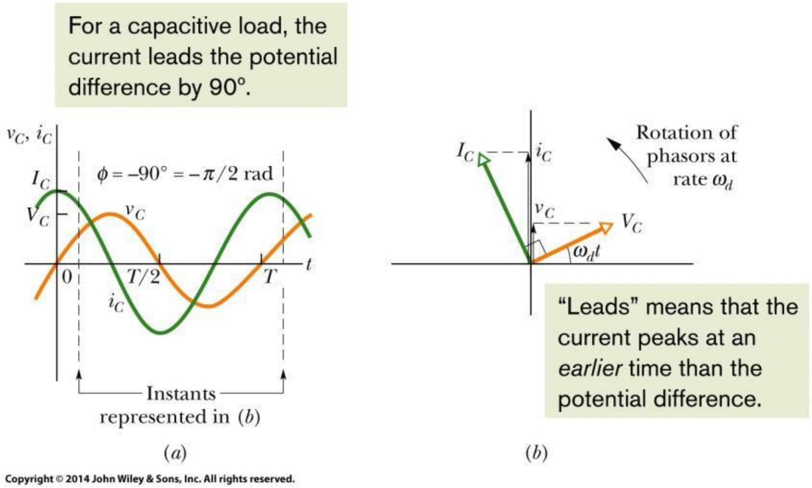
$$= -V_p \cos(\omega t) \quad \text{where} \quad V_p = \frac{I_p}{\omega C}$$

$$= V_p \sin\left(\omega t - \frac{\pi}{2}\right)$$

– Diagram HW Ch. 31

Alternating Current (AC) – Capacitive Reactance:





V is 90° behind *I* (or *V* lags by 90°).

Considering the term $V_p = \frac{I_p}{\omega C}$, we can write

$$V_p = I_p \frac{1}{\omega C} = I_p X_C$$

This is similar to Ohm's Law.

Here

$$X_C = \frac{1}{\omega C}$$

is the capacitive reactance in Ohms (Ω).

As $\omega \rightarrow 0$, $X_C \rightarrow \infty$ (blocks dc)

As $\omega \rightarrow \infty$, $X_C \rightarrow 0$ (short circuit for ac)

Capacitors pass high frequency current and block low frequency i.e. a high pass filter.

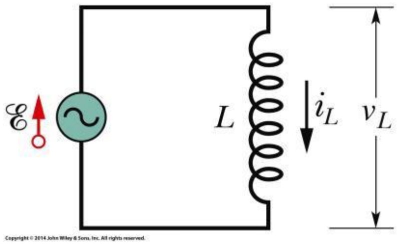
– Diagram HW Ch. 31

Alternating Current (AC) – inductor and inductive Reactance:

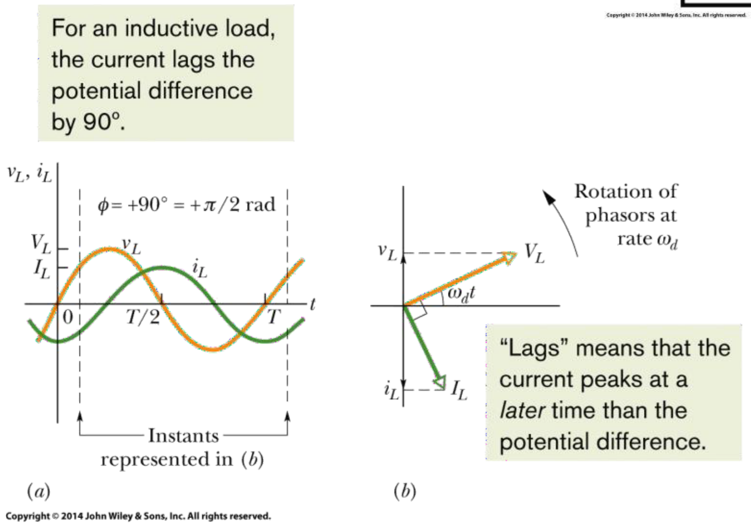
$$V = L\frac{di}{dt} = L\frac{d}{dt}\left(I_p \sin \omega t\right)$$

$$= LI_p\omega \cos \omega t \quad \text{where} \quad V_p = I_p\omega L$$

$$= V_p \sin\left(\omega t + \frac{\pi}{2}\right)$$



In an inductor V leads I by 90°.



Considering the term $V_p = I_p\omega L$, we can write

$$V_p = I_p X_L$$

which is similar to Ohm's Law.

Here

$$X_L = \omega L$$

is the inductive reactance in Ohms (Ω).

As $\omega \to 0$, $X_L \to 0$ (short circuit for dc)

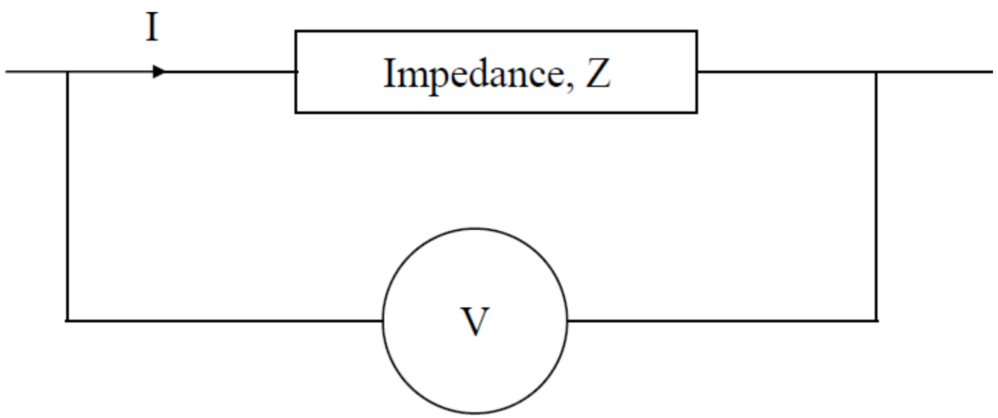
As $\omega \to \infty$, $X_L \to \infty$ (blocks ac)

Inductors pass low frequency current and block high frequency i.e. a low pass filter. Used in power supplies and radio interference filters to block high frequencies and let dc to pass.

– Diagram HW Ch. 31

General Case:

- All three components (L, C, and R) in one circuit.
- Their combined opposition to current flow is **impedance**, *Z*.

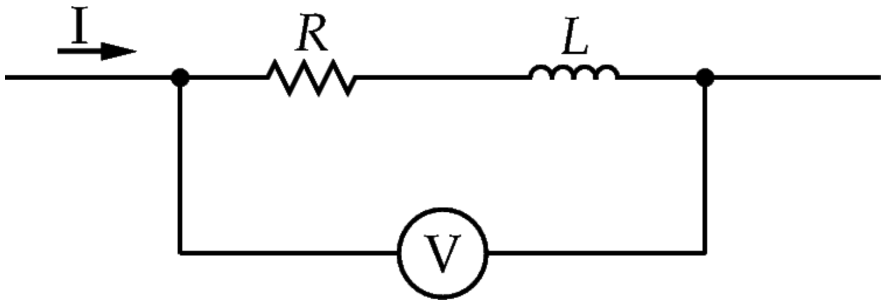


$$i = I_p \sin \omega t \qquad v = V_p \sin(\omega t + \phi)$$

"AC Ohm's Law":

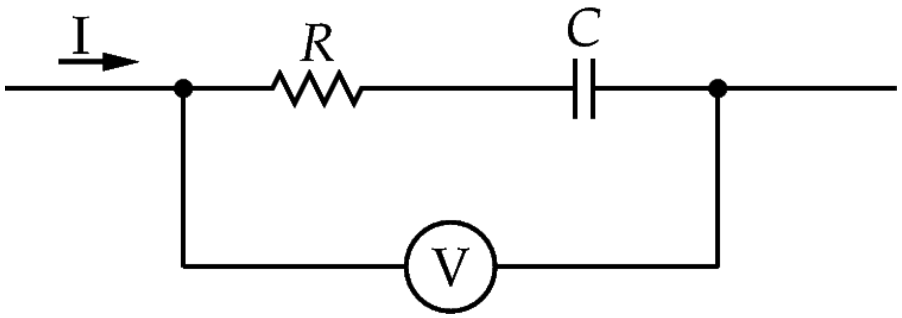
$$V_p = I_p Z \quad \rightarrow Z = \frac{V_p}{I_p}$$

Example:



Find the impedance for the R-L combination illustrated when the current is AC i.e. $i = I_p \sin \omega t$

Example:

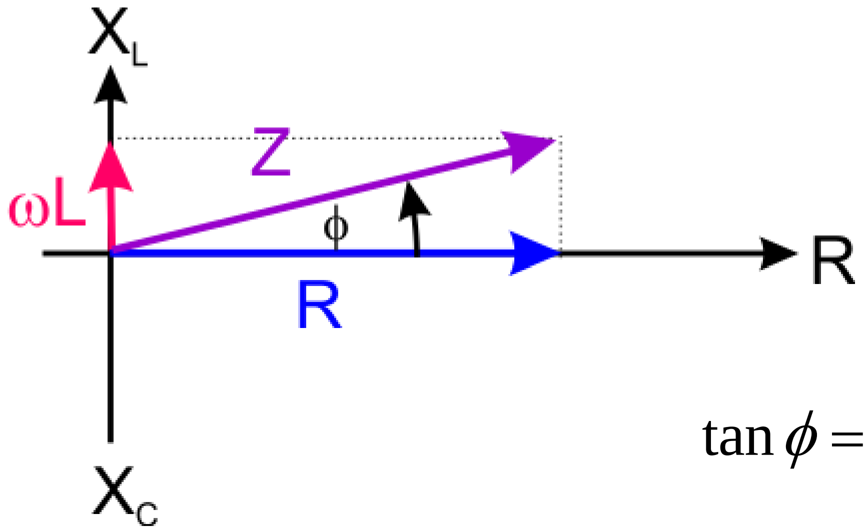


Find the impedance for the R-C combination illustrated when the current is AC i.e. $i = I_p \sin \omega t$

Impedance and Vector approach:

We can think of impedance as a vector made up of components owed to R, X_C, and X_L.

Case R and L



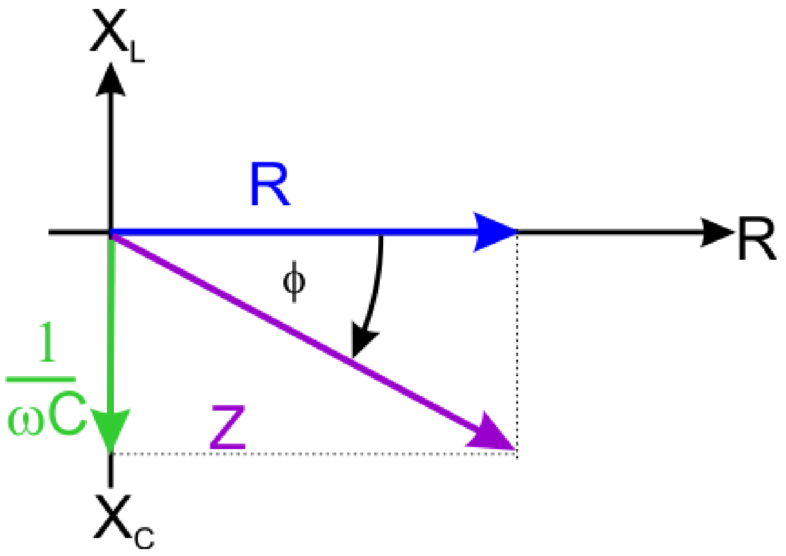
$$\tan\phi = \frac{\omega L}{R}$$

$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

– Diagram HW Ch. 31

Impedance and Vector approach:

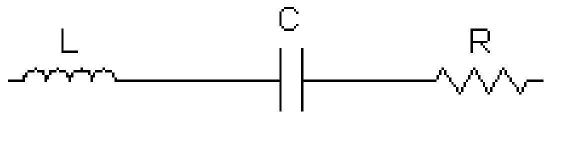
Case R and C

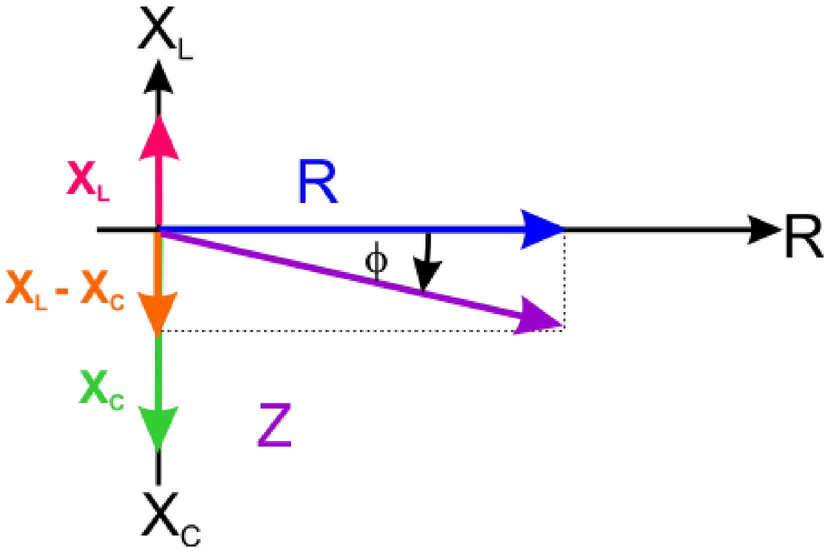


$$\tan\phi = -\frac{1}{\omega CR}$$

$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

Case L, C and R





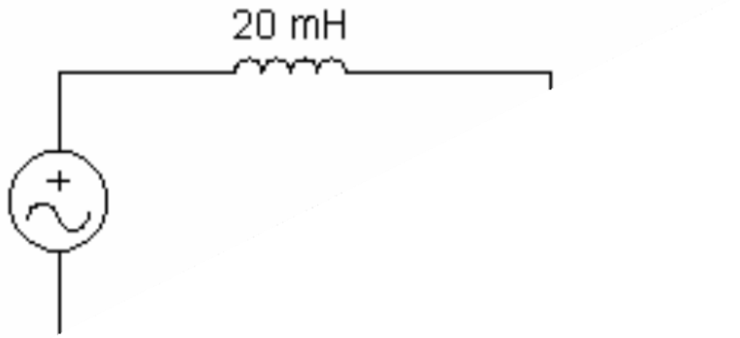
$$\tan\phi = \frac{\omega L - \dfrac{1}{\omega C}}{R} = \frac{X_L - X_C}{R}$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

– Diagram HW Ch. 31

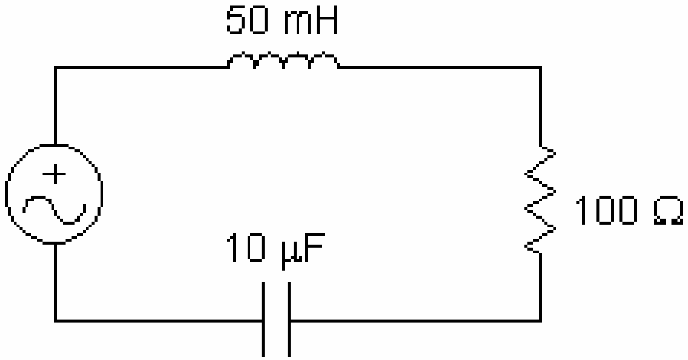
Example:

The circuit shown has a source given by $V = 10 \sin 1000t$.
What current flows in the circuit?



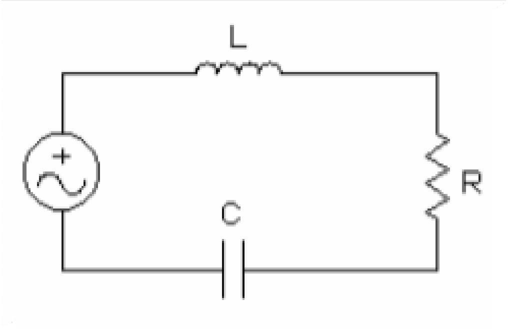
Example:

The circuit shown has a source given by V = 10 sin 1000t.
Find the circuit current and V_R, V_L, and V_C ?



Resonance in series ac circuits:

Consider the circuit where $v = V_p \sin \omega t$



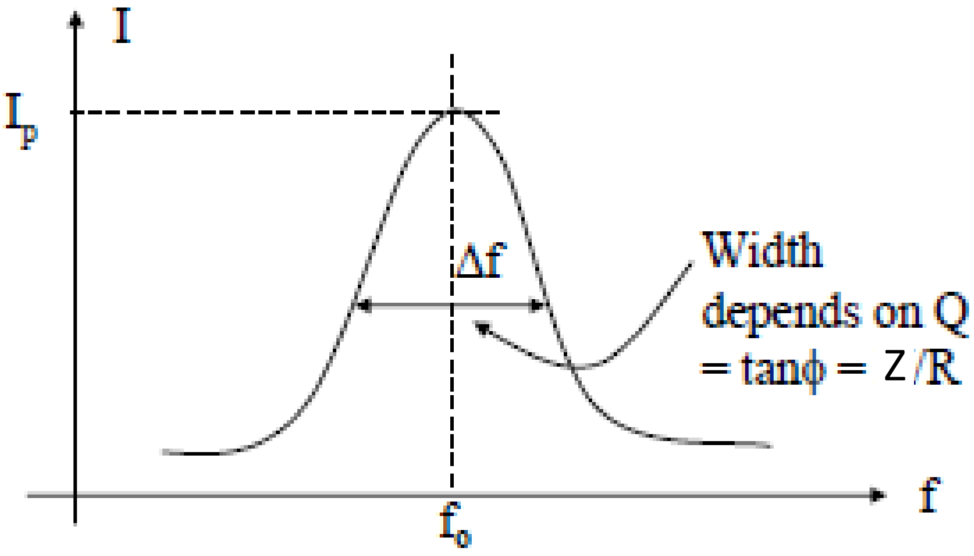
$$I_p = \frac{V_p}{Z} = \frac{V_p}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

This is a max. when $\omega L = \frac{1}{\omega C}$ (i.e. $X_L = X_C$).

A circuit like this with

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{or} \quad f_0 = \frac{1}{2\pi\sqrt{LC}}$$

is said to be a resonant circuit, and the frequency f_0 is the resonant frequency. At this frequency, the impedance Z is purely resistive, and $\phi = 0$ (meaning that V and I are in phase).



– Diagram HW Ch. 31

Example:

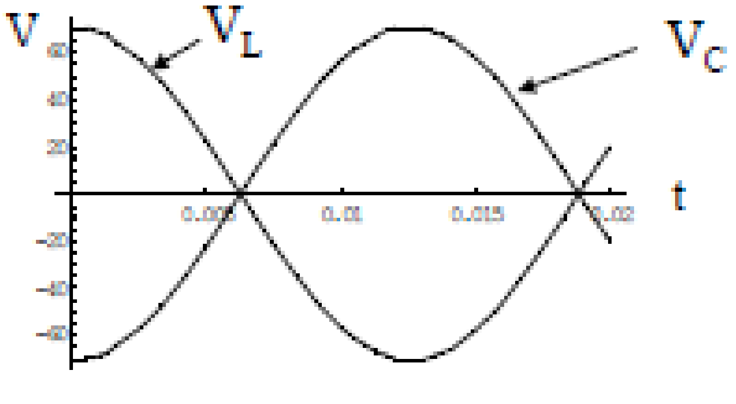
(a)At what frequency would the reactances of an 8μF capacitor, C, and 2 H inductor, L, be equal?

If L and C are in series and carry a peak current of 0.141 mA at the above frequency, find

(b) the voltages across L and C

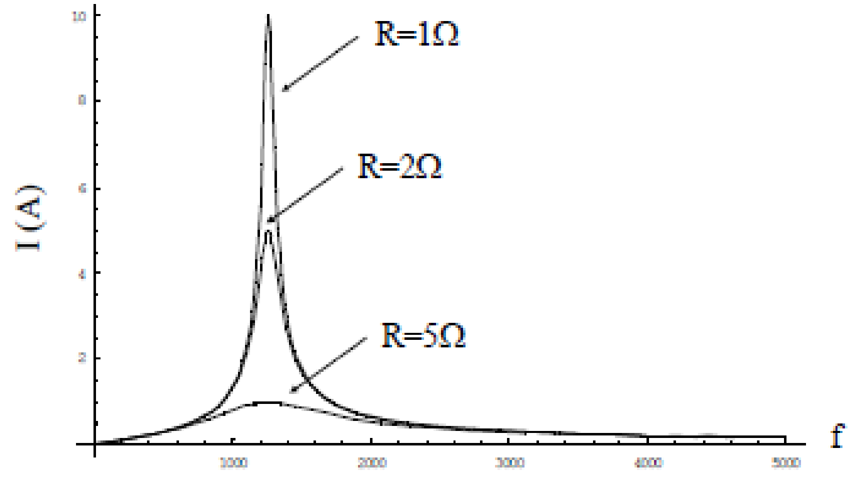
(c)the phase angle between V_L and V_C.

(d)the voltage across the series combination.



Example:

A series LCR circuit with L=2mH and C=8μF is connected across a 10V(peak) ac supply of variable frequency. Calculate and plot the peak current as a function of frequency for R=1Ω, 2Ω and 5Ω.



Root mean square values (RMS)

To relate ac current to an equivalent dc current, we compare their joule heating effects:

Instantaneous power dissipation $= I^2\ R$

Energy dissipated in time $dt\ = dE = I^2\ Rdt$

In one cycle of ac, $E = \int_0^T I^2 Rdt$

Hence average power dissipation,

$$P_{av} = \frac{E}{T} = \frac{1}{T}\int_0^T I^2\, R\, dt$$

e.g. Suppose $i = I_p \sin \omega t$ then

$$P_{av} = \frac{1}{T}\int_0^T I_p^2\, R\sin^2 \omega t\, dt = \frac{1}{2} I_p^2 R$$

The equivalent dc current ($= I_{RMS}$"root mean square") is

$$P = I_{RMS}^2 R = \frac{1}{2} I_p^2 R$$

Thus,

$$I_{RMS} = \frac{I_p}{\sqrt{2}}$$

similarly we can find

$$V_{RMS} = \frac{V_p}{\sqrt{2}}$$

For mains supply (Australia)

$V_{RMS} = 240V$, with $V_p = 240V\sqrt{2} = 340V$

In general

$$V_{RMS} = \frac{1}{T}\int_0^T V^2 dt \quad or \quad I_{RMS} = \frac{1}{T}\int_0^T I^2 dt$$

– Diagram HW Ch. 31

Power Factor and Average Power

For a general LCR circuit with ac input,

$$I(t) = I_p \sin \omega t$$

$$V(t) = V_p \sin(\omega t + \phi)$$

$$P_{inst.} = VI$$

$$= V_p I_p \sin \omega t \sin(\omega t + \phi)$$

$$= V_p I_p \sin \omega t (\sin(\omega t)\cos(\phi) + \cos(\omega t)\sin(\phi))$$

$$= V_p I_p (\sin^2(\omega t)\cos(\phi) + \sin(\omega t)\cos(\omega t)\sin(\phi))$$

This is very messy! Look at average power instead.

$$P_{av} = \frac{1}{2} \int_0^T I_p V_p \sin \omega t \sin(\omega t + \phi)\, dt$$

$$= V_{RMS} I_{RMS} \cos \phi$$

$\cos \phi$ is the power factor.

At resonance, ϕ=0, ∴ $P_{av} =$ maximum value.

Also, in a circuit with ϕ = 90° (purely inductive) or ϕ = -90° (purely capacitive), cos ϕ = 0 and so the average power dissipation is zero.

– Diagram HW Ch. 31