COLLEGE OF SCIENCE AND ENGINEERING

MA1000

Mathematical Foundations

LECTURE NOTES
(Including Tutorial Exercises)

Mathematics Discipline, James Cook University.

Contents

1 Algebra & Analytic Geometry

1.1 Numbers, powers, exponentials, logarithms

1.1.1 Number Sets and Notation

There are many different types of numbers that make up the set of real numbers. Some of these are as follows;

Natural Numbers,	$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$; positive whole numbers
Cardinal Numbers,	$= \{0, 1, 2, 3, 4, \ldots\}$;
Integers,	$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$; positive and negative whole numbers including 0. (Note that if we divide one whole number by another, the result may not be a whole number)
Rationals,	$\mathbb{Q} = \left\{ \frac{a}{b}, \text{ where } a, b \text{ are integers and } b \neq 0 \right\}$;

Irrationals — any number that cannot be expressed as a fraction.
That is, a non-terminating, non-repeating decimal.
Examples are $\sqrt{2}$, π, e. Note that $\pi \neq \frac{22}{7}$!!

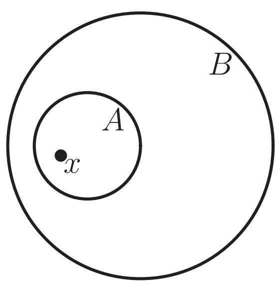
The Reals, $\mathbb{R}$, include all of the above numbers.

Set Notation

A lot of mathematics is built using the notion of the *set*. The extent to which you study mathematics will determine where and how much of this you will see this. In MA1000 you need to be familiar with the concept of and notation used for the *element* and the *subset*, as described below.

$x \in A$ means "x is an element of the set A".
$A \subset B$ means "set A is contained in set B" or "A is a subset of B".

These two statements can be represented using a Venn diagram:



$A \subseteq B$ means "set A is contained in or equal to set B".

EXERCISE 0:

1. Are integers also rational numbers? Explain.
2. The natural numbers, $\mathbb{N}$ are a subset of the integers, $\mathbb{Z}$. That is $\mathbb{N} \subset \mathbb{Z}$. Write down as many relationships of this type as you can between the natural numbers, $\mathbb{N}$, the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$ and the real numbers $\mathbb{R}$.
3. Is there a symbol for the set of irrational numbers? (You might like to do a bit of an internet search.)
4. Can a real number belong to both the set of rational numbers and the set of irrational numbers? Explain.
5. What is the significance of the different sets of numbers introduced in this section? Try to think of of a reason (and/or do some research) as to why this distinction is important . Don't be too concerned if you don't get much on your first attempt - this is quite a deep question. Add to your answer as things come to mind through the semester.

1.1.2 Index Laws

Although calculations of any difficulty can be performed with calculators, it remains important to have a clear understanding of the underlying algebraic rules. Here we introduce some basic terminology and notation that are important to algebra and examine the ways of working that the notation introduces to mathematics.

We often have a number multiplied by itself several times. Rather than write the number over and over, we use a shorthand notation.

Definition 1. $a^n = \underbrace{a \times a \times a \times \cdots \times a}_{n \text{ factors}}$

Here a is a any real number, called the base, and the number n, which has to be an integer for this definition to make sense, is called the index (exponent). The expression a^n is read as the "n^{th} power of a".

Examples : $(-5)(-5)(-5)(-5) = (-5)^4$

$$\left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3$$

Positive Indices

We now state the index laws for positive integer exponents. Note that we restrict ourselves to positive integer exponents because that is how the concept of an exponent was motivated above, as a shorthand for multiplication of a number by itself. At this stage that is what we are restricted to. For example it doesn't really make sense to say that $a^{2.5}$ is a shorthand for a product involving two and a half factors. Later we will look at ways of relaxing the requirement for positive integer exponents, that is we will look to *generalise* the definition of an index, ultimately so that the exponent can be any real number.

Index Laws for positive indices.
Let $m, n \in \mathbb{N}$ and $a, b \in \mathbb{R}$.

IL1. $a^m \times a^n = a^{m+n}$

IL2. $\dfrac{a^m}{a^n} = a^{m-n}$

IL3. $\left(a^m\right)^n = \left(a^n\right)^m = a^{nm}$

IL4. $(ab)^n = a^n b^n$

IL5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Example 1: $a^3 \times a^5 = a^{3+5} = a^8$ (applying IL1)

This can be shown using the definition of an exponent from above:

$$\text{LHS} = a^3 \times a^5 = \underbrace{(a \times a \times a)}_{3 \text{ factors}} \times \underbrace{(a \times a \times a \times a \times a)}_{+\ 5 \text{ factors}} = a^8 = \text{RHS}$$

EXAMPLE 2: $\dfrac{a^5}{a^3} = a^{5-3} = a^2$ (applying IL2)

This can be shown using the definition of an exponent and cancelling common factors:

$$\text{LHS} = \frac{a^5}{a^3} = \frac{\not{a} \times \not{a} \times \not{a} \times a \times a}{\not{a} \times \not{a} \times \not{a}} = a^2 = \text{RHS}$$

EXAMPLE 3: $(a^5)^3 = a^{5\times 3} = a^{15}$ (applying IL3)

This can be shown by using the definition of an exponent on the power 3 and then using IL1:

$$\text{LHS} = (a^5)^3 = (a^5)(a^5)(a^5) = a^{5+5+5} = a^{15} = \text{RHS}$$

Zero and Negative Indices

We now extend the definition of a^n to allow for zero and negative indices. That is we allow $n, m \in \mathbb{Z}$. In each case we make a deduction about the properties that zero and negative indicies must have in order for our working to be consistent with the earlier index laws and the properties of our number system.
Firstly we determine how to interpret an index of zero.

Start with IL1. $a^m \times a^n = a^{m+n}$
Let $m = 0$, $a^0 \times a^n = a^n$
so $a^0 = \dfrac{a^n}{a^n} = 1,$ provided $a \neq 0$.

IL6. $a^0 = 1, \quad (a \neq 0)$

EXAMPLES : $5^0 = 1$
$(2x)^0 = 1$
$(ax + b)^0 = 1$
$(a^2b^0c)^2 = a^4b^0c^2 = a^4(1)c^2 = a^4c^2$

We can now determine how to interpret a negative index.

Start with IL2. $\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$

Let $m = 0$, and using IL6 on the numerator $\dfrac{a^0}{a^n} = \dfrac{1}{a^n} = a^{0-n} = a^{-n}$

IL7. $\dfrac{1}{a^n} = a^{-n}, \quad (a \neq 0)$

Similarly: $\dfrac{1}{a^{-n}} = a^n$

EXAMPLES: Express the following with positive indices.

(a) $4^{-2} = \dfrac{1}{4^2} \qquad \left(= \dfrac{1}{(2^2)^2} = \dfrac{1}{2^4} \right)$

(b) $\left(\dfrac{1}{2}\right)^{-3} = \left(2^{-1}\right)^{-3} = 2^3$

(c) $\dfrac{3^{-2} \times 6^3}{9^{-3} \times 2^{-1}} = \dfrac{3^{-2} \times (3 \times 2)^3}{(3^2)^{-3} \times 2^{-1}} = 3^{-2} \times 3^3 \times 2^3 \times 3^6 \times 2^1 = 2^4 \times 3^7$

Be sure to identify each index law used in these examples. Record that information next to each of them. Also look to see if these examples could be done in a different way. For example, when using more than one index law, could they be used in a different order.

EXERCISE 1: In these exercises explain your working and reference the index laws. Indicate where each has been used. Look for multiple ways of doing these problems, and try to determine if one way is superior over the others.

1. Simplify, expressing with positive indices:
   (a) $\dfrac{3^{-2}x^{-1}}{3^{-3}x^{-2}}$
   (b) $\dfrac{a^2b^3c^{-4}}{a^4b^{-1}c^{-5}}$
   (c) $\dfrac{2^n4^{n+1}}{8^{n-2}}$
   (d) $\left(5^0x^2y^{-1}\right)^{-1}$
   (e) $\dfrac{\left(2x^{-2}y^3\right)^2}{8x^{-4}y}$
   (f) $\left(\dfrac{3a^2}{4b^{-1}}\right)^{-3}\left(\dfrac{4}{a}\right)^{-5}$
   (g) $\dfrac{2x^{-2} - 2y^{-2}}{(xy)^{-3}}$
   (h) $\dfrac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}}$
   (i) $\dfrac{ax^{-2} + a^{-2}x}{a^{-1} + x^{-1}}$
2. Evaluate:
   (a) $\dfrac{2^{-1}4^{-3}6^3}{3^32^{-3}}$
   (b) $\dfrac{(-3)^4(-3)^{-3}}{(-3)^{-2}}$
3. Show that:
   (a) $\dfrac{2^{n+1}4^n}{8^{n-1}} = 16$
   (b) $\dfrac{25^{2n}5^{1-n}}{(5^3)^n} = 5$
4. Simplify: $\dfrac{x^{-2} + x^{-3} - 2x^{-1}}{2x^{-1} + x^{-2}}$

Using Index Laws to Solve Equations

The index laws form part of the fundamental algebraic processes which are often required to solve equations. Some of the techniques for solving equations using the index laws are demonstrated in the following examples.

Example 4: Find x, given $5^x = 125$.

$$\begin{aligned} 5^x &= 125 \\ \therefore\ 5^x &= 5^3 \implies x = 3. \end{aligned}$$

We can interpret this example as asking us to determine what 125 is as a power of 5. So you need to know $125 = 5^3$. Without that you cannot proceed.

The example also uses the property of our number system that if two numbers are equal and written as powers with the same base, then the exponents have to be equal.

Example 5: Find x, given $x^{-3} = 8$.

$$\begin{aligned} x^{-3} &= 8 \\ \text{Using IL7} \qquad \frac{1}{x^3} &= 8 = 2^3 \\ \text{Using IL5} \qquad \left(\frac{1}{x}\right)^3 &= 2^3 \\ \therefore \qquad \frac{1}{x} &= 2 \implies x = \frac{1}{2} \end{aligned}$$

The path to the solution relies on knowing that $2^3 = 8$. After application of IL7 and IL5 the problem can be stated as: What number do I have to raise to the power 3 to get 8? The approach here uses a different property of our number system to that in the previous example. In this example we use that fact that if two numbers are equal and written to the same power (in this case 3) then their bases are either identical or differ only in sign. If the powers are equal and odd then the bases have to be the same.

The next example gives a case when the bases may differ in sign.

Example 6: Find x, given $x^{-2} = 81$.

$$\begin{aligned} x^{-2} &= 81 \\ \text{Using IL7} \qquad \frac{1}{x^2} &= 81 = 9^2 \\ \text{Using IL5} \qquad \left(\frac{1}{x}\right)^2 &= 9^2 \\ \therefore \qquad \frac{1}{x} &= \pm 9 \implies x = \pm\frac{1}{9} \end{aligned}$$

In this case we needed to know that $9^2 = 81$. After the application of the index laws we had a statement that the square of $\frac{1}{x}$ was equal to the square of 9. So the two numbers must be equal or differ only in sign.

Another strategy is to rearrange the equation until the LHS and RHS have a single power with the same base.

EXAMPLE 7: Given $\dfrac{2^{x+4}}{4^{2x-1}} = 1$, find x.

$$\begin{aligned} \frac{2^{x+4}}{4^{2x-1}} &= 1 \\ \therefore \quad 2^{x+4} &= 4^{2x-1} \\ \therefore \quad 2^{x+4} &= \left(2^2\right)^{2x-1} = 2^{4x-2} \end{aligned}$$

Equating indices, we get

$$\begin{aligned} x+4 &= 4x-2 \\ \therefore \quad 6 &= 3x \\ \therefore \quad x &= 2. \end{aligned}$$

A general assumption is that the equation is true for all values of x. This allows us to equate powers and coefficientxs. The next example uses IL1, IL3 and IL7.

EXAMPLE 8: If $\left(3x^n\right)^3\left(3x\right)^{n-6} = ax^2$, then find the constants n and a.

$$\begin{aligned} 3^3x^{3n}3^{n-6}x^{n-6} &= ax^2 \\ \therefore \quad 3^{n-3}x^{4n-6} &= ax^2 \end{aligned}$$

$\therefore$ we must have $3^{n-3} = a$ and $x^{4n-6} = x^2$

$$\begin{aligned} \therefore \quad 4n-6 &= 2 \\ \therefore \quad 4n &= 8 \\ \therefore \quad n &= 2 \end{aligned}$$

If $n = 2$, then $3^{2-3} = 3^{-1} = a$ $\qquad \therefore \quad a = \dfrac{1}{3}$

EXERCISE 2: Be sure to explain each step of working.

1. Solve for x:
   (a) $3^{x-1} = 27$
   (b) $\dfrac{2^{x-3}}{4^{1-x}} = 1$
   (c) $3^{2x+1} = \dfrac{1}{9}$
   (d) $\left(5^x - 25\right)\left(3^x - 27\right) = 0$
   (e) $\left(2^x - 1\right)\left(3^x - \dfrac{1}{9}\right) = 0$
   (f) $\left(3^x\right)^2 + 6\left(3^x\right) - 27 = 0$
   (g) $2^{2x} - 5\left(2^x\right) + 4 = 0$
2. Solve for x: $5\left(1-2x\right)^4 = 80$
3. Show that the following can be written as a quadratic equation: $2^x - 1 = \dfrac{2}{2^x}$
   Show that this quadratic has only one real solution at $x = 1$.
4. Find all the values of n such that: $\dfrac{\left(4^{3n}\right)^2}{2^{32}} = 2^{n^2}$
5. If $3^m = 2$ and $4^n = 27$, use index laws to show that the product: $m \times n = \dfrac{3}{2}$.

Rational Indices

The index laws can be seen to generalise to hold for rational indices ($a \in \mathbb{Q}$) provided $a > 0$ through the following reasoning. We know the solution of $x^q = a$ is written as $x = \sqrt[q]{a}$. This is the q^{th} root of a. That is

$$\left(\sqrt[q]{a}\,\right)^q = a.$$

Now suppose we can write $\sqrt[q]{a}$ as a power of a. Then $\sqrt[q]{a} = a^p$. Using IL3 and the property that two equal numbers of the same base have the same exponent:

$$\begin{aligned}
\left(a^p\right)^q &= a^{pq} = a = a^1 \\
\therefore \quad pq &= 1 \\
\therefore \quad p &= \frac{1}{q} \\
\therefore \text{ we can write } a^{1/q} &= \sqrt[q]{a}\,.
\end{aligned}$$

IL8. $a^{1/q} = \sqrt[q]{a}\,, \qquad (q \neq 0)$

Using IL3, note that if $q \neq 0$ then $a^{p/q} = (a^{1/q})^p = (a^p)^{1/q}$ can also be written as $\left(\sqrt[q]{a}\,\right)^p$, or as$\left(\sqrt[q]{(a^p)}\,\right)$.

EXAMPLES :

$$\begin{aligned}
a^{1/2} &= \sqrt{a} \\
a^{1/3} &= \sqrt[3]{a} \\
a^{1/5} &= \sqrt[5]{a} \\
a^{2/3} &= \sqrt[3]{a^2} = \left(\sqrt[3]{a}\right)^2
\end{aligned}$$

Thus $\quad 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4.$

Furthermore, we can define $a^{-p/q} = \dfrac{1}{a^{p/q}}$, which is consistent with $a^{-n} = \dfrac{1}{a^n}$, from before.

We can generalise - the index laws presented above hold for all real indices, a^r, $r \in \mathbb{R}$ provided $a > 0$.

Fill in the detail for each of the following examples by explaining each step.

EXAMPLES: Simplify 1) $32^{2/5}$, 2) $125^{-2/3}$, and 3) $\left(\dfrac{9}{49}\right)^{-1/2}$.

1) $\quad 32^{2/5} = (2^5)^{2/5} = 2^{5\times(2/5)} = 2^2 = 4$

2) $\quad 125^{-2/3} = \dfrac{1}{125^{2/3}} = \dfrac{1}{(5^3)^{2/3}} = \dfrac{1}{5^2} = \dfrac{1}{25}$

3) $\quad \left(\dfrac{9}{49}\right)^{-1/2} = \left(\dfrac{49}{9}\right)^{1/2} = \dfrac{7}{3}$

EXERCISE 3: As always, explain every step of your working in each of questions 1 to 6.

1. Simplify, expressing with positive indices:
   (a) $2^{\frac{2}{3}}4^{\frac{1}{6}}$
   (b) $243^{-\frac{2}{5}}$
   (c) $\dfrac{3^{-1}a^{-\frac{1}{2}}}{4^{-\frac{1}{2}}b} \div \dfrac{9^{\frac{1}{2}}a^{-\frac{1}{3}}}{2b^{-\frac{1}{4}}}$
   (d) $\dfrac{3^{\frac{1}{3}} \times 3^{\frac{5}{6}} \times 3^{-\frac{1}{6}}}{3^0 \times 3}$
   (e) $\dfrac{2^{\frac{1}{2}} \times 3^{\frac{1}{2}} \times 4^{\frac{1}{2}} \times 12^{\frac{1}{4}}}{3^{\frac{3}{4}}}$

2. (a) Find the product of the following two factors: $\left(6^{\frac{1}{2}} - 5^{\frac{1}{2}}\right)\left(6^{\frac{1}{2}} + 5^{\frac{1}{2}}\right)$
   (b) Factorise: $x - 5$

3. If $a = 2^3$, $b = 2^{-3}$, $c = 6^2$, and $d = 3^{-1}$ find the value of: $\dfrac{a^2b}{c^{\frac{1}{2}}d}$.

4. Without using a calculator, show that: $\left(\dfrac{1}{2}\right)^{\frac{1}{2}} = \left(\dfrac{1}{4}\right)^{\frac{1}{4}}$

5. Find x given: $32^{-0.4x} = 64^{\frac{2}{3}}$

6. Show that

$$a^{\frac{1}{2}} \times \left(a^2b^{-\frac{1}{2}}\right)^3 \times \left(\frac{c^{\frac{1}{2}}}{b^{-6}a^6}\right)^{\frac{1}{2}} \div \left(\frac{a^5c^{\frac{1}{2}}}{b^{-1}}\right)^{\frac{1}{2}} = ab$$

7. **Research question.** Find out how real indicies are catered for in the index laws. That is, what is the justification that the use of the index laws can be generalised in this way?

1.1.3 Surds (or Radicals)

A frequently occurring term is $x^{1/2} = \sqrt{x}$, or the "square root of x". It is the positive number that when squared is equal to x. Thus $(\sqrt{x})^2 = x$ *and* $\sqrt{x} \geq 0$. The detail about $\sqrt{x}$ being non-negative is important. If the root is an irrational quantity (i.e. does not belong to the set of rational numbers, $\mathbb{Q}$) it is called a *surd*.

Some examples are; $\sqrt{2}\ (= 2^{1/2})$, $\sqrt{x-1}$, $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$

Properties of surds:	
P1.	$\sqrt{a}\sqrt{b} = \sqrt{ab}$
P2.	$\sqrt{a} \div \sqrt{b} = \dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$
P3.	$a\sqrt{c} \pm b\sqrt{c} = (a \pm b)\sqrt{c}$
P4.	$\sqrt{a^2} = \lvert a \rvert$. The usual convention is that $\sqrt{a}$ means the positive square root of a.

NOTE: $\sqrt{a+b} \equiv \sqrt{(a+b)}$ — This means that the addition under the $\sqrt{\ }$ sign must be performed before taking the square root. More generally all operations under a $\sqrt{\ }$ sign must be performed before taking the root.

EXAMPLES:

(*a*) $\sqrt{16+9} = \sqrt{25} = 5$

(*b*) $\sqrt{2^2+6^2} = \sqrt{4+36} = \sqrt{40} = \sqrt{4 \times 10} = 2\sqrt{10}.$

(*c*) using P1: $2\sqrt{7} \times 3\sqrt{2} = (2 \times 3)\sqrt{7 \times 2} = 6\sqrt{14}$

(*d*) using P2: $\sqrt{6} \div \sqrt{3} = \sqrt{\dfrac{6}{3}} = \sqrt{2}$

$$2\sqrt{5} \div 3\sqrt{3} = \frac{2}{3}\sqrt{\frac{5}{3}}$$

Simplest Form

When operating with surd expressions, the surds should first be simplified. e.g. $\sqrt{a^2b^2cd} = \lvert ab \rvert\sqrt{cd}$. Thus in example b above the convention is to express the surd as $2\sqrt{10}$ rather than $\sqrt{40}$.

Addition and Subtraction

In order to add or subtract surds, first express the surds in simplest form. Then only similar surds can be added or subtracted.

EXAMPLE 1: Simplify $6\sqrt{7}-\sqrt{28}+3\sqrt{63}$.

$$\begin{aligned} 6\sqrt{7}-\sqrt{28}+3\sqrt{63} &= 6\sqrt{7}-\sqrt{7\times 4}+3\sqrt{9\times 7} \\ &= 6\sqrt{7}-2\sqrt{7}+3(3\sqrt{7}) \\ &= 6\sqrt{7}-2\sqrt{7}+9\sqrt{7} \qquad \text{— similar surds} \\ &= 13\sqrt{7} \end{aligned}$$

EXAMPLE 2: Simplify $\sqrt{5}+2\sqrt{3}-5\sqrt{5}$.

$$\begin{aligned} \sqrt{5}+2\sqrt{3}-5\sqrt{5} &= 2\sqrt{3}+\underbrace{\sqrt{5}-5\sqrt{5}}_{\text{similar surds}} \\ &= 2\sqrt{3}-4\sqrt{5} \qquad \text{no further simplification possible (no similar surds present)} \end{aligned}$$

EXAMPLE 3: Simplify $3\sqrt{125}-\sqrt{20}+\sqrt{27}$.

$$\begin{aligned} 3\sqrt{125}-\sqrt{20}+\sqrt{27} &= 3\sqrt{25\times 5}-\sqrt{4\times 5}+\sqrt{9\times 3} \\ &= 3(5\sqrt{5})-2\sqrt{5}+3\sqrt{3} \\ &= \underbrace{15\sqrt{5}-2\sqrt{5}}_{\text{similar surds}}+3\sqrt{3} \\ &= 13\sqrt{5}+3\sqrt{3} \end{aligned}$$

EXERCISE 4: Be sure to explain each step of working. In particular which of the properties of surds have you used to answer each question?

1. Simplify, expressing in simplest form:
   (a) $\sqrt{3}\sqrt[3]{81}$ (b) $\sqrt{5}+\sqrt{20}$ (c) $2\sqrt{20}-\sqrt{125}-\sqrt{45}$
2. Expand and simplify
   (a) $\sqrt{2}(\sqrt{3}-\sqrt{2})$ (b) $(\sqrt{3}-\sqrt{2})^2$ (c) $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$
3. Without using a calculator, show whether each of the following is true or false:
   (a) Half of $\sqrt{12}$ is $\sqrt{6}$. (b) The square root of $5+2\sqrt{6}$ is $\sqrt{3}+\sqrt{2}$. (c) $4\sqrt{\frac{1}{2}}=\sqrt{8}$
4. Write the following in ascending order of magnitude: $4\sqrt{5}$, $2\sqrt{21}$, 9, $3\sqrt{10}$.
5. Solve for x
   (a) $\sqrt{32}+\sqrt{50}-\sqrt{8}=x\sqrt{2}$ (b) $\sqrt{52}+\sqrt{117}=5\sqrt{x}$
6. Find x and y if
   (a) $(2\sqrt{6}-3\sqrt{2})^2=x+y\sqrt{3}$ (c) $x+y+\sqrt{x-y}=2+\sqrt{6}$
   (b) $x+\sqrt{y-3}=4+\sqrt{5}$ (d) $2x-y+\sqrt{4x-y}=x-2y+3+\sqrt{x+5}$

   Be explicit about any assumptions you make in your working.
7. Justify each of the properties P1 to P3 for surds by reference to the index laws or other properties of the real numbers.
8. Explain why $\sqrt{(-10)^2}=10$ and not -10.

Rationalising the denominator

It is standard practice to write numbers such as $\frac{1}{\sqrt{5}}$, $\frac{2}{\sqrt{3}}$, etc. in a form which contains only rational numbers in the denominator. i.e.,

$$\begin{aligned}\frac{1}{\sqrt{5}} &= \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \qquad \left(\text{since } \frac{\sqrt{5}}{\sqrt{5}} = 1\right)\\ &= \frac{\sqrt{5}}{5}\end{aligned}$$

This process is called "*rationalising the denominator*".

EXAMPLES :

$$\frac{1}{2\sqrt{7}} = \frac{1}{2\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{7}}{2 \times 7} = \frac{\sqrt{7}}{14}$$

$$\frac{5\sqrt{2}}{2\sqrt{3}} = \frac{5\sqrt{2}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{6}}{2 \times 3} = \frac{5\sqrt{6}}{6}$$

Conjugate Pairs

Recall that the difference of two squares can be written as $a^2 - b^2 = (a + b)(a - b)$. The product $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b$ is again the very familiar "difference of two squares"; a and b being the squares of $\sqrt{a}$ and $\sqrt{b}$ respectively. The two expressions $\sqrt{a} - \sqrt{b}$ and $\sqrt{a}+\sqrt{b}$ are called conjugate surds; each being the conjugate of the other (note the change in sign). Together they form a "conjugate pair". The usefulness of conjugate pairs lies in the fact that their product *does not contain any surds*. e.g., $\sqrt{2} - \sqrt{3}$ and $\sqrt{2} + \sqrt{3}$ are conjugates and $(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3}) = 2 - 3 = -1$.

We use this property to rationalise the denominator of expressions like

$$\begin{aligned}\frac{3}{2\sqrt{7} - \sqrt{5}} &= \frac{3}{2\sqrt{7} - \sqrt{5}} \times \frac{2\sqrt{7} + \sqrt{5}}{2\sqrt{7} + \sqrt{5}}\\ &= \frac{6\sqrt{7} + 3\sqrt{5}}{2^2 \times 7 - 5}\\ &= \frac{6\sqrt{7} + 3\sqrt{5}}{23}.\end{aligned}$$

EXERCISE 5:

1. Rationalise the denominator:

   (a) $\dfrac{3}{2\sqrt{5}}$ (b) $\dfrac{\sqrt{5}}{\sqrt{7}}$ (c) $\dfrac{3}{\sqrt{3}-\sqrt{2}}$ (d) $\dfrac{3\sqrt{6}}{2\sqrt{2}+\sqrt{3}}$

2. If $x = \dfrac{2\sqrt{7}+\sqrt{5}}{2\sqrt{7}-\sqrt{5}}$, find the value of $x + \dfrac{1}{x}$

3. Find the reciprocal of $2 + \sqrt{3}$ and rationalise.

4. Rationalise the denominator: $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$.

   Try doing this in two ways, firstly using (i) $1 + \sqrt{2} - \sqrt{3}$ and then (ii) $1 - \sqrt{2} + \sqrt{3}$.

   Show your working. Justify each step.

   Be careful multiplying the products of the form $(1 + \sqrt{2} + \sqrt{3})(1 + \sqrt{2} - \sqrt{3})$. How can you ensure you get all terms in the product?

   Which approach ((i) or (ii)) is faster? Why? Would this behaviour (for rationalising denominators of three terms), with one approach being quicker than another always be observed? Discuss in one or two sentences.

5. Note that the concept of a surd generalises beyond square roots. Anything that is expressed as an n^{th}-root of a number is a surd (assuming it can't be simplified to a rational). Express the product $\sqrt[3]{2} \times \sqrt[2]{3}$ as a single surd i.e. $\sqrt[x]{N}$. That is determine both x and N that satisfies the equation:

$$\sqrt[3]{2} \times \sqrt[2]{3} = \sqrt[x]{N}$$

6. Expand

$$\left(\sqrt{3\sqrt{2}+\sqrt{3}}\right)\left(\sqrt{3\sqrt{2}-\sqrt{3}}\right)$$

7. Show without using a calculator that

$$\sqrt{3+2\sqrt{2}} - \sqrt{3-2\sqrt{2}} = 2$$

8. Generalising the special case in the previous question. Can you write $\sqrt{a+b} - \sqrt{a-b}$ as the square root of a difference?

1.1.4 Logarithms

The logarithm of a number is simply the index when the number is expressed as a power of some other number. Thus we rewrite statements such as $2^3 = 8$, using logarithms. In this case we write $3 = \log_2 8$,

i.e.,	$2^3 = 8$	$\Longleftrightarrow$ equivalent expression	$3 = \log_2 8$
Also	$3^2 = 9$	$\Longleftrightarrow$	$2 = \log_3 9$
	$10^4 = 10,000$	$\Longleftrightarrow$	$4 = \log_{10} 10,000$
	$16 \ = 4$	$\Longleftrightarrow$	$= \log_{16} 4$

The expression $\log_a n$ is read as "logarithm of n to the base a."

In general, if $a \in \mathbb{R}^+$ (positive real) and $x \in \mathbb{R}$, then the statements $a^x = n$ and $\log_a n = x$ are equivalent.

$$\boxed{a^x = n \quad \Longleftrightarrow \quad x = \log_a n}$$

Notes

1. In order to compute a logarithm a base must be chosen. Thus writing $\log 8$ on its own is a meaningless statement. If we want to be absolutely clear what is meant then a base needs to be quoted.

2. While the above statement is true, convention dictates that in some disciplines the base need not be quoted. In these cases the base is usually either 10 or Euler's number, e.

3. We can evaluate (without a calculator) expressions such as

(i)	$\log_2 16$	(ii)	$\log_5 125$
Put	$n = \log_2 16$	Put	$n = \log_5 125$
Then	$2^n = 16 = 2^4$	Then	$5^n = 125 = 5^3$
$\therefore$	$n = 4.$	$\therefore$	$n = 3.$

   These are the exception, rather than the rule.

4. We can get a feel for the logarithm operation and the quantities it will produce by considering the equivalent statement using an index:

$\log_2 8 = x$	means	$8 = 2^x$, so $x = 3$
but $\log_2 9 = x$	means	$9 = 2^x$, so $x = 3.169925\ldots$ is irrational
and $\log_2 10 = x$	means	$10 = 2^x$, so $x = 3.321928\ldots$ is irrational

   The logarithm operation only produces an integer when the number whose logarithm is being calculated is an integer power of the base. This is unlikely most of the time, as the preceding calculations of $\log_2 8$, $\log_2 9$ and $\log_2 10$ would suggest.

   To get more familiar with the output of a logarithm, we further explore the case of a logarithms to the base 8. This base is chosen for no particular reason.

$$\begin{aligned}
\log_8 8 = x &\quad \text{means} \quad 8 = 8^x, \text{ so } x = 1 \\
\log_8 64 = x &\quad \text{means} \quad 64 = 8^x, \text{ so } x = 2 \\
\log_8 10 = x &\quad \text{means} \quad 10 = 8^x, \text{so } x \text{ must be between 1 and 2} \\
\log_8 65 = x &\quad \text{means} \quad 65 = 8^x, \text{ so } x \text{ must be between 2 and 3 and closer to 2} \\
\log_8 1 = x &\quad \text{means} \quad 1 = 8^x, \text{ so } x = 0 \\
\log_8 \frac{1}{8} = x &\quad \text{means} \quad \frac{1}{8} = 8^x, \text{ so } x = -1 \\
\log_8 \frac{1}{2} = x &\quad \text{means} \quad \frac{1}{2} = 8^x, \text{ so } x = \text{ must be between } -1 \text{ and } 0
\end{aligned}$$

We can now generalise to make statements about $\log_8 a$ where a is any number:

$$\begin{aligned}
\log_8 a > 1 &\quad \text{when} \quad a > 8 \\
0 < \log_8 a < 1 &\quad \text{when} \quad 1 < a < 8 \\
\log_8 a < 0 &\quad \text{when} \quad 0 < a < 1
\end{aligned}$$

Can you generalise further? In particular, how do these statements change as the base of the logarithm changes?

Note that you should have some intuition as to the size of the numbers produced by the logarithm operation at least for smaller numbers with smaller bases. For example you should be able to state that $\log_{10} 50$ lies between 1 and 2.

From the index laws given earlier, we can now deduce the laws of logarithms.

$$a^0 = 1 \iff \boxed{0 = \log_a 1} \tag{LL1}$$

$$a^1 = a \iff \boxed{1 = \log_a a} \tag{LL2}$$

LL1 clearly comes from IL6. LL2 comes from the definition for a^n that was introduced at the beginning of the section on index laws.

Let $m = a^x$ and $n = a^y$
Then $\log_a m = x$ and $\log_a n = y$ by definition of the logarithm.

From IL1 (product of exponentials with same base), we have

$$\begin{aligned}
& m \times n = a^x \times a^y = a^{x+y} \\
\iff & \log_a(mn) = x + y = \log_a m + \log_a n. \\
\text{i.e.} \quad & \boxed{\log_a(mn) = \log_a m + \log_a n}
\end{aligned} \tag{LL3}$$

From IL2 (quotient involving exponentials with same base), we have

$$\begin{aligned}
& \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y} \\
\iff & \log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n. \\
\text{i.e.} \quad & \boxed{\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n}
\end{aligned} \tag{LL4}$$

If $m = 1$, then LL4 gives

$$\log_a \left(\frac{1}{n}\right) = \log_a 1 - \log_a n$$
$$= 0 - \log_a n.$$

i.e. $$\boxed{\log_a \left(\frac{1}{n}\right) = -\log_a n} \qquad \text{(LL5)}$$

Note that this can also be derived from IL7.

Using IL3 (exponential to a power), we have

$$m^p = \big(a^x\big)^p = \big(a^p\big)^x = a^{px}$$
$$\iff \quad \log_a(m^p) = px = p\log_a m.$$

i.e. $$\boxed{\log_a(m^p) = p\log_a m} \qquad \text{(LL6)}$$

Finally, if $a^x = m$ then $x = \log_a m$.

Also starting with $a^x = m$ we can take log to base b of both sides.

This gives $\log_b(a^x) = \log_b m$

and using LL6 $x\log_b a = \log_b m$

$$\therefore \quad x = \frac{\log_b m}{\log_b a}.$$

This gives us the "<u>change of base law</u>" for logarithms,

i.e. $$\boxed{\log_a m = \frac{\log_b m}{\log_b a}} \qquad \text{(LL7)}$$

The following are routine application of logarithm laws:

1. $\log_a 5 + \log_a 3 = \log_a(5 \times 3)$ using (LL3)
   $= \log_a 15$
2. $\log_a 25 - \log_a 5 = \log_a\left(\frac{25}{5}\right)$ using (LL4)
   $= \log_a 5$
3. $\log_a 81 = \log_a 3^4 = 4\log_a 3$ using (LL6)
4. $\log_5 560 = \dfrac{\log_{10} 560}{\log_{10} 5}$ using (LL7)
   $\simeq 3.932$ (using calculator)

More difficult worked examples:

EXAMPLE 1: Simplify $3\log_{10}2+\log_{10}18-2\log_{10}\left(\frac{6}{5}\right)$.

$$\begin{aligned}3\log_{10}2+\log_{10}18-2\log_{10}\left(\tfrac{6}{5}\right) &= \underset{\text{(LL6)}}{\log_{10}2^3}+\log_{10}18+\underset{\text{(LL5)}}{2\log_{10}\left(\tfrac{5}{6}\right)} \\ &= \log_{10}2^3+\log_{10}18+\underset{\text{(LL6)}}{\log_{10}\left(\tfrac{25}{36}\right)} \\ &= \log_{10}\left(8\times 18\times \tfrac{25}{36}\right) && \text{using (LL3)} \\ &= \log_{10}100=\underset{\text{(LL6)}}{\log_{10}10^2=2\log_{10}10}\underset{\text{(LL2)}}{=2}.\end{aligned}$$

EXAMPLE 2: For what value of x is $\log_2(x+1)-\log_2(x-1)=3$?

We have $\log_2\left(\dfrac{x+1}{x-1}\right)=3$ using (LL4)

$$\begin{aligned}\therefore\quad \frac{x+1}{x-1} &= 2^3=8 && \text{by definition} \\ \therefore\quad x+1 &= 8x-8 \\ 9 &= 7x \\ \therefore\quad x &= \frac{9}{7}.\end{aligned}$$

EXERCISE 6: You must be able to identify which logarithm laws have been used and your reasoning for each step of your working. Do this for at least three or four of these questions.

1. Simplify:
   (a) $\log_2 32$
   (b) $\log_3 81^{-1}$
   (c) $\log_2 16-\log_2 8$
   (d) $(\log_2 16)(\log_2 4)$
   (e) $3\log_5 2-2\log_5 4$
   (f) $\log_{10}4+2\log_{10}5$
   (g) $3^{2\log_3 5}$
   (h) $\log_2\left(\dfrac{1}{2}\right)^{-18}$
2. Solve for x:
   (a) $5\log_{32}x=-3$
   (b) $\log_{10}10^x=5$
   (d) $\log_2 x+\log_2(x+2)=3$
   (e) $\log_2\left(\dfrac{1}{x}\right)=2$
   (c) $\log_{10}2+5\log_{10}x-\log_{10}5-\log_{10}\left(x^3\right)=\log_{10}40.$

Example 3: If $x = A^2\sqrt{b^3}\,c = A^2b^{3/2}c$, express $\log_a x$ in terms of $\log_a A$, $\log_a b$, and $\log_a c$.

Take $\log_a$ of both sides

$$\begin{aligned}\log_a x &= \log_a(A^2b^{3/2}c)\\ &= \log_a A^2 + \log_a b^{3/2} + \log_a c && \text{using (LL3)}\\ &= 2\log_a A + \tfrac{3}{2}\log_a b + \log_a c. && \text{using (LL6)}\end{aligned}$$

In the next example try not to be too concerned about the chemistry context. If you follow the notation you will see that this is a very simple application of the logarithm laws.

Example 4: Consider the reaction for the dissociation of the water molecule

$$H_2O \rightleftharpoons H^+ + OH^-$$

Notation: chemists use$[H^+]$ to represent the concentration of H^+
Notation: chemists use$[OH^-]$ to represent the concentration of OH^-
In chemistry, the pH and pOH are defined as

$$\text{pH} = -\log_{10}[H^+] \text{ and } \text{pOH} = -\log_{10}[OH^-].$$

It is a known fact of chemistry that the product of the concentrations is $[H^+][OH^-] = 10^{-14}$.
Question. How are the pH and pOH related?

Using $[H^+][OH^-] = 10^{-14}$, take $\log_{10}$ of both sides, to give

$$\begin{aligned}\log_{10}\big([H^+][OH^-]\big) &= \log_{10} 10^{-14}\\ \underbrace{\log_{10}[H^+] + \log_{10}[OH^-]}_{\text{(LL3)}} &= \underbrace{-14\log_{10} 10}_{\text{(LL6)}}\\ \log_{10}[H^+] + \log_{10}[OH^-] &= \underbrace{-14}_{\text{(LL2)}}\\ -\log_{10}[H^+] - \log_{10}[OH^-] &= 14\\ \text{i.e.}\quad \text{pH} + \text{pOH} &= 14. && \text{(using definitions)}\end{aligned}$$

The next example also comes from a chemistry context. The notation follows from the previous example.

EXAMPLE 5: Consider the reaction

$$\mathrm{HA} \rightleftharpoons \mathrm{H}^+ + \mathrm{A}^-$$

The acidity constant is $K_a = \dfrac{[\mathrm{H}^+][\mathrm{A}^-]}{[\mathrm{HA}]}$. If HA is the only acid present then $[\mathrm{A}^-] = [\mathrm{H}^+]$ and $K_a = \dfrac{[\mathrm{H}^+]^2}{[\mathrm{HA}]}$.

Question: If the pK_a is defined as $-\log_{10} K_a$ and pHA is $-\log_{10}[\mathrm{HA}]$, find an expression for the pH.

Take $\log_{10}$ of both sides of the expression for K_a:

$$\begin{aligned}
\log_{10} K_a &= \log_{10} \frac{[\mathrm{H}^+]^2}{[\mathrm{HA}]} \\
&= \log_{10}[\mathrm{H}^+]^2 - \log_{10}[\mathrm{HA}] && \text{using (LL4)} \\
&= 2\log_{10}[\mathrm{H}^+] - \log_{10}[\mathrm{HA}]. && \text{using (LL6)} \\
\therefore \quad \log_{10} K_a + \log_{10}[\mathrm{HA}] &= 2\log_{10}[\mathrm{H}^+]. \\
\therefore \quad -\mathrm{p}K_a - \mathrm{pHA} &= -2\mathrm{pH}. \\
\therefore \quad \mathrm{pH} &= \tfrac{1}{2}(\mathrm{pHA} + \mathrm{p}K_a).
\end{aligned}$$

Any positive number can be used as a base for logarithms. Of all the possibilities, the number e (where $e \simeq 2.71828\cdots$) is a special choice. (We will see the reason for this later). The function $\log_e x$ is called the natural logarithm of x. It is given the particular notation

$$\log_e x = \ln x = \log x.$$

Note that, in mathematics, if the base of a logarithm is not given it is assumed to be e. The bases of other log functions bases must be given explicitly. e.g., $\log_2 x$, $\log_{10} x$, etc.

On most calculators, the natural log function is denoted by $\ln x$. e is commonly known as Euler's number (pronounced like "oiler") and is irrational.

Often we need to be able to transform formulae and move between index and logarithmic representations. We often do this by taking the logarithm of both sides of an equation or taking an exponential of both sides of an equation as suggested by the following observations:

We know that $a^x = y \iff x = \log_a y$. It follows that

$$\boxed{a^{\log_a y} = y} \quad \text{and} \quad \boxed{\log_a a^x = x}$$

In particular, if $a = e$, we have $e^x = y \iff x = \ln y$ and

$$\boxed{e^{\ln y} = y} \quad \text{and} \quad \boxed{\ln e^x = x}$$

EXAMPLE 6: If $y = K \times 10^x$, express x in terms of the other symbols.

We have $$\frac{y}{K} = 10^x$$

Take $\log_{10}$ of both sides, giving

$$\begin{aligned} \log_{10}\left(\frac{y}{K}\right) &= \log_{10} 10^x \\ &= x \log_{10} 10 && \text{using (LL6)} \\ &= x && \text{using (LL2)} \\ \therefore \quad x = \log_{10}\left(\frac{y}{K}\right) &= \log_{10} y - \log_{10} K. && \text{using (LL4)} \end{aligned}$$

EXAMPLE 7: The charge q on a capacitor is given by $q = q_0(1 - e^{-at})$, where q_0 is the final charge, a is a constant, and t is time. Solve for t.

$$\begin{aligned} \frac{q}{q_0} &= 1 - e^{-at} \\ \therefore \quad e^{-at} &= 1 - \frac{q}{q_0} \end{aligned}$$

Take the natural logarithm ($\log_e = \ln$) of both sides

$$\begin{aligned} \ln e^{-at} &= \ln\left(1 - \frac{q}{q_0}\right) \\ \therefore \quad -at \ln e &= \ln\left(1 - \frac{q}{q_0}\right) \\ \therefore \quad -at &= \ln\left(1 - \frac{q}{q_0}\right) \\ \therefore \quad t &= -\frac{1}{a} \ln\left(1 - \frac{q}{q_0}\right). \end{aligned}$$

EXAMPLE 8: An equation relating the distance s through which a falling object moves and its velocity is $\ln s + \ln 2g = 2 \ln v$, where g is acceleration due to gravity. Solve for s.

$$\begin{aligned} \ln s + \ln(2g) &= 2 \ln v \\ \therefore \quad \ln s &= \ln v^2 - \ln(2g) = \ln\left(\frac{v^2}{2g}\right) \end{aligned}$$

Take the exponential of both sides (since $e^{\ln y} = y$)

$$\begin{aligned} e^{\ln s} &= e^{\ln(v^2/(2g))} \\ \therefore \quad s &= \frac{v^2}{2g} \end{aligned}$$

EXAMPLE 9: Given the equation $\ln Q = Q_0 t + \ln kv$, solve for v.

$$\ln Q - Q_0 t = \ln kv$$

Take the exponential of both sides

$$\begin{aligned} e^{(\ln Q - Q_0 t)} &= e^{\ln kv} \\ \therefore \quad e^{\ln Q} \times e^{-Q_0 t} &= e^{\ln kv} \\ \therefore \quad Qe^{-Q_0 t} &= kv \\ \therefore \quad v &= \frac{Q}{k} e^{-Q_0 t}. \end{aligned}$$

EXAMPLE 10: A satellite loses 0.1% of its remaining power each week. An equation relating the power P, the initial power P_0, and the time t in weeks is $\ln P = P_0 + t \ln 0.999$. Solve for P.

$$\ln P = P_0 + \ln(0.999)^t$$

Take the exponential of both sides

$$\begin{aligned} e^{\ln P} &= e^{P_0 + \ln(0.999)^t} \\ &= e^{P_0} \times e^{\ln(0.999)^t} \\ \therefore \quad P &= e^{P_0}(0.999)^t. \end{aligned}$$

EXERCISE 7: Be sure to indicate your reasoning for each step of your working, referring to index laws or log laws as appropriate.

1. Solve for x: $3 \ln 2x = 2$
2. Simplify: $\log_e \left(\frac{1}{e} \right)$
3. If $\log_e x = 0.6$ and $\log_e y = 0.2$, evaluate $\log_e \left(\frac{x^2}{\sqrt{y}} \right)$.
4. If $y = ae^{4t}$, express t in terms of a and y.
5. If $\ln A = bt + \ln P$, express P in terms of the other symbols.
6. The law governing radioactive decay is $p = p_0 e^{-kt}$, where p is the intensity at time t and p_0 is the initial intensity. Show that if $p = \frac{p_0}{2}$ when $t = h$ then the time taken for the initial radioactivity to decay 99% is $2h \log_2 10$.

1.2 Algebra

1.2.1 Expanding and Factorising

The following results should be familiar to you. They follow from repeated use of the Distributive Law: $a(b+c) = ab + ac$.

Two factors

Binomial (Quadratic) Expansions:

$(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd$

$$\left.\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\(a-b)^2 &= a^2 - 2ab + b^2\end{aligned}\right\}\quad \text{Perfect Squares of a sum and difference}$$

$(a+b)(a-b) = a^2 - b^2$ Difference of Two Squares

Basic Factorising: The above results can be used to factorise expressions such as;

$a^2 + 2ab + b^2 = (a+b)^2$
$a^2 - 2ab + b^2 = (a-b)^2$
$a^2 - b^2 = (a+b)(a-b)$

Noting that we are working left to right in these identities.
Also note that $a^2 + b^2$ has no REAL FACTORS.

Many algebraic expressions can be factorised:

Examples:

1. The first is a straightforward application of the square of a sum identity:

   $x^2 + 6x + 9 = (x+3)^2$

2. This one is a bit trickier. There are two lots of factoring in this one example. First use $a = x^2$ and $b = y^2$ in the square of a difference identity. After the first equals sign the quantity in brackets is a difference of squares, so it is further factorised.

   $$x^4 - 2x^2y^2 + y^4 = \left(x^2 - y^2\right)^2 = [(x+y)(x-y)]^2 = (x+y)^2(x-y)^2.$$

3. This example shows how we may apply the factoring of a difference of squares in the circumstance where the quantities a and b are seen to be expressions involving another variable.

   $(5x-2)^2 - (2x-3)^2$ (difference of two squares)
   Let $a = 5x - 2$ and $b = 2x - 3$, so that
   $$\begin{aligned}(5x-2)^2 - (2x-3)^2 &= a^2 - b^2 = (a+b)(a-b)\\&= [(5x-2)-(2x-3)][(5x-2)+(2x-3)]\\&= (3x+1)(7x-5).\end{aligned}$$

4. This example is similar to the previous - it shows an expression that is quadratic in a linear expression. The $x - 5$ is the linear expression. This is replaced by the variable a. The quadratic (in a) is then factorised. Finally the variable a is replaced by $x - 5$ again.

   $(x-5)^2 + 10(x-5) + 24$
   Let $a = x - 5$, then $a^2 = (x-5)^2$ and
   $$\begin{aligned}(x-5)^2 + 10(x-5) + 24 &= a^2 + 10a + 24 = (a+6)(a+4)\\ &= (x-5+6)(x-5+4) = (x+1)(x-1).\end{aligned}$$

5. This example shows the application of two of the identities – factoring the square of a difference and then the difference of squares. What is a and what is b in the difference of squares?

   $$\begin{aligned}\underbrace{p^2 - 2pq + q^2}_{\text{perfect square}} - r^2 &= (p-q)^2 - r^2 \qquad \text{(difference of two squares)}\\ &= [(p-q)+r][(p-q)-r]\\ &= (p-q+r)(p-q-r).\end{aligned}$$

EXERCISE 8: Expand the following. Identify the cases that are examples of a perfect square or a difference of two squares:

1. $3a^2(2a+7)$
2. $(4x-5y)(2x+3y)$
3. $(x-2)(x+4)$
4. $(7x-2)(7x+2)$
5. $(3x+10y)(3x-10y)$
6. $(3x+10y)(3x+10y)$
7. $(2x-3)^2$
8. $(2x^2-3)(x^2+5)$
9. $(3-2x)(5-x)$
10. $\left(\frac{3}{x} - 2x\right)^2$

1.2.2 Factors of a Quadratic Polynomial: $\mathbf{ax^2 + bx + c,\ a \neq 0}$ over $\mathbb{R}$

Recall that a linear polynomial is of the form $ax + b$, and a quadratic polynomial is of the form $ax^2 + bx + c,\ a \neq 0$. The quadratic polynomial $x^2 - x - 6 = (x+2)(x-3)$ has two linear factors which can be identified by looking for factors of -6 that sum to -1 that are also integers. However the quadratic polynomial $x^2 - 4x + 1$ cannot be factorised in this way. More precisely, this quadratic has no factors over the field of Rationals, $\mathbb{Q}$; but it may have factors over the field of real numbers, $\mathbb{R}$. This means the numbers inside the brackets can't be rational numbers, but may be irrational - involving square roots.

A general quadratic, $ax^2 + bx + c$ $(a \neq 0)$, can be expressed as the "difference of two squares" and then factorised, provided $\boxed{b^2 - 4ac \geqslant 0}$. The quantity $b^2 - 4ac$ is called the **discriminant** and is denoted by the Greek letter delta, Δ. The value and sign of the discriminant can tell us how many factors the quadratic will have:

- $\Delta > 0 \implies$ two different factors
- $\Delta = 0 \implies$ one repeated factor
- $\Delta < 0 \implies$ no factors

Note: If $\sqrt{\Delta} = \sqrt{b^2 - 4ac}$ is an integer, the quadratic has 2 linear factors involving rational numbers (integers or fractions), but not irrational numbers.

Example: Factorise: $x^2 - 2x - 24$ $\quad (a = 1, b = -2, c = -24)$ $\quad \therefore \Delta = b^2 - 4ac = 100$

$\sqrt{\Delta} = \sqrt{100} = 10$ $\quad \therefore x^2 - 2x - 24$ has 2 linear factors involving rational numbers.

We can assume the two linear factors will be of the form: $(x + A)(x + B)$.

To find A and B, we consider two numbers that multiply to give -24 and add to give -2

i.e. $-6 \times 4 = -24$ and $-6 + 4 = -2$. We may use the following array to assist:

$$\begin{array}{ccccc} \underline{\quad} & \times & \underline{\quad} & = & -24 \\ \underline{\quad} & + & \underline{\quad} & = & -2 \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} \underline{-6} & \times & \underline{4} & = & -24 \\ \underline{-6} & + & \underline{4} & = & -2 \end{array}$$

Therefore: $x^2 - 2x - 24 = (x - 6)(x + 4)$

Example: Factorise: $3x^2 - 2x + 5$ $\quad (a = 3, b = -2, c = 5)$

$\Delta = b^2 - 4ac = (-2)^2 - 4 \times 3 \times 5 = -56 < 0$ $\quad \therefore \quad 3x^2 - 2x + 5$ cannot be factorised over $\mathbb{R}$.

The existence of real factors can be determined by expressing the quadratic in the "completing the square form". If we can express the polynomial in the form of a "difference of perfect squares", it can be factored using the identity $a^2 - b^2 = (a + b)(a - b)$.

EXAMPLES: Find linear factors over $\mathbb{R}$ for the following quadratics.

1. $$\begin{aligned} x^2 - 4x + 1 &= (x^2 - 4x + 4) - 4 + 1 = (x-2)^2 - 3 \\ &= (x-2)^2 - (\sqrt{3}\,)^2 \\ &= (x - 2 + \sqrt{3}\,)(x - 2 - \sqrt{3}\,). \end{aligned}$$

2. $$\begin{aligned} 2x^2 + 5x - 1 &= 2\big[x^2 + \tfrac{5}{2}x - \tfrac{1}{2}\big] \\ &= 2\big[\big(x^2 + \tfrac{5}{2}x + \tfrac{25}{16}\big) - \tfrac{25}{16} - \tfrac{1}{2}\big] \end{aligned}$$

$$\begin{array}{ccc} \downarrow & \nwarrow & \uparrow \\ \tfrac{1}{2}\big(\tfrac{5}{2}\big) & \underset{\text{square}}{\longrightarrow} & \big(\tfrac{25}{16}\big) \end{array}$$

$$\begin{aligned} &= 2\big[\big(x + \tfrac{5}{4}\big)^2 - \tfrac{33}{16}\big] \\ &= 2\big(x + \tfrac{5}{4} + \tfrac{\sqrt{33}}{4}\big)\big(x + \tfrac{5}{4} - \tfrac{\sqrt{33}}{4}\big). \end{aligned}$$

using $a^2 - b^2 = (a+b)(a-b)$

The procedure used here is called "*completing the square*".

To factorise a general quadratic $ax^2 + bx + c$ $(a \neq 0)$ by completing the square, follow the steps outlined below

Step 1: Factor out a

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

Step 2: Add and subtract the square of half the coefficient of x

$$= a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{c}{a}\right]$$

$$\begin{array}{ccc} \downarrow & \nwarrow & \uparrow \\ \dfrac{1}{2}\left(\dfrac{b}{a}\right) & \underset{\text{square}}{\longrightarrow} & \left(\dfrac{b^2}{4a^2}\right) \end{array}$$

Step 3: Rewrite in the "difference of two squares form"

$$\begin{aligned} &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right] \end{aligned}$$

Step 4: To factorise the quadratic, apply the knowledge that $a^2 - b^2 = (a+b)(a-b)$

$$= a\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right).$$

Can we factorise the quadratic $3x^2 - 2x + 5$?

$$\begin{aligned}3x^2 - 2x + 5 &= 3\big[x^2 - \tfrac{2}{3}x + \tfrac{5}{3}\big]\\ &= 3\big[\big(x^2 - \tfrac{2}{3}x + \tfrac{1}{9}\big) - \tfrac{1}{9} + \tfrac{5}{3}\big]\\ &= 3\big[\big(x - \tfrac{1}{3}\big)^2 + \tfrac{14}{9}\big]\end{aligned}$$

This is in the form of $a^2 + b^2$ and cannot be expressed in the form $a^2 - b^2$, so we cannot proceed any further, since $a^2 + b^2$ has no real factors.

More than two factors

Moving onto expressions with more than two factors (expansion of cubic polynomials).

Expand the product $(x + 2)(x - 3)(x - 2)$.

$$\begin{aligned}(x + 2)(x - 3)(x - 2) &= (x + 2)(x^2 - 5x + 6)\\ &= x(x^2 - 5x + 6) + 2(x^2 - 5x + 6)\\ &= x^3 - 5x^2 + 6x + 2x^2 - 10x + 12\\ &= x^3 - 3x^2 - 4x + 12.\end{aligned}$$

By repeated use of the distributive law we can expand;

$$\left.\begin{aligned}(a + b)^3 &= (a + b)(a + b)(a + b)\\ &= a^3 + 3a^2b + 3ab^2 + b^3\\ (a - b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3\end{aligned}\right\}\ \text{Useful Identities}$$

Examples:

$$\begin{aligned}(x + 2)^3 &= (x)^3 + 3(x)^2 2 + 3x(2)^2 + (2)^3\\ &= x^3 + 6x^2 + 12x + 8.\\ (2p - 3)^3 &= (2p)^3 + 3(2p)^2(-3) + 3(2p)(-3)^2 + (-3)^3\\ &= 8p^3 - 36p^2 + 54p - 27.\end{aligned}$$

Futher Useful Identities: Cubic Factors.

$$\begin{aligned}a^3 - b^3 &= (a - b)(a^2 + ab + b^2) &&\text{the difference of two cubes}\\ a^3 + b^3 &= (a + b)(a^2 - ab + b^2) &&\text{the sum of two cubes}\end{aligned}$$

Examples: Factorise the following expressions.

1. $x^3 - 8 = x^3 - 2^3 = (x - 2)(x^2 + 2x + 4)$
2. $$\begin{aligned}64x^3 + 27y^3 &= (4x)^3 + (3y)^3\\ &= (4x + 3y)\big((4x)^2 - (4x)(3y) + (3y)^2\big)\\ &= (4x + 3y)(16x^2 - 12xy + 9y^2).\end{aligned}$$

As an exercise, try to factorise $x^6 + 1$ by first writing it as $\big(x^3\big)^2 + 1^2$, then by writing it as $\big(x^2\big)^3 + 1^3$.

EXERCISE 9: Factorise the following, where possible:

1. $x^2 - 5x + 6$
2. $x^2 + 7x + 12$
3. $2x^2 + 5x - 3$
4. $12x^2 + x - 1$
5. $x^2 + 9$
6. $(x + 4)^2 + 2(x + 4) + 1$
7. $x^2 - 6$
8. $p^2 + 16p - 22$
9. $5v^2 - 10v + 21$
10. $x^2 - 14x + 49$
11. $81 + 18y^2 + y^4$
12. $x^2 - 64$
13. $16x^2 - 49y^2$
14. $x^3 - 27$
15. $y^3 + 8$
16. $x^2 - 8$
17. $r^2 + 2r - 33$
18. $n^2 + 13n + 29$
19. $50x^2 - 40x + 8$
20. $16y^2 - x^2 - 6x - 9$
21. $(2x + y)^2 - (3x - 2y)^2$
22. $e^{2t} + 3e^t + 2$
23. $x^6 - 1$
24. $3x^2 - 9x$
25. $m^2 - 12m + 26$
26. $r^2 - 9r - 29$

1.2.3 Polynomials

The function $P(x) = ax^2 + bx + c,\ a \neq 0$ is a quadratic polynomial or a *polynomial of degree* 2. A function $P(x)$ is called a *polynomial of degree n* if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,\ a_n \neq 0$$

where n is a **non-negative integer**, $a_0, a_1, a_2, \ldots, a_n$ are constants (called coefficients), and the leading coefficient $a_n \neq 0$.

$$\begin{aligned}
P(x) &= ax + b && \text{is of degree 1, a linear function.}\\
P(x) &= 3x^7 + 4x^5 - 2 && \text{is a polynomial of degree 7.}\\
f(x) &= 6x^{3/2} - \frac{2}{x} + 4 && \text{is not a polynomial.}
\end{aligned}$$

($3/2$: not an integer; $\frac{2}{x}$: negative power)

Once $P(x)$ is defined, the coefficients will be fixed, but the value of x can alter. To highlight this point, consider evaluating the polynomial at different $x-$values:

$$\begin{aligned}
\text{If} \qquad P(x) &= 3x^2 + 7x + 4\\
\text{then} \qquad P(0) &= 3(0)^2 + 7(0) + 4 = 4\\
P(1) &= 3(1)^2 + 7(1) + 4 = 14\\
P(a) &= 3a^2 + 7a + 4\\
P(-1) &= 3(-1)^2 + 7(-1) + 4 = 0.
\end{aligned}$$

1.2.4 Manipulating Polynomials

Polynomials can be added, subtracted, multiplied, and divided. Addition (subtraction) is performed by adding (subtracting) the coefficients of like terms. e.g.,

$$\begin{aligned}
P(x) &= 7x^3 + 4x^2 + x - 2\\
Q(x) &= 3x^4 + 5x^3 - 8x^2 - 9x + 1\\
\text{then,} \qquad P(x) + Q(x) &= 3x^4 + (7+5)x^3 + (4-8)x^2 + (1-9)x + (-2+1)\\
&= 3x^4 + 12x^3 - 4x^2 - 8x - 1.
\end{aligned}$$

Multiplication: $P(x) \times Q(x)$. If the polynomials are large, then set up in the table form

$$\begin{array}{rrrrrrrr}
 & & & 3x^4+ & 5x^3- & 8x^2- & 9x+ & 1\\
 & & & & 7x^3+ & 4x^2+ & x- & 2\\
\hline
 & & & -6x^4 & -10x^3 & +16x^2 & +18x & -2\\
 & & 3x^5+ & 5x^4- & 8x^3- & 9x^2+ & x & \\
 & 12x^6 & +20x^5 & -32x^4 & -36x^3+ & 4x^2 & & \\
21x^7 & +35x^6 & -56x^5 & -63x^4+ & 7x^3 & & & \\
\hline
21x^7 & +47x^6 & -33x^5 & -96x^4 & -47x^3 & +11x^2 & +19x & -2
\end{array}$$

Division of Polynomials: We will look at two methods;

1. $\dfrac{P(x)}{Q(x)}$ for any polynomials — long division algorithm.
2. $\left.\begin{array}{l}\text{Remainder Theorem}\\ \text{Factor Theorem}\end{array}\right\}$ — when dividing by linear factors.

1.2.5 The Long Division Algorithm

The operation of division of polynomials is carried out in the same way as "long division" of integers. Recall the method for calculating $214679 \div 53$.

$$\begin{array}{rr}
 & 4\,0\,5\,0 \\
5\,3\,\big) & 2\,1\,4\,6\,7\,9 \\
- & \underline{2\,1\,2}\downarrow\downarrow\downarrow \\
 & 2\,6\,7\downarrow \\
- & \underline{2\,6\,5}\downarrow \\
 & 2\,9
\end{array}
\qquad \text{remainder} \neq 0, \ \therefore 53 \text{ is not a factor of } 4050$$

53 is divided into the first 3 digits, '214', to give 4. The remainder is calculated by subtracting 4×53 (212) from 214 to get 2. The next step is to 'bring down' the next digit, '6', and repeat the procedure. Note that, in this example, 53 is larger than 26, so the quotient is entered as '0' and the next digit, '7' is brought down. When the remainder is less than the divisor, the procedure stops.

So we can write $214679 = (4050 \times 53) + 29$, or

$$\frac{214679}{53} = 4050 + \frac{29}{53}$$

Examples of Polynomial Division:

1. $(x^2 + 9x + 4) \div (x + 1)$.

The method is similar for polynomial division. In this case, x is divided into the first term, x^2, to give x. The remainder, $8x$, is calculated by multiplying x by $x+1$ and subtracting the result from $x^2 + 9x$. Then the next term, 4, is brought down and the procedure is repeated. When the degree of the remainder is less than the degree of the divisor, the procedure must stop.

$$\begin{array}{rr}
 & x \ + 8 \\
x+1\,\big) & x^2 + 9x + 4 \\
- & \underline{x^2 + \ \ x} \quad \downarrow \\
 & 8x + 4 \\
- & \underline{8x + 8} \\
 & -4
\end{array}
\qquad \text{remainder } \neq 0, \ \therefore (x+1) \text{ is not a factor of } x^2 + 9x + 4$$

Thus, $\dfrac{x^2 + 9x + 4}{x + 1} = x + 8 + \dfrac{-4}{x+1}$.

Or, $x^2 + 9x + 4 = (x + 8)(x + 1) - 4$

2. $\dfrac{x^3+4x^2+3x-4}{x^2+2x+1}$.

$$
\begin{array}{rl}
 & x+2 \\
x^2+2x+1 \,\big) & x^3+4x^2+3x-4 \\
- & \underline{x^3+2x^2+x} \ \downarrow \\
 & 2x^2+2x-4 \\
- & \underline{2x^2+4x+2} \\
 & -2x-6
\end{array}
$$

Thus, $\dfrac{x^3+4x^2+3x-4}{x^2+2x+1} = x+2-\dfrac{2x+6}{x^2+2x+1}$.

3. $(2x^4+6x+11) \div (x-2)$ (missing powers of x)

Set the problem out as follows

$$
\begin{array}{rl}
 & 2x^3+4x^2+8x\ \ +22 \\
x-2\,\big) & 2x^4+0x^3+0x^2+\ 6x+11 \\
- & \underline{2x^4-4x^3} \quad \downarrow \quad \downarrow \quad \downarrow \\
 & 4x^3+0x^2 \quad \downarrow \quad \downarrow \\
- & \underline{4x^3-8x^2} \quad \downarrow \quad \downarrow \\
 & 8x^2+\ 6x \quad \downarrow \\
- & \underline{8x^2-16x} \quad \downarrow \\
 & 22x+11 \\
- & \underline{22x-44} \\
 & 55
\end{array}
$$

Thus, $\dfrac{2x^4+6x+11}{x-2} = 2x^3+4x^2+8x+22+\dfrac{55}{x-2}$

or $2x^4+6x+11 = (x-2)(2x^3+4x^2+8x+22)+55$.

EXERCISE 10: Divide the following polynomials:

1. $\dfrac{2x^3+3x^2-2x+1}{x-1}$
2. $\dfrac{x^3+4x^2+3x-4}{x^2+2x-1}$
3. $\dfrac{x^3+2x^2-x-2}{x-1}$
4. $\dfrac{x^3+2x+8}{x+1}$
5. $\dfrac{3x^4-9x^3-x^2+5x-10}{x-3}$
6. $\dfrac{10x^3-25x^2+12x-5}{2x-5}$
7. $\dfrac{2x^3-9x^2+13x}{x^2-3x+4}$
8. $\dfrac{(x-1)(4x^2+2x+2)}{2x+1}$

Can we avoid having to use long division? Yes, but only when the denominator of the division operation is linear.

1.2.6 Remainder Theorem

We will now present a known fact about the remainder term which comes out of polynomial long division in the case where the divisor is the linear polynomial $(x-a)$. Firstly the result is presented in the form of a theorem and then the theorem is justified through mathematical proof. Mathematics is often presented in a theorem-proof framework. The mathematical statements that get labelled as a theorem are statements of significance in mathematics. Thus it is the important results of mathematics that get called theorems. A proof of a theorem presents a justification for the result. The proof should use only known results and properties of mathematics. Preceding this is a preamble that leads in to the statement of the theorem - it gives the observations that lead to the theorem and its proof.

Note from example 1 in the previous section that

$x^2 + 9x + 4 = (x + 1)(x + 8) - 4.$

Choosing $x = -1$ yields $1 - 9 + 4 = -4$. Also note from example 3 in the previous section that

$2x^4 + 6x + 11 = (x - 2)(2x^3 + 4x^2 + 8x + 22) + 55.$

Choosing $x = 2$ yields $32 + 12 + 11 = 55$.

In each case the original polynomial can be expressed as a product of the divisor and a second polynomial, plus a constant. The constant can be determined by evaluating the original polynomial at the value of x that makes the divisor zero.

Theorem. If the polynomial $P(x)$ is divided by the linear polynomial $(x - a)$ until a remainder, R is obtained that does not involve any power of x, then $R = P(a)$.

Proof. When the polynomial $P(x)$ is divided by the linear polynomial $(x - a)$ until a remainder, R is obtained that does not involve any power of x, then we must have

$P(x) = (x - a)Q(x) + R$

for some polynomial $Q(x)$ and constant, R. If we substitute $x = a$ in this equation, we get

$P(a) = (a - a)Q(a) + R = 0 \times Q(a) + R = R$ (the remainder)

It follows that the remainder, R, is just $P(a)$ and the theorem is proved.

This theorem enables us to find the remainder without actually performing a long division.

Example 1 (again): $(x^2 + 9x + 4) \div (x + 1)$.

Let $P(x) = x^2 + 9x + 4$.

We are dividing by $x + 1$, so we use $a = -1$ in the formula.

$P(-1) = (-1)^2 + 9(-1) + 4 = 1 - 9 + 4 = -4$ as before.

This is the remainder after dividing $P(x)$ by $x + 1$.

Example 3 (again): $(2x^4 + 6x + 11) \div (x - 2)$.

Let $P(x) = 2x^4 + 6x + 11$.
We divide by $x - 2$, so we use $a = 2$ in the formula.

$$P(2) = 2(2)^4 + 6 \times 2 + 11 = 21 + 12 + 11 = 55 \qquad \text{as before.}$$

Following on from the Remainder Theorem; consider $14 \div 7 = 2 + 0$ (remainder). Since the remainder is equal to zero, we say that 7 is a factor of 14. If a polynomial $P(x)$ is divided by $x - a$ and there is no reminder ($R = 0$), then $x - a$ is a factor of $P(x)$. We have a special case of the Remainder Theorem, which is important enough to have its own named theorem.

1.2.7 The Factor Theorem

If, for a polynomial $P(x)$; $P(a) = 0$; then $(x - a)$ is a factor of $P(x)$, and a is called a *zero* or *root* of $P(x)$. Thus by finding the zeros of a polynomial, we can find the factors of the polynomial.

> **Factor Theorem.** If for $x = a$, $P(a) = 0$, then $x = a$ is a root and $(x - a)$ is a factor of $P(x)$.

EXAMPLE 1: Find the zeros of $P(x) = x^3 - x^2 - 14x + 24$ then find the linear factors.

There are at most 3 factors — $(x-a)(x-b)(x-c)$. The constant term is 24, so the product abc must be equal to 24. Therefore, we look for factors of 24, namely $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$ and check if any of these is a root of $P(x)$.

Now, $P(1) \neq 0$ and $P(-1) \neq 0$, but $P(2) = 0$ $\quad \therefore (x - 2)$ is a factor of $P(x)$.

Using long division, Factor Theorem, or otherwise, we can then find the remaining factors.

$$\begin{aligned} P(x) = x^3 - x^2 - 14x + 24 &= (x - 2)(Ax^2 + Bx + C) \\ &= (x - 2)(x^2 + x - 12) \\ &= (x - 2)(x - 3)(x + 4). \end{aligned}$$

The factors of $P(x)$ are $(x - 2)(x - 3)(x + 4)$. The zeros of $P(x)$ are $x = 2, 3, -4$.
EXAMPLE 2: Factorise $P(x) = 2x^3 + 7x^2 - 10x - 24$.

$$\begin{aligned} &P(1) = 2 + 7 - 10 - 24 \neq 0 \\ &P(2) = 2 \times 8 + 7 \times 4 - 10 \times 2 - 24 = 16 + 28 - 20 - 24 = 0 \end{aligned}$$

$\therefore (x - 2)$ is a factor of $P(x)$.

$$\begin{aligned} 2x^3 + 7x^2 - 10x - 24 &= (x - 2)(Ax^2 + Bx + C) \\ &= (x - 2)(2x^2 + 11x + 12) \\ &= (x - 2)(2x + 3)(x + 4). \end{aligned}$$

EXERCISE 11:

1. If $P(x) = 3x^3 + 13x^2 + 6x - 12$, without using long division, find the remainder when $P(x)$ is divided by
   (a) $x - 2$ (b) $x + 3$

2. Use the **Factor Theorem,** or otherwise, to find linear factors for the following
   (a) $x^3 + 2x^2 - 41x - 42$ (b) $2x^3 - 3x^2 - 11x + 6$

3. (a) Show that the remainder when $6x^3 + 7x^2 - 3x + 2$ is divided by $2x - 1$ is 3.
   Hint: Recall factors of $2x - 1$ are $2\left(x - \frac{1}{2}\right)$.
   (b) Find the remainder when $3x^3 - x$ is divided by $2x + 1$.

4. In each of the following, find the quotient, $Q(x)$, and the remainder, $R(x)$, when the first polynomial, $P(x)$, is divided by the second polynomial, $D(x)$.
   (a) $P(x) = x^3 - x + 6$, $D(x) = x + 2$ (b) $P(x) = x^3 - 6x^2 - x - 8$, $D(x) = x - 4$

5. Use the factor theorem to determine whether or not the first polynomial is a factor of the second
   (a) $x - 1$; $2x^2 + x - 3$ (b) $x - 2$; $x^3 - 3x + 2$ (c) $3x + 1$; $3x^3 + 4x^2 + 4x + 1$

6. Factorise the following
   (a) $x^3 - 6x^2 + 11x - 6$ (b) $2x^3 - 7x^2 + 7x - 2$ (c) $x^3 + x^2 + x + 1$

7. In each of the following, find the value of k if $D(x)$ is a factor of $P(x)$
   (a) $P(x) = x^3 + kx^2 + 2x - 5$, $D(x) = x - 1$
   (b) $P(x) = 2x^3 + x^2 + kx + 2k^2$, $D(x) = x + 1$
   (c) $P(x) = 2x^3 - x^2 + (k - 1)x + 1$, $D(x) = 2x + 1$

8. When $x^3 + x^2 + kx - 3$ is divided by $x - 3$, the remainder is 30. Find k.

9. When $3x^3 - ax^2 - bx + 1$ is divided by $x - 2$ the remainder is 15. If $x - 1$ is a factor of the given polynomial, find the values of a and b.

10. If $x + 1$ is a factor of $a_0x^3 + a_1x^2 + a_2x + a_3$, show that $a_0 + a_2 = a_1 + a_3$.

1.2.8 Solution of Polynomial Equations

Quadratic equations

To solve quadratic equations, $ax^2 + bx + c = 0$ $(a \neq 0)$, we factorise the LHS into linear factors (if possible) and then use the principle that if $p \times q = 0$ then either $p = 0$ or $q = 0$.

Example: Solve $3x^2 + 5x = 0$.

$$3x^2 + 5x = x(3x + 5) = 0.$$

Either $x = 0$ or $3x + 5 = 0$, so that $x = 0$ and $x = -\frac{5}{3}$ are solutions of $3x^2 + 5x = 0$.

The Quadratic Formula

If the factors cannot be found by trial and error, we use the quadratic formula;
If $ax^2 + bx + c = 0$, then $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$.

$$\therefore \quad \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right) = 0 \qquad \text{complete the square}$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2 - 4ac}{4a^2}\right)$$

$$\therefore \quad \left(x + \frac{b}{2a}\right) = \pm\sqrt{\left(\frac{b^2 - 4ac}{4a^2}\right)}$$

$$\therefore \quad x = -\frac{b}{2a} \pm \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right).$$

Thus $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

These are the roots, solutions or zeros of the quadratic equation $ax^2 + bx + c = 0$. There may be zero, one or two of them depending on the whether $b^2 - 4ac$ is negative, zero or positive.

Note that an alternate approach to this derivation uses the factors of a difference of squares after completion of the square.

Cubic and higher order polynomials

To solve cubics or higher order polynomials, use the Remainder and Factor Theorems to find the factors, then the zeros.

Example 1: Solve $f(x) = 3x + 7x^2 - 6x^3 = 0$.

We have $\quad x(3 + 7x - 6x^2) = 0$

$$x(3x + 1)(2x - 3) = 0$$

so $x = 0, -\dfrac{1}{3}, \dfrac{3}{2}$.

Example 2: Solve $P(x) = x^3 + 3x^2 - 4x - 12 = 0$.

Check if a factor of 12 is a root of the cubic.

$$\begin{aligned} P(1) &= 1 + 3 - 4 - 12 \neq 0. \\ P(2) &= 8 + 3(4) - 4(2) - 12 = 0 && \therefore\ (x-2) \text{ is a factor.} \\ x^3 + 3x^2 - 4x - 12 &= (x-2)(x^2 + 5x + 6) \\ &= (x-2)(x+2)(x+3). && \text{(these are the factors)} \end{aligned}$$

So the solutions of $P(x)$ are $x = -3, \pm 2$.

EXERCISE 12:

1. Solve for **Real** x:
   (a) $x^2 - 7x + 6 = 0$
   (b) $2x^2 + 5x - 12 = 0$
   (c) $(2x-1)(x-2) = 5$
   (d) $9x^2 + 8x + 1 = 0$
2. Find **rational** solutions for $f(x) = 0$, where
   (a) $f(x) = 2x^3 + 7x^2 - 17x - 10$
   (b) $f(x) = x^4 + 4x^3 + 4x^2$
3. Solve the following
   (a) $x(2x-1) = 3$
   (b) $x^2 = 20(x-5)$
   (c) $2x(x-1) = 3$
   (d) $\frac{x}{4} + \frac{x-6}{x} = \frac{1}{2}$
4. Django thinks of a positive number between 1 and 10, then adds to it the square of itself to reach the sum of 72. What number did Django think of?
5. The length of a large room is three times its width. What is the length if the area is 192 m^2?
6. A cyclist rides 120km from Townsville to Ingham at a uniform rate (speed). If she had ridden 3 km/hr slower, her trip would have taken 2 hours longer. At what rate (speed) did she ride?

1.2.9 Introduction to Partial Fractions

A rational function is of the form

$$f(x) = \frac{P(x)}{Q(x)}, \qquad Q(x) \neq 0$$

where $P(x)$ and $Q(x)$ are polynomials. e.g., $\dfrac{6}{x-7}$ and $\dfrac{2x-1}{x^3+6x+1}$ are rational functions.

We can combine two rational functions into a single fraction by addition. e.g.,

$$\begin{aligned}\frac{6}{x-2} + \frac{3}{x-5} &= \frac{6(x-5)+3(x-2)}{(x-2)(x-5)} \qquad ((x-2)(x-5) \text{ is a common denominator})\\ &= \frac{6x-30+3x-6}{(x-2)(x-5)}\\ &= \frac{9x-36}{(x-2)(x-5)}.\end{aligned}$$

We are often required to reverse this process and express a rational function as a sum of simpler functions. For example, to express $\dfrac{9x-6}{(x-2)(x-5)}$ as a sum of simple terms, write

$$\frac{9x-36}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}.$$

We then need to find A and B. This technique is called the method of Partial Fractions. The methods is sometimes described as *decomposing* the rational function into partial fractions.

Before we launch into a description of how partial fractions are computed it would be useful for you to look at the process of finding a common denominator in adding/subtracting rational functions. Compute each of the following:

1. $\dfrac{1}{x} + \dfrac{1}{x+1} =$
2. $\dfrac{1}{2x} + \dfrac{3}{x+1} =$
3. $\dfrac{1}{2(x-3)} - \dfrac{2}{x+1} =$
4. $\dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} =$
5. $\dfrac{1}{x} - \dfrac{x}{x^2+4} =$
6. $\dfrac{3}{x-2} + \dfrac{5}{x^2+4x+10} =$
7. $\dfrac{-2}{(x-4)^2} + \dfrac{4}{x^2+4x+10} =$
8. $\dfrac{1}{x+2} + \dfrac{1}{(x+2)^3} =$

In decomposing into partial fractions, we commence with a rational function, composed of polynomials on the numerator and denominator. The polynomials have the property that the highest power of the variable on the numerator is lower than the highest power of the variable on the denominator. (If this isn't the case we can perform polynomial division to place the rational function in that form.) In order to decompose into partial fractions, it must be the case that the denominator can be factorised. If that cannot happen then there is no partial fraction decomposition.

Linear Factors

In this first case, the denominator is factorised to give unique linear factors.

Example 1: Express $\dfrac{5x+1}{x^2+x-2}$ as a sum of simple fractions.

METHOD 1: Firstly we must factorize the denominator:

$$\frac{5x+1}{x^2+x-2} = \frac{5x+1}{(x+2)(x-1)}$$

Each factor in the denominator gives rise to a term in the expansion. Thus we write

$$\begin{aligned}\frac{5x+1}{(x+2)(x-1)} &= \frac{A}{x+2} + \frac{B}{x-1}\\ &= \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}\end{aligned}$$

We now multiply both sides by the denominator to get

$$\begin{aligned}5x+1 &= A(x-1)+B(x+2)\\ &= Ax - A + Bx + 2B\\ &= (A+B)x - A + 2B.\end{aligned}$$

Now this equation must be true for *all* values of x. This means that the coefficient of x on both sides must be the same and the constant terms must also be the same. Equating coefficient of powers of x gives

$$\underline{x} \qquad 5 = A + B \tag{1.1}$$

$$\underline{1} \qquad 1 = -A + 2B \tag{1.2}$$

We can add equations (1.1) and (1.2) to eliminate A. This gives

$$6 = 3B \qquad \therefore \boxed{B = 2}$$

Substitute into (1.1) $\qquad 5 = A + 2 \qquad \therefore \boxed{A = 3}$

$$\therefore \quad \frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}.$$

METHOD 2: (COVER-UP METHOD)

$$\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

Again, multiply through by $(x+2)(x-1)$ to get

$$5x+1 = A(x-1) + B(x+2)$$

This is true for all x, so we choose values of x which simplify our problem.
i.e., $x = 1$ and $x = -2$.

$\underline{x=1}$ $\quad 5+1 = 0+3B \quad \therefore \boxed{B=2}$

$\underline{x=-2}$ $\quad -10+1 = -3A+0 \quad \therefore \boxed{A=3}$

$$\therefore \quad \frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}.$$

Note that each chosen value of x makes one of the unknowns disappear from the equation. This is why it is called the cover-up method.

EXAMPLE 2: Expand $\dfrac{54}{(x^2+x-20)(x-1)}$ using partial fractions.

Factorizing the denominator gives

$$\frac{54}{(x+5)(x-4)(x-1)} = \frac{A}{x+5} + \frac{B}{x-4} + \frac{C}{x-1}$$

Multiplying through by $(x+5)(x-4)(x-1)$ we get

$$54 = A(x-4)(x-1) + B(x+5)(x-1) + C(x+5)(x-4)$$

Let $x = 4$, $\quad 54 = 9 \times 3 \times B \quad \therefore \boxed{B=2}$

$x = 1$, $\quad 54 = 6 \times (-3) \times C \quad \therefore \boxed{C=-3}$

$x = -5$, $\quad 54 = (-9) \times (-6) \times A \quad \therefore \boxed{A=1}$

$$\therefore \quad \frac{54}{(x+5)(x-4)(x-1)} = \frac{1}{x+5} + \frac{2}{x-4} - \frac{3}{x-1}.$$

Non-Linear factors

Example: Expand $\dfrac{2x^2-2x+3}{x^3-x^2-x-2}$ using partial fractions.

$$\frac{2x^2-2x+3}{x^3-x^2-x-2} = \frac{2x^2-2x+3}{(x-2)(x^2+x+1)} \qquad \text{(long division)}$$

There are two factors in the denominator. Each will give a term in the expansion which will have the form

$$\frac{2x^2-2x+3}{(x-2)(x^2+x+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1}$$

Note: For a quadratic denominator we use a linear numerator i.e., the top line is always 1 degree less than the bottom. e.g. If the denominator was a quartic, the numerator would be Bx^3+Cx^2+Dx+E (or cubic).

$$\therefore \quad \frac{2x^2-2x+3}{(x-2)(x^2+x+1)} = \frac{A(x^2+x+1)+(Bx+C)(x-2)}{(x-2)(x^2+x+1)}$$

$$\therefore \quad 2x^2-2x+3 = A(x^2+x+1)+(Bx+C)(x-2)$$

Note: The cover-up method doesn't help here, so we expand the brackets.

$$= Ax^2+Ax+A+Bx^2-2Bx+Cx-2C$$

Equating the coefficients of powers of x

$\underline{x^2}$ $\qquad 2 = A+B \qquad (1.3)$

$\underline{x}$ $\qquad -2 = A-2B+C \qquad (1.4)$

$\underline{1}$ $\qquad 3 = A \quad -2C \qquad (1.5)$

This system of 3 equations with 3 unknowns can be reduced to a system of 2 equations with 2 unknowns by eliminating C in (1.4) using equation (1.5).

$2\times(1.4)$ $\qquad -4 = 2A-4B+2C \qquad (1.6)$

$(1.6)+(1.5)$ $\qquad -1 = 3A-4B \qquad (1.7)$

Now equations (1.3) and (1.7) have only A and B. Hence we now have two equations with two unknowns. Eliminating B from equation (1.7)

$4\times(1.3)+(1.7)$ $\qquad 8-1 = 4A+3A$

$\therefore \quad 7 = 7A \qquad$ *i.e.* $\boxed{A=1}$

Substitute into (1.3) $\qquad 2 = 1+B \qquad \therefore \boxed{B=1}$

Substitute into (1.4) $\qquad -2 = 1-2+C \qquad \therefore \boxed{C=-1}$

$$\therefore \quad \frac{2x^2-2x+3}{(x-2)(x^2+x+1)} = \frac{1}{x-2} + \frac{x-1}{x^2+x+1}.$$

Repeated Factors

When we have repeated factors (i.e. factors raised to a power), we must take every possibility into account when calculating the solution.

EXAMPLE: Expand $\dfrac{3x+4}{x^3+5x^2+8x+4}$ using partial fractions.

Factorizing the denominator and separating the factors gives

$$\frac{3x+4}{(x+2)^2(x+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+1}$$

Note that both $(x+2)$ and $(x+2)^2$ are included as possible denominators.

Multiplying through by $(x+2)^2(x+1)$ we get

$$3x+4 = A(x+2)(x+1) + B(x+1) + C(x+2)^2$$

We use the standard cover-up method to determine C

$$\text{Let } x=-1, \qquad -3+4 = 0+0+C \qquad \therefore \boxed{C=1}$$

Normally we would now multiply out the brackets and equate the coefficients of powers of x to calculate A and B. Here we will demonstrate an alternate method. As we know the left hand side of the equation is equal to the right hand side, we can substitute any value for x into both sides to determine another equation. For example:

$$\begin{aligned} &\text{Let } x=0, \quad && 4 = 2A+B+4 \qquad && (1) \qquad \text{Note that } C=1 \\ &\text{Let } x=1, \quad && 7 = 6A+2B+9 \qquad && (2) \end{aligned}$$

We now solve these simultaneous equations as before to determine A and B.
Equations (1) and (2) simplify to

$$\begin{aligned} 2A+B &= 0 \qquad && (3) \\ 6A+2B &= -2 \qquad && (4) \end{aligned}$$

$$\begin{aligned} (4) - 2\times(3) \qquad 6A-4A+2B-2B &= -2+0 \\ 2A &= -2 \\ \therefore A &= -1 \end{aligned}$$

Substituting A into (3) gives

$$\begin{aligned} -2+B &= 0 \\ \therefore B &= 2 \end{aligned}$$

$$\therefore \frac{3x+4}{x^3+5x^2+8x+4} = \frac{-1}{x+2} + \frac{2}{(x+2)^2} + \frac{1}{x+1}$$

EXERCISE 13: Express each of the following as partial fractions:

1. $\dfrac{5x+1}{(x-1)(x+2)}$

2. $\dfrac{x+5}{(x+1)(x+3)}$

3. $\dfrac{2x^2-3x-11}{(x+2)(x-1)(x+1)}$

4. $\dfrac{7x-x^2}{(x-1)^2(x+2)}$

5. $\dfrac{15x+28}{3x^2+25x+8}$

6. $\dfrac{5x^2+26x+29}{x^3+6x^2+11x+6}$

7. $\dfrac{x^4+x^3-x^2-x+1}{x^3-x}$

8. $\dfrac{-2x^2+3x-5}{(x^2+4)(x+1)}$

9. $\dfrac{2x^3+9x^2+4}{x^2(x^2+4)(x-1)}$

1.2.10 Inequalities

There are many examples where we need to deal with inequalities (rather than equalities). The following notation is used.

$a < b$ is read as "a is less than b"
$a > b$ is read as "a is greater than b"
$a \leqslant b$ is read as "a is less than or equal to b"
$a \geqslant b$ is read as "a is greater than or equal to b"

Interval Notation

Certain sets of real numbers can be represented by intervals of the real number line. The following three statements are equivalent.

(i) $x \geqslant a$

(ii) $\{x\colon\ x \geqslant a,\ x \in \mathbb{R}\}$ or $\{x\colon\ x \geqslant a\}$

(iii) $x \in [a, \infty)$.

The set $[a, \infty)$ is called an **interval** and corresponds geometrically to a portion of the real number line. Intervals can be **open** or **closed** or neither. For instance, the following is an example of an **open interval**.

$$\{x\colon\ a < x < b\} = (a, b)$$

This set consists of all numbers between a and b, and is represented on the number line in the following way

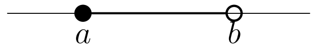
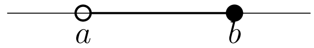
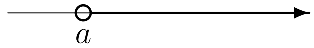
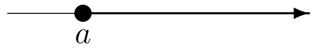
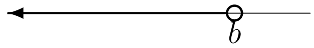
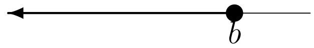
Notice the endpoints are excluded. This is indicated by the round brackets ( ) and by the open dots on the diagram.

The **closed interval** from a to b is the set

$$\{x\colon\ a \leqslant x \leqslant b\} = [a, b]$$

The endpoints are included here. This is indicated by the square brackets [ ] and by the solid dots on the diagram.

We can also consider infinite intervals or intervals which include only one endpoint. The possible types of intervals are listed below.

Notation	Set Description	Picture
(a, b)	$\{x\colon\ a < x < b\}$	
$[a, b]$	$\{x\colon\ a \leqslant x \leqslant b\}$	
$[a, b)$	$\{x\colon\ a \leqslant x < b\}$	
$(a, b]$	$\{x\colon\ a < x \leqslant b\}$	
(a, ∞)	$\{x\colon\ x > a\}$	
$[a, \infty)$	$\{x\colon\ x \geqslant a\}$	
$(-\infty, b)$	$\{x\colon\ x < b\}$	
$(-\infty, b]$	$\{x\colon\ x \leqslant b\}$	
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

Note: The set $[a, \infty]$ does not exist. This is because the notation $[a, \infty]$ means the set of all numbers greater than or equal to a and less than or equal to infinity. However, infinity is considered to be a quantity greater than any fixed number, and so is not a number itself.

The following rules apply when working with inequalities.

(1) If $a < b$, then $a + c < b + c$ and $a - c < b - c$

(2) If $a < b$ and $c < d$, then $a + c < b + d$

(3) If $a < b$ and $c > 0$, then $ac < bc$ and $\dfrac{a}{c} < \dfrac{b}{c}$

(4) If $a < b$ and $c < 0$, then $ac > bc$ and $\dfrac{a}{c} > \dfrac{b}{c}$

(5) If $0 < a < b$, then $\dfrac{1}{a} > \dfrac{1}{b}$

In particular, to summarise (4) and (5):

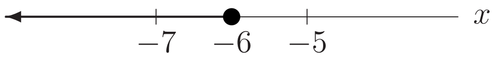
When multiplying or dividing an inequality by a negative number, or taking the reciprocal, reverse the direction of the inequality sign.

Solving Linear Inequalities

The solution of an inequality consists of the set (or sets) of values for the variable which make the inequality a true statement. The procedure used is similar to that used in solving standard equations.

EXAMPLE 1: Solve $3 - 2x \geqslant 15$.

$$\begin{aligned} 3 - 2x &\geqslant 15 \\ \therefore \quad -2x &\geqslant 12 \\ \therefore \quad x &\leqslant -6 \qquad \text{(as we are dividing by a negative number)} \\ \text{or} \quad x &\in (-\infty, -6]. \end{aligned}$$

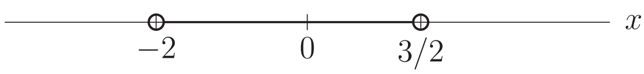


EXAMPLE 2: Solve $\frac{3}{2}(1 - x) > \frac{1}{4} - x$.

$$\begin{aligned} \frac{3}{2}(1 - x) &> \frac{1}{4} - x \\ \therefore \quad 6(1 - x) &> 1 - 4x \\ \therefore \quad 6 - 6x &> 1 - 4x \\ \therefore \quad 5 &> 2x \\ \text{i.e.} \quad x < \frac{5}{2} \quad &\text{or} \quad x \in (-\infty, 5/2). \end{aligned}$$

EXAMPLE 3: Solve $-1 < 2x + 3 < 6$.

$$\begin{aligned} -1 < 2x + 3 &< 6 \\ \therefore \quad -4 < 2x &< 3 \\ \therefore \quad -2 < \ x &< \frac{3}{2} \\ \text{or} \quad x &\in (-2, 3/2). \end{aligned}$$



Example 4: Solve $2x < x - 4 \leqslant 3x + 8$.

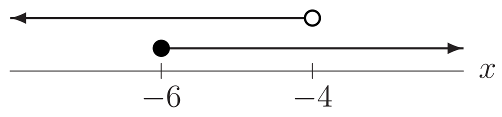
We can treat this problem as two separate inequalities

$$2x < x - 4 \qquad \text{and} \qquad x - 4 \leqslant 3x + 8$$

We now solve each inequality separately but our final answer must satisfy both inequalities

$$\begin{aligned} 2x &< x - 4 \qquad &\text{and} \qquad x - 4 &\leqslant 3x + 8 \\ x &< -4 & -2x &\leqslant 12 \\ & & x &\geqslant -6. \end{aligned}$$

The diagram of these two solutions is



We need the values of x that satisfy both inequalities, so the solution is $-6 \leqslant x < -4$ or $x \in [-6, -4)$.

1.2.11 Modulus or Absolute Value

Definition 1. The modulus or absolute value of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line.

Distances are always positive (or zero) thus $|a| \geqslant 0$ for all numbers a.

Definition 2. Let $a \in \mathbb{R}$. The modulus or absolute value $|a|$ is defined by

$$|a| = \begin{cases} a & \text{if } a \geqslant 0 \\ -a & \text{if } a < 0. \end{cases}$$

Definition 3. $|a| = \sqrt{a^2}$, where $\sqrt{\ }$ denotes the positive square root.

Properties of Absolute Values

Suppose $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}$, then

(i) $|ab| = |a||b|$

(ii) $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

(iii) $|a^n| = |a|^n$.

You may find some of the following identities useful when dealing with absolute values.

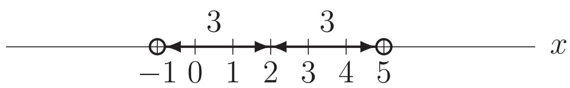
1. $|x| = a$ if and only if $x = \pm a$
2. $|x| < a$ if and only if $-a < x < a$
3. $|x| > a$ if and only if $x < -a$ or $x > a$.

EXAMPLE 1: Solve $|x - 2| < 3$.

From identity 2 we get

$$\begin{aligned} & -3 < x - 2 < 3 \\ \therefore \quad & -1 < \quad x \quad < 5 \end{aligned}$$

Alternatively we could have solved the problem geometrically by interpreting $|x - a|$ as *the distance of x from a*. Therefore $|x - 2| < 3$ can be interpreted as "the distance of x from 2 is less than 3". From the diagram, we get the same result.

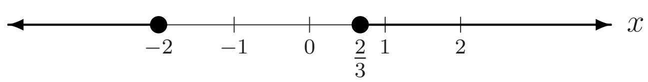


$$\therefore \quad -1 < x < 5.$$

EXAMPLE 2: Solve $|3x + 2| \geqslant 4$.

This is of the form of inequality 3 above, hence

$$\begin{aligned} 3x + 2 \leqslant -4 \quad & \text{or} \quad 3x + 2 \geqslant 4 \\ \therefore \quad 3x \leqslant -6 \quad & \quad \therefore \quad 3x \geqslant 2 \\ \therefore \quad x \leqslant -2 \quad & \quad \therefore \quad x \geqslant 2/3 \end{aligned}$$



The solution is $x \leqslant -2$ or $x \geqslant 2/3$. i.e. $x \in (-\infty, -2] \cup [2/3, \infty)$.

It is sometimes convenient to eliminate the modulus sign by squaring both sides of the inequality. However this can only be done (without affecting the inequality) if both sides are positive.

EXAMPLE 3: $4 < 6$ so $4^2 < 6^2$

but $-6 < -4$ and $(-6)^2 > (-4)^2$

Example 4: Solve $\left|\dfrac{x+3}{x+1}\right| \geqslant 1$.

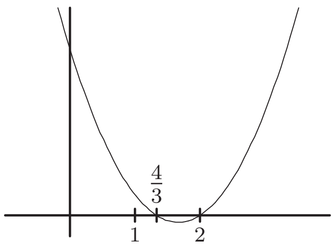
$$
\begin{aligned}
\therefore \quad & \frac{|x+3|}{|x+1|} \geqslant 1 && \text{as } \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \\
\therefore \quad & |x+3| \geqslant |x+1| && \text{as } |x+1| \geqslant 0 \\
\therefore \quad & (x+3)^2 \geqslant (x+1)^2 && \text{as both } |x+1| \geqslant 0 \text{ and } |x+3| \geqslant 0 \\
\therefore \quad & x^2+6x+9 \geqslant x^2+2x+1 \\
\therefore \quad & 4x \geqslant -8 \\
\therefore \quad & x \geqslant -2
\end{aligned}
$$

Example 5: Solve $|2x-3| \leqslant |x-1|$.

Squaring both sides

$$
\begin{aligned}
& (2x-3)^2 \leqslant (x-1)^2 \\
\therefore \quad & 4x^2-12x+9 \leqslant x^2-2x+1 \\
\therefore \quad & 3x^2-10x+8 \leqslant 0 \\
\therefore \quad & (3x-4)(x-2) \leqslant 0
\end{aligned}
$$

A graph of the left hand side is as follows.

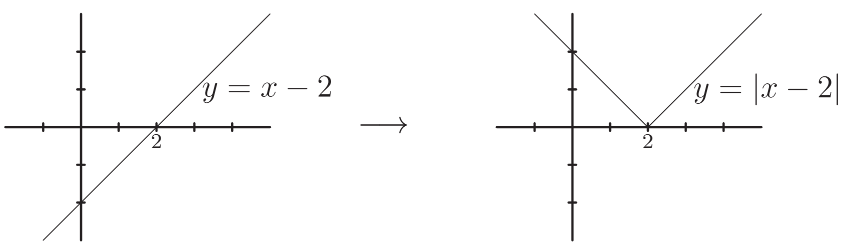


From this we can see that the inequality is satisfied for

$$\frac{4}{3} \leqslant x \leqslant 2.$$

The graph $y=|f(x)|$ can be sketched by firstly sketching $y=f(x)$ and then "reflecting in the x axis" all those parts of the graph which are negative.

Example 6: Sketch the curve $y=|x-2|$.



EXERCISE 14: Determine the intervals in which the following inequalities are satisfied:

1. $2 - x > 2(x + 4)$
2. $x + 4 \geq 3(x - 3)$
3. $-5 \leq 3 - 2x \leq 9$
4. $-1 \leq 2x + 1 < 3$
5. $3x + 1 \leq 1 - 3x \leq x + 5$
6. $2x - 3 \leq x - 5 \leq 3x - 3$
7. $2x < x - 1 \leq 3x + 5$
8. $|x + 1| \leq |x - 2|$
9. $\left|\dfrac{x + 4}{x - 2}\right| \leq 1$
10. $(x - 1)(x + 4) \geq 0$
11. $6 + x > x^2$
12. $|1 - x| - x \geq 0$
13. $|2x - 5| \leq |x + 4|$
14. $\dfrac{x - 1}{x + 2} > 0$
15. $(x - 2)^2(x + 1)(x - 3) < 0$
16. $(x - 2)(x + 4)(x - 5) \geq 0$
17. $\dfrac{(x - 1)(x + 2)}{3 - x} < 0$
18. $\dfrac{x + 1}{x(1 - x)} > 1$
19. $|x^2 - 2| < 1$

1.2.12 Variation (Proportionality)

Direct Variation

A quantity A is *directly proportional to* or varies (directly) with another quantity B if $A = kB$, where k is a constant, called the *constant of proportionality*. We denote (direct) proportionality by $A \propto B$.

ILLUSTRATION: A travelling salesman receives an allowance of 10 cents per km.
The actual allowance varies according to the number of kilometres travelled.
The allowance is "directly proportional" to the number of kilometres travelled.
The variables are: A — allowance
n — number of kilometres travelled
A varies directly as n. This is denoted by

$$\begin{aligned} & A \propto n \\ \text{i.e.} \quad & A = kn \end{aligned}$$

where k is the *constant of proportionality*.

More generally, direct proportion occurs between variables A and B when $A = kB^p$ where k is the constant of proportionality and p is any power such that $p > 0$. The next example includes a power, p. The constant of proportionality is determined using extra information given and the relationship is used to determine the value of one variable given a value of the other.

EXAMPLE: y is directly proportional to the cube of x. If $y = 16$ when $x = 2$, what is the value of x when $y = 54$?

$$\begin{aligned} & y \propto x^3 \\ \therefore \quad & y = kx^3 \end{aligned} \qquad (*)$$

Solve for k using $x = 2$ when $y = 16$

$$\begin{aligned} 16 &= k \times 2^3 \\ \therefore \quad k &= 2 \end{aligned}$$

Substitute into $(*)$ giving $y = 2x^3$. Now find x when $y = 54$.

$$\begin{aligned} 54 &= 2x^3 \\ \therefore \quad 27 &= x^3 \\ \therefore \quad x &= 3. \end{aligned}$$

Inverse Variation

Inverse proportion occurs between variables A and B when $A = kB^p$ where k is the constant of proportionality and p is any power such that $p < 0$.

We denote (indirect) proportionality by $A \propto \dfrac{1}{B}$.

Example 1: The current c flowing in an electric circuit is inversely proportional to the resistance R. If $c = 6$ when $R = 40$, find R when $c = \frac{1}{2}$.

If c is inversely proportional to R then it is proportional to the reciprocal of R.

$$\text{i.e.} \quad c \propto \frac{1}{R}$$
$$\therefore \quad c = k/R \qquad (**)$$

Find k using $c = 6$ when $R = 40$

$$6 = k/40$$
$$\therefore \quad k = 240$$

Subsitute into $(**)$ so that $c = \dfrac{240}{R}$. Now find R when $c = \dfrac{1}{2}$.

$$\frac{1}{2} = \frac{240}{R}$$
$$\therefore \quad R = 2 \times 240 = 480.$$

In the next example proportional relationships between two pairs of variables are indicated. One variable is the same in each pair. The information given is then used to determine the relationship between the two of the variables whose relationship was not given explicitly.

Example 2: y varies as x^2 and x varies inversely as $z^{1/3}$. How does y vary with z?

$$y \propto x^2 \quad \text{and} \quad x \propto \frac{1}{z^{1/3}}$$
$$\therefore \quad y = K_1 x^2 \quad \text{and} \quad x = \frac{K_2}{z^{1/3}}$$
$$\therefore \quad y = K_1 \left(\frac{K_2}{z^{1/3}}\right)^2 = \frac{K_1 {K_2}^2}{z^{2/3}} = \frac{C}{z^{2/3}} \qquad \text{as } K_1, K_2, C \text{ are constants.}$$

Note: We often have one variable that is proportional to a number of different variables. We can combine all relationships into one equation.

In the final example we are given the proportional change in two independent variables that results in a proportional change in a third dependent variable. Note that in this example the constant of proportionality does not need to be calculated.

Example 3: The electrical resistance R of a conductor varies directly as its length and inversely as the square of the diameter d. Find the percentage change in R if the length is increased by 8% and the diameter is decreased by 25%.

$$\text{i.e.} \quad R \propto \ell/d^2 \quad \text{or} \quad R = k\ell/d^2.$$

Initially, let $R = R_1$, $\ell = \ell_1$, and $d = d_1$, so that

$$R_1 = \frac{k\ell_1}{{d_1}^2}$$

Later we have $R = R_2$, $\ell = \ell_2 = 1.08\ell_1$ (increased 8%), and $d = d_2 = 0.75d_1$ (decreased 25%).

$$\begin{aligned} R_2 = \frac{k\ell_2}{{d_2}^2} &= \frac{k(1.08\ell_1)}{(0.75d_1)^2} \\ &= \frac{1.08}{(0.75)^2} \times \frac{k\ell_1}{{d_1}^2} \\ &= 1.92R_1 \end{aligned}$$

$\therefore$ Percentage increase in R is $0.92 \times 100\% = 92\%$.

EXERCISE 15:

1. One end of a string is fixed to a point on a smooth horizontal table and a mass at the other end of the string is moving on the table with uniform speed in a circle. The tension of the string varies directly as the square of the speed and inversely as the radius of the circle. The tension is 21.6 *kg weight* when the speed is 24 *m/sec* and radius 2.5 *m*. Find the radius corresponding to a speed of 20 *m/sec* and a tension of 18 *kg weight*.

2. The gravitational attraction F between two bodies is directly proportional to the product of their masses m and M respectively, and inversely proportional to the square of their distance D apart. What would be the percentage change in F if m were increased by 50%, M by 40% and D decreased by 20% ?

3. The height of a steel column necessary to support a load without buckling varies directly as the square of the diameter of the column and inversely as the square root of the load. What percentage increase in the load is possible if the same height of steel column is used but the diameter is increased by 10%?

4. A bank statement exactly 30 years old is discovered. It states:
   "This 10-year-old account is now worth 185.03 and pays 4% interest compounded annually."
   An investment with annual compound interest varies directly as $1 + r$ to the power n, where r is the interest rate expressed as a decimal and n is the number of years of compounding.
   What was the value of the original investment, and what is it worth now?

1.2.13 The Binomial Theorem

Expressions of the form $(a+b)^n$ where $a, b \in \mathbb{R}$ (are real numbers) and $n \in \mathbb{N}$ (a positive integer) are called binomial expansions and occur frequently in mathematics. Expressions such as

$$(x-1)^2, \quad (x+2)^3, \quad (a-b)^4, \quad \text{etc.}$$

can be expanded by repeated multiplication. However, for more complicated expressions such as

$$(x-4)^{10}, \quad (a+b)^8, \quad \text{etc.}$$

repeated multiplication becomes tedious. Note that ultimately we will want to relax the condition on the exponent, so later we will allow $n \in \mathbb{R}$.

Our first approach is to expand using Pascal's Triangle. The first two rows of the triangle consist of a 1 and then two 1s. Subsequent rows are generated so that each term in the row is the sum of the two nearest terms in the line above.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Each row gives the coefficients in the expansion of $(a+b)^n$ for some value of n.

NOTE: The expression $(a+b)^n$ has $n+1$ terms in its expansion.

EXAMPLE: Expand $(a+b)^3$.

As $n = 3$ there are 4 terms in this expansion. Hence the coefficients of $(a+b)^3$ are given by row 4 of Pascal's triangle.

$$\text{i.e.} \qquad \begin{aligned} (a+b)^3 &= 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3. \end{aligned}$$

Note the patterns. The powers of a decrease from one term to the next, while the powers of b increase from one term to the next. In each term the sum of the exponent of a and the exponent of b fixed.

Pascal's triangle is useful, but we need to obtain an explicit expression for $(a+b)^n$ for any positive value of the integer n. To obtain this, we must introduce factorial notation and binomial coefficient notation. We will introduce the notation and illustrate some properties of binomial coefficients before returning to a statement of the binomial theorem.

Definition 1. $n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1$ is the product of the first n natural numbers. We also define $0! = 1$ and $1! = 1$.

EXAMPLE: $2! = 2 \times 1 = 2, \quad 3! = 3 \times 2 \times 1, \quad \text{etc.}$

Definition 2. "n choose k" is the **binomial coefficient** and is defined as:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

NOTE: $\binom{n}{k}$ is sometimes written as ${}^nC_k$.

EXAMPLE 1: Binomial Coefficients.

$$\begin{aligned}
\binom{5}{2} &= \frac{5!}{(5-2)!\,2!} = \frac{5!}{3!\,2!} = \frac{5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{3} \times \cancel{2} \times \cancel{1} \times 2 \times 1} \\
&= \frac{5 \times 4}{2 \times 1} \qquad = 10. \\
\binom{5}{3} &= \frac{5!}{(5-3)!\,3!} = \frac{5!}{2!\,3!} = \frac{5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{2 \times 1 \times \cancel{3} \times \cancel{2} \times \cancel{1}} \\
&= \frac{5 \times 4}{2 \times 1} \qquad = 10.
\end{aligned}$$

In general, it can be shown that $\binom{n}{k} = \binom{n}{n-k}$.

EXAMPLE 2: If $n = 5$ and $k = 2$ then

$$\binom{5}{2} = \binom{5}{5-2} = \binom{5}{3}$$

which was shown in Example 1.

EXAMPLE 3: $\displaystyle\binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)(n-2)\ldots 3 \times 2 \times 1}{(n-2)\ldots 3 \times 2 \times 1 \times 2 \times 1} = \frac{n(n-1)}{2 \times 1}$

NOTE: By definition $\binom{n}{n} = 1 \quad \text{and} \quad \binom{n}{0} = 1.$

These identities can be be obtained by noting that $0! = 1$ by definition.

It is useful to note that by cancelling common factors the binomial coefficients can be written as in the following:

EXAMPLE 4:
$$\begin{aligned}
\binom{n}{k} = \frac{n!}{(n-k)!\,k!} &= \frac{n(n-1)(n-2)\ldots 3 \times 2 \times 1}{(n-k)\ldots 3 \times 2 \times 1 \times k(k-1)\ldots 2 \times 1} \\
&= \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots 2 \times 1}
\end{aligned}$$

EXERCISE 16:

1. Evaluate:

   (a) $\dfrac{8!}{7!}$ (b) $\dbinom{6}{3}$ (c) $\dbinom{20}{4}$ (d) $\dbinom{56}{2}$ (e) $\dbinom{3}{0}$ (f) $\dbinom{9}{9}$

2. Show that $\dbinom{n}{r} = \dbinom{n}{n-r}$ and hence evaluate $\dbinom{40}{38}$

3. Show that $\dbinom{n}{r} + \dbinom{n}{r+1} = \dbinom{n+1}{r+1}$

4. Verify Vandermonde's identity. That is, show that $\dbinom{n}{r} = \dbinom{n-1}{r} + \dbinom{n-1}{r-1}$

5. Find x if:

   (a) $\dbinom{16}{x} = \dbinom{16}{7}$ (b) $\dbinom{x}{5} = \dbinom{x}{11}$

We can now expand a general formula for $(a+b)^n$ for any positive integer, n, via. the binomial theorem.

> If $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, then
>
> $$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n\end{aligned}$$

where $\dbinom{n}{k}$ is the binomial coefficient. Note that we have not proved the theorem, but simply stated it. The proof is beyond the scope of MA1000, but you need to know the theorem and how to use it.

EXAMPLE 5: Binomial Theorem.

$$\begin{aligned}(a+b)^3 &= \sum_{k=0}^{3} \binom{3}{k} a^{3-k} b^k \\ &= \binom{3}{0} a^{3-0} b^0 + \binom{3}{1} a^{3-1} b^1 + \binom{3}{2} a^{3-2} b^2 + \binom{3}{3} a^{3-3} b^3\end{aligned}$$

The coefficients: $\dbinom{3}{0} = 1$ $\dbinom{3}{1} = \dfrac{3!}{2!\,1!} = 3$ $\dbinom{3}{2} = 3$ $\dbinom{3}{3} = 1$

$$\therefore \quad (a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

Using the observation we made at example 5 above, the <u>binomial theorem</u> can also be written as

$$\boxed{(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n}$$

EXAMPLE 6: Expand $(x-3)^5$.

$n = 5$, $a = x$, and $b = -3$

$$\begin{aligned}
(x-3)^5 &= [x+(-3)]^5 \\
&= x^5 + 5x^{5-1}(-3)^1 + \frac{5\times \overset{2}{\cancel{4}}}{\cancel{2}\times 1}x^{5-2}(-3)^2 + \frac{5\times \overset{2}{\cancel{4}}\times \cancel{3}}{\cancel{3}\times \cancel{2}\times 1}x^{5-3}(-3)^3 \\
&\quad + \frac{5\times \cancel{4}\times \cancel{3}\times \cancel{2}}{\cancel{4}\times \cancel{3}\times \cancel{2}\times 1}x^{5-4}(-3)^4 + \frac{\cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times 1}{\cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times 1}x^{5-5}(-3)^5 \\
&= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243
\end{aligned}$$

EXERCISE 17:

1. Expand and simplify using the Binomial Theorem:
   (a) $(2x+3)^5$ (b) $(3a-1)^4$
2. How many terms will each of the following expansions have? Find the first four terms:
   (a) $(x^2+2)^9$ (b) $(1-b^2)^{12}$
3. Consider the Binomial expansion of $(1+1)^n$.
   Use this to show that $\displaystyle\sum_{r=0}^{n}\binom{n}{r} = 2^n$.
4. Write down the 9th term in the expansion of $(a+b)^{13}$.
5. Find the middle term of the expansion of $(x+\frac{1}{x})^{10}$.
6. Find the term which is independent of x in the expansion $(2x-\frac{3}{x})^4$.
7. Find the term containing y^6 in the expansion of $(3xy^2-z^2)^7$.

1.2.14 The Binomial Series

If we put $a = 1$ and $b = x$ in the binomial expansion then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

becomes

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

which is a polynomial of degree n. This leads to the extension of the binomial theorem to the case where n is no longer a positive integer. If $n \in \mathbb{R}$, then $(1+x)^n$ is no longer a finite sum; it becomes an <u>infinite series</u>.

> **<u>The Binomial Series</u>**
> If n is any real number and $|x| < 1$, then
>
> $$\begin{aligned}(1+x)^n &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \\ &= \sum_{r=0}^{\infty} \binom{n}{r} x^r\end{aligned}$$
>
> which is an <u>infinite series</u>.

Note that in generalising over the exponent ($n \in \mathbb{N}$ became $n \in \mathbb{R}$) we have lost some flexibility in the numbers that can appear in the binomial - we are restricted to $|x| < 1$.

The expansion of $(1+x)^n$ is an infinite series which may be truncated, resulting in a finite sum which can be used as an *approximation* to $(1+x)^n$. The more terms retained in this finite sum, the better the approximation is.

The significance of the the binomial series is that it allows us to write down an expression for the power of any binomial in terms of positive integer powers. In general positive integer powers are regarded as being easier to work with mathematically.

EXAMPLE 1: Expand $\dfrac{1}{\sqrt{4-x}}$.

We must first convert this to the form $(1+x)^n$.

$$\begin{aligned}\frac{1}{\sqrt{4-x}} &= \frac{1}{\sqrt{4(1-x/4)}} \\ &= \frac{1}{2\sqrt{1-x/4}} \\ &= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} \\ &= \frac{1}{2}\left[1+\left(-\frac{x}{4}\right)\right]^{-1/2}\end{aligned}$$

Using the binomial expansion with $n = -1/2$ and x replaced by $(-x/4)$ we get

$$\begin{aligned}\frac{1}{\sqrt{4-x}} &= \frac{1}{2}\left[1 + (-\tfrac{1}{2})\left(-\frac{x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{x}{4}\right)^2 \right.\\ &\qquad\qquad \left. + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{x}{4}\right)^3 + \ldots\right] \\ &= \frac{1}{2}\left[1 + \frac{x}{8} + \frac{3x^2}{128} + \frac{5x^3}{16\times 64} + \ldots\right]\end{aligned}$$

Remember that for the expansion of $(1+x)^n$ we require $|x| < 1$. Therefore, in this case, we require

$$\begin{aligned}&\left|-\frac{x}{4}\right| = \left|\frac{x}{4}\right| < 1 \\ &\text{i.e.} \quad -1 < \frac{x}{4} < 1 \qquad \text{or} \qquad -4 < x < 4.\end{aligned}$$

Hence $\dfrac{1}{\sqrt{4-x}} \simeq \dfrac{1}{2}\left[1 + \dfrac{x}{8} + \dfrac{3x^2}{128} + \dfrac{5x^3}{1024}\right]$ if $-4 < x < 4$ where we have *truncated* the infinite series after the first four terms.

Linear Approximation for $(1+x)^n$

If x is very small (i.e., $|x| \ll 1$, which is read "x is very much smaller than 1"), we may ignore the terms containing $x^2, x^3, x^4, \ldots$ in the binomial series, so that

$$(1+x)^n \simeq 1 + nx.$$

Example 2: $\dfrac{1}{(1+x)^2} = (1+x)^{-2} \simeq 1 - 2x.$

The linear approximation is $1-2x$, provided $|x| < 1$. If $x = 0.01$ then $1-2x = 0.98$, whereas $\dfrac{1}{(1+x)^2}$ is actually equal to $0.98029\cdots$.

Example 3: Find the linear approximation for $(4+7x)^{3/2}$.

We must firstly put this into the form $(1+x)^n$,

$$\begin{aligned}\therefore \qquad (4+7x)^{3/2} &= 4^{3/2}\left(1 + \frac{7}{4}x\right)^{3/2} \\ &= 8\left(1 + \frac{7x}{4}\right)^{3/2}\end{aligned}$$

Now using $(1+x)^n \simeq 1 + nx$ we get

$$\begin{aligned}8\left(1+\frac{7x}{4}\right)^{3/2} &\simeq 8\left[1 + \left(\frac{3}{2}\right)\left(\frac{7x}{4}\right)\right] \qquad\qquad \text{if } \left|\frac{7x}{4}\right| < 1 \text{ or } |x| < \frac{4}{7}. \\ &= 8 + 21x.\end{aligned}$$

Higher Order Approximations for $(1+x)^n$

If we require a quadratic (second order) approximation to $(1+x)^n$, we only include the terms involving powers of x up to x^2. For a cubic (third order) approximation to $(1+x)^n$, we would only include terms up to x^3, etc.

Example: Given $\dfrac{2x^2-x+3}{(1+x^2)(1-x)}$, find a quadratic approximation.

Using partial fractions:

$$\begin{aligned}\frac{2x^2-x+3}{(1+x^2)(1-x)} &= \frac{A}{1-x}+\frac{Bx+C}{1+x^2}\\ \therefore\quad 2x^2-x+3 &= A(1+x^2)+(1-x)(Bx+C)\\ &= A+Ax^2+Bx+C-Bx^2-Cx\end{aligned}$$

Equating coefficients:

$$\begin{aligned}x^2:&\qquad 2 = A-B\\ x:&\qquad -1 = B-C\\ 1:&\qquad 3 = A+C\end{aligned}$$

Solving these gives $A=2$, $B=0$, and $C=1$.

$$\therefore\quad \frac{2x^2-x+3}{(1+x^2)(1-x)} = \frac{2}{1-x}+\frac{1}{1+x^2}$$

We may now find the binomial expansions for each of these terms

1. $$\begin{aligned}\frac{2}{1-x} &= 2[1+(-x)]^{-1}\\ &= 2\left[1+(-1)(-x)+\frac{(-1)(-2)}{2\times 1}(-x)^2+\cdots\right]\\ &= 2[1+x+x^2+\cdots]\end{aligned}$$

2. $$\begin{aligned}\frac{1}{1+x^2} &= (1+x^2)^{-1}\\ &= 1+(-1)\big(x^2\big)+\frac{(-1)(-2)}{2\times 1}\big(x^2\big)^2+\cdots\\ &= 1-x^2+x^4-\cdots\end{aligned}$$

Hence $$\begin{aligned}\frac{2x^2-x+3}{(1+x^2)(1-x)} &= \frac{2}{1-x}+\frac{1}{1+x^2}\\ &= 2[1+x+x^2+\cdots]+[1-x^2+x^4-\cdots]\\ &\simeq 3+2x+x^2. \qquad \text{Quadratic Approximation}\end{aligned}$$

EXERCISE 18:

1. Find at least the first four terms of the *Binomial Series* for:
   (a) $(1-x)^{-2}$ (b) $\sqrt{1+x}$ (c) $\dfrac{1}{\sqrt{1+x}}$

2. (a) Use your answer to 1.(b) to find a second order approximation to the Binomial Series for $\sqrt{1+x}$
   (b) Use your answer to 1.(c) to find the Binomial Series for $\dfrac{x}{\sqrt{1+x}}$

3. Find the first four terms of the *Binomial Series* for:
   (a) $\sqrt{4+x^2}$ (b) $(9-9x)^{-\frac{1}{2}}$

4. Find the first non-zero term of the Binomial Series expansion for
   $\sqrt{1+x} - \sqrt{1-x} - x, \quad |x| < 1$

5. The charge Q on a leaking capacitor is given by: $Q = \dfrac{2Q_0}{(1+t)(2+t)}$, where t is the time (seconds) and Q_0 is the initial charge (farads).
   (a) Express Q in partial fractions;
   (b) Find the Binomial Series expansion for your answer to (a);
   (c) Show that

$$Q \approx \left(1 - \frac{3}{2}t + \frac{7}{4}t^2\right) Q_0, \quad \text{provided that } t \text{ is small.}$$

6. The field strength H of a magnet at a point on the x–axis at a distance x from the centre is given by

$$H = \frac{M}{2a}\left[\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right]$$

   where M is the moment and $2a$ is the length of the magnet.
   Show that if x is large compared to a then

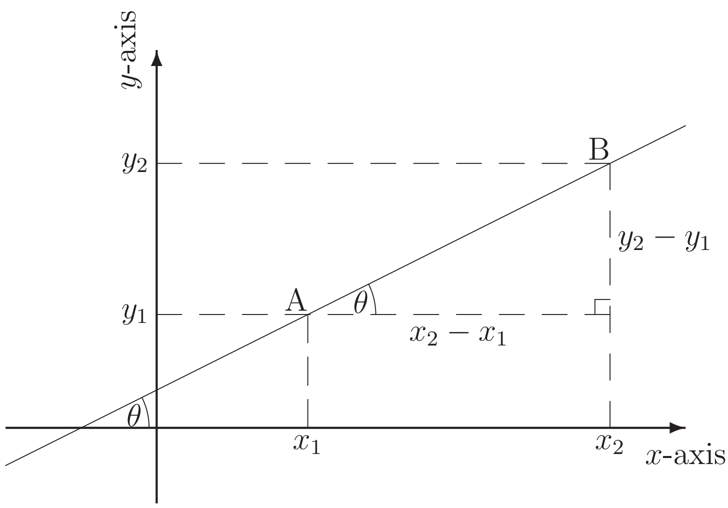
$$H \approx \frac{2M}{x^3}.$$

1.3 Coordinate Geometry

Coordinate geometry is the study of geometric objects through algebraic representation and manipulation of equations describing their properties (position, characteristic features, etc.). Objects are described by pairs of points (in two dimensional space) and triples of points (in three dimensional space). In MA1000 we focus on two-dimensional space and we work on a coordinate plane which is constructed using axes that are perpendicular to each other. Points on the plane are identified using (x, y) coordinates with the x-coordinate indicating the horizontal position of a point and the y-coordinate indicating the vertical position of a point.

1.3.1 The Straight Line

The first geometric object we discuss is the straight line. Consider two fixed points, $\mathrm{A} = (x_1, y_1)$ and $\mathrm{B} = (x_2, y_2)$ that lie on the line. Diagrammatically the line looks like this.

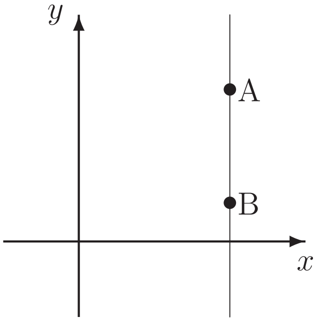


Gradient: This is one of the characteristics of a straight line. The gradient (or slope), m, of the line with segment AB is defined by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{if } x_2 \neq x_1$$
$$= \tan\theta$$

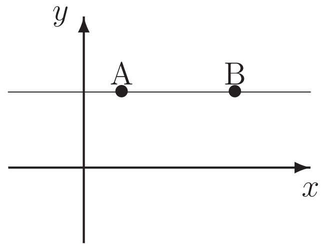
where θ is measured from the positive $x-$axis in an anticlockwise direction.

NOTE: Vertical Lines



If $x_1 = x_2$, the gradient is not defined.

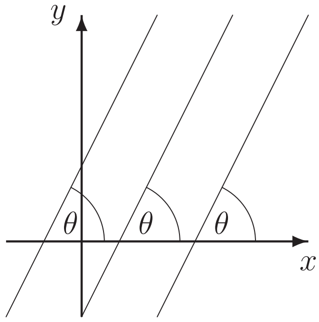
Horizontal Lines



If $y_1 = y_2$, the gradient is zero.

Parallel Lines

Parallel lines make the same angle with the $x-$axis and hence have the same gradient.

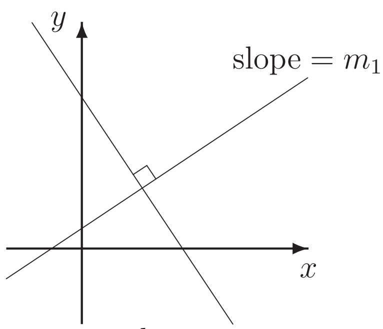


Perpendicular Lines

The product of the gradients of perpendicular lines is -1. We will see a proof of this later.

i.e. $\boxed{m_1 m_2 = -1}$

or $\boxed{m_1 = -\frac{1}{m_2}.}$



EXAMPLE 1: The vertices of a triangle are A = $(-2, 1)$, B = $(3, 2)$, and C = $(4, -3)$.
(1) Show that the line AB is perpendicular ($\perp$) to BC.

(2) Find the angle AB makes with the x-axis.

1. Gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

If m_{AB} is the gradient of AB and m_{BC} is the gradient of BC, we need to show that $m_{\text{AB}} \times m_{\text{BC}} = -1$.

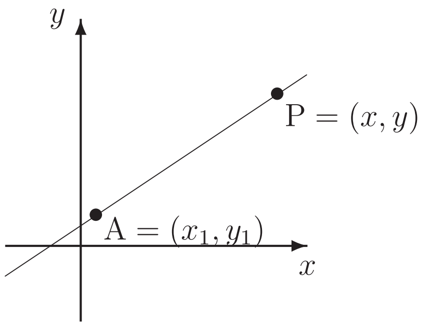
$$m_{\text{AB}} = \frac{2-1}{3-(-2)} = \frac{1}{5}$$
$$m_{\text{BC}} = \frac{-3-2}{4-3} = -5$$
$$\therefore \quad m_{\text{AB}} \times m_{\text{BC}} = \frac{1}{5} \times -5 = -1 \qquad \therefore \quad \text{AB} \perp \text{BC}$$

2. $$m_{\mathrm{AB}} = \tan\theta = \frac{1}{5}$$

$$\therefore \quad \theta = \tan^{-1}\frac{1}{5} \simeq 11.3°$$

Equation of a Straight Line

We will find the equation to the straight line with slope m, passing through the point (x_1, y_1). That is, we assume that the slope is known and so is one point on the line.



Let $\mathrm{A} = (x_1, y_1)$ be a fixed point on the line and $\mathrm{P} = (x, y)$ be any other point on the same line.

As slope of the line = slope of AP

then $$m = \frac{y - y_1}{x - x_1}$$

$$\therefore \quad \boxed{y - y_1 = m(x - x_1)}$$

This is the equation to a straight line through the point (x_1, y_1) with a slope m. This equation can also be written in the form

$$\boxed{y = mx + c}$$

where m is the slope and c is the y-intercept. Remember the y-intercept is obtained by letting $x = 0$ and can be determined in terms of x_1, y_1 and m.

EXAMPLE 2: Find the equation of the line passing through the point $(2, -3)$ and (1) parallel to and (2) perpendicular to the line $3x + 4y - 5 = 0$.

(1) $$3x + 4y - 5 = 0$$

$$\therefore \quad 4y = -3x + 5$$

$$\therefore \quad y = -\frac{3}{4}x + \frac{5}{4}$$

The required line is parallel to this line. Therefore it has the same gradient, i.e. $m = -3/4$. Our line also passes through the point $(2, -3)$

$$\begin{aligned}
\therefore \qquad y - y_1 &= m(x - x_1) \\
\text{becomes} \qquad y - (-3) &= -\frac{3}{4}(x - 2) \\
\therefore \qquad y + 3 &= -\frac{3}{4}x + \frac{3}{2} \\
\therefore \qquad y &= -\frac{3}{4}x - \frac{3}{2}.
\end{aligned}$$

(2) The required line is perpendicular to the line $y = -\dfrac{3x}{4} + \dfrac{5}{4}$

$$\begin{aligned}
\therefore \qquad m &= \text{negative reciprocal of } -3/4 \\
\textit{i.e.} \qquad m &= 4/3
\end{aligned}$$

The line also passes through the point $(2, -3)$,

$$\begin{aligned}
\therefore \qquad y - y_1 &= m(x - x_1) \\
\text{becomes } y + 3 &= \frac{4}{3}(x - 2) \\
&= \frac{4x}{3} - \frac{8}{3} \\
\therefore \qquad y &= \frac{4}{3}x - \frac{17}{3}.
\end{aligned}$$

EXERCISE 19:

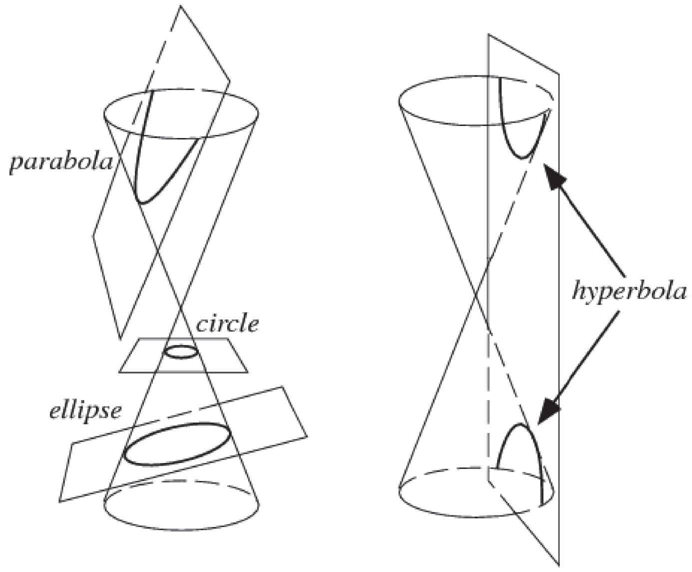
1. Find the equation of the following straight lines:
   (a) gradient $\frac{3}{4}$, passing through $(-6, 5)$;
   (b) passing through $(2, -8)$ and $(7, 2)$
   (c) passing through $(-5, -2)$ and making an angle of 135 with the x–axis
   (d) parallel to the x–axis and passing through the point $(5, 2)$
   (e) parallel to the y–axis and passing through the point $(-2, -4)$
   (f) x–intercept -3, y–intercept -2
   (g) containing the point $(2, -3)$ and parallel to the line $3x + 2y - 6 = 0$
   (h) containing the point $(2, -3)$ and perpendicular to the line $3x + 2y - 6 = 0$

2. The co-ordinates of A are $(0, -2)$ and B are $(3, 0)$. The x–coordinate of a point C on AB is 6. Find
   (a) the equation of AB;
   (b) the angle AB makes with the positive x–axis;
   (c) the equation of the line containing the point C and perpendicular to AB.

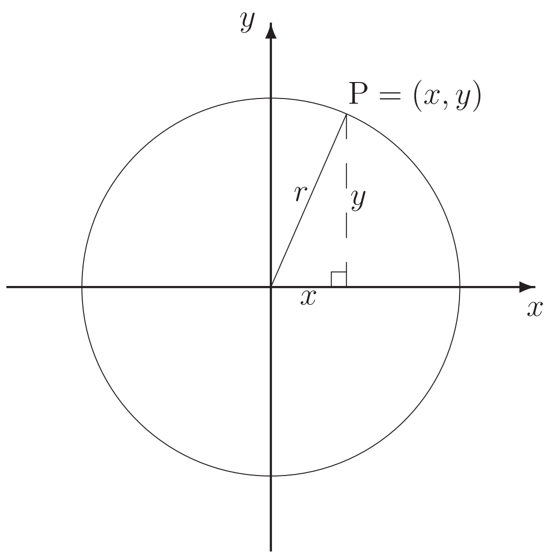
3. Find a proof that the product of the gradients of perpendicular lines is -1. What are the essential features of the proof? That is, what mathematical concepts and procedures does the proof rely on?

1.3.2 Conic Sections: (The Circle, Ellipse, Hyperbola, and Parabola.)

The next geometric objects we consider are called conic sections or simply conics. These arise when a plane intersects a double cone (two cones sharing the same axis and arranged apex to apex). This arrangement is indicated in the figure below. If the plane is perpendicular to the axis then a circle results. If the plane is other than perpendicular and intersects one cone only an ellipse or parabola results. Finally if the plane is arranged such that it intersects both cones then a hyperbola results. See here for more information: http://mathworld.wolfram.com/ConicSection.html



1.3.3 The Circle



If (x, y) is a point on the circle with radius r and centre at the origin, then by Pythagoras:

$$\boxed{x^2 + y^2 = r^2}$$

For a circle with centre (h, k) and radius r, the equation is

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

This is the "standard form" of the equation of a circle and immediately identifies the coordinates of the centre and radius of a circle.

Example 1: Find the centre and the radius of $(x+2)^2 + (y-1)^2 = 9$.

Centre is $(-2, 1)$ radius $= \sqrt{9} = 3$.

Example 2: Given that a circle has radius of 6 and a centre located at $(-3, 4)$, find its equation.

$$(x+3)^2 + (y-4)^2 = 36$$

If we expand this we would get

$$x^2 + 6x + y^2 - 8y - 11 = 0$$

We should be able to recognise this as the equation of a circle. The "General Equation" for a circle is

$$\boxed{x^2 + y^2 + Ax + By + C = 0}$$

This can always be transformed into standard form by completing the square on both x and y.

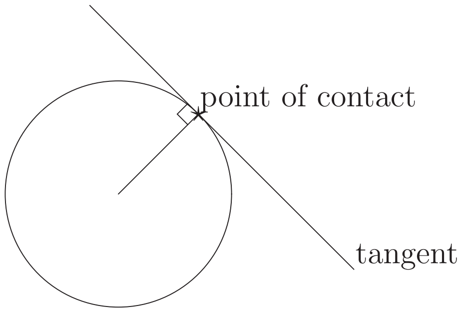
Example 3: Describe the curve whose equation is $x^2 + y^2 - 4x + 10y + 14 = 0$.

$$\begin{aligned} & x^2 + y^2 - 4x + 10y + 14 = 0 \\ \therefore \quad & x^2 - 4x + y^2 + 10y + 14 = 0 \\ \therefore \quad & (x-2)^2 - 4 + (y+5)^2 - 25 + 14 = 0 \\ \therefore \quad & (x-2)^2 + (y+5)^2 = 15 \end{aligned}$$

This is a circle, centre $(2, -5)$ and radius $\sqrt{15}$.

1.3.4 Tangents to Circles

A tangent to a circle is a straight line which touches the circle at only one point (the point of contact). The line joining the centre of the circle to the point of contact is perpendicular to the tangent.

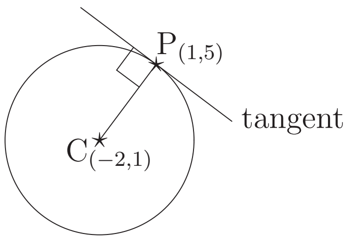


EXAMPLE 1: Find the equation of the tangent to the circle $x^2 + y^2 + 4x - 2y - 20 = 0$ at the point $(1, 5)$ on the circle.

Completing the square:

$$(x+2)^2 - 4 + (y-1)^2 - 1 - 20 = 0$$
$$(x+2)^2 + (y-1)^2 = 25$$

This is the equation of a circle, centre $(-2, 1)$ and radius 5.



Gradient of CP is: $m_{\text{CP}} = \dfrac{5-1}{1+2} = \dfrac{4}{3}$. Hence the gradient of the tangent is $-\dfrac{3}{4}$ (negative reciprocal). The tangent also passes through the point of contact, $(1, 5)$

Hence $\quad y - y_1 = m(x - x_1)$

becomes $\quad y - 5 = -\dfrac{3}{4}(x - 1)$

$\therefore \quad y - 5 = -\dfrac{3x}{4} + \dfrac{3}{4}$

$\therefore \quad y = -\dfrac{3x}{4} + \dfrac{23}{4}$

is the equation to the tangent.

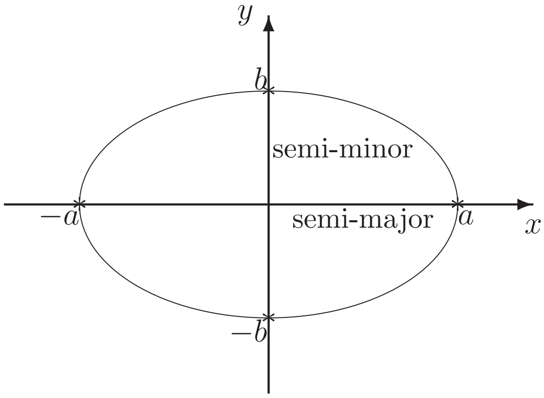
EXERCISE 20:

1. Show that the point $(4, -3)$ does not lie on the circle $x^2 + y^2 - 5x + 3y + 2 = 0$
2. Find the equation of the circle which touches the x–axis at $(4, 0)$ and touches the y–axis at $(0, 4)$
3. (a) Find the equation of the circle whose centre is the point $(-1, -2)$ and radius is of length 5 units
   (b) What is the length of the intercept cut off by this circle on the x–axis?
   (c) Find the length of the tangent to this circle from the point $(6, 4)$
4. Consider a circle with centre $(3, -1)$ and radius 5
   (a) Find the length of the tangent to the circle from the point $(2, 6)$
   (b) What is the equation to the tangent?
5. Find the equation to the circle such that the points A(-3,5) and B(4,-2) form the ends of a diameter.
6. Find the coordinates of the points in which the line $y - 2x = 1$ cuts the circle $x^2 + y^2 - 2x + 4y - 45 = 0$.
7. If the line $y = 2x + k$ is a tangent to the circle $x^2 + y^2 - 6x + 2y - 10 = 0$, find the value of k.
8. Find a proof that the line joining the centre of the circle to the point of contact of the tangent is perpendicular to the tangent at that point.

1.3.5 Ellipses

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

is the *standard form* for an ellipse centred at the origin. If $a > b > 0$ then the ellipse has a semi-major axis (the longer of its two axes) with length a along the $x-$axis and semi-minor axis (the shorter of its two axes) with length b along the y-axis as indicated in the figure below.



If $b > a > 0$ then the ellipse has the semi-major axis with length b along the $y-$axis and the semi-minor axis with length a along the $x-$axis.

The standard form for the equation for an ellipse with centre (h, k) is

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

The semi-major and semi-minor axes are parallel to the axes of the coordinate system. The general equation for an ellipse is

$$Ax^2 + By^2 + Cx + Dy + E = 0.$$

Such equations can always be transformed into standard form by completing the square on both x and y.

Notes:

1. Ellipses with axes that are not parallel to the coordinate axes do occur, but we do not consider them in MA1000.

2. Ellipses do not have rotational symmetry about their centre, but a circle does. This means if we rotate an ellipse about its centre it won't look the same (unless we rotate through an integer multiple of 180°), but any rotation of a circle about its centre leaves the circumference of the circle in the same place.

3. If $a = b$ then the ellipse becomes a circle. That is, a circle is a special case of an ellipse.

EXAMPLE 1: Describe the curve whose equation is $4x^2 + y^2 - 16x - 10y + 21 = 0$.

$$\begin{aligned}
4x^2 + y^2 - 16x - 10y + 21 &= 0 \\
4x^2 - 16x + y^2 - 10y + 21 &= 0 \\
4(x^2 - 4x) + (y-5)^2 - 25 + 21 &= 0 \\
4[(x-2)^2 - 4] + (y-5)^2 - 4 &= 0 \\
4(x-2)^2 - 16 + (y-5)^2 - 4 &= 0 \\
4(x-2)^2 + (y-5)^2 &= 20 \\
\frac{(x-2)^2}{5} + \frac{(y-5)^2}{20} &= 1.
\end{aligned}$$

This is an ellipse with centre $(2, 5)$ with the semi-major axis of length $\sqrt{20}$ aligned parallel to the y-axis and semi-minor axis of length $\sqrt{5}$ aligned parallel to the x-axis.

1.3.6 Hyperbolas

The *standard form* of the equation of a hyperbola with its centre at the origin is either

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

or

$$\boxed{\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1}$$

We will work with the first of these and then make comments about how the second differs from the first.

The working that follows assumes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. To find the x intercepts put $y = 0$, so that

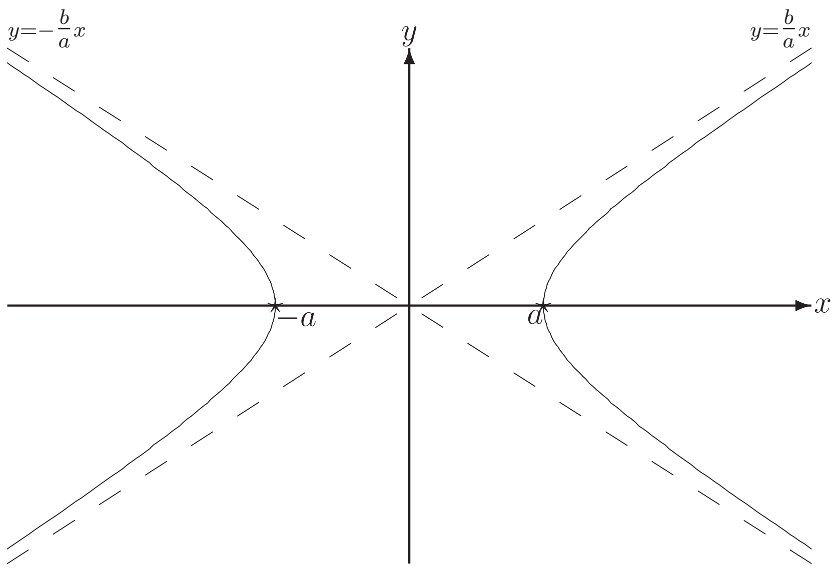
$$\frac{x^2}{a^2} = 1, \qquad \therefore\ x = \pm a$$

To find the y intercepts put $x = 0$, so that

$$-\frac{y^2}{b^2} = 1, \qquad \therefore y^2 = -b^2$$

Therefore there are no y intercepts.

When x is large, $y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) \simeq \frac{b^2}{a^2}x^2$. The asymptotes are straight lines given by $y = \pm\frac{b}{a}x$. These are lines that hyperbola gets very close to (but never quite reaches) as x and y values get large.



We see that the hyperbola has vertices on the $x-$axis. These are the points $(x = \pm a)$ on the hyperbola that are closest to its centre. Also note that there are no points on the hyperbola for which $x \in (-a, a)$.

If the hyperbola has the form $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ then the hyperbola has the same asymptotes as the previous case, but the vertices are now on the $y-$axis at $y = \pm b$. In this case there are no points on the hyperbola for which $y \in (-b, b)$.

The equation for a hyperbola with centre (h, k) is

$$\boxed{\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1}$$

or

$$\boxed{\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1}$$

depending on whether the vertices are on an axis parallel to the $x-$axis or an axis parallel to the $y-$axis. Irrespective of the axis on which the vertices lie, the *general equation* of a hyperbola is

$$Ax^2 - By^2 + Cx + Dy + E = 0.$$

The equations of the asymptotes are:

$$y = k \pm \frac{b}{a}(x - h)$$

Example 1: Describe the curve whose equation is $4x^2 - 9y^2 + 8x + 18y = 41$.

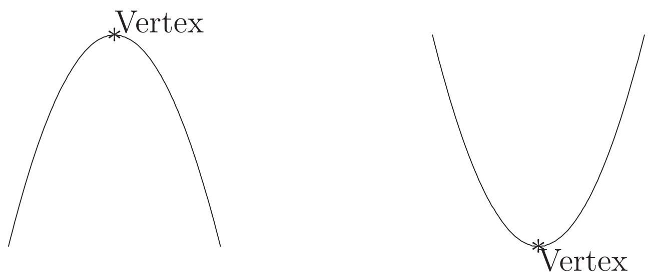
Completing the square on both x and y gives

$$\begin{aligned} 4(x^2 + 2x) - 9(y^2 - 2y) &= 41 \\ 4[(x+1)^2 - 1] - 9[(y-1)^2 - 1] &= 41 \\ 4(x+1)^2 - 4 - 9(y-1)^2 + 9 &= 41 \\ 4(x+1)^2 - 9(y-1)^2 &= 36 \\ \frac{(x+1)^2}{9} - \frac{(y-1)^2}{4} &= 1. \end{aligned}$$

This is the equation for a hyperbola with centre at $(-1, 1)$ and $a = 3$, $b = 2$.

1.3.7 Parabolas

The quadratic function $y = ax^2 + bx + c$, $a \neq 0$ has a graph which is parabolic. If $a > 0$, the graph opens upwards, whereas if $a < 0$ the graph opens downwards.



$a < 0$ $\qquad$ $a > 0$

The turning point of the parabola $y = f(x)$ is called its vertex and it has the coordinates $\left[-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right]$.

Example 1: Sketch the graph $y = f(x) = 2x^2 - 4x + 1$.

<u>y-intercept:</u> $x = 0 \quad \Rightarrow \quad y = 0 - 0 + 1 = 1$

<u>x-intercept:</u> $y = 0 \quad \Rightarrow \quad 0 = 2x^2 - 4x + 1$

$$\begin{aligned} \therefore \quad x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{16 - 8}}{4} \\ &= \frac{4 \pm \sqrt{8}}{4} \\ &= \frac{4 \pm 2\sqrt{2}}{4} \\ &= \frac{2 \pm \sqrt{2}}{2} \end{aligned}$$

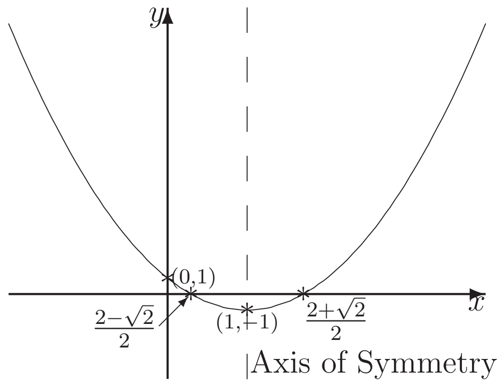
Recall: The value and sign of the discriminant ($\Delta = b^2 - 4ac$) will tell us how many solutions there are to the equation $f(x) = 0$, i.e. how many $x-$intercepts there are:

- $\Delta > 0 \implies$ two $x-$intercepts
- $\Delta = 0 \implies$ one $x-$intercept
- $\Delta < 0 \implies$ no $x-$intercepts

Vertex: $x = -\dfrac{b}{2a} = \dfrac{4}{4} = 1 \qquad \therefore y = f\left(-\dfrac{b}{2a}\right) = f(1) = 2 \times 1^2 - 4 \times 1 + 1 = -1$

$\therefore$ the vertex lies at $(1, -1)$.

The graph of $y = f(x) = 2x^2 - 4x + 1$ is



EXERCISE 21:

1. Identify the type of curve given by the following equations. Identify lengths of semi-major, semi-minor axes and equations of asymptotes where appropriate.
   (a) $4x^2 + 9y^2 - 16x + 54y + 61 = 0$ (b) $y^2 - 4x^2 + 16x + 6y = 23$

2. A stone is projected vertically upwards from the ground. The height $h(t)$ (in metres) above the ground is a function of time t ($t \geq 0$), with rule

   $$h(t) = 12.5t - 5t^2.$$

   Find the greatest height reached.

3. A symmetrical road bridge has the shape of half an ellipse. Its span is 30 *metres* and its height is 20 *metres*. Determine the height at a distance of 12 *metres* from the axis of symmetry.

4. A symmetrical parabolic bridge has a height of 4 *metres* and a span of 8 *metres*. A vehicle is 4 *metres* broad and has a height of just over 3 *metres*.

   (a) Can the vehicle pass under the bridge?

   (b) Determine the maximum head height which a vehicle 3 *metres* wide can have to pass under the bridge without contact.

5. (a) Find the values of x where the line $y = m(x - \sqrt{6})$ intersects the ellipse $\frac{x^2}{2} + y^2 = 1$ (Note: your answer will depend on m).

   (b) Use this to find the equations to the tangents to the ellipse from the point $(\sqrt{6}, 0)$.

6. If the line $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $c^2 = b^2 + a^2m^2$.

7. The coordinates of one end of a focal chord of the parabola $y^2 = 8x$ are (8,-8). Find the coordinates of the other end.

8. For what values of m is line $y = mx$ tangent to the hyperbola $x^2 - y^2 = 1$?

2 Functions and their Graphs

In this chapter we will use graphs to help identify the important properties and features of a function or relation.

2.1 Definitions

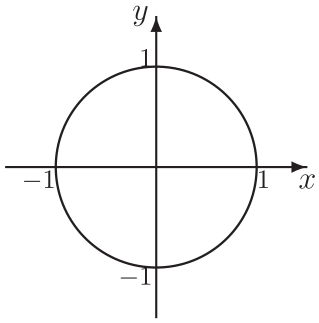
Definition 1. A relation is a set of ordered pairs (x, y) and is usually defined by a property or rule, i.e. the equation.

The words "ordered pair" mean that the positioning of the objects x and y in the brackets is significant. We have come across these many times, usually in the context of graphing where x represents a horizontal coordinate and y a vertical coordinate.

Definition 2. The domain of the relation is the set of all possible x values (the first elements of the ordered pairs).

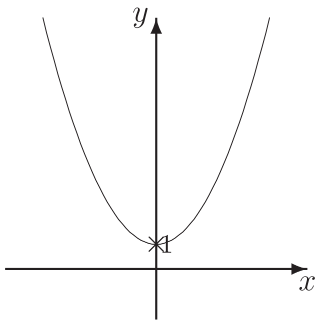
Definition 3. The range of the relation is the set of all possible y values (the second elements of the ordered pairs).

EXAMPLE 1: $x^2 + y^2 = 1$ is a relation describing the unit circle. $(h, k) = (0, 0)$, $r = 1$



Domain: $-1 \leqslant x \leqslant 1$
Range: $-1 \leqslant y \leqslant 1$

EXAMPLE 2: $y = x^2 + 1$ is a relation describing a parabola.



Domain: $x \in \mathbb{R}$ (all real numbers)
Range: $y \geqslant 1$

Definition 4. A function is a relation with the property that no two ordered pairs have the same x coordinate.

EXAMPLE 3: Consider the unit circle, $x^2 + y^2 = 1$. This is not a function as there are points on the curve that have the same x value but two different y values.

$$\text{i.e.}\quad \text{If } x = 0, \text{ then} \qquad 0^2 + y^2 = 1$$
$$\therefore \quad y = \pm 1.$$

EXAMPLE 4: The parabola $y = x^2 + 1$ is a function.

> **The Vertical Line Test**: A curve is a function if no vertical line cuts the curve more than once.

In the next few sections we will present some terminology that is used to characterise functions. In particular we will talk about the domain, range and sign of functions and whether they are increasing or decreasing.

2.2 Function Notation

We can write $f(x) = x^2 + 2x - 3$ instead of $y = x^2 + 2x - 3$. i.e. $f(x)$ is the value of y for the given value of x. y changes according to the value of x. Therefore x is called the *independent* variable and y is called the *dependent* variable.

2.2.1 Domain

As stated earlier, the domain of a function is the set of all possible values of x allowed for that function. Functions are often given merely by a rule, without the domain written in full. Where no domain is specified, it will be assumed that the maximal domain is intended.

EXAMPLE 1: The domain of the function $f(x) = \sqrt{x}$ is $x \geqslant 0$ (as you cannot take the square root of a negative real number).

EXAMPLE 2: The domain of the function $f(x) = \dfrac{1}{x}$ is $x \neq 0$ (all real numbers, except $x = 0$).

EXAMPLE 3: The domain of the function $f(x) = \dfrac{1}{\sqrt{x}}$ is $x > 0$. (Note: the point $x = 0$ is excluded because you cannot divide by zero.)

EXAMPLE 4: Find the domain of $f(x) = \sqrt{x - 3}$.

This function is defined if $x - 3 \geqslant 0$, i.e. $x \geqslant 3$.

$\therefore$ Domain of $f = D_f$: $x \geqslant 3$.

EXAMPLE 5: Find the domain of $f(x) = \dfrac{x + 2}{(x - 1)(x + 1)}$.

This function is only defined if $x \neq 1$ and $x \neq -1$.

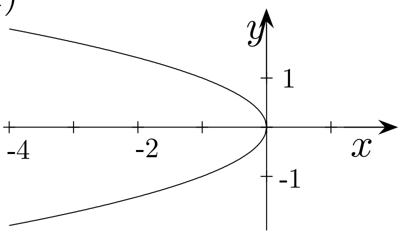
$\therefore$ Domain of $f = D_f$: $x \neq -1$ and $x \neq 1$.

i.e. all real values of x excluding $x = \pm 1$.

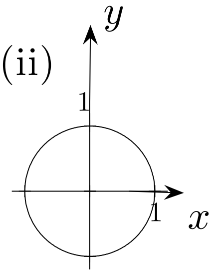
EXERCISE 22:

1. For each of the following graphs (i)–(iv), state
   (a) the domain and range, and
   (b) whether the relation is a function or not.

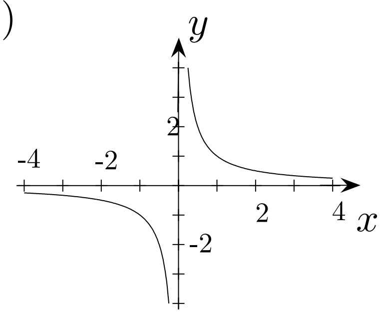
(i)



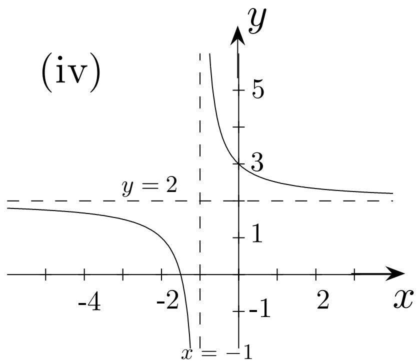
(ii)



(iii)



(iv)



2. Find the domains of the following functions:

   (a) $f(x) = \dfrac{10}{x}$ (b) $h(x) = \dfrac{x^2}{x-1}$ (c) $f(x) = \sqrt{1-x}$ (d) $g(x) = \dfrac{4}{\sqrt{4-2x}}$

3. For each function, construct a table of values and sketch the function. From the sketch, state the domain and range.

   (a) $y = 3 - x$ (b) $y = (x-1)^2$ (c) $y = \sqrt{x+3}$ (d) $y = \dfrac{1}{x-2}$

4. By considering the restrictions on the values of x and y, find the domain and range of the following.

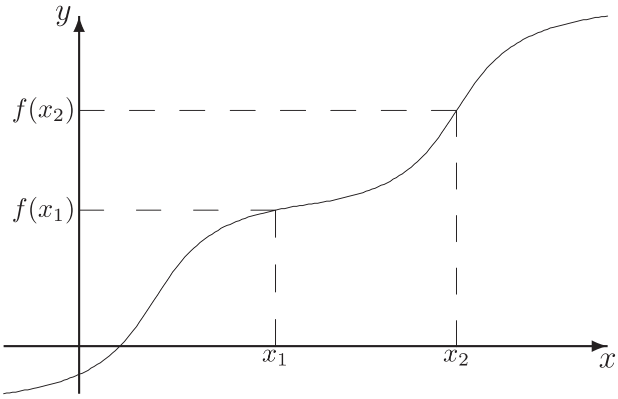
   (a) $y = \sqrt{x^2}$ (b) $y = \dfrac{3}{1-x^2}$ (c) $y = \sqrt{1-\sqrt{x}}$ (d) $y = \dfrac{1}{\sqrt{x^2-25}}$

5. A plumber melts and mixes x grams of solder (a metal alloy, with a low melting point, used to join metals with higher melting points) that is 40% tin with y grams of another brand of solder that is 20% tin to yield a final solder mixture that contains a total of 200 grams of tin. Find the domain of x and the range of y.

2.3 Increasing and Decreasing Functions

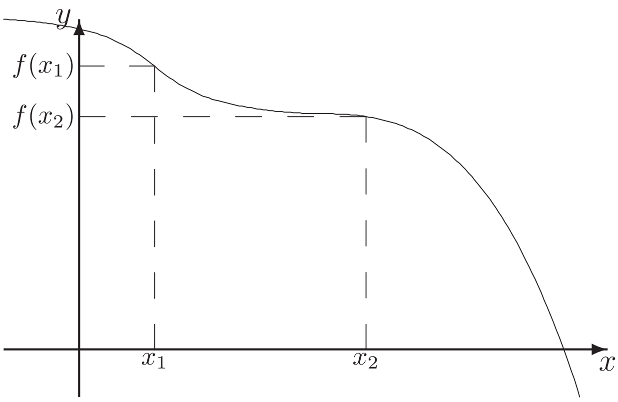
A function is called *increasing* on an interval I if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$



A function is called decreasing on I if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$



2.4 Sign of Functions

EXAMPLE 1: For $f(x) = 2x^2 - 3x - 5$, find (i) where $f(x) = 0$, (ii) where $f(x) > 0$, (iii) where $f(x) < 0$, and (iv) the intervals on which $f(x)$ is increasing (decreasing).

(i) $f(x) = 0$:

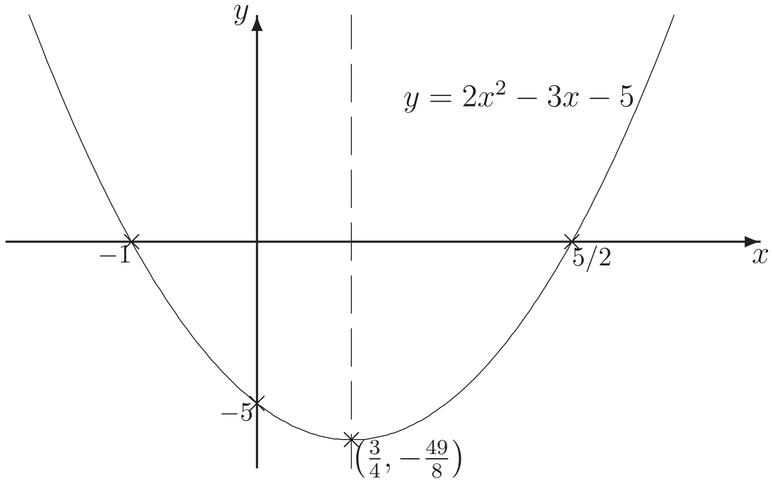
$$\begin{aligned} 2x^2 - 3x - 5 &= 0 \\ (2x-5)(x+1) &= 0 \\ \therefore \quad x &= \frac{5}{2}, -1 \end{aligned}$$

We recognise $f(x)$ to be the equation for a parabola (opening upwards). To graph this parabola we must also find its vertex.
Recall that the coordinates of the vertex are $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. In this case,

$$\begin{aligned} x &= -\frac{b}{2a} = \frac{3}{4} \\ y &= f\left(-\frac{b}{2a}\right) = f\left(\frac{3}{4}\right) \\ &= 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) - 5 = -\frac{49}{8} \end{aligned}$$

$$\boxed{\text{Vertex:} \quad \left(\frac{3}{4}, -\frac{49}{8}\right)}$$

We can now draw (sketch) the graph of $f(x)$.



(ii) From the graph: $f(x) > 0$ when $x < -1$ or $x > 5/2$.
(iii) From the graph: $f(x) < 0$ when $-1 < x < 5/2$.
(iv) From the graph:
The function is increasing for $x > 3/4$.
The function is decreasing for $x < 3/4$.

2.5 Cubics

Recall that the general equation for a cubic is $f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$.

Orientation: If $a > 0$, the curve is similar to 
If $a < 0$, the curve is similar to 

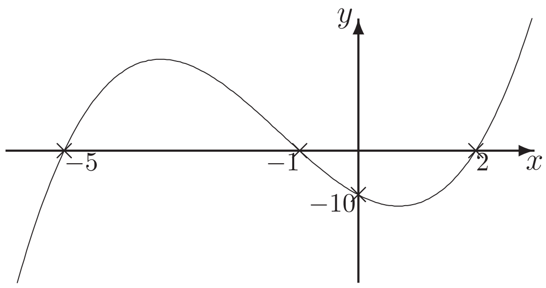
When sketching a cubic you may need to solve $ax^3 + bx^2 + cx + d = 0$ to find the $x-$intercepts. This can usually be done using the Factor theorem or Remainder theorem.

EXAMPLE 1: Sketch the curve $f(x) = x^3 + 4x^2 - 7x - 10$.

Orientation: $a = 1 > 0$. Therefore the shape is 
$y-$ intercepts: $(x = 0) \quad \therefore \quad y = -10$.
$x-$ intercepts: $(y = 0) \quad \therefore \quad 0 = x^3 + 4x^2 - 7x - 10$.
The factor theorem and long division gives

$$x^3 + 4x^2 - 7x - 10 = (x + 1)(x + 5)(x - 2) = 0$$
$$\therefore \quad x = -1, -5, 2.$$

Sketch the graph:



Note that we can often get a good idea of what the graph of a function looks like *without* plotting points. In fact this approach often gives a better picture of the function, because a plot of a function can often miss features (e.g., if the plotted points are too far apart). To *sketch* the graph of a function, we look for features such as the x and y intercepts. For the next group of functions, we will also need to look for the asymptotes of the function. Later we will see how to find the turning points of a function.

These features are usually sufficient to get a good idea of what a function looks like.

EXERCISE 23:

1. Sketch the graphs of the following quadratics. In doing so, find the x and y intercepts, places where the functions are positive or negative and where the function is increasing or decreasing.
   (a) $f(x) = x^2 - 4x + 2$ (c) $f(x) = 2x^2 - 7x - 15$
   (b) $f(x) = x^2 - 4x + 6$ (d) $f(x) = -4x^2 + 4x + 2$
2. Sketch the graphs of the following cubics. In doing so, find the x and y intercepts and places where the functions are positive or negative.
   (a) $f(x) = x^3 + 2x^2 - x - 2$ (c) $f(x) = 2x^3 - 8x^2 + 10x - 4$
   (b) $f(x) = x^3 - 3x + 2$ (d) $f(x) = x^3 - 6x^2 + 14x - 12$
3. Graphing a function is easier if information regarding the *slope* of a function is available, as well as information about the function itself. In particular this allows *turning points* to be determined. These are points where the slope of a function changes sign, from positive to negative or vice versa. In each of the following $f(x)$ is a function to be graphed and $s(x)$ is its slope. Plot $f(x)$ and $s(x)$ on the same axes. Note how turning points for the function $f(x)$ occur where $s(x) = 0$. Is it the case that if $s(x) = 0$ then a turning point occurs on the graph for $f(x)$? Explain.
   (a) $f(x) = x^2,\ s(x) = 2x$ (c) $f(x) = x^3 + 2x^2 - x - 2,\ s(x) = 3x^2 + 4x - 1$
   (b) $f(x) = x^3,\ s(x) = 3x^2$ (d) $f(x) = x^3 - 3x + 2,\ s(x) = 3x^2 - 3$
   (e) $f(x) = x^4 - 5x^2 + 4,\ s(x) = 4x^3 - 10x$

   In (d) what do you notice about the value of the function at the turning point further to the right on the graph? Can you predict when a zero of a polynomial will correspond to a turning point?

2.6 Limits

In this section we consider how to characterise functions in terms of their behaviour near places where they are undefined - that is behaviour at the edge of a domain. This will lead to the concept of a limit. In a mathematical sense the word "limit" means:

> "The value of $f(x)$ as x approaches some particular value, a."

In the case where we are at the edge of the domain the function it is often the case that the function is not defined at $x = a$.

EXAMPLE 1: The following functions have restricted domains because they each have a denominator that is zero at one point.

1. $f(x) = \dfrac{3}{x-2}$
2. $f(x) = \dfrac{x}{2x}$
3. $f(x) = \dfrac{2\sin x}{x}$
4. $f(x) = \dfrac{x^2}{e^x - 1}$
5. $f(x) = \dfrac{5x+10}{x+2}$

Exercise In each of the functions in example 1, state the value of x that is excluded from the domain. We will inspect the behaviour of each of these functions near the point that is excluded from the domain, by evaluating each one of them at a carefully chosen set of points.

Analysis of behaviour of a function at the edge of its domain

We look at the first of the functions in example 1. Here $f(x) = \dfrac{3}{x-2}$ is undefined at $x = 2$. We consider a sequence of x values less than 2, increasing from $x = 1$ toward $x = 2$ in such a way that $2 - x$ decreases by a factor of 10 each time. We can think of this as focusing a microscope at the place where the function is undefined.

x	$f(x) = \dfrac{3}{x-2}$
1	-3
1.5	-6
1.75	-12
1.9	-30
1.99	-300
1.999	-3000
1.9999	-30000
1.99999	-300000
1.999999	-3000000
2	undefined

Of interest in this analysis is how the value of the function changes as the point where the function is undefined is approached. In this example the function values are negative and it appears as though they become arbitrarily large in magnitude as that point is approached.

We can perform a similar analysis for values of x decreasing from $x = 3$ toward $x = 2$. Complete the table below:

x	$f(x) = \dfrac{3}{x-2}$
3	3
2.5	
2.25	
2.1	
2.01	
2.001	
2.0001	
2.00001	
2.000001	
2	undefined

Describe the behaviour of the function as x approaches 2 from the values of $x > 2$. Is it the same behaviour as when x approaches 2 from the values of $x < 2$?

Exercise On the following pages complete the tables for the other four functions in example 1.

2. $f(x) = \dfrac{x}{2x}$

x	$f(x) = \dfrac{x}{2x}$	x	$f(x) = \dfrac{x}{2x}$
1	0.5	-1	0.5
0.5		-0.5	
0.25		-0.25	
0.1			
0.01			
0.001			
0.0001			
0.00001			
0.000001			
0	undefined	0	undefined

Describe the behaviour of the function as x approaches 0 from each side. Is the behaviour the same from each side?

3. $f(x) = \dfrac{2\sin x}{x}$

x	$f(x) = \dfrac{2\sin x}{x}$	x	$f(x) = \dfrac{2\sin x}{x}$
1	1.68294197	-1	1.68294197
0.5			
0.25			
0.1			
0.01			
0.001			
0.0001			
0.00001			
0.000001			
0	undefined	0	undefined

Describe the behaviour of the function as x approaches 0 from each side. Is the behaviour the same from each side?

4. $f(x) = \dfrac{x^2}{e^x - 1}$

x	$f(x) = \dfrac{x^2}{e^x - 1}$	x	$f(x) = \dfrac{x^2}{e^x - 1}$
1	0.58197671	-1	-1.58197671
0.5			
0.25			
0.1			
0.01			
0.001			
0.0001			
0.00001			
0.000001			
0	undefined	0	undefined

Describe the behaviour of the function as x approaches 0 from each side. Is the behaviour the same from each side?

5. $f(x) = \dfrac{5x + 10}{x + 2}$

x	$f(x) = \dfrac{5x + 10}{x + 2}$	x	$f(x) = \dfrac{5x + 10}{x + 2}$
-3	5	-1	5
-2.5			
-2.25			
-2.1			
-2.01			
-2.001			
-2.0001			
-2.00001			
-2.000001			
-2	undefined	-2	undefined

Describe the behaviour of the function as x approaches -2 from each side. Is the behaviour the same from each side?

We can compare the behaviour of each of the five functions by completing the following table:

Function	Behaviour
$\dfrac{3}{x-2}$	As x approaches 2, the point at which the function is undefined, $f(x)$ gets larger in magnitude, apparently without bound. The behaviour is in opposites: from values lower than 2 the function is negative, but from values higher than 2 the function is positive.
$\dfrac{x}{2x}$	As x approaches 0, the point at which the function is undefined, $f(x)$ is constant, in fact $f(x) = 1/2$. The behaviour is
$\dfrac{2\sin x}{x}$	As x approaches 0, the point at which the function is undefined, $f(x)$ is almost constant, in fact $f(x) = 2$. The behaviour is
$\dfrac{x^2}{e^x - 1}$	As x approaches 0, the point at which the function is undefined, $f(x)$ is almost constant, in fact $f(x) = 0$. The behaviour is
$\dfrac{5x+10}{x+2}$	As x approaches -2, the point at which the function is undefined, $f(x)$ is constant, in fact $f(x) = 5$. The behaviour is

In four out of the five cases we have seen the function approach a constant value, while in the other (the first case) the functions values became very large in magnitude, with different values depending on the direction of approach.

In those cases where the function values approach the same value from either side of the point where the function is undefined we can summarise this behavior as:

$$\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{x}{2x} &= \frac{1}{2} \\
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{2\sin x}{x} &= 2 \\
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{x^2}{e^x - 1} &= 0 \\
\lim_{x\to -2} f(x) &= \lim_{x\to -2} \frac{5x+10}{x+2} &= 5
\end{aligned}$$

The notation $\lim_{x\to 0} f(x)$ is read as "the limit as x tends to zero of $f(x)$". This is a shorthand to say that we have established the value of the function as x gets arbitrarily close to 0. Note that this notation does not communicate any information about the actual function value at $x = 0$ (which we would denote $f(0)$ if we could compute it). What we can see is that in each of these examples, if we were to substitute the limiting value of x directly into the function a ratio $0/0$ is observed. We call this an indeterminate form, because we cannot predict the limiting value by direct substitution into the function - and as we see any limiting value of $f(x)$ is possible.

In the case of the functions $f(x) = \dfrac{x}{2x}$ and $f(x) = \dfrac{5x+10}{x+2}$ it is possible to determine the limiting value of the function by cancelling common factors. In fact

$$f(x) = \frac{x}{2x} = \frac{1}{2}$$

at every value of x where $f(x)$ is defined, remembering that $f(x)$ is undefined at $x = 0$ because the denominator is zero there. Similarly

$$f(x) = \frac{5x+10}{x+2} = 5$$

at every value of x where $f(x)$ is defined, remembering that $f(x)$ is undefined at $x = -2$ because the denominator is zero there. This type of analysis is not possible on the other functions because there are no common factors to cancel.

In the remaining example, $f(x) = \dfrac{3}{x-2}$, the function values became infinitely large in magnitude as the point where $f(x)$ was undefined was approached. On one side they were negative and the other they were positive. For this reason we would say:

$$\lim_{x\to 2} f(x) \text{ is undefined.}$$

We can however make a statement about the limit as we approach from just one side of the point where the function is undefined. Thus in the first example we would write

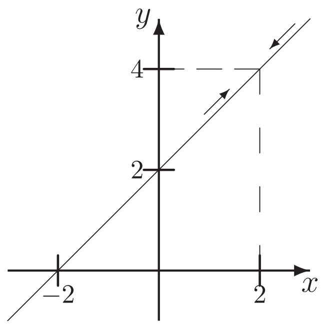
$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} \frac{3}{x-2} &= -\infty \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} \frac{3}{x-2} &= \infty \end{aligned}$$

In each case the superscript "$-$" and superscript "+" means that the point is approached only from the left or right respectively. These are known as "one-sided" limits. Note also that the notation "∞", the "infinity" symbol represents an unbounded large number.

Application of the concept of a limit. We used the analysis of functions where the domain was restricted by a zero in the denominator as motivation for the investigation above, however the concept of the limit is not restricted to circumstances in which a function has a domain that excludes one point. In fact the limit is a concept that occurs frequently in mathematics and it is cornerstone of calculus.

EXAMPLE 2: Find $\lim\limits_{x\to 2} x + 2$.

From the graph:



From the left: As x approaches 2 from the left y approaches 4.

Notation: $\lim_{x \to 2^-} x + 2 = 4$. (Left-Hand Limit)

From the right: As x approaches 2 from the right y approaches 4.

Notation: $\lim_{x \to 2^+} x + 2 = 4$. (Right-Hand Limit)

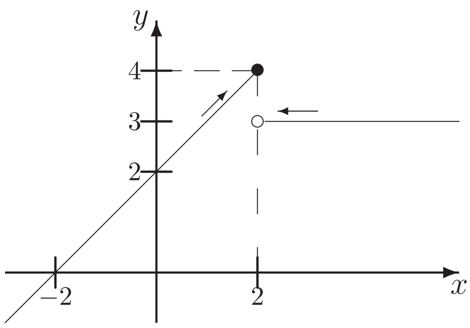
We have $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 4$.

As the left-hand limit equals the right-hand limit, we can say that the limit exists and $\lim_{x \to 2} f(x) = 4$.

Example 3: Find $\lim_{x \to 2} f(x)$ when

$$f(x) = \begin{cases} x + 2 & , \quad x \leqslant 2 \\ 3 & , \quad x > 2 \end{cases}$$

From the graph:



From the left: $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x + 2 = 4$

From the right: $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3 = 3$

Here $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

i.e. Left-Hand Limit $\neq$ Right-Hand Limit

$\therefore \quad \lim_{x \to 2} f(x)$ does not exist.

Thus for a limit to exist, we must have

$$\text{Left-Hand Limit} = \text{Right-Hand Limit}.$$

Only then can we say

$$\lim_{x \to a} f(x) = L, \quad \text{where } L \text{ is the limit.}$$

2.6.1 Limit Laws

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then we have the following properties

1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

3. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$

2.6.2 Evaluating Limits

We need to be able to evaluate limits like

$$\lim_{x\to\infty} f(x), \quad \lim_{x\to 0} g(x), \quad \lim_{x\to a} h(x),$$

In some cases, there are no problems. For example we can often evaluate a limit by simply substituting the limiting value of x into the function:

$$\begin{aligned}
\lim_{x\to 1} x^2 + 2x + 1 &= \lim_{x\to 1} x^2 + \lim_{x\to 1} 2x + \lim_{x\to 1} 1 \\
&= 1 + 2 + 1 \\
&= 4, \\
\lim_{x\to 0} \cos(x^2) &= \cos 0 \\
&= 1, \\
\lim_{x\to\infty} \frac{1}{x^2} &= \left(\lim_{x\to\infty} \frac{1}{x}\right)^2 \\
&= 0, \\
\lim_{x\to\infty} x^2 &= \left(\lim_{x\to\infty} x\right)^2 \\
&= \infty.
\end{aligned}$$

In the last case, x is obviously unbounded, and there is no limit—we say the limit does not exist, although we often use the language that "the function tends to infinity" or that "the limit is infinity". (Note that this is the only time that you can say something is equal to infinity—that is, when the limit is infinity).

In general, for powers of x:

$$\lim_{x\to\infty} x^q = \begin{cases} 0, & q < 0 \\ 1, & q = 0 \\ \infty, & q > 0 \end{cases}.$$

Sometimes this result is expressed using the inverse:

$$\lim_{x\to\infty} \frac{1}{x^p} = \begin{cases} 0, & p > 0 \\ 1, & p = 0 \\ \infty, & p < 0 \end{cases}.$$

For functions that may be discontinuous functions, we need to check the left and right limits. For example, find $\lim_{x\to 1} f(x)$, where

$$f(x) = \begin{cases} 2x+1, & x \le 1 \\ x^2+2, & x > 1 \end{cases}.$$

Now,

$$\lim_{x\to 1^-} f(x) = 2\times 1 + 1 = 3, \qquad \lim_{x\to 1^+} f(x) = 1^2 + 2 = 3.$$

Both left and right limits exist and are equal.

$$\therefore \lim_{x\to 1} f(x) = 3.$$

Sometimes we have to be careful. For example, consider $\lim_{x\to 1} f(x)$, where

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \ne 1 \\ 3, & x = 1 \end{cases}.$$

Now, as we saw earlier we can cancel common factors,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)} = x+1,$$

so that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+1) = 2 = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1).$$

Therefore the left and right limits of $f(x)$ at $x = 1$ exist and are equal:

$$\lim_{x\to 1} f(x) = 2.$$

Note however that $f(1) = 3$. The function is *discontinuous*, yet the limit exists at $x = 1$:

$$\lim_{x\to 1} f(x) \ne f(1).$$

Definition: A **continuous function** is one for which everywhere in its domain, function values are equal to the limit:

$$\lim_{x\to a} f(x) = f(a) \qquad \text{for all } a \in D_f.$$

All polynomial, trigonometric and exponential functions have this property. An intuitive definition for continuity is that a function is continuous if its graph can be drawn without lifting the pen off the page.

Limits that can't be evaluated by simple substitution. Suppose that:

$$\lim_{x\to a} f(x) = 0, \quad \lim_{x\to a} g(x) = 0, \quad \lim_{x\to a} h(x) \to \infty, \quad \lim_{x\to b} f(x) \to \infty, \quad \lim_{x\to b} g(x) \to \infty$$

In the following cases further analysis is needed to evaluate the limit. Usually we need to change the form of the function:

1. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$. Looks like 0/0.
2. $\lim\limits_{x\to b} \dfrac{f(x)}{g(x)}$. Looks like ∞/∞.
3. $\lim\limits_{x\to a} f(x)h(x)$. Looks like $0 \times \infty$.
4. $\lim\limits_{x\to b} f(x) - g(x)$. Looks like $\infty - \infty$.

To illustrate the type of analysis we can do, we will consider ratios of polynomials. For example, find $\lim\limits_{x\to\infty} f(x)$, where

$$f(x) = \frac{x^2 + 2x + 5}{x^2 + 3x + 4}$$

{Note: there are no common factors. If there are, analysis is often aided by cancelling them.}

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x^2 + 2x + 5}{x^2 + 3x + 4} \quad \text{(we try dividing top and bottom by the highest power of } x\text{)} \\
&= \lim_{x\to\infty} \frac{\cancel{x^2}\left(1 + \frac{2}{x} + \frac{5}{x^2}\right)}{\cancel{x^2}\left(1 + \frac{3}{x} + \frac{4}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{\left(1 + \frac{2}{x} + \frac{5}{x^2}\right)}{\left(1 + \frac{3}{x} + \frac{4}{x^2}\right)} \\
&= \frac{1 + 0 + 0}{1 + 0 + 0} \\
&= 1.
\end{aligned}$$

Hence the limit exists, and

$$\lim_{x\to\infty} f(x) = 1.$$

Note: Some standard limits to remember:

- $\lim\limits_{x\to 0} \dfrac{1}{x} \to \infty$
- $\lim\limits_{x\to\infty} \dfrac{1}{x} \to 0$
- $\lim\limits_{x\to 0} \dfrac{c}{x} \to \infty$
- $\lim\limits_{x\to\infty} \dfrac{x}{c} \to \infty$

As another example, find $\lim_{x\to\infty} g(x)$, where $g(x) = \dfrac{x^3+1}{x^4+3x+4}$

Now,

$$\begin{aligned}
\lim_{x\to\infty} g(x) &= \lim_{x\to\infty} \frac{x^3+1}{x^4+3x+4} \quad \text{(divide top and bottom by the highest power of } x\text{)} \\
&= \lim_{x\to\infty} \frac{\not{x}^4\left(\frac{1}{x}+\frac{1}{x^4}\right)}{\not{x}^4\left(1+\frac{3}{x^3}+\frac{4}{x^4}\right)} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^4}}{1+\frac{3}{x^3}+\frac{4}{x^4}} \\
&= \frac{0}{1} = 0
\end{aligned}$$

How do we determine if the limit does not exist? For example, consider:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3+4x^2}{x^2+1} &= \lim_{x\to\infty} \frac{\not{x}^3\left(1+\frac{4}{x}\right)}{\not{x}^3\left(\frac{1}{x}+\frac{1}{x^3}\right)} \\
&= \lim_{x\to\infty} \frac{1+\frac{4}{x}}{\frac{1}{x}+\frac{1}{x^3}} \\
&= \frac{1+0}{0+0} = \frac{1}{0} \to \infty \\
&\therefore \quad \lim_{x\to\infty} \frac{x^3+4x^2}{x^2+1} \to \infty \quad \text{and the limit does not exist}
\end{aligned}$$

In general, if

$$\begin{aligned}
p(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0 \\
q(x) &= b_m x^m + b_{m-1}x^{n-1} + \cdots + b_1 x + b_0, \qquad b_m \neq 0,
\end{aligned}$$

then

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} \infty, & n > m \\ \frac{a_n}{b_m}, & n = m \\ 0, & n < m \end{cases}.$$

Limits that go to ∞ can be converted to limits that go to 0 and vice versa by a substitution of the form: $x = \dfrac{1}{y}$. For example, consider:

$$\lim_{x\to\infty} \frac{x^2+2x+5}{x^2+3x+4}$$

Let $y = \dfrac{1}{x}$ $\left(\text{or } x = \dfrac{1}{y}\right)$: then, as $x \to \infty$, $y \to 0$

$$\begin{aligned}
\frac{x^2+2x+5}{x^2+3x+4} &= \frac{\frac{1}{y^2}+2\frac{1}{y}+5}{\frac{1}{y^2}+3\frac{1}{y}+4} \quad \text{(multiply top and bottom by the highest power of } y) \\
&= \frac{y^2\left(\frac{1}{y^2}+2\frac{1}{y}+5\right)}{y^2\left(\frac{1}{y^2}+3\frac{1}{y}+4\right)} \\
&= \frac{1+2y+5y^2}{1+3y+4y^2}
\end{aligned}$$

Noting that $\lim\limits_{x\to\infty}\frac{1}{x} = \lim\limits_{y\to 0} y = 0$,

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^2+2x+5}{x^2+3x+4} &= \lim_{y\to 0}\frac{1+2y+5y^2}{1+3y+4y^2} \\
&= \frac{1+0+0}{1+0+0} \\
&= 1
\end{aligned}$$

In the following example the limit is of the form $\infty \times 0$ so we need other methods to change the form of the function. Consider:

$$\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x+a}-\sqrt{x}\right)$$

We need to re-arrange this into a format that allows us to determine the limit, if it exists.

$$\begin{aligned}
\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x+a}-\sqrt{x}\right) &= \lim_{x\to\infty}\sqrt{x}\left(\sqrt{x}\left(1+\frac{a}{x}\right)^{\frac{1}{2}}-\sqrt{x}\right) \\
&= \lim_{x\to\infty} x\left(\left(1+\frac{a}{x}\right)^{\frac{1}{2}}-1\right)
\end{aligned}$$

Using the Binomial series,

$$\begin{aligned}
\left(1+\frac{a}{x}\right)^{\frac{1}{2}} &= 1+\frac{1}{2}\left(\frac{a}{x}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{a}{x}\right)^2 + +\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{a}{x}\right)^3+\cdots \\
&= 1+\frac{a}{2x}-\frac{a^2}{8x^2}+\frac{a^3}{16x^3}-\cdots
\end{aligned}$$

$$\begin{aligned}
\text{Thus } \lim_{x\to\infty}\sqrt{x}\left(\sqrt{x+a}-\sqrt{x}\right) &= \lim_{x\to\infty} x\left(\not{1}+\frac{a}{2x}-\frac{a^2}{8x^2}+\frac{a^3}{16x^3}-\cdots-\not{1}\right) \\
&= \lim_{x\to\infty}\left(\not{x}.\frac{a}{2\not{x}}-\not{x}.\frac{a^2}{8x^{\not{2}}}+\not{x}.\frac{a^3}{16x^{\not{3}2}}-\cdots\right) \\
&= \frac{a}{2}+\lim_{x\to\infty}\left(-\frac{a^2}{8x}+\frac{a^3}{16x^2}-\cdots\right) \qquad **\\
&= \frac{a}{2}+0 \\
&= \frac{a}{2}
\end{aligned}$$

** Note that all the terms in the brackets contain $\frac{1}{x^k}$, where $k = 1, 2, \cdots$, and all of these will go to zero, as $x \to \infty$.

EXERCISE 24: Evaluate the following limits, if they exist:

1. $\lim_{x\to\infty} \frac{x^3+5x+1}{2x^3+7x-25}$
2. $\lim_{x\to 0} \frac{x^3+5x}{2x^3+7x}$
3. $\lim_{x\to\infty} \frac{x^2+3+(2/x)}{3x^2+4x+2+(1/x)}$
4. $\lim_{x\to\infty} \frac{x^4+3x^2+2x+1}{3x^3+2x}$
5. $\lim_{x\to 0} \frac{x^4+3x^2+2x+1}{3x^3+2x}$
6. $\lim_{x\to\infty} \frac{4x^4+x^2+x+1}{4x^3+x^2+x+1}$
7. $\lim_{x\to\infty} \frac{4x^3+x^2+x+1}{4x^4+x^2+x+1}$
8. $\lim_{x\to\infty} \frac{x^3+2x+2}{3x^3+4x^2+2x+1}$
9. $\lim_{x\to 0} \frac{x^3+2x^2+2}{3x^3+4x^2+2x+1}$
10. $\lim_{x\to 2} \frac{3x+12}{x+4}$
11. $\lim_{x\to -4} \frac{3x+12}{x+4}$
12. $\lim_{x\to -4} \frac{3x+11}{x+4}$
13. $\lim_{x\to 1} \frac{(x-1)(x+1)}{x-1}$
14. $\lim_{x\to 5} \frac{x^2-6x+5}{x-5}$
15. $\lim_{x\to -3} \frac{x^2+5x+6}{x^2+11x+24}$
16. $\lim_{x\to 1} \frac{x^2-2x+1}{x^3+x^2-x-1}$
17. $\lim_{x\to\infty} \sqrt{x^2+x}-x$
18. $\lim_{x\to -\infty} \sqrt{x^2+x}-x$

2.7 Rational Functions

Recall that a rational function is of the form

$$\boxed{f(x) = \frac{P(x)}{Q(x)}}$$

where $P(x)$ and $Q(x)$ are polynomials and $\boxed{Q(x) \neq 0}$. We will consider some special cases of these, namely functions of the form

$$f(x) = \frac{1}{x}, \quad f(x) = \frac{a}{x-h} + k, \quad f(x) = \frac{1}{x^2}, \quad \text{and} \quad f(x) = \frac{a}{(x-h)^2} + k.$$

Through the questions at the end of the section we will make some comments about the case when the $P(x)$ and $Q(x)$ are any quadratic. We will ultimately draw graphs of these functions. In order to do this we will draw together some of the ways of characterising functions that we have introduced in this chapter.

2.7.1 Asymptotes

Definition 1. (Horizontal Asymptote) The line $y = L$ is called a *horizontal asymptote* of $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

For example, $y = \dfrac{1}{x}$ has a horizontal asymptote at $y = 0$ (the $x-$axis) as:

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

(Vertical Asymptote) The line $x = a$ is called a *vertical asymptote* of $y = f(x)$ if at least one of the following statements is true.

$$\lim_{x\to a} f(x) = \infty; \quad \lim_{x\to a^-} f(x) = \infty; \quad \lim_{x\to a^+} f(x) = \infty;$$

$$\lim_{x\to a} f(x) = -\infty; \quad \lim_{x\to a^-} f(x) = -\infty; \quad \lim_{x\to a^+} f(x) = -\infty.$$

For *Rational Functions*, vertical asymptotes are located by equating the denominator to zero, after cancelling common factors. (This method does not apply for other functions.)

2.7.2 Graphs of rational functions

We will refer to the case where $Q(x)$ is a linear function as a simple rational function. In the examples that follow the numerator is either constant or linear. We can cater for more general numerators by using long division, there are some examples of these in the exercises.

$$\boxed{\text{A graph of } y = f(x) = \frac{1}{x}}$$

We begin by exploring the characteristic features of this function. The graph follows by putting this information together.

Domain of $f(x)$: Because the denominator is zero at $x = 0$ the domain excludes this point: $D_f = \{x \in \mathbb{R} \ : \ x \neq 0\}$.

Sign of $f(x)$: Because the numerator is positive, when $x > 0$, $f(x) > 0$,
and when $x < 0$, $f(x) < 0$.

Increasing/decreasing: We have when $x_2 > x_1 > 0$ that $\dfrac{1}{x_2} < \dfrac{1}{x_1}$ so $f(x)$ is decreasing when $x > 0$.
And when $x_1 < x_2 < 0$ that $\dfrac{1}{x_1} > \dfrac{1}{x_2}$ so $f(x)$ is also decreasing when $x < 0$.

Intercepts: x intercepts: these occur at $y = 0$ so solve $0 = 1/x$ $\quad\therefore$ no x intercepts.
y intercepts: these occur at $x = 0$. This point is excluded from the domain, so there are no y intercepts.

Vertical Asymptotes: Since the denominator is zero at $x = 0$ in the factorised form of the function, there is a *vertical asymptote* at $x = 0$. Taking account of the information we have about when $f(x)$ is positive or negative we have:

$$\lim_{x\to 0^-} f(x) = -\infty$$

$$\lim_{x\to 0^+} f(x) = \infty$$

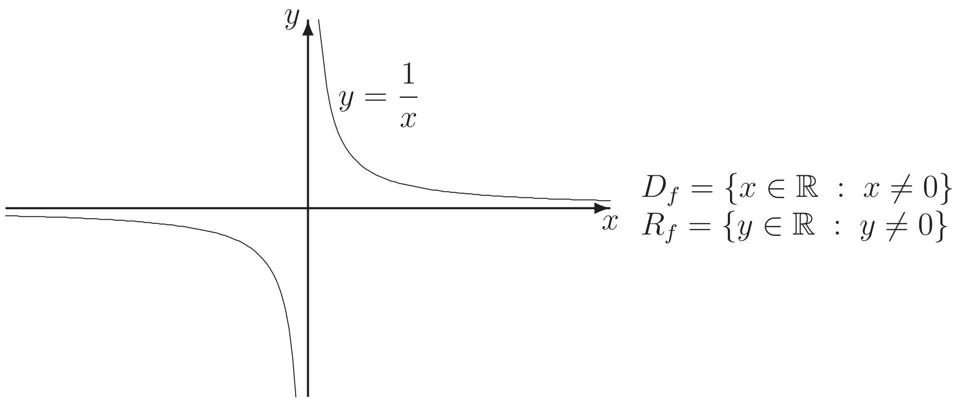
Horizontal Asymptotes:
$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{1}{x} = 0^-$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to \infty} \frac{1}{x} = 0^+$$

Hence $y = 0$ is a *horizontal asymptote*.

Recall from our work on limits that the symbols 0^- and 0^+ represent approaching the point 0 from values less than zero or greater than zero respectively.

The graph of $f(x) = \dfrac{1}{x}$ (a *rectangular hyperbola*) can now be sketched.



EXAMPLE 1: Sketch the graph of $f(x) = \dfrac{1}{x-2}$.

Domain of $f(x)$: Because the denominator is zero at $x = 2$ the domain excludes this point: $D_f = \{x \in \mathbb{R} \ : \ x \neq 2\}$.

Sign of $f(x)$: Because the numerator is positive, when $x > 2$, $f(x) > 0$,
and when $x < 2$, $f(x) < 0$.

Increasing/decreasing: We have when $x_2 > x_1 > 2$ that $\dfrac{1}{x_2 - 2} < \dfrac{1}{x_1 - 2}$ so $f(x)$ is decreasing when $x > 2$.

And when $x_1 < x_2 < 2$ that $\dfrac{1}{x_1 - 2} > \dfrac{1}{x_2 - 2}$ so $f(x)$ is also decreasing when $x < 2$.

Intercepts: x intercepts: these occur at $y = 0$ so solve $\quad 0 = 1/(x-2) \quad \therefore$ no x intercepts.
y intercepts: these occur at $x = 0$. Thus $y = 1/(0-2) = -1/2$ is the y intercept.

Vertical Asymptotes: Since the denominator is zero at $x = 2$ in the factorised form of the function, there is a *vertical asymptote* at $x = 2$. Taking account of the information we have about when $f(x)$ is positive or negative we have:

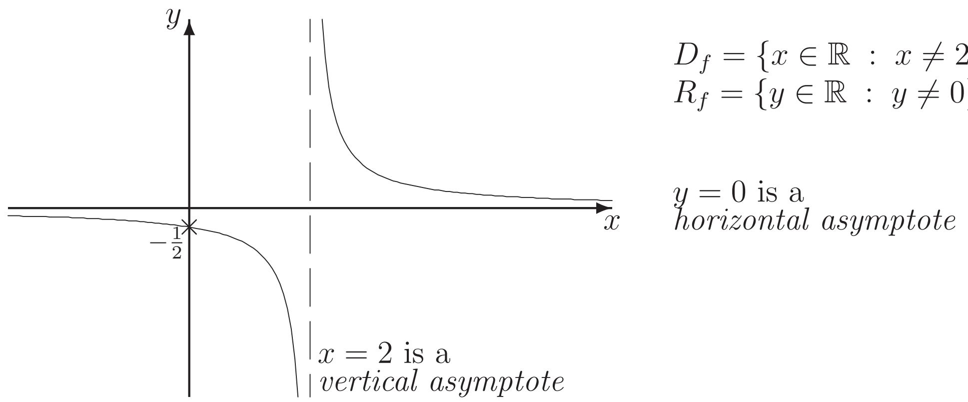
$$\lim_{x \to 2^-} f(x) = -\infty$$

$$\lim_{x \to 2^+} f(x) = \infty$$

Horizontal Asymptotes: $\displaystyle\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x-2} = 0^-$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \frac{1}{x-2} = 0^+$$

Hence $y = 0$ is a *horizontal asymptote*.



Note that the graph of $f(x) = \dfrac{1}{x-2}$ is identical to that of $f(x) = \dfrac{1}{x}$, but it has been shifted to 2 units to the right.

2.7.3 Translations of rational functions

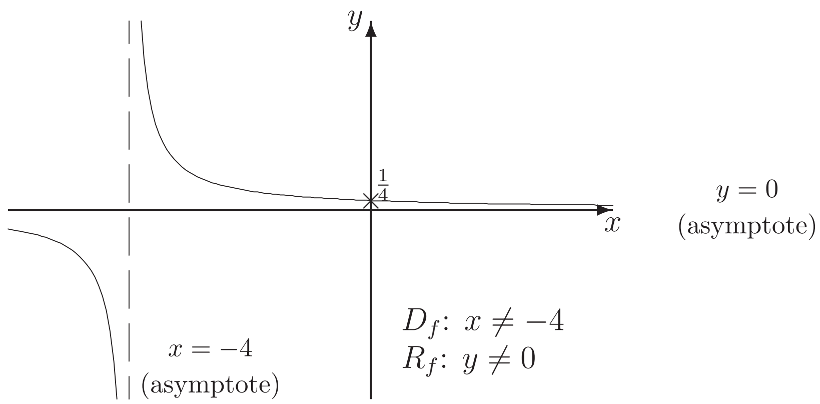
The analysis above allows us to generalise: the graph of $f(x) = \dfrac{1}{x-h}$ is identical to the graph of $f(x) = \dfrac{1}{x}$, but translated h units to the right. This is a **horizontal translation**.

Example 2: Graph the function $f(x) = \dfrac{1}{x+4}$.

We identify this as being of the form $f(x) = \dfrac{1}{x-h}$ and will determine the quantity h.

$$f(x) = \frac{1}{x+4} = \frac{1}{x-(-4)}$$
$$\therefore \quad h = -4.$$

Hence the graph of $f(x) = \dfrac{1}{x+4}$ is identical to that of $f(x) = \dfrac{1}{x}$, but it has been shifted -4 units to the right. i.e. 4 units to the left. Using this approach there is no need to identify characteristic features of the function $f(x) = 1/(x+4)$ - they have been identified through the connection to the function $f(x) = 1/x$. The graph is thus:



Note that to aid in the detail we can compute intercepts:

<u>x intercepts</u>: these occur at $y = 0$ so solve $\quad 0 = 1/(x+4) \quad \therefore$ no x intercepts.
<u>y intercepts</u>: these occur at $x = 0$. Thus $y = 1/(0+4) = 1/4$ is the y intercept.

Example 3: Graph $f(x) = \frac{1}{x} + 1$

Domain of $f(x)$: As the denominator is zero at $x = 0$ the domain excludes this point:

$$D_f = \{x \in \mathbb{R} \ : \ x \neq 0\}$$

Sign of $f(x)$: Consider: $f(x) = \frac{1}{x} + 1 = \frac{x+1}{x}$.

$f(x) > 0$ when $x < -1$ or when $x > 0$ and $f(x) < 0$ when $-1 < x < 0$

Increasing/decreasing:

For $x_2 > x_1 > 2$ we have $\frac{1}{x_2} < \frac{1}{x_1}$ so $f(x)$ is decreasing when $x > 0$.

For $x_1 < x_2 < 2$ we have $\frac{1}{x_1} > \frac{1}{x_2}$ so $f(x)$ is also decreasing when $x < 2$.

x intercept: $y = 0$

$$0 = \frac{1}{x} + 1$$
$$\therefore \quad \frac{1}{x} = -1$$
$$\therefore \quad x = -1$$

y intercept: $x = 0$ This point is excluded from the domain, $\therefore$ no y intercepts.

Vertical Asymptotes:
Since the denominator is zero at $x = 0$ in the factorised form of the function there is a vertical asymptote when $x = 0$.Taking account of the information we have about when $f(x)$ is positive or negative we have:

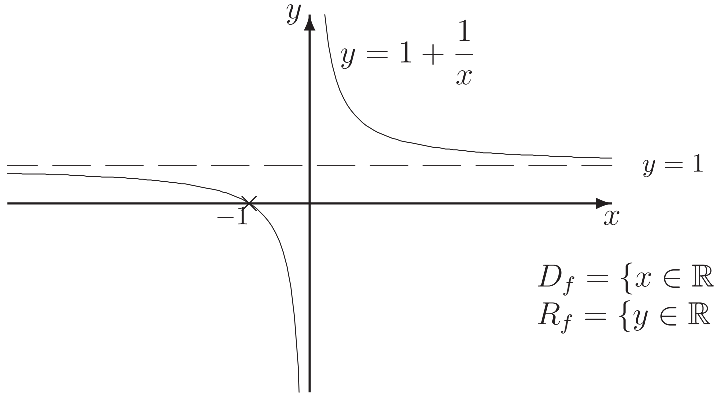
$$\lim_{x \to 0^-} f(x) = -\infty$$

$$\lim_{x \to 0^+} f(x) = \infty$$

Horizontal Asymptotes: $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} + 1 = 0^- + 1 = 1^-$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \frac{1}{x} + 1 = 0^+ + 1 = 1^+$$

Hence $y = 1$ is a horizontal asymptote.

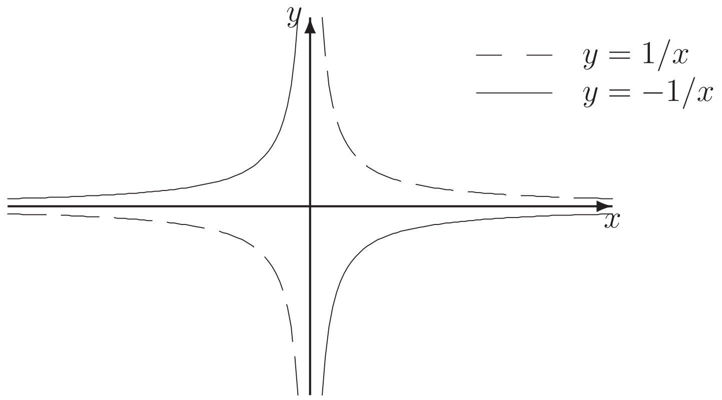


$$D_f = \{x \in \mathbb{R} \ : \ x \neq 0\}$$
$$R_f = \{y \in \mathbb{R} \ : \ y \neq 1\}$$

The graph of $f(x) = \dfrac{1}{x} + 1$ is the same shape as the graph of $f(x) = \dfrac{1}{x}$ but it has been translated vertically upward by one unit. This is a **vertical translation.**

In general: The graph of $f(x) = \dfrac{1}{x-h} + k$ is the same shape as $f(x) = \dfrac{1}{x}$, *translated h units along the x-axis* and k *units up the y-axis.*

Example 4: Graph $y = -\dfrac{1}{x}$.



The graph of $f(x) = -\dfrac{1}{x}$ is the reflection in the $x-$axis of the graph $f(x) = \dfrac{1}{x}$.

Exercise: Describe how the graph of $y = \dfrac{a}{x}$ is related to the graph of $y = \dfrac{1}{x}$.

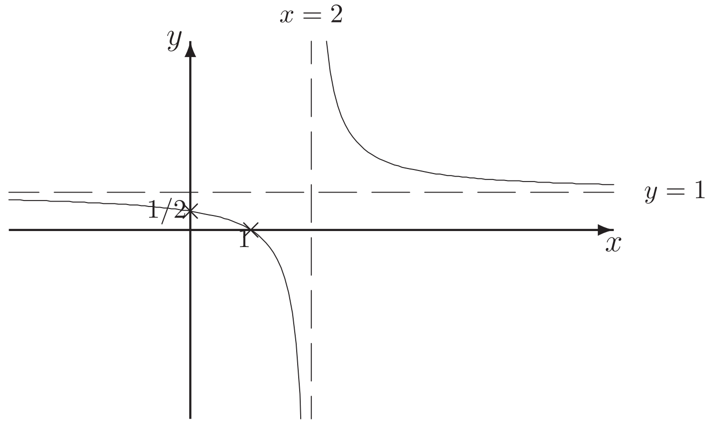
Example 5: Sketch the graph of $f(x) = \dfrac{x-1}{x-2}$.

We can approach this in one of two ways. The first approach is to do a complete analysis such as for example 1, determining the characteristic features of the function and synthesising this information to come up with the graph. The second approach is to place the function in the form of $f(x) = \dfrac{1}{x-h} + k$. We can then identify the graph as a translation of $f(x) = \dfrac{1}{x}$.

We take the second approach and use long division.

$$\begin{array}{r} 1 \\ x-2 \overline{)\, x-1} \\ \underline{x-2} \\ 1 \end{array}$$

$\therefore \quad f(x) = 1 + \dfrac{1}{x-2}$ is now in the required form. Hence we must translate $f(x) = \dfrac{1}{x}$ two units to the right and one unit vertically (upward).



$\underline{x \text{ intercepts}}$: $(y = 0) \quad \therefore \quad x = 1$
$\underline{y \text{ intercepts}}$: $(x = 0) \quad \therefore \quad y = -1/2$
$\underline{\text{Domain/Range}}$: $D_f = \{x \in \mathbb{R} \, : \, x \neq 2\}$
$R_f = \{y \in \mathbb{R} \, : \, y \neq 1\}$

Be sure to repeat this example using a complete analysis such as in example 1. This involves identifying the domain of the function, where the function is positive, where the function is negative (using inequalities), where the function is increasing, where the function is decreasing (using properties of the $1/x$ function) and the location of the vertical and horizontal asymptotes using limits.

EXERCISE 25:

1. Sketch the graphs of the following. Be sure that you can do these either (i) using a complete analysis or by (ii) recognising the translation of the $1/x$ function. State the domain and range, and any horizontal and vertical asymptotes:
   (a) $f(x) = \dfrac{2}{3-x}$ (b) $f(x) = \dfrac{x-2}{x+2}$
2. In each of the following graph the function. Using partial fractions should help in all of these, while in (b) and (c) long division should be helpful in identifying some features of the graph. In each case a complete analysis should be conducted - although note that determining intervals on which these functions are increasing/decreasing may be difficult.
   (a) $f(x) = \dfrac{x+1}{(x+3)(x-1)}$ (c) $f(x) = \dfrac{x(x-2)}{(x-1)(x+1)}$
   (b) $f(x) = \dfrac{x^2}{x^2-4}$
3. (a) Sketch the graph of $y = x^2 - 1$
   (b) On the same set of axes, sketch its reciprocal $y = \dfrac{1}{x^2-1}$ stating its domain and range and any horizontal and vertical asymptotes.
4. On the same set of axes, sketch the graphs of $y = (x+1)(2-x)$ and its reciprocal $y = \dfrac{1}{(x+1)(2-x)}$

2.7.4 Even and Odd Functions

We now present another way of characterising functions. This is in terms of their symmetry.

An *even* function satisfies the condition

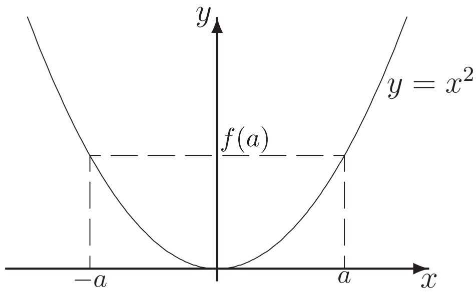
$$\boxed{f(-x) = f(x)}$$

The graphs of even functions are symmetric about the y-axis.

EXAMPLE: Consider $f(x) = x^2$.

To show that this is an even function, we calculate $f(-x)$.

$f(-x) = (-x)^2$ and $(-x)^2 = x^2$ which is equal to $f(x)$. $\quad\therefore\quad f(x) = x^2$ is an even function.



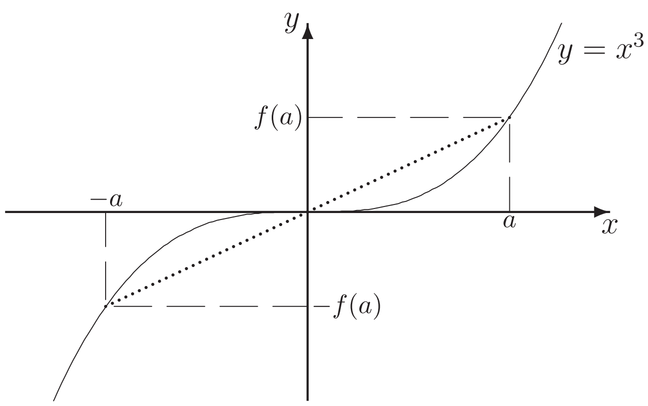
An *odd* function satisfies the condition

$$\boxed{f(-x) = -f(x)}$$

The graphs of odd functions are symmetric through the origin.

EXAMPLE: Consider $f(x) = x^3$.

$f(-x) = (-x)^3 = -x^3 = -f(x) \quad\therefore\quad f(x) = x^3$ is an odd function.



Some functions are neither even nor odd.

EXAMPLE: Consider $f(x) = x^2 + x$.

$f(-x) = (-x)^2 + (-x) = x^2 - x$ which is not the same as either $f(x)$ or $-f(-x)$. Therefore $f(x) = x^2 + x$ is neither even nor odd.

NOTE: Knowing whether a function is odd or even is useful when you are sketching some of the more difficult graphs.

EXERCISE 26:

1. Are the following functions even, odd or neither?

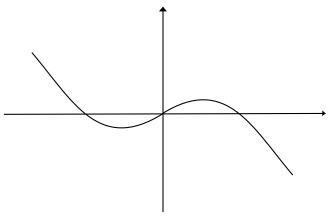
   (a) $f(x) = x^4 + x$

   (b) $g(x) = x^3 - 2x$

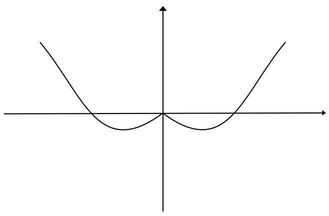
   (c) $h(x) = x^{-2}$

   (d) $p(x) = \dfrac{1}{x^3}$

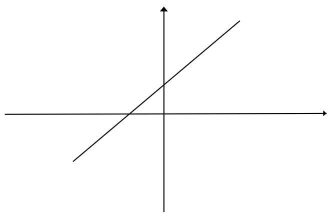
   (e)



   (f)



   (g)



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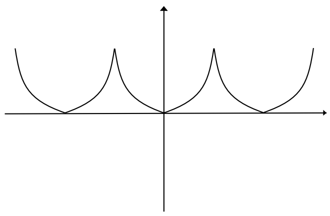


2. Prove the following:

   (a) The sum and the product of even functions are both even.

   (b) The sum of odd functions is odd and the product of two odd functions is even.

   (c) If $f(x)$ is odd then $|f(x)|$ and $f(x)^2$ are even.

   (d) If $f(x)$ is even, then $|f(x)|$ and $f(x)^2$ are even.

3. Represent the function $f(x) = \dfrac{1}{(1-x)^2}$ as the sum of an even and an odd function.

2.7.5 Graphs of rational functions with an exact square in the denominator

In the previous sections we have been concerned with cases where the denominator of the rational function has been linear. In the exercises (Ex 25, Q2) we saw that cases in which the denominator is a quadratic that can be factorised can be approached by using partial fractions to express the rational function as the sum of rational functions each with linear denominator. In this section we look at the case where the denominator involves an exact square and cannot be factorised.

A graph of $y = f(x) = \dfrac{1}{x^2}$

Domain of $f(x)$: Because the denominator is zero at $x = 0$ the domain excludes this point: $D_f = \{x \in \mathbb{R} \, : \, x \neq 0\}$.

Sign of $f(x)$: Both the numerator and denominator are always positive, $f(x) > 0$ for all x.

Increasing/decreasing: We have when $x_2 > x_1 > 0$ that $\dfrac{1}{x_2^2} < \dfrac{1}{x_1^2}$ so $f(x)$ is decreasing when $x > 0$.

And when $x_1 < x_2 < 0$ that $\dfrac{1}{x_1^2} < \dfrac{1}{x_2^2}$ so $f(x)$ is increasing when $x < 0$.

Even/Odd: $f(-x) = \dfrac{1}{(-x)^2} = \dfrac{1}{x^2} = f(x) \quad \therefore \quad f(x)$ is even.
As the function is even, then it is symmetric about the y-axis.

x intercepts: $(y = 0)$ Need to solve $0 = \dfrac{1}{x^2}$. Can't $\therefore$ no x intercepts.

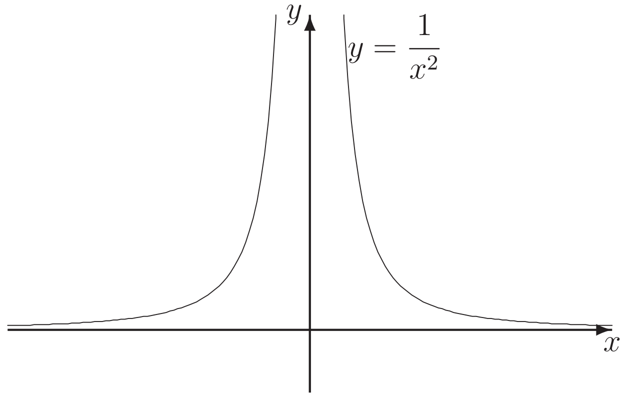
y intercepts: As $x = 0$ is not included in the domain there are no y intercepts

Vertical Asymptotes: Since the denominator is zero at $x = 0$ in the factorised form of the function there is a vertical asymptote when $x = 0$.Taking account of the information we have about when $f(x)$ being positive and even we have:
$\lim_{x \to 0} f(x) = \infty$

Horizontal Asymptotes: $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \dfrac{1}{x^2} = 0^+$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x^2} = 0^+$$

Hence $y = 0$ is a horizontal asymptote.



Domain/Range: $D_f = \{x \in \mathbb{R} \, : \, x \neq 0\} \qquad R_f = \{y \in \mathbb{R} \, : \, y > 0\}$

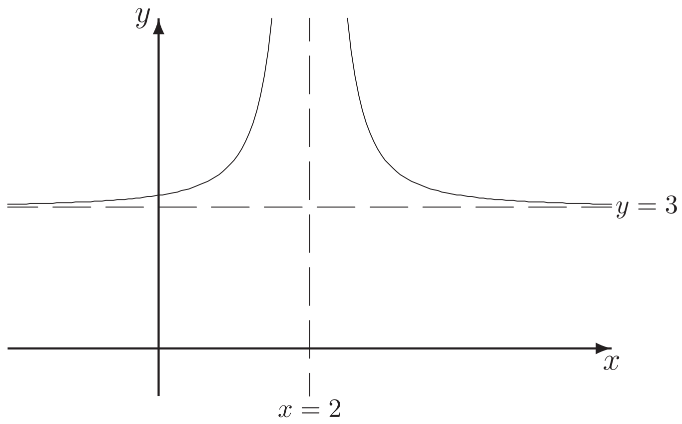
We saw that when a rational function had a linear denominator that we could often identify the graph as being a translation of $f(x) = 1/x$. The same ideas apply here. In particular the graph of $y = \dfrac{a}{(x-h)^2} + k$ is obtained by translating the graph of $y = \dfrac{a}{x^2}$ by h *units to the right* and k units vertically (upwards).

EXAMPLE: Graph the curve $y = \dfrac{1}{(x-2)^2} + 3$.

Here we regonise this as a translation of the graph of $y = \dfrac{1}{x^2}$ two units to the right and one unit up.

x intercepts: Solve $(y = 0)$ $\quad 0 = \dfrac{1}{(x-2)^2} + 3 \quad \therefore \quad \dfrac{1}{(x-2)^3} = -3 \quad \therefore \quad$ no x intercepts.

y intercepts: Compute $(x = 0)$ $\quad y = \dfrac{1}{(-2)^2} + 3 \quad \therefore \quad y = 3\frac{1}{4}$.



Domain/Range: $D_f = \{x \in \mathbb{R} \ : \ x \neq 2\}$
$R_f = \{y \in \mathbb{R} \ : \ y > 3\}$

Be sure that you can complete this example using a complete analysis such as in the earlier examples. This involves identifying the domain of the function, where the function is positive, where the function is negative (using inequalities), where the function is increasing, where the function is decreasing (using properties of the $1/x^2$ function) and the location of the vertical and horizontal asymptotes using limits.

EXERCISE 27:

1. Sketch the graphs of the following functions, stating domain, range, intercepts and any horizontal or vertical asymptotes. Be sure that you can do these questions in two ways (i) identifying the function as a translation of $f(x) = 1/x^2$ and by conducting a complete analysis as in earlier sections.

   (a) $f(x) = \dfrac{1}{x^2} + 1$ (c) $h(x) = -\dfrac{1}{(x-3)^2}$

   (b) $g(x) = \dfrac{1}{(x-3)^2}$ (d) $p(x) = \dfrac{1}{(x+1)^2} + 2$

2. Sketch the graph of the following functions, stating domain, range, intercepts and any horizontal or vertical asymptotes.

   (a) $f(x) = \dfrac{1}{x^2+1}$ (c) $h(x) = \dfrac{2x+3}{x^2+6x+10}$

   (b) $g(x) = \dfrac{1}{(x-3)^2-3}$ (d) $p(x) = \dfrac{2x+3}{x^2+6x+5}$

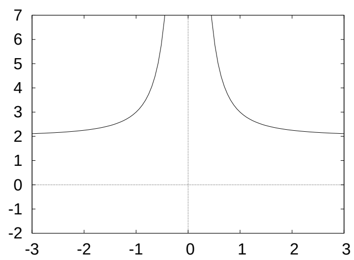
3. How do the examples in the previous question differ from the special case $f(x) = \frac{a}{(x-h)^2} + k$ presented earlier?

4. Which of the following transformations were applied to the graph $y = \dfrac{1}{x^2}$ to obtain each of the graphs shown.

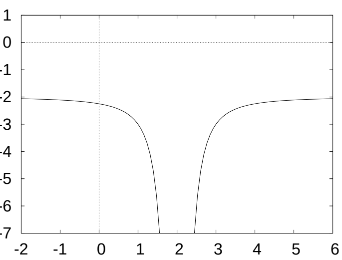
   (A) Translation to the right   (C) Translation up   (E) Reflection in the x-axis

   (B) Translation to the left   (D) Translation down

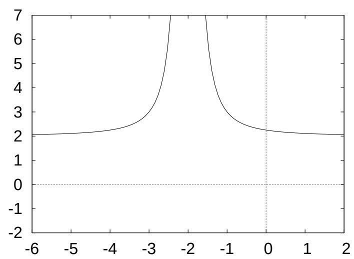
   (a)

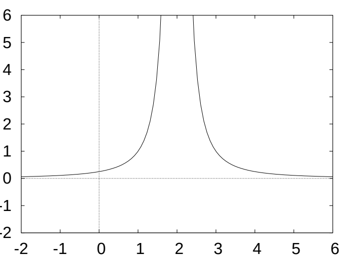
   (c)

   (b)

   (d)

5. If a function is given by $f(x) = \dfrac{1}{x^2}$, sketch each of the following stating domain, range, intercepts and any horizontal or vertical asymptotes.

   (a) $f(x+2)$ (c) $-f(x-1) - 1$

   (b) $f(x-2) + 2$ (d) $f(x+2) - 2$

2.8 Composition of Functions

Definition 1. The composition of two function $f \circ g$ is defined to be

$$\boxed{(f \circ g)(x) = f[g(x)]}$$

Example 1: If $f(x) = 2x + 4$ and $g(x) = x^2$, then find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$\begin{aligned}(f \circ g)(x) &= f[g(x)]\\ &= f[x^2]\\ &= 2x^2 + 4.\\ (g \circ f)(x) &= g[f(x)]\\ &= g[2x+4]\\ &= (2x+4)^2.\end{aligned}$$

In general $(f \circ g)(x) \neq (g \circ f)(x)$.

Example 2: If $f(x) = \sqrt{x-1}$, $x > 1$ and $g(x) = \dfrac{1}{x}$, $x \neq 0$, then find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$\begin{aligned}(f \circ g)(x) &= f[g(x)]\\ &= f\left[\frac{1}{x}\right]\\ &= \sqrt{\frac{1}{x} - 1}.\\ (g \circ f)(x) &= g[f(x)]\\ &= g[\sqrt{x-1}]\\ &= \frac{1}{\sqrt{x-1}}.\end{aligned}$$

EXERCISE 28:

1. For the following functions f and g, find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$.
   (a) $f(x) = 2x + 3, \quad g(x) = 4x - 1$
   (b) $f(x) = 2x^2 - x, \quad g(x) = 3x + 2$
   (c) $f(x) = \sqrt{x^2 - 1}, \quad g(x) = \sqrt{1 - x}$
2. If $y = f \circ g$ is given as follows, find $f(x)$ and $g(x)$.
   (a) $y = \sqrt{2x + 3}$
   (b) $y = (x^2 + 5x + 6)^3$
   (c) $y = (2x)^{-3}$
3. Use composite functions to solve the following:
   (a) $(x+2)^2 - 5(x+2) + 6 = 0$
   (b) $(x-5)^2 - 16 = 0$
   (c) $(2^x)^2 - 5(2^x) + 4 = 0$
   (d) $3(4x^2 - 9) - 2 = 19$

2.9 Inverse Functions

The inverse of a function $f(x)$ must satisfy the following condition:

$$\boxed{(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x}$$

where $f^{-1}(x)$ is the inverse of $f(x)$.

Note the inverse function is *not* the same as the recriprocal of $f(x)$, i.e.

$$\boxed{f^{-1}(x) \neq \frac{1}{f(x)}}$$

Example 1: Find the inverse function $f^{-1}(x)$ for the following:

$$\begin{aligned} f(x) &= \frac{x}{2} - 3 \\ \textit{i.e.} \quad y &= \frac{x}{2} - 3 \end{aligned}$$

We require that $f[f^{-1}(x)] = x$, i.e. $\frac{1}{2}f^{-1}(x) - 3 = x$. Now, solve for $f^{-1}(x)$

$$\begin{aligned} \frac{1}{2}f^{-1}(x) &= x + 3 \\ \therefore \quad f^{-1}(x) &= 2x + 6. \end{aligned}$$

Another method to obtain the inverse is to swap the positions of x and y in the original equation and solve for y. Thus

$$\begin{aligned} y &= \frac{x}{2} - 3 \\ \text{becomes} \quad x &= \frac{y}{2} - 3. \end{aligned}$$

Solving for y gives

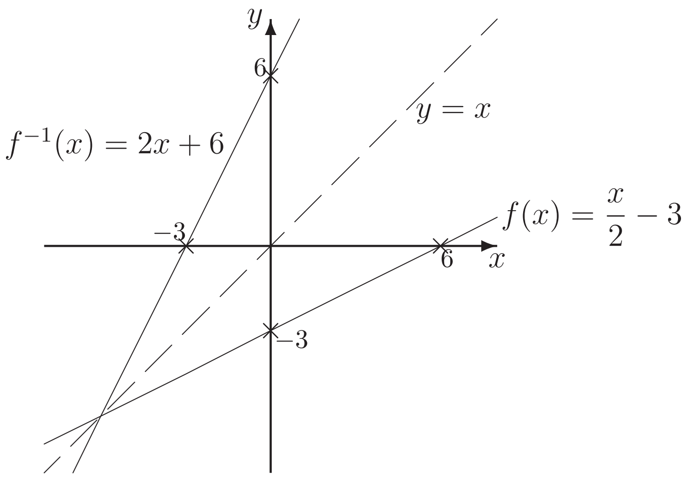
$$\begin{aligned} 2x &= y - 6 \\ \therefore \quad y &= 2x + 6 \end{aligned}$$

i.e. $f^{-1}(x) = 2x + 6$, as before.

We can check our result by verifying that $(f^{-1} \circ f)(x) = x$.

$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}[f(x)] \\ &= f^{-1}\left[\frac{x}{2} - 3\right] \\ &= 2\left[\frac{x}{2} - 3\right] + 6 \\ &= x - 6 + 6 = x \qquad \text{as required.} \end{aligned}$$

The relationship between $f^{-1}(x)$ and $f(x)$ is best illustrated graphically.

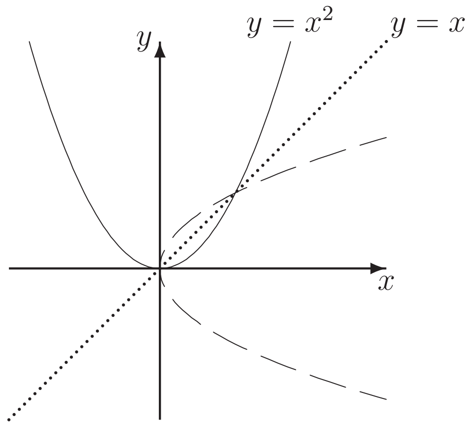


Note that the graphs of $f(x)$ and $f^{-1}(x)$ are actually reflections of each other in the line $y = x$. Also, the domain of f^{-1} is the same as the range of f and vice-versa. i.e.

$$\boxed{\begin{array}{l} D_f = R_{f^{-1}} \\ R_f = D_{f^{-1}} \end{array}}$$

2.9.1 Existence of Inverse Functions

Consider the function $y = x^2$



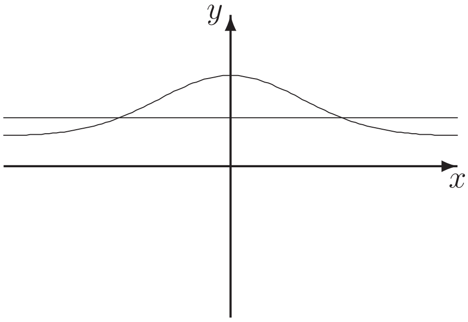
The function $f(x) = x^2$ does not have an inverse function because if you reflect $f(x) = x^2$ in the line $y = x$ the result is not a function (as the resulting curve has two y values for each value of $x > 0$). For this reason a function must be one-to-one to have an inverse.

2.9.2 One-to-One Functions

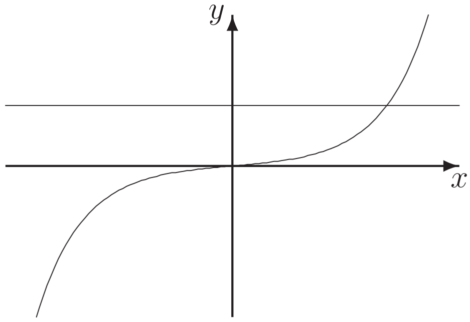
A function is said to be one-to-one if, for each y value in the range, there is only one corresponding x value in the domain.

2.9.3 Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects the graph more than once.



not one-to-one



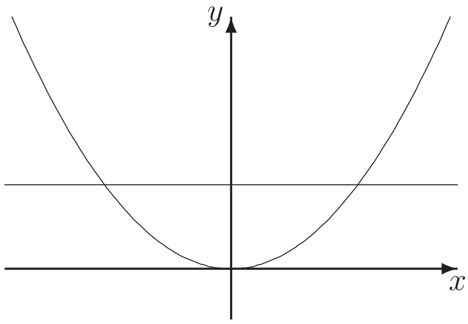
one-to-one

2.9.4 Restricting Domains

Most functions can be made one-to-one by restricting the domain, i.e. we only consider a part of the domain.

EXAMPLE: Consider $f(x) = x^2$.

From sketching the curve $y = x^2$ and using the horizontal line test we see that $f(x)$ is not one-to-one.

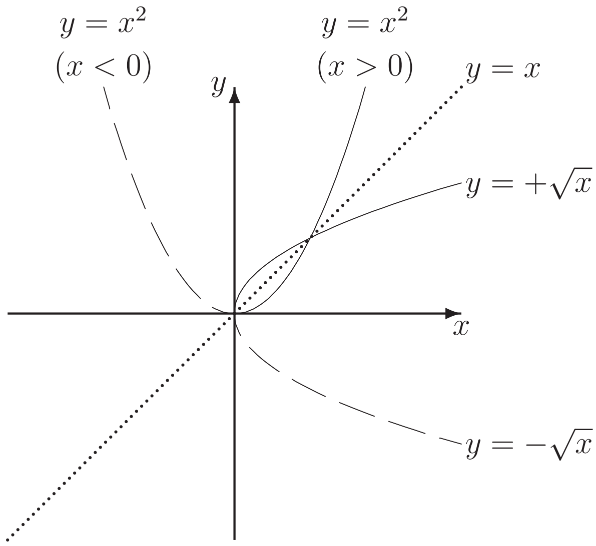


If we now try to get the inverse our result will *not* be a function.

$$f(x) = y = x^2$$

<u>Inverse</u>: Swap x and y.

$$\begin{aligned} x &= y^2 \\ \therefore \quad y &= \pm\sqrt{x} \end{aligned}$$



From the graph: $y = x^2$ has no inverse function
but $f(x) = x^2,\ x > 0$ has an inverse $f^{-1}(x) = +\sqrt{x}$.
and $g(x) = x^2,\ x < 0$ has an inverse $g^{-1}(x) = -\sqrt{x}$.

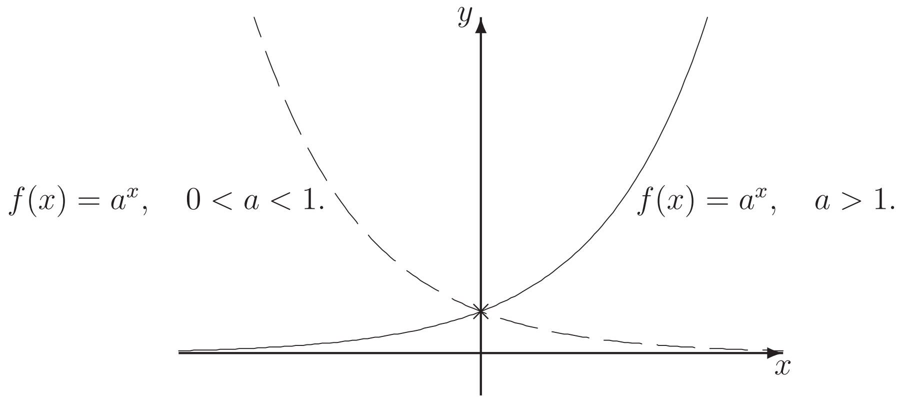
Hence if we restrict the domain of $f(x)$ so that $f(x)$ is one-to-one then we can get an inverse.

EXERCISE 29:

1. Determine if the given functions are *one-to-one*
   (a) $f(x) = 7x - 3$
   (b) $g(x) = x^2 - 2x + 5$
   (c) $h(x) = |x|$
   (d) $p(x) = x^4 + 5, \quad 0 \leq x \leq 2$
2. Find the inverse (if it exists) for the following functions:
   (a) $f(x) = 4x + 6$
   (b) $f(x) = \dfrac{x-2}{x+2}, \quad x \neq -2$
   (c) $f(x) = \dfrac{1+3x}{5-2x}, \quad x \neq \dfrac{5}{2}$
   (d) $f(x) = \sqrt{2+5x}, \quad x \geq -\dfrac{2}{5}$
   (e) $f(x) = x^2 + x, \quad x \geq -\dfrac{1}{2}$
3. (a) Sketch the graph of $f(x) = x^2 - 2x + 1$.
   (b) State two domains for which $f(x) = x^2 - 2x + 1$ defines one-to-one functions.
   (c) Find and graph the two functions which are the inverse of the functions you defined in (b).

2.10 The Exponential and Logarithmic Functions

The function $f(x) = a^x$, where $a \in \mathbb{R}^+,\ a \neq 1$, is an exponential function. i.e. the base a is any positive real number excluding 1. If $a = 1$ then $f(x) = 1$, since $1^x = 1$ (trivial case).



The graph of a^x increases as x increases when $a > 1$, but decreases as x increases when $0 < a < 1$. The domain and range in each case are D_f: $x \in \mathbb{R}$ and R_f: $y > 0$ respectively.

The graph of $f(x) = a^x$ has the following properties:

(i) the curve passes through $(0, 1)$,

(ii) each curve is asymptotic to the x-axis,

(iii) and each curve lies above the x-axis.

If for example, $a = \frac{1}{2}$, then $\left(\frac{1}{2}\right)^x = 2^{-x}$, so we can consider this graph to be of the form $f(x) = a^{-x}, \quad a > 1$.

Exercise: If $a = 1$, what does the graph of $f(x) = a^x$ look like?

If $a = e$, Euler's number, an irrational number where

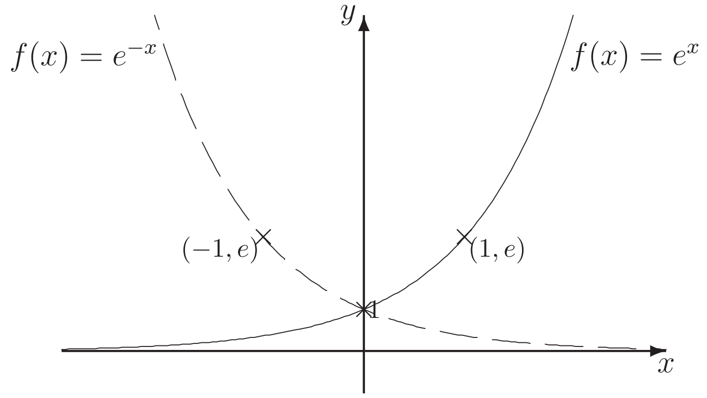
$$e \simeq 2.71828\cdots$$

we obtain "THE" exponential function

$$f(x) = e^x$$

which is sometimes written as $\exp(x)$. The graphs of e^x and e^{-x} have the same domain, range, and properties as graphs of a^x and a^{-x}.

The graphs of e^x and e^{-x} are



You will be expected to recognise and draw these graphs.

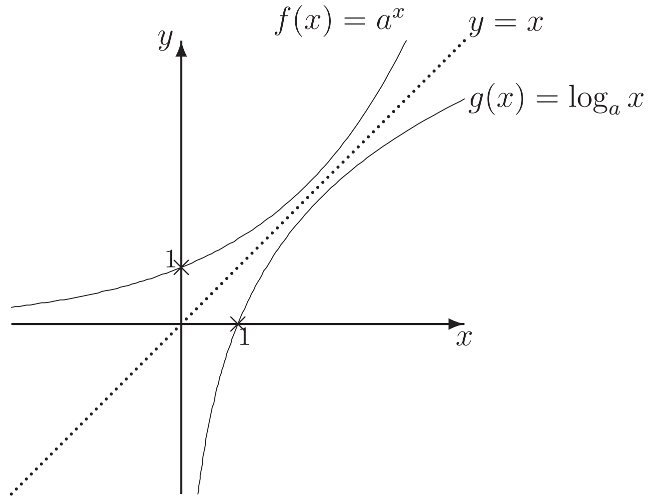
Recall from earlier that $y = a^x$ and $\log_a y = x$ are equivalent statements.

EXAMPLE 1: For $f(x) = a^x$, find the corresponding logarithm function.

Let $y = f(x) = a^x$, $x \in \mathbb{R}$ and $y > 0$. Now interchange x and y, so that $x = a^y$, $y \in \mathbb{R}$ and $x > 0$. The latter is equivalent to $y = g(x) = \log_a x$.

Note on the graphs below that $f(x)$ and $g(x)$ are reflections of each other in the line $y = x$. We can see that $f(x)$ and $g(x)$ are *inverses* of each other.

The graphs of $f(x) = a^x$ and $g(x) = \log_a x$ are



$f(x) = a^x, \quad a > 1$	$g(x) = \log_a x$
$D_f\colon\ x \in \mathbb{R}$	$D_g\colon\ x > 0$
$R_f\colon\ y > 0.$	$R_g\colon\ y \in \mathbb{R}.$

Properties of the log function:

(a) The log function is defined only for $x > 0$, i.e. we can't take the log of a negative.

(b) The y-axis is a vertical asymptote.

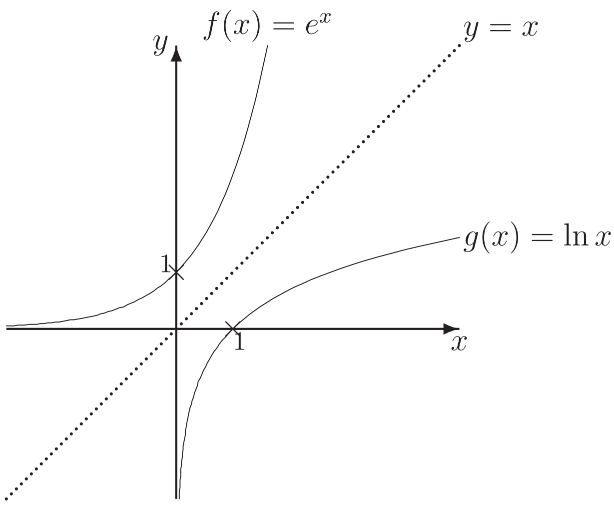
(c) Since $\log_a 1 = 0$ regardless of the value of a, it always passes through the point $(1, 0)$.

The properties of the exponential and logarithm functions give the following results

$$\begin{aligned} \log_a(a^x) &= x \qquad \text{for } x \in \mathbb{R} \\ a^{\log_a x} &= x \qquad \text{for all } x > 0. \end{aligned}$$

The 'power of a' and $\log_a$ effectively 'cancel' each other.

If $a = e$, using the notation above, $f(x) = e^x$, and $g(x) = \log_e x = \log x = \ln x$ (the natural logarithm). The graphs of e^x and $\ln x$ are



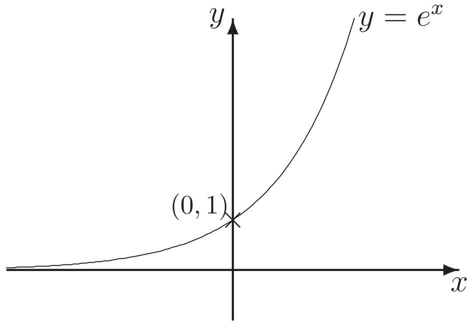
The above 'cancellation' laws then become

$$\begin{aligned} \ln(e^x) &= x \qquad \text{for } x \in \mathbb{R} \\ e^{\ln x} &= x \qquad \text{for all } x > 0. \end{aligned}$$

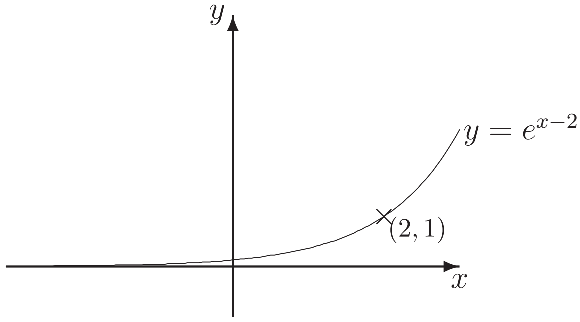
2.10.1 Variations on basic Exponential and Logarithm Functions

The exponential function $f(x) = e^{x-h} + k$ shifts the graph of e^x by h units to the right (along the x-axis) and k units upward (along the y-axis).

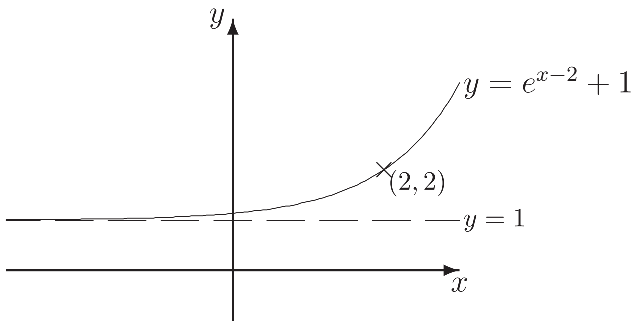
EXAMPLE 2: Graph $f(x) = e^{x-2} + 1$.



$\downarrow$ translation (2 units to the right)



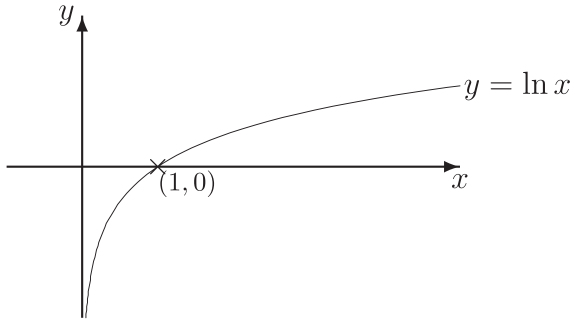
$\downarrow$ translation (1 unit up the y-axis)



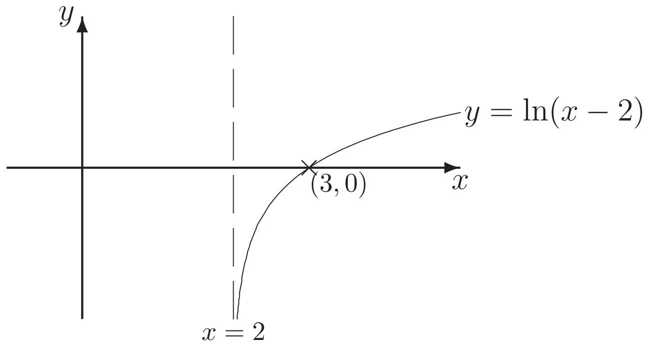
The domain and range of $f(x) = e^{x-2} + 1$ are D_f: $x \in \mathbb{R}$, and R_f: $y > 1$.

The logarithm function $f(x) = \log_a(x - h) + k$ shifts the graph of $\log_a x$ by h units to the right (along the x-axis) and k units upward (along the y-axis).

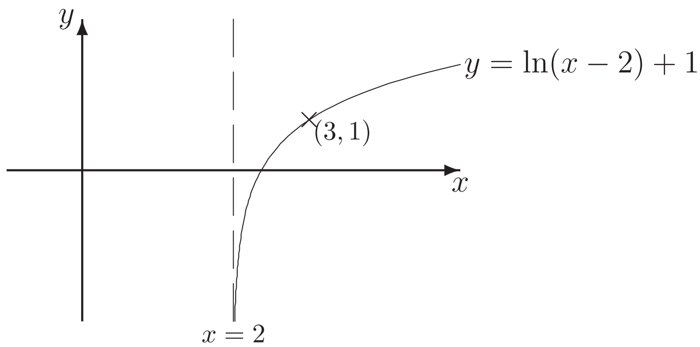
EXAMPLE 3: Graph $f(x) = \log(x - 2) + 1$.



$\downarrow$ translation (2 units to the right)



$\downarrow$ translation (1 unit up the y-axis)



The domain and range of $f(x) = \log(x - 2) + 1$ are D_f: $x > 2$, and R_f: $y \in \mathbb{R}$.

EXERCISE 30:

1. Sketch the graphs of the following, stating the domain and range, and any intercepts, horizontal and vertical asymptotes.
   (a) $f(x) = \ln(x-1)$
   (b) $g(x) = 1 + \log_{10} x$
   (c) $h(x) = e^{x-2}$
   (d) $p(x) = e^x - 2$
   (e) $q(x) = e^{-x} - 2$
2. Show that the graphs of $y = 2^x$ and $y = 2^{x-1}$ cut off line segments of equal length on all lines parallel to the x-axis where $y > 0$.
3. A vertical line cuts the graphs of $y = \log_2 x$ and $y = \log_2(2x)$ at points **P** and **Q** respectively. Find the length of **PQ**, the line joining the two points.

3 Trigonometry

3.1 Measuring angles: Radians and Degrees

Angles measure rotation about an axis. Equivalently rotation can be thought of as "angular displacement". The figures below show two angles. We can get a conceptual view of an angle that results from rotation by imagining an axis of rotation that is perpendicular to the page on which the figures are drawn. The angle is the rotation that we undertake in standing at the intersection of the two lines (point O in the second of the figures) and moving from an orientation where we look down one of the lines (such as OF in the second figure) to an orientation where we look down the second of the lines (such as OD in the second figure). Often the algebraic symbol θ is used to represent the angle between two straight lines. Sometimes the magnitude of the displacement is what we are concerned with (as in the figure on the left) and other times the angular displacement has a direction associated with it. The usual conventions is that anti-clockwise rotation results in a positive angular displacement and clockwise rotation a negative angular displacement.

The size of an angle can be measured in degrees or radians. The angle of a full circle is 360 degrees (360°) or 2π radians ($2\pi^c$). Thus

$$360^\circ = 2\pi^c$$

One half of this is 180° or π^c. Note that this does <u>not</u> mean that π and 180 are in any way the same. It is only when we put in the <u>units</u> that we get $180^\circ = \pi^c$. Note that we also have

$$\begin{aligned} 1^\circ &= (\pi/180)^c \\ \text{or} \quad 1^c &= (180/\pi)\circ \end{aligned}$$

Note that, whenever an angle is measured in degrees it is important to include the units (the degree sign) explicitly. Often though, if the measurement is in radians, the units can be omitted. Thus if the units are not specified, it is understood that the angle is in radians. Therefore, $\sin x$ will always mean "the sine of x radians".

EXAMPLE 1: Convert the following angles from degrees to radians.

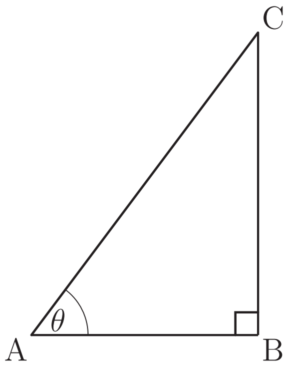
$$30^\circ = 30(\pi/180)^c = (\pi/6)^c$$
$$45^\circ = 45(\pi/180)^c = (\pi/4)^c$$
$$90^\circ = 90(\pi/180)^c = (\pi/2)^c$$

EXAMPLE 2: Convert the following angles from radians to degrees.

$$\frac{3\pi^c}{2} = \frac{3\pi}{2} \times \frac{180^\circ}{\pi} = 270^\circ$$
$$\frac{4\pi^c}{5} = \frac{4\pi}{5} \times \frac{180^\circ}{\pi} = 144^\circ$$

3.2 Trigonometry from right-angled triangles: ratios from an angle

Trigonometric functions are usually first encountered in considering geometric properties of right-angled triangles. It is the ratios of the lengths of sides of a right-angled triangle that feature in their definition.

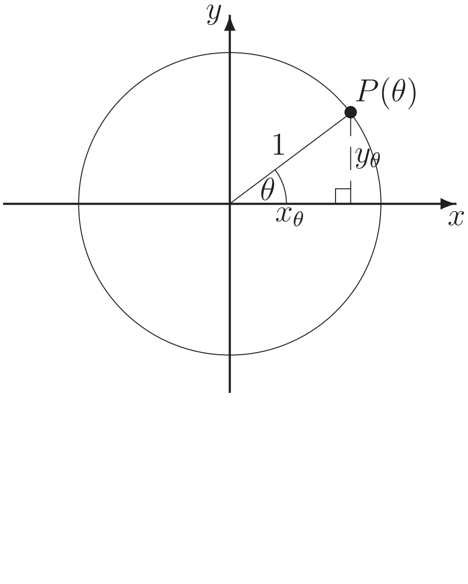


The longest side is is the *hypotenuse*. Here this is $\overline{\mathrm{AC}}$. With respect to angle θ, $\overline{\mathrm{BC}}$ is the side *opposite* and $\overline{\mathrm{AB}}$ is the side *adjacent*.

Then
$$\cos\theta = \frac{\mathrm{ADJ}}{\mathrm{HYP}}$$
$$\sin\theta = \frac{\mathrm{OPP}}{\mathrm{HYP}}$$
$$\tan\theta = \frac{\mathrm{OPP}}{\mathrm{ADJ}} = \frac{\sin\theta}{\cos\theta}.$$

3.3 Trigonometry from a circle: coordinates on the unit circle

We can define trigonometric functions as functions of *real numbers* and for this pupose, we consider a circle of unit radius (i.e. $r = 1$); centred on the origin: $x^2 + y^2 = 1$. This circle is so important it is called *the* unit circle.



At the origin $(0, 0)$ an angle θ can be formed between the positive x-axis and a radial line that intersects the unit circle at a point P. For each value of θ the point on the unit circle is unique. As θ changes the position of P varies and so the values of the (x, y) coordinates of P depend on θ. We label them x_θ and y_θ .

From the trig function definitions, we have that when the point P lies such that $0 \leqslant \theta \leqslant \pi/2$ then $\cos\theta = \dfrac{x_\theta}{1}$ and $\sin\theta = \dfrac{y_\theta}{1}$

$$\therefore \quad \boxed{x_\theta = \cos\theta} \quad \text{and} \quad \boxed{y_\theta = \sin\theta}$$

Once we extend beyond the interval $0 \leq \theta \leq \pi/2$ we can *define* the trigonometric functions by reference to the point P on the unit circle for any value of $\theta \in \mathbb{R}$:

$$\begin{aligned} \text{Thus} \qquad \cos\theta &= \text{the } x \text{ coordinate of } P \\ \sin\theta &= \text{the } y \text{ coordinate of } P \\ \tan\theta &= \frac{\sin\theta}{\cos\theta} = \frac{\text{the } y \text{ coordinate of } P}{\text{the } x \text{ coordinate of } P} \quad \text{when } x \neq 0. \end{aligned}$$

Now, as on the unit circle $-1 \leqslant x \leqslant 1$ the range of cosine must satisfy $-1 \leqslant \cos\theta \leqslant 1$, and as on the unit circle $-1 \leqslant y \leqslant 1$ the range of cosine must satisfy $-1 \leqslant \sin\theta \leqslant 1$.

We also define the reciprocal functions

$$\boxed{\begin{aligned} \sec\theta &= \frac{1}{\cos\theta}, \quad \cos\theta \neq 0 \\ \operatorname{cosec}\theta &= \frac{1}{\sin\theta}, \quad \sin\theta \neq 0 \\ \cot\theta &= \frac{1}{\tan\theta}, \quad \tan\theta \neq 0 \end{aligned}}$$

3.3.1 Two different representations of the unit circle

The equation of the unit circle is $x^2 + y^2 = 1$. We can also identify the circle through *parametric equations*

$$\boxed{x = \cos\theta, \quad y = \sin\theta, \quad \text{where} \quad 0 \leqslant \theta \leqslant 2\pi.}$$

Here θ is a *parameter*. In fact so long as θ is allowed to vary over any interval that is at least 2π in magnitude then the parametric equations are an equivalent representation of the unit circle.

Now that we have two different ways of representing the same circle we need to be able to distinguish between them. The original form of the equation of a circle is called the *cartesian* form of the circle: $x^2 + y^2 = 1$

3.3.2 Trigonometric identities from the unit circle

We can substitute for x and y from the parametric equations for the circle into the Cartesian form to get **the unit circle identity**:

$$\boxed{\cos^2\theta + \sin^2\theta = 1} \qquad \text{for all } \theta$$

This is an important identity for the trigonometric functions. You need to remember it!

Dividing this identity by $\cos^2\theta$, we get

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

or dividing by $\sin^2\theta$, we get

$$\boxed{\cot^2\theta + 1 = \operatorname{cosec}^2\theta}$$

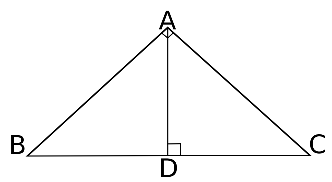
These identities are also important. As an example of how these identities can be used, consider the following.

EXAMPLE: Simplify $(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2$.

$$\begin{aligned}
&(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 \\
&\quad = \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta \\
&\quad = 2\cos^2\theta + 2\sin^2\theta \\
&\quad = 2(\cos^2\theta + \sin^2\theta) \\
&\quad = 2 \times 1 = 2.
\end{aligned}$$

EXERCISE 31:

1. Find the five remaining trigonometric ratios in each of the following cases:
   (a) $\cos\theta = 3/5, \quad 0 < \theta < \pi/2$
   (b) $\sec\alpha = -3/2, \quad \pi/2 < \alpha < \pi$
   (c) $\cot\beta = 3, \quad \pi < \beta < 3\pi/2$
2. Prove the following trigonometric identities:
   (a) $\tan x + \cot x = \sec x \operatorname{cosec} x$
   (b) $\dfrac{1+\cos x}{\sin x} = \dfrac{\sin x}{1-\cos x}$
3. Show that $\sec^2 x$ can be written as $1 + \sin^2 x + \sin^4 x + \sin^6 x + \cdots$.
4. If A is acute and $\sin A = \dfrac{3}{5}$, find a value for $\operatorname{cosec}^2 A - \cot^2 A$.
5. In the diagram given, show that $\dfrac{BD}{DC} = \cot^2 B$.



6. Prove that $\sin^2 x\cos^2 y - \cos^2 x\sin^2 y = \sin^2 x - \sin^2 y$.

7. If $x \sec A = y \tan A$, prove that $\tan A \sec A = \dfrac{xy}{y^2 - x^2}$.
8. Express $\cot^2 A$ in terms of $\sin^2 A$.
9. Eliminate θ to find a relationship between x and y given that

$$x = 2 + 3\cos\theta \qquad\qquad y = 5 - 2\sin\theta$$

   What type of geometric shape do these parametric equations represent?

10. If $\tan\theta = t$, show that $\sin\theta\cos\theta = \dfrac{1}{1+t^2}$

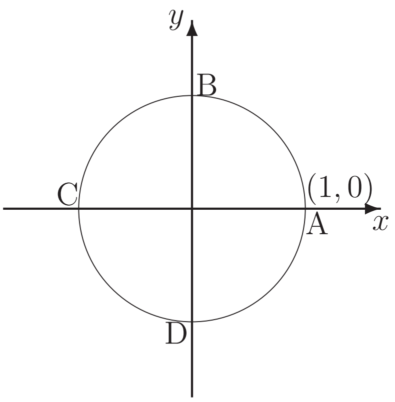
3.4 Quadrants and the CAST Diagram

In this section by using the symmetry and patterns evident in the unit circle we make some deductions about the properties of trigonometric functions.

On the unit circle we have, $x = \cos\theta$ and $y = \sin\theta$.

3.4.1 Values of sine and cosine at the quadrant boundaries

We know where the unit circle intersects the coordinate axes on the cartesian plane and the angles θ at which these intersections occur: $\theta = 0, \pi/2, \pi$ and $3\pi/2$



At the point A $= (1, 0)$, $\theta = 0$.
$\therefore \quad \cos 0 = x = 1, \quad \sin 0 = y = 0.$

i.e. $\boxed{\begin{array}{l}\cos 0 = 1 \\ \sin 0 = 0\end{array}}$

At the point B $= (0, 1)$, $\theta = \dfrac{\pi}{2}$.

$\therefore \quad \boxed{\begin{array}{l}\cos\dfrac{\pi}{2} = 0 \\ \sin\dfrac{\pi}{2} = 1\end{array}}$

At C $= (-1, 0)$, $\theta = \pi$.

$\therefore \quad \boxed{\begin{array}{l}\cos\pi = -1 \\ \sin\pi = 0\end{array}}$

At D $= (0, -1)$, $\theta = \dfrac{3\pi}{2}$.

$\therefore \quad \boxed{\begin{array}{l}\cos\dfrac{3\pi}{2} = 0 \\ \sin\dfrac{3\pi}{2} = -1\end{array}}$

3.4.2 Periodic property of the trigonometric functions

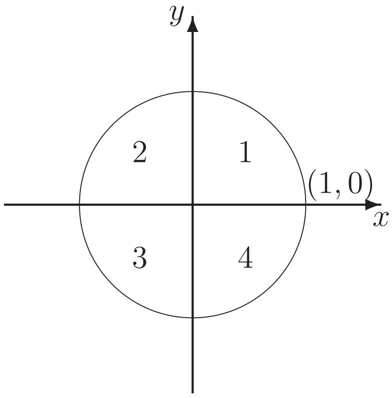
If $P(\theta)$ is the point on a circle at angle θ, then the points $P(\theta)$, $P(\theta + 2\pi)$, $P(\theta + 4\pi)$, etc. all coincide. Therefore, in general

$$\boxed{\begin{aligned}\cos\theta &= \cos(\theta + 2k\pi)\\ \sin\theta &= \sin(\theta + 2k\pi)\end{aligned}}$$

for any integer value of k. This property of the trigonometric functions is one of their most distinguishing features. We say the trigonometric functions are *periodic*. Their *period* is 2π because this is the smallest quantity we can add to the argument to get the same image (output).

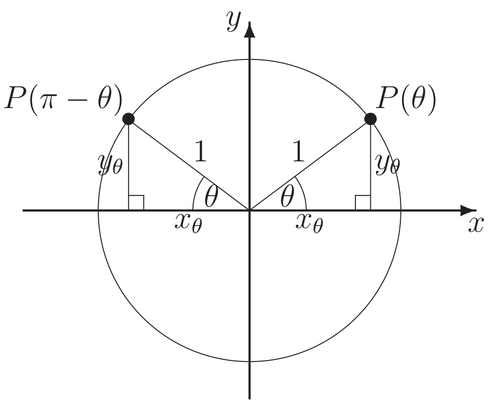
3.4.3 Symmetry of the unit circle - relating angles to the first quadrant

The coordinate axes divide the unit circle into four quadrants.



In this section we show how to determine $\cos\theta$, $\sin\theta$ etc for values of θ outside the interval $0 < \theta < \pi/2$. We do this using the symmetry of the unit circle and relate back to the first quadrant where $0 < \theta < \pi/2$.

Second Quadrant: We can use symmetry properties of the unit circle to relate the values of trigonometric functions when $\pi/2 < \theta < \pi$ to values of these functions in the first quadrant $(0 < \theta < \pi/2)$.



Consider the point $P(\theta)$ in quadrant 1 and the point $P(\pi - \theta)$ in quadrant 2. $P(\theta)$ has coordinates (x_θ, y_θ) which are both positive. $P(\pi - \theta)$ has coordinates $(x_{\pi-\theta}, y_{\pi-\theta})$

By symmetry, the y coordinates are equal and the x coordinates are equal in magnitude but opposite in sign.

Hence, $\cos(\pi - \theta) = x_{\pi-\theta} = -x_\theta$

$\therefore$ $\boxed{\cos(\pi - \theta) = -\cos(\theta)}$

$\sin(\pi - \theta) = y_{\pi-\theta} = y_\theta$

$\therefore$ $\boxed{\sin(\pi - \theta) = \sin(\theta)}$

$$\tan(\pi - \theta) = \frac{y_{\pi-\theta}}{x_{\pi-\theta}} = \frac{y_\theta}{-x_\theta} = -\frac{y_\theta}{x_\theta}$$

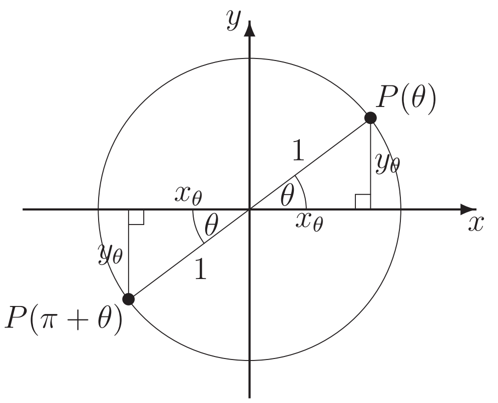
$\therefore$ $\boxed{\tan(\pi - \theta) = -\tan(\theta)}$

Example 3: Rewrite the following trigonometric expressions with arguments in the first quadrant.

(1) $\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3}$

(2) $\cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4}$

Third Quadrant: We can use symmetry properties of the unit circle to relate the values of trigonometric functions when $\pi < \theta < 3\pi/2$ to values of these functions in the first quadrant $(0 < \theta < \pi/2)$.



Consider the point $P(\theta)$ with coordinates (x_θ, y_θ) in quadrant 1 and the point $P(\pi + \theta)$ with coordinates $(x_{\pi+\theta}, y_{\pi+\theta})$ in quadrant 3. By symmetry, their coordinates are equal in magnitude but opposite in sign.

Hence, $\cos(\pi + \theta) = x_{\pi+\theta} = -x_\theta$

$\therefore$ $\boxed{\cos(\pi + \theta) = -\cos(\theta)}$

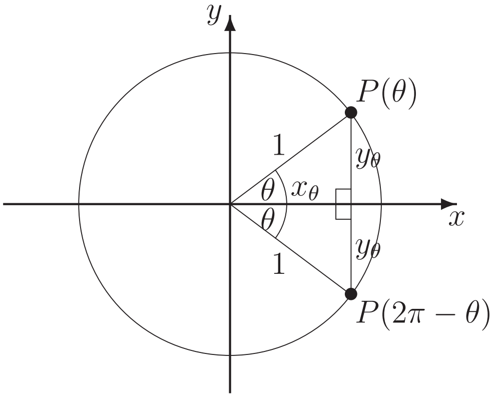
$\sin(\pi + \theta) = y_{\pi+\theta} = -y_\theta$

$\therefore$ $\boxed{\sin(\pi + \theta) = -\sin(\theta)}$

$$\tan(\pi + \theta) = \frac{\sin(\pi + \theta)}{\cos(\pi + \theta)} = \frac{-y_\theta}{-x_\theta} = \frac{y_\theta}{x_\theta}$$

$\therefore$ $\boxed{\tan(\pi + \theta) = \tan(\theta)}$

Fourth Quadrant: We can use symmetry properties of the unit circle to relate the values of trigonometric functions when $3\pi/2 < \theta < 2\pi$ to values of these functions in the first quadrant ($0 < \theta < \pi/2$).



Consider the point $P(\theta)$ with coordinates (x_θ, y_θ) in the first quadrant and the point $P(2\pi - \theta)$ with coordinates $(x_{2\pi-\theta}, y_{2\pi-\theta})$ in the fourth quadrant. By symmetry, their x coordinates are equal and their y coordinates are equal in magnitude but opposite in sign.

Hence, $\quad \cos(2\pi - \theta) = x_{2\pi-\theta} = x_\theta$

$$\therefore \quad \boxed{\cos(2\pi - \theta) = \cos(\theta)}$$

$$\sin(2\pi - \theta) = y_{2\pi-\theta} = -y_\theta$$

$$\therefore \quad \boxed{\sin(2\pi - \theta) = -\sin(\theta)}$$

$$\tan(2\pi - \theta) = \frac{\sin(2\pi - \theta)}{\cos(2\pi - \theta)} = \frac{-y_\theta}{x_\theta} = -\frac{y_\theta}{x_\theta}$$

$$\therefore \quad \boxed{\tan(2\pi - \theta) = -\tan(\theta)}$$

Example 4: Rewrite the following trigonometric expressions with angles in the first quadrant.

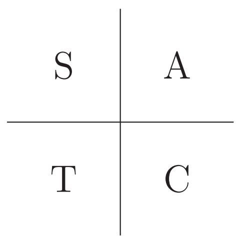
(1) $\quad \sin\frac{5\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) = -\sin\frac{\pi}{4}$

(2) $\quad \cos\frac{7\pi}{4} = \cos\left(2\pi - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$

These identities can be summarized by remembering that any angle between 0^c and $2\pi^c$ can be related to an angle, θ, in the first quadrant by writing it as

$$\begin{aligned} \pi - \theta &\quad (2^{\text{nd}} \text{ quadrant}) \\ \pi + \theta &\quad (3^{\text{rd}} \text{ quadrant}) \\ 2\pi - \theta &\quad (4^{\text{th}} \text{ quadrant}). \end{aligned}$$

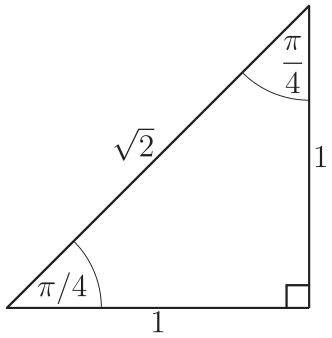
The sign of the function can be deduced using the following **CAST diagram**.



Cos is positive in the fourth quadrant.
All functions are positive in the first quadrant.
Sin is positive in the second quadrant.
Tan is positive in the third quadrant.

3.5 Common Triangles

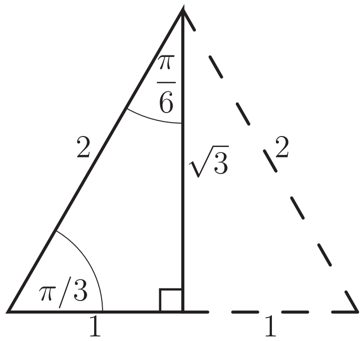
There are particular angles for which we should know values of the trigonometric functions. These values can be obtained from the following triangles.



$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\tan\frac{\pi}{4} = 1$$



$$\cos\frac{\pi}{3} = \frac{1}{2} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\tan\frac{\pi}{3} = \sqrt{3} \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

Examples :

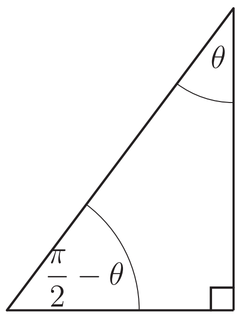
$$\cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

$$\sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\tan\frac{7\pi}{4} = \tan\left(2\pi - \frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1$$

3.6 Complementary Angles

Complementary angles add to $\pi/2$ radians.



$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

In general if the angle for the trigonometric function is of the form $\frac{\pi}{2} \pm \theta$ or $\frac{3\pi}{2} \pm \theta$, the trigonometric function changes: i.e. $\sin \to \cos \quad \cos \to \sin \quad \tan \to \cot$

These relationships can also be identified using symmetry arguments based on the unit circle. Can you see what these arguments are?

EXAMPLE 1: $\sin\left(\frac{\pi}{2}+\frac{\pi}{4}\right) = +\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$

EXAMPLE 2: $\tan\left(\frac{3\pi}{2}+\frac{\pi}{6}\right) = -\cot\frac{\pi}{6} = \sqrt{3}.$

EXERCISE 32:

1. Calculate sine, cosine and tangent for the following angles:
   (a) $\frac{3\pi}{4}$ (b) $\frac{4\pi}{3}$ (c) $\frac{3\pi}{4}$ (d) $\frac{11\pi}{4}$ (e) π (f) $\frac{7\pi}{4}$ (g) $\frac{5\pi}{6}$
2. Without using tables or a calculator, evaluate $\sin\frac{\pi}{6}\cos^2\frac{\pi}{3}+\sin\frac{\pi}{4}\cos\frac{\pi}{4}$
3. Find a simple alternative expression for the following:
   (a) $\sin(\frac{\pi}{2}+A)$ (c) $\text{cosec}(-C)$ (e) $\tan(E+2\pi)$
   (b) $\sin(2\pi - B)$ (d) $\cos(\frac{3\pi}{2}+D)$
4. Prove that $\tan(\frac{\pi}{2}-\theta)\sec(\pi+\theta)\cos(\frac{\pi}{2}+\theta)=1$
5. In a triange ABC, show that $\cos(A+C) = -\cos B$

3.7 Addition Formulae

The remaining trigonometric identities are obtained from the addition formulae. These are presented below this paragraph without proof. You need to remember these identities and be able to quote those that follow. For many people remembering *how* to obtain the subsequent results is easier than rote learning those subsequent results.

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

From these we can obtain the formulae for $\sin(x-y)$ and $\cos(x-y)$

$$\begin{aligned}\sin(x-y) &= \sin[x+(-y)] \\ &= \sin x \cos(-y) + \cos x \sin(-y)\end{aligned}$$

$$\therefore \quad \boxed{\sin(x-y) = \sin x \cos y - \cos x \sin y}$$

$$\begin{aligned}\cos(x-y) &= \cos[x+(-y)] \\ &= \cos x \cos(-y) - \sin x \sin(-y)\end{aligned}$$

$$\therefore \quad \boxed{\cos(x-y) = \cos x \cos y + \sin x \sin y}$$

$$\begin{aligned}\tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} \\ &= \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}\end{aligned}$$

Dividing top and bottom by $\cos x \cos y$, we get

$$\tan(x+y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$\text{i.e.} \quad \boxed{\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}}$$

Replacing y with $-y$, we have

$$\boxed{\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}}$$

Example 1: Expand $\sin\left(\frac{3\pi}{4}+\alpha\right)$.

$$\begin{aligned}\sin\left(\frac{3\pi}{4}+\alpha\right) &= \sin\frac{3\pi}{4}\cos\alpha+\cos\frac{3\pi}{4}\sin\alpha\\ &= \sin\frac{\pi}{4}\cos\alpha-\cos\frac{\pi}{4}\sin\alpha\\ &= \frac{1}{\sqrt{2}}\cos\alpha-\frac{1}{\sqrt{2}}\sin\alpha.\end{aligned}$$

Example 2: If $\quad \cos A=\frac{1}{2}, \quad \frac{3\pi}{2}<A<2\pi$

and $\quad \sin B=\frac{3}{5}, \quad \frac{\pi}{2}<B<\pi$

Find (a) $\sin(A+B)$, and
(b) $\cos(A-B)$ without using a calculator.

First, recall the diagram:

S	A
T	C

The expansions of $\sin(A+B)$ and $\cos(A-B)$ require values for $\sin A$ and $\cos B$. We are given that $3\pi/2<A<2\pi$, which indicates that A lies in quadrant 4, so that $\sin A<0$. We have also been given that $\pi/2<B<\pi$, which indicates that B lies in quadrant 2, so that $\cos B<0$.

First using $\cos A=\frac{1}{2}$ and $\frac{3\pi}{2}<A<2\pi$, find $\sin A$:

Using: $\cos^2 A+\sin^2 A=1$. Rearranging and solving for $\sin A$ gives

$$\sin A=\pm\sqrt{1-\cos^2 A}.$$

Substituting the value of $\cos A$ into this expression yields two possible values for $\sin A$:

$$\sin A=\pm\sqrt{1-\frac{1}{4}}=\pm\frac{\sqrt{3}}{2}.$$

As noted above the angle A lies in the fourth quadrant, so the sine has to be negative. Thus:

$$\sin A=-\frac{\sqrt{3}}{2}.$$

Second, using $\sin B=\frac{3}{5}$, and $\frac{\pi}{2}<B<\pi$ find $\cos B$:

Using: $\cos^2 B+\sin^2 B=1$. Rearranging and solving for $\cos B$ gives

$$\cos B=\pm\sqrt{1-\sin^2 B}.$$

Substituting the value of $\sin B$ into this expression yields two possible values for $\cos B$:

$$\cos B = \pm\sqrt{1 - \frac{9}{25}} = \pm\frac{4}{5}.$$

As noted above the angle B lies in the second quadrant, so the cosine is negative. Thus:

$$\cos B = -\frac{4}{5}.$$

With this information we can now use the trigonometric identities to solve the problem:

$$(a) \qquad \begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B \\ &= \frac{-\sqrt{3}}{2} \times \frac{-4}{5} + \frac{1}{2} \times \frac{3}{5} \\ &= \frac{4\sqrt{3}+3}{10}.\end{aligned}$$

$$(b) \qquad \begin{aligned}\cos(A-B) &= \cos A\cos B + \sin A\sin B \\ &= \frac{1}{2} \times \frac{-4}{5} + \frac{-\sqrt{3}}{2} \times \frac{3}{5} \\ &= \frac{-4-3\sqrt{3}}{10}.\end{aligned}$$

EXERCISE 33:

1. Use addition formulae to simplify the following:
   (a) $\sin 70^\circ \cos 20^\circ + \cos 70^\circ \sin 20^\circ$
   (b) $\cos 65^\circ \cos 25^\circ - \sin 65^\circ \sin 25^\circ$
   (c) $\sin 2x \cos x + \cos 2x \sin x$
   (d) $\cos 2\alpha \cos \alpha + \sin 2\alpha \sin \alpha$
   (e) $\dfrac{\tan 75^\circ - \tan 45^\circ}{1 + \tan 75^\circ \tan 45^\circ}$
   (f) $\dfrac{\tan 15^\circ + \tan 30^\circ}{1 - \tan 15^\circ \tan 30^\circ}$
   (g) $\dfrac{\tan x - \tan 2y}{1 + \tan x \tan 2y}$
2. If $\tan A = 3$ and $\tan B = 2$, find the values of:
   (a) $\tan(A+B)$ (b) $\tan(B-A)$
3. If $\cos\alpha = 3/5$, where $0 < \alpha < \pi/2$ and $\cos\beta = -4/5$, where $\pi/2 < \beta < \pi$, find
   (a) $\sin\alpha, \tan\alpha$ (c) $\sin(\alpha+\beta)$ (e) $\tan(\alpha-\beta)$
   (b) $\sin\beta, \tan\beta$ (d) $\cos(\alpha-\beta)$ (f) $\tan(\beta+\pi/4)$
4. Prove that $\tan 75^\circ = 2 + \sqrt{3}$ using an appropriate addition formula.
5. Find $\tan 15^\circ$ in the simplest surd form.
6. Simplify without expanding each term:
   $\cos(\alpha+\beta)\cos(\alpha-\beta) - \sin(\alpha+\beta)\sin(\alpha-\beta)$
7. Use the identities
   $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \quad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$
   $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$
   to find expressions for $\sin\alpha\cos\beta$, $\cos\alpha\cos\beta$ and $\sin\alpha\sin\beta$.

8. If $\sin(A+B) = \cos(A+B)$ show that $\tan A = \dfrac{1-\tan B}{1+\tan B}$.
9. For any triangle, show that the **sum** of the tan of the three angles equals the **product** of the tan of the three angles.

3.8 Double Angle Formulae

If we let $y = x$ in the addition formula we get identities for the trig functions of multiples of angles

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

we get

$$\sin(x+x) = \sin x \cos x + \cos x \sin x$$

$$\text{i.e.} \quad \boxed{\sin 2x = 2 \sin x \cos x}$$

Similarly, $\cos(x+y) = \cos x \cos y - \sin x \sin y$
becomes $\cos(x+x) = \cos x \cos x - \sin x \sin x$

$$\text{i.e.} \quad \boxed{\cos 2x = \cos^2 x - \sin^2 x}$$

Also, $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

becomes $\boxed{\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}}$

By using the identity $\cos^2 x + \sin^2 x = 1$, we obtain the alternative forms of the double angle formula for $\cos 2x$

$$\text{i.e.} \quad \boxed{\begin{array}{l} \cos 2x = 1 - 2\sin^2 x \\ \cos 2x = 2\cos^2 x - 1 \end{array}}$$

Example 1: If $\sin\alpha = \dfrac{4}{5}$, $\dfrac{\pi}{2} < \alpha < \pi$, find (a) $\sin 2\alpha$, (b) $\cos 2\alpha$, and (c) $\tan 2\alpha$.

First, recall the diagram: $\begin{array}{c|c} \text{S} & \text{A} \\ \hline \text{T} & \text{C} \end{array}$

The expansion of $\sin 2\alpha$, $\cos 2\alpha$, and $\tan 2\alpha$ will require values for $\cos\alpha$. We are given that $\pi/2 < \alpha < \pi$, which indicates that α lies in quadrant 2, so that $\cos\alpha < 0$.

Now using the identity $\cos^2\alpha + \sin^2\alpha = 1$ we can solve for $\cos\alpha$:

$$\cos\alpha = \pm\sqrt{1 - \sin^2\alpha} = \pm\sqrt{1 - \left(\frac{4}{5}\right)^2} = \pm\frac{3}{5}$$

Our knowledge that α lies in the second quadrant means we take the minus sign:

$$\cos\alpha = -\frac{3}{5}.$$

Then:

$$\text{(a)} \qquad \sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$= 2\times\frac{4}{5}\times\frac{-3}{5} = -\frac{24}{25}.$$
$$\text{(b)} \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}.$$
$$\text{(c)} \qquad \tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$
$$= \frac{-24/25}{-7/25} = \frac{24}{7}.$$

EXAMPLE 2: Simplify $2\sin 2\alpha\cos 2\alpha$.

$$2\sin 2\alpha\cos 2\alpha = \sin 2(2\alpha)$$
$$= \sin 4\alpha.$$

EXAMPLE 3: Simplify $\cos^2 15\ - \sin^2 15\ $.

$$\cos^2 15\ - \sin^2 15\ = \cos 2(15\ )$$
$$= \cos 30\ = \frac{\sqrt{3}}{2}.$$

EXAMPLE 4: If $\cos 2\alpha = -\frac{7}{9}$ and α is an acute angle (i.e. less than 90), find (i) $\cos\alpha$, (ii) $\sin\alpha$, (iii) $\tan\alpha$.

$$\text{(i)} \qquad \cos 2\alpha = 2\cos^2\alpha - 1$$
$$\therefore \quad \cos^2\alpha = \frac{\cos 2\alpha + 1}{2}$$
$$\therefore \quad \cos\alpha = \pm\sqrt{\frac{\cos 2\alpha + 1}{2}}$$

Take the positive square root, as we are in quadrant 1.

$$\therefore \quad \cos\alpha = +\sqrt{\frac{\cos 2\alpha + 1}{2}}$$
$$= \sqrt{\frac{-7/9 + 1}{2}}$$
$$= \sqrt{\frac{2/9}{2}} = \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

(ii) As $\cos 2\alpha = 1 - 2\sin^2\alpha$

$$\therefore \quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\therefore \quad \sin\alpha = +\sqrt{\frac{1-\cos 2\alpha}{2}} \qquad \text{(Quadrant 1)}$$

$$= \sqrt{\frac{1+7/9}{2}} = \sqrt{\frac{16/9}{2}}$$

$$= \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}.$$

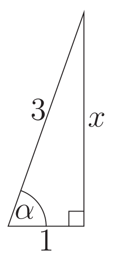
Note that as you have $\cos\alpha$ from part (i) you could also have made use of the identity $\sin^2\alpha + \cos^2\alpha = 1$.

$$\therefore \quad \sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(\frac{1}{9}\right)^2 = \frac{8}{9}$$

$$\therefore \quad \sin\alpha = +\sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}. \qquad \text{(Quadrant 1)}$$

Another approach would involve using $\cos\alpha$ from part (i), then drawing up a triangle to get a value for $\sin\alpha$ as follows. Note that the lengths of the sides in the triangle are arbitrary - that is we could choose any lengths we like, so long as the ratios of the lengths have the property that with respect to the angle α:

$$\frac{\text{length of adjacent side}}{\text{length of hypotenuse}} = \frac{1}{3}.$$



Using Pythagoras' theorem we find $x = \sqrt{8}$,

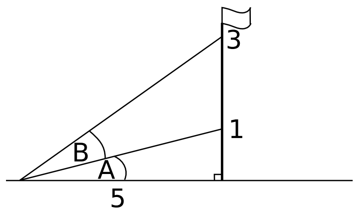
$$\therefore \quad \sin\alpha = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}.$$

(iii) $$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$= \frac{2\sqrt{2}/3}{1/3} = 2\sqrt{2}.$$

EXERCISE 34:

1. Express the following in terms of 2θ:
   (a) $2\sin\theta\cos\theta$
   (b) $4\cos\theta\sin\theta$
   (c) $2\cos^2\theta - 1$
   (d) $\cos^2\theta - \sin^2\theta$
   (e) $\sin^2\theta - \cos^2\theta$
   (f) $\dfrac{2\tan\theta}{1-\tan^2\theta}$
2. If $\cos\beta = -1/3$ and $\pi < \beta < 3\pi/2$, find the values of:
   (a) $\sin 2\beta$ (b) $\cos 2\beta$ (c) $\tan 2\beta$
3. If $\cos 2\theta = -7/9$ and θ is acute, find the values of:
   (a) $\cos\theta$ (b) $\sin\theta$ (c) $\tan\theta$
4. Prove the following trigonometric identities:
   (a) $\cos^4 x - \sin^4 x = \cos 2x$
   (b) $\sin 3x = 3\sin x - 4\sin^3 x$
   (c) $\cos 3x = 4\cos^3 x - 3\cos x$
5. Two ladders, one of which is **twice** as long as the other, rest on a floor and reach the same vertical height on the wall. The shorter ladder makes an angle of 60 with the floor. What angle does the longer ladder make with the floor?
6. Show that $\sin 8x = 8\sin x\cos x\cos 2x\cos 4x$
7. A flag pole 4m high has two hooks at the 1m mark and at the top. They are connected to the ground by rope which is staked in the ground 5m from the base of the flag pole as shown. Show that $\tan B = \dfrac{15}{29}$.



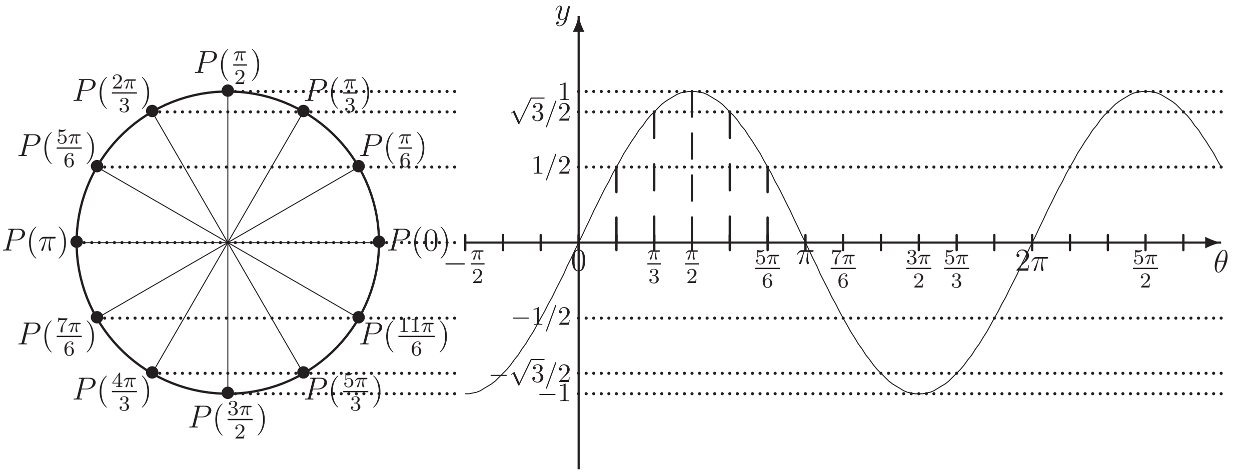
8. Show that $\cos 2\theta = \dfrac{1-t^2}{1+t^2}$ where $t = \tan\theta$

3.9 Trigonometric Graphs

Consider the graph of $y = f(\theta) = \sin\theta$.

Table of values:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0



The graph of $y = f(\theta) = \sin\theta$ continues indefinitely in each direction, beyond the values show in the table. The basic shape,  being repeated over and over again. This curve is called '**the sine curve**' or '**the sine wave**'. This function is said to be '**periodic**' with '**period** 2π' because the length of one complete wave is 2π.

3.9.1 Graphs of sine and cosine with different Amplitude and Angular Frequency

Graphs of: $f(\theta) = a\sin(n\theta)$ and $f(\theta) = a\cos(n\theta)$

The constants a and n in these formulae cause transformations on the basic sine and cosine curves. $\sin\theta$ and $\cos\theta$ have a maximum value of 1 and a minimum value of -1. So it follows that $a\sin\theta$ and $a\cos\theta$ have a maximum value of a and a minimum vaue of $-a$. The effect of the a is to dilate the graph parallel to the y-axis (i.e. either stretch or compress).

$|a|$ **is called the amplitude of the graph**.

Excercise: If a is negative, what effect would it have on the curve?

We have seen from the graphs of $\sin\theta$ and $\cos\theta$, that they repeat themselves over intervals in θ greater than their period 2π. We can state this fact by noting that the following equation for T

$$\sin(\theta + T) = \sin\theta$$

has a smallest non-zero solution of $T = 2\pi$. T is the symbol used for the period.

Now the graphs of $f(\theta) = a\sin n\theta$ and $f(\theta) = a\cos n\theta$ have a period T that satisfies

$$\sin(n(\theta + T)) = \sin(n\theta + nT) = \sin n\theta$$

Thus we require $nT = 2\pi$ and so the period of these functions is $T = \dfrac{2\pi}{n}$. The constant n has the effect of dilating the graph parallel to the θ-axis.

$$\textbf{The period of the sine and cosine curves is } T = \frac{2\pi}{n}\textbf{.}$$

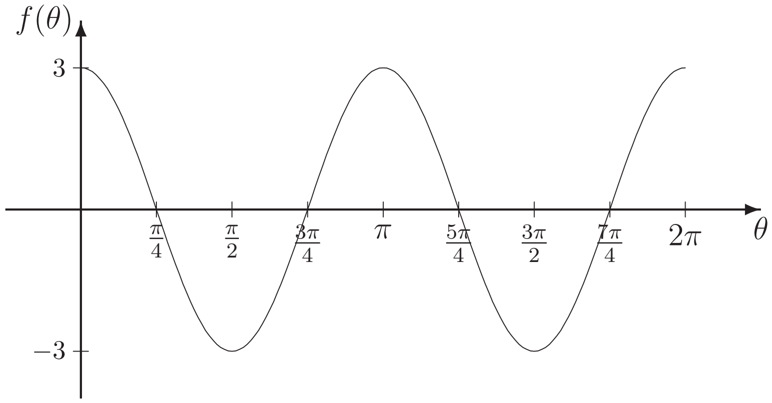
When trigonometric functions are used to represent quantities that have periodic behaviour over time the multiplier of the time variable is known as the **angular frequency**. Thus in

$$f(t) = a \sin nt$$

where t measures time, n is the angular frequency. It is the time taken for the argument of the sine function, nt, to increase by 2π.

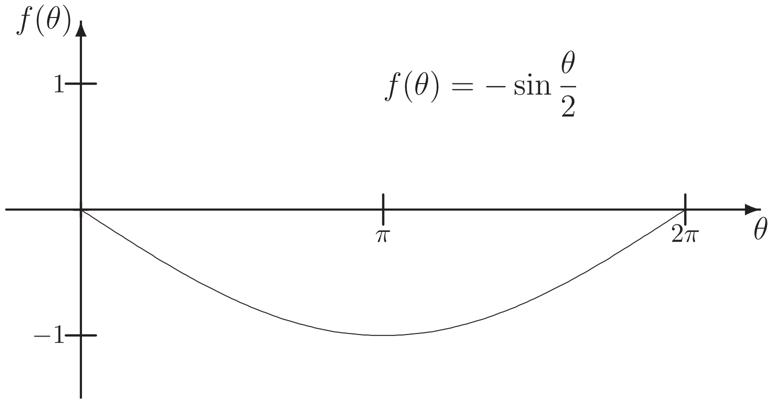
EXAMPLE 1: Sketch the graph of $f(\theta) = 3\cos 2\theta$, $0 \leqslant \theta \leqslant 2\pi$.

Here $a = 3$ and $n = 2$. Hence the amplitude is $|a| = 3$ and the period is $T = \dfrac{2\pi}{n} = \dfrac{2\pi}{2} = \pi$. That is, the curve completes one cycle in a distance of π. For $0 \leqslant \theta \leqslant 2\pi$, the curve completes two cycles.



EXAMPLE 2: Sketch the graph of $f(\theta) = -\sin\dfrac{\theta}{2}$, $0 \leqslant \theta \leqslant 2\pi$.

Here $a = -1$ and $n = \dfrac{1}{2}$. Hence the amplitude is $|a| = 1$ and the period is $T = \dfrac{2\pi}{n} = \dfrac{2\pi}{1/2} = 4\pi$. That is, the curve completes one cycle in a distance of 4π. For $0 \leqslant \theta \leqslant 2\pi$, we have only half a cycle. Note the effect of the negative number a is to reflect the graph of $f(\theta) = \sin\dfrac{\theta}{2}$ in the $x-$axis.



3.9.2 Horizontal Translations of the graphs of sine and cosine

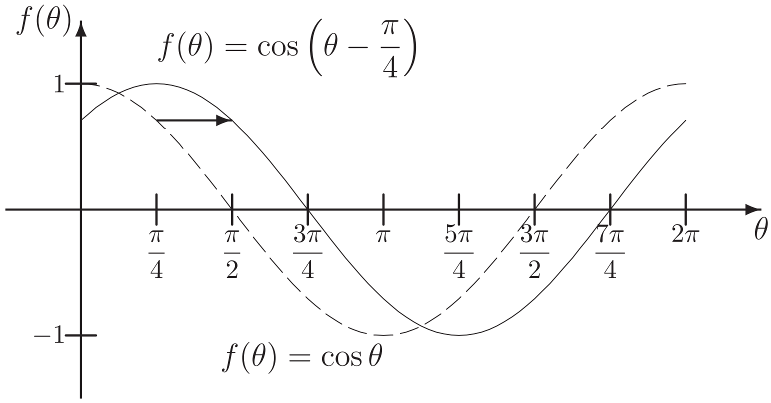
Graphs of: $a \sin n(\theta - \alpha)$ and $a \cos n(\theta - \alpha)$

We now have θ replaced by $\theta - \alpha$. This has the effect of translating the graph of $a \sin n\theta$ or $a \cos n\theta$ α units to the right along the θ-axis.

α is called the **horizontal shift** or **phase**.

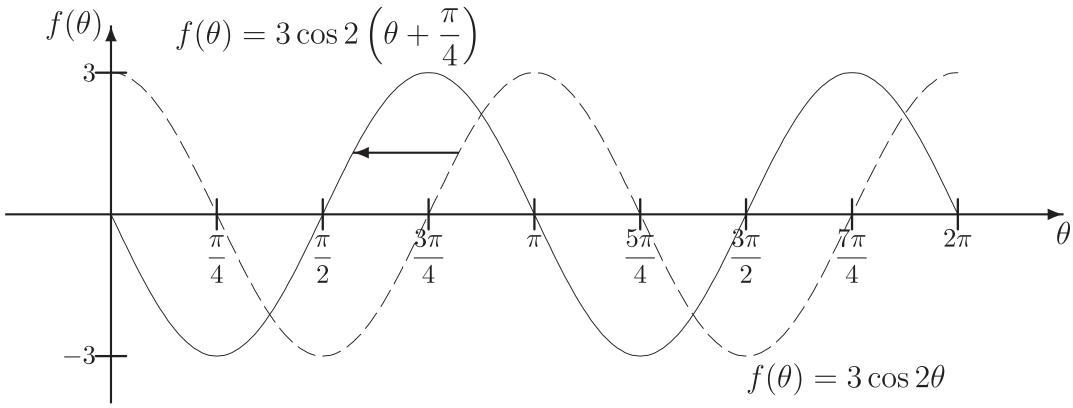
EXAMPLE 3: Sketch the graph of $f(\theta) = \cos\left(\theta - \frac{\pi}{4}\right)$, $0 \leqslant \theta \leqslant 2\pi$.

Here $a = 1$ and $n = 1$. Hence the amplitude is $|a| = 1$ and the period is $T = \frac{2\pi}{n} = \frac{2\pi}{1} = 2\pi$. The phase is $\alpha = \frac{\pi}{4}$, which has the effect of translating the basic cosine curve $\frac{\pi}{4}$ units to the right.



EXAMPLE 4: Sketch the graph of $f(\theta) = 3\cos 2\left(\theta + \frac{\pi}{4}\right)$, $0 \leqslant \theta \leqslant 2\pi$.

Here $a = 3$ and $n = 2$. Hence the amplitude is $|a| = 3$ and the period is $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$. The phase is $\alpha = \frac{\pi}{4}$, which has the effect of translating the curve of $f(\theta) = 3\cos 2\theta$, $\frac{\pi}{4}$ units to the left.



3.9.3 Vertical Translations of graphs of sine and cosine

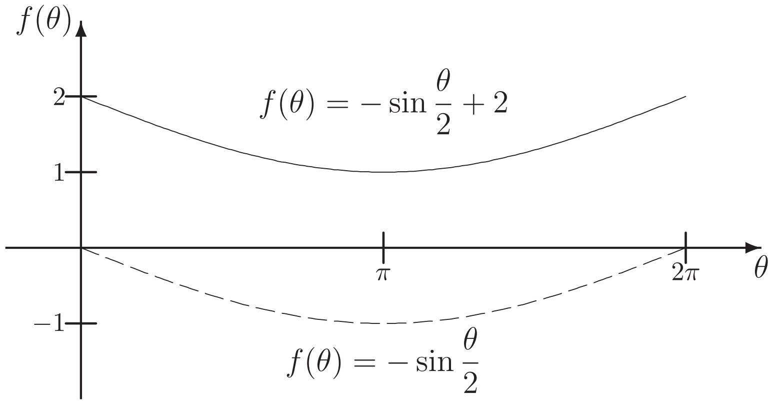
Graphs of: $a \sin n(\theta - \alpha) + b$ and $a \cos n(\theta - \alpha) + b$.

The effect of the constant b is to translate the curve b units up or down the y-axis according to whether b is positive or negative. It does not alter the period or the amplitude of the curve.

The range of these functions is $\{y \in \mathbb{R} : b - |a| \leq y \leq b + |a|\}$.

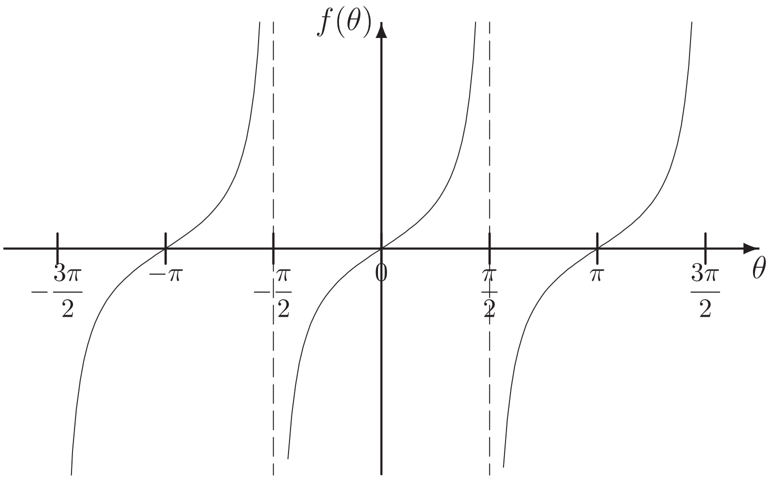
EXAMPLE 5: Sketch the graph of $f(\theta) = -\sin \frac{\theta}{2} + 2$, $0 \leqslant \theta \leqslant 2\pi$.

The constant 2 raises the graph of $f(\theta) = -\sin \frac{\theta}{2}$, two units up the y-axis. The range is $0 \leqslant f(\theta) \leqslant 2$, the amplitude is $|a| = 1$, and the period is 4π. Note that only half of the cycle needs to be drawn as $0 \leqslant \theta \leqslant 2\pi$.



3.9.4 The Tangent Function

To sketch the graph of $f(\theta) = \tan\theta$ we first locate the zeros of the function then the vertical asymptotes. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, the graph will cross the θ axis where $\sin\theta = 0$ (i.e. at $\theta = n\pi$). The asymptotes will occur at $\cos\theta = 0$ or $\theta = \frac{\pi}{2} + n\pi$. (Note that $\sin\theta \neq 0$ at these points). It follows that the graph looks like

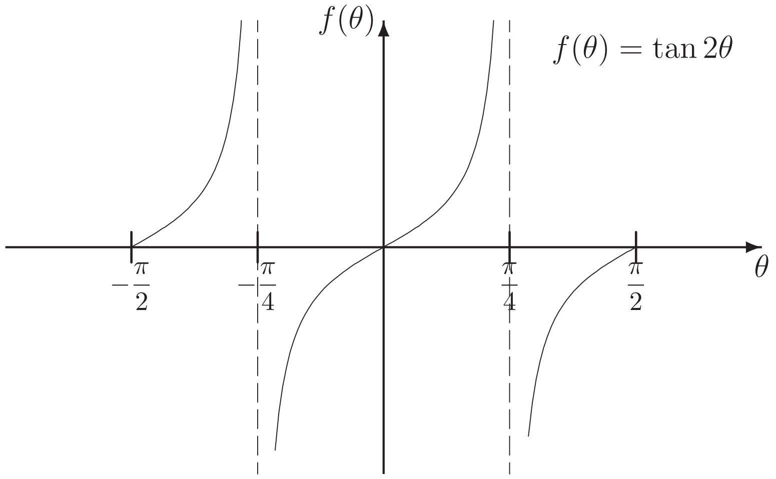


Note that the period of the graph is $T = \pi$.

In general the period, T, of the **tangent** function with angular frequency n is given by $T = \dfrac{\pi}{n}$. i.e., the function repeats itself every $\dfrac{\pi}{n}$ units.

EXAMPLE 6: Sketch the graph of $f(\theta) = \tan 2\theta$, $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$.

Here $n = 2$, so the period $T = \dfrac{\pi}{n} = \dfrac{\pi}{2}$.



EXERCISE 35:

1. State (i) the amplitude, (ii) angular frequency, (iii) horizontal translation, and (iv) vertical translation for
   (a) $\cos\left[3\left(t + \dfrac{\pi}{4}\right)\right] - 1$ (b) $\cos\left(\dfrac{t}{2} - \pi\right) - 3$ (c) $2\sin(4t - \pi) + 1$

2. Write down the trigonometric function described by:
   (a) the cosine function with amplitude $\dfrac{1}{2}$, angular frequency 3, horizontal translation $\dfrac{2\pi}{3}$ units to the right and vertical translation 2 units downwards
   (b) the sine function with amplitude 10, angular frequency 5 and horizontal translation $\dfrac{\pi}{10}$ units to the left.

3. A mass attached to the end of a spring oscillates so that its displacement s *cm* from a central position is given by: $s = 6\sin(3\pi t)$, where t is the time in seconds.
   (a) What is the maximum displacement from the central position?
   (b) Where is the object 5 seconds into the motion?
   (c) Where is the object 17.3 seconds into the motion?

4. Graph the functions:
   (a) $y = \cos\left(x - \dfrac{\pi}{3}\right)$ (b) $y = \tan\left(\dfrac{x}{2}\right)$ (c) $y = \sin\left(x + \dfrac{\pi}{4}\right) + 2$

5. A fisherman notes that the height of the tide in the harbour can be found by using the equation $h = 5 + 2\cos\dfrac{\pi}{6}t$ where h metres is the height of the tide and t is the number of hours after midnight.

(a) What is the height of the high tide and when does it occur in the first 24 hours?

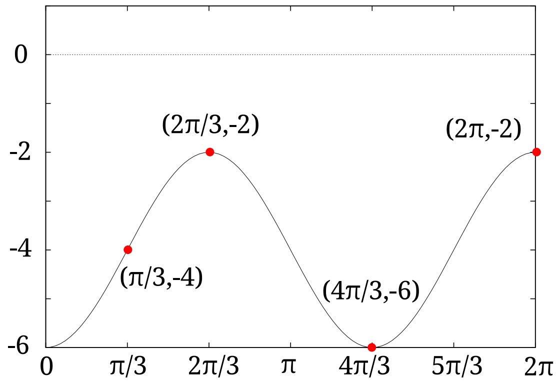
(b) What is the height difference between high and low tides?

(c) Sketch the graph of h for $0 \le t \le 24$

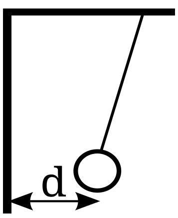
(d) The man knows that his trawler needs a depth of 6 metres to enter the harbour. Between what hours is he able to bring his boat back into the harbour?

6. Sketch the curve $y = \pi\cos(\pi x + \pi) + \pi$ for $0 \le x \le 5$ and $0 \le y \le 3\pi$.

7. This graph shows a function of the form $y = A\sin(Bx + C) + D$. Determine the values of A, B, C and D.



8. A pendulum hands from a ceiling as shown. As the pendulum swings, the distance **d** centimetres from one wall of the room depends on the time **t** seconds since it was set in motion. The equation for the distance d as a function of t is given as $d = 30\cos(\frac{\pi}{3}t) + 80, t \ge 0$. Find:



(a) The distance from the pendulum to the wall after 2 seconds.

(b) The maximum distance of the pendulum from the wall.

9. When between 12:00 noon and 1:00pm are the minute and hour hands of a clock 180° apart?

10. The times of sunrise in Townsville are given by $T = 340 - 50\cos((n+10) \times \frac{2\pi}{365})$ where T is the number of minutes after midnight and n is the number of the day of the year. If a student starts studying at 6:15am each day, how many days do they start studying before sunrise?

3.10 Solving Trigonometric Equations

Generally, trigonometric equations have an infinite number of solutions unless the domain is restricted in some way.

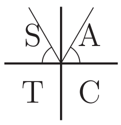
3.10.1 Equations involving the sine function

Solving: $\sin\theta = a$

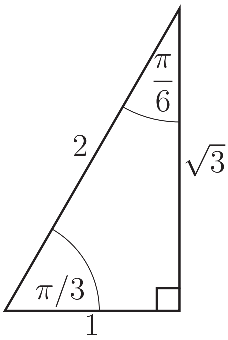
EXAMPLE 1: Solve $2\sin\theta - \sqrt{3} = 0$.

Note that no domain is specified. Hence it is assumed we are solving for $\theta \in \mathbb{R}$. Also, x is measured in radians.

Rewrite the equation as $\sin\theta = \dfrac{\sqrt{3}}{2}$.



Sine is positive in quadrants 1 and 2. We recognise the quantity $\sqrt{3}/2$ as one associated with one of the "common triangles" from section 3.5. Thus we draw the triangle below



and we recall

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Thus $\theta = \dfrac{\pi}{3}$ is a solution in the first quadrant.

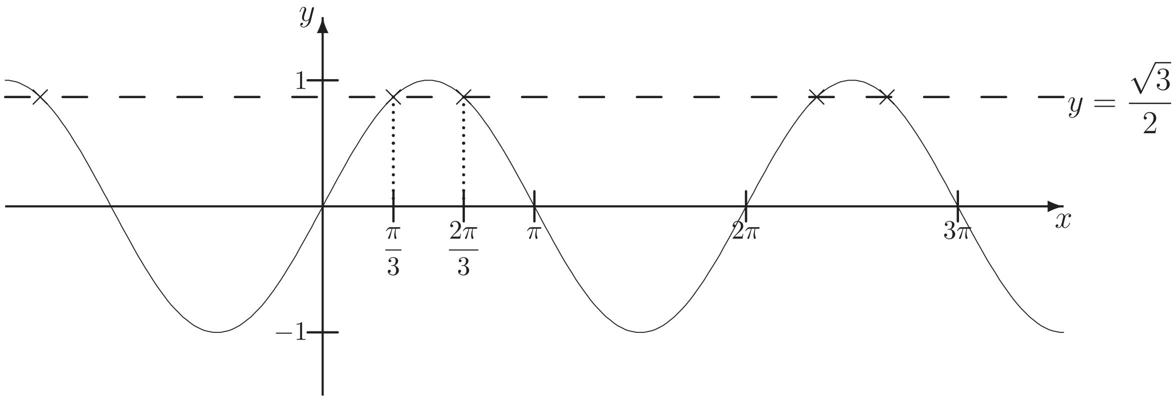
A solution in the second quadrant is found when we consider the symmetry of the unit circle. From section 3.4.3 we had that $\sin(\pi-\alpha) = \sin\alpha$ for any α. Using $\alpha = \pi/3$ in that statement means

$$\sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

so $\theta = \dfrac{2\pi}{3}$ is a solution to the equation in the second quadrant. Sine has period 2π and so each of the solutions in the first and second quadrant can have integer multiples of 2π added to them to obtain more solutions. In fact the equation has an infinite number of solutions.

$$\therefore \quad \theta = \begin{cases} \dfrac{\pi}{3} + 2n\pi \\ \dfrac{2\pi}{3} + 2n\pi \end{cases} \qquad (n \text{ is an integer})$$

Graphically the solutions of the equation occur at the points of intersection of the curves $y = \sin\theta$ and $y = \dfrac{\sqrt{3}}{2}$.



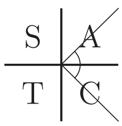
3.10.2 Equations involving the cosine function

Solving: $\cos\theta = a$

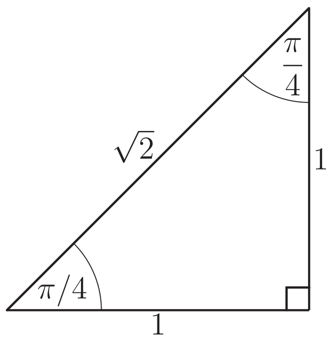
Example 2: Solve $\sqrt{2}\cos\theta - 1 = 0$.

Note that no domain is specified, so we solve for all $\theta \in \mathbb{R}$.

Rewrite the equations as $\cos\theta = \dfrac{1}{\sqrt{2}}$



Cosine is positive in quadrants 1 and 4. We recognise the quantity $1/\sqrt{2}$ as one associated with one of the "common triangles" from section 3.5. Thus we draw the triangle below



and we recall

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Thus $\theta = \dfrac{\pi}{4}$ is a solution in the first quadrant.

A solution in the fourth quadrant is found when we consider the symmetry of the unit circle. From section 3.4.3 we had that $\cos(2\pi - \alpha) = \cos\alpha$ for any α. Using $\alpha = \pi/4$ in that statement means

$$\cos\left(2\pi - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

so $\theta = \dfrac{7\pi}{4}$ is a solution to the equation in the fourth quadrant. But since cosine has period 2π we have an infinite number of solutions. Thus

$$\theta = \begin{cases} \dfrac{\pi}{4} + 2n\pi \\ \dfrac{7\pi}{4} + 2n\pi = 2\pi - \dfrac{\pi}{4} + 2n\pi = -\dfrac{\pi}{4} + 2(n-1)\pi \end{cases} \qquad (n \text{ is an integer})$$

Alternately we can record the solutions this way:

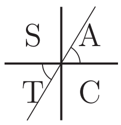
$$\theta = \begin{cases} \dfrac{\pi}{4} + 2n\pi \\ -\dfrac{\pi}{4} + 2n\pi \end{cases} \qquad (n \text{ is an integer})$$

3.10.3 Equations involving the tangent function

Solving: $\tan\theta = a$

EXAMPLE 3: Solve: $3\tan\theta - 5 = 0$.

$$\therefore \quad \tan\theta = \frac{5}{3}$$



The tangent function is positive in quadrants 1 and 3. This ratio doesn't correspond to one of the standard triangles, so we must use a calculator to find θ (in radians). We find

$$x = \tan^{-1}\left(\frac{5}{3}\right) \simeq 1.030^{c} \qquad \text{is a solution in the first quadrant.}$$

A solution in the third quadrant is found when we consider the symmetry of the unit circle. From section 3.4.3 we had that $\tan(\pi + \alpha) = \tan\alpha$ for any α. Using $\alpha = 1.030^{c}$ in that statement means

$$\tan(\pi + 1.030^{c}) = \tan 1.030^{c} = \frac{5}{3}$$

Thus $\theta \simeq \pi + 1.030^{c}$ is a solution in the third quadrant.

The general solution is therefore

$$\theta = \begin{cases} 1.030 + n\pi \\ \pi + 1.030 + n\pi \end{cases} \qquad (n \text{ is an integer})$$

This result could also be written as $\theta = 1.030 + n\pi$.

3.10.4 Restricted Domain

Consider the example for $\cos\theta = a$ again.

EXAMPLE 4: Solve $\sqrt{2}\cos\theta - 1 = 0$, $0 \leqslant \theta \leqslant 2\pi$.

This time our domain is restricted. We only want solutions that lie in the range $0 \leqslant \theta \leqslant 2\pi$. From before: $\theta = 2n\pi \pm \dfrac{\pi}{4}$.

$$
\begin{array}{llll}
\text{If } n = 0 & & \theta = \dfrac{\pi}{4} & \text{(inside range)} \\
 & \text{or} & \theta = -\dfrac{\pi}{4} & \text{(outside range)} \\
n = 1 & & \theta = 2\pi + \dfrac{\pi}{4} = \dfrac{9\pi}{4} & \text{(outside range)} \\
 & \text{or} & \theta = 2\pi - \dfrac{\pi}{4} = \dfrac{7\pi}{4} & \text{(inside range)}
\end{array}
$$

For the domain specified we have only two solutions, namely

$$\theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{7\pi}{4}.$$

EXERCISE 36:

1. Solve for x:
   (a) $2\cos x - 1 = 0, \quad x \in [0, 4\pi]$
   (b) $\sin x = 0.9962, \quad x \in [0, 360°]$
   (c) $\tan x + 1 = 0, \quad x \in [0, 2\pi]$
   (d) $\sin\left(2x - \dfrac{\pi}{6}\right) = 1, \quad x \in [0, 2\pi)$
   (e) $\tan(3x + 80°) = -0.8390, \quad x \in [0, 180°)$
2. Solve for θ where $\theta \in [0, 2\pi]$:
   (a) $\sin\theta + \cos\theta = 1$
   (b) $\cos^2\theta + \sin\theta + 1 = 0$
   (c) $\sin\theta + \cos\theta = \dfrac{1}{2}$
   (d) $\sin^2\theta + \cos^2\theta = 1$
   (e) $\tan\theta = \cot(2\theta)$
   (f) $\sin(2\theta) = \sin(7\theta)$
3. Solve for x in the following quadratic equation in $\sin x$ $2\sin^2 x + 7\sin x - 4 = 0$
4. A particle moves in a straight line so that its distance, x metres, from a point O is given by the equation $x = 3 + 4\sin 2t$, where t is the time in seconds after the particle begins to move.
   (a) Find the distance from O when the particle began to move.
   (b) Find the time when the particle reaches O, correct to two decimal places.
5. Solve the simultaneous equation:

$$\sqrt{3}\tan x = 2\sin x \ (1) \qquad 5\tan^2 x + 13 = 11\sec^2 x \ (2)$$

3.11 Limits involving trigonometric functions

Limits involving trigonometric functions sometimes arise in calculus. One example is

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta}.$$

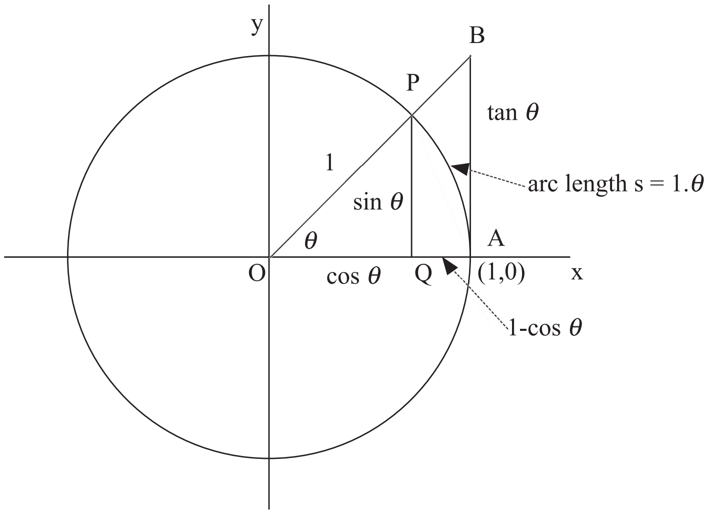
First, we need to state (without proof) a theorem and then prove a lemma, before we can prove this result. (A lemma is small, preliminary theorem.)
Squeeze Theorem: If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$, and

$$f(x) \leqslant h(x) \leqslant g(x) \quad \text{for } a - \delta < x < a + \delta$$

where δ is a positive, non-zero number (δ can be arbitrarily small), then

$$\lim_{x \to a} h(x) = L.$$

We also need to prove the following.
Lemma: $\lim_{\theta \to 0} \sin \theta = 0$, $\lim_{\theta \to 0} \cos \theta = 1$.
Proof:
Consider the following diagram (we will use this later, also):



In radian measure, $\theta = \dfrac{s}{r}$, where s is the arc length, and r is the radius. As $r = 1$, $s = \theta$. Noting that length $PA < s = \theta$ and using Pythagoras theorem,

$$PQ^2 + QA^2 = AP^2 < \theta^2 \qquad \text{i.e.} \quad \sin^2 \theta + (1 - \cos \theta)^2 < \theta^2$$

As $\sin^2 \theta > 0$ and $(1 - \cos \theta)^2 > 0$ and their sum is less that θ^2, each individually must be less than θ^2:

$$\begin{aligned} \sin^2 \theta &< \theta^2, & (1 - \cos \theta)^2 < \theta^2 \\ \text{That is,} \quad |\sin \theta| &< |\theta|, & |1 - \cos \theta| < |\theta| \\ \therefore -\theta < |\sin \theta| &< \theta, & -\theta < |1 - \cos \theta| < \theta \end{aligned}$$

Now, noting that $\lim_{\theta\to 0}\theta = 0$, and using the result from the previous theorem,

$$\begin{array}{rrcl}
\text{From the first inequality} & \lim_{\theta\to 0}|\sin\theta| & = & \lim_{\theta\to 0}\theta = 0. \\
\text{Because the absolute value tends to zero} & \lim_{\theta\to 0}\sin\theta & = & 0. \\
\text{Then from the second inequality} & \lim_{\theta\to 0}(1-\cos\theta) & = & \lim_{\theta\to 0}\theta \\
 & & = & 0 \\
\text{Thus also because} & \lim_{\theta\to 0}(1-\cos\theta) & = & 1-\lim_{\theta\to 0}\cos\theta = 0 \\
\text{we have the result} & \lim_{\theta\to 0}\cos\theta & = & 1.
\end{array}$$

□

Next we prove the main result.
Theorem: $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1.$

Proof:
From the previous figure, note that

$$\text{Area } \triangle\ AOP < \text{Area sector } AOP < \text{Area } \triangle\ AOB$$

We will determine an expression for each of the three areas in this inequality.

Firstly, the area of the sector AOP as a proportion of the area of the circle is the same as the length of the arc AP as a proportion of the circumference of the circle. That is,

$$\begin{array}{lrcl}
 & \dfrac{\text{Area sector } AOP}{\text{Area Circle}} & = & \dfrac{\text{length of sector} AP}{\text{circumference of circle}} \\
\text{thus} & \dfrac{\text{Area sector } AOP}{\pi.1^2} & = & \dfrac{1.\theta}{2\pi.1} \\
\therefore & \text{Area sector } AOP & = & \dfrac{\not\pi\theta}{\not\pi 2} = \dfrac{\theta}{2}.
\end{array}$$

Now the areas of the triangles at each end of the inequality are given by

$$\text{Area } \triangle\ AOP = \frac{1}{2}.\sin\theta.1, \qquad \text{Area } \triangle\ AOB = \frac{1}{2}.\tan\theta.1,$$

and we obtain

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}, \qquad 0 < \theta < \frac{\pi}{2}.$$

Dividing through by $\sin\theta (> 0)$ and multiplying through by 2, we have

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

The same inequality holds when $\theta < 0$. To see this, note that

$$\frac{-\theta}{\sin(-\theta)} = \frac{-\theta}{-\sin\theta} = \frac{\theta}{\sin\theta}, \qquad \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta}.$$

So, for $0 < |\theta| < \frac{\pi}{2}$,

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

We can re-arrange this inequality, after noting that $\frac{\theta}{\sin\theta} > 0$, $\frac{1}{\cos\theta} > 0$ is always true, whether $\theta > 0$ or $\theta < 0$. First, consider

$$\begin{aligned} 1 &< \frac{\theta}{\sin\theta} \\ \sin\theta &< \theta \\ \frac{\sin\theta}{\theta} &< 1 \end{aligned}$$

Next, consider

$$\begin{aligned} \frac{\theta}{\sin\theta} &< \frac{1}{\cos\theta} \\ \cos\theta\frac{\theta}{\sin\theta} &< 1 \\ \theta\cos\theta &< \sin\theta \\ \cos\theta &< \frac{\sin\theta}{\theta} \end{aligned}$$

Combining these inequalities, we obtain

$$\cos\theta < \frac{\sin\theta}{\theta} < 1, \qquad |\theta| < \frac{\pi}{2}.$$

Now, applying the result from the previous lemma and theorem,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\cos\theta = 1 \qquad \square$$

The strategy in evaluating limits involving trigonometric functions is to manipulate expressions so that the form

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

appears in the working.

Example. Evaluate: $\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta}$

The first approach demonstrated is to place the expression in the form for which the limit derived above can be used.

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta} &= 2\lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} \\
&= 2\lim_{2\theta\to 0} \frac{\sin 2\theta}{2\theta} \qquad \text{when } \theta \to 0 \text{ then } 2\theta \to 0 \text{ also} \\
&= 2\lim_{\phi\to 0} \frac{\sin\phi}{\phi} \qquad \text{where} \qquad \phi = 2\theta \\
&= 2 \times 1 = 2
\end{aligned}$$

The second approach is to first use a trigonometric identity and then use the limit derived above.

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta} &= \lim_{\theta\to 0} \frac{2\sin\theta\cos\theta}{\theta} \\
&= 2\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \lim_{\theta\to 0} \cos\theta \\
&= 2 \times 1 \times 1 \\
&= 2
\end{aligned}$$

EXERCISE 37:

1. Show that the angle of intersection of the two tangents to the curve $y = x - \frac{1}{x}$ at the points where $x = -1$ and $x = 3$ is $\tan^{-1} \frac{8}{29}$.
2. Using the fact that $|\sin x| < 1$, show that: $\lim_{x\to\infty} \frac{\sin x}{x} = 0$
3. Evaluate:
   (a) $\lim_{x\to 0} \frac{\sin ax}{x}$, a is const. (b) $\lim_{x\to 0} \frac{\sin x}{ax}$, a is const.
   (c) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ Hint: use the double angle formula for $\cos 2\left(\frac{x}{2}\right)$.
   (d) $\lim_{x\to 0} \frac{\sin^{-1}(2x)}{x}$ (e) $\lim_{x\to \frac{\pi}{2}} (\sec x - \tan x)$
4. Show that the $\lim_{x\to 3} \frac{\sin(x^2-3x)}{x^2-9}$ is $\frac{1}{2}$.
5. Evaluate: $\lim_{x\to 0} \frac{\cos x - 1}{x}$
   Hint: use $\cos x = \sqrt{1-\sin^2 x}$ and then multiply the numerator and denominator by the conjugate of the numerator (i.e. $\sqrt{1-\sin^2 x} + 1$).

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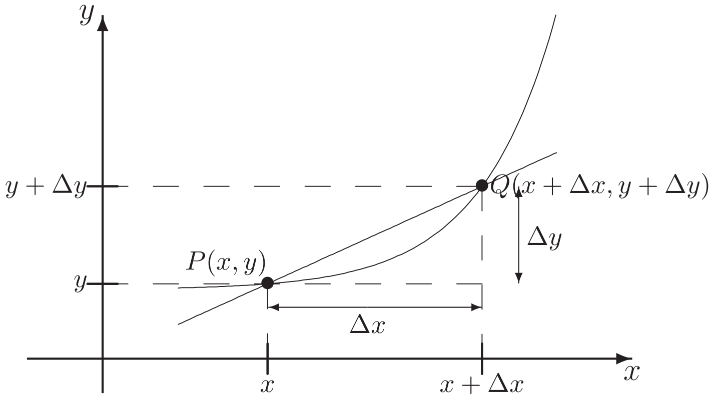
4 Differential Calculus

4.1 The Gradient of a Curve

The gradient at any point on a curve is defined as the gradient of the tangent to the curve at that point. Whereas the gradient of a straight line is constant, the gradient of a curve is constantly changing.

Consider $y = f(x)$, with $P(x, y)$ being any point on the curve and Q a neighbouring point, also lying on the curve. The coordinates of Q will be $(x + \Delta x, y + \Delta y)$ where $y + \Delta y = f(x + \Delta x)$. Thus $\Delta y = f(x + \Delta x) - f(x)$.



The chord (straight line) through PQ has gradient

$$m_{PQ} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

As we take Q closer to P then eventually the slope of PQ will approach the actual slope of the tangent at P. The gradient of the tangent at P is defined as the limit of the gradient of the chord PQ as $\Delta x \to 0$.

i.e. Gradient of tangent at P

$$\begin{aligned} &= \lim_{\Delta x \to 0} m_{PQ} \\ &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \text{Derivative of } f(x) \text{ at } P. \end{aligned}$$

$$\text{i.e. } \boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

$f'(x)$ is a new function giving the slope/gradient of the curve $y = f(x)$ at P. The derivative is often represented by the symbol $\frac{dy}{dx}$. Thus if $y = f(x)$, $\frac{dy}{dx} = f'(x)$ gives the slope of the graph at the point x. This symbol can also be written as $\frac{d}{dx}(y)$. Thus $\frac{d}{dx}(y)$ means to take the derivative of the function, y, with respect to x.

4.2 Rules for Differentiation

It is possible to prove the following rules for differentiation. Each of them can be deduced by evaluating the limit:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

1. If c is a constant then $\dfrac{dc}{dx} = 0$
2. $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for all $x \in \mathbb{R}$
3. $\dfrac{d}{dx}(cx^n) = cnx^{n-1}$
4. $\dfrac{d}{dx}(f+g) = \dfrac{df}{dx} + \dfrac{dg}{dx}$

EXAMPLE 1:

$$\begin{aligned} y &= 5x^3 + 6\sqrt{x} - \frac{1}{x^3} + \frac{2}{x^{3/2}} + 4 \\ &= 5x^3 + 6x^{1/2} - x^{-3} + 2x^{-3/2} + 4 \\ \therefore \quad \frac{dy}{dx} &= 15x^2 + 6 \times \frac{1}{2}x^{-1/2} - (-3)x^{-4} + 2 \times \frac{-3}{2}x^{-5/2} + 0 \\ &= 15x^2 + \frac{3}{\sqrt{x}} + \frac{3}{x^4} - \frac{3}{x^{5/2}}. \end{aligned}$$

EXAMPLE 2: Find the equation of the tangent to $f(x) = x^3 - 5x - 1$ at the point $(-2, 1)$.

Since $\quad f(x) = x^3 - 5x - 1$

then $\quad f'(x) = 3x^2 - 5$

$\therefore \quad f'(-2) = 3(-2)^2 - 5 = 7$

The tangent at $(-2, 1)$ has $m = 7$, therefore the equation to the tangent is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \text{i.e.} \quad y - 1 &= 7(x+2) \\ \text{i.e.} \quad y &= 7x + 15. \end{aligned}$$

EXAMPLE 3: Find the coordinates of any points on the following curve where the tangent is horizontal.

$$\begin{aligned} f(x) &= x^3 + 3x^2 - 9x + 5. \\ \therefore \quad f'(x) &= 3x^2 + 6x - 9. \end{aligned}$$

The tangent is horizontal if it has zero slope. i.e., $m = f'(x) = 0$.

$$\therefore \quad 3x^2 + 6x - 9 = 0.$$
$$\therefore \quad x^2 + 2x - 3 = 0.$$
$$\therefore \quad (x+3)(x-1) = 0.$$
$$\therefore \quad x = 1, -3.$$

When $x = 1$, $f(1) = 1+3-9+5 = 0$. When $x = -3$, $f(-3) = (-3)^3+3(-3)^2-9(-3)+5 = 32$. Therefore, at the points $(1, 0)$ and $(-3, 32)$ the tangents have zero gradient.

EXERCISE 38:

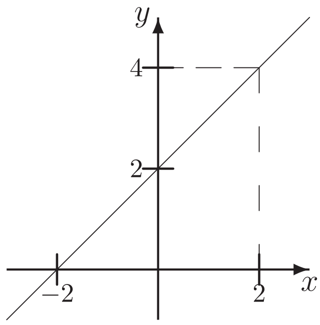
1. If $f(x) = x^3 + 2x^2 - 3x + 1$, find $f'(x)$ and evaluate $f'(0)$, $f'(1)$, $f'(2)$, $f'(-1)$.
2. Find the equation of the tangent to the curve $f(x) = x^3 - 2x + 1$ at $(2, 5)$.
3. Find the co-ordinates of the point on $f(x) = \sqrt{x}$ where the tangent has slope $\frac{1}{2}$.
4. Find a if $f(x) = \frac{a}{x}$ has a tangent with slope 3 at the point $x = 3$.
5. Two curves are said to be *tangential* to each other when they have a common tangent at a point of intersection. Show that the graphs of $y = \frac{1}{x}$ and $y = 3x - 2x^2$ are tangential.
6. For what value of c is the line $y = x + c$ a tangent to the curve $y = x^2 + 3x + 2$?
7. Differentiate each of the following:
   (a) $y = x^2 + \pi^2$ (b) $y = 5x^3 - \frac{1}{5x^3}$ (c) $y = 7\sqrt{x} + x\sqrt{7}$
8. If $f(x) = x^{10}$ and $g(x) = x^2 - 4$ are two functions of x show that: *The derivative of a product is **not** the product of the derivatives.*

4.3 Continuity

Definition: A function is continuous at $x = a$ if (i) $f(x)$ has a definite value, $f(a)$, at $x = a$, and (ii) $\lim_{x \to a} f(x) = f(a)$.

EXAMPLE 4: Consider

$$f(x) = x + 2$$



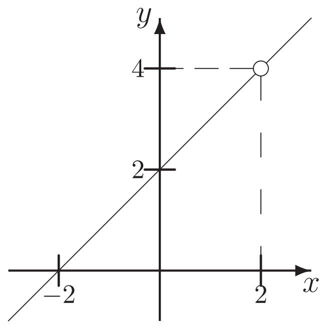
(i) $f(2) = 4$.

(ii) $\lim_{x \to 2} f(x) = 4$.

Therefore the function is continuous at $x = 2$.

EXAMPLE 5: Consider

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x+2)(x-2)}{x-2}$$



(i) $f(2)$ does not exist.

(ii) $\lim_{x \to 2} f(x) = 4$.

Therefore the function is not continuous at $x = 2$.

4.4 Differentiability

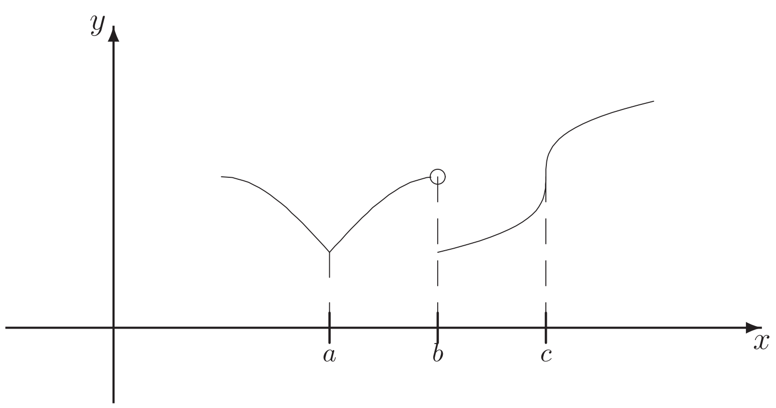
A function fails to be differentiable at $x = a$ if any of the following occur:

(a) it is discontinuous,

(b) it has "sharp spikes", or

(c) it has a vertical tangent.

EXAMPLE: The following curve is not differentiable at a, b, or c.



4.5 Derivative of a Product of Functions

If $u(x)$ and $v(x)$ are differentiable functions then

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}} \qquad \text{or} \qquad \boxed{(uv)' = uv' + u'v}$$

This is the **Product Rule**.

EXAMPLE 6: Differentiate $y = (3x+2)(2x^2 - 3x + 4)$.

Let $u = 3x + 2$ and $v = 2x^2 - 3x + 4$

$\therefore \quad \frac{du}{dx} = 4$ and $\frac{dv}{dx} = 4x - 3$

$$\begin{aligned} \frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= (3x+2)(4x-3) + (2x^2 - 3x + 4)3 \\ &= 18x^2 - 10x + 6. \end{aligned}$$

4.6 Chain Rule

If $y = g(u)$ and $u = h(x)$, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

Note: There is no dash notation for the chain rule as we have two "different" derivatives.

EXAMPLE 7: Differentiate $y = (2x+1)^3$.

Let $u = 2x + 1$, so that $y = u^3$. We have $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 2$.

Chain Rule:
$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 3u^2 \times 2 \\ &= 6(2x+1)^2. \end{aligned}$$

EXAMPLE 8: Differentiate $y = \frac{1}{(x^2-1)^2}$.

Let $u = x^2 - 1$, so that $y = u^{-2}$. We have $\frac{dy}{du} = -2u^{-3}$ and $\frac{du}{dx} = 2x$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -\frac{2}{u^3} \times 2x \\ &= -\frac{4x}{(x^2-1)^3}. \end{aligned}$$

4.7 Derivative of a Quotient

If $u = g(x)$, $v = h(x)$ and $y = u/v$, then

$$\boxed{\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}} \qquad \text{or} \qquad \boxed{\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}}$$

This is the **Quotient Rule**.

NOTE: If you write $y = u/v = uv^{-1}$, then you can use the product rule to differentiate functions of this type.

EXAMPLE 9: Differentiate: $y = \dfrac{2x+1}{4x-3}$.

Let $u = 2x + 1$ and $v = 4x - 3$, $\therefore \dfrac{du}{dx} = 2$, $\dfrac{dv}{dx} = 4$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{(4x-3)2 - (2x+1)4}{(4x-3)^2} \\
&= -\frac{10}{(4x-3)^2}.
\end{aligned}$$

EXAMPLE 10: Differentiate: $y = \dfrac{x}{1+x^2}$.

Let $u = x$ and $v = 1 + x^2$, $\therefore \dfrac{du}{dx} = 1$, $\dfrac{dv}{dx} = 2x$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{(1+x^2)\times 1 - x \times 2x}{(1+x^2)^2} \\
&= \frac{1-x^2}{(1+x^2)^2}.
\end{aligned}$$

EXERCISE 39: Differentiate the following, showing your working:

1. $y = \dfrac{x^2+x}{x}$
2. $y = (x^2 - 1)(2x + 3)$
3. $f(x) = x^3(4x^5 + x)$
4. $f(x) = \dfrac{x^2+3}{3x^3}$
5. $y = \dfrac{1}{1+x^2}$
6. $y = \dfrac{x+1}{\sqrt{x}}$
7. $f(x) = \dfrac{x}{(1-x)^{\frac{1}{2}}}$
8. $f(x) = (1 + x^2)(1 - 2x)^{\frac{1}{2}}$
9. $\dfrac{x}{1+2x^2}$
10. $\sqrt{x^2+2}$
11. $\dfrac{x}{1+\sqrt{x}}$

4.8 Derivatives of Exponential Functions and Logarithms

We can show that the elementary functions have the derivatives shown.

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad \text{Note: } \ln x \text{ is only defined for } x > 0.$$

In general, if $y = e^u$ where $u = u(x)$, then by the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \frac{du}{dx}.$$

Example 1: Differentiate: $y = e^{6x}$.

Let $u = 6x$, so that $y = e^u$. We have $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 6$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times 6 = 6e^{6x}.$$

Example 2: Differentiate: $y = e^{x^2}$.

Let $u = x^2$, so that $y = e^u$. We have $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 2x$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times 2x = 2xe^{x^2}.$$

In general, if $y = \ln u$ where $u = u(x)$, then by the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

Example 3: Differentiate:: $y = \ln(6x^2 - 4x + 1)$.

Let $u = 6x^2 - 4x + 1$, so that $y = \ln u$. We have $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 12x - 4$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{u} \times (12x - 4) = \frac{12x - 4}{6x^2 - 4x + 1}.$$

EXERCISE 40: Find the derivative of each of the following functions:

1. $g(t) = \dfrac{e^t + 1}{e^t - 1}$
2. $f(x) = e^{-x^2}$
3. $f(x) = \log(1 - x + x^2)$
4. $y = \ln(x^2)$
5. $y = \ln(x^2 + 7x)$
6. $y = x \ln x$
7. $y = e^{x^2+3x}$
8. $y = xe^{2x}$

4.9 Derivatives of Trigonometric functions

We can show that the elementary functions have the derivatives shown.

$$\boxed{\begin{aligned}\frac{d}{dx}\sin x &= \cos x\\ \frac{d}{dx}\cos x &= -\sin x\\ \frac{d}{dx}\tan x &= \sec^2 x\end{aligned}}$$

EXAMPLE 1: Differentiate $y = \sin(2x-6)$.

Let $u = 2x - 6$, so that $y = \sin u$. We have $\dfrac{dy}{du} = \cos u$ and $\dfrac{du}{dx} = 2$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times 2 = 2\cos(2x-6).$$

EXAMPLE 2: Differentiate $y = \tan 3x$.

Let $u = 3x$, so that $y = \tan u$. We have $\dfrac{dy}{du} = \sec^2 u$ and $\dfrac{du}{dx} = 3$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \sec^2 u \times 3 = 3\sec^2(3x).$$

EXAMPLE 3: Differentiate $y = \sqrt{\cos x}$.

Let $u = \cos x$, so that $y = \sqrt{u} = u^{1/2}$. We have $\dfrac{dy}{du} = \dfrac{1}{2}u^{-1/2} = \dfrac{1}{2\sqrt{u}}$ and $\dfrac{du}{dx} = -\sin x$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times (-\sin x) = -\frac{\sin x}{2\sqrt{\cos x}}.$$

4.10 Derivatives of Reciprocal Functions

The reciprocal trigonometric functions can be differentiated using the Chain Rule.

EXAMPLE: Differentiate $y = \sec x$.

$$y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1}.$$

Let $u = \cos x$, so that $y = u^{-1}$. We have $\dfrac{dy}{du} = -u^{-2} = -\dfrac{1}{u^2}$ and $\dfrac{du}{dx} = -\sin x$.

$$\begin{aligned}\therefore \quad \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{u^2} \times (-\sin x) = \frac{\sin x}{\cos^2 x}\\ &= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}\\ &= \sec x \tan x.\end{aligned}$$

Exercise: Show that $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$, and

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$$

Example 1: Differentiate $\ln(\sin x)$.

Let $u = \sin x$, so that $y = \ln u$. We have $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = \cos x$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times \cos x = \frac{\cos x}{\sin x} = \cot x.$$

EXERCISE 41: Find the derivative of each of the following functions:

1. $f(x) = \sin 6x$
2. $f(x) = x^2 \sin x$
3. $f(x) = \cos(x^3)$
4. $f(x) = x^2 \sec x$
5. $f(x) = \tan(x^5)$
6. $f(x) = e^x \cos x$
7. $f(x) = e^{2x} \sin x$
8. $f(x) = \ln(\sin 2x)$
9. $f(x) = \sin x \cos 2x$
10. $y = e^{\log(\sin x)}$
11. $y = \ln(\cos x)$
12. $y = e^{\cos x}$
13. $y = \cos^3 x$
14. $y = x \sin(x^2 + 2x)$
15. $y = \cos(x + \sin x)$
16. $y = (\sin x)^x$. Hint: Take logarithm of both sides first.
17. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ given the relation: $y = \sin(2x + 3y)$

4.11 Higher Order Derivatives

$f'(x)$ is the derivative of $f(x)$, (the change in y with respect to x).

$f'(x)$ may have a derivative of its own, denoted by

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' \quad \text{(the change in } f'(x) \text{ with respect to } x\text{)}$$

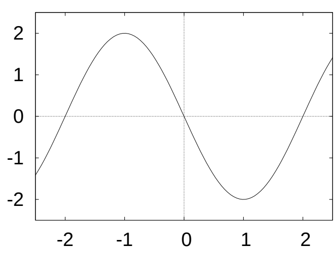
Similarly the third derivative is $f'''(x) = \dfrac{d^3y}{dx^3}$, etc.

EXAMPLE:

$$\begin{aligned} f(x) &= x^3 - 6x^2 + 4x - 1 \\ f'(x) &= 3x^2 - 12x + 4 \\ f''(x) &= 6x - 12 \\ f'''(x) &= 6. \end{aligned}$$

EXERCISE 42:

1. Find $f'(x)$ and $f''(x)$ for the following:
   (a) $f(x) = \dfrac{x}{1+x}$ (b) $f(x) = e^{x^2}$ (c) $f(x) = \dfrac{1}{2}\ln(1+x^2)$
2. If $f(x) = ax^3 + bx^2$, evaluate the following differential equation: $x^2 f''(x) - 4xf'(x) + 6f(x)$.
3. On the same set of axes, draw graphs of the given function (y) and its derivative ($\dfrac{dy}{dx}$).
   (a) $y = x^2$ (b) $y = x^2 + x$ (c) $y = 3x - x^3$
4. Draw the graph of $y = x^3 - 6x^2 + 9x$. On the same set of axes draw graphs for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.
5. The graph of the derivative $f'(x)$ of a function $f(x)$ is shown.



   (a) If $f(0) = 0$, sketch a possible graph of $f(x)$.
   (b) Also sketch the graph of $f''(x)$.
6. If $f(x) = e^x - ex$, find:
   (a) $f'(0)$
   (b) $f''(0)$
   (c) The value of x for which $f'(x) = 0$.
   (d) The value of x for which $f''(x) = 0$.

4.12 Implicit Differentiation

The functions we have dealt with so far can all be expressed explicitly. (i.e., in the form $y = f(x)$) Some functions are defined implicitly by a relation between x and y such as

$$x^2y + y^3 = 6xy \quad \text{or} \quad \mathrm{e}^{(x+y)} = y^2 \ln x.$$

Here, we cannot express y explicitly in terms of x. To differentiate such expressions we use the method of implicit differentiation. We differentiate both sides of the equation w.r.t. (with respect to) x using the Chain Rule and then solve for $\dfrac{dy}{dx}$.

EXAMPLE 1: If $x^2 + y^2 = 30$, find $\dfrac{dy}{dx}$.

Differentiate w.r.t. x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(30)$$

Note that y is a function of x, so we must use the Chain Rule:

$\left(\text{That gives } \dfrac{d}{dx}(y^2) = \dfrac{d}{dy}(y^2)\dfrac{dy}{dx} = 2y\dfrac{dy}{dx}\right)$

$$\begin{aligned} \therefore \quad 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

EXAMPLE 2: If $x^3 + y^3 = 6xy$, find $\dfrac{dy}{dx}$.

Differentiate w.r.t. x:

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(6xy)$$

Note that the product rule is required for the evaluation of $\dfrac{d}{dx}(6xy)$.

$$\therefore \quad 3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}.$$

Solving for $\dfrac{dy}{dx}$,

$$\begin{aligned} 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} &= 6y - 3x^2 \\ (3y^2 - 6x)\frac{dy}{dx} &= 6y - 3x^2 \\ \therefore \quad \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}. \end{aligned}$$

Higher Order Derivatives

EXAMPLE 3: Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in term of x and y, when

$$2x^3 - 3y^2 = 7.$$

$$\begin{aligned} \frac{d}{dx}(2x^3) - \frac{d}{dx}(3y^2) &= \frac{d}{dx}(7). \\ \therefore \quad 6x^2 - 6y\frac{dy}{dx} &= 0. \\ \therefore \quad y\frac{dy}{dx} &= x^2 \\ \text{or} \quad \frac{dy}{dx} &= \frac{x^2}{y}. \end{aligned}$$

To find $\dfrac{d^2y}{dx^2}$ use the Quotient Rule:

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x^2}{y}\right) &= \frac{y\frac{d}{dx}(x^2) - x^2\frac{dy}{dx}}{y^2} \\ &= \frac{2xy - x^2\frac{dy}{dx}}{y^2}. \end{aligned}$$

However, $\dfrac{dy}{dx} = \dfrac{x^2}{y}$.

$$\begin{aligned} \therefore \quad \frac{d^2y}{dx^2} &= \frac{2xy - x^2\frac{x^2}{y}}{y^2} \\ &= \frac{2xy^2 - x^4}{y^3}. \end{aligned}$$

EXERCISE 43:

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ given the relations:

   (a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$

   (b) $x^5 + 4xy^3 - 3y^5 = 2$

   (c) $xy + y^3 = 8$

   (d) $y = xe^y$

2. For the function $3x^3 + 2xy^2 = 2y + 3$, show that $\frac{dy}{dx} = \frac{9x^2 + 2y^2}{2(1 - 2xy)}$

3. Find the gradient and the equation of the tangent to the curve $x^3 + 3xy = y^3 + 37$ at the point (3,2).

4. Differentiate the equations implicitly with respect to x:

   (a) $\ln y = e^x$

   (b) $x^2 \ln y - x^3 = 10$

5. If a thermal nuclear reactor is built in the shape of a right circular cylinder of radius r and height h, then, according to neutron diffusion theory, r and h must satisfy the equation:
   $\left(\frac{2.4048}{r}\right)^2 + \left(\frac{\pi}{h}\right)^2 = k = constant$. Find $\frac{dr}{dh}$.

6. The output of a certain Townsville factory is given by $Q = 2x^3 + x^2y + y^3$ kg, where x is the number of hours of skilled labour and y is the number of hours of unskilled labour; the current labour force consists of 30 hours of skilled labour and 20 hours of unskilled labour. Using implicit differentiation, estimate the change in unskilled labour, y, that should be made to offset a one-hour increase in skilled labour, x, so that output will be maintained at its current level.

4.13 Related Rates

The quantity $\dfrac{dy}{dx}$ can be interpreted as: "the rate at which y varies w.r.t. x"

We now look at problems which deal with the rate of change of one variable in terms of the rate of change of another variable.

Example 1: The area, A, of an oil spill is in the shape of a circle and increasing with time. Find the rate at which the area is increasing w.r.t. time (i.e. $\dfrac{dA}{dt}$) if the radius of the circle is increasing at a rate of 2m/min (i.e. $\dfrac{dr}{dt} = 2\text{m/min}$) when $r = 200\text{m}$.

We must firstly relate $\dfrac{dA}{dt}$ and $\dfrac{dr}{dt}$ using the Chain Rule.

Note that A depends on r, and r depends on t. Therefore A depends on t. From the chain rule,

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$$

But for a circle: $A = \pi r^2$

$$\therefore \quad \frac{dA}{dr} = 2\pi r$$

$$\therefore \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

We are asked to find $\dfrac{dA}{dt}$ when $r = 200\text{m}$ and $\dfrac{dr}{dt} = 2\text{m/min}$.

$$\therefore \quad \frac{dA}{dt} = 2\pi \times 200 \times 2 = 800\pi\ \text{m}^2/\text{min}.$$

Hence the area of the circle is increasing at a rate of $800\pi\ \text{m}^2/\text{min}$.

Example 2: A spherical balloon is being inflated at a rate of $2\text{m}^3/\text{min}$ (i.e. $\dfrac{dV}{dt} = 2$). Find the rate at which the radius is increasing w.r.t. time (i.e. $\dfrac{dr}{dt}$) when $r = 3\text{m}$.

We must firstly relate $\dfrac{dV}{dt}$ and $\dfrac{dr}{dt}$ using the chain rule:

$$\text{i.e. } \frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$

For a sphere $\quad V = \frac{4}{3}\pi r^3$

$$\therefore \quad \frac{dV}{dr} = 4\pi r^2$$

$$\therefore \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

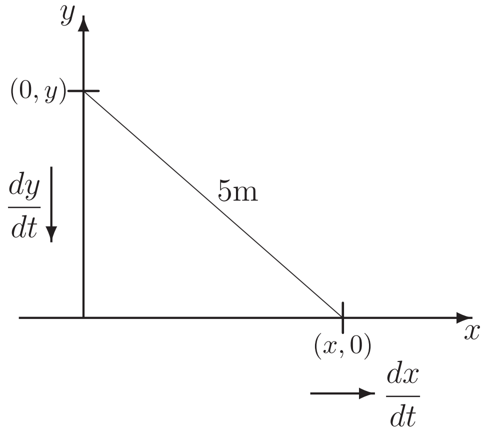
We are asked to find $\frac{dr}{dt}$ when $\frac{dV}{dt} = 2\text{m}^3/\text{min}$ and $r = 3\text{m}$.

$$\therefore \quad 2 = 4\pi \times 9 \times \frac{dr}{dt}$$

$$\therefore \quad \frac{dr}{dt} = \frac{2}{36\pi} = \frac{1}{18\pi} \simeq 0.0177\text{m/min}.$$

Hence, when the radius is 3m, the radius is changing at a rate of 0.0177m/min.

EXAMPLE 3: A 5m long ladder rests against a vertical wall. The foot of the ladder is drawn away from the wall at a rate of 1m/s. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 4m from the wall.



We need to find $\frac{dy}{dt}$ when $x = 4\text{m}$ and $\frac{dx}{dt} = 1\text{m/s}$. We relate x and y from the triangle:

$$x^2 + y^2 = 25$$

We now differentiate implicitly w.r.t. t, giving

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2y\frac{dy}{dt} = -2x\frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

Now when $x = 4$, we have $4^2 + y^2 = 25$

$$\therefore \quad y^2 = 9$$

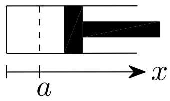
$$\therefore \quad y = 3$$

Substituting $x = 4\text{m}$, $y = 3\text{m}$, and $\frac{dx}{dt} = 1\text{m/s}$ into the expression for $\frac{dy}{dt}$ gives

$$\frac{dy}{dt} = -\frac{4}{3} \times 1 = -\frac{4}{3}\text{m/s}.$$

EXERCISE 44:

1. The pressure P inside a cylinder is given by $P = \dfrac{k}{\pi a^2 x}$, where a and k are constant.



   Find the rate of change of pressure with respect to time t in terms of the velocity of the piston. If initially $x = a$ and $P = P_0$, find $\dfrac{dP}{dt}$ when x has doubled its initial value and the piston is travelling at 1 m/sec.

2. The electric resistance of a resistor is given by $R = 4.000 + 0.003T^2$, where R is measured in ohms and T is Temperature in degrees Celsius ($°C$). If the temperature is increasing at $0.100° C/second$, find how fast the resistance changes when $T = 150° C$.

3. Fatty deposits have increased the circular cross-sectional opening of a person's artery. A drug reduces fat such that the radius of the opening decreases at the rate of 0.020 *mm/month*. Find the rate at which the area decreases when $r = 1.2$ *mm*.

4. Two cars start from point A at the same time, one travelling west at 60km/h and the other travelling north at 45km/h. How fast is the distance between them increasing three hours later?

5. A block of ice in the form of a cube has 10cm long edges. It is melting evenly so that its dimensions decrease at the rate of 1mm per second (i.e. the block always remains a cube). At what rate is the volume decreasing:

   (a) Initially? (b) When edges are 5cm?

6. A man walks along a straight path at a speed of 1.5m/s. A searchlight is located on the ground 6m from the path where the man starts his walk and is kept focused on the man. Show that after the man walks 8m, the searchlight is rotating at a rate of 0.16 radians/sec.

7. Given that $y = x^x$, find the rate of change of y with respect to time (t) when x has a value of 1 and is changing at a rate of 2 units/sec.

8. A large tank in the form of an inverted right circular cone with altitude 6m and a base radius 3m is being filled with water at a rate of 4000L/min. How fast is the water rising when the tank is half full? (Leave π in your answer).

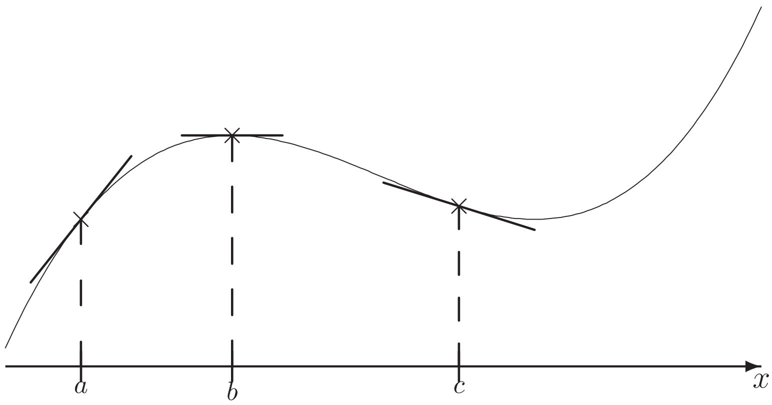
9. A man 1.80m tall approaches a street light 4.5m above the ground at the rate of 1.5m/s. How fast is the end of the man's shadow moving when he is 3m from the base of the light?

10. One end of a rope is tied to a bucket of cement. The other end is passed over a pulley 5m above the floor and tied 1m above the floor to the back of a tractor. If the rope is taut and the tractor moves slowly away at 0.5m/s, show that the bucket is rising at a speed of 0.3m/s when the tractor is 3m from the plumbline through the pulley.

4.14 Using the Derivative

The derivative, $f'(x)$, of a function, $y = f(x)$, is the rate at which y changes w.r.t. x. It defines the slope of the curve at x.

Consider the following curve:



At a: The slope of the tangent is positive. Therefore $f(x)$ is an increasing function here and $f'(x) > 0$.

At c: The slope of the tangent is negative. Therefore $f(x)$ is a decreasing function here and $f'(x) < 0$.

At b: The slope of the tangent is zero. i.e., $f'(x) = 0$.

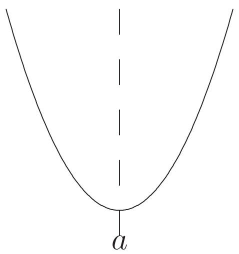
4.15 Turning Points, Critical Points, Stationary Points

Points on the curve where $f'(x) = 0$ are called critical points or stationary points. Critical points may be local maxima, local minima, or points of inflection. If the derivative changes sign at a critical point it is either a local maximum or local minimum. If the derivative does not change sign at the critical point then it is a point of inflection. To determine the nature of critical points we solve $f'(x) = 0$ and then apply the following test.

4.15.1 First Derivative Test

One of the following situations occurs if $f'(x) = 0$.

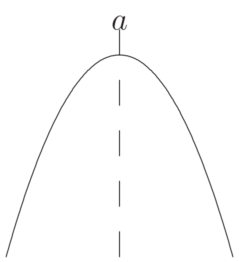
1.



$$x < a,\ f'(x) < 0$$
$$x = a,\ f'(x) = 0$$
$$x > a,\ f'(x) > 0$$

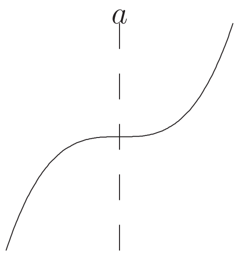
then at the point $(a, f(a))$ we have a local minimum.

2.

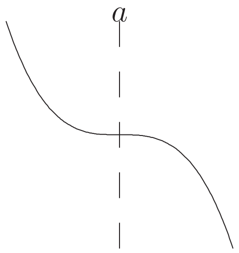


$$x < a,\ f'(x) > 0$$
$$x = a,\ f'(x) = 0$$
$$x > a,\ f'(x) < 0$$

then at the point $(a, f(a))$ we have a local maximum.

3. 

$x < a,\ f'(x) > 0$
$x = a,\ f'(x) = 0$
$x > a,\ f'(x) > 0$

OR 

$x < a,\ f'(x) < 0$
$x = a,\ f'(x) = 0$
$x > a,\ f'(x) < 0$

then at the point $(a, f(a))$ we have a point of horizontal inflection.

This information is helpful if we wish to sketch the graph of a function.

EXAMPLE 1: Sketch the curve $f(x) = x^3 - 3x$.

<u>y intercepts</u>: $x = 0 \quad \therefore \quad y = 0$
<u>x intercepts</u>: $y = 0 \quad \therefore \quad 0 = x^3 - 3x$

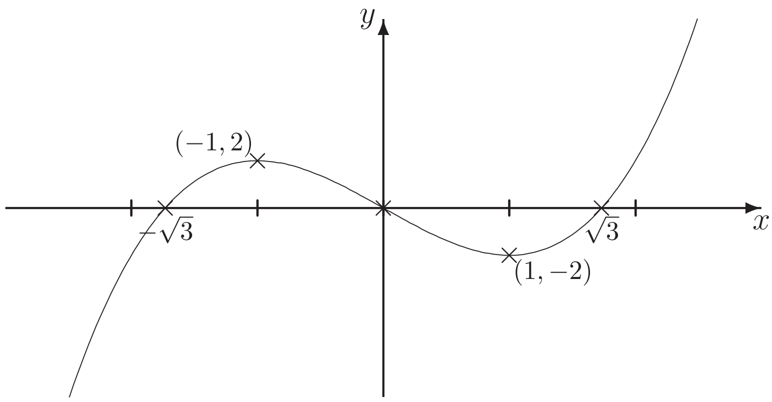
$$= x(x^2 - 3)$$
$$= x(x - \sqrt{3})(x + \sqrt{3})$$
$$\therefore \quad x = 0, \pm\sqrt{3}.$$

<u>Critical Points</u>: $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$
Set $f'(x)$ equal to zero: $0 = 3x^2 - 3$

$$\therefore \quad 0 = x^2 - 1$$
$$\therefore \quad x = \pm 1$$

When $x = 1$, $y = 1 - 3 = -2$. When $x = -1$, $y = -1 + 3 = 2$.
Therefore we have critical points at $(1, -2)$ and $(-1, 2)$.
<u>Nature of Critical Points</u>:
For $x = 1$: $\quad x < 1 \quad f'(x) < 0$
$\quad x > 1 \quad f'(x) > 0$
Therefore there is a local minimum at $(1, -2)$.
For $x = -1$: $\quad x < -1 \quad f'(x) > 0$
$\quad x > -1 \quad f'(x) < 0$
Therefore there is a local maximum at $(-1, 2)$.
<u>Other Information</u>:
As $x \to +\infty, \quad y \to +\infty$.
As $x \to -\infty, \quad y \to -\infty$.

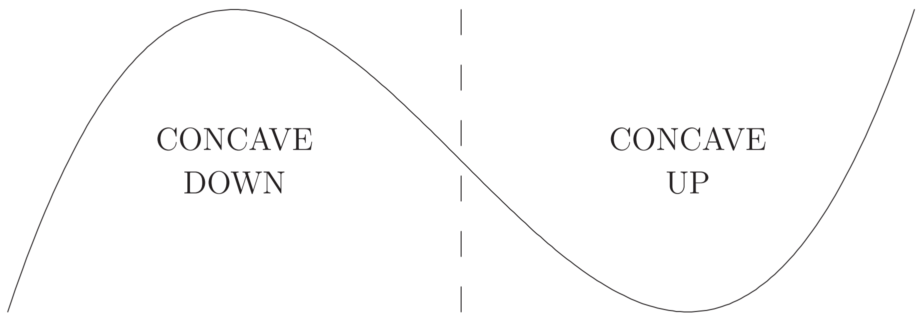
We can use this information to produce the following sketch of the graph of the function.



4.15.2 Second Derivative

In the same way that $\dfrac{df}{dx}$ is defined as the rate of change of f, $\dfrac{d^2f}{dx^2}$ is defined as the rate of change of $\dfrac{df}{dx}$. i.e., the rate of change of the slope.

Concavity:



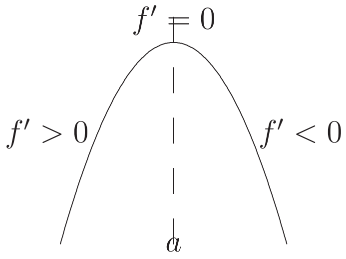
The slope of $f'(x)$ is decreasing from left to right. Therefore $f''(x)$, the rate of change of slope, is negative. i.e. $f''(x) < 0$.

The slope of $f'(x)$ is increasing from left to right $- \to +$. Therefore $f''(x) > 0$.

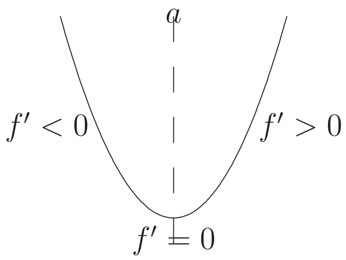
If $f''(x) < 0$ the curve is concave down.
If $f''(x) > 0$ the curve is concave up.

We can also use this information in determining the nature of critical points.

4.15.3 Second Derivative Test



If $f'(a) = 0$ and $f''(a) < 0$ then we have a local maximum at $(a, f(a))$.



If $f'(a) = 0$ and $f''(a) > 0$ then we have a local minimum at $(a, f(a))$.

EXAMPLE 1: Returning to the previous example, where $f(x) = x^3 - 3x$.

We will now use the second derivative test to check for local maxima/minima at $x = \pm 1$.

At $x = 1$ $\quad f'(x) = 3x^2 - 3$

$\therefore \quad f''(x) = 6x$

$\therefore \quad f''(1) = 6 > 0 \quad \therefore$ minimum at $(1, -2)$

At $x = -1$ $\quad f''(x) = 6x$

$\therefore \quad f''(-1) = -6 < 0 \quad \therefore$ maximum at $(-1, 2)$

EXERCISE 45:

1. Sketch graphs of the following showing all turning points, intercepts and asymptotes:
   (a) $x^3 - 6x^2 + 9x + 1$ (b) $\dfrac{x}{(x+3)^2}$

2. In an underwater telephone cable the ratio of the radius of the core to the thickness of the protective sheath is denoted by x. The speed v at which the signal is transmitted is proportional to $x^2 \ln\left(\dfrac{1}{x}\right)$. Show that $\dfrac{dv}{dx} = Kx\left(2\ln\left(\dfrac{1}{x}\right) - 1\right)$ where K is the constant of proportionality, and hence deduce the turning points of v. Distinguish between these turning points and show that the speed is greatest when $x = \dfrac{1}{\sqrt{e}}$.

3. If $f(x) = x^3 - 6x$, find for what values of x:
   (a) $f'(x)$ is negative? (b) $f'(x)$ is zero? (c) $f'(x)$ is positive?

   Interprete these results geometrically on a diagram.

4. Find the intervals of x for which the functions below are *increasing* or *decreasing*, then sketch the curves for each function marking any turning points.
   (a) $y = x^4$ (b) $y = x^3 - x^2 - x + 4$ (c) $y = \dfrac{x+1}{x^2+3}$

5. If $f(x) = (x+1)^3(x-2)^4$, show that $f'(x) = (x+1)^2(x-2)^3(7x-2)$. Hence, find the turning points of $f(x)$ and distinguish between them. Sketch the curve. For what values of x is $f(x)$:
   (a) increasing (b) decreasing

6. Sketch the graph of $y = (x-1)^3(x+2)^2$.

7. Find the turning points of the function $y = 3x^4 - 4x^3 - 12x^2$. Sketch the curve.

8. If a curve whose equation is $12y = ax^3 + bx^2 + cx + d$ has the following properties, find a, b, c, d and sketch the curve.

   (a) it passes through the origin, and the tangent there makes an angle of 45 with the x-axis.

   (b) there are stationary points at $x = 1$ and $x = 2$.

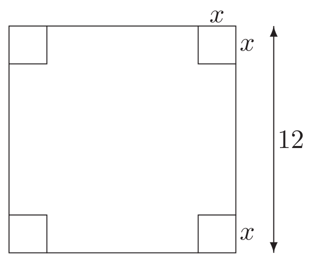
9. Find the points on the curve $y = x^4 - 2x^3$ where the tangent is parallel to the x-axis. Sketch the graphs of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ on the same set of axes.

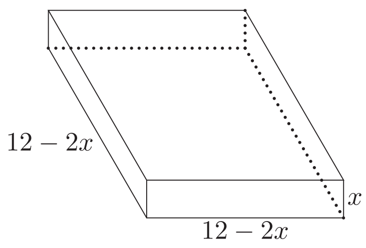
10. Given $f(x) = \dfrac{(x+1)(x-2)}{x(x-1)}$. For what value of x is:
    (a) $f(x) > 0$ (b) $f(x) < 0$ (c) $f'(x) > 0$ (d) $f'(x) < 0$

4.16 Maxima/Minima Problems

The methods we have covered in this section for finding maximum and minimum values have many real applications. e.g., minimum cost, maximum profit, minimum surface area, etc.

Example 1: Square corners are cut from a piece of tin-plate 12cm × 12cm, which is then bent to form an open box. What size squares should be cut away if the volume is to be a maximum?





Suppose we cut squares of length $x$cm.

$$\begin{aligned}\text{Volume} &= \text{length} \times \text{breadth} \times \text{height} \\ V &= (12-2x)^2 x \\ &= x(144 - 48x + 4x^2) \\ &= 144x - 48x^2 + 4x^3 \\ \frac{dV}{dx} &= 144 - 96x + 12x^2 \\ &= 12(12 - 8x + x^2) \\ &= 12(x-6)(x-2) \\ &= 0 \quad \text{if } x = 6 \text{ or } x = 2.\end{aligned}$$

Second Derivative Test:

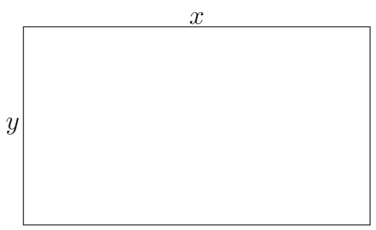
$$\frac{d^2V}{dx^2} = 24x - 96$$

For $x = 2$, $\quad \dfrac{d^2V}{dx^2} = 48 - 96 = -48 < 0 \quad$ (maximum).

For $x = 6$, $\quad \dfrac{d^2V}{dx^2} = 144 - 96 = 48 > 0 \quad$ (minimum).

Thus, to maximize the volume, we remove squares of side 2cm.

Example 2: A rectangular field is enclosed with 1600m of fencing. Find the maximum possible area of the field.



Perimeter: $\quad 1600 = 2x + 2y$

$$\therefore \quad 800 = x + y \qquad (1)$$

The area of the field is

$$A = xy \tag{2}$$

Substitute (1) into (2)
$$\begin{aligned} A &= x(800 - x) \\ &= 800x - x^2. \end{aligned} \tag{3}$$

Differentiate this expression

$$\frac{dA}{dx} = 800 - 2x$$

Set $\frac{dA}{dx} = 0$ and solve for x:

$$\begin{aligned} 0 &= 800 - 2x \\ \therefore \quad x &= 400\text{m}. \end{aligned}$$

We now check that this is indeed a maximum by using the second derivative test:

$$\begin{aligned} \frac{d^2A}{dx^2} &= -2 < 0 \quad \text{(maximum)} \\ \therefore \quad x &= 400\text{m} \end{aligned}$$

Substitute into (3), so that $A = 800 \times 400 - 400^2 = 160,000\text{m}^2$.

EXERCISE 46:

1. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.
2. A block of metal is cast into a solid cylinder whose volume is $10cm^2$. Find the radius of the base that will minimise the surface area.
3. A piece of wire 12cm long is bent to form a rectangle. Find the dimensions of the figure so that its area is a maximum.
4. Show that the minimum value of:
   (a) $\frac{x}{lnx}$ is e.   (b) $x\ln x$ is $-\frac{1}{e}$.
5. Find the maximum volume of a cylinder in which the sum of the height and the base radius is 3m.
6. Find the coordinates of the point(s) on the parabola $y = 3.5 - x^2$ closest to the fixed point (0,2).
7. Prove that $\ln x + \frac{1}{x}$ is never less than 1.
8. A cone of radius r and height h is inscribed in a sphere of radius R. Show that if the volume of the cone is a maximum, its height is $\frac{4R}{3}$.
9. Dale, a potato grower, can deliver 8 tonnes of potatoes to the markets at a profit of 2.50 per tonne. For each week he delays his delivery, he can add 4 tonnes to his shipment but his profit would be reduced by 25c per tonne per week. How long should Dale wait in order to make the maximum profit?
10. Sean is to drive a truck, 400km from Townsville to Cairns. When he is travelling at an average speed of $x$km/hr, the truck consumes fuel at the rate of $\frac{1}{400}(\frac{1600}{x} + x)$ litres per km. If Sean is paid d dollars per hour plus a fixed commission of c dollars, and fuel costs f dollars per litre, show that the most economical speed is $20\sqrt{\frac{(4f+d)}{f}}$.

5 Integral Calculus

5.1 Indefinite Integrals

In some cases, the process of integration is an anti-derivative. The derivative of x^3 is $3x^2$. Integration tells us the function that has a derivative of $3x^2$, i.e. x^3 is one of the possible answers. There are infinitely many solutions to this problem;

$$x^3 + 1, \quad x^3 - 5, \quad x^3 + 20, \quad \text{etc.}$$

All possible answers are of the form

$$x^3 + c \qquad \text{where } c \in \mathbb{R}.$$

We say that $x^3 + c$ is the anti-derivative or integral of $3x^2$ and write

$$\int 3x^2\,dx = x^3 + c$$

which is read as "the integral of $3x^2$ w.r.t. x". Here c is an arbitrary constant called the **constant of integration.**

In general if $F(x)$ and $f(x)$ are functions such that $F'(x) = f(x)$, then

(i) $f(x)$ is the derivative of $F(x)$: $\quad \dfrac{d}{dx}F(x) = f(x)$

(ii) $F(x)$ is the integral (anti-derivative) of $f(x)$: $\quad \displaystyle\int f(x)\,dx = F(x) + c$

Thus the statement $F'(x) = f(x)$ is equivalent to the statement $\int f(x)\,dx = F(x) + c$, where $f(x)$ **is the integrand** and dx indicates that x is the variable of integration. The set of all antiderivatives of $f(x)$ is called the indefinite integral of $f(x)$ w.r.t. x.

5.2 Rules for Integrating

These follow from the corresponding rules for differentiation.

$$\int k\,dx = kx + c \qquad \text{where } k \text{ is a constant.}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Here, c is the constant of integration.

Example 1:

$$\int (x^3 + 3x + 4)\,dx = \frac{x^4}{4} + \frac{3x^2}{2} + 4x + c.$$

Example 2: $\displaystyle\int \left(\frac{4}{\sqrt{x}} + \frac{2}{x^{3/2}} - 2x^{1/2} + 6 \right) dx$

$$\begin{aligned}
&= \int (4x^{-1/2} + 2x^{-3/2} - 2x^{1/2} + 6)\,dx \\
&= \frac{4x^{1/2}}{1/2} + \frac{2x^{-1/2}}{(-1/2)} - \frac{2x^{3/2}}{3/2} + 6x + c \\
&= 8x^{1/2} - 4x^{-1/2} - \frac{4}{3}x^{3/2} + 6x + c.
\end{aligned}$$

To determine the value of the integration constant we need extra information.

Example 3: If $f'(x) = 4x - 3$ and $f(x)$ passes through the point $(2, 4)$, find $f(x)$.

$$\begin{aligned}
f(x) &= \int (4x - 3)\,dx \\
&= 4\frac{x^2}{2} - 3x + c \\
&= 2x^2 - 3x + c
\end{aligned}$$

When $x = 2$, $y = 4$ $\quad \therefore \quad 4 = 2 \times 4 - 3 \times 2 + c$

$$\therefore \quad c = 2$$

$$\therefore \quad f(x) = 2x^2 - 3x + 2.$$

From the knowledge of differential calculus we can obtain the integrals of some fundamental functions.

$$\begin{aligned}
\int e^x\,dx &= e^x + c \\
\int \frac{1}{x}\,dx &= \ln|x| + c \\
\int \cos x\,dx &= \sin x + c \\
\int \sin x\,dx &= -\cos x + c \\
\int \sec^2 x\,dx &= \tan x + c \\
\int \sec x \tan x\,dx &= \sec x + c
\end{aligned}$$

Example 4:

$$\int e^x + \cos x\,dx = e^x + \sin x + c.$$

Determining integrals involving trigonometric functions may involve the use of trigonometric identities.

Example 5: $\displaystyle\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx \qquad \text{as } 1+\tan^2 x = \sec^2 x$

$$= \tan x - x + c.$$

5.3 Substitution

This method may be regarded as the converse of differentiating via the Chain Rule. The aim of substitution is to transform a difficult integral into one which involves a standard result.

Example 1: Calculate $\displaystyle\int e^{6x}\,dx$.

Let $u = 6x$, $\therefore\ \dfrac{du}{dx} = 6$ or $dx = \dfrac{du}{6}$.

$$\begin{aligned}\therefore \quad \int e^{6x}\,dx &= \int e^u\,\frac{du}{6}\\ &= \frac{1}{6}e^u + c\\ &= \frac{e^{6x}}{6} + c.\end{aligned}$$

Note: The substitution transforms the integrand from a function of x to a function of u. Therefore we must also transform the differentials from dx to du – we use the derivative du/dx to do this.

Example 2: Calculate $\displaystyle\int \sin 3x\,dx$.

Let $u = 3x$, $\therefore\ \dfrac{du}{dx} = 3$ or $dx = \dfrac{du}{3}$.

$$\begin{aligned}\therefore \quad \int \sin 3x\,dx &= \int \sin u\,\frac{du}{3}\\ &= \frac{1}{3}(-\cos u) + c\\ &= -\frac{\cos 3x}{3} + c.\end{aligned}$$

EXAMPLE 3: Calculate $\int x(1+x^2)^3\,dx$.

Let $u = 1+x^2$, $\quad \therefore\ \dfrac{du}{dx} = 2x$ or $x\,dx = \dfrac{du}{2}$.

$$\begin{aligned}\therefore \quad \int x(1+x^2)^3\,dx &= \int u^3\,\frac{du}{2} \\ &= \frac{1}{2}\frac{u^4}{4} + c \\ &= \frac{(1+x^2)^4}{8} + c.\end{aligned}$$

EXAMPLE 4:

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx \qquad \text{as } \cos 2x = 1 - 2\sin^2 x \\ &= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c.\end{aligned}$$

EXAMPLE 5: Calculate $\int \dfrac{\sin x}{\cos^2 x}\,dx$.

Let $u = \cos x$, then $\dfrac{du}{dx} = -\sin x$ or $\sin x\,dx = -du$.

$$\begin{aligned}\therefore \quad \int \frac{\sin x}{\cos^2 x}\,dx &= -\int \frac{1}{u^2}\,du \\ &= -\int u^{-2}\,du \\ &= -\frac{u^{-1}}{-1} + c \\ &= \frac{1}{u} + c = \frac{1}{\cos x} + c.\end{aligned}$$

EXAMPLE 6: Consider the integral $I = \int (x^2+1)^{3/2}\,dx$.

This example shows that the method of substitution doesn't always help us.

Try $u = x^2 + 1$, so that $\dfrac{du}{dx} = 2x$ or $dx = \dfrac{du}{2x}$.

But $u = x^2 + 1$, therefore $x = \sqrt{u-1}$ and $dx = \dfrac{du}{2\sqrt{u-1}}$.

$$\therefore \quad I = \int u^{3/2}\frac{du}{2\sqrt{u-1}}.$$

In this case, the choice of substitution was not helpful.

EXERCISE 47: Determine the following indefinite integrals:

1. $\int x^{-4} - \frac{5}{x^2}\ dx$
2. $\int x^{-\frac{2}{3}} - x^{-\frac{4}{3}}\ dx$
3. $\int x^{-1} - 4\sqrt{x}\ dx$
4. $\int 2e^{-3x}\ dx$
5. $\int \frac{4}{x} + 2x^{-\frac{1}{2}}\ dx$
6. $\int \frac{1}{x+1}\ dx$
7. $\int \frac{1}{4-x}\ dx$
8. $\int e^{6x}\ dx$
9. $\int 4e^{12x} + 4e^{-12x}\ dx$
10. $\int 2xe^{-x^2}\ dx$
11. $\int \sin 3x\ dx$
12. $\int \sec^2 3x\ dx$
13. $\int \sin 2x - \cos 2x\ dx$
14. $\int \sec x \tan x\ dx$
15. $\int 2\text{cosec}x \cot x\ dx$
16. $\int \sin\frac{x}{2}\cos\frac{x}{2}\ dx$
17. $\int 2\cos^2 x\ dx$

5.4 Definite Integrals

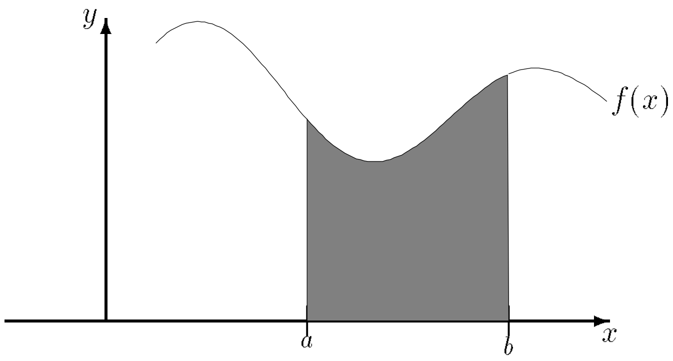
The set of all antiderivatives of $f(x)$ is called the indefinite integral of $f(x)$ w.r.t. x. All the integrals we have dealt with so far have been indefinite integrals. We now need to define a definite integral.

The definite integral of $f(x)$ from $x = a$ to $x = b$ is

$$\boxed{\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)}$$

Here, a is the lower limit of integration and b is the upper limit of integration.

NOTE: This can be interpreted geometrically as the area under the curve $f(x)$ between $x = a$ and $x = b$.



EXAMPLE 1: Evaluate the definite integral $\int_1^2 x^3\,dx$.

$$\begin{aligned}\int_1^2 x^3\,dx &= \left[\frac{x^4}{4}\right]_1^2 \\ &= \frac{2^4}{4} - \frac{1^4}{4} \\ &= \frac{16}{4} - \frac{1}{4} \\ &= \frac{15}{4}.\end{aligned}$$

Note that the constant of integration does not appear in the result of a definite integral.

EXAMPLE 2: Evaluate the integral $\int_1^2 x^2\,dx$.

$$\begin{aligned}\int_1^2 x^2\,dx &= \left[\frac{x^3}{3}\right]_1^2 \\ &= \left(\frac{2^3}{3}\right) - \left(\frac{1^3}{3}\right) \\ &= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.\end{aligned}$$

5.5 Properties of Definite Integrals

1. $$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

2. $$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \qquad \text{where } k \text{ is a constant}$$

3. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example 1:

$$\begin{aligned}\int_{-2}^{-1} \frac{x^2+4}{x^2}\,dx &= \int_{-2}^{-1} \frac{x^2}{x^2} + \frac{4}{x^2}\,dx \\ &= \int_{-2}^{-1} (1+4x^{-2})\,dx \\ &= \left[x + \frac{4x^{-1}}{-1}\right]_{-2}^{-1} \\ &= \left[x - \frac{4}{x}\right]_{-2}^{-1} \\ &= \left(-1 - \frac{4}{-1}\right) - \left(-2 - \frac{4}{-2}\right) \\ &= 3 - 0 = 3.\end{aligned}$$

Example 2:

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx &= \Big[\ln|x|\Big]_1^2 \\ &= \ln 2 - \ln 1 = \ln 2.\end{aligned}$$

Example 3: Evaluate the definite integral $\displaystyle\int_0^2 \frac{1}{2x+1}\,dx$.

Let $u = 2x+1$, $\therefore\ \dfrac{du}{dx} = 2$ or $dx = \dfrac{du}{2}$.

Again, we change the limits of integration.

When $x = 0$, $u = 0 + 1 = 1$. When $x = 2$, $u = 4 + 1 = 5$.

$$\begin{aligned}\therefore \quad \int_0^2 \frac{1}{2x+1}\,dx &= \int_1^5 \frac{1}{u}\,\frac{du}{2} \\ &= \left[\frac{1}{2}\ln u\right]_1^5 \\ &= \frac{1}{2}\ln 5 - \frac{1}{2}\ln 1 \\ &= \frac{\ln 5}{2}.\end{aligned}$$

EXAMPLE 4: Calculate $\displaystyle\int_0^1 \left(x^3+1\right)^{-1/2} \times x^2\,dx$.

Let $u = x^3 + 1$, $\quad\therefore\ \dfrac{du}{dx} = 3x^2$ or $x^2\,dx = \dfrac{du}{3}$.

When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.

$$\begin{aligned}
\therefore\quad \int_0^1 \left(x^3+1\right)^{-1/2} \times x^2\,dx &= \int_1^2 u^{-1/2}\frac{du}{3} \\
&= \left[\frac{1}{3}\frac{u^{1/2}}{1/2}\right]_1^2 \\
&= \left[\frac{2}{3}u^{1/2}\right]_1^2 \\
&= \left(\frac{2}{3}2^{1/2}\right) - \left(\frac{2}{3}1^{1/2}\right) \\
&= \frac{2}{3}(\sqrt{2}-1).
\end{aligned}$$

EXAMPLE 5: Evaluate the definite integral $\displaystyle\int_0^{\pi/4} \sin 2x\,dx$.

Let $u = 2x$, $\quad\therefore\ \dfrac{du}{dx} = 2$ or $dx = \dfrac{du}{2}$.

When changing the variable of integration, we also need to change the limits of integration:

When $x = 0$, $u = 2 \times 0 = 0$. When $x = \pi/4$, $u = 2 \times \pi/4 = \pi/2$.

$$\begin{aligned}
\therefore\quad \int_0^{\pi/4} \sin 2x\,dx &= \int_0^{\pi/2} \sin u\,\frac{du}{2} \\
&= \frac{1}{2}\big[-\cos u\big]_0^{\pi/2} \\
&= \left(-\frac{\cos(\pi/2)}{2}\right) - \left(-\frac{\cos 0}{2}\right) \\
&= 0 + \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

EXERCISE 48: Evaluate the following definite integrals:

1. $\displaystyle\int_1^2 x^2 + 5\ dx$
2. $\displaystyle\int_1^9 \frac{3}{x} + x^{-\frac{1}{2}}\ dx$
3. $\displaystyle\int_0^1 \frac{1}{4x+1}\ dx$
4. $\displaystyle\int_0^1 e^{3x} - e^{-3x}\ dx$
5. $\displaystyle\int_0^{\pi/4} \sin 2x\ dx$
6. $\displaystyle\int_0^{\pi/4} \sin 2x + \cos 2x\ dx$
7. $\displaystyle\int_0^{\pi/4} \tan^2 x\ dx$
8. $\displaystyle\int_0^{\pi/2} \sin^2 x\ dx$

EXERCISE 1:

1. (a) $3x$ (b) $\dfrac{b^4c}{a^2}$ (c) 2^8 (d) $\dfrac{y}{x^2}$ (e) $\dfrac{y^5}{2}$ (f) $\dfrac{1}{2^4 3^3 ab^3}$ (g) $2xy\left(y^2-x^2\right)$

(h) $x^{-1}+y^{-1}=\dfrac{x+y}{xy}$ (i) $ax^{-1}-1+a^{-1}x$ 2. (a) $\dfrac{1}{2}$ (b) -27 4. $\dfrac{1-x}{x}$

EXERCISE 2:

1. (a) $x=4$ (b) $x=\dfrac{5}{3}$ (c) $x=-\dfrac{3}{2}$ (d) $x=2$ or $x=3$ (e) $x=0$ or $x=-2$

(f) $x=1$ (g) $x=0$ or $x=2$ 2. $x=-\dfrac{1}{2}$ 4. $n=4$ or $n=8$

EXERCISE 3:

1. (a) 2 (b) $\dfrac{1}{9}$ (c) $\dfrac{2^2}{3^2a^{\frac{1}{6}}b^{\frac{5}{4}}}$ (d) 1 (e) 4 2. (a) 1 (b) $(x^{\frac{1}{2}}-5^{\frac{1}{2}})(x^{\frac{1}{2}}+5^{\frac{1}{2}})$

3. 4 5. $x=-2$

EXERCISE 4:

1. (a) $3^{\frac{11}{6}}$ (b) $3\sqrt{5}$ (c) $-4\sqrt{5}$ 2. (a) $\sqrt{6}-2$ (b) $5-2\sqrt{6}$ (c) 1

3. (a) FALSE (b) TRUE (c) TRUE 4. $4\sqrt{5}, 9, 2\sqrt{21}, 3\sqrt{10}$

5. (a) $x=7$ (b) $x=13$

6. (a) $x=42, y=-24$ (b) $x=4, y=8$ (c) $x=4, y=-2$ (d) $x=2, y=1$

EXERCISE 5:

1. (a) $\dfrac{3\sqrt{5}}{10}$ (b) $\dfrac{\sqrt{35}}{7}$ (c) $3\left(\sqrt{2}+\sqrt{3}\right)$ (d) $\dfrac{3\left(4\sqrt{3}-3\sqrt{2}\right)}{5}$

2. $\dfrac{66}{23}$ 3. $2-\sqrt{3}$ 4. $\dfrac{\sqrt{2}+2-\sqrt{6}}{4}$ 5. $\sqrt[6]{108}$ 6. $\sqrt{15}$

EXERCISE 6:

1. (a) 5 (b) -4 (c) 1 (d) 8 (e) $-\log_5 2$ (f) 2 (g) 25 (h) 18

2. (a) $x=2^{-3}=\dfrac{1}{8}$ (b) $x=5$ (c) 10 (d) $x=-4$ (N.P.) or $x=2$ (e) $x=\dfrac{1}{4}$

EXERCISE 7:

1. $x=\dfrac{e^{\frac{2}{3}}}{2}$ 2. -1 3. 1.1 4. $t=\dfrac{1}{4}\ln\left(\dfrac{y}{a}\right)$ 5. $P=Ae^{-bt}$

EXERCISE 8:

1. $6a^3+21a^2$ 2. $8x^2+2xy-15y^2$ 3. x^2+2x-8 4. $49x^2-4$ 5. $9x^2-100y^2$

6. $9x^2+60xy+100y^2$ 7. $4x^2-12x+9$ 8. $2x^4+7x^2-15$ 9. $15-13x+2x^2$

10. $\dfrac{9}{x^2}+12+4x^2$

EXERCISE 9:

1. $(x-2)(x-3)$ 2. $(x+3)(x+4)$ 3. $(2x-1)(x+3)$ 4. $(3x+1)(4x-1)$
5. not possible 6. $(x+5)^2$ 7. $(x-\sqrt{6})(x+\sqrt{6})$ 8. $(p+8-\sqrt{86})(p+8+\sqrt{86})$
9. $5\left(v-1-\dfrac{\sqrt{130}}{5}\right)\left(v-1+\dfrac{\sqrt{130}}{5}\right)$ 10. $(x-7)^2$ 11. $(y^2+9)^2$ 12. $(x-8)(x+8)$
13. $(4x-7y)(4x+7y)$ 14. $(x-3)(x^2+3x+9)$ 15. $(y+2)(y^2-2y+4)$
16. $(x-\sqrt{8})(x+\sqrt{8})$ 17. $(r+1-\sqrt{34})(r+1+\sqrt{34})$ 18. $\left(n+\dfrac{13+\sqrt{53}}{2}\right)\left(n+\dfrac{13-\sqrt{53}}{2}\right)$
19. $2(5x-2)^2$ 20. $(4y-x-3)(4y+x+3)$ 21. $(3y-x)(5x-y)$ 22. $(e^t+1)(e^t+2)$
23. $(x+1)(x-1)(x^2-x+1)(x^2+x+1)$ 24. $3x(x-3)$ 25. $(m-6-\sqrt{10})(m-6+\sqrt{10})$
26. $\left(r-\dfrac{9+\sqrt{197}}{2}\right)\left(r-\dfrac{9-\sqrt{197}}{2}\right)$

EXERCISE 10:

1. $2x^2+5x+3+\dfrac{4}{x-1}$ 2. $x+2-\dfrac{2}{x^2+2x-1}$ 3. x^2+3x+2 4. $x^2-x+3+\dfrac{5}{x+1}$
5. $3x^3-x+2-\dfrac{4}{x-3}$ 6. $5x^2+6+\dfrac{25}{2x-5}$ 7. $2x^2-3x+\dfrac{4x}{x^2-3x+4}$
8. $2x^2-2x+1-\dfrac{3}{2x+1}$

EXERCISE 11:

1. (a) 76 (b) 6 2. (a) $(x-6)(x+1)(x+7)$ (b) $(x-3)(x+2)(2x-1)$
3. (b) $\dfrac{1}{8}$ 4. (a) $Q(x)=x^2-2x+3, R=0$ (b) $Q(x)=x^2-2x-9, R=-44$
5. (a) YES (b) NO (c) YES 6. (a) $(x-1)(x-2)(x-3)$ (b) $(x-1)(x-2)(2x-1)$
(c) $(x+1)(x^2+1)$ 7. (a) $k=2$ (b) $k=-\dfrac{1}{2},1$ (c) $k=2$ 8. $k=-1$
9. $a=1, b=3$

EXERCISE 12:

1. (a) $1,6$ (b) $\dfrac{3}{2},-4$ (c) $3,-\dfrac{1}{2}$ (d) $\dfrac{-4\pm\sqrt{7}}{9}$ 2. (a) $2,-5,-\dfrac{1}{2}$ (b) $0,-2$
3. (a) $-1,\dfrac{3}{2}$ (b) $10,10$ (c) $\dfrac{1\pm\sqrt{7}}{2}$ (d) $4,-6$ 4. 8 5. 24 m 6. 15 km/hr

EXERCISE 13:

1. $\dfrac{3}{x+2}+\dfrac{2}{x-1}$ 2. $\dfrac{2}{x+1}-\dfrac{1}{x+3}$ 3. $\dfrac{1}{x+2}+\dfrac{3}{x+1}-\dfrac{2}{x-1}$ 4. $\dfrac{1}{x-1}+\dfrac{3}{x+2}+\dfrac{2}{(x-1)^2}$
5. $\dfrac{3}{3x+1}+\dfrac{4}{x+8}$ 6. $\dfrac{-2}{x+3}+\dfrac{3}{x+2}+\dfrac{4}{x+1}$ 7. $\dfrac{1}{2(x+1)}+\dfrac{1}{2(x-1)}-\dfrac{1}{x}+x+1$
8. $\dfrac{3}{x^2+4}-\dfrac{2}{x+1}$ 9. $-\dfrac{1}{x}-\dfrac{1}{x^2}-\dfrac{2x}{x^2+4}+\dfrac{3}{x-1}$

EXERCISE 14:

1. $x < -2$ 2. $x \leq \frac{13}{2}$ 3. $-3 \leq x \leq 4$ 4. $-1 \leq x < 1$ 5. $-1 \leq x \leq 0$

6. no solution 7. $-3 \leq x < -1$ 8. $x \leq \frac{1}{2}$ 9. $x \leq -1$ 10. $x \leq -4$ or $x \geq 1$

11. $-2 < x < 3$ 12. $x \leq \frac{1}{2}$ 13. $\frac{1}{3} \leq x \leq 9$ 14. $x < -2$ or $x > 1$

15. $-1 < x < 3,\ x \neq 2$ 16. $-4 \leq x \leq 2$ or $x \geq 5$ 17. $-2 < x < 1$ or $x > 3$

18. $0 < x < 1$ 19. $-\sqrt{3} < x < -1$ or $1 < x < \sqrt{3}$

EXERCISE 15:

1. $2.08\dot{3}\ m$ 2. 228% 3. 46% 4. Original investment 125, current value 600.13

EXERCISE 16:

1. (a) 8 (b) 20 (c) 4845 (d) 1540 (e) 1 (f) 1
5. (a) $x = 7$ or 9 (b) $x = 16$

EXERCISE 17:

1. (a) $32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243$ (b) $81a^4 - 108a^3 + 54a^2 - 12a + 1$
2. (a) 10 terms: $x^{18}+18x^{16}+144x^{14}+672x^{12}+\cdots$ (b) 13 terms: $1-12b^2+66b^4-220b^6+\cdots$
4. $1287a^5b^8$ 5. 252 6. 216 7. $945x^3y^6z^8$

EXERCISE 18:

1. (a) $1+2x+3x^2+4x^3+\cdots$ (b) $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\cdots$ (c) $1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}+\cdots$

2. (a) $1+\frac{x}{2}-\frac{x^2}{8}$ (b) $x-\frac{x^2}{2}+\frac{3x^3}{8}-\frac{5x^4}{16}+\cdots$ 3. (a) $2+\frac{x^2}{4}-\frac{x^4}{64}+\frac{x^6}{512}$
(b) $\frac{1}{3}+\frac{x}{6}+\frac{x^2}{8}+\frac{5x^3}{48}$ 4. $\frac{x^3}{8}$ 5. (a) $\frac{2Q_0}{(1+t)}-\frac{2Q_0}{(2+t)}$ (b) $Q_0(1-\frac{3}{2}t+\frac{7}{4}t^2-\frac{15}{8}t^3+\cdots)$

EXERCISE 19:

1. (a) $y = \frac{3x}{4} + \frac{19}{2}$ (b) $y = 2x - 12$ (c) $y = -x - 7$ (d) $y = 2$ (e) $x = -2$

(f) $y = -\frac{2x}{3} - 2$ (g) $y = -\frac{3x}{2}$ (h) $y = \frac{2x}{3} - \frac{13}{3}$ 2. (a) $y = \frac{2x}{3} - 2$ (b) 33.69°
(c) $y = \frac{-3x}{2} + 11$

EXERCISE 20:

2. $(x-4)^2 + (y-4)^2 = 16$ 3. (a) $(x+1)^2 + (y+2)^2 = 25$ (b) $2\sqrt{21}$ (c) $\sqrt{60}$

4. (a) 5 (b) $y = \frac{4x}{3} + \frac{10}{3}$ or $y = -\frac{3x}{4} + \frac{15}{2}$ 5. $x^2 + y^2 - x - 3y - 22 = 0$

6. (2,5) and (-4,-7)   7. $k = 7$

EXERCISE 21:

1. (a) $\dfrac{(x-2)^2}{9} + \dfrac{(y+3)^2}{4} = 1$;   Ellipse: length of axes 2 and 3

(b) $\dfrac{(y+3)^2}{16} - \dfrac{(x-2)^2}{4} = 1$;   Hyperbola: asymptotes given by $y = -3 \pm 2(x-2)$

2. 7.8125   3. 12 m 4. (a) No   (b) 3.4375 m

5. (a) $x = \dfrac{2\sqrt{6}m^2 \pm \sqrt{2-8m^2}}{1+2m^2}$ (b) $y = \pm\frac{1}{2}(x - \sqrt{6})$ 7. $(\frac{1}{2}, 2)$

EXERCISE 22:

1. (i) (a) Domain: $x \leq 0$; Range: $y \in \mathcal{R}$ (b) not a function
(ii) (a) Domain: $-1 \leq x \leq 1$; Range: $-1 \leq y \leq 1$ (b) not a function
(iii) (a) Domain: $x \neq 0$; Range: $y \neq 0$ (b) is a function
(iv) (a) Domain: $x \neq -1$; Range: $y \neq 2$ (b) is a function
2. (a) $x \neq 0$ (b) $x \neq 1$ (c) $x \leq 1$ (d) $x < 2$
3. (a) Domain: $x \in \mathcal{R}$; Range: $y \in \mathcal{R}$ (b) Domain: $x \in \mathcal{R}$; Range: $y \geq 0$
(c) Domain: $x \geq -3$; Range: $y \geq 0$ (d) Domain: $x \neq 2$; Range: $y \neq 0$
4. (a) Domain: $x \in \mathcal{R}$; Range: $y \geq 0$ (b) Domain: $x \neq \pm 1$; Range: $y < 0, y \geq 3$
(c) Domain: $0 \leq x \leq 1$; Range: $0 \leq y \leq 1$ (d) Domain: $x < -5, x > 5$; Range: $y > 0$
5. Domain: $0 \leq x \leq 500$; Range: $0 \leq y \leq 1000$

EXERCISE 24:

1. $\frac{1}{2}$ 2. $\frac{5}{7}$ 3. $\frac{1}{3}$ 4. ∞ 5. ∞ 6. ∞ 7. 0   8. $\frac{1}{3}$ 9. 2

EXERCISE 25:

1. (a) Domain: $x \neq 3$; Range: $y \neq 0$; H.A.: $y = 0$; V.A.: $x = 3$.
(b) Domain: $x \neq -2$; Range: $y \neq 1$; H.A.: $y = 1$; V.A.: $x = -2$.
2. (a) Domain: $x \neq -3$ and $x \neq 1$; Range: $y \in \mathcal{R}$; H.A.: $y = 0$; V.A.: $x = -3$ and $x = 1$.
(b) Domain: $x \neq -2, x \neq 2$; Range: $y \leq 0$ or$y > 1$; H.A.: $y = 1$; V.A.: $x = -2$ and $x = 2$.
(c) Domain: $x \neq -1$ and $x \neq 1$; Range: $y \in \mathcal{R}$; H.A.: $y = 1$; V.A.: $x = -1$ and $x = 1$.
3. (b) Domain: $x \neq -1, x \neq 1$; Range: $y \leq -1, y > 0$; H.A.: $y = 0$; V.A.: $x = -1, x = 1$.
4. Domain: $x \neq -1, x \neq 2$; Range: $y \leq -1, y > 0$; H.A.: $y = 0$; V.A.: $x = -1, x = 1$.

EXERCISE 26:

1. (a) neither   (b) odd   (c) even   (d) odd   (e) odd   (f) even   (g) neither
(h) even   3. $\dfrac{1+x^2}{(1-x^2)^2} + \dfrac{2x}{(1-x^2)^2}$

EXERCISE 27:

1. (a) Domain: $x \neq 0$; Range: $y > 1$; x-int: none; y-int: none; Asym.: $x = 0$, $y = 1$
(b) Domain: $x \neq 3$; Range: $y > 0$; x-int: none; y-int: $\frac{1}{9}$; Asym.: $x = 3$, $y = 0$
(c) Domain: $x \neq 3$; Range: $y < 0$; x-int: none; y-int: $-\frac{1}{9}$; Asym.: $x = 3$, $y = 0$
(d) Domain: $x \neq -1$; Range: $y > 2$; x-int: none; y-int: 3;Asym.: $x = -1$, $y = 2$
4. (a) C (b) B & C (c) A, D & E (d) A
5. (a) Domain: $x \neq -2$; Range: $y > 0$; x-int: none; y-int: $\frac{1}{4}$; Asym.: $x = -2$, $y = 0$
(b) Domain: $x \neq 2$; Range: $y > 2$; x-int: none; y-int: $\frac{9}{4}$; Asym.: $x = 2$, $y = 2$
(c) Domain: $x \neq 1$; Range: $y < -1$; x-int: none; y-int: -2; Asym.: $x = 1$, $y = -1$
(d) Domain: $x \neq -2$; Range: $y > -2$; x-int: -2.707,-1.293; y-int: $-\frac{7}{4}$; Asym.: $x = -2$, $y = -2$

EXERCISE 28:

1. (a) $f \circ g(x) = 8x + 1$, $g \circ f(x) = 8x + 11$, $f \circ f(x) = 4x + 9$, $g \circ g(x) = 16x - 5$
(b) $f \circ g(x) = 18x^2 + 21x + 6$, $g \circ f(x) = 6x^2 - 3x + 2$, $f \circ f(x) = 8x^4 - 8x^3 + x$, $g \circ g(x) = 9x + 8$
(c) $f \circ g(x) = \sqrt{-x}$, $g \circ f(x) = \sqrt{1 - \sqrt{x^2 - 1}}$, $f \circ f(x) = \sqrt{x^2 - 2}$, $g \circ g(x) = \sqrt{1 - \sqrt{1 - x}}$
2. (a) $g(x) = 2x + 3, f(x) = \sqrt{x}$ (b) $g(x) = x^2 + 5x + 6, f(x) = x^3$
(c) $g(x) = 2x, f(x) = x^{-3}$ 3. (a) 0,1 (b) 1,9 (c) 0,2 (d) ± 2

EXERCISE 29:

1. (a) yes (b) no (c) no (d) yes 2. (a) $f^{-1}(x) = \dfrac{x - 6}{4}$ (b) $f^{-1}(x) = \dfrac{2x + 2}{1 - x}$
(c) $f^{-1}(x) = \dfrac{5x - 1}{2x + 3}$ (d) $f^{-1}(x) = \dfrac{x^2 - 2}{5}$ (e) $f^{-1}(x) = -\dfrac{1}{2} + \sqrt{x + \dfrac{1}{4}}$
3. (b) $x > 1$ *and* $x < 1$ (c) The inverse of $f(x) = x^2 - 2x + 1,\ x < 1$, is $f^{-1}(x) = 1 - \sqrt{x}$
The inverse of $f(x) = x^2 - 2x + 1,\ x > 1$, is $f^{-1}(x) = 1 + \sqrt{x}$

EXERCISE 30:

1. (a) D_f: $x > 1$; R_f: $y \in \mathcal{R}$; x-int: 2; y-int: none; Asym.: $x = 1$
(b) D_f: $x > 0$; R_f: $y \in \mathcal{R}$; x-int: $\frac{1}{10}$; y-int: none; Asymp.: $x = 0$
(c) D_f: $x \in \mathcal{R}$; R_f: $y > 0$; x-int: none; y-int: e^{-2}; Asym.: $y = 0$
(d) D_f: $x \in \mathcal{R}$; R_f: $y > -2$; x-int: $\ln 2$; y-int: -1; Asym.: $y = -2$
(e) D_f: $x \in \mathcal{R}$; R_f: $y > -2$; x-int: $-\ln 2$; y-int: -1; Asym.: $y = -2$
3. 1

EXERCISE 31:

1. (a) $\sin\theta = \frac{4}{5}$, $\tan\theta = \frac{4}{3}$, $\cot\theta = \frac{3}{4}$, $\sec\theta = \frac{5}{3}$, $\text{cosec}\,\theta = \frac{5}{4}$
(b) $\sin\alpha = \dfrac{\sqrt{5}}{3}$, $\cos\alpha = -\dfrac{2}{3}$, $\tan\alpha = -\dfrac{\sqrt{5}}{2}$, $\cot\alpha = -\dfrac{2}{\sqrt{5}}$, $\text{cosec}\,\alpha = \dfrac{3}{\sqrt{5}}$

(c) $\sin\beta = -\dfrac{1}{\sqrt{10}},\ \cos\beta = -\dfrac{3}{\sqrt{10}},\ \tan\beta = \dfrac{1}{3},\ \sec\beta = -\dfrac{\sqrt{10}}{3},\ \operatorname{cosec}\beta = -\sqrt{10}$

4. 1 8. $\dfrac{1-\sin^2 A}{\sin^2 A}$ 9. $\dfrac{(x-2)^2}{9} + \dfrac{(y-5)^2}{4} = 1$

EXERCISE 32:

1. These answers for the tangent only. (a)-1 (b) $\sqrt{3}$ (c) -1 (d) -1 (e) 0
(f) -1 (g) $-\dfrac{1}{\sqrt{3}}$ 2. $\dfrac{5}{8}$
3. (a) $\cos A$ (b) $-\sin B$ (c) $-\operatorname{cosec} C$ (d) $\sin D$ (e) $\tan E$

EXERCISE 33:

1. (a) 1 (b) 0 (c) $\sin 3x$ (d) $\cos\alpha$ (e) $\dfrac{1}{\sqrt{3}}$ (f) 1 (g) $\tan(x-2y)$
2. (a) -1 (b) $-\dfrac{1}{7}$ 3. (a) $\sin\alpha = \dfrac{4}{5},\ \tan\alpha = \dfrac{4}{3}$ (b) $\sin\beta = \dfrac{3}{5},\ \tan\beta = -\dfrac{3}{4}$
(c) $-\dfrac{7}{25}$ (d) 0 (e) not defined (f) $\dfrac{1}{7}$ 5. $2-\sqrt{3}$ 6. $\cos 2\alpha$
7. $\sin\alpha\cos\beta = \dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$, $\cos\alpha\cos\beta = \dfrac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$,
$\sin\alpha\sin\beta = \dfrac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$.

EXERCISE 34:

1. (a) $\sin 2\theta$ (b) $2\sin 2\theta$ (c) $\cos 2\theta$ (d) $\cos 2\theta$ (e) $-\cos 2\theta$ (f) $\tan 2\theta$
2. (a) $\dfrac{4\sqrt{2}}{9}$ (b) $-\dfrac{7}{9}$ (c) $-\dfrac{4\sqrt{2}}{7}$ 3. (a) $\dfrac{1}{3}$ (b) $\dfrac{2\sqrt{2}}{3}$ (c) $2\sqrt{2}$ 5. 25.66°

EXERCISE 35:

1. (b) (i) 1 (ii) $\dfrac{1}{2}$ (iii) 2π units to right (iv) 3 units down
(c) (i) 2 (ii) 4 (iii) $\dfrac{\pi}{4}$ units to right (iv) 1 unit up
2. (a) $\dfrac{1}{2}\cos 3\left(t-\dfrac{2\pi}{3}\right)-2$ (b) $10\sin 5\left(t+\dfrac{\pi}{10}\right)$ 3. (a) 6 (b) 0 (c) -1.8540
5. (a) $h=7$ at midnight (t=0), noon (t=12), midnight (t=24) (b) 4m
(d) Before 2am, between 10am and 2pm, between 10pm and 2am the next morning.
7. $A=2, B=1.5, C=-\dfrac{\pi}{2}, D=-4$ 8. (a) 65cm (b) 110cm 9. 12:32:44 pm
10. 93 days

EXERCISE 36:

1. (a) $\dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}, \dfrac{11\pi}{3}$ (b) $85.003^\circ,\ 94.997^\circ$ (c) $\dfrac{3\pi}{4}, \dfrac{7\pi}{4}$ (d) $\dfrac{\pi}{3}, \dfrac{4\pi}{3}$
(e) $20^\circ,\ 80^\circ,\ 140^\circ$

2. (a) $0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, 2\pi$ (b) $\dfrac{3\pi}{2}$ (c) $0.425, 5.858, 1.995, 4.288$ (d) All θ in $[0, 2\pi]$

(e) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ (f) $0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi, \frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}, \frac{7\pi}{9}, \pi, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{5\pi}{3}, \frac{17\pi}{9}$

3. $x = \frac{\pi}{6}, \frac{5\pi}{6}$ 4. t=2 5. $\frac{\pi}{3}, \frac{5\pi}{3}$

EXERCISE 37:

3. (a) a (b) $\frac{1}{a}$ (c) $\frac{1}{2}$ (d) $-\frac{2}{3}$ (e) 2 (f) 0

EXERCISE 38:

1. $f'(0) = -3,\ f'(1) = 4,\ f'(2) = 17,\ f'(-1) = -4$ 2. $y = 10x - 15$ 3. (1,1)

4. $a = -27$ 6. $c = 1$ 7. (a) $2x$ (b) $15x^2 + \frac{3}{5x^4}$ (c) $\frac{7}{2\sqrt{x}} + \sqrt{7}$

EXERCISE 39:

1. 1 2. $6x^2 + 6x - 2$ 3. $32x^7 + 4x^3$ 4. $\frac{-1}{3x^2} - \frac{3}{x^4}$ 5. $\frac{-2x}{(1+x^2)^2}$

6. $\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

7. $\frac{2-x}{2\sqrt{(1-x)^3}}$ 8. $\frac{-1+2x-5x^2}{\sqrt{1-2x}}$ 9. $\frac{1-2x^2}{(1+2x^2)^2}$ 10. $\frac{x}{\sqrt{x^2+2}}$ 11. $\frac{2+\sqrt{x}}{2(1+\sqrt{x})^2}$

EXERCISE 40:

1. $\frac{-2e^t}{(e^t-1)^2}$ 2. $-2xe^{-x^2}$ 3. $\frac{2x-1}{1-x+x^2}$ 4. $\frac{2}{x}$ 5. $\frac{2x+7}{x^2+7x}$ 6. $\ln x + 1$

7. $(2x+3)e^{x^2+3x}$ 8. $e^{2x} + 2xe^{2x}$

EXERCISE 41:

1. $6\cos 6x$ 2. $x^2\cos x + 2x\sin x$ 3. $-3x^2\sin x^3$ 4. $2x\sec x + x^2\sec x\tan x$

5. $5x^4\sec^2 x^5$ 6. $e^x\cos x - e^x\sin x$ 7. $e^{2x}\cos x + 2e^{2x}\sin x$ 8. $2\cot 2x$

9. $\cos x\cos 2x - 2\sin x\sin 2x$ 10. $\cos x$ if $\sin x > 0$ 11. $-\tan x$ 12. $-\sin x e^{\cos x}$

13. $-3\cos^2 x\sin x$ 14. $\sin(x^2+2x) + 2x(x+1)\cos(x^2+2x)$ 15. $-(1+\cos x)\sin(x+\sin x)$

16. $(\ln(\sin x) + x\cot x)(\sin x)^x$ 17. $\frac{dy}{dx} = \frac{2\cos(2x+3y)}{1-3\cos(2x+3y)}$

EXERCISE 42:

1. (a) $f'(x) = \frac{1}{(1+x)^2},\quad f''(x) = \frac{-2}{(1+x)^3}$ (b) $f'(x) = 2xe^{x^2},\quad f''(x) = 2e^{x^2} + 4x^2e^{x^2}$

(c) $f'(x) = \frac{x}{1+x^2},\quad f''(x) = \frac{1-x^2}{(1+x^2)^2}$ 2. 0 6. (a) 1-e (b) 1 (c) $x = 1$

(d) not possible

EXERCISE 43:

1. (a) $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$ (b) $\frac{dy}{dx} = \frac{-5x^4 - 4y^3}{12xy^2 - 15y^4}$ (c) $\frac{dy}{dx} = \frac{-y}{x+3y^2}$, $\frac{d^2y}{dx^2} = \frac{2xy}{(x+3y^2)^3}$

(d) $\frac{dy}{dx} = \frac{e^y}{1 - xe^y}$ 3. $m = 11,\ y = 11x - 31$ 4. (a) $e^x y$ (b) $3y - \frac{2lny}{x}$

5. $-0.1729\frac{r^5}{h^3}$ 6. Decreased by 3.14 hours

EXERCISE 44:

1. $\frac{dP}{dt} = \frac{-k}{\pi a^2 x^2}v$, $\frac{dP}{dt} = \frac{-P_0}{4a}$ 2. $\frac{dR}{dt} = 0.09\ ohms/sec$ 3. $\frac{dA}{dt} = 0.1508\ mm^2/month$

4. 75 km/h 5. (a) $30cm^3/sec$ (b) $7.5cm^3/sec$ 7. 2 8. $\frac{16}{9\pi}$m/min 9. 2.5m/s

EXERCISE 45:

1. (a) maximum at $(1,5)$; minimum at $(3,1)$; $y-$intercept: 1

(b) Vertical Asymptote at $x = -3$; Horizontal asymptote at $y = 0$. Maximum at $\left(3, \frac{1}{12}\right)$.

3. (a) $-\sqrt{2} < x < \sqrt{2}$ (b) $x = \pm\sqrt{2}$ (c) $x < -\sqrt{2}$ or $x > \sqrt{2}$

4. (a) decreasing for $x < 0$, increasing for $x > 0$, stationary for $x = 0$.

(b) increasing for $x < 0$, decreasing for $x > 0$, stationary for $x = 0$.

(c) decreasing for $x < -2$, $x > 1$, increasing for $-2 < x < 1$, stationary for $x = -2$, $x = 1$.

5. HI at (-1,0), MIN at (2,0), MAX at $(\frac{2}{7}, \frac{2^8 3^8}{7^6})$

(a) increasing for $x < \frac{2}{7}$ but $x \neq -1$ and $x > 2$. (b) decreasing for $\frac{2}{7} < x < 2$

10. (a) $x < -1$, $x > 2$ and $0 < x < 1$

(b) $-1 < x < 0$ and $1 < x < 2$ (c) $x > 0.5$ (d) $x < 0.5$

EXERCISE 46:

1. $24cm \times 36cm$ 2. $r = \sqrt[3]{\frac{5}{\pi}}$ 3. 3cm × 3cm 5. 4πm^3

6. $(-1, 2.5)$ and $(1, 2.5)$ 9. 4 weeks

EXERCISE 47:

1. $-\frac{1}{3x^3} + \frac{5}{x} + c$ 2. $3x^{\frac{1}{3}} + \frac{3}{x^{\frac{1}{3}}} + c$ 3. $\ln|x| - \frac{8x^{\frac{3}{2}}}{3} + c$ 4. $-\frac{2}{3}e^{-3x} + c$

5. $4\ln|x| + 4\sqrt{x} + c$ 6. $\ln|x+1| + c$ 7. $-\ln|4 - x| + c$ 8. $\frac{e^{6x}}{6} + c$

9. $\frac{e^{12x}}{3} - \frac{e^{-12x}}{3} + c$ 10. $-e^{-x^2} + c$ 11. $-\frac{\cos 3x}{3} + c$ 12. $\frac{\tan 3x}{3} + c$

13. $-\frac{\cos 2x}{2} - \frac{\sin 2x}{2} + c$ 14. $\sec x + c$ 15. $-2\text{cosec}x + c$ 16. $-\frac{\cos x}{2} + c$

17. $\frac{\sin 2x}{2} + x + c$

EXERCISE 48:

1. $\frac{22}{3}$ 2. $3\ln 9 + 4$ 3. $\frac{\ln 5}{4}$ 4. $\frac{e^3 + e^{-3} - 2}{3}$ 5. $\frac{1}{2}$ 6. 1 7. $1 - \frac{\pi}{4}$ 8. $\frac{\pi}{4}$