THE UNIVERSITY OF SYDNEY

Metric Spaces

Lecture Notes for MATH3961

Daniel Daners

School of Mathematics and Statistics
The University of Sydney

Semester 1, 2011

THE UNIVERSITY OF SYDNEY

Lecture Notes for MATH3961

Metric Spaces (Advanced)

Daniel Daners

School of Mathematics and Statistics

The University of Sydney
NSW 2006, Australia

Acknowledgement: These notes are a summary of the theory of metric spaces from the functional analysis lecture notes of the author and originally designed as revision material for students taking fourth year functional analysis.

Preliminary Edition Semester 1, 2011

Contents

Chapter I

Metric Spaces

The theory of metric spaces is a natural generalisation of the notion of Euclidean distance. We only keep the basic properties of symmetry of distance and the triangle inequality. Then we can still define open and closed sets the same way as done in $\mathbb{R}^N$. Similarly we can look at properties of sequences, continuity, uniform convergence of functions and more. Euclidean distance is closely tied to the geometry of $\mathbb{R}^N$. This can be lifted to a more abstract level as well and leads to the theory of inner product spaces and Hilbert spaces, which is the second part of these notes.

1 The Axiom of Choice and Zorn's Lemma

Suppose that A is a set, and that for each $\alpha \in A$ there is a set X_α. We call $(X_\alpha)_{\alpha\in A}$ a family of sets indexed by A. The set A may be finite, countable or uncountable. We then consider the *Cartesian product* of the sets X_α:

$$\prod_{\alpha\in A} X_\alpha$$

consisting of all "collections" $(x_\alpha)_{\alpha\in A}$, where $x_\alpha \in X_\alpha$. More formally, $\prod_{\alpha\in A} X_\alpha$ is the set of functions

$$x\colon A \to \bigcup_{\alpha\in A} X_\alpha$$

such that $x(\alpha) \in X_\alpha$ for all $\alpha \in A$. We write x_α for $x(\alpha)$ and $(x_\alpha)_{\alpha\in A}$ or simply (x_α) for a given such function x. Suppose now that $A \neq \emptyset$ and $X_\alpha \neq \emptyset$ for all $\alpha \in A$. Then there is a fundamental question:

Is $\prod_{\alpha\in A} X_\alpha$ nonempty in general?

Here some brief history about the problem, showing how basic and difficult it is:

- Zermelo (1904) (see [7]) observed that it is not obvious from the existing axioms of set theory that there is a procedure to select a single x_α from each X_α in general. As a consequence he introduced what we call the *axiom of choice*, asserting that $\prod_{\alpha \in A} X_\alpha \neq \emptyset$ whenever $A \neq \emptyset$ and $X_\alpha \neq \emptyset$ for all $\alpha \in A$.

  It remained open whether his axiom of choice could be derived from the other axioms of set theory. There was an even more fundamental question on whether the axiom is consistent with the other axioms!

- Gödel (1938) (see [5]) proved that the axiom of choice is consistent with the other axioms of set theory. The open question remaining was whether it is independent of the other axioms.

- P.J. Cohen (1963/1964) (see [2, 3]) finally showed that the axiom of choice is in fact independent of the other axioms of set theory, that is, it cannot be derived from them.

The majority of mathematicians accept the axiom of choice, but there is a minority which does not. Many very basic and important theorems in functional analysis cannot be proved without the axiom of choice.

We accept the axiom of choice.

There are some non-trivial equivalent formulations of the axiom of choice which are useful for our purposes. Given two sets X and Y recall that a *relation* from X to Y is simply a subset of the Cartesian product $X \times Y$. We now explore some special relations, namely *order relations*.

1.1 Definition (partial ordering) A relation $\prec$ on a set X is called a *partial ordering* of X if

- $x \prec x$ for all $x \in X$ (reflexivity);
- $x \prec y$ and $y \prec z$ imply $x \prec z$ (transitivity);
- $x \prec y$ and $y \prec x$ imply $x = y$ (anti-symmetry).

We also write $x \succ y$ for $y \prec x$. We call $(X, \prec)$ a *partially ordered set*.

1.2 Examples (a) The usual ordering $\leq$ on $\mathbb{R}$ is a partial ordering on $\mathbb{R}$.

(b) Suppose $\mathcal{S}$ is a collection of subsets of a set X. Then inclusion is a partial ordering. More precisely, if $S, T \in \mathcal{S}$ then $S \prec T$ if and only if $S \subseteq T$. We say $\mathcal{S}$ is partially ordered by inclusion.

(c) Every subset of a partially ordered set is a partially ordered set by the induced partial order.

There are more expressions appearing in connection with partially ordered sets.

1.3 Definition Suppose that $(X, \prec)$ is a partially ordered set. Then

(a) $m \in X$ is called a *maximal element in* X if for all $x \in X$ with $x \succ m$ we have $x \prec m$;

(b) $m \in X$ is called an *upper bound* for $S \subseteq X$ if $x \prec m$ for all $x \in S$;

(c) A subset $C \subseteq X$ is called a *chain in* X if $x \prec y$ or $y \prec x$ for all $x, y \in C$;

(d) If a partially ordered set $(X, \prec)$ is a chain we call it a *totally ordered set*.

(e) If $(X, \prec)$ is partially ordered and $x_0 \in X$ is such that $x_0 \prec x$ for all $x \in X$, then we call x_0 a *first element*.

There is a special class of partially ordered sets playing a particularly important role in relation to the axiom of choice as we will see later.

1.4 Definition (well ordered set) A partially ordered set $(X, \prec)$ is called a *well ordered set* if every subset has a first element.

1.5 Examples (a) $\mathbb{N}$ is a well ordered set, but $\mathbb{Z}$ or $\mathbb{R}$ are not well ordered with the usual order.
(b) $\mathbb{Z}$ and $\mathbb{R}$ are totally ordered with the usual order.

1.6 Remark Well ordered sets are always totally ordered. To see this assume $(X, \prec)$ is well ordered. Given $x, y \in X$ we consider the subset $\{x, y\}$ of X. By definition of a well ordered set we have either $x \prec y$ or $y \prec x$, which shows that $(X, \prec)$ is totally ordered. The converse is not true as the example of $\mathbb{Z}$ given above shows.

There is another, highly non-obvious but very useful statement appearing in connection with partially ordered sets:

1.7 Zorn's Lemma *Suppose that $(X, \prec)$ is a partially ordered set such that each chain in X has an upper bound. Then X has a maximal element.*

There is a non-trivial connection between all the apparently different topics we discussed so far. We state it without proof (see for instance [4]).

1.8 Theorem *The following assertions are equivalent*

(i) *The axiom of choice;*

(ii) *Zorn's Lemma;*

(iii) *Every set can be well ordered.*

The axiom of choice may seem "obvious" at the first instance. However, the other two equivalent statements are certainly not. For instance take $X = \mathbb{R}$, which we know is not well ordered with the usual order. If we accept the axiom of choice then it follows from the above theorem that there exists a partial ordering making $\mathbb{R}$ into a well ordered set. This is a typical "existence proof" based on the axiom of choice. It does not give us any hint on *how to find* a partial ordering making $\mathbb{R}$ into a well ordered set. This reflects Zermelo's observation that it is not obvious how to choose precisely one element from each set when given an arbitrary collection of sets. Because of the *non-constructive* nature of the axiom of choice and its equivalent counterparts, there are some mathematicians rejecting the axiom. These mathematicians have the point of view that everything should be "constructible," at least in principle, by some means (see for instance [1]).

2 Elementary Properties of Metric Spaces

Metric spaces are sets in which we can measure distances between points. We expect such a "distance function," called a *metric*, to have some obvious properties, which we postulate in the following definition.

2.1 Definition (Metric Space) Suppose X is a set. A map $d\colon X \times X \to \mathbb{R}$ is called a *metric* on X if the following properties hold:

(i) $d(x, y) \geq 0$ for all $x, y \in x$;

(ii) $d(x, y) = 0$ if and only if $x = y$;

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality).

We call (X, d) a *metric space*. If it is clear what metric is being used we simply say X is a metric space.

2.2 Example The simplest example of a metric space is $\mathbb{R}$ with $d(x, y) := |x - y|$. The standard metric used in $\mathbb{R}^N$ is the *Euclidean metric* given by

$$d(x, y) = |x - y|_2 := \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2}$$

for all $x, y \in \mathbb{R}^N$.

2.3 Remark If (X, d) is a metric space, then every subset $Y \subseteq X$ is a metric space with the metric restricted to Y. We say the metric on Y is *induced* by the metric on X.

2.4 Definition (Open and Closed Ball) Let (X, d) be a metric space. For $r > 0$ we call

$$B(x, r) := \{y \in X : d(x, y) < r\}$$

the *open ball* about x with radius r. Likewise we call

$$\bar{B}(x, r) := \{y \in X : d(x, y) \leq r\}$$

the *closed ball* about x with radius r.

Using open balls we now define a "topology" on a metric space.

2.5 Definition (Open and Closed Set) Let (X, d) be a metric space. A subset $U \subseteq X$ is called *open* if for every $x \in X$ there exists $r > 0$ such that $B(x, r) \subseteq U$. A set U is called *closed* if its complement $X \setminus U$ is open.

2.6 Remark For every $x \in X$ and $r > 0$ the open ball $B(x, r)$ in a metric space is open. To prove this fix $y \in B(x, r)$. We have to show that there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq B(x, r)$. To do so note that by definition $d(x, y) < r$. Hence we can choose $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < r - d(x, y)$. Thus, by property (iv) of a metric, for $z \in B(y, \varepsilon)$ we have $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r$. Therefore $z \in B(x, r)$, showing that $B(y, \varepsilon) \subseteq B(x, r)$.

Next we collect some fundamental properties of open sets.

2.7 Theorem *Open sets in a metric space (X, d) have the following properties.*

(i) *X, $\emptyset$ are open sets;*

(ii) *arbitrary unions of open sets are open;*

(iii) *finite intersections of open sets are open.*

Proof. Property (i) is obvious. To prove (ii) let U_α, $\alpha \in A$ be an arbitrary family of open sets in X. If $x \in \bigcup_{\alpha \in A} U_\alpha$ then $x \in U_\beta$ for some $\beta \in A$. As U_β is open there exists $r > 0$ such that $B(x, r) \subseteq U_\beta$. Hence also $B(x, r) \subseteq \bigcup_{\alpha \in A} U_\alpha$, showing that $\bigcup_{\alpha \in A} U_\alpha$ is open. To prove (iii) let U_i, $i = 1, \dots, n$ be open sets. If $x \in \bigcap_{i=1}^n U_i$ then $x \in U_i$ for all $i = 1, \dots, n$. As the sets U_i are open there exist $r_i > 0$ such that $B(x, r_i) \subseteq U_i$ for all $i = 1, \dots, n$. If we set $r := \min_{i=1,\dots,n} r_i$ then obviously $r > 0$ and $B(x, r) \subseteq \bigcap_{i=1}^n U_i$, proving (iii). ■

2.8 Remark There is a more general concept than that of a metric space, namely that of a "topological space." A collection $\mathcal{T}$ of subsets of a set X is called a *topology* if the following conditions are satisfied

(i) X, $\emptyset \in \mathcal{T}$;

(ii) arbitrary unions of sets in $\mathcal{T}$ are in $\mathcal{T}$;

(iii) finite intersections of sets in $\mathcal{T}$ are in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets*, and $(X, \mathcal{T})$ a *topological space*. Hence the open sets in a metric space form a topology on X.

2.9 Definition (Neighbourhood) Suppose that (X, d) is a metric space (or more generally a topological space). We call a set U a *neighbourhood* of $x \in X$ if there exists an open set $V \subseteq U$ with $x \in V$.

Now we define some sets associated with a given subset of a metric space.

2.10 Definition (Interior, Closure, Boundary) Suppose that U is a subset of a metric space (X, d) (or more generally a topological space). A point $x \in U$ is called an *interior point* of U if U is a neighbourhood of x. We call

(i) $\mathring{U} := \operatorname{Int}(U) := \{x \in U \colon x \text{ interior point of } U\}$ the *interior* of U;

(ii) $\overline{U} := \{x \in X \colon U \cap V \neq \emptyset \text{ for every neighbourhood } V \text{ of } x\}$ the *closure* of U;

(iii) $\partial U := \bar{U} \setminus \operatorname{Int}(U)$ the *boundary* of U.

2.11 Remark A set is open if and only if $\mathring{U} = U$ and closed if and only if $\overline{U} = U$. Moreover, $\partial U = \overline{U} \cap \overline{X \setminus U}$.

Sometimes it is convenient to look at products of a (finite) number of metric spaces. It is possible to define a metric on such a product as well.

2.12 Proposition *Suppose that (X_i, d_i), $i = 1, \dots, n$ are metric spaces. Then $X = X_1 \times X_2 \times \cdots \times X_n$ becomes a metric space with the metric d defined by*

$$d(x, y) := \sum_{i=1}^{n} d_i(x_i, y_i)$$

for all $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in X.

Proof. Obviously, $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. Moreover, as $d_i(x_i, y_i) \geq 0$ we have $d(x, y) = 0$ if and only if $d_i(x_i, y_i) = 0$ for all $i = 1, \dots, n$. As d_i are metrics we get $x_i = y_i$ for all $i = 1, \dots, n$. For the triangle inequality note that

$$\begin{aligned} d(x, y) = \sum_{i=1}^{n} d(x_i, y_i) &\leq \sum_{i=1}^{n} \bigl(d(x_i, z_i) + d(z_i, y_i)\bigr) \\ &= \sum_{i=1}^{n} d(x_i, z_i) + \sum_{i=1}^{n} d(z_i, y_i) = d(x, y) + d(z, y) \end{aligned}$$

for all $x, y, z \in X$. ■

2.13 Definition (Product space) The space and metric introduced in Proposition 2.12 is called a *product space* and a *product metric*, respectively.

3 Limits

Once we have a notion of "closeness" we can discuss the asymptotics of sequences and continuity of functions.

3.1 Definition (Limit) Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X, d), or more generally a topological space. We say x_0 is a *limit* of (x_n) if for every neighbourhood U of x_0 there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. We write

$$x_0 = \lim_{n\to\infty} x_n \qquad \text{or} \qquad x_n \to x \text{ as } n \to \infty.$$

If the sequence has a limit we say it is *convergent*, otherwise we say it is *divergent*.

3.2 Remark Let (x_n) be a sequence in a metric space (X, d) and $x_0 \in X$. Then the following statements are equivalent:

(1) $\lim_{n\to\infty} x_n = x_0$;

(2) for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_0) < \varepsilon$ for all $n \geq n_0$.

Proof. Clearly (1) implies (2) by choosing neighbourhoods of the form $B(x, \varepsilon)$. If (2) holds and U is an arbitrary neighbourhood of x_0 we can choose $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq U$. By assumption there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_0) < \varepsilon$ for all $n \geq n_0$, that is, $x_n \in B(x_0, \varepsilon) \subseteq U$ for all $n \geq n_0$. Therefore, $x_n \to x_0$ as $n \to \infty$. ■

3.3 Proposition *A sequence in a metric space (X, d) has at most one limit.*

Proof. Suppose that (x_n) is a sequence in (X, d) and that x and y are limits of that sequence. Fix $\varepsilon > 0$ arbitrary. Since x is a limit there exists $n_1 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon/2$ for all $n > n_1$. Similarly, since y is a limit there exists $n_2 \in \mathbb{N}$ such that $d(x_n, y) < \varepsilon/2$ for all $n > n_2$. Hence $d(x, y) \leq d(x, x_n) + d(x_n, y) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $n > \max\{n_1, n_2\}$. Since $\varepsilon > 0$ was arbitrary it follows that $d(x, y) = 0$, and so by definition of a metric $x = y$. Thus (x_n) has at most one limit. ■

We can characterise the closure of sets by using sequences.

3.4 Theorem *Let U be a subset of the metric space (X, d) then $x \in \overline{U}$ if and only if there exists a sequence (x_n) in U such that $x_n \to x$ as $n \to \infty$.*

Proof. Let $U \subseteq X$ and $x \in \overline{U}$. Hence $B(x,\varepsilon) \cap U) \neq \emptyset$ for all $\varepsilon > 0$. For all $n \in \mathbb{N}$ we can therefore choose $x_n \in U$ with $d(x, x_n) < 1/n$. By construction $x_n \to x$ as $n \to \infty$. If (x_n) is a sequence in U converging to x then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(x,\varepsilon)$ for all $n \geq n_0$. In particular, $B(x,\varepsilon) \cap U \neq \emptyset$ for all $\varepsilon > 0$, implying that $x \in \overline{U}$ as required. ■

There is another concept closely related to convergence of sequences.

3.5 Definition (Cauchy Sequence) Suppose (x_n) is a sequence in the metric space (X, d). We call (x_n) a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$.

Some sequences may not converge, but they accumulate at certain points.

3.6 Definition (Point of Accumulation) Suppose that (x_n) is a sequence in a metric space (X, d) or more generally in a topological space. We say that x_0 is a *point of accumulation* of (x_n) if for every neighbourhood U of x_0 and every $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ such that $x_n \in U$.

3.7 Remark Equivalently we may say x_0 is an accumulation point of (x_n) if for every $\varepsilon > 0$ and every $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ such that $d(x_n, x_0) < \varepsilon$. Note that it follows from the definition that every neighbourhood of x_0 contains infinitely many elements of the sequence (x_n).

3.8 Proposition *Suppose that (X, d) is a metric space and (x_n) a sequence in that space. Then $x \in X$ is a point of accumulation of (x_n) if and only if*

$$x \in \bigcap_{k=1}^{\infty} \overline{\{x_j : j \geq k\}}. \tag{3.1}$$

Proof. Suppose that $x \in \bigcap_{k=1}^{\infty} \overline{\{x_j : j \geq k\}}$. Then $x \in \overline{\{x_j : j \geq k\}}$ for all $k \in \mathbb{N}$. By Theorem 3.4 we can choose for every $k \in \mathbb{N}$ an element $x_{n_k} \in \{x_j : j \geq k\}$ such that $d(x_{n_k}, x) < 1/k$. By construction $x_{n_k} \to x$ as $k \to \infty$, showing that x is a point of accumulation of (x_n). If x is a point of accumulation of (x_n) then for all $k \in \mathbb{N}$ there exists $n_k \geq k$ such that $d(x_{n_k}, x) < 1/k$. Clearly $x_{n_k} \to x$ as $k \to \infty$, so that $x \in \overline{\{x_{n_j} : j \geq k\}}$ for all $k \in \mathbb{N}$. As $\{x_{n_j} : j \geq k\} \subseteq \{x_j : j \geq k\}$ for all $k \in \mathbb{N}$ we obtain (3.1). ■

In the following theorem we establish a connection between Cauchy sequences and converging sequences.

3.9 Theorem *Let (X, d) be a metric space. Then every convergent sequence is a Cauchy sequence. Moreover, if a Cauchy sequence (x_n) has an accumulation point x_0, then (x_n) is a convergent sequence with limit x_0.*

Proof. Suppose that (x_n) is a convergent sequence with limit x_0. Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_0) < \varepsilon/2$ for all $n \geq n_0$. Now

$$d(x_n, x_m) \leq d(x_n, x_0) + d(x_0, x_m) = d(x_n, x_0) + d(x_m, x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n, m \geq n_0$, showing that (x_n) is a Cauchy sequence. Now assume that (x_n) is a Cauchy sequence, and that $x_0 \in X$ is an accumulation point of (x_n). Fix $\varepsilon > 0$ arbitrary. Then by definition of a Cauchy sequence there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon/2$ for all $n, m \geq n_0$. Moreover, since x_0 is an accumulation point there exists $m_0 \geq n_0$ such that $d(x_{m_0}, x_0) < \varepsilon/2$. Hence

$$d(x_n, x_0) \leq d(x_n, x_{m_0}) + d(x_{m_0}, x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq n_0$. Hence by Remark 3.2 x_0 is the limit of (x_n). ■

In a general metric space not all Cauchy sequences have necessarily a limit, hence the following definition.

3.10 Definition (Complete Metric Space) A metric space is called *complete* if every Cauchy sequence in that space has a limit.

One property of the real numbers is that the intersection of a nested sequence of closed bounded intervals whose lengths shrinks to zero have a non-empty intersection. This property is in fact equivalent to the "completeness" of the real number system. We now prove a counterpart of that fact for metric spaces. There are no intervals in general metric spaces, so we look at a sequence of nested closed sets whose diameter goes to zero. The diameter of a set K in a metric space (X, d) is defined by

$$\operatorname{diam}(K) := \sup_{x,y \in K} d(x, y).$$

3.11 Theorem (Cantor's Intersection Theorem) *Let (X, d) be a metric space. Then the following two assertions are equivalent:*

(i) *(X, d) is complete;*

(ii) *For every sequence of closed sets $K_n \subseteq X$ with $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$*

$$\operatorname{diam}(K_n) := \sup_{x,y \in K_n} d(x, y) \to 0$$

as $n \to \infty$ we have $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$.

Proof. First assume that X is complete and let K_n be as in (ii). For every $n \in \mathbb{N}$ we choose $x_n \in K_n$ and show that (x_n) is a Cauchy sequence. By assumption $K_{n+1} \subseteq K_n$ for all $n \in \mathbb{N}$, implying that $x_m \in K_m \subseteq K_n$ for all $m > n$. Since $x_m, x_n \in K_n$ we have

$$d(x_m, x_n) \leq \sup_{x,y \in K_n} d(x, y) = \operatorname{diam}(K_n)$$

for all $m > n$. Since $\operatorname{diam}(K_n) \to 0$ as $n \to \infty$, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\operatorname{diam}(K_{n_0}) < \varepsilon$. Hence, since $K_m \subseteq K_n \subseteq K_{n_0}$ we have

$$d(x_m, x_n) \leq \operatorname{diam}(K_n) \leq \operatorname{diam}(K_{n_0}) < \varepsilon$$

for all $m > n > n_0$, showing that (x_n) is a Cauchy sequence. By completenes of S, the sequence (x_n) converges to some $x \in X$. We know from above that $x_m \in K_n$ for all $m > n$. As K_n is closed $x \in K_n$. Since this is true for all $n \in \mathbb{N}$ we conclude that $x \in \bigcap_{n \in \mathbb{N}} K_n$, so the intersection is non-empty as claimed.

Assume now that (ii) is true and let (x_n) be a Cauchy sequence in (X, d). Hence there exists $n_0 \in X$ such that $d(x_{n_0}, x_n) < 1/2$ for all $n \geq n_0$. Similarly, there exists $n_1 > n_0$ such that $d(x_{n_1}, x_n) < 1/2^2$ for all $n \geq n_1$. Continuing that way we construct a sequence (n_k) in $\mathbb{N}$ such that for every $k \in \mathbb{N}$ we have $n_{k+1} > n_k$ and $d(x_{n_k}, x_n) < 1/2^{k+1}$ for all $n > n_k$. We now set $K_k := \overline{B(x_k, 2^{-k}))}$. If $x \in K_{k+1}$, then since $n_{k+1} > n_k$

$$d(x_{n_k}, x) \leq d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x) < \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k}.$$

Hence $x \in K_k$, showing that $K_{k+1} \subseteq K_k$ for all $k \in \mathbb{N}$. By assumption (ii) we have $\bigcap_{k \in \mathbb{N}} K_k \neq \emptyset$, so choose $x \in \bigcap_{k \in \mathbb{N}} K_k \neq \emptyset$. Then $x \in K_k$ for all $k \in \mathbb{N}$, so $d(x_{n_k}, x) \leq 1/2^k$ for all $k \in \mathbb{N}$. Hence $x_{n_k} \to x$ as $k \to \infty$. By Theorem 3.9 the Cauchy sequence (x_n) converges, proing (i). ∎

We finally look at product spaces defined in Definition 2.13. The rather simple proof of the following proposition is left to the reader.

3.12 Proposition *Suppose that (X_i, d_i), $i = 1, \dots, n$ are complete metric spaces. Then the corresponding product space is complete with respect to the product metric.*

4 Compactness

We start by introducing some additional concepts, and show that they are all equivalent in a metric space. They are all generalisations of "finiteness" of a set.

4.1 Definition (Open Cover, Compactness) Let (X, d) be a metric space. We call a collection of open sets $(U_\alpha)_{\alpha\in A}$ an *open cover* of X if $X \subseteq \bigcup_{\alpha\in A} U_\alpha$. The space X is called *compact* if for every open cover $(U_\alpha)_{\alpha\in A}$ there exist finitely many $\alpha_i \in A$, $i = 1, \dots, m$ such that $(U_{\alpha_i})_{i=1,\dots,m}$ is an open cover of X. We talk about a finite sub-cover of X.

4.2 Definition (Sequential Compactness) We call a metric space (X, d) *sequentially compact* if every sequence in X has an point of accumulation.

4.3 Definition (Total Boundedness) We call a metric space X *totally bounded* if for every $\varepsilon > 0$ there exist finitely many points $x_i \in X$, $i = 1, \dots, m$, such that $(B(x_i, \varepsilon))_{i=1,\dots,m}$ is an open cover of X.

It turns out that all the above definitions are equivalent, at least in metric spaces (but not in general topological spaces).

4.4 Theorem *For a metric space (X, d) the following statements are equivalent:*

(i) *X is compact;*

(ii) *X is sequentially compact;*

(iii) *X is complete and totally bounded.*

Proof. To prove that (i) implies (ii) assume that X is compact and that (x_n) is a sequence in X. We set $C_n := \overline{\{x_j : j \geq n\}}$ and $U_n := X \setminus C_n$. Then U_n is open for all $n \in \mathbb{N}$ as C_n is closed. By Proposition 3.8 the sequence (x_n) has a point of accumulation if

$$\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset,$$

which is equivalent to

$$\bigcup_{n\in\mathbb{N}} U_n = \bigcup_{n\in\mathbb{N}} X \setminus C_n = X \setminus \bigcap_{n\in\mathbb{N}} C_n \neq X$$

Clearly $C_0 \supset C_1 \supset \cdots \supset C_n \neq \emptyset$ for all $n \in \mathbb{N}$. Hence every finite intersection of sets C_n is nonempty. Equivalently, every finite union of sets U_n is strictly smaller than X, so that X cannot be covered by finitely many of the sets U_n. As X is compact it is impossible that $\bigcup_{n\in\mathbb{N}} U_n = X$ as otherwise a finite number would cover X already, contradicting what we just proved. Hence (x_n) must have a point of accumulation.

Now assume that (ii) holds. If (x_n) is a Cauchy sequence it follows from (ii) that it has a point of accumulation. By Theorem 3.9 we conclude that it has a limit, showing that X is complete. Suppose now that X is not totally bounded. Then, there exists $\varepsilon > 0$ such that X cannot be covered by finitely many balls of radius ε. If we let x_0 be arbitrary we can therefore choose $x_1 \in X$ such that $d(x_0, x_1) > \varepsilon$.

By induction we may construct a sequence (x_n) such that $d(x_j, x_n) \geq \varepsilon$ for all $j = 1, \dots, n-1$. Indeed, suppose we have $x_0, \dots, x_n \in X$ with $d(x_j, x_n) \geq \varepsilon$ for all $j = 1, \dots, n-1$. Assuming that X is not totally bounded $\bigcup_{j=1}^n B(x_j, \varepsilon) \neq X$, so we can choose x_{n+1} not in that union. Hence $d(x_j, x_{n+1}) \geq \varepsilon$ for $j = 1, \dots, n$. By construction it follows that $d(x_n, x_m) \geq \varepsilon/2$ for all $n, m \in \mathbb{N}$, showing that (x_n) does not contain a Cauchy subsequence, and thus has no point of accumulation. As this contradicts (ii), the space X must be totally bounded.

Suppose now that (iii) holds, but X is not compact. Then there exists an open cover $(U_\alpha)_{\alpha \in A}$ not having a finite sub-cover. As X is totally bounded, for every $n \in \mathbb{N}$ there exist finite sets $F_n \subseteq X$ such that

$$X = \bigcup_{x \in F_n} B(x, 2^{-n}). \tag{4.1}$$

Assuming that $(U_\alpha)_{\alpha \in A}$ does not have a finite sub-cover, there exists $x_1 \in F_1$ such that $B(x_1, 2^{-1})$ and thus $K_1 := \overline{B(x_1, 3 \cdot 2^{-1})}$ cannot be covered by finitely many U_α. By (4.1) it follows that there exists $x_2 \in F_2$ such that $B(x_1, 2^{-1}) \cap B(x_2, 2^{-2})$ and therefore $K_2 := \overline{B(x_2, 3 \cdot 2^{-2})}$ is not finitely covered by $(U_\alpha)_{\alpha \in A}$. We can continue this way and choose $x_{n+1} \in F_{n+1}$ such that $B(x_n, 2^{-n}) \cap B(x_{n+1}, 2^{-(n+1)})$ and therefore $K_{n+1} := \overline{B(x_2, 3 \cdot 2^{-(n+1)})}$ is not finitely covered by $(U_\alpha)_{\alpha \in A}$. Note that $B(x_n, 2^{-n}) \cap B(x_{n+1}, 2^{-(n+1)}) \neq \emptyset$ since otherwise the intersection is finitely covered by $(U_\alpha)_{\alpha \in A}$. Hence if $x \in K_{n+1}$, then

$$d(x_n, x) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{3}{2^{n+1}} = \frac{6}{2^{n+1}} = \frac{3}{2^n},$$

implying that $x \in K_n$. Also $\operatorname{diam} K_n \leq 3 \cdot 2^{n-1} \to 0$. Since X is complete, by Cantor's intersection Theorem 3.11 there exists $x \in \bigcap_{n \in \mathbb{N}} K_n$. As (U_α) is a cover of X we have $x \in U_{\alpha_0}$ for some $\alpha_0 \in A$. Since U_{α_0} is open there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U_{\alpha_0}$. Choose now n such that $6/2^n < \varepsilon$ and fix $y \in K_n$. Since $x \in K_n$ we have $d(x, y) \leq d(x, x_n) + d(x_n, y) \leq 6/2^n < \varepsilon$. Hence $K_n \subseteq B(x, \varepsilon) \subseteq U_{\alpha_0}$, showing that K_n is covered by U_{α_0}. However, by construction K_n cannot be covered by finitely many U_α, so we have a contradiction. Hence X is compact, completing the proof of the theorem. ∎

The last part of the proof is modelled on the usual proof of the Heine-Borel theorem asserting that bounded and closed sets are the compact sets in $\mathbb{R}^N$. Hence it is not a surprise that the Heine-Borel theorem easily follows from the above characterisations of compactness.

4.5 Theorem (Heine-Borel) *A subset of $\mathbb{R}^N$ is compact if and only if it is closed and bounded.*

Proof. Suppose $A \subseteq \mathbb{R}^N$ is compact. By Theorem 4.4 the set A is totally bounded, and thus may be covered by finitely many balls of radius one. A finite union of such balls is clearly bounded, so A is bounded. Again by Theorem 4.4, the set A is complete, so in particular it is closed. Now assume A is closed and bounded. As $\mathbb{R}^N$ is complete it follows that A is complete. Next we show that A is totally bounded. We let M be such that A is contained in the cube $[-M, M]^N$. Given $\varepsilon > 0$ the interval $[-M, M]$ can be covered by $m := [2M/\varepsilon] + 1$ closed intervals of length $\varepsilon/2$ (here $[2M/\varepsilon]$ is the integer part of $2M/\varepsilon$). Hence $[-M, M]^N$ can be covered by m^N cubes with edges $\varepsilon/2$ long. Such cubes are contained in open balls of radius ε, so we can cover $[-M, M]^N$ and thus A by a finite number of balls of radius ε. Hence A is complete and totally bounded. By Theorem 4.4 the set A is compact. ■

We can also look at subsets of metric spaces. As they are metric spaces with the metric induced on them we can talk about compact subsets of a metric space. It follows from the above theorem that compact subsets of a metric space are always closed (as they are complete). Often in applications one has sets that are not compact, but their closure is compact.

4.6 Definition (Relatively Compact Sets) We call a subset of a metric space *relatively compact* if its closure is compact.

4.7 Proposition *Closed subsets of compact metric spaces are compact.*

Proof. Suppose $C \subseteq X$ is closed and X is compact. If $(U_\alpha)_{\alpha\in A}$ is an open cover of C then we get an open cover of X if we add the open set $X \setminus C$ to the U_α. As X is compact there exists a finite sub-cover of X, and as $X \setminus C \cap C = \emptyset$ also a finite sub-cover of C. Hence C is compact. ■

Next we show that finite products of compact metric spaces are compact.

4.8 Proposition *Let (X_i, d_i), $i = 1, \dots, n$, be compact metric spaces. Then the product $X := X_1 \times \cdots \times X_n$ is compact with respect to the product metric introduced in Proposition 2.12.*

Proof. By Proposition 3.12 it follows that the product space X is complete. By Theorem 4.4 is is therefore sufficient to show that X is totally bounded. Fix $\varepsilon > 0$. Since X_i is totally bounded there exist $x_{ik} \in X_i$, $k = 1, \dots m_i$ such that X_i is covered by the balls B_{ik} of radius ε/n and centre x_{ik}. Then X is covered by the balls of radius ε with centres $(x_{1k_1}, \dots, x_{ik_i}, \dots x_{nk_n})$, where $k_i = 1, \dots m_i$. Indeed, suppose that $x = (x_1, x_2, \dots, x_n) \in X$ is arbitrary. By assumption, for every $i = 1, \dots n$ there exist $1 \leq k_i \leq m_i$ such that $d(x_i, x_{ik_i}) < \varepsilon/n$. By definition of the product metric the distance between $(x_{1k_1}, \dots, x_{nk_n})$ and x is no larger than $d(x_1, x_{1k_1}) + \cdots + d(x_n, x_{nk_n}) \leq n\varepsilon/n = \varepsilon$. Hence X is totally bounded and thus X is compact. ■

5 Continuous Functions

We give a brief overview on continuous functions between metric spaces. Throughout, let $X = (X, d)$ denote a metric space. We start with some basic definitions.

5.1 Definition (Continuous Function) A function $f : X \to Y$ between two metric spaces is called continuous at a point $x \in X$ if for every neighbourhood $V \subseteq Y$ of $f(x)$ there exists a neighbourhood $U \subseteq X$ of x such that $f(U) \subseteq V$. The map $f : X \to Y$ is called continuous if it is continuous at all $x \in X$. Finally we set

$$C(X, Y) := \{f : X \to Y \mid f \text{ is continuous}\}.$$

The above is equivalent to the usual ε-δ definition.

5.2 Theorem *Let X, Y be metric spaces and $f : X \to Y$ a function. Then the following assertions are equivalent:*

(i) *f is continuous at $x \in X$;*

(ii) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y\big(f(x), f(y)\big) \leq \varepsilon$ for all $y \in X$ with $d_X(x, y) < \delta$;*

(iii) *For every sequence (x_n) in X with $x_n \to x$ we have $f(x_n) \to f(x)$ as $n \to \infty$.*

Proof. Taking special neighbourhoods $V = B(f(x), \varepsilon)$ and $U := B(x, \delta)$ then (ii) is clearly necessary for f to be continuous. To show the (ii) is sufficient let V be an arbitrary neighbourhood of $f(x)$. Then there exists $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq V$. By assumption there exists $\delta > 0$ such that $d_Y\big(f(x), f(y)\big) \leq \varepsilon$ for all $y \in X$ with $d_X(x, y) < \delta$, that is, $f(U) \subseteq V$ if we let $U := B(x, \delta)$. As U is a neighbourhood of x it follows that f is continuous. Let now f be continuous and (x_n) a sequence in X converging to x. If $\varepsilon > 0$ is given then there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ for all $y \in X$ with $d_X(x, y) < \delta$. As $x_n \to x$ there exists $n_0 \in \mathbb{N}$ such that $d_X(x, x_n) < \delta$ for all $n \geq n_0$. Hence $d_Y(f(x), f(x_n)) < \varepsilon$ for all $n \geq n_0$. As $\varepsilon > 0$ was arbitrary $f(x_n) \to f(x)$ as $n \to \infty$. Assume now that (ii) does not hold. Then there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $x_n \in X$ with $d_X(x, x_n) < 1/n$ but $d_Y(f(x), f(x_n)) \geq \varepsilon$ for all $n \in \mathbb{N}$. Hence $x_n \to x$ in X but $f(x_n) \not\to f(x)$ in Y, so (iii) does not hold. By contrapositive (iii) implies (ii), completing the proof of the theorem. ■

Next we want to give various equivalent characterisations of continuous maps (without proof).

5.3 Theorem (Characterisation of Continuity) *Let X, Y be metric spaces. Then the following statements are equivalent:*

(i) *$f \in C(X, Y)$;*

(ii) *$f^{-1}[O] := \{x \in X : f(x) \in O\}$ is open for every open set $O \subseteq Y$;*

(iii) *$f^{-1}[C]$ is closed for every closed set $C \subseteq Y$;*

(iv) *For every $x \in X$ and every neighbourhood $V \subseteq Y$ of $f(x)$ there exists a neighbourhood $U \subseteq X$ of x such that $f(U) \subseteq V$;*

(v) *For every $x \in X$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y\big(f(x), f(y)\big) < \varepsilon$ for all $y \in X$ with $d_X(x, y) < \delta$.*

5.4 Definition (Distance to a Set) Let A be a nonempty subset of X. We define the distance between $x \in X$ and A by

$$\operatorname{dist}(x, A) := \inf_{a \in A} d(x, a)$$

5.5 Proposition *For every nonempty set $A \subseteq X$ the map $X \to \mathbb{R}$, $x \mapsto \operatorname{dist}(x, A)$, is continuous.*

Proof. By the properties of a metric $d(x, a) \leq d(x, y) + d(y, a)$. By first taking an infimum on the left hand side and then on the right hand side we get $\operatorname{dist}(x, A) \leq d(x, y) + \operatorname{dist}(y, A)$ and thus

$$\operatorname{dist}(x, A) - \operatorname{dist}(y, A) \leq d(x, y)$$

for all $x, y \in X$. Interchanging the roles of x and y we get $\operatorname{dist}(y, A) - \operatorname{dist}(x, A) \leq d(x, y)$, and thus

$$|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y),$$

implying the continuity of $\operatorname{dist}(\cdot\,, A)$. ■

We continue to discuss properties of continuous functions on compact sets.

5.6 Theorem *If $f \in C(X, Y)$ and X is compact then the image $f(X)$ is compact in Y.*

Proof. Suppose that (U_α) is an open cover of $f(X)$ then by continuity $f^{-1}[U_\alpha]$ are open sets, and so $(f^{-1}[U_\alpha])$ is an open cover of X. By the compactness of X it has a finite sub-cover. Clearly the image of that finite sub-cover is a finite sub-cover of $f(X)$ by (U_α). Hence $f(X)$ is compact. ■

Continuous functions on compact sets have other nice properties.

5.7 Definition (Uniform Continuity) We say a function $f : X \to Y$ is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y\big(f(x), f(y)\big) < \varepsilon$ for all $x, y \in X$ satisfying $d_X(x, y) < \delta$.

The difference to continuity is that δ does not depend on the point x, but can be chosen to be the same for *all* $x \in X$, that is *uniformly* with respect to $x \in X$.

5.8 Theorem *If X is compact, then every function $f \in C(X, Y)$ is uniformly continuous.*

Proof. Suppose that X is compact and f not uniformly continuous. Then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exist $x_n, y_n \in X$ with $d(x_n, y_n) < 1/n$ and

$$d(f(x_n), f(y_n)) \geq \varepsilon. \tag{5.1}$$

As X is compact and thus sequentially compact there exists a subsequence x_{n_k} converging to some $x \in X$ as $k \to \infty$ (see Theorem 4.4). Now

$$d(x, y_{n_k}) \leq d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) \leq d(x, x_{n_k}) + \frac{1}{n_k} \xrightarrow{k\to\infty} 0,$$

so that $y_{n_k} \to x$ as well. By the continuity of f and the triangle inequality

$$d(f(x_{n_k}), f(y_{n_k})) \leq d(f(x_{n_k}), f(x)) + d(f(y_{n_k}), f(x)) \xrightarrow{k\to\infty} 0,$$

contradicting our assumption (5.1). Hence f must be uniformly continuous. ∎

One could give an alternative proof of the above theorem using the covering property of compact sets. We complete this section by an important property of real valued continuous functions.

5.9 Theorem *Suppose that X is a compact metric space and $f \in C(X, \mathbb{R})$. Then f attains its maximum and minimum, that is, there exist $x_1, x_2 \in X$ such that $f(x_1) = \inf\{f(x) : x \in X\}$ and $f(x_2) = \sup\{f(x) : x \in X\}$.*

Proof. By Theorem 5.6 the image of f is compact, and so by the Heine-Borel theorem (Theorem 4.5) closed and bounded. Hence the image $f(X) = \{f(x) : x \in X\}$ contain its infimum and supremum, that is, x_1 and x_2 as required exist. ∎

Chapter II

Hilbert Spaces

Preliminary Remarks

Hilbert spaces are in some sense a direct generalisation of finite dimensional Euclidean spaces, where the norm has some geometric meaning and angles can be defined by means of the dot product. The dot product can be used to define the norm and prove many of its properties. Hilbert space theory is doing this in a similar fashion, where an *inner product* is a map with properties similar to the dot product in Euclidean space. We will emphasise the analogies and see how useful they are to find proofs in the general context of inner product spaces.

6 Inner Product Spaces

Throughout we let E denote a vector space over $\mathbb{K}$, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

6.1 Definition (Inner product, inner product space) A function $(\cdot \mid \cdot)\colon E \times E \to \mathbb{K}$ is called an *inner product* or *scalar product* if

(i) $(u \mid v) = \overline{(v \mid u)}$ for $u, v \in E$,

(ii) $(u \mid u) \geq 0$ for all $u \in E$ and $(u \mid u) = 0$ if and only if $u = 0$.

(iii) $(\alpha u + \beta v \mid w) = \alpha(u \mid w) + \beta(v \mid w)$ for all $u, v, w \in E$ and $\alpha, \beta \in \mathbb{K}$,

We say that E equipped with $(\cdot \mid \cdot)$ is an *inner product space*.

6.2 Remark As an immediate consequence of the above definition, inner products have the following properties:

(a) By property (i) we have $(u \mid u) = \overline{(u \mid u)}$ and therefore $(u \mid u) \in \mathbb{R}$ for all $u \in E$. Hence property (ii) makes sense.

(b) Using (i) and (iii) we have

$$(u \mid \alpha v + \beta w) = \overline{\alpha}(u \mid v) + \overline{\beta}(u \mid w)$$

for all $u, v, w \in E$ and $\alpha, \beta \in \mathbb{K}$. In particular we have

$$(u \mid \lambda v) = \overline{\lambda}(u \mid v)$$

for all $u, v \in E$ and $\lambda \in \mathbb{K}$.

Next we give some examples of Hilbert spaces.

6.3 Examples (a) The space $\mathbb{C}^N$ equipped with the Euclidean scalar product given by

$$(x \mid y) := x \cdot y = \sum_{i=1}^{N} x_i \overline{y}_i$$

for all $x := (x_1, \dots, x_N), y := (y_1, \dots, y_N) \in \mathbb{C}^N$ is an inner product space. More generally, if we take a positive definite Hermitian matrix $A \in \mathbb{C}^{N \times N}$, then

$$(x \mid y)_A := x^T A \bar{y}$$

defines an inner product on $\mathbb{C}^N$.

(b) An infinite dimensional version is ℓ_2. An inner product is defined by

$$(x \mid y) := \sum_{i=1}^{\infty} x_i \overline{y}_i$$

for all $(x_i), (y_i) \in \ell_2$. The series converges absolutely by the Cauchy Schwarz inequality.

(c) If $(a, b) \subseteq \mathbb{R}$ we let

$$L_2((a, b)) := \{f : (a, b) \to \mathbb{C} \mid \int_a^b |f(t)|^2 \, dt < \infty\}$$

For $u, v \in L_2((a, b))$ we let

$$(u \mid v) := \int_a^b u(t)\overline{v(t)} \, dx.$$

The Euclidean norm on $\mathbb{C}^N$ is defined by means of the dot product, namely by $\|x\| = \sqrt{x \cdot x}$ for $x \in \mathbb{C}^N$. We make a similar definition in the context of general inner product spaces.

6.4 Definition (induced norm) If E is an inner product space with inner product $(\cdot \mid \cdot)$ we define

$$\|u\| := \sqrt{(u \mid u)} \tag{6.1}$$

for all $u \in E$.

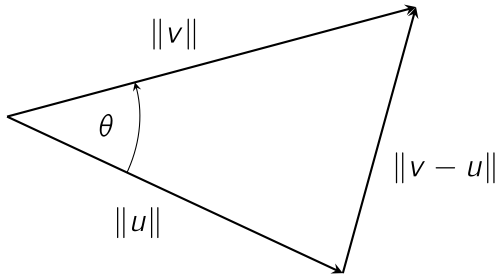

Figure 6.1: Triangle formed by u, v and $v-u$.

Note that from Remark 6.2 we always have $(x \mid x) \geq 0$, so $\|x\|$ is well defined. We call $\|\cdot\|$ a "norm," but at the moment we do not know whether it really is a norm. We now want to work towards a proof that $\|\cdot\|$ is a norm on E. On the way we look at some geometric properties of inner products and establish the Cauchy-Schwarz inequality.

By the algebraic properties of the inner products in a space over $\mathbb{R}$ and the definition of the norm we get

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2(u \mid v).$$

On the other hand, by the law of cosines we know that for vectors $u, v \in \mathbb{R}^2$

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta.$$

if we form a triangle from u, v and $v-u$ as shown in Figure 6.1. Therefore

$$u \cdot v = \|u\|\|v\| \cos\theta$$

and thus

$$|u \cdot v| \leq \|u\|\|v\|.$$

The latter inequality has a counterpart in general inner product spaces. We give a proof *inspired* by (but not relying on) the geometry in the plane. All arguments used purely depend on the algebraic properties of an inner product and the definition of the induced norm.

6.5 Theorem (Cauchy-Schwarz inequality) *Let E be an inner product space with inner product $(\cdot \mid \cdot)$. Then*

$$|(u \mid v)| \leq \|u\|\|v\| \tag{6.2}$$

for all $u, v \in E$ with equality if and only if u and v are linearly dependent.

Proof. If $u = 0$ or $v = 0$ the inequality is obvious and u and v are linearly dependent. Hence assume that $u \neq 0$ and $v \neq 0$. We can then define

$$n = v - \frac{(u \mid v)}{\|u\|^2}\, u.$$

Note that the vector

$$p := \frac{(u \mid v)}{\|u\|^2}\, u$$

is the projection of v in the direction of u, and n is the projection of v orthogonal to u as shown in Figure 6.2. Using the algebraic rules for the inner product and

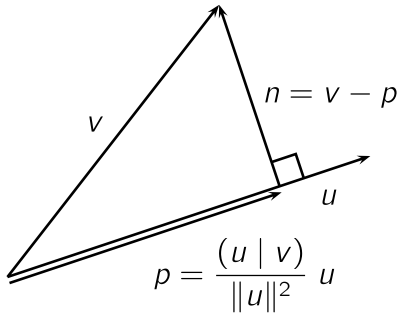

Figure 6.2: Geometric interpretation of n.

the definition of the norm we get

$$\begin{aligned} 0 \leq \|n\|^2 &= v \cdot v - 2\frac{(u \mid v)(v \mid u)}{\|u\|^2} + \frac{(u \mid v)\overline{(u \mid v)}}{\|u\|^4}(u \mid u) \\ &= \|v\|^2 - 2\frac{|(u \mid v)|^2}{\|u\|^2} + \frac{|(u \mid v)|^2}{\|u\|^4}\|u\|^2 = \|v\|^2 - \frac{|(u \mid v)|^2}{\|u\|^2}. \end{aligned}$$

Therefore $|(u \mid v)|^2 \leq \|u\|^2\|v\|^2$, and by taking square roots we find (6.2). Clearly equality holds if and only if $\|n\| = 0$, that is, if

$$v = \frac{(u \mid v)}{\|u\|^2}\, u.$$

Hence we have equality in (6.2) if and only if u and v are linearly dependent. This completes the proof of the theorem. ■

As a consequence we get a different characterisation of the induced norm.

6.6 Corollary *If E is an inner product space and $\|\cdot\|$ the induced norm, then*

$$\|u\| = \sup_{\|v\|\leq 1} |(u \mid v)| = \sup_{\|v\|=1} |(u \mid v)|$$

for all $u \in E$.

Proof. If $u = 0$ the assertion is obvious, so assume that $u \neq 0$. If $\|v\| \leq 1$, then $|(u \mid v)| \leq \|u\|\|v\| = \|u\|$ by the Cauchy-Schwarz inequality. Hence

$$\|u\| \leq \sup_{\|v\|\leq 1} |(u \mid v)|.$$

Choosing $v := u/\|u\|$ we have $|(u \mid v)| = \|u\|^2/\|u\| \leq \|u\|$, so equality holds in the above inequality. Since the supremum over $\|v\| = 1$ is larger or equal to that over $\|v\| \leq 1$, the assertion of the corollary follows. ■

Using the Cauchy-Schwarz inequality we can now prove that $\|\cdot\|$ is in fact a norm.

6.7 Theorem *If E is an inner product space, then* (6.1) *defines a norm on E.*

Proof. By property (ii) of an inner product (see Definition 6.1 we have $\|u\| = \sqrt{(u \mid u)} \geq 0$ with equality if and only if $u = 0$. If $u \in E$ and $\lambda \in \mathbb{K}$, then

$$\|\lambda u\| = \sqrt{(\lambda u \mid \lambda u)} = \sqrt{\lambda\bar{\lambda}(u \mid u)} = \sqrt{|\lambda|^2\|u\|^2} = |\lambda|\|u\|$$

as required. To prove the triangle inequality let $u, v \in E$. By the algebraic properties of an inner product and the Cauchy-Schwarz inequality we have

$$\begin{aligned}\|u+v\|^2 &= (u+v \mid u+v) = \|u\|^2 + (u \mid v) + (v \mid u) + \|v\|^2 \\ &\leq \|u\|^2 + 2|(u \mid v)|^2 + \|v\|^2 \leq \|u\|^2 + 2\|u\|^2\|v\|^2 + \|v\|^2 = \big(\|u\| + \|v\|\big)^2.\end{aligned}$$

Taking square roots the triangle inequality follows. Hence $\|\cdot\|$ defines a norm. ■

As a matter of convention we always consider inner product spaces as normed spaces.

6.8 Convention Since every inner product induces a norm we will always assume that an inner product space is a normed space with the *norm induced by the inner product*.

Once we have a norm we can talk about convergence and completeness. Note that not every inner product space is complete, but those which are play a special role.

6.9 Definition (Hilbert space) An inner product space which is complete with respect to the induced norm is called a *Hilbert space*.

The inner product is a map on $E \times E$. We show that this map is continuous with respect to the induced norm.

6.10 Proposition (Continuity of inner product) *Let E be an inner product space. Then the inner product $(\cdot \mid \cdot) \colon E \times E \to \mathbb{K}$ is continuous with respect to the induced norm.*

Proof. If $x_n \to x$ and $y_n \to y$ in E (with respect to the induced norm), then using the Cauchy-Schwarz inequality

$$\begin{aligned}|(x_n \mid y_n) - (x \mid y)| &= |(x_n - x \mid y_n) + (x \mid y_n - y)| \\ &\leq |(x_n - x \mid y_n)| + |(x \mid y_n - y)| \leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \to 0\end{aligned}$$

as $n \to \infty$. Note that we also use the continuity of the norm in the above argument to conclude that $\|y_n\| \to \|y\|$. Hence the inner product is continuous. ■

The lengths of the diagonals and edges of a parallelogram in the plane satisfy a relationship. The norm in an inner product space satisfies a similar relationship, called the parallelogram identity. The identity will play an essential role in the next section.

6.11 Proposition (Parallelogram identity) *Let E be an inner product space and $\|\cdot\|$ the induced norm. Then*

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \tag{6.3}$$

for all $u, v \in E$.

Proof. By definition of the induced norm and the properties of an inner product

$$\begin{aligned}\|u+v\|^2 + \|u-v\|^2 = \|u\|^2 + (u \mid v) + (v \mid u) + \|v\|^2 \\ + \|u\|^2 - (u \mid v) - (v \mid u) + \|v\|^2 = 2\|u\|^2 + 2\|v\|^2\end{aligned}$$

for all $u, v \in E$ as required. ■

It turns out that the converse is true as well. More precisely, if a norm satisfies (6.3) for all $u, v \in E$, then there is an inner product inducing that norm (see [6, Section I.5] for a proof).

7 Projections and Orthogonal Complements

In this section we discuss the existence and properties of "nearest point projections" from a point onto a set, that is, the points that minimise the distance from a closed set to a given point. If (X, d) is a metric space and $M \subseteq X$ we define the distance of a point $x \in X$ to M by $\operatorname{dist}(x, M) := \inf\{d(x, y) : y \in M\}$.

7.1 Definition (Projection) Let E be a normed space and M a non-empty closed subset. We define the *set of projections of x onto M* by

$$P_M(x) := \{m \in M : \|x - m\| = \operatorname{dist}(x, M)\}.$$

The meaning of $P_M(x)$ is illustrated in Figure 7.1 for the Euclidean norm in the plane. If the set is not convex, $P_M(x)$ can consist of several points, if it is convex, it is precisely one.

We now look at some example. First we look at subsets of $\mathbb{R}^N$, and show that then $P_M(x)$ is never empty.

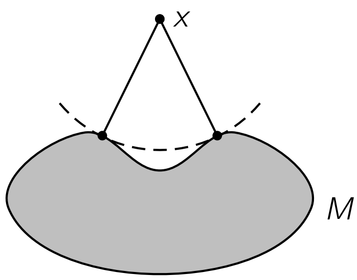

Figure 7.1: The set of nearest point projections $P_M(x)$.

7.2 Example Suppose that $M \subset \mathbb{R}^N$ is non-empty and closed. If we fix $\alpha > \operatorname{dist}(x, M)$ and $x \in \mathbb{R}^N$, then the set $K := M \cap \overline{B(x, \alpha)}$ is a closed and bounded, and $\operatorname{dist}(x, M) = \operatorname{dist}(x, K)$. We know that the distance function $x \mapsto \operatorname{dist}(x, K)$ is continuous. Since K is compact by the Heine-Borel theorem, the continuous map $y \mapsto d(x, y)$ attains a minimum on K. Hence there exists $y \in K$ such that $d(x, y) = \inf_{z \in K} d(y, z) = \operatorname{dist}(x, K) = \operatorname{dist}(x, M)$, which means that $y \in P_M(x)$. Hence $P_M(x)$ is non-empty if $M \subset \mathbb{R}^N$. The same applies to any finite dimensional space.

The argument to prove that $P_M(x)$ is non-empty used above very much depends on the set K to be compact. Since bounded and closed sets in an infinite dimensional space are not necessarily compact we have to use a different argument for the existence of a nearest point. The main idea is to use the parallelogram identity from Proposition 6.11.

7.3 Theorem (Existence and uniqueness of projections) *Let H be a Hilbert space and $M \subset H$ non-empty, closed and convex. Then $P_M(x)$ contains precisely one element which we also denote by $P_M(x)$.*

Proof. Let $M \subset H$ be non-empty, closed and convex. If $x \in M$, then $P_M(x) = x$, so there is existence and also uniqueness of an element of $P_M(x)$. Hence we assume that $x \notin M$ and set

$$\alpha := \operatorname{dist}(x, M) = \inf_{m \in M} \|x - m\|.$$

Since M is closed and $x \notin M$ we have $\alpha > 0$. From the parallelogram identity Proposition 6.11 we get

$$\begin{aligned}\|m_1 - m_2\|^2 &= \|(m_1 - x) - (m_2 - x)\|^2 \\ &= 2\|m_1 - x\|^2 + 2\|m_1 - x\|^2 - \|(m_1 - x) + (m_2 - x)\|^2.\end{aligned}$$

If $m_1, m_2 \in M$, then $\|m_i - x\| \geq \alpha$ for $i = 1, 2$ and by the convexity of M we have $(m_1 + m_2)/2 \in M$. Hence

$$\|(m_1 - x) + (m_2 - x)\| = \|m_1 + m_2 - 2x\| = 2\left\|\frac{m_1 + m_2}{2} - x\right\| \geq 2\alpha.$$

and by using the above

$$\|m_1 - m_2\|^2 \leq 2\|m_1 - x\|^2 + 2\|m_1 - x\|^2 - 4\alpha^2. \tag{7.1}$$

for all $m_1, m_2 \in M$. We can now prove uniqueness. Given $m_1, m_2 \in P_M(x)$ we have by definition $\|m_i - x\| = \alpha$ $(i = 1, 2)$, and so by (7.1)

$$\|m_1 - m_2\|^2 \leq 4\alpha^2 - 4\alpha^2 = 0.$$

Hence $\|m_1 - m_2\| = 0$, that is, $m_1 = m_2$ proving uniqueness. As a second step we prove the existence of an element in $P_M(x)$. By definition of an infimum there exists a sequence (x_n) in M such that

$$\|x_n - x\| \to \alpha := \operatorname{dist}(x, M).$$

This obviously implies that (x_n) a bounded sequence in H, but since H is not necessarily finite dimensional, we cannot conclude it is converging without further investigation. We show that (x_n) is a Cauchy sequence and therefore converges by the completeness of H. Fix now $\varepsilon > 0$. Since $\alpha \leq \|x_n - x\| \to \alpha$ there exists $n_0 \in \mathbb{N}$ such that

$$\alpha \leq \|x_n - x\| \leq \alpha + \varepsilon$$

for all $n > n_0$. Hence using (7.1)

$$\|x_k + x_n\|^2 \leq 2\|x_k - x\|^2 + 2\|x_n - x\|^2 - 4\alpha^2 \leq 4(\alpha + \varepsilon)^2 - 4\alpha^2 = 4(2\alpha + \varepsilon)\varepsilon$$

for all $n, k > n_0$. Hence (x_n) is a Cauchy sequence as claimed. ∎

We next derive a geometric characterisation of the projection onto a convex set. If we look at a convex set M in the plane and the nearest point projection m_x from a point x onto M, then we expect the angle between $x - m_x$ and $m_x - m$ to be larger or equal than $\pi/2$. This means that the inner product $(x - m_x \mid m_x - m) \leq 0$. We also expect the converse, that is, if the angle is larger or equal to $\pi/2$ for all $m \in M$, then m_x is the projection. Look at Figure 7.2 for an illustration. A similar fact remains true in an arbitrary Hilbert space, except that we have to be careful in a complex Hilbert space because $(x - m_x \mid m_x - m)$ does not need to be real.

7.4 Theorem *Suppose H is a Hilbert space and $M \subset H$ a non-empty closed and convex subset. Then for a point $m_x \in M$ the following assertions are equivalent:*

(i) $m_x = P_M(x)$;

(ii) $\operatorname{Re}(m - m_x \mid x - m_x) \leq 0$ *for all* $m \in M$.

Proof. By a translation we can assume that $m_x = 0$. Assuming that $m_x = 0 = P_M(x)$ we prove that $\operatorname{Re}(m \mid x) \leq 0$ for all $m \in M$. By definition of $P_M(x)$ we have

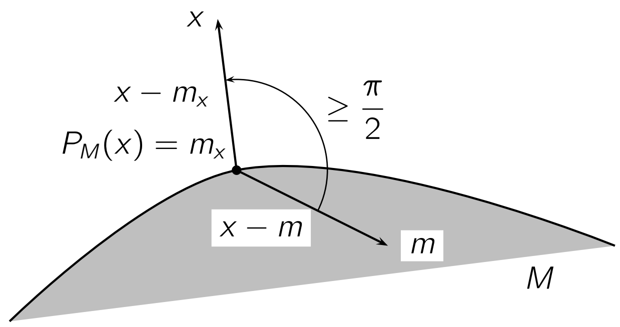

Figure 7.2: Projection onto a convex set

$\|x\| = \|x - 0\| = \inf_{m \in M} \|x - m\|$, so $\|x\| \leq \|x - m\|$ for all $m \in M$. As $0, m \in M$ and M is convex we have

$$\|x\|^2 \leq \|x - tm\|^2 = \|x\|^2 + t^2\|m\|^2 - 2t \operatorname{Re}(m \mid x)$$

for all $m \in M$ and $t \in (0, 1]$. Hence

$$\operatorname{Re}(m \mid x) \leq \frac{t}{2}\|m\|^2$$

for all $m \in M$ and $t \in (0, 1]$. If we fix $m \in M$ and let t go to zero, then $\operatorname{Re}(m \mid x) \leq 0$ as claimed. Now assume that $\operatorname{Re}(m \mid x) \leq 0$ for all $m \in M$ and that $0 \in M$. We want to show that $0 = P_M(x)$. If $m \in M$ we then have

$$\|x - m\|^2 = \|x\|^2 + \|m\|^2 - 2 \operatorname{Re}(x \mid m) \geq \|x\|^2$$

since $\operatorname{Re}(m \mid x) \leq 0$ by assumption. As $0 \in M$ we conclude that

$$\|x\| = \inf_{m \in M} \|x - m\|,$$

so $0 = P_M(x)$ as claimed. ■

Every vector subspace M of a Hilbert space is obviously convex. If it is closed, then the above characterisation of the projection can be applied. Due to the linear structure of M it simplifies and the projection turns out to be linear. From Figure 7.3 we expect that $(x - m_x \mid m) = 0$ for all $m \in M$ if m_x is the projection of x onto M and vice versa. The corollary also explains why P_M is called the *orthogonal projection* onto M.

7.5 Corollary *Let M be a closed subspace of the Hilbert space H. Then $m_x = P_M(x)$ if and only if $m_x \in M$ and $(x - m_x \mid m) = 0$ for all $m \in M$. Moreover, $P_M \colon H \to M$ is linear.*

Proof. By the above theorem $m_x = P_M(x)$ if and only if $\operatorname{Re}(m_x - x \mid m - m_x) \leq 0$ for all $m \in M$. Since M is a subspace $m + m_x \in M$ for all $m \in M$, so using $m + m_x$ instead of m we get that

$$\operatorname{Re}(m_x - x \mid (m + m_x) - m_x) = \operatorname{Re}(m_x - x \mid m) \leq 0$$

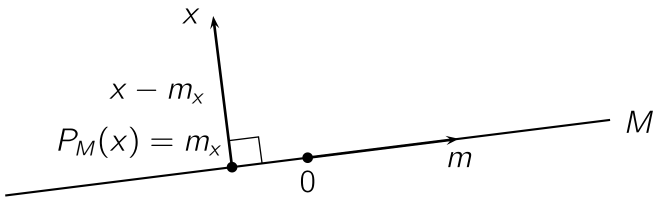

Figure 7.3: Projection onto a convex set

for all $m \in M$. Replacing m by $-m$ we get $-\operatorname{Re}(m_x - x \mid m) = \operatorname{Re}(m_x - x \mid -m) \leq 0$, so we must have $\operatorname{Re}(m_x - x \mid m) = 0$ for all $m \in M$. Similarly, replacing $m = \pm im$ if H is a complex Hilbert space we have

$$\pm \operatorname{Im}(m_x - x \mid im) = \operatorname{Re}(m_x - x \mid \pm m) \leq 0,$$

so also $\operatorname{Im}(m_x - x \mid m) = 0$ for all $m \in M$. Hence $(m_x - x \mid m) = 0$ for all $m \in M$ as claimed. It remains to show that P_M is linear. If $x, y \in H$ and $\lambda, \mu \in \mathbb{R}$, then by what we just proved

$$0 = \lambda(x - P_M(x) \mid m) + \mu(x - P_M(y) \mid m) = (\lambda x + \mu y - (\lambda P_M(x) + \mu P_M(y)) \mid m)$$

for all $m \in M$. Hence again by what we proved $P_M(\lambda x + \mu y) = \lambda P_M(x) + \mu P_M(y)$, showing that P_M is linear. ■

We next connect the projections discussed above with the notion of orthogonal complements.

7.6 Definition (Orthogonal complement) For an arbitrary non-empty subset M of an inner product space H we set

$$M^\perp := \{x \in H \colon (x \mid m) = 0 \text{ for all } m \in M\}.$$

We call $M^\perp$ the *orthogonal complement of M in H.*

We now establish some elementary but very useful properties of orthogonal complements.

7.7 Lemma *Suppose M is a non-empty subset of the inner product space H. Then $M^\perp$ is a closed subspace of H and $M^\perp = \overline{M}^\perp = (\operatorname{span} M)^\perp = (\operatorname{span} \overline{M})^\perp$.*

Proof. If $x, y \in M^\perp$ and $\lambda, \mu \in \mathbb{K}$, then

$$(\lambda x + \mu y \mid m) = \lambda(x \mid m) + \mu(y \mid m) = 0,$$

for all $m \in M$, so $M^\perp$ is a subspace of H. If x is from the closure of $M^\perp$, then there exist $x_n \in M^\perp$ with $x_n \to x$. By the continuity of the inner product

$$(x \mid m) = \lim_{n\to\infty} (x_n \mid m) = \lim_{n\to\infty} 0 = 0$$

for all $m \in M$. Hence $x \in M^\perp$, showing that $M^\perp$ is closed. We next show that $M^\perp = \overline{M}^\perp$. Since $M \subset \overline{M}$ we have $\overline{M}^\perp \subset M^\perp$ by definition the orthogonal complement. Fix $x \in M^\perp$ and $m \in \overline{M}$. Then there exist $m_n \in M$ with $m_n \to m$. By the continuity of the inner product

$$(x \mid m) = \lim_{n\to\infty} (x \mid m_n) = \lim_{n\to\infty} 0 = 0.$$

Hence $x \in \overline{M}^\perp$ and thus $\overline{M}^\perp \supset M^\perp$, showing that $\overline{M}^\perp = M^\perp$. Next we show that $M^\perp = (\operatorname{span} M)^\perp$. Clearly $(\operatorname{span} M)^\perp \subset M^\perp$ since $M \subset \operatorname{span} M$. Suppose now that $x \in M^\perp$ and $m \in \operatorname{span} M$. Then there exist $m_i \in M$ and $\lambda_i \in \mathbb{K}$, $i = 1, \dots, n$, such that $m = \sum_{i=1}^n \lambda_i m_i$. Hence

$$(x \mid m) = \overline{\lambda}_i \sum_{i=1}^{n} (x \mid m_i) = 0,$$

and thus $x \in (\operatorname{span} M)^\perp$. Therefore $(\operatorname{span} M)^\perp \supset M^\perp$ and so $(\operatorname{span} M)^\perp = M^\perp$ as claimed. The last assertion of the lemma follows by what we have proved above. Indeed we know that $M^\perp = \overline{M}^\perp$ and that $\overline{M}^\perp = (\operatorname{span} \overline{M})^\perp$. ■

We are now ready to prove the main result on orthogonal projections. It is one of the most important and useful facts on Hilbert spaces.

7.8 Theorem (orthogonal complements) *Suppose that M is a closed subspace of the Hilbert space H. Then*

(i) $H = M \oplus M^\perp$;

(ii) *P_M is the projection of H onto M parallel to $M^\perp$ (that is, $P_M(M^\perp) = \{0\}$)*

(iii) *$P_M \in \mathcal{L}(H, M)$ with $\|P_M\|_{\mathcal{L}(H,M)} \leq 1$.*

Proof. (i) By Corollary 7.5 we have $(x - P_M(x) \mid m) = 0$ for all $x \in H$ and $m \in M$. Hence $x - P_M(x) \in M^\perp$ for all $x \in H$ and therefore

$$x = P_M(x) + (I - P_M)(x) \in M + M^\perp,$$

and thus $H = M + M^\perp$. If $x \in M \cap M^\perp$, then $(x \mid x) = 0$, so $x = 0$, showing that $H = M \oplus M^\perp$ is a direct sum.

(ii) By Corollary 7.5 the map P_M is linear. Since $P_M(x) = x$ for $x \in M$ we have $P_M^2 = P_M$ and $P_M(M^\perp) = \{0\}$. Hence P_M is a projection.

(iii) By (i) we have $(P_M(x) \mid x - P_M(x)) = 0$ and so

$$\begin{aligned}\|x\|^2 &= \|P_M(x) + (I - P_M)(x)\|^2 \\ &= \|P_M(x)\|^2 + \|x - P_M(x)\|^2 + 2\operatorname{Re}(P_M(x) \mid x - P_M(x)) \geq \|P_M(x)\|^2\end{aligned}$$

for all $x \in H$. Hence $P_M \in \mathcal{L}(H, M)$ with $\|P_M\|_{\mathcal{L}(H,M)} \leq 1$ as claimed. ■

The above theorem can be used to prove some properties of orthogonal complements. The first is a very convenient criterion for a subspace of a Hilbert space to be dense.

7.9 Corollary *A subspace M of a Hilbert space H is dense in H if and only if $M^\perp = \{0\}$.*

Proof. Since $M^\perp = \overline{M}^\perp$ by Lemma 7.7 it follows from Theorem 7.8 that

$$H = \overline{M} \oplus M^\perp$$

for every subspace M of H. Hence if M is dense in H, then $\overline{M} = H$ and so $M^\perp = \{0\}$. Conversely, if $M^\perp = \{0\}$, then $\overline{M} = H$, that is, M is dense in H. ■

We finally use Theorem 7.8 to get a characterisation of the second orthogonal complement of a set.

7.10 Corollary *Suppose M is a non-empty subset of the Hilbert space H. Then*

$$M^{\perp\perp} := (M^\perp)^\perp = \overline{\operatorname{span} M}.$$

Proof. By Lemma 7.7 we have $M^\perp = (\operatorname{span} M)^\perp = (\overline{\operatorname{span} M})^\perp$. Hence by replacing M by $\overline{\operatorname{span} M}$ we can assume without loss of generality that M is a closed subspace of H. We have to show that $M = M^{\perp\perp}$. Since $(x \mid m) = 0$ for all $x \in M$ and $m \in M^\perp$ we have $M \subset M^{\perp\perp}$. Set now $N := M^\perp \cap M^{\perp\perp}$. Since M is a closed subspace it follows from Theorem 7.8 that $M^{\perp\perp} = M \oplus N$. By definition $N \subset M^\perp \cap M^{\perp\perp} = \{0\}$, so $N = \{0\}$, showing that $M = M^{\perp\perp}$. ■

8 Orthogonal Systems

In $\mathbb{R}^N$, the standard basis or any other basis of mutually orthogonal vectors of length one play a special role. We look at generalisations of such bases. Recall that two u, v of an inner product space are called *orthogonal* if $(u \mid v) = 0$.

8.1 Definition (orthogonal systems) Let H be an inner product space with inner product $(\cdot \mid \cdot)$ and induced norm $\|\cdot\|$. Let $M \subset H$ be a non-empty subset.

(i) M is called an *orthogonal system* if $(u \mid v) = 0$ for all $u, v \in M$ with $u \neq v$.

(ii) M is called an *orthonormal system* if it is an orthogonal system and $\|u\| = 1$ for all $u \in M$.

(iii) M is called a *complete orthonormal system* or *orthonormal basis* of H if it is an orthogonal system and $\overline{\operatorname{span} M} = H$.

Note that the notion of orthogonal system depends on the particular inner product, so we always have to say with respect to which inner product it is orthogonal.

8.2 Example (a) The standard basis in $\mathbb{K}^N$ is a complete orthonormal system in $\mathbb{K}^N$ with respect to the usual dot product.

(b) The set

$$M := \{(2\pi)^{-1/2}e^{inx} \colon n \in \mathbb{Z}\}$$

forms an orthonormal system in $L_2((-\pi,\pi),\mathbb{C})$. Indeed,

$$\|(2\pi)^{-1/2}e^{inx}\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dx = 1$$

for all $n \in \mathbb{N}$. Moreover, if $n \neq m$, then

$$\begin{aligned}\Big(\frac{1}{\sqrt{2\pi}}e^{inx}\Big|\frac{1}{\sqrt{2\pi}}e^{imx}\Big) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \frac{1}{2\pi}\frac{1}{i(n-m)}e^{i(n-m)x}\Big|_{-\pi}^{\pi} = 0\end{aligned}$$

since the exponential function is $2\pi i$-periodic. Using the Weierstrass approximation theorem one can show that this system forms a *complete* orthonormal system.

(c) The set of real valued functions

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx, \qquad n \in \mathbb{N}\setminus\{0\}$$

forms an orthonormal system on $L_2((-\pi,\pi),\mathbb{R})$. Again it turns out that this system is complete. The proof of the orthogonality is a consequence of the trigonometric identities

$$\begin{aligned}\sin mx \sin nx &= \frac{1}{2}\big(\cos(m-n)x - \cos(m+n)x\big) \\ \cos mx \cos nx &= \frac{1}{2}\big(\cos(m-n)x + \cos(m+n)x\big) \\ \sin mx \cos nx &= \frac{1}{2}\big(\sin(m-n)x + \sin(m+n)x\big)\end{aligned}$$

which easily follow from using the standard addition theorems for $\sin(m \pm n)x$ and $\cos(m \pm n)x$

We next show that orthogonal systems are linearly independent if we remove the zero element. Recall that by definition an infinite set is linearly independent if every finite subset is linearly independent. We also prove a generalisation of Pythagoras' theorem.

8.3 Lemma (Pythagoras theorem) *Suppose that H is an inner product space and M an orthogonal system in H. Then the following assertions are true:*

(i) *$M \setminus \{0\}$ is linearly independent.*

(ii) *If (x_n) is a sequence in M with $x_n \neq x_m$ for $n \neq m$ and H is complete, then $\sum_{k=0}^{\infty} x_k$ converges if and only if $\sum_{k=0}^{\infty} \|x_k\|^2$ converges. In that case*

$$\Big\|\sum_{k=0}^{\infty} x_k\Big\|^2 = \sum_{k=0}^{\infty} \|x_k\|^2. \tag{8.1}$$

Proof. (i) We have to show that every finite subset of $M \setminus \{0\}$ is linearly independent. Hence let $x_k \in M \setminus \{0\}$, $k = 1, \dots, n$ be a finite number of distinct elements. Assume that $\lambda_k \in \mathbb{K}$ are such that

$$\sum_{k=0}^{n} \lambda_k x_k = 0.$$

If we fix x_m, $m \in \{0, \dots, n\}$, then by the orthogonality

$$0 = \Big(\sum_{k=0}^{n} \lambda_k x_k \Big| x_m\Big) = \sum_{k=0}^{n} \lambda_k (x_k \mid x_m) = \lambda_m \|x_m\|^2.$$

Since $x_m \neq 0$ it follows that $\lambda_m = 0$ for all $m \in \{0, \dots, n\}$, showing that $M \setminus \{0\}$ is linearly independent.

(ii) Let (x_n) be a sequence in M with $x_n \neq x_m$. (We only look at the case of an infinite set because otherwise there are no issues on convergence). We set $s_n := \sum_{k=1}^{n} x_k$ and $t_n := \sum_{k=1}^{n} \|x_k\|^2$ the partial sums of the series under consideration. If $1 \leq m < n$, then by the orthogonality

$$\begin{aligned}\|s_n - s_m\|^2 = \Big\|\sum_{k=m+1}^{n} x_k\Big\|^2 &= \Big(\sum_{k=m+1}^{n} x_k \Big| \sum_{j=m+1}^{n} x_j\Big) \\ &= \sum_{k=m+1}^{n} \sum_{j=m+1}^{n} (x_k \mid x_j) = \sum_{k=m+1}^{n} \|x_k\|^2 = |t_n - t_m|.\end{aligned}$$

Hence (s_n) is a Cauchy sequence in H if and only if t_n is a Cauchy sequence in $\mathbb{R}$, and by the completeness they either both converge or diverge. The identity (8.1) now follows by setting $m = 0$ in the above calculation and then letting $n \to \infty$. ■

In the case of $H = \mathbb{K}^N$ and the standard basis e_i, $i = 1, \dots, N$, we call $x_i = (x \mid e_i)$ the components of $x \in \mathbb{K}^N$. The Euclidean norm is given by

$$\|x\|^2 = \sum_{k=1}^{n} |x_i|^2 = \sum_{k=1}^{n} |(x \mid e_i)|^2.$$

If we do not sum over the full standard basis we may only get an inequality, namely

$$\sum_{k=1}^{m} |(x \mid e_i)|^2 \leq \sum_{k=1}^{n} |(x \mid e_i)|^2 \leq \|x\|^2.$$

if $m \leq n$. We now prove a similar inequality replacing the standard basis by an arbitrary orthonormal system M in an inner product space H. From the above reasoning we expect that

$$\sum_{m \in M} |(x \mid m)|^2 \leq \|x\|$$

for all $x \in M$. The definition of an orthonormal system M does not make any assumption on the cardinality of M, so it may be uncountable. However, if M is uncountable, it is not clear what the series above means. To make sense of the above series we define

$$\sum_{m \in M} |(x \mid m)|^2 := \sup_{N \subset M \text{ finite}} \sum_{m \in N} |(x \mid m)|^2 \tag{8.2}$$

We now prove the expected inequality.

8.4 Theorem (Bessel's inequality) *Let H be an inner product space and M an orthonormal system in H. Then*

$$\sum_{m \in M} |(x \mid m)|^2 \leq \|x\|^2 \tag{8.3}$$

for all $x \in H$. Moreover, the set $\{m \in M \colon (x \mid m) \neq 0\}$ is at most countable for every $x \in H$.

Proof. Let $N = \{m_k \colon k = 1, \dots n\}$ be a finite subset of the orthonormal set M in H. Then, geometrically,

$$\sum_{k=1}^{n} (x \mid m_k) m_k$$

is the projection of x onto the span of N. By Pythagoras theorem (Lemma 8.3) and since $\|m_k\| = 1$ we have

$$\Big\|\sum_{k=1}^{n} (x \mid m_k) m_k\Big\|^2 = \sum_{k=1}^{n} |(x \mid m_k)|^2 \|m_k\|^2 = \sum_{k=1}^{n} |(x \mid m_k)|^2.$$

We expect the norm of the projection to be smaller than the norm of $\|x\|$. To see that we use the properties of the inner product and the above identity to get

$$
\begin{aligned}
0 \leq \Big\|x - \sum_{k=1}^{n}(x \mid m_k)m_k\Big\|^2 &= \|x\|^2 + \Big\|\sum_{k=1}^{n}(x \mid m_k)m_k\Big\|^2 \\
&\quad - \sum_{k=1}^{n}\overline{(x \mid m_k)}(x \mid m_k) - \sum_{k=1}^{n}(x \mid m_k)(m_k \mid x) \\
&= \|x\|^2 + \sum_{k=1}^{n}|(x \mid m_k)|^2 - 2\sum_{k=1}^{n}|(x \mid m_k)|^2 \\
&= \|x\|^2 - \sum_{k=1}^{n}|(x \mid m_k)|^2.
\end{aligned}
$$

Hence we have shown that

$$\sum_{m\in N}|(x \mid m)|^2 \leq \|x\|^2$$

for every finite set $N \subset M$. Taking the supremum over all such finite sets (8.3) follows. To prove the second assertion note that for every given $x \in H$ the sets $M_n := \{m \in M\colon |(x \mid m)| \geq 1/n\}$ is finite for every $n \in \mathbb{N}$ as otherwise (8.3) could not be true. Since countable unions of finite sets are countable, the set

$$\{m \in M\colon (x \mid m) \neq 0\} = \bigcup_{n\in\mathbb{N}} M_n$$

is countable as claimed. ■

8.5 Remark Since for every x the set $\{m \in M\colon (x \mid m) \neq 0\}$ is countable we can choose an arbitrary enumeration and write $M_x := \{m \in M\colon (x \mid m) \neq 0\} = \{m_k\colon k \in \mathbb{N}\}$. Since the series $\sum_{k=1}^{\infty}|(x \mid m_k)|^2$ has non-negative terms and every such sequence is unconditionally convergent we have

$$\sum_{m\in M}^{\infty}|(x \mid m)|^2 = \sum_{k=1}^{\infty}|(x \mid m_k)|^2$$

no matter which enumeration we take. Recall that unconditionally convergent means that a series converges, and every rearrangement also converges to the same limit. We make this more precise in the next section.

9 Abstract Fourier Series

If x is a vector in $\mathbb{K}^N$ and e_i the standard basis, then we know that

$$\sum_{k=1}^{n}(x \mid e_k)e_k$$

is the orthogonal projection of x onto the subspace spanned by $e_1, \dots, e_n$ if $n \leq N$, and that

$$x = \sum_{k=1}^{N} (x \mid e_k) e_k.$$

We might therefore expect that the analogous expression

$$\sum_{m \in M} (x \mid m) m \tag{9.1}$$

is the orthogonal projection onto $\operatorname{span} M$ if M is an orthonormal system in a Hilbert space H. However, there are some difficulties. First of all, M does not need to be countable, so the sum does not necessarily make sense. Since we are not working in $\mathbb{R}$, we cannot use a definition like (8.2). On the other hand, we know from Theorem 8.4 that the set

$$M_x := \{m \in M \colon (x \mid M) \neq 0\} \tag{9.2}$$

is at most countable. Hence M_x is finite or its elements can be enumerated. If M_x is finite (9.1) makes perfectly good sense. Hence let us assume that m_k, $k \in \mathbb{N}$ is an enumeration of M_x. Hence, rather than (9.1), we could write

$$\sum_{k=0}^{\infty} (x \mid m_k) m_k.$$

This does still not solve all our problems, because the limit of the series may depend on the particular enumeration chosen. The good news is that this is not the case, and that the series is unconditionally convergent, that is, the series converges and for every bijection $\sigma \colon \mathbb{N} \to \mathbb{N}$ we have

$$\sum_{k=0}^{\infty} (x \mid m_k) m_k = \sum_{k=0}^{\infty} (x \mid m_{\sigma(k)}) m_{\sigma(k)}.$$

Recall that the series on the right hand side is called a rearrangement of the series on the left. We now show that (9.1) is actually a projection, not onto $\operatorname{span} M$, but onto its closure.

9.1 Theorem *Suppose that M is an orthonormal system in a Hilbert space H and set $N := \overline{\operatorname{span} M}$. Let $x \in H$ and m_k, $k \in \mathbb{N}$ an enumeration of M_x. Then $\sum_{k=0}^{\infty} (x \mid m_k) m_k$ is unconditionally convergent, and*

$$P_N(x) = \sum_{k=0}^{\infty} (x \mid m_k) m_k, \tag{9.3}$$

where $P_N(x)$ is the orthogonal projection onto N as defined in Section 7.

Proof. Fix $x \in H$. By Theorem 8.4 the set M_x is either finite or countable. We let m_k, $k \in \mathbb{N}$ an enumeration of M_x, setting for convenience $m_k := 0$ for k larger than the cardinality of M_x if M_x is finite. Again by Theorem 8.4

$$\sum_{k=0}^{\infty} |(x \mid m_k)|^2 \leq \|x\|^2,$$

so by Lemma 8.3 the series

$$y := \sum_{k=0}^{\infty} (x \mid m_k) m_k$$

converges in H since H is complete. We now use the characterisation of projections from Corollary 7.5 to show that $y = P_N(x)$. For $m \in M$ we consider

$$s_n(m) := \Big(\sum_{k=0}^{n} (x \mid m_k) m_k - x \Big| m \Big) = \sum_{k=0}^{n} (x \mid m_k)(m_k \mid m) - (x \mid m).$$

Since the series is convergent, the continuity of the inner product shows that

$$(y - x \mid m) = \lim_{n \to \infty} s_n(m) = \sum_{k=0}^{\infty} (x \mid m_k)(m_k \mid m) - (x \mid m)$$

exists for all $m \in M$. If $m \in M_x$, that is, $m = m_j$ for some $j \in \mathbb{N}$, then by the orthogonality

$$(y - x \mid m) = (x \mid m_j) - (x \mid m_j) = 0.$$

If $m \in M \setminus M_x$, then $(x \mid m) = (m_k \mid m) = 0$ for all $k \in \mathbb{N}$ by definition of M_x and the orthogonality. Hence again $(y - x \mid m) = 0$, showing that $y - x \in M^{\perp}$. By Lemma 7.7 it follows that $y - x \in \overline{\operatorname{span} M}^{\perp}$. Now Corollary 7.5 implies that $y = P_N(x)$ as claimed. Since we have worked with an arbitrary enumeration of M_x and $P_N(x)$ is independent of that enumeration, it follows that the series is unconditionally convergent. ∎

We have just shown that (9.3) is unconditionally convergent. For this reason we can make the following definition, giving sense to (9.1).

9.2 Definition (Fourier series) Let M be an orthonormal system in the Hilbert space H. If $x \in H$ we call $(x \mid m)$, $m \in M$, the *Fourier coefficients* of x with respect to M. Given an enumeration m_k, $k \in \mathbb{N}$ of M_x as defined in (9.2) we set

$$\sum_{m \in M} (x \mid m) m := \sum_{k=0}^{\infty} (x \mid m_k) m_k$$

and call it the *Fourier series* of x with respect to M. (For convenience here we let $m_k = 0$ for k larger than the cardinality of M_x if it is finite.)

With the above definition, Theorem 9.1 shows that

$$\sum_{m\in M}(x \mid m)m = P_N(x)$$

for all $x \in H$ if $N := \overline{\operatorname{span} M}$. As a consequence of the above theorem we get the following characterisation of complete orthonormal systems.

9.3 Theorem (orthonormal bases) *Suppose that M is an orthonormal system in the Hilbert space H. Then the following assertions are equivalent:*

(i) *M is complete;*

(ii) *$x = \sum_{m\in M}(x \mid m)m$ for all $x \in H$ (Fourier series expansion);*

(iii) *$\|x\|^2 = \sum_{m\in M} |(x \mid m)|^2$ for all $x \in H$ (Parseval's identity).*

Proof. (i)⇒(ii): If M is complete, then by definition $N := \overline{\operatorname{span} M} = H$ and so by Theorem 9.1

$$x = P_N(x) = \sum_{m\in M}(x \mid m)m$$

for all $x \in H$, proving (ii).

(ii)⇒(iii): By Lemma 8.3 and since M_x is countable we have

$$\|x\|^2 = \Big\|\sum_{m\in M}(x \mid m)m\Big\|^2 = \sum_{m\in M}|(x \mid m)|^2$$

if (ii) holds, so (iii) follows.

(iii)⇒(i): Let $N := \overline{\operatorname{span} M}$ and fix $x \in N^\perp$. By assumption, Theorem 7.8 and 9.1 as well as Lemma 8.3 we have

$$0 = \|P_N(x)\|^2 = \Big\|\sum_{m\in M}(x \mid m)m\Big\|^2 = \sum_{m\in M}|(x \mid m)|^2 = \|x\|^2.$$

Hence $x = 0$, showing that $\overline{\operatorname{span} M}^\perp = \{0\}$. By Corollary 7.9 $\overline{\operatorname{span} M} = H$, that is, M is complete, proving (i). ∎

We next provide the connection of the above "abstract Fourier series" to the "classical" Fourier series you may have seen elsewhere. To do so we look at the expansions with respect to the orthonormal systems considered in Example 8.2.

9.4 Example (a) Let e_i be the standard basis in $\mathbb{K}^N$. The Fourier "series" of $x \in \mathbb{K}^N$ with respect to e_i is

$$x = \sum_{i=1}^{N}(x \mid e_i)e_i.$$

Of course we do not usually call this a "Fourier series" but say $x_i := (x \mid e_i)$ are the components of the vector x and the above sum the representation of x with respect to the basis e_i. The example should just illustrate once more the parallels of Hilbert space theory to various properties of Euclidean spaces.

(b) The Fourier coefficients of $u \in L_2((-\pi, \pi), \mathbb{C})$ with respect to the orthonormal system

$$\frac{1}{\sqrt{2\pi}} e^{inx}, \qquad n \in \mathbb{Z},$$

are given by

$$c_n := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} u(x)\, dx.$$

Hence the Fourier series of u with respect to the above system is

$$u = \sum_{n \in \mathbb{Z}} c_n e^{inx} = \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} u(x)\, dx e^{inx}.$$

This is precisely the complex form of the classical Fourier series of u. Our theory tells us that the series converges in $L_2((-\pi, \pi), \mathbb{C})$, but we do not get any information on pointwise or uniform convergence.

(c) We now look at $u \in L_2((-\pi, \pi), \mathbb{R})$ and its expansion with respect to the orthonormal system given by

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx, \qquad n \in \mathbb{N} \setminus \{0\}.$$

The Fourier coefficients are

$$\begin{aligned} a_0 &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(x)\, dx \\ a_n &= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} u(x) \cos nx\, dx \\ b_n &= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} u(x) \sin nx\, dx \end{aligned}$$

Hence the Fourier series with respect to the above system is

$$u = a_0 + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx),$$

which is the classical cosine-sine Fourier series. Again convergence is guaranteed in $L_2((-\pi, \pi), \mathbb{R})$, but not pointwise or uniform.

Orthonormal bases in linear algebra come from diagonalising symmetric matrices associated with a particular problem from applications or otherwise. Similarly, orthogonal systems of functions come by solving partial differential equations by separation of variables. There are many such systems like Legendre and Laguerre polynomials, spherical Harmonics, Hermite functions, Bessel functions and so on. They all fit into the framework discussed in this section if we choose the right Hilbert space of functions with the appropriate inner product.

9.5 Remark One can also get orthonormal systems from any finite or countable set of linearly independent elements of an inner product space by means of the *Gram-Schmidt orghogonalisation process* as seen in second year algebra.

We have mentioned the possibility of uncountable orthonormal systems or bases. They can occur, but in practice all orthogonal bases arising from applications (like partial differential equations) are countable. Recall that a metric space is separable if it has a countable dense subset.

9.6 Theorem *A Hilbert space is separable if and only if it has a countable orthonormal basis.*

Proof. If the space H is finite dimensional and e_i, $i = 1, \dots N$, is an orthonormal basis of H, then the set

$$\operatorname{span}_{\mathbb{Q}}\{e_1, \dots, e_N\} := \Bigl\{\sum_{k=1}^{N} \lambda_k e_k \colon \lambda_k \in \mathbb{Q}(+i\mathbb{Q})\Bigr\}$$

is dense in H since $\mathbb{Q}$ is dense in $\mathbb{R}$, so every finite dimensional Hilbert space is separable. Now assume that H is infinite dimensional and that H has a complete countable orthonormal system $M = \{e_k \colon k \in \mathbb{N}\}$. For every $N \in \mathbb{N}$ we let $H_N := \operatorname{span}\{e_1, \dots, e_N\}$. Then $\dim H_N = N$ and by what we just proved, H_N is separable. Since countable unions of countable sets are countable it follows that countable unions of separable sets are separable. Hence

$$\operatorname{span} M = \bigcup_{N \in \mathbb{N}} H_N$$

is separable. Since M is complete $\operatorname{span} M$ is dense. Hence any dense subset of $\operatorname{span} M$ is dense in H as well, proving that H is separable. Assume now that H is a separable Hilbert space and let $D := \{x_k \colon k \in \mathbb{N}\}$ be a dense subset of H. We set $H_n := \operatorname{span}\{x_k \colon k = 1, \dots, n\}$. Then H_n is a nested sequence of finite dimensional subspaces of H whose union contains D and therefore is dense in H. We have $\dim H_n \leq \dim H_{n+1}$, possibly with equality. We inductively construct a basis for $\operatorname{span} D$ by first choosing a basis of H_1. Given a basis for H_n we extend it to a basis of H_{n+1} if $\dim H_{n+1} > \dim H_n$, otherwise we keep the basis we had. Doing that inductively from $n = 1$ will give a basis for H_n for each $n \in \mathbb{N}$. The union of all these bases is a countable linearly independent set spanning $\operatorname{span} D$. Applying the Gram-Schmidt orthonormalisation process we can get a countable orthonormal system spanning $\operatorname{span} D$. Since $\operatorname{span} D$ is dense, it follows that H has a complete countable orthonormal system. ■

Using the above theorem we show that there is, up to an isometric isomorphism, there is only one separable Hilbert space, namely ℓ_2. Hence ℓ_2 plays the same role as $\mathbb{K}^N$ is isomorphic to an arbitrary N-dimensional space.

9.7 Corollary *Every separable infinite dimensional Hilbert space is isometrically isomorphic to* ℓ_2.

Proof. Let H be a separable Hilbert space. Then by Theorem 9.6 H has a countable orthonormal basis $\{e_k : k \in \mathbb{N}\}$. We define a linear map $T : H \to \ell_2$ by setting

$$(Tx)_i := (x \mid e_i)$$

for $x \in H$ and $i \in \mathbb{N}$. (This corresponds to the components of x in case $H = \mathbb{K}^N$.) By Parseval's identity from Theorem 9.3 we have

$$\|x\|^2 = \sum_{i=1}^{\infty} |(x \mid e_i)|^2 = |Tx|_2^2$$

Hence T is an isometry. Hence it remains to show that T is surjective. Let $(\xi_i) \in \ell_2$ and set

$$x := \sum_{i=1}^{\infty} \xi_k e_i$$

Since $(\xi_i) \in \ell_2$ we have

$$\sum_{i=1}^{\infty} |\xi_i|^2 \|e_i\|^2 = \sum_{i=1}^{\infty} |\xi_i|^2 < \infty$$

By Lemma 8.3 the series defining x converges in H. Also, by orthogonality, $(x \mid e_i) = \xi_i$, so $Tx = (\xi_i)$. Hence T is surjective and thus an isometric isomorphism between H and ℓ_2. ∎

Bibliography

[1] E. Bishop and D. Bridges. *Constructive analysis*, volume 279 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1985.

[2] P. Cohen. The independence of the continuum hypothesis. *Proc. Nat. Acad. Sci. U.S.A.*, 50:1143–1148, 1963.

[3] P. J. Cohen. The independence of the continuum hypothesis. II. *Proc. Nat. Acad. Sci. U.S.A.*, 51:105–110, 1964.

[4] J. Dugundji. *Topology*. Allyn and Bacon Inc., Boston, Mass., 1966.

[5] K. Gödel. The consistency of the axiom of choice and of the generalized continuum-hypothesis. *Proc. Nat. Acad. Sci. U.S.A.*, 24:556–557, 1938.

[6] K. Yosida. *Functional analysis*, volume 123 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 6th edition, 1980.

[7] E. Zermelo. Beweis, daß jede Menge wohlgeordnet werden kann. *Math. Ann.*, 59:514–516, 1904.