MATH3968 – Lecture 1 Informal Overview, Parameterised Curves

Dr. Emma Carberry

- What is differential geometry and why is it useful?
- Course information
- Parameterised curves

What is Differential Geometry?

Tough question. An (informal) attempt:

Differential geometry is

- the study of shape
- the study of smooth curved objects
- calculus on curved spaces

Curvature

Curvature seems important in these notions. What is it?

We would agree that a sheet of paper is flat.

The curvature of a piece of paper should be 0.

If we roll it up into say a cylinder or a cone, does its curvature change?

We can make the piece of paper "curvy" but there is always a direction in which the paper is straight.

So does the rolled up piece of paper have 0 curvature?

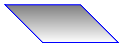
For surfaces we will define two notions of curvature.

One of these changes when we roll the piece of paper; the other one doesn't.

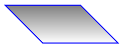
We prefer notions that don't change under distance-preserving transformations like "rolling up". They are *intrinsic*.

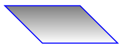
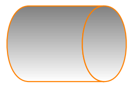
Extrinsic vs Intrinsic, Local vs Global

To begin with, we will define everything extrinsically.

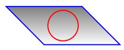
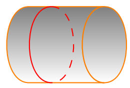
 extrinsically different from 

They are described by different equations.

 intrinsically the same as 

 extrinsically and intrinsically different from 

For example, in the cylinder we can draw a closed loop which we cannot contract to a point.

Differential geometry has two flavours: beautiful intrinsic constructions/results, and coordinate-based computations.

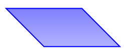
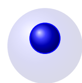
Geometry vs Topology

Have you ever tried to wrap a spherical present, such as a soccer ball?

Even locally, a sphere is intrinsically different to a plane (e.g. the sphere has positive curvature).

You cannot wrap even part of a sphere without folding the wrapping paper.

However,

 is topologically equivalent to 

Topology allows wrapping "paper" made from stockings.

What is Differential Geometry good for?

Where shall I start? (or stop!)

Optimisation Problems: it teaches us to do calculus on anything smooth. Examples:

- shortest path between two points: what path should a plane follow from Sydney to San Francisco?

- least energy: what shape soap film would you get if you dip a closed wire loop in soapy water? What path should the space shuttle take to Mars?

Applications: Physics

Differential geometry was/is used

- by Einstein to generalise special relativity to the case where the frames of reference are accelerating. He interpreted gravity as curvature.
- to estimate the mass of black holes
- in attempts to reconcile general relativity and quantum mechanics.

Applications: Mechanical Systems

- harmonic oscillator (e.g. pendulum)
- motion of particles within crystals
- What movements should a diver make to follow a particular path?

Connections with other mathematics?

If we agree the four basic pillars of mathematics are

Analysis, topology, geometry and algebra:

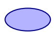
Analysis: huge overlap. Optimisation usually means solving differential equations, for which we need analysis.

Topology: topology underlies geometry.

The Poincaré conjecture was perhaps THE outstanding problem in topology, until it was solved in 2002/2003 by Perelman using differential geometry.

Poincaré conjecture (theorem!)

Suppose M is a surface which

1. is connected
2. is bounded (contained in some finite box)
3. has no boundary e.g. , not 
4. is such that any closed loop can be continuously deformed to a point

then M is a (possibly stretched) sphere.

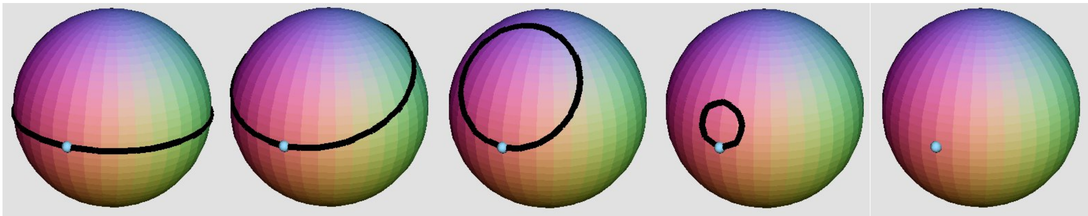


Image copyright 2006 Richard Morris, used by permission.

Poincaré says that the same is true for three-dimensional spaces.

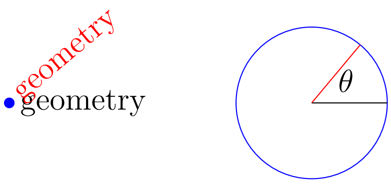
Algebra

Much overlap, especially with Lie groups, which have both an algebraic and a geometric structure. Algebra: group; geometry: curved space.

For example, the space of linear transformations of $\mathbb{R}^2$ which preserve length and orientation is a Lie group, denoted by $SO(2)$.

It is the space of rotation matrices,

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\} \cong S^1 \quad \text{(a circle)}$$



Course Information

Required text "Differential Geometry of Curves and Surfaces", by Manfredo do Carmo, available at the student Co-op.

Office hours

Carslaw 723, Thursdays 10:30 - 11:30AM

Assignments

There will be 2 assignments, worth 10% each, due Monday 12th of September and Monday 17th of October.

Expectations

Keep up to Date Look over your lecture notes as soon as possible after class, and definitely before the next lecture.

Do the Problems Mathematics is not a spectator sport. Doing problems is even more important than learning theory.

Ask Questions I don't acknowledge the concept of a "stupid question". Confused? Ask!

Tutorials

Each Friday afternoon tutorial will approximately cover material from the Monday and Thursday of that week and the Friday of the previous week. Tutorials will run weeks 1–13.

You should have attempted all of the required tutorial problems BEFORE coming to the tutorial.

Assessment

Final Exam 70%

Assignments 10% There will be 2 assignments, worth 10% each

Participation in Tutorials 10% More information on the tutorial handout.

OR the exam counts for 100%, whichever method gives you the better mark.

Tutorial Structure

Please divide into small groups at the first tutorial. There will be

- **Problems to write up:** Bring to the tutorial solutions to these problems. Then please provide peer feedback within your group, including a mark.
- **Discussion problems:** For discussing amongst your group, with help from the tutor.
- **Problems for presentation:** You will be emailed in advance if you are presenting. Marks are given for mathematics and presentation quality.

The emphasis is on interaction and a positive supportive atmosphere in which to learn.

Tutorial Sheets

Tutorial sheets have required and recommended problems. You should do all the required problems, which are divided into

1. problems to write up
2. problems for presentations/group discussion
3. other required problems.

The recommended problems are additional study material.

Parameterised Curves

- We can define a curve in $\mathbb{R}^n$ either by giving equations or by giving a parameterisation.
- We will begin with parameterised curves.

Definition 1. A *parameterised smooth curve* in $\mathbb{R}^n$ is a smooth (i.e. infinitely differentiable) map $\alpha : I \to \mathbb{R}^n$ from an open interval $I = (a, b)$ into $\mathbb{R}^n$.

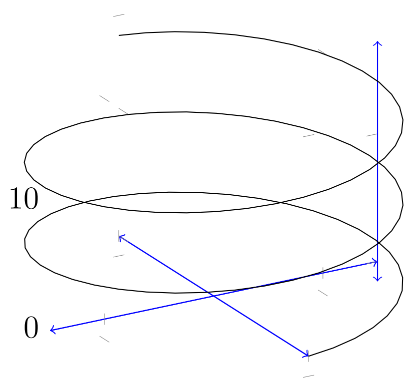
Note that two parameterised curves with the same trace $\alpha(I)$ but different parameterisations are NOT considered to be the same.

Definition 2. Let $\alpha : I \to \mathbb{R}^n$, $\beta : I' \to \mathbb{R}^n$ be parameterised smooth curves. β is a *reparameterisation* of α if there is a smooth function $\phi : I' \to I$ with smooth inverse so that

$$\beta = \alpha \circ \phi.$$

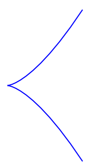
Example 3.

$$\alpha : \mathbb{R} \to \mathbb{R}^3, \quad \alpha(t) = (2\cos(t), 3\sin(t), t)$$
$$\alpha : \mathbb{R} \to \mathbb{R}^3, \quad \alpha(t) = (2\cos(2t), 3\sin(2t), 2t)$$

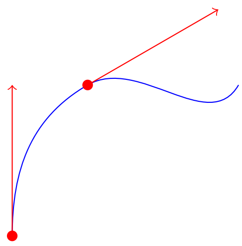


Example 3 (continued).

$$\alpha : \mathbb{R} \to \mathbb{R}^2, \quad \alpha(t) = (t^2, t^3)$$



Definition 4. $\alpha'(t)$ is called the *velocity vector* of α at t. If $\alpha'(t) \neq 0$, then there is a unique line in $\mathbb{R}^n$ that contains the point $\alpha(t)$ and is parallel to $\alpha'(t)$; we call this the *tangent line* of α at t.



Definition 5. The *arc-length* of a parameterised smooth curve $\alpha : (a, b) \to \mathbb{R}^n$ after time $t \in (a, b)$ is

$$s(t) = \int_a^t |\alpha'(u)| du;$$

$s(b)$ is simply called the arc-length of α.

Note that

$$s'(t) = |\alpha'(t)|$$

(the Fundamental Theorem of Calculus).

A particularly nice parameterisation is when a curve is parameterised by arc-length.

The trace of the curve is then being traversed at unit speed, $|\alpha'(s)| = 1$.

Active Learning

Question 6. Can every parameterised smooth curve be reparameterised by arc-length?

i.e. Given smooth $\alpha : (a, b) \to \mathbb{R}^n$, is $t \mapsto s(t)$ smooth with a smooth inverse?

Definition 8. A parameterised smooth curve $\alpha : I \to \mathbb{R}^n$ is *regular* if $\alpha'(t) \neq 0$ for all $t \in I$.

We have just seen that being regular is necessary in order that α may be reparameterised by arc-length.

Is it sufficient?

Theorem 9 (Inverse Function Theorem).

- *Let $W \subset \mathbb{R}^n$ be an open set, and*

$$W \to \mathbb{R}^n$$
$$x = (x_1, \ldots, x_n) \mapsto (f^1(x), \ldots, f^n(x))$$

be a smooth map. Suppose that at $a = (a_1, \ldots, a_n) \in W$,

$$df(a) := \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(a) & \frac{\partial f^1}{\partial x_2}(a) & \cdots & \frac{\partial f^1}{\partial x_n}(a) \\ \frac{\partial f^2}{\partial x_1}(a) & \frac{\partial f^2}{\partial x_2}(a) & \cdots & \frac{\partial f^2}{\partial x_n}(a) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^n}{\partial x_1}(a) & \frac{\partial f^n}{\partial x_2}(a) & \cdots & \frac{\partial f^n}{\partial x_n}(a) \end{pmatrix}$$

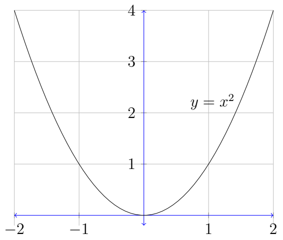
is invertible.

- *Then there are open neighbourhoods U of a and V of $b = f(a)$ so that $f|_U : U \to V$ is invertible with smooth inverse f^{-1}.*

Active Learning

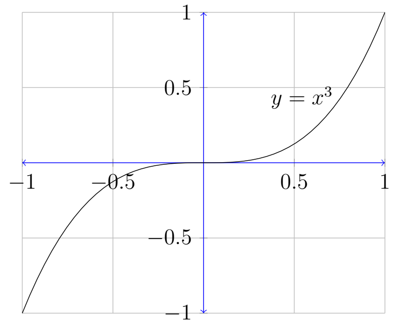
Question 10. Let $y = f(x)$ where

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$



State the inverse function theorem specifically for this function. Does the function have a local inverse near each of the points $x = 2, x = -2, x = 0$?

What about $f(x) = x^3$ near $x = 0$?



What does the inverse function theorem say for the arc length $s : (a, b) \to (c, d)$?