MATH3968 – Lecture 2 Inverse Function Theorem, Curvature, Torsion

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Last Lecture

Definition 1. The *arc-length* of a parameterised smooth curve $\alpha : (a, b) \to \mathbb{R}^n$ after time $t \in (a, b)$ is

$$s(t) = \int_a^t |\alpha'(u)| \, du;$$

$s(b)$ is simply called the arc-length of α.

Note that

$$s'(t) = |\alpha'(t)|$$

Definition 2. Let $\alpha : I \to \mathbb{R}^n$, $\beta : I' \to \mathbb{R}^n$ be parameterised smooth curves. β is a *reparameterisation* of α if there is a smooth function $\phi : I' \to I$ with smooth inverse so that

$$\beta = \alpha \circ \phi.$$

We saw last lecture that not every parameterised smooth curve can be reparameterised by arc length.

Definition 3. A parameterised smooth curve $\alpha : I \to \mathbb{R}^n$ is *regular* if $\alpha'(t) \neq 0$ for all $t \in I$.

We have just seen that being regular is necessary in order that α may be reparameterised by arc-length.

Is it sufficient?

Theorem 4 (Inverse Function Theorem). • *Let $W \subset \mathbb{R}^n$ be an open set, and*

$$W \to \mathbb{R}^n$$
$$x = (x_1, \ldots, x_n) \mapsto (f^1(x), \ldots, f^n(x))$$

be a smooth map. Suppose that at $a = (a_1, \ldots, a_n) \in W$*,*

$$df(a) := \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(a) & \frac{\partial f^1}{\partial x_2}(a) & \cdots & \frac{\partial f^1}{\partial x_n}(a) \\ \frac{\partial f^2}{\partial x_1}(a) & \frac{\partial f^2}{\partial x_2}(a) & \cdots & \frac{\partial f^2}{\partial x_n}(a) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^n}{\partial x_1}(a) & \frac{\partial f^n}{\partial x_2}(a) & \cdots & \frac{\partial f^n}{\partial x_n}(a) \end{pmatrix}$$

is invertible, where $f^i_j = \frac{\partial f^i}{\partial x_j}$.

- *Then there are open neighbourhoods* U *of* a *and* V *of* $b = f(a)$ *so that* $f|_U : U \to V$ *is invertible with smooth inverse* f^{-1}.

In particular, what does this say for $s : (a, b) \to (c, d)$?

If $s : (a, b) \to (c, d)$ is a smooth map and $s'(x) \neq 0$ for all $x \in (a, b)$, the inverse function theorem tells us that for each x there is an open interval about x on which s has a smooth inverse.

Patching this together, we obtain a smooth inverse for s on (a, b).

That is, it tells us that a regular curve can be reparameterised by arc length.

However, just as we often cannot explicitly anti-differentiate a given function, we will often not be able to carry out the reparameterisation by arc-length explicitly.

Example 5.

$$\begin{aligned} \alpha : \quad & \mathbb{R} \to \mathbb{R}^2 \\ & t \mapsto (a\cos t, b\sin t), \quad a, b > 0 \end{aligned}$$

Computing arc length directly:

$$s(t) = \int_{t_0}^{t} \sqrt{a^2\cos^2 u + b^2 \sin^2 u}\, du$$

cannot be done in terms of elementary functions for general a, b. Finding the arc length of an ellipse can be done using elliptic integrals, which you may learn about in a Riemann surfaces class.

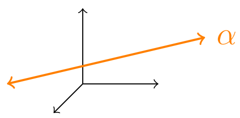
- Let $\alpha : (a, b) \to \mathbb{R}^n$ be a regular smooth curve, parameterised by arc-length s.
- Note that

  $$|\alpha'(s)| = 1;$$

  parameterisation by arc-length means that one is travelling at unit speed.

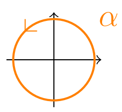
- So $\alpha'(s)$ is the unit length tangent vector to α at s.
- We will write $\mathbf{t}(s) = \alpha'(s)$ ("unit length velocity vector").
- Write
$$\alpha''(s) =: k(s)\mathbf{n}(s),$$
where
$$k(s) = |\alpha''(s)|, \quad \mathbf{n}(s) = \frac{\alpha''(s)}{|\alpha''(s)|}$$
- $\mathbf{n}(s)$ ("unit length acceleration vector with respect to arc length") is defined only if $k(s) \neq 0$; we shall restrict ourselves to this (generic) situation
- Then $k(s)$ is the magnitude of the rate of change of the unit tangent vector $\mathbf{t}$ at s, and we call it the *curvature of* α *at* s.
- The function k is called the *curvature* of α.

Example 6.
$$\alpha(u) = (a_1, \dots, a_n)u + (b_1, \dots, b_n)$$
(if unspecified, assume the domain of definition is $\mathbb{R}$).



$$\mathbf{t}(u) \equiv \frac{1}{\sqrt{a_1^2 + \cdots + a_n^2}}(a_1, \dots, a_n), \quad k \equiv 0, \quad \mathbf{n} \text{ not defined}$$

Example 7.
$$\alpha(u) = (r\cos(u), r\sin(u))$$



$$\alpha'(u) = (-r\sin(u), r\cos(u))$$
$$\mathbf{t}(u) = (-\sin(u), \cos(u)).$$

To compute $\mathbf{n}$ and the curvature, either

1. we parameterise α in terms of arc-length, $\alpha(s) = \left(r\cos(\frac{s}{r}), r\sin(\frac{s}{r})\right)$; or
2. use the chain rule to compute $\alpha''(s)$ without computing s.

Example 7 (continued).

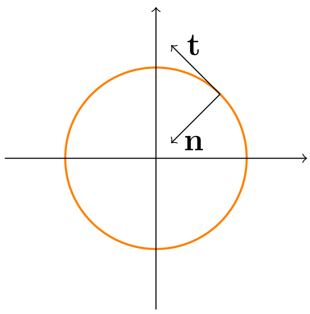
$$\alpha'(u) = (-r\sin(u), r\cos(u))$$
$$\frac{ds}{du} = |\alpha'(u)| = r$$
$$\frac{du}{ds} = \frac{1}{r}$$
$$\mathbf{t}(u) = (-\sin(u), \cos(u))$$

hence

$$\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{du}\frac{du}{ds} = \frac{1}{r}(-\cos(u), -\sin(u)),$$

so

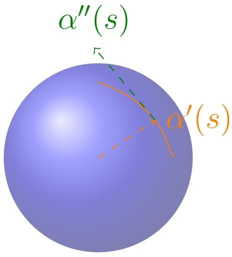
$$\mathbf{n}(u) = -(\cos(u), \sin(u)), \quad k(u) = \frac{1}{r}.$$



Example 7 (continued).

In the above examples we had $\mathbf{t} \cdot \mathbf{t}' = 0$. This is always true since $\mathbf{t}(s) = \alpha'(s)$ has constant length, it and its derivative are orthogonal.

Geometric argument:



Since $\mathbf{t}(s)$ has constant length, it traces out a path on the surface of a sphere and so is perpendicular to its derivative.

Differentiating inner product/norm

Recall that for $v(t) = (v_1(t), \ldots, v_n(t))$ and $w(t) = (w_1(t), \ldots, w_n(t))$,

$$v(t) \cdot w(t) = v_1(t)w_1(t) + \cdots v_n(t)w_n(t)$$

so by the product rule

$$
\begin{aligned}
\frac{d}{dt}\left(v(t)\cdot w(t)\right) &= v_1'(t)\cdot w_1(t) + v_1(t)\cdot w_1'(t) + \cdots \\
&\qquad + v_n'(t)\cdot w_n(t) + v_n(t)\cdot w_n'(t) \\
&= v'(t)\cdot w(t) + v(t)\cdot w'(t).
\end{aligned}
$$

To differentiate $|v(t)|$, view it as $\sqrt{v(t)\cdot v(t)}$.

$$
\begin{aligned}
\frac{d}{dt}\left(|v(t)|\right) &= \frac{1}{2}\left(v(t)\cdot v(t)\right)^{-\frac{1}{2}}\frac{d}{dt}\left(v(t)\cdot v(t)\right) \\
&= \frac{v'(t)\cdot v(t) + v(t)\cdot v'(t)}{2|v(t)|} \\
&= \frac{v'(t)\cdot v(t)}{|v(t)|}
\end{aligned}
$$

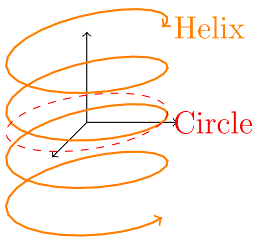
Now we can give the algebraic argument as to why the constant length vector $\mathbf{t}(s)$ is orthogonal to its derivative:

$$
\begin{aligned}
\mathbf{t}\cdot\mathbf{t} &= \text{constant} \\
\mathbf{t}'\cdot\mathbf{t} + \mathbf{t}\cdot\mathbf{t}' &= 0 \\
\mathbf{t}'\cdot\mathbf{t} &= 0
\end{aligned}
$$

ie $\mathbf{t}$ and $\mathbf{n}$ are orthogonal, $\mathbf{t}\cdot\mathbf{n} = 0$.

Definition 8. If $\mathbf{n}(s) \neq 0$, then the plane spanned by $\mathbf{t}(s)$ and $\mathbf{n}(s)$ is called the *osculating plane* of α at s.

You can think of this as the plane which "best fits" the curve α at the point $\alpha(s)$



Question 9.

How should the curvatures of a circle and a helix with the same radius compare?

Answer 10. The curvature of the helix should be less. The fact that it is stretched vertically means that there is less turning per unit length.

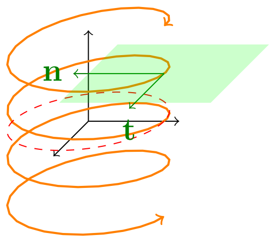
A parameterisation by arc-length is

$$
\alpha(s) = \left(r\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), r\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{as}{\sqrt{r^2+a^2}}\right)
$$

and

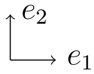
$$\mathbf{t}(s) = \frac{1}{\sqrt{r^2+a^2}}\left(-r\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), r\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), a\right),$$
$$\mathbf{n}(s) = -\left(\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), \sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), 0\right),$$
$$k(s) = \frac{r}{r^2+a^2}$$

Answer 10 (continued).



For curves in the plane, $\mathbb{R}^2$, curvature can be given a sign.

Instead of choosing a unit normal $\mathbf{n}_0(s)$ pointing in the direction of $\alpha''(s)$, we choose it so that the basis $(\mathbf{t}, \mathbf{n}_0)$ of $\mathbb{R}^2$ has the same orientation as the standard basis (e_1, e_2).



This means that the change of basis matrix M,

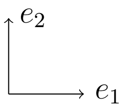
$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n}_0 \end{pmatrix} = M \begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

has positive determinant.

$\mathbf{n}_0(s)$ agrees with the old definition except possibly by a sign.

Definition 11. The *signed curvature* $k_0(s)$ is given by

$$\alpha''(s) =: k_0(s)\mathbf{n}_0(s)$$



curvature $= \frac{1}{\text{radius}}$

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n}_0 \end{pmatrix} = M \begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

det $(M) > 0$; in fact $M \in SO(2)$.

- $O(n) = \{n \times n$ matrices M with $MM^t = I,\ M^tM = I\}$; elements are called orthogonal; they preserve the standard inner product.

- $SO(n) = \{M \in O(n) |\det(M) = 1\}$; elements are called special orthogonal; they preserve the standard inner product and orientation.

- $SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$.