MATH3968 – Lecture 3

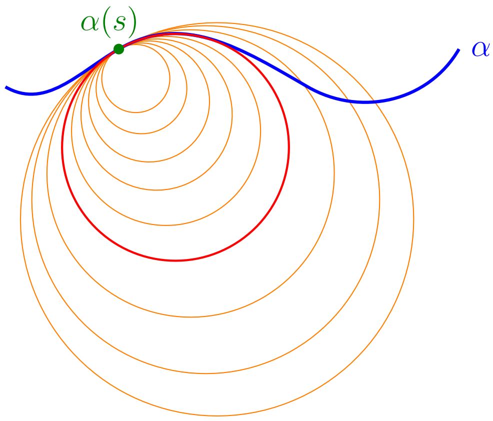
Fundamental Theorem, Frenet Equations

Dr. Emma Carberry

Last Lecture

- A parameterised smooth curve $\alpha : I \to \mathbb{R}^n$ is *regular* if $\alpha'(t) \neq 0$ for all $t \in I$.
- regularity is necessary and sufficient in order that α may be reparameterised by arc-length.
- the curvature of a regular curve at a point is the magnitude of the acceleration needed to transverse it at unit speed.
- signed curvature is positive for , negative for .

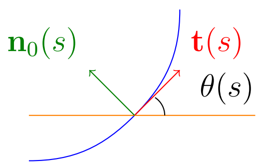
We can think geometrically of finding the curvature of a plane curve by considering the family of circles tangent to it at s.



All of these circles fit the curve α to first-order at s. $k(s)$ is the curvature of the (unique) circle that fits α to second order at s.

For a curve α in the plane, its unit tangent is given by

$$t(s) = \alpha'(s) = (\cos(\theta(s)), \sin(\theta(s))).$$



Then

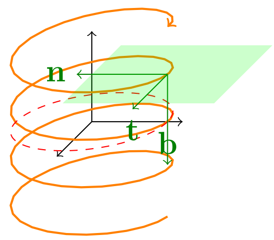
$$\mathbf{n}_0(s) = \left(\cos\left(\theta(s) + \frac{\pi}{2}\right), \sin\left(\theta(s) + \frac{\pi}{2}\right)\right) = (-\sin\theta(s), \cos\theta(s))$$

$$\alpha''(s) = \frac{d\alpha'(s)}{d\theta}\frac{d\theta}{ds} = \frac{d\theta}{ds}(-\sin\theta(s), \cos\theta(s))$$

so

$$k_0(s) = \frac{d\theta}{ds}.$$

- Henceforth, we shall assume that our curves are at most in $\mathbb{R}^3$.
- Define
$$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s).$$
- Note: do Carmo uses $\wedge$ for $\times$.
- $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ (tangent, normal, bi-normal) is called the *Frenet frame* of α.



Lemma 1.

$$\mathbf{b}'(s) = \tau(s)\mathbf{n}(s)$$

for some function τ, which we shall call the torsion of α at s.

Proof. Differentiating $\mathbf{b} \cdot \mathbf{b} = 1$ gives $\mathbf{b}'(s) \cdot \mathbf{b}(s) = 0$. Differentiating $0 = \mathbf{t} \cdot \mathbf{b}$ gives

$$0 = \mathbf{t}' \cdot \mathbf{b} + \mathbf{t} \cdot \mathbf{b}' = k\mathbf{n} \cdot \mathbf{b} + \mathbf{t} \cdot \mathbf{b}' = \mathbf{t} \cdot \mathbf{b}'.$$ □

$\mathbf{n}' = A\mathbf{t} + B\mathbf{n} + C\mathbf{b}$ for some functions A, B, C.

Differentiating $\mathbf{n} \cdot \mathbf{n} = 1$ gives $\mathbf{n}' \cdot \mathbf{n} = 0$ so $B = 0$.

Differentiating $\mathbf{t} \cdot \mathbf{n} = 0$ gives $A = \mathbf{t} \cdot \mathbf{n}' = -\mathbf{t}' \cdot \mathbf{n} = -k$.

Differentiating $\mathbf{b} \cdot \mathbf{n} = 0$ gives $C = \mathbf{b} \cdot \mathbf{n}' = -\mathbf{b}' \cdot \mathbf{n} = -\tau$.

The *Frenet equations* summarise these relations:

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} & k & \\ -k & & -\tau \\ & \tau & \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}.$$

Theorem 2 (Fundamental Existence and Uniqueness Theorem for Curves)**.** *1. If $k : I \to \mathbb{R}^+$ and $\tau : I \to \mathbb{R}$ are smooth functions, then there exists a regular parameterised curve $\alpha : I \to \mathbb{R}^3$ with curvature k and torsion τ.*

2. α is unique up to rigid motion; if $\tilde{\alpha}$ is another regular parameterised curve with the same curvature and torsion, then they differ by a rigid motion of $\mathbb{R}^3$,

$$\textit{i.e.}\ \tilde{\alpha} = \rho \cdot \alpha + c$$

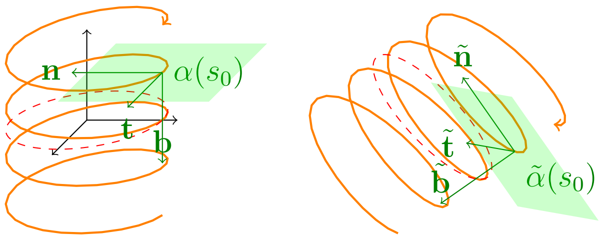
for some special orthogonal transformation $\rho : R^3 \to R^3$ and constant vector c.

- Orthogonal: preserves inner product; represented by an $O(3)$ matrix.
- Special orthogonal: also preserves orientation, i.e. has positive determinant; represented by an $SO(3)$ matrix ($\det = 1$).

Proof

Do 1.5, exercise 6 (p23), do Carmo to convince yourself that **arc-length, curvature and torsion are invariant under rigid motions**.

Uniqueness. Suppose $\alpha, \tilde{\alpha} : I \to \mathbb{R}^3$ have the same curvature and torsion. Fix $s_0 \in I$; **there is a rigid motion which transforms $\alpha(s_0)$ to $\tilde{\alpha}(s_0)$ and takes the Frenet frame of α at s_0 to that of $\tilde{\alpha}$ at s_0.**



i.e. Solve

$$(\tilde{\mathbf{t}}\ \tilde{\mathbf{n}}\ \tilde{\mathbf{b}}) = A(\mathbf{t}\ \mathbf{n}\ \mathbf{b})$$

using

$$A = (\tilde{\mathbf{t}}\ \tilde{\mathbf{n}}\ \tilde{\mathbf{b}})(\mathbf{t}\ \mathbf{n}\ \mathbf{b})^{-1}$$

and compose with the translation $c = \tilde{\alpha}(s_0) - A\alpha(s_0)$.

Then our rigid motion is

$$\begin{aligned} \psi : \quad & \mathbb{R}^3 \to \mathbb{R}^3 \\ & x \mapsto Ax + c. \end{aligned}$$

We show: $\psi\alpha(s) = \tilde{\alpha}(s)$ **for all** $s \in I$**.**

Use the Frenet equations to prove that **the norm of the difference between these two frames is constant, and therefore must vanish** since it is zero at $s = s_0$:

$$\begin{aligned} & \frac{d}{ds}(|A(\mathbf{t}) - \tilde{\mathbf{t}}|^2 + |A(\mathbf{n}) - \tilde{\mathbf{n}}|^2 + |A(\mathbf{b}) - \tilde{\mathbf{b}}|^2) \\ &= 2(A(\mathbf{t}) - \tilde{\mathbf{t}}) \cdot (A(\mathbf{t}) - \tilde{\mathbf{t}})' + 2(A(\mathbf{n}) - \tilde{\mathbf{n}}) \cdot (A(\mathbf{n}) - \tilde{\mathbf{n}})' \\ &\quad + 2(A(\mathbf{b}) - \tilde{\mathbf{b}}) \cdot (A(\mathbf{b}) - \tilde{\mathbf{b}})' \\ &= 2\big[k(A(\mathbf{t}) - \tilde{\mathbf{t}}) \cdot (A(\mathbf{n}) - \tilde{\mathbf{n}}) - k(A(\mathbf{n}) - \tilde{\mathbf{n}}) \cdot (A(\mathbf{t}) - \tilde{\mathbf{t}}) \\ &\qquad - \tau(A(\mathbf{n}) - \tilde{\mathbf{n}}) \cdot (A(\mathbf{b}) - \tilde{\mathbf{b}}) + \tau(A(\mathbf{b}) - \tilde{\mathbf{b}}) \cdot (A(\mathbf{n}) - \tilde{\mathbf{n}})\big] \\ &= 0. \end{aligned}$$

Therefore $A(\mathbf{t}(s)) = \tilde{\mathbf{t}}(s)$, $A(\mathbf{n}(s)) = \tilde{\mathbf{n}}(s)$ and $A(\mathbf{b}(s)) = \tilde{\mathbf{b}}(s)$.

Similarly **the difference** $\psi\alpha(s) - \tilde{\alpha}(s)$ **must vanish**:

$$\frac{d}{ds}\left(\psi\alpha(s) - \tilde{\alpha}(s)\right) = A(\mathbf{t}) - \tilde{\mathbf{t}} = 0,$$

so $\psi\alpha(s) - \tilde{\alpha}(s)$ is constant, and hence equal to $\psi\alpha(s_0) - \tilde{\alpha}(s_0) = 0$.

Existence. The proof uses the existence (and uniqueness) theorem for solutions of ordinary differential equations and we omit it. Section 1.5, exercise 9 (p24), do Carmo does this result in the plane and he has the full proof in the appendix to chapter 4. □

Lecture Summary

- Geometric definition of curvature using circles;
- $k_0(s) = \dfrac{d\theta(s)}{ds}$;
- Frenet equations giving derivative of Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$;
- There exists a regular curve with any given smooth curvature and torsion, and this curve is unique up to rigid motions.