MATH3968 Lecture 4

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Last Lecture

- Geometric definition of curvature using circles;
- $k_0(s) = \dfrac{d\theta(s)}{ds}$;
- Frenet equations giving derivative of Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$;
- There exists a regular curve with any given smooth curvature and torsion, and this curve is unique up to rigid motions.

We return now to curves in the plane.

- $$\alpha : (a, b) \to \mathbb{R}^2,$$
  parameterised by arc length s.
- As above, write
  $$\mathbf{t}(s) = \alpha'(s) = (\cos(\theta(s)), \sin(\theta(s)))$$
  where θ is a continuous choice of angle for t.
- The *total curvature* of α is
  $$\begin{aligned} k_{\text{tot}} &= \int_a^b k_o(s)ds \\ &= \int_a^b \frac{d\theta}{ds} ds \\ &= \theta(b) - \theta(a). \end{aligned}$$

Definition 1. $\alpha : [a, b] \to \mathbb{R}^2$ is *closed* if $\alpha(a) = \alpha(b)$.

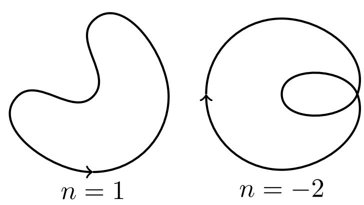
If $\alpha : (a, b) \to \mathbb{R}^2$ is a closed curve, then there is a least value L for which

$$\alpha(a + L) = \alpha(a).$$

Definition 2. Then

$$\int_a^{a+L} k_o(s)ds = \theta(a + L) - \theta(a) = 2\pi n,$$

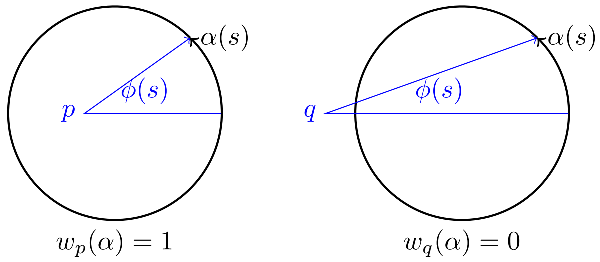
for some $n \in \mathbb{Z}$, since $\mathbf{t}(a + L) = \mathbf{t}(a)$. n is called the *rotation index* of α.



Definition 3. We can also define the *winding number* $w_p(\alpha)$ of α with respect to $p \notin \alpha(0,b)$ by

$$w_p(\alpha) = \frac{1}{2\pi}(\phi(L) - \phi(0)).$$

where $\phi(s)$ is a continuous choice of angle for the vector $\alpha(s) - p$.



Can you see why we require $p \notin \alpha(0,b)$?

Terminology Warning This is the standard definition of winding number, and agrees with do Carmo section 5.7. However, the class notes use the words “winding number” for what we have called “rotation index”.

The winding number $w_p(\alpha)$ is preserved under continuous deformations of α in which it does not “cross” p. This is because it is a continous function under such deformations, but takes discrete values (so can't “jump”, eg from 1 to 2).

By “continuous deformations” we mean the following:

Definition 4. Let $\alpha, \beta : [0,l] \to \mathbb{R}^n$ be continuous, with $\alpha(0) = \beta(0) = p$ and $\alpha(l) = \beta(l) = q$. We say that α and β are *homotopic* if there is a continuous map $H : [0,l] \times [0,1] \to \mathbb{R}^n$ such that

1. $H(s,0) = \alpha(s)\, H(s,1) = \beta(s)$ for all $s \in [0,l]$
2. $H(0,t) = p$, $H(l,t) = q$ for all $t \in [0,1]$.

we call H a *homotopy* from α to β.

Think of t as a time parameter; we continuously deform from α $(t = 0)$ to β $(t = 1)$.

Equivalently, the winding number $w_p(\alpha)$ is invariant under moving p along an arc which does not intersect α.

The rotation index n is defined from the tangent $\mathbf{t}$, rather than from α directly. It is invariant under continuous deformations of $\mathbf{t}$, again because it is a continous function under such deformations, but takes discrete values (so can't “jump”, eg from 1 to 2).

In particular, n is preserved under smooth deformations of α that keep it regular as these result in continuous deformations of $\mathbf{t}$.

Definition 5. If a closed curve α has no self-intersections, it is called a *simple closed curve.*

Theorem 6 (Theorem of Turning Tangents For Regular Curves (p396, do Carmo))**.** *Let $\alpha : [0,b] \to \mathbb{R}^2$ be a regular simple closed curve, oriented anticlockwise. Then the rotation index of α is 1.*

Proof

- Take a horizontal line which is below the curve α, and translate it upwards until it is first tangent to α. Let p be a point of tangency.
- Reparameterise α if necessary so that $\alpha(0) = p$.
- Let T be the triangle $\{(u,v) \in [0,l] \times [0,l] : 0 \le u \le v \le l\}$ and define $\psi : T \to S^1$ by

$$\psi(u,v) = \frac{\alpha(v) - \alpha(u)}{|\alpha(v) - \alpha(u)|}, \text{ if } u \neq v \text{ and } (u,v) \neq (0,l)$$
$$\psi(u,u) = \frac{\alpha'(u)}{|\alpha'(u)|}$$
$$\psi(0,l) = -\frac{\alpha'(0)}{|\alpha'(0)|}$$

- Set $A = (0,0)$, $B = (0,l)$, $C = (l,l)$
- Note that along AC, the restriction of ψ is $\mathbf{t}$, the tangent map
- The tangent map is thus homotopic to the restriction of ψ to the sides AB, BC
- $\psi(A) = \frac{\alpha'(0)}{|\alpha'(0)|}$, along the interior of AB, the map ψ gives the vector in the direction $\overrightarrow{p\alpha(u)}$ and $\psi(B) = -\frac{\alpha'(0)}{|\alpha'(0)|}$
- Thus $\psi|_{AB}$ covers the top half of the unit circle, anti-clockwise

  $$\text{angle}(\psi|_{AB}(B)) - \text{angle}(\psi|_{AB}(A)) = \pi.$$

  (assuming a continuous choice of angle)
- Similarly, $\psi|_{BC}$ covers the bottom half of the unit circle, anti-clockwise

  $$\text{angle}(\psi|_{BC}(C)) - \text{angle}(\psi|_{BC}(B)) = \pi.$$

- Hence, recalling that we write θ for a continuous choice of angle for $\mathbf{t}$,

  $$\theta(l) - \theta(0) = 2\pi$$

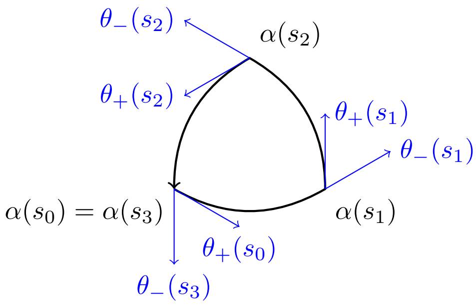
  so the rotation index is 1

Definition 7. If α is regular except at finitely many points, it is said to be *piecewise-regular*.

Let $\alpha : [0, L] :\to \mathbb{R}^2$ be a simple closed curve, regular except at points s_i,

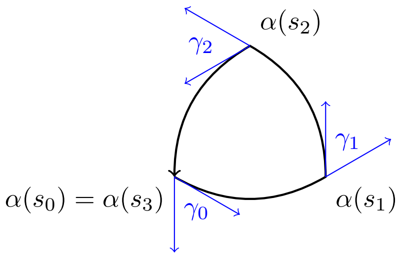
$$0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = L$$

(we are assuming that the curve is parameterised by arc length).



The total curvature is

$$\begin{aligned}\int_0^L k_o ds &= \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} d\theta \\ &= \sum_{i=0}^{n-1} (\theta_-(s_{i+1}) - \theta_+(s_i)) \\ &= \sum_{i=1}^{n-1} (\theta_-(s_i) - \theta_+(s_i)) + \theta_-(s_n) - \theta_+(s_0).\end{aligned}$$



- Write γ_i for the angle ($\in (-\pi, \pi)$) from the vector $\lim_{s \to s_i^-} t(s)$ to the vector $\lim_{s \to s_i^+} t(s)$, $1 \leq i \leq n-1$
- Write γ_0 for the angle ($\in (-\pi, \pi)$) from the vector $\lim_{s \to s_n^-} t(s)$ to the vector $\lim_{s \to s_0^+} t(s)$,
- Then
$$\theta_-(s_i) - \theta_+(s_i) = -\gamma_i$$
$$\theta_-(s_n) - \theta_+(s_0) = -\gamma_0 + 2\pi n$$
for some $n \in \mathbb{Z}$.
- We call n the *rotation index* of the piecewise-regular curve α, and from above
$$\int k_o \, ds + \sum_{i=0}^{n-1} \gamma_i = 2\pi n.$$

Theorem 8 (Theorem of Turning Tangents)**.** *Let $\alpha : [0, b] \to \mathbb{R}^2$ be a piecewise-regular simple closed curve, oriented anticlockwise. Then the rotation index of α is 1.*