MATH3968 – Lecture 5

Basic topology

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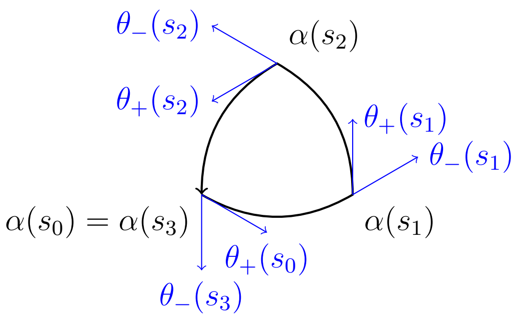
Definition 1. If α is regular except at finitely many points, it is said to be *piecewise-regular*.

Let $\alpha : [0, L] :\to \mathbb{R}^2$ be a simple closed curve, regular except at points s_i,

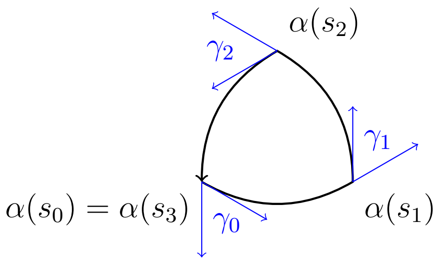
$$0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = L$$

(we are assuming that the curve is parameterised by arc length).



The total curvature is

$$\begin{aligned}\int_0^L k_0 ds &= \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} d\theta \\ &= \sum_{i=0}^{n-1} (\theta_-(s_{i+1}) - \theta_+(s_i)) \\ &= \theta_-(s_n) - \theta_+(s_0) + \sum_{i=1}^{n-1} (\theta_-(s_i) - \theta_+(s_i)).\end{aligned}$$



Write γ_i for the angle ($\in (-\pi, \pi)$) from the vector $\lim_{s \to s_i^-} t(s)$ to the vector $\lim_{s \to s_i^+} t(s)$, $1 \leq i \leq n-1$.

Write γ_0 for the angle ($\in (-\pi, \pi)$) from the vector $\lim_{s \to s_n^-} t(s)$ to the vector $\lim_{s \to s_0^+} t(s)$.

Then

$$\theta_-(s_i) - \theta_+(s_i) = -\gamma_i$$

$$\theta_-(s_n) - \theta_+(s_0) = -\gamma_0 + 2\pi n,$$

for some $n \in \mathbb{Z}$.

We call n the *rotation index* of the piecewise-regular curve α, and from above

$$\int k_o \, ds + \sum_{i=0}^{n-1} \gamma_i = 2\pi n.$$

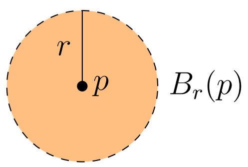
Theorem 2 (Theorem of Turning Tangents)**.** *Let $\alpha : [0, L] \to \mathbb{R}^2$ be a piecewise-regular simple closed curve, oriented anticlockwise. Then the rotation index of α is 1.*

Continuity

Read do Carmo, Appendix A, p118, and Appendices A p456 and C p466.

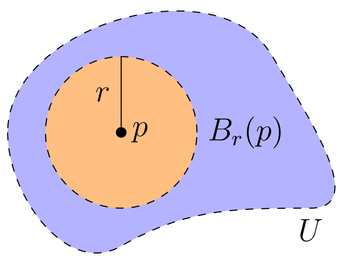
Definition 3. The *open ball* $B_r(p)$ in $\mathbb{R}^n$ centred at $p = (p_1, \ldots p_n)$ of radius r is

$$B_r(p) = \{x \in \mathbb{R}^n : |x - p| < r\}.$$



Definition 4. $U \subset \mathbb{R}^n$ is *open* if for each $p \in U$, there is an $r > 0$ so that

$$B_r(p) \subset U.$$



Definition 5. An open set containing $p \in \mathbb{R}^n$ is said to be a *(open) neighbourhood* of p.

Proposition 6. *Open sets in $\mathbb{R}^n$ have the following properties:*

1. *$\emptyset$ and $\mathbb{R}^n$ are open sets*
2. *the union of an arbitrary collection of open sets is open*
3. *the intersection of finitely many open sets is open.*

Proof. See MATH2962, or exercise. □

Definition 7. A set $U \subset \mathbb{R}^n$ is *closed* if its complement $U^c = \mathbb{R}^n \setminus U$ is open

Since

$$\left(\bigcap_{i\in I} U_i\right)^c = \bigcup_{i\in I} U_i^c, \qquad \left(\bigcup_{i\in I} U_i\right)^c = \bigcap_{i\in I} U_i^c$$

we have

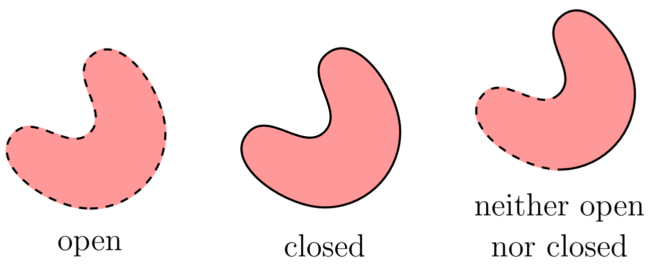
Proposition 8. 1. *$\emptyset$ and $\mathbb{R}^n$ are closed sets*

2. *the intersection of an arbitrary collection of closed sets is closed*
3. *the union of finitely many closed sets is closed*

Proposition 9. *$U \subset \mathbb{R}^n$ is closed if and only if every convergent sequence $\{x_k\}$ with $x_k \in U$ has its limit in U*

Proof. See MATH2962, or exercise. □

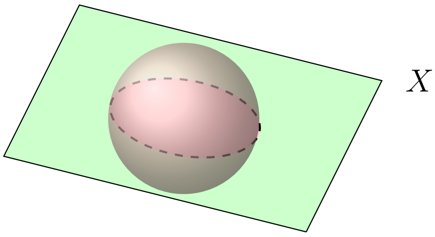
Example 10.



Definition 11. Let $X \subset \mathbb{R}^n$. A subset U of X is *relatively open in X* if for each $p \in U$ there exists $r > 0$ such that $B_r(p) \cap X \subset U$.

Proposition 12. *$U \subset X \subset \mathbb{R}^n$ is relatively open in X if and only if it is the intersection of X with an open set in $\mathbb{R}^n$.*

Proof. See MATH2962, or exercise □



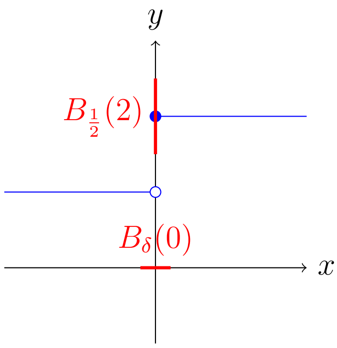
Definition 13. $f : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous at* $p \in \mathbb{R}^n$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(B_\delta(p)) \subset B_\epsilon(f(p)).$$

Example 14. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x < 0 \\ 2, & x \geq 0 \end{cases}$$

is NOT continuous at $x = 0$, since there is no $\delta > 0$ such that $f(B_\delta(0)) \subset B_{\frac{1}{2}}(f(0))$.



Definition 15. Let $U \subset \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}^m$ is *continuous* if it is continuous at every point in U.

Definition 16. Let $A \subset \mathbb{R}^n$ be an arbitrary set. $f : A \to \mathbb{R}^m$ is *continuous* if it is the restriction of a continuous function on an open set containing A.

Proposition 17. *Let $U \subset \mathbb{R}^n$ be open. $f : U \to \mathbb{R}^m$ is continuous if and only if whenever $V \subset \mathbb{R}^m$ is open, $f^{-1}(V)$ is also open.*

Proof. ($\Rightarrow$) Let $V \subset \mathbb{R}^m$ be open, and take $p \in f^{-1}(V)$.

Since V is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(p)) \subset V$.

Since f is continuous, there is a $\delta > 0$ such that $f(B_\delta(p)) \subset B_\epsilon(f(p))$. But then $B_\delta(p) \subset f^{-1}(V)$.

(⇐) Take $p \in U$, and $\epsilon > 0$.

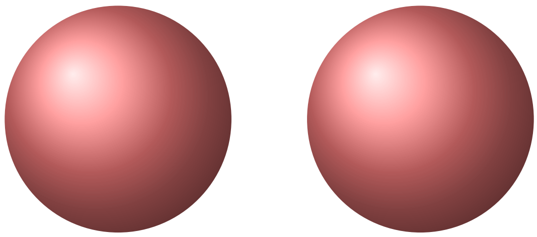
Then by assumption, $f^{-1}(B_\epsilon(f(p)))$ is open, so there is some $\delta > 0$ so that $B_\delta(p) \subset f^{-1}(B_\epsilon(f(p)))$ and hence $f(B_\delta(p)) \subset B_\epsilon(f(p))$.

□

Proposition 18. *Let $U \subset \mathbb{R}^n$ be open. $f : U \to \mathbb{R}^m$ is continuous if and only if all the component functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $f = (f_1, \ldots, f_m)$, are continuous.*

Proof. Exercise. □

Definition 19. A set $U \subset \mathbb{R}^n$ is *disconnected* if it may be written as the union of two disjoint nonempty relatively open sets. It is *connected* it is not disconnected.



disconnected

Theorem 20 (Heine-Borel)**.** *Let $[a, b]$ be a closed interval, and I_γ be a collection of open intervals so that*

$$[a, b] \subset \bigcup_\gamma I_\gamma.$$

Then there is a finite subcollection $I_{\gamma_1}, \ldots, I_{\gamma_n}$ so that

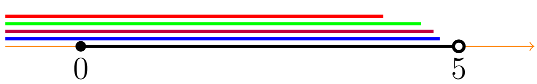
$$[a, b] \subset I_{\gamma_1} \cup \cdots \cup I_{\gamma_n}$$

.

Example 21. The interval $[0, 5)$ does NOT have this property.

$$[0, 5) \subset \bigcup_{n=1}^{\infty} \left(-1,\ 5 - \frac{1}{n}\right),$$

but no finite sub-collection of these intervals will cover $[0, 5)$.



Theorem 22 (General Heine-Borel)**.** *More generally in $\mathbb{R}^n$, if K is a closed and bounded subset, and $\{U_\alpha\}$ is a collection of open sets that cover K,*

$$K \subseteq \bigcup_\alpha U_\alpha$$

then there is a finite subcollection that also covers K (ie K is compact),

$$K \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$$

Differentiability

Definition 23. Let $U \subset \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}^m$ is *smooth* if all of its component functions have continuous partial derivatives of all orders.

Definition 24. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be smooth. The *differential of f at $p \in U$* is a linear map

$$df_p : \mathbb{R}^n \to \mathbb{R}^m$$

defined as follows: take $w \in \mathbb{R}^n$, and let $\alpha : (-\epsilon, \epsilon) \to U$ be a smooth curve so that $\alpha(0) = p$ and $\alpha'(0) = w$. Then

$$df_p(w) = (f \circ \alpha)'(0).$$

Proposition 25. *df_p is a well-defined linear map, and with respect to the standard bases is given by the matrix*

$$df_p := \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(p) & \frac{\partial f^1}{\partial x_2}(p) & \cdots & \frac{\partial f^1}{\partial x_n}(p) \\ \frac{\partial f^2}{\partial x_1}(p) & \frac{\partial f^2}{\partial x_2}(p) & \cdots & \frac{\partial f^2}{\partial x_n}(p) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^m}{\partial x_1}(p) & \frac{\partial f^m}{\partial x_2}(p) & \cdots & \frac{\partial f^m}{\partial x_n}(p) \end{pmatrix}.$$

Proposition 26 (Chain Rule)**.** *If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ are smooth mappings, then so is $g \circ f$, and for $p \in \mathbb{R}^n$,*

$$d(g \circ f)_p = dg_{f(p)} df_p.$$

Theorem 27 (Implicit Function Theorem)**.**

$$\begin{array}{rccc} F: & W \subset \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n & \to & \mathbb{R}^n \\ & (x_1, \ldots, x_m, y_1, \ldots, y_n) & \mapsto & (F^1(x,y), \ldots, F^n(x,y)) \end{array}$$

Suppose F is a smooth map, and that for $(a, b) \in W$,

$$\begin{pmatrix} \frac{\partial F^1}{\partial y_1}(a,b) & \frac{\partial F^1}{\partial y_2}(a,b) & \cdots & \frac{\partial F^1}{\partial y_n}(a,b) \\ \frac{\partial F^2}{\partial y_1}(a,b) & \frac{\partial F^2}{\partial y_2}(a,b) & \cdots & \frac{\partial F^2}{\partial y_n}(a,b) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^n}{\partial y_1}(a,b) & \frac{\partial F^n}{\partial y_2}(a,b) & \cdots & \frac{\partial F^n}{\partial y_n}(a,b) \end{pmatrix}$$

is invertible. Write $c = F(a,b)$. Then there are open neighbourhoods $U \subset \mathbb{R}^m$ of a and $V \subset \mathbb{R}^n$ of b and a smooth map $g : U \to V$ so that for $(x,y) \in U \times V$,

$$F(x,y) = c \qquad \Leftrightarrow \qquad y = g(x).$$

If F is linear, the Implicit Function Theorem is a familiar statement from linear algebra:

Theorem 28 (Linear Implicit Function Theorem)**.** *Given n linear equations in $m+n$ variables, if the rank of coefficient matrix is n then we have m free variables and we can solve uniquely for the remaining n pivot variables in terms of the m free variables.*

The x_i above are free variables, and the y_i are pivot variables.

The Implicit Function Theorem tells us that given n "smooth equations"

$$\begin{aligned} F^1(x_1, \ldots, x_m, y_1, \ldots, y_n) &= c_1 \\ \vdots \quad &\quad \vdots \\ F^n(x_1, \ldots, x_m, y_1, \ldots, y_n) &= c_n \end{aligned}$$

and $p \in \mathbb{R}^{m+n}$, if dF_p has rank n, in particular if

$$\left[\frac{\partial F^i(p)}{\partial y_j}\right]$$

is invertible, then we can solve uniquely for $y_1, \ldots, y_n$ in terms of $x_1, \ldots, x_m$ and moreover the map

$$g : x_1, \ldots, x_m \mapsto y_1, \ldots, y_n$$

is smooth.

Active Learning

Question 29. What does the Implicit Function Theorem tell you about the function

$$\begin{aligned} F : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x,y) &\mapsto x^2 + y^2. \end{aligned}$$

at the point $(0,1)$?

Theorem 30 (Implicit Function Theorem)**.**

$$\begin{array}{rccc} F: & W \subset \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n & \to & \mathbb{R}^n \\ & (x_1, \ldots, x_m, y_1, \ldots, y_n) & \mapsto & (F^1(x,y), \ldots, F^n(x,y)) \end{array}$$

Suppose F is a smooth map, and that for $(a,b) \in W$,

$$\begin{pmatrix} \frac{\partial F^1}{\partial y_1}(a,b) & \frac{\partial F^1}{\partial y_2}(a,b) & \cdots & \frac{\partial F^1}{\partial y_n}(a,b) \\ \frac{\partial F^2}{\partial y_1}(a,b) & \frac{\partial F^2}{\partial y_2}(a,b) & \cdots & \frac{\partial F^2}{\partial y_n}(a,b) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^n}{\partial y_1}(a,b) & \frac{\partial F^n}{\partial y_2}(a,b) & \cdots & \frac{\partial F^n}{\partial y_n}(a,b) \end{pmatrix}$$

is invertible (note then that $dF_{(a,b)}$ is surjective). Write $c = F(a,b)$. Then there are open neighbourhoods $U \subset \mathbb{R}^m$ of a and $V \subset \mathbb{R}^n$ of b and a smooth map $g : U \to V$ so that for $(x,y) \in U \times V$,

$$F(x,y) = c \qquad \Leftrightarrow \qquad y = g(x).$$