MATH3968 – Lecture 6

Regular surfaces

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Last Lecture

Definition 1. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be smooth. The *differential of f at $p \in U$* is a linear map

$$df_p : \mathbb{R}^n \to \mathbb{R}^m$$

defined as follows: take $w \in \mathbb{R}^n$, and let $\alpha : (-\epsilon, \epsilon) \to U$ be a smooth curve so that $\alpha(0) = p$ and $\alpha'(0) = w$. Then

$$df_p(w) = (f \circ \alpha)'(0).$$

Proposition 2. *df_p is a well-defined linear map, and with respect to the standard bases is given by the matrix*

$$df_p := \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(p) & \frac{\partial f^1}{\partial x_2}(p) & \cdots & \frac{\partial f^1}{\partial x_n}(p) \\ \frac{\partial f^2}{\partial x_1}(p) & \frac{\partial f^2}{\partial x_2}(p) & \cdots & \frac{\partial f^2}{\partial x_n}(p) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^m}{\partial x_1}(p) & \frac{\partial f^m}{\partial x_2}(p) & \cdots & \frac{\partial f^m}{\partial x_n}(p) \end{pmatrix}.$$

Proposition 3 (Chain Rule). *If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ are smooth mappings, then so is $g \circ f$, and for $p \in \mathbb{R}^n$,*

$$d(g \circ f)_p = dg_{f(p)} df_p.$$

Recall from linear algebra that if

$$A : \mathbb{R}^m \to \mathbb{R}^n$$

is linear, then writing $[A]$ for the matrix of A with respect to the standard bases,

rank $[A]$

$$\begin{aligned} &= \text{number of linearly independent columns of [A]} \\ &= \text{number of linearly independent rows of [A]} \\ &= \text{dimension of the image of } A \text{ (=column space of } [A]) \end{aligned}$$

To prove that these things are equivalent, reduce the matrix to row echelon form and check that

1. each of the above are invariant under row operations
2. the above are equal for a matrix in row echelon form

In particular, then

Proposition 4. *A linear map* $A : \mathbb{R}^{n+m} \to \mathbb{R}^n$ *is surjective if and only if* $[A]$ *has rank* n*.*

Recall also the

Theorem 5 (Rank–Nullity Theorem)**.** *Suppose*

$$A : \mathbb{R}^m \to \mathbb{R}^n$$

is linear, then

$$\textit{dim image } (A) + \textit{dim ker } (A) = m$$

or

$$\textit{rank + nullity = number of columns.}$$

The rank–nullity theorem immediately gives us

Proposition 6. *A linear map* $A : \mathbb{R}^n \to \mathbb{R}^{n+m}$ *is injective if and only if* $[A]$ *has rank* n*, where* $[A]$ *is the matrix of* A *with respect to the standard bases.*

This proposition enables us in particular to check when the differential $d\phi_q$ is injective.

Definition 7. A *parametrised surface* in $\mathbb{R}^3$ is a smooth mapping

$$\phi : U \to \mathbb{R}^3$$

where U is an open set in $\mathbb{R}^2$. ϕ is *regular* if

$$d\phi_p : \mathbb{R}^2 \to \mathbb{R}^3$$

is *injective* for all $p \in U$, ie $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$ are linearly independent where in terms of coordinates, ϕ takes the form

$$\phi(u, v) = (\phi_1(u, v),\ \phi_2(u, v),\ \phi_3(u, v)) \ .$$

The regularity condition guarantees the existence of a tangent plane at each point.

Unlike with curves, this will not be the basic definition of "surface" that we shall use.

The reason is that it does not allow us to deal properly with global properties of surfaces.

Indeed, one of the most basic surfaces—the sphere—is not properly treated with this definition, because there is not a regular ϕ whose image is the whole sphere.

We need to be able to use more than one such map ϕ in order to cover the sphere.

Definition 8. Let $U \subset \mathbb{R}^n$ be open. We say that $\phi : U \to \mathbb{R}^m$ is a *homeomorphism* onto its image if it is continuous and has continuous inverse.

(Note that this requires ϕ to be one-to-one.)

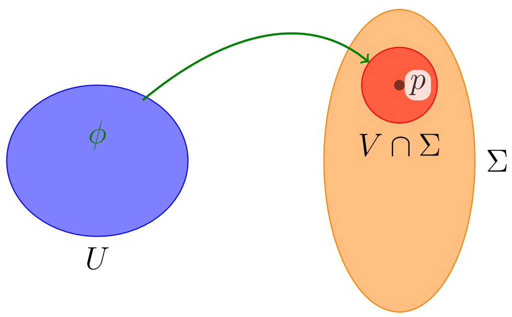
Definition 9 (Regular Surface)**.** A subset $\Sigma \subset \mathbb{R}^3$ is a *regular surface* if for each $p \in \Sigma$ there exists a neighbourhood V of p in $\mathbb{R}^3$ and a map

$$\phi : U \to V \cap \Sigma$$

of an open set $U \subset \mathbb{R}^2$ onto $V \cap \Sigma$ so that

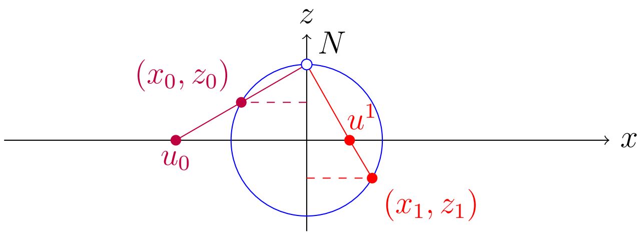
1. ϕ is smooth;
2. ϕ is a homeomorphism;
3. for each $q \in U$, $d\phi_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one (regularity condition).

ϕ is called a *local parameterisation* or *local coordinate* or *coordinate chart* near p, and $V \cap \Sigma$ is called a *coordinate neighbourhood* of p.



Example 10 (Stereographic Projection from the North Pole).

$$S^2 := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$



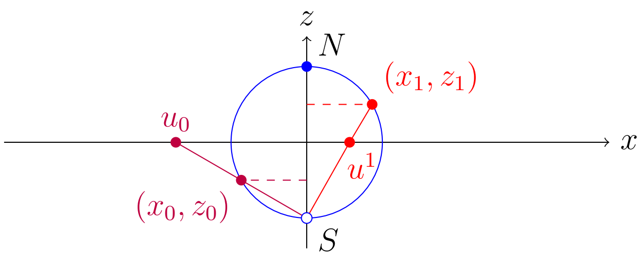
$$(u, v) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

Example 10 (continued)*.* Use inverse of stereographic projection from the North Pole N to obtain a local parameterisation of $S^2 \setminus \{N\}$, given by

$$x = \frac{2u}{u^2 + v^2 + 1}$$
$$y = \frac{2v}{u^2 + v^2 + 1}$$
$$z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}$$

(exercise).

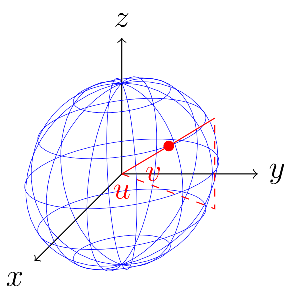
Example 10 (continued)*.* Use inverse of stereographic projection from the South Pole S (or any point other than N) to obtain a local parameterisation near N.



Example 11*.* Alternatively, we can try

$$\phi(u, v) = (\cos u \cos v, \sin u \cos v, \sin v),$$

$0 < u < 2\pi$, $-\pi/2 < v < \pi/2$. Here u is *longitude* and v is *latitude.*



Note that $u = \theta$, $v = \pi/2 - \varphi$ where θ, φ are the usual spherical coordinates.

Example 11 (continued). ϕ is smooth and one-to-one, hence a homeomorphism if it is everywhere regular (Inverse Function Theorem).

It misses the North and South Poles, and the "international dateline".

The Jacobian matrix is

$$d\phi_{u,v} = \begin{pmatrix} -\sin u \cos v & -\cos u \sin v \\ \cos u \cos v & -\sin u \sin v \\ 0 & \cos v \end{pmatrix}.$$

The columns are linearly dependent only when $\cos v = 0$, which does not occur in our domain.

We can cover the whole sphere by, for example, interchanging the roles of x and z, and choosing a domain such that arcs "missed" by the two charts do not intersect.

Proposition 12. *Let $U \subset \mathbb{R}^2$ be open, and $f : U \to \mathbb{R}$ be smooth. Then the graph of f, namely*

$$\{(x, y, f(x, y)) : (x, y) \in U\}$$

is a regular surface.

Proof: Define

$$\phi : U \to \mathbb{R}^3$$
$$(u, v) \mapsto (u, v, f(u, v)).$$

1. ϕ is smooth.
2. ϕ is one-to-one, and its inverse is given by projection onto the first two co-ordinates, the restriction of a continuous map, and hence is continuous.
3. The first two rows of the Jacobian matrix $d\phi_{u,v}$ are the identity, so it is one-to-one.

□

Definition 13 (Regular/Critical Points and Values)**.** Let $F : U \subset \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a smooth map on the open set U. $p \in U$ is a *regular point* / critical point of F if

$$dF_p : \mathbb{R}^{n+m} \to \mathbb{R}^n$$

is *surjective* / is not surjective. The image $F(p)$ of a critical point is called a critical value of F; a point in $F(U) \subset \mathbb{R}^n$ which is not the image of any critical point is called a *regular value* of F.

Theorem 14 (Implicit Function Theorem)**.**

$$\begin{array}{rrcl} F: & W \subset \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n & \to & \mathbb{R}^n \\ & (x_1, \ldots, x_m, y_1, \ldots, y_n) & \mapsto & (F^1(x,y), \ldots, F^n(x,y)) \end{array}$$

Suppose F is a smooth map, and that for $(a,b) \in W$,

$$\begin{pmatrix} \frac{\partial F^1}{\partial y_1}(a,b) & \frac{\partial F^1}{\partial y_2}(a,b) & \cdots & \frac{\partial F^1}{\partial y_n}(a,b) \\ \frac{\partial F^2}{\partial y_1}(a,b) & \frac{\partial F^2}{\partial y_2}(a,b) & \cdots & \frac{\partial F^2}{\partial y_n}(a,b) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^n}{\partial y_1}(a,b) & \frac{\partial F^n}{\partial y_2}(a,b) & \cdots & \frac{\partial F^n}{\partial y_n}(a,b) \end{pmatrix}$$

is invertible (note then that $dF_{(a,b)}$ is surjective). Write $c = F(a,b)$. Then there are open neighbourhoods $U \subset \mathbb{R}^m$ of a and $V \subset \mathbb{R}^n$ of b and a smooth map $g : U \to V$ so that for $(x,y) \in U \times V$,

$$F(x,y) = c \qquad \Leftrightarrow \qquad y = g(x).$$

Proposition 15. *If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.*

Proof: Use the Implicit Function Theorem:

Take $p = (x_0, y_0, z_0) \in f^{-1}(a)$, and by renaming the axes if necessary assume that $\dfrac{\partial f}{\partial z}(p) \neq 0$.

By the Implicit Function Theorem, there are open neighbourhoods $U \subset \mathbb{R}^2$ of (x_0, y_0) and $V \subset \mathbb{R}$ of z_0 together with a smooth map

$$g : U \to V$$

such that $g(x_0, y_0) = z_0$ and

$$f(x, y, g(x,y)) = a$$

for all $(x,y) \in U$.

Near p, the surface $f^{-1}(a)$ is given by the graph of g, which we proved above to be a regular surface.

That is, we checked that

$$\begin{array}{rccl} \phi: & U & \to & \mathbb{R}^3 \\ & (x,y) & \mapsto & (x, y, g(x,y)). \end{array}$$

gives a local coordinate about p.

Since $p \in f^{-1}(a)$ was arbitrary, $f^{-1}(a)$ is a regular surface. $\square$

Active Learning

Question 16. Let $f(x, y, z) = z^2$.

1. Is 0 a regular value of f?
2. Is $f^{-1}(0)$ a regular surface?

Example 18 (Sphere). We can define the sphere implicitly as

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

It is the level set $F(x, y, z) = 1$ of the function $F(x, y, z) = x^2 + y^2 + z^2$.

$$dF_{(x,y,z)} = (2x, 2y, 2z),$$

so the only critical point of F is $(0, 0, 0)$, and hence its only critical value is 0.

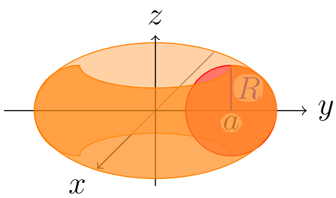
In particular, this shows that the sphere is a regular surface.

Example 19 (Torus). We can define a torus of internal radius R and external radius $a > R$ by rotating the circle

$$(y - a)^2 + z^2 = R^2$$

in the yz plane about the z axis, to give

$$\left(\sqrt{x^2 + y^2} - a\right)^2 + z^2 = R^2.$$



Example 19 (continued). Let $f(x, y, z) = (\sqrt{x^2 + y^2} - a)^2 + z^2$; then

$$df_{(x,y,z)} = \left(\frac{2x(\sqrt{x^2 + y^2} - a)}{\sqrt{x^2 + y^2}}, \frac{2y(\sqrt{x^2 + y^2} - a)}{\sqrt{x^2 + y^2}}, 2z\right).$$

The differential only fails to have maximal rank when $\sqrt{x^2 + y^2} = a$ and $z = 0$.

In particular, R^2 is a regular value of $f(x, y, z) = (\sqrt{x^2 + y^2} - a)^2 + z^2$, so the torus is a regular surface.