MATH3968 – Lecture 7

Change of parameters

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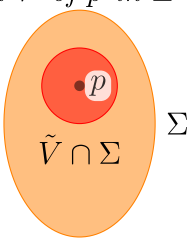
The Inverse Function Theorem also tells us that locally every regular surface can be realised as a graph:

Proposition 1. *Let $\Sigma \subset \mathbb{R}^3$ be a regular surface and $p \in \Sigma$. Then there is an open neighbourhood V of p in Σ so that V is the graph of a smooth function which has one of the following three forms:*

$$z = f(x, y),\ y = g(x, z),\ \text{or } x = h(y, z).$$

Definition 2. By an *open neighbourhood V of p in Σ* we mean the intersection $\tilde{V} \cap \Sigma$ of Σ with a neighbourhood $\tilde{V}$ of p in $\mathbb{R}^3$.



Theorem 3 (Inverse Function Theorem)**.** *Let $W \subset \mathbb{R}^n$ be an open set, and*

$$W \to \mathbb{R}^n$$
$$x = (x_1, \ldots, x_n) \mapsto (f^1(x), \ldots, f^n(x))$$

be a smooth map. Suppose that at $a = (a_1, \ldots, a_n) \in W$,

$$df(a) := \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(a) & \frac{\partial f^1}{\partial x_2}(a) & \cdots & \frac{\partial f^1}{\partial x_n}(a) \\ \frac{\partial f^2}{\partial x_1}(a) & \frac{\partial f^2}{\partial x_2}(a) & \cdots & \frac{\partial f^2}{\partial x_n}(a) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^n}{\partial x_1}(a) & \frac{\partial f^n}{\partial x_2}(a) & \cdots & \frac{\partial f^n}{\partial x_n}(a) \end{pmatrix}$$

is invertible.

Then there are open neighbourhoods U of a and V of $b = f(a)$ so that $f|_U : U \to V$ is invertible with smooth inverse f^{-1}.

Definition 4 (Diffeomorphism)**.** A map $f : U \to V$ between open sets is a *diffeomorphism* if it is smooth and has smooth inverse.

The conclusion of the inverse function theorem is that $f|_U$ is a diffeomorphism onto its image.

Proof of Proposition: Let

$$\begin{aligned} \phi : \quad & U \to W = \tilde{W} \cap \Sigma \\ & (u, v) \mapsto (x, y, z) = \phi(u, v) \end{aligned}$$

be a local coordinate near $p = \phi(u_0, v_0)$.

Then by the regularity condition, $d\phi_{(u_0,v_0)}$ has rank 2. Assume

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

is invertible (this will give us z as a function of (x, y)).

Let

$$\begin{aligned} \pi : \qquad & \mathbb{R}^3 \to \mathbb{R}^2 \\ & (x, y, z) \mapsto (x, y). \end{aligned}$$

$$d\pi_{(x,y,z)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ for all } (x, y, z),$$

so by the chain rule,

$$d(\pi \circ \phi)_{(u_0,v_0)} = d\pi_{\phi(u_0,v_0)} \circ d\phi_{(u_0,v_0)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}.$$

By the inverse function theorem, there is a neighbourhood $U_0 \subset U$ of (u_0, v_0) in $\mathbb{R}^2$ such that $\pi \circ \phi$ is a diffeomorphism onto its image $V_0 = \pi \circ \phi(U_0)$.

So for $(x, y) \in V_0$, we have u, v as smooth functions of x, y:

$$(u(x, y), v(x, y)) = (\pi \circ \phi)^{-1}(x, y).$$

Then on $V = \phi(U_0)$,

$$(x, y, z) = (x, y, z(u(x, y), v(x, y))),$$

so V is the graph of $z \circ (\pi \circ \phi)^{-1} : (x, y) \mapsto z$.

The other cases are similar. □

In the study of regular curves, there was a canonical parameterisation, namely parameterisation by arc-length.

There is no canonical way to give a a local parameterisation of a regular surface.

We will often have to make definitions using a local parameterisation, and it is important that the definition does not depend upon the choice of parameterisation.

For example,

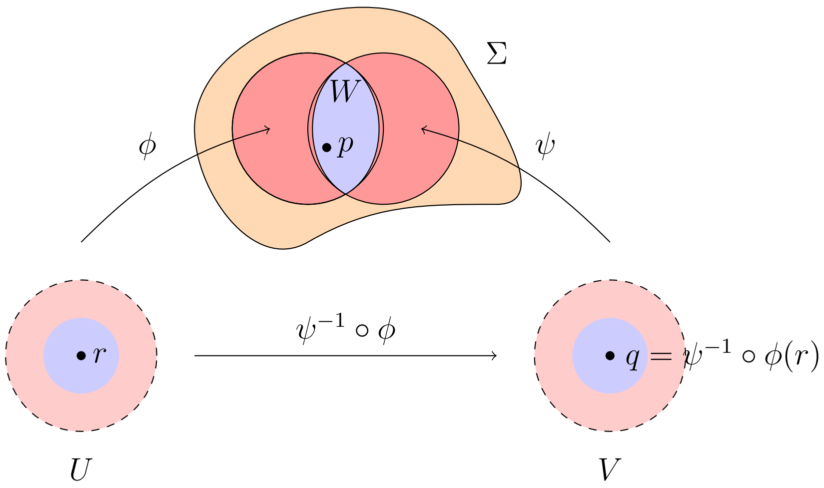
Definition 5 (Smooth function)**.** Let Σ be a regular surface and $W \subset \Sigma$ an open subset. A function $f : W \to \mathbb{R}$ is *smooth* at $p \in W$ if for some coordinate chart $\phi : U \subset \mathbb{R}^2 \to \Sigma$ with $p \in \phi(U) \cap W$, the composition $f \circ \phi$ is smooth at $\phi^{-1}(p)$

We need to know that if p is in the image of *two* such local parameterisations, ϕ and ψ, then $f \circ \psi$ is smooth at $\psi^{-1}(p)$ if and only if $f \circ \phi$ is smooth at $\phi^{-1}(p)$. This follows from

Proposition 6 (Change of Parameters)**.** *Let $\Sigma \subset \mathbb{R}^3$ be a regular surface, and suppose that $p \in \Sigma$ is in the image of two local parameterisations, $\phi : U \subset \mathbb{R}^2 \to \Sigma$, and $\psi : V \subset \mathbb{R}^2 \to \Sigma$. Write $W = \phi(U) \cap \psi(V)$. Then*

$$\psi^{-1} \circ \phi : \phi^{-1}(W) \to \psi^{-1}(W)$$

is a diffeomorphism.



Proof:

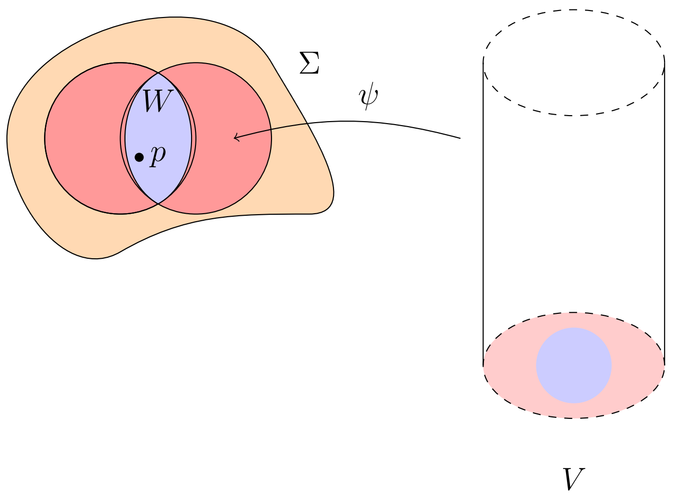
$\psi^{-1} \circ \phi$ is the composition of homeomorphisms and hence a homeomorphism.

We would like to argue similarly that it is a diffeomorphism, but ψ^{-1} is not defined on an open set in $\mathbb{R}^3$, so we have no definition of smoothness for it.

Currently $\psi : V \underset{\text{open}}{\subset} \mathbb{R}^2 \to \Sigma \underset{\text{not open}}{\subset} \mathbb{R}^3$

We will extend ψ to a map $V \times \mathbb{R} \underset{\text{open}}{\subset} \mathbb{R}^3 \to \mathbb{R}^3$ whose image is open in $\mathbb{R}^3$.

This extension will locally have smooth inverse.



Choose $r \in \phi^{-1}(W)$, and set $q = \psi^{-1} \circ \phi(r)$.

We will show that $\psi^{-1} \circ \phi$ is smooth at r.

Write $\psi(u, v) = (\psi^1(u, v), \psi^2(u, v), \psi^3(u, v))$.

Renaming axes if necessary, we may assume

$$\frac{\partial(\psi^1, \psi^2)}{\partial(u, v)}(q) = \det \begin{pmatrix} \frac{\partial \psi^1}{\partial u}(q) & \frac{\partial \psi^1}{\partial v}(q) \\ \frac{\partial \psi^2}{\partial u}(q) & \frac{\partial \psi^2}{\partial v}(q) \end{pmatrix} \neq 0.$$

Define

$$\begin{aligned} \Psi : \quad V \times \mathbb{R} &\to \mathbb{R}^3 \\ (u, v, t) &\mapsto (\psi^1(u, v), \psi^2(u, v), \psi^3(u, v) + t) \end{aligned}$$

The last column of $d\Psi_{(u,v,t)}$ is $(0, 0, 1)^T$, so

$$\det d\Psi_{(u,v,t)} = \frac{\partial(\psi^1, \psi^2)}{\partial(u, v)}(q) \neq 0.$$

Hence, by the inverse function theorem, there are (open) neighbourhoods A of q and B of $p = \Psi(q, 0) = \psi(q)$ in $\mathbb{R}^3$ such that

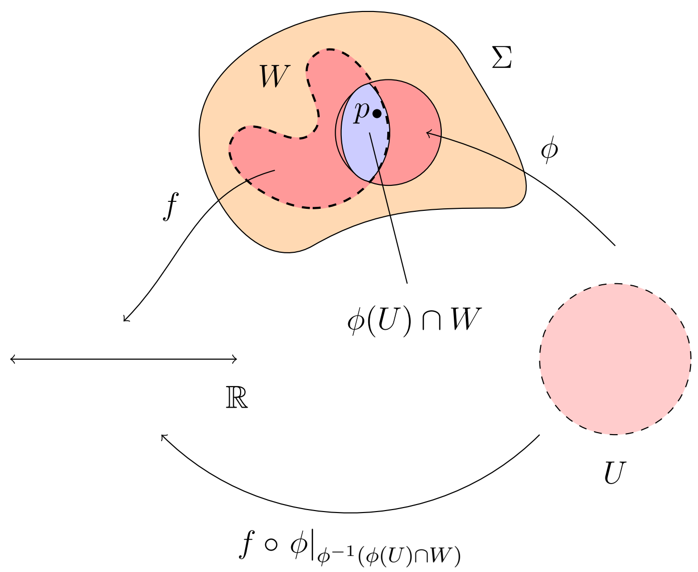
$$\Psi : A \to B$$

is a diffeomorphism.

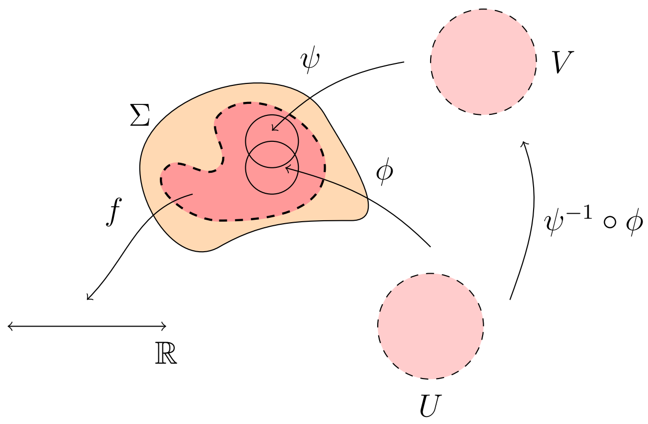
Since ϕ is continuous, $\phi^{-1}(B) \subset U \subset \mathbb{R}^2$ is open

$\psi^{-1} \circ \phi|_{\phi^{-1}(B)} = \Psi^{-1} \circ \phi|_{\phi^{-1}(B)}$ is a combination of smooth maps and hence smooth, as required. $\square$

Definition 7. Let Σ be a regular surface and $W \subset \Sigma$ an open subset. A function $f : W \to \mathbb{R}$ is *smooth* at $p \in W$ if for some coordinate chart $\phi : U \subset \mathbb{R}^2 \to \Sigma$ with $p \in \phi(U) \cap W$, the composition $f \circ \phi$ is smooth at $\phi^{-1}(p)$



This definition is independent of the choice of local coordinate chart ϕ.



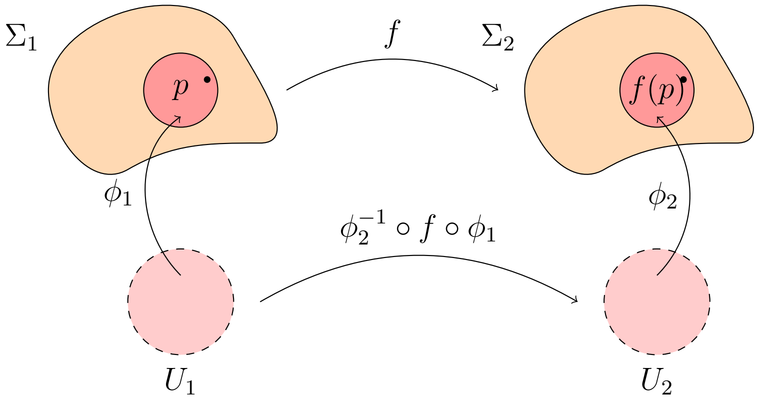
Since by the above proposition $\psi^{-1} \circ \phi$ is a diffeomorphism, $f \circ \phi|_{\phi^{-1}(\phi(U) \cap \psi(V))}$ is smooth if and only if $f \circ \psi|_{\psi^{-1}(\phi(U) \cap \psi(V))}$ is smooth.

Similarly,

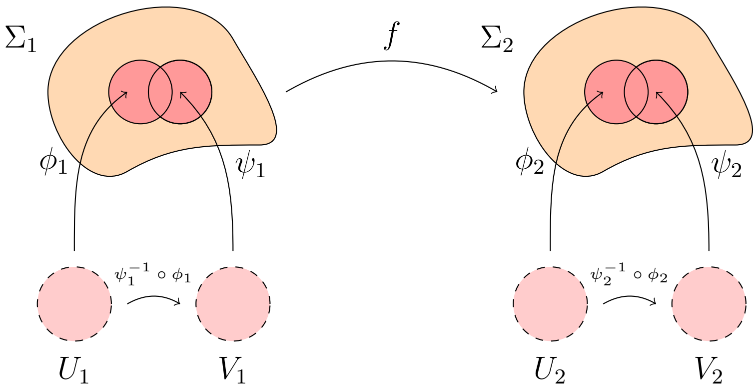
Definition 8. A map $f : \Sigma_1 \to \Sigma_2$ is *smooth* at $p \in \Sigma_1$ if for local coordinates $\phi_1 : U_1 \subset \mathbb{R}^2 \to \Sigma_1$ near p, and $\phi_2 : U_2 \subset \mathbb{R}^2 \to \Sigma_2$ near $f(p)$ with $f(\phi_1(U_1)) \subset \phi_2(U_2)$,

$$\phi_2^{-1} \circ f \circ \phi_1$$

is smooth at $\phi_1^{-1}(p)$.



This definition is also independent of the choice of local coordinate systems.



$$\psi_2^{-1} \circ f \circ \psi_1|_{\psi_1^{-1}(\phi_1(U_1) \cap \psi_1(V_1))}$$

is smooth if and only if

$$\phi_2^{-1} \circ f \circ \phi_1|_{\phi_1^{-1}(\phi_1(U_1) \cap \psi_1(V_1))}$$

is smooth, since both $\psi_1^{-1} \circ \phi_1$ and $\psi_2^{-1} \circ \phi_2$ are diffeomorphisms.

Thus we can extend our notion of diffeomorphism to maps between surfaces (they are smooth maps with smooth inverse).