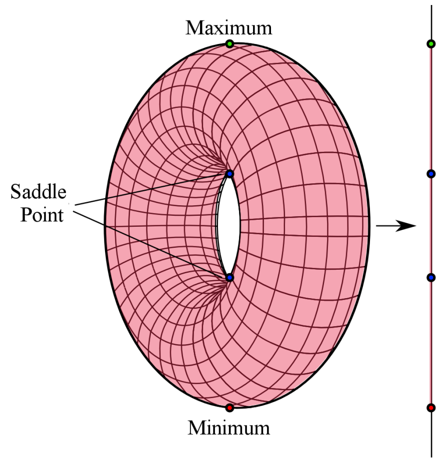
MATH3968 – Lecture 8

The tangent plane

Dr. Emma Carberry



Example 1 (Height function on a torus).

How would you prove that these critical points are of the types claimed?

Example 2. Local coordinates are diffeomorphisms.

By definition, a local coordinate $\phi : U \to \Sigma$ is smooth and invertible.

For each $p \in \phi(U)$, if $\psi : V \to \Sigma$ is another local coordinate about p then $\phi^{-1} \circ \psi|_{\psi^{-1}(\psi(V)\cap\phi(U))}$ is smooth.

This is the definition of what it means for ϕ^{-1} to be smooth at p.

Active Learning

Question 3. Let Σ be the paraboloid $z = x^2 + y^2$.

1. Show that Σ is a regular surface.

2. Show that Σ is diffeomorphic to a plane (that is, there is a diffeomorphism between Σ and a plane).

Surfaces of Revolution

A number of interesting surfaces can be obtained as surfaces of revolution.

Let Σ be the surface in $\mathbb{R}^3$ obtained by rotating the regular plane curve

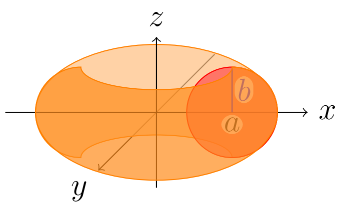
$$x = f(v), \quad z = g(v)$$

about the z-axis, where we assume that the curve does not intersect the z-axis.

$$\phi(u, v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$$

defines local coordinates. By changing the range of angles (u, v) for which the map ϕ is defined we can cover the entire surface of revolution.

Example 5 (Torus). Let $x = a + b\cos v$, $z = b\sin v$, $b < a$.



Example 5 (continued).

$$\begin{aligned}\phi_1 :&(0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3 \\ &(u, v) \mapsto ((a + b\cos v)\cos u, \\ &\qquad (a + b\cos v)\sin u, b\sin v)\end{aligned}$$

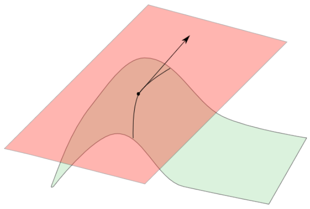
is one local coordinate; together with

$$\begin{aligned}\phi_2 :&(-\frac{\pi}{2}, \frac{3\pi}{2}) \times (-\frac{\pi}{2}, \frac{3\pi}{2}) \to \mathbb{R}^3 \\ \phi_3 :&(-\pi, \pi) \times (-\pi, \pi) \to \mathbb{R}^3\end{aligned}$$

both using the same formula as above for their respective local coordinates, we have an atlas for the torus.

Tangent Plane

Definition 6. Let $\Sigma \subset \mathbb{R}^3$ be a regular surface, and take $p \in \Sigma$. A *tangent vector* to Σ at p is the velocity vector $\alpha'(0)$ of some smooth parametrised curve $\alpha : (-\epsilon, \epsilon) \to \Sigma$ with $\alpha(0) = p$.



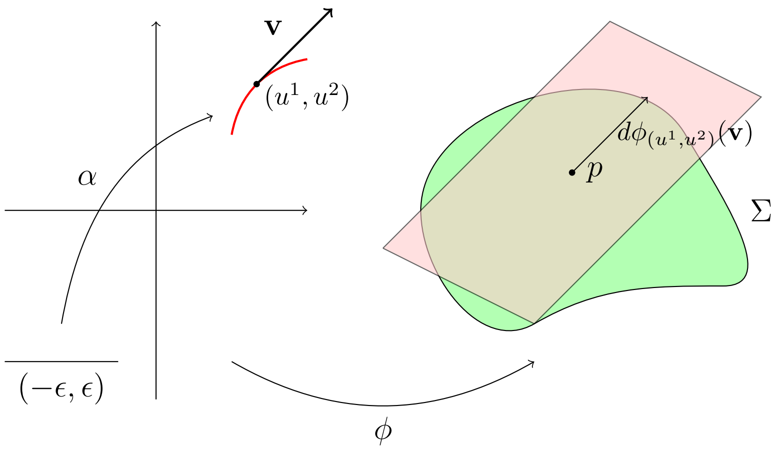
Definition 7. The set of all tangent vectors to Σ at p is called the *tangent plane to* Σ *at* p, and is denoted by $T_p\Sigma$.

Proposition 8. *Let Σ be a regular surface, and take $p \in \Sigma$. The tangent plane $T_p\Sigma$ is a 2-dimensional linear subspace of $\mathbb{R}^3$, and is equal to $d\phi_{u^1,u^2}(\mathbb{R}^2)$, for any local parameterisation $\phi : U \subset \mathbb{R}^2 \to \Sigma$ with $\phi(u^1, u^2) = p$.*

Proof: The first statement follows from the second one, so we show that given a local parameteristion ϕ of the specified form, we have $d\phi_{u^1,u^2}(\mathbb{R}^2) = T_p\Sigma$.

$d\phi_{u^1,u^2}(\mathbb{R}^2) \subset T_p\Sigma$.

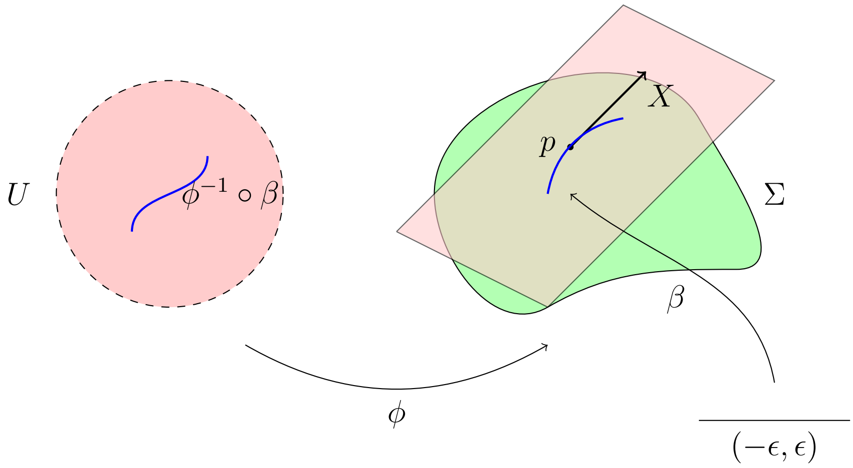
- Take $\mathbf{v} \in \mathbb{R}^2$, and choose $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^2$ such that $\alpha(0) = (u^1, u^2)$ and $\alpha'(0) = \mathbf{v}$.
- By definition, $d\phi_{(u^1,u^2)}(\mathbf{v}) = (\phi \circ \alpha)'(0)$, so $d\phi_{(u^1,u^2)}(\mathbf{v}) \in T_p\Sigma$.



$T_p\Sigma \subset d\phi_{u^1,u^2}(\mathbb{R}^2)$

- Take $X \in T_p\Sigma$, and choose $\beta : (-\epsilon, \epsilon) \to \Sigma$ so that $\beta(0) = p$, $\beta'(0) = X$.
- The local coordinate charts ϕ are diffeomorphisms, so $\phi^{-1} \circ \beta : (-\epsilon, \epsilon) \to U$ are smooth curves.
- $d\phi_{(u^1,u^2)}((\phi^{-1} \circ \beta)'(0)) = (\phi \circ \phi^{-1} \circ \beta)'(0) = \beta'(0)$, so $X \in d\phi_{(u^1,u^2)}(\mathbb{R}^2)$.

$\square$



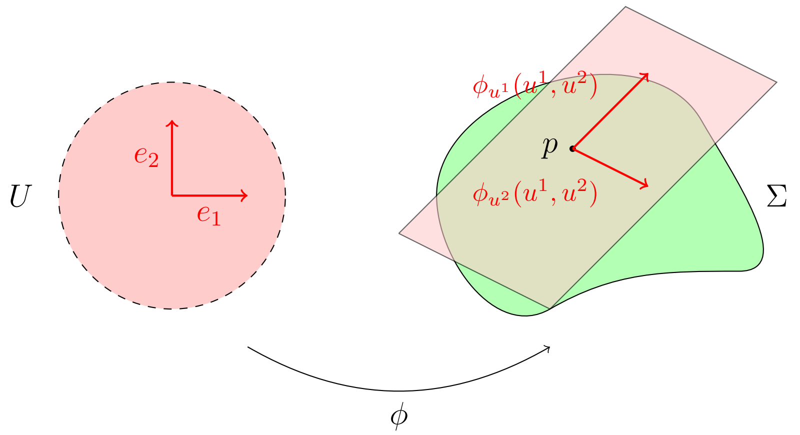
Definition 9. Given a local parameterisation $\phi : U \to \Sigma$ with $\phi(u^1, u^2) = p$, write e_1, e_2 for the standard basis of $\mathbb{R}^2$. Then

$$\phi_{u^1}(u^1, u^2) = \frac{\partial \phi}{\partial u^1}(u^1, u^2) = d\phi_{(u^1,u^2)}(e_1),$$
$$\phi_{u^2}(u^1, u^2) = \frac{\partial \phi}{\partial u^2}(u^1, u^2) = d\phi_{(u^1,u^2)}(e_2)$$

is called the *basis of* $T_p\Sigma$ *associated to* ϕ, and if $X \in T_p\Sigma$ is given by

$$X = a\phi_{u^1}(u^1, u^2) + b\phi_{u^2}(u^1, u^2)$$

we call (a, b) the *coordinates* of X with respect to ϕ.



Take $X \in T_p\Sigma$, and let (a, b) be the coordinates of X with respect to ϕ, $\phi(u_0^1, u_0^2) = p$.

$$X = a\phi_{u^1}(u_0^1, u_0^2) + b\phi_{u^2}(u_0^1, u_0^2)$$

Let $\alpha : (-\epsilon, \epsilon) \to \Sigma$ be a smooth curve with $\alpha(0) = p$, $\alpha'(0) = X$.

Then writing $\phi^{-1} \circ \alpha(t) = (u^1(t), u^2(t))$,

$$\begin{aligned} a\phi_{u^1}(u_0^1, u_0^2) + b\phi_{u^2}(u_0^1, u_0^2) &= \alpha'(0) \\ &= (\phi \circ (\phi^{-1} \circ \alpha))'(0) \\ &= d\phi_{(u_0^1, u_0^2)}({u^1}'(0)e_1 + {u^2}'(0)e_2) \\ &= {u^1}'(0)\phi_{u^1}(u_0^1, u_0^2) + {u^2}'(0)\phi_{u^2}(u_0^1, u_0^2) \end{aligned}$$

so

$$(a, b) = ({u^1}'(0), {u^2}'(0)),$$

i.e.

$$X = {u^1}'(0)\phi_{u^1}(u_0^1, u_0^2) + {u^2}'(0)\phi_{u^2}(u_0^1, u_0^2).$$

Active Learning

Question 10. Let $\phi : U \subset \mathbb{R}^2 \to \Sigma \subset \mathbb{R}^3$ be a local parameterisation of a regular surface Σ, and for $(u_0^1, u_0^2) \in U$, consider

$$d\phi_{(u_0^1, u_0^2)} : \mathbb{R}^2 \to T_{\phi(u_0^1, u_0^2)}\Sigma.$$

What is the matrix of $d\phi_{(u_0^1, u_0^2)}$ with respect to

- the standard basis e_1, e_2 on $\mathbb{R}^2$;
- the basis $\phi_{u^1}(u_0^1, u_0^2), \phi_{u^2}(u_0^1, u_0^2)$ of $T_{\phi(u_0^1, u_0^2)}\Sigma$?

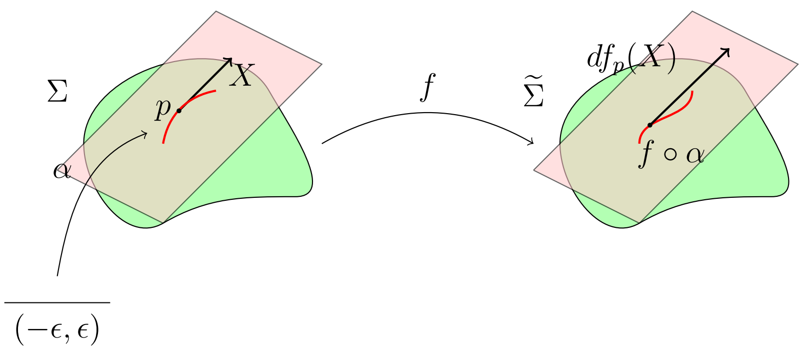
Differential of a smooth map

Definition 12. Let $f : \Sigma \to \widetilde{\Sigma}$ be a smooth map with $f(p) = \widetilde{p}$. The *differential* df_p *of* f *at* p is a linear map

$$df_p : T_p\Sigma \to T_{\widetilde{p}}\widetilde{\Sigma};$$

to define $df_p(X)$, take a smooth curve $\alpha : (-\epsilon, \epsilon) \to \Sigma$ with $\alpha(0) = p$ and $\alpha'(0) = X$ and set

$$df_p(X) = (f \circ \alpha)'(0).$$



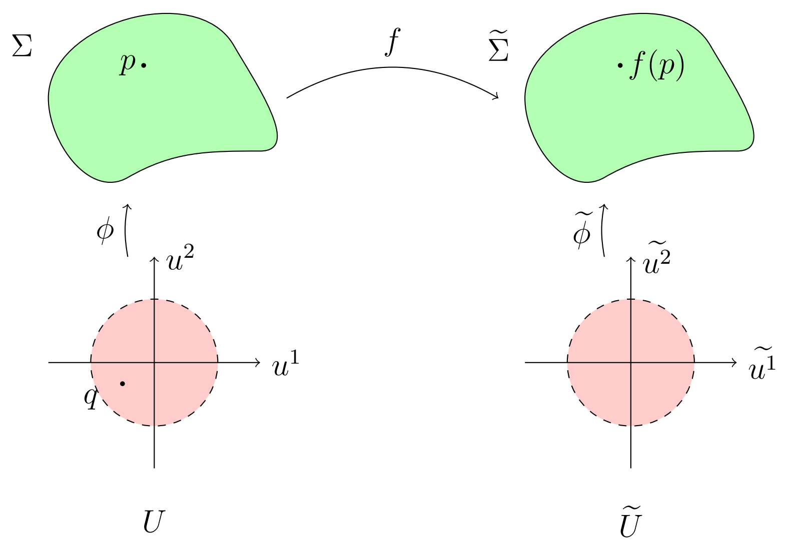
Proposition 13. *Let $f : \Sigma \to \widetilde{\Sigma}$ be a smooth map and take $p \in \Sigma$.*

1. *The differential df_p defined above is a well-defined linear map.*

2. *Let $\phi, \widetilde{\phi}$ be local parameterisations of $\Sigma, \widetilde{\Sigma}$ about $p, f(p)$. Writing $\phi^{-1} = (u^1, u^2)$, $\widetilde{\phi}^{-1} = (\widetilde{u}^1, \widetilde{u}^2)$ and $\widetilde{\phi}^{-1} \circ f = (f^1, f^2)$, the matrix of df_p with respect to the bases $(\phi)_{u^1}$ and $(\widetilde{\phi})_{\widetilde{u}^1}$ is*

$$\begin{pmatrix} \dfrac{\partial f^1}{\partial u^1}(q) & \dfrac{\partial f^1}{\partial u^2}(q) \\ \dfrac{\partial f^2}{\partial u^1}(q) & \dfrac{\partial f^2}{\partial u^2}(q) \end{pmatrix}$$

   where $q = \phi^{-1}(p)$.

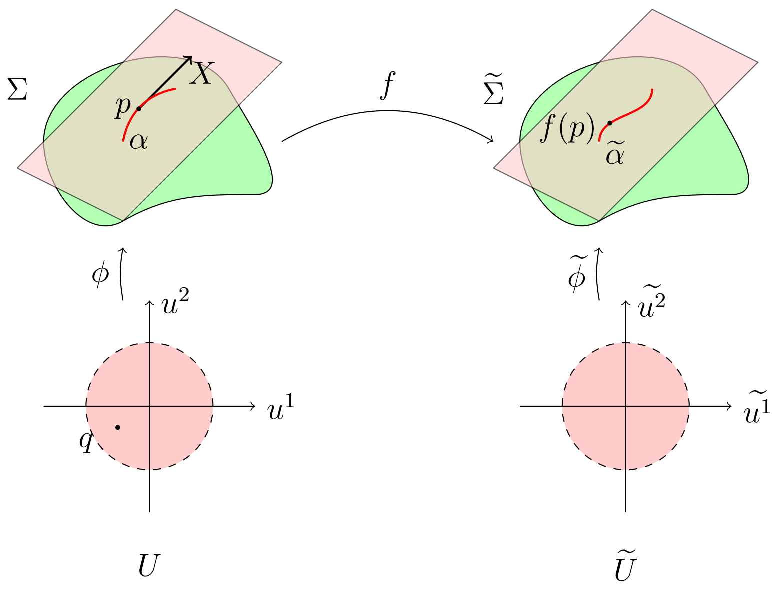


Proof: The first statement follows from the second. Take $X \in T_p\Sigma$, and write

$$X = (u^1)'\phi_{u^1}(q) + (u^2)'\phi_{u^2}(q).$$

Let $\alpha : (-\epsilon, \epsilon) \to \Sigma$ be a smooth curve with $\alpha(0) = p$ and $\alpha'(0) = X$.

Then $\widetilde{\alpha} = f \circ \alpha$ is a smooth curve in $\widetilde{\Sigma}$.



By the chain rule

$$\begin{aligned}(\widetilde{\phi}^{-1} \circ f \circ \alpha)'(0) &= \frac{df^1}{dt}(0)\widetilde{e}_1 + \frac{df^2}{dt}(0)\widetilde{e}_2 \\ &= \left(\frac{\partial f^1}{\partial u^1}(q)\frac{du^1}{dt}(0) + \frac{\partial f^1}{\partial u^2}(q)\frac{du^2}{dt}(0)\right)\widetilde{e}_1 \\ &\quad + \left(\frac{\partial f^2}{\partial u^1}(q)\frac{du^1}{dt}(0) + \frac{\partial f^2}{\partial u^2}(q)\frac{du^2}{dt}(0)\right)\widetilde{e}_2.\end{aligned}$$

Hence

$$\begin{aligned}(f \circ \alpha)'(0) &= \left(\frac{\partial f^1}{\partial u^1}(q)\frac{du^1}{dt}(0) + \frac{\partial f^1}{\partial u^2}(q)\frac{du^2}{dt}(0)\right)\widetilde{\phi}_{\widetilde{u}^1} \\ &\quad + \left(\frac{\partial f^2}{\partial u^1}(q)\frac{du^1}{dt}(0) + \frac{\partial f^2}{\partial u^2}(q)\frac{du^2}{dt}(0)\right)\widetilde{\phi}_{\widetilde{u}^2}\end{aligned}$$

so the matrix of df_p is as claimed.

Observe the coordinates of $(f \circ \alpha)'(0)$ with respect to $\widetilde{\phi}$ depend only on the coordinates of $\alpha'(0)$ with respect to ϕ, and hence only on $X = \alpha'(0)$, not on the choice of α.