MATH3968 – Lecture 9 Riemannian metrics

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Definition 1. A *parametrised surface* is a smooth map $\phi : U \to \mathbb{R}^3$ where $U \subset \mathbb{R}^2$ is open.

If the differential $d\phi_{(u^1,u^2)}$ is not one-to-one (i.e., has rank < 2), we say that (u^1, u^2) is a *singular point* of ϕ.

If the differential $d\phi_{(u^1,u^2)}$ is one-to-one (i.e., has rank 2), we say that (u^1, u^2) is a *regular point* of ϕ.

The parametrised surface is *regular* if all $(u^1, u^2) \in U$ are regular points of ϕ.

Notice that we have NOT required that the map ϕ be one-to-one.

Example 2.

$$\begin{array}{cccc} \phi : & \mathbb{R}^2 & \to & \mathbb{R}^3 \\ & (u,v) & \mapsto & (u^2 - 1,\ v,\ u(u^2-1)) \end{array}$$

The differential at (u, v) is

$$d\phi_{(u,v)} = \begin{pmatrix} 2u & 0 \\ 0 & 1 \\ 3u^2 - 1 & 0 \end{pmatrix}$$

Since $2u$ and $3u^2 - 1$ cannot both be zero,

$$d\phi_{(u,v)} : \mathbb{R}^2 \to \mathbb{R}^3$$

is for each $(u, v) \in \mathbb{R}^2$ a one-to-one linear mapping.

Example 2 (continued). However, the trace of this regular parametrised surface is NOT a regular surface.

ϕ is not one-to-one, since

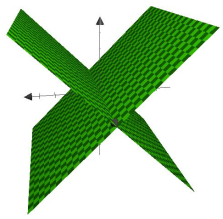
$$((u^1)^2 - 1,\ v^1,\ u^1((u^1)^2 - 1)) = ((u^2)^2 - 1,\ v^2,\ u^2((u^2)^2 - 1))$$

has solution $(u^1, v^1) = (1, k)$, $\quad (u^2, v^2) = (-1, k)$, $k \in \mathbb{R}$.

For each $v \in \mathbb{R}$, there is no open neighbourhood V of $\phi(1, v) = (0, v, 0)$ in $\mathbb{R}^3$ such that $V \cap \phi(\mathbb{R}^2)$ can be parametrised by a coordinate chart – otherwise $T_{(0,v,0)}(\phi(\mathbb{R}^2))$ would exist and be a 2-dimensional linear subspace of $\mathbb{R}^3$.

Example 2 (continued). However we see that the space of velocity vectors to curves through $(0, v, 0)$ is given by the plane $z = x$, which is the image of $d\phi_{(1,k)}$ and the plane $z = -x$, which is the image of $d\phi_{(-1,k)}$, since the respective matrices are

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}.$$



Example 3 (Torus). Recall that we made the torus T^2:

$$z^2 + \left(\sqrt{x^2 + y^2} - a\right)^2 = b, \quad b < a$$

into a regular surface by defining

$$\phi : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3$$
$$(u, v) \mapsto ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v)$$

as one local coordinate; together with functions given by the same formula but on different domains

$$\psi : \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \to \mathbb{R}^3$$
$$\varphi : \left(\frac{\pi}{2}, \frac{5\pi}{2}\right) \times \left(\frac{\pi}{2}, \frac{5\pi}{2}\right) \to \mathbb{R}^3$$

we have an atlas for the torus.

Example 3 (continued). Define $f : T^2 \to T^2$ to be the reflection in the yz-plane, namely $f(x, y, z) = (-x, y, z)$.

1. Describe $df_{(x,y,z)}$.

2. Calculate the matrix of the differential $df_{(-\frac{a+b}{\sqrt{2}}, \frac{a+b}{\sqrt{2}}, 0)}$ with respect to the parameterisation ψ near $(\frac{a+b}{\sqrt{2}}, \frac{a+b}{\sqrt{2}}, 0)$ and $(-\frac{a+b}{\sqrt{2}}, -\frac{a+b}{\sqrt{2}}, 0)$.

Example 3 (continued). 1. Take $X \in T_{(x,y,z)}$ and let $\alpha : (-\epsilon, \epsilon) \to T^2$ be a smooth curve with $g(0) = (x, y, z)$, $\alpha'(0) = X$.

Let $R_x : \mathbb{R}^3 \to \mathbb{R}^3$ denote reflection in the yz-plane.

$$\begin{aligned} df_{(x,y,z)}(X) &= (f \circ \alpha)'(0) \\ &= \frac{d}{dt}\left(R_x \circ \alpha(t)\right)\big|_{t=0} \\ &= R_x(\alpha'(0)) \\ &= R_x(X). \end{aligned}$$

Example 3 (continued). 2. $(\frac{a+b}{\sqrt{2}}, \frac{a+b}{\sqrt{2}}, 0) = \psi(\frac{\pi}{4}, 0)$ and $(-\frac{a+b}{\sqrt{2}}, \frac{a+b}{\sqrt{2}}, 0) = \psi(\frac{3\pi}{4}, 0)$

Near $(\frac{a+b}{\sqrt{2}}, \frac{a+b}{\sqrt{2}}, 0)$,

$$\psi^{-1} \circ R_x \circ \psi(u, v) = (\pi - u, v)$$

which has differential

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Relevant Linear Algebra

Let V be a vector space over the real numbers.

Definition 4. A *bilinear form* B on V is a map $B : V \times V \to \mathbb{R}$ which is linear in each component, i.e.

1. $B(a_1v^1 + a_2v^2, w) = a_1B(v^1, w) + a_2B(v^2, w)$ for $a_1, a_2 \in \mathbb{R}$, $v^1, v^2, w \in V$, and
2. $B(v, a_1w_1 + a_2w_2) = a_1B(v, w_1) + a_2B(v, w_2)$ for $a_1, a_2 \in \mathbb{R}$, $v, w_1, w_2 \in v$.

Definition 5. The bilinear form B is *symmetric* if $B(v, w) = B(w, v)$ for all $v, w \in V$.

Definition 6. A bilinear form B on V is *positive definite* if $B(v, v) \geq 0$ for all $v \in V$, with equality if and only if $v = 0$.

Definition 7. A *inner product* on V is a positive definite symmetric bilinear form.

Definition 8. A *quadratic form* on an n-dimensional real vector space V is given by a homogeneous polynomial of degree 2 satisfying an additional symmetry condition.

We shall view them as maps:

Definition 9. A *quadratic form* on V is a map $Q : V \to \mathbb{R}$ such that

1. $Q(av) = a^2Q(v)$ for all $a \in \mathbb{R}$ and $v \in V$, and
2. the map $B : V \times V \to \mathbb{R}$ defined by $B(v, w) = \frac{1}{2}(Q(v + w) - Q(v) - Q(w))$ is a (symmetric) bilinear form.

Conversely, a symmetric bilinear form B on V defines a quadratic form Q via

$$Q(v) = B(v, v).$$

Assume V is n-dimensional with basis $e_1, \dots, e_n$, and write $v \in V$ as $v = \sum_{i=1}^n v^i e_i$.

A bilinear form B is represented with respect to this basis by a matrix A, where

$$B(v, w) = [v^1, \dots, v^n] \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix}.$$

B is symmetric if and only if the matrix A is symmetric: $A^t = A$.

The associated quadratic form Q is represented by the same matrix

$$Q(v) = [v^1, \dots, v^n] \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

Note that $Q(v)$ is given by a homogeneous polynomial

$$Q(v) = \sum_{i,j=1}^n a_{ij} v^i v^j$$

of degree 2 in the coefficients v^i.

We could alternatively define a quadratic form on V to be a map $Q : V \to \mathbb{R}$ such that if $e_1, \dots, e_n$ is a basis of V and we write $v \in V$ as $v = \sum_{i=1}^n v^i e_i$ then

$$Q(v) = \sum_{i,j=1}^n a_{ij} v^i v^j$$

to for some a_{ij} (independent of v) satisfying $a_{ij} = a_{ji}$.

Riemannian Metric

Let Σ be a regular surface, and

$$\begin{array}{rccc} \phi : & U & \to & \mathbb{R}^3 \\ & (u^1, u^2) & \mapsto & \phi(u^1, u^2) \end{array}$$

a local parameterisation near $p \in \Sigma$.

Notation

Write

$$\boldsymbol{E_1}(p) = \frac{\partial \phi}{\partial u^1}(\phi^{-1}(p)), \quad \boldsymbol{E_2}(p) = \frac{\partial \phi}{\partial u^2}(\phi^{-1}(p)).$$

The restriction $\langle \cdot, \cdot \rangle_p$ of the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ to $T_p\Sigma \subset \mathbb{R}^3$ varies smoothly with p in the sense that the

$$g_{ij}(p) = \langle E_i(p), E_j(p) \rangle = E_i(p) \cdot E_j(p)$$

are smooth functions $U \to \mathbb{R}$ for every coordinate neighbourhood $U \subset \Sigma$.

We denote this inner product also by $g(p)(X, Y)$ or $X \cdot Y$, $X, Y \in T_p\Sigma$ and frequently omit the p.

Definition 10. We call the smoothly varying inner product $\langle \cdot, \cdot \rangle_p$ a *Riemannian metric* on Σ.

We shall often simply write $\langle \cdot, \cdot \rangle$.

Warning: A better name would be “Riemannian inner product”. The word metric is traditional, but don’t think of metric topology!

The 2×2 matrix

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

is symmetric and for $X = X^1\boldsymbol{E_1} + X^2\boldsymbol{E_2}, Y = Y^1\boldsymbol{E_1} + Y^2\boldsymbol{E_2} \in T_p\Sigma$, defines a smoothly varying inner product on the tangent spaces of Σ by

$$g(X, Y) = \langle X, Y \rangle = (X^1, X^2) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} = \sum_{ij} g_{ij} X_i Y_j$$

Definition 11. The associated quadratic form on $T_p\Sigma$ is denoted I_p and is called the *first fundamental form of the regular surface* Σ *at* p. The smoothly varying quadratic form I is called the *first fundamental form* of Σ.

The functions $g_{11}, g_{12} = g_{21}, g_{22} : \Sigma \to \mathbb{R}$ are called the *coefficients* of the first fundamental form.

What is the point?

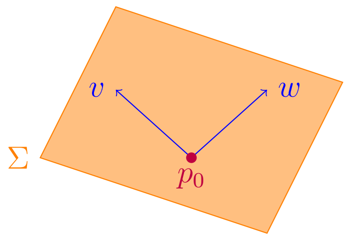
The Riemannian metric (smoothly varying inner product) $g = \langle \cdot, \cdot \rangle$ and the first fundamental form (smoothly varying quadratic form) are equivalent.

Geometrically, they give us a notion of length and angle in every tangent plane.

This enables us to define area.

It also enables us to distinguish between different "geometries" (more later).

Example 12. Let Σ be the plane in $\mathbb{R}^3$ through the point p_0 and containing the orthonormal vectors v, w.



Find the coefficients of the first fundamental form with respect to the global parameterisation $\phi(u^1, u^2) = p_0 + u^1 v + u^2 w$. For any $p \in \Sigma$, recall the notation

$$\boldsymbol{E_1}(p) = \frac{\partial \phi}{\partial u^1}(\phi^{-1}(p)), \;\; \boldsymbol{E_2}(p) = \frac{\partial \phi}{\partial u^2}(\phi^{-1}(p)).$$

Example 12 (continued).

$$\boldsymbol{E_1}(p) = v, \; \boldsymbol{E_2}(p) = w$$

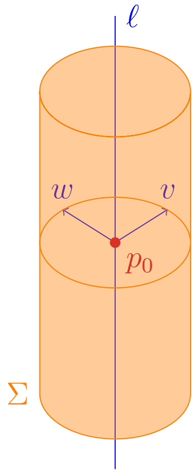
so since these are orthonormal,

$$\begin{aligned} g_{11}(\phi(u^1, u^2)) &= \langle \boldsymbol{E_1}, \boldsymbol{E_1} \rangle = 1, \\ g_{12}(\phi(u^1, u^2)) &= \langle \boldsymbol{E_1}, \boldsymbol{E_2} \rangle = 0 = \langle \boldsymbol{E_2}, \boldsymbol{E_1} \rangle = g_{21}(\phi(u^1, u^2)), \\ g_{22}(\phi(u^1, u^2)) &= \langle \boldsymbol{E_2}, \boldsymbol{E_2} \rangle = 1 \end{aligned}$$

If the coordinate chart ϕ is understood, we may write $\boldsymbol{E_1}, \boldsymbol{E_2}, g_{ij}$ directly as functions of (u^1, u^2).

Active Learning

Question 13. Let $v, w \in \mathbb{R}^3$ be orthonormal, ℓ be the line with direction vector $v \times w$ through the point p_0, and Σ be the cylinder of radius 1 about the line ℓ.



Question 13 (continued)**.** The local parameterisations

$$\begin{aligned}\phi : (0, 2\pi) \times \mathbb{R} &\to \Sigma \\ (u^1, u^2) &\mapsto p_0 + \cos(u^1)v + \sin(u^1)w + u^2(v \times w)\end{aligned}$$

and

$$\begin{aligned}\psi : (-\pi, \pi) \times \mathbb{R} &\to \Sigma \\ (u^1, u^2) &\mapsto p_0 + \cos(u^1)v + \sin(u^1)w + u^2(v \times w)\end{aligned}$$

give Σ the structure of a regular surface.

Compute the coefficients of the first fundamental form with respect to these parameterisations.

The geometric "reason" why we can find local parameterisations of the plane and the cylinder with the same g_{ij} is the fact that locally we can transform one into the other without any "stretching".