Texts and Readings in Mathematics

S. Kesavan

Functional Analysis

Second Edition

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Preface

Functional analysis studies linear spaces provided with suitable topological structures and (continuous) linear transformations between such spaces. It has far reaching applications in several disciplines. For instance, the modern theory of (partial) differential equations and the numerical approximation of their solutions rely a lot on functional analytic techniques.

Functional analysis is now an integral part of the curriculum in any post graduate course in Mathematics. Ideally it should be taught after having covered courses in linear algebra, real and complex analysis, and topology. An introduction to the theory of measure and integration is also helpful since the richest examples in functional analysis come from function spaces whose study demands such a knowledge.

The present book grew out of notes prepared by myself while lecturing to graduate students at the Tata Institute of Fundamental Research (Bangalore Centre) and the Institute of Mathematical Sciences, Chennai. The material presented in this book is standard and is ideally suited for a course which can be followed by masters students who have covered the necessary prerequisites mentioned earlier. While covering all the standard material, I have also tried to illustrate the use of various theorems via examples taken from differential equations and the calculus of variations, either through brief sections or through the exercises. In fact, this book is well suited for students who would like to pursue a research career in the applications of mathematics. In particular, familiarity with the material presented in this book will facilitate studying my earlier book, published nearly two decades ago, 'Topics in Functional Analysis and Applications' (Wiley Eastern, now called New Age International), which serves as a functional analytic introduction to the theory of partial differential equations.

Chapter 1 gives a rapid revision of linear algebra, topology and measure theory. Important definitions, examples and results are recalled and no proofs are given. At the end of each section, the reader is referred to a standard text on that topic. This chapter has been included only for reference purposes and it is not intended that it be covered in a course based on this book.

Chapter 2 introduces the notion of a normed linear space and that of continuous linear transformations between such spaces.

Chapter 3 studies the analytic and geometric versions of the Hahn-Banach theorem and presents some applications of these.

Chapter 4 is devoted to the famous 'trinity' in functional analysis—the Banach-Steinhaus, the open mapping and the closed graph theorems, which are all consequences of Baire's theorem for complete metric spaces. Several applications are discussed. The notion of an 'unbounded linear mapping' is introduced.

In my opinion, most texts do not emphasize the importance of weak topologies in a course on functional analysis. That a bounded sequence in a reflexive space admits a weakly convergent sequence is the corner stone of many an existence proof in the theory of (partial) differential equations. These topologies also provide nice counterexamples to show the inadequacy of sequences in a general topological space. For instance, we will see that two topologies on a set could be different while having the same convergent sequences. We will also see an example of a compact topological space in which a sequence does not have any convergent subsequence. Chapter 5 deals with weak and weak* topologies and their applications to the notions of reflexivity, separability and uniform convexity.

Chapter 6 introduces the Lebesgue spaces and also presents the theory of one of the simplest classes of Sobolev spaces. Chapter 7 is devoted to the study of Hilbert spaces. Chapter 8 studies compact operators and their spectra.

Much of the fun in learning mathematics comes from actually doing it! Every chapter from Chapters 2 through 8 has a fairly large collection of exercises at the end. These illustrate the results of the text, show the optimality of the hypotheses of the various theorems via various examples or counterexamples, or develop simple versions of theories not elaborated upon in the text. They are of varying degrees of difficulty. Occasionally, some hints for the solution are provided. It is hoped that the students will benefit by solving them.

Since this is meant to be a first course on functional analysis, I have kept the bibliographic references to a minimum, merely citing important texts where the reader may find further details of the topics covered in this book.

No claim of originality is made with respect to the material presented in this book. My treatment of this subject has been influenced by the writings of authors like Simmons (whose book was my first introduction to Functional Analysis) and Rudin. These works figure in the bibliography of this book. I would also like to mention a charming book, hardly known outside the francophonic mathematical community, viz. 'Analyse Fonctionnelle' by H. Brézis.

The preparation of this manuscript would not have been possible but for the excellent facilities provided by the Institute of Mathematical Sciences, Chennai, and I wish to place on record my gratitude. I also thank the Hindustan Book Agency and the editor of the TRIM Series, Prof. R. Bhatia, for their kind cooperation in bringing out this volume. I must thank the anonymous referees who painstakingly went through the first draft of the manuscript. I have tried to incorporate many of their constructive suggestions in the current version.

Finally, I thank my family for its constant support and encouragement.

Chennai, India S. Kesavan
May 2008

Preface to the Second Edition

It is now more than a decade since the first edition of this book appeared and I am gratified by the reception accorded to it by mathematics students and teachers in this country and also in many places abroad. The book has been used as a text for Functional Analysis courses in many institutions and also as the main reference text for several refresher courses on this subject.

I myself have used it to teach a course on Functional Analysis in institutions like the Chennai Mathematical Institute (CMI) and the Indian Institute of Technology Madras (IITM). I have also used it to deliver a course on this subject under the NPTEL banner. While giving these lectures, I have faced questions from students, prepared exercises for weekly assignments and examinations and have also improved my own appreciation of the finer points of the subject. These factors encouraged me to prepare this revised version of the book, and I hope it will be more useful and user-friendly than the first edition.

In the present (second) edition, I have completely overhauled the presentation without changing the basic structure of the book. The statements of results, definitions and remarks have been modified wherever necessary, and many proofs have been rewritten, in view of greater clarity of the exposition. Some examples have been added to illustrate the results proved in the text. Several exercises have been added to the existing collection in each chapter. It is hoped that students will have fun solving them.

Section 3.1, which originally dealt with the Hahn-Banach theorem and introduced the notion of reflexivity, has now been bifurcated into two sections. In the new framework, Sect. 3.1 deals with the Hahn-Banach theorem (extension version) and its immediate consequences. Section 3.2 is devoted to reflexivity. Several examples of continuous linear functionals, which do not realize their norm on the unit sphere, are presented, thereby proving the non-reflexivity of many familiar Banach spaces. Similarly, Sect. 8.3, which originally introduced the notion of the spectrum of an operator and described that of a compact operator, has been bifurcated into two sections. Section 8.3 introduces the notion of the spectrum. Several examples are presented, and the spectra of some special operators on a Hilbert space are characterized. Section 8.4 deals with the spectrum of a compact operator.

The final preparation of the manuscript was done while I was working from home, partly due to the COVID-19 pandemic, and this was possible because of the help rendered by the system administration team of the Institute of Mathematical Sciences (IMSc). In particular, I thank Mr. G. S. Vasan for his help. I also thank my colleagues Profs. Amritanshu Prasad and S. Viswanath for introducing me to the joys of teaching on the NPTEL platform. I use this opportunity to thank the NPTEL team at IIT Madras, headed by Ms. Bharathi and ably assisted by Ms. Lakshmi Priya, for their support and help when preparing my NPTEL courses. Finally, I thank Shri J. K. Jain of the Hindustan Book Agency and Prof. Rajendra Bhatia, the chief editor of the TRIM Series, for encouraging me to bring out this new edition.

Chennai, India S. Kesavan
April 2022

Notations

Certain general conventions followed throughout the text regarding notations are described below. All other specific notations are explained as and when they appear in the text.

- The set of natural numbers is denoted by the symbol $\mathbb{N}$, the integers by $\mathbb{Z}$, the rationals by $\mathbb{Q}$, the reals by $\mathbb{R}$ and the complex numbers by $\mathbb{C}$.
- Sets (including vector spaces and their subspaces) and also linear transformations between vector spaces are denoted by uppercase Latin letters.
- Elements of sets (and, therefore, vectors as well) are denoted by lowercase Latin letters.
- Scalars are denoted by lowercase Greek letters.
- To distinguish between the scalar zero and the null (or zero) vector, the latter is denoted by the zero in boldface, i.e. $\mathbf{0}$. This is also used to denote the zero linear transformation and the zero linear functional.
- Column vectors in Euclidean space are denoted by lowercase Latin letters in boldface, and matrices are denoted by uppercase Latin letters in boldface.
- Elements of L^p spaces (cf. Chap. 6) are equivalence classes of functions under the equivalence relation of 'equality almost everywhere'. To emphasize this fact, all elements of L^p spaces (and hence of Sobolev spaces as well) are denoted by lowercase Latin letters in the san serif font. A generic representative of that equivalence class is denoted by the same lowercase Latin letter (in italics). Thus, if $\mathsf{f} \in L^1(0, 1)$, a generic representative of this class will be denoted by f and will feature in all computations involving this element. For instance, we have

$$\|\mathsf{f}\|_1 = \int_0^1 |f(t)|\, dt.$$

- The norm in a normed linear space V will be denoted by $\|.\|$, or by $\|.\|_V$, if we wish to distinguish it from other norms that may be entering the argument. Similarly, the inner product in a Hilbert space H will, in general, be denoted by $(.,.)$ (or by $(.,.)_H$ in case we wish to stress the role played by H).

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S. Kesavan is Adjunct Professor at the Indian Institute of Technology Madras, Chennai, India. He received his Docteur-es-Sciences Mathematiques from the Universite Pierre et Marie Curie (Paris VI) in 1979 for the thesis entitled "Sur l'approximation de problèmes lineaires et nonlineaires de valeurs propres", supervised by Profs. J. L. Lions and P. G. Ciarlet. Earlier, he was a Deputy Director of Chennai Mathematical Institute (CMI) during 2007–2010; TIFR Centre for Applicable Mathematics, Bengaluru, India. His research areas include partial differential equations, homogenization, control theory and isoperimetric inequalities.

Author of five books, Prof. Kesavan has published more than 50 papers in national and international journals and several contributions to conference proceedings. He was elected Vice-President of the Ramanujan Mathematical Society in 2016 (served from 2016–2019); Fellow of the Indian Academy of Sciences, Bengaluru; Fellow of the National Academy of Sciences, India; Member of the National Board of Higher Mathematics (NBHM) (from 2000–2019); and Secretary, Grants Selection; Commission for Developing Countries, International Mathematical Union (from 2011–2018). He is Recipient of the C. L. Chandna Award for Outstanding Contributions to Mathematics Research and Teaching (1999) and Tamil Nadu Scientist Award, awarded by the Tamil Nadu State Council for Science and Technology, in Mathematical Sciences (1998).

Chapter 1
Preliminaries

1.1 Linear Spaces

Functional analysis is the study of vector spaces endowed with topological structures (that are compatible with the linear structure of the space) and of (linear) mappings between such spaces. Throughout this book we will be working with vector spaces whose underlying field is the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

For completeness, we will recall some basic definitions and some important results.

Definition 1.1.1 A **vector space** or a **linear space** over a field $\mathbb{F}$ (whose elements are called **scalars**) is a set V, whose elements are called **vectors**, on which two operations—**addition** and **scalar multiplication**—are defined such that the following properties hold:
Addition: $(x, y) \in V \times V \mapsto x + y \in V$ such that

(i) (commutativity) for all x and y in V, we have

$$x + y = y + x;$$

(ii) (associativity) for all x, y and z in V, we have

$$x + (y + z) = (x + y) + z;$$

(iii) there exists a unique vector $\mathbf{0} \in V$, called the **zero** or the **null vector**, such that, for every $x \in V$,

$$x + \mathbf{0} = x;$$

(iv) for every $x \in V$, there exists a unique vector $-x \in V$ such that

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$$x + (-x) = \mathbf{0}.$$

Scalar multiplication: $(\alpha, x) \in \mathbb{F} \times V \mapsto \alpha x \in V$ such that

(v) for every $x \in V$, $1x = x$ where 1 is the multiplicative identity in $\mathbb{F}$;

(vi) for all α and β in $\mathbb{F}$ and for every $x \in V$,

$$\alpha(\beta x) = (\alpha\beta)x;$$

(vii) for all α and β in $\mathbb{F}$ and for every $x \in V$,

$$(\alpha + \beta)x = \alpha x + \beta x;$$

(viii) for every $\alpha \in \mathbb{F}$ and for all x and y in V,

$$\alpha(x + y) = \alpha x + \alpha y.$$ ■

Remark 1.1.1 The conditions (i)–(iv) above imply that V is an abelian group with respect to vector addition. The conditions (vii) and (viii) above are known as the *distributive laws*. ■

Example 1.1.1 Let $N \geq 1$ be a positive integer. Define

$$\mathbb{R}^N = \{x = (x_1, \ldots, x_N) \ |x_i \in \mathbb{R} \text{ for all } 1 \leq i \leq N\}.$$

We define addition and scalar multiplication componentwise, *i.e.* if $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ are elements of $\mathbb{R}^N$ and if $\alpha \in \mathbb{R}$, we define

$$x + y = (x_1 + y_1, \ldots, x_N + y_N)$$

and

$$\alpha x = (\alpha x_1, \ldots, \alpha x_N).$$

It is now easy to see that $\mathbb{R}^N$ is a vector space over $\mathbb{R}$ with the zero vector being that element in $\mathbb{R}^N$ with all its components zero. In the same way, we can define $\mathbb{C}^N$ as a vector space over $\mathbb{C}$.

Setting $N = 1$, we see that $\mathbb{R}$ (respectively $\mathbb{C}$) is a vector space over itself, the scalar multiplication being the usual multiplication operation. ■

Definition 1.1.2 Let V be a vector space and let $W \subset V$. Then W is said to be a **subspace** of V if W is a vector space in its own right for the same operations of addition and scalar multiplication. ■

Definition 1.1.3 Let V be a vector space and let $x_1, \ldots, x_n$ be vectors in V. A **linear combination** of these vectors is any vector of the form $\alpha_1 x_1 + \cdots + \alpha_n x_n$, where the α_i, $1 \leq i \leq n$ are scalars. A **linear relation** between these vectors is an equation of the form

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = \mathbf{0}. \qquad \blacksquare$$

Definition 1.1.4 A finite set of vectors in a vector space is said to be **linearly independent** if there does not exist any linear relation between them other than the trivial one, i.e. when all the scalar coefficients are zero. If there exists a non-trivial linear relation, then the set of vectors is said to be **linearly dependent**. An infinite set of vectors is said to be linearly independent if there does not exist any finite linear relation amongst vectors in that set. ■

Given a vector space V and a set S of vectors, the collection of all finite linear combinations of vectors from S will form a subspace of V. In fact, it is clear that this subspace is the smallest subspace containing S and is called the *linear span* of the set S or the *subspace generated by* S. This subspace is denoted by $\text{span}\{S\}$.

Definition 1.1.5 A maximal linearly independent subset of a vector space is called a **basis**. ■

In other words, a basis is a linearly independent subset such that, if any other vector is adjoined to the set, the enlarged set becomes linearly dependent. This implies that every vector in the space can be expressed as a (finite) linear combination of the members of the basis. Thus, the vector space is generated by its basis.

Proposition 1.1.1 (i) *Every vector space has a basis;*
(ii) *any two bases of a vector space have the same cardinality.* ■

The above proposition leads us to the following definition.

Definition 1.1.6 A vector space V is said to be **finite dimensional** if it admits a basis with a finite number of elements. Otherwise, it is said to be **infinite dimensional**. The **dimension** of a vector space is the number of elements in a basis, if it is finite dimensional, and infinity if it is infinite dimensional and is denoted $\dim(V)$. ■

Example 1.1.2 The space $\mathbb{R}^N$ (respectively, $\mathbb{C}^N$) has a basis which is defined as follows. Let $1 \leq i \leq N$. Let $\mathbf{e}_i$ be the vector whose ith component is unity and all other components are zero. Then $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ is a basis for $\mathbb{R}^N$ (respectively, $\mathbb{C}^N$) and is called the **standard basis**. ■

Example 1.1.3 Let $\mathcal{P}$ denote the collection of all polynomials in one variable with real coefficients. Let $\mathsf{p}(x) = \sum_{i=0}^{n} a_i x^i$ and $\mathsf{q}(x) = \sum_{i=0}^{m} b_i x^i$ where the a_i and b_i are real numbers and x is the variable. Let $\alpha \in \mathbb{R}$. Assume, without loss of generality, that $m \leq n$. Define

$$(\mathsf{p} + \mathsf{q})(x) = \sum_{i=0}^{n} (a_i + b_i) x^i$$

where we set $b_i = 0$ for $m < i \leq n$ if $m < n$. Define

$$(\alpha \mathsf{p})(x) = \sum_{i=0}^{n} \alpha a_i x^i.$$

With these operations, $\mathcal{P}$ becomes a vector space over $\mathbb{R}$. It is easy to check that the collection of monomials $\{\mathsf{p}_i\}_{i=0}^{\infty}$ where $\mathsf{p}_0(x) \equiv 1$ and $\mathsf{p}_i(x) = x^i$ for $i > 1$ forms a basis for $\mathcal{P}$. Thus, $\mathcal{P}$ is an infinite dimensional vector space. ■

We will come across numerous examples of infinite dimensional vector spaces in the sequel (cf. Sect. 2.2, for instance).

Let V be a vector space and let W_i, $1 \leq i \leq n$ be subspaces. The span of the W_i is the subspace of all vectors of the form

$$v = w_1 + \cdots + w_n$$

where $w_i \in W_i$ for each $1 \leq i \leq n$. The spaces are said to be independent if an element in the span is zero if, and only if each, $w_i = \mathbf{0}$. In particular, it follows that, if the W_i are independent, then for all $1 \leq i, j \leq n$ such that $i \neq j$, we have $W_i \cap W_j = \{\mathbf{0}\}$. Further, every element in the span will have a *unique* decomposition into vectors from the spaces W_i.

Definition 1.1.7 Let V be a vector space and let W_i, $1 \leq i \leq n$ be subspaces. Then, V is said to be the **direct sum** of the W_i if the spaces W_i are independent and their span is the space V. In this case we write

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_n = \oplus_{i=1}^{n} W_i. \quad ■$$

We now study mappings between vector spaces which preserve the linear structure.

Definition 1.1.8 Let V and W be vector spaces (over the same base field). A **linear transformation**, or **linear operator**, is a mapping $T : V \to W$ such that for all x and $y \in V$ and for all scalars α and β, we have

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

If W is the base field (which is a vector space over itself), then a linear transformation from V into W is called a **linear functional** on V. ■

Definition 1.1.9 Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. The image of T is a subspace of W and is called the **range** of T. The dimension of the range is called the **rank** of T. The set

$$\{x \in V | T(x) = \mathbf{0}\}$$

is a subspace of V and is called the **null space** or **kernel** of T. ■

Definition 1.1.10 Let V and W be vector spaces and T a linear transformation between them. The transformation is said to be **invertible** if T is a bijection. ■

It is easy to see that a linear transformation which is an injection maps a linearly independent set onto a linearly independent set. In particular, if $T : V \to W$ is an injection, then, necessarily, $\dim(V) \leq \dim(W)$. On the other hand, if $T : V \to W$ is a surjection, clearly, $\dim(V) \geq \dim(W)$. Thus, if T is invertible, then the two spaces must have the same dimension.

We now focus our attention on finite dimensional spaces. Let V be a space of dimension n with basis $\{v_1, \ldots, v_n\}$ and W a space of dimension m with basis $\{w_1, \ldots, w_m\}$. A linear map $T : V \to W$ is completely defined, once it is defined on a basis. So let us write

$$T(v_j) = \sum_{i=1}^{m} t_{ij} w_i,\ 1 \leq j \leq n. \tag{1.1.1}$$

The coefficients (t_{ij}) in the above relation form a **matrix** with m rows and n columns. Such a matrix is referred to as an $m \times n$ matrix. The j-th column of the matrix represents the coefficients in the expansion of $T(v_j)$ in terms of the basis $\{w_i\}_{i=1}^{m}$ of W. Of course, if we change the bases for V and W, the same linear transformation will be given by another matrix. In particular, let $\dim(V) = n$ and let $T : V \to V$ be a linear operator. Let T be represented by the $n \times n$ matrix (also known as a square matrix of order n) $\mathbf{T} = (t_{ij})$ with respect to a given basis. If we change the basis, then T will be represented by another $n \times n$ matrix $\widetilde{\mathbf{T}} = (\widetilde{t_{ij}})$ and the two will be connected by a relation of the form:

$$\mathbf{T} = \mathbf{P}\widetilde{\mathbf{T}}\mathbf{P}^{-1}$$

where $\mathbf{P}$ is called the change of basis matrix and represents the linear transformation which maps one basis to another. The matrix $\mathbf{P}^{-1}$ represents the inverse of this change of basis mapping and is the inverse matrix of $\mathbf{P}$. In this case, the matrices $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ are said to be **similar**. The identity matrix $\mathbf{I}$ represents the identity mapping $x \mapsto x$ for all $x \in V$ for any fixed basis of V. For a given basis, if $T : V \to V$ is invertible, then the matrix representing T^{-1} will be the inverse of the matrix representing T.

A square matrix is said to be **diagonal** if all its off-diagonal entries are zero. A $n \times n$ square matrix $\mathbf{A} = (a_{ij})$ is said to be **upper triangular** (respectively, **lower triangular**) if $a_{ij} = 0$ for all $1 \leq j < i \leq n$ (respectively, $a_{ij} = 0$ for all $1 \leq i < j \leq n$). If $\mathbb{F} = \mathbb{C}$, it can be shown that every matrix is similar to an upper triangular matrix. A matrix is said to be **diagonalizable** if it is similar to a diagonal matrix.

Given an $m \times n$ matrix and two vector spaces of dimensions n and m, respectively, along with a basis for each of them, the matrix can be used, as in relation (1.1.1), to define a linear transformation between these two spaces. Thus, there is a one-to-one correspondence between matrices and linear transformations between vector spaces of appropriate dimension, once the bases are fixed.

Definition 1.1.11 If $\mathbf{T} = (t_{ij})$ is an $m \times n$ matrix, then the $n \times m$ matrix $\mathbf{T}' = (t_{ji})$, formed by interchanging the rows and the columns of the matrix $\mathbf{T}$, is called the **transpose** of the matrix $\mathbf{T}$. If $\mathbf{T} = (t_{ij})$ is an $m \times n$ matrix with complex entries, then the $n \times m$ matrix $\mathbf{T}^* = (t^*_{ij})$ where $t^*_{ij} = \bar{t}_{ji}$ (the bar denoting complex conjugation), is called the **adjoint** of the matrix $\mathbf{T}$. ■

If x and $y \in \mathbb{R}^n$ (respectively, $\mathbb{C}^n$), then $y'x$ (respectively, y^*x) represents the 'usual' scalar product of vectors in $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) given by $\sum_{i=1}^n x_i y_i$ (respectively, $\sum_{i=1}^n x_i \bar{y}_i$). If the scalar product is zero, we say that the vectors are **orthogonal** to each other and write $x \perp y$. If W is a subspace and x is a vector orthogonal to all vectors in W, we write $x \perp W$.

Definition 1.1.12 Let $\mathbf{T}$ be an $m \times n$ matrix. Then its **row rank** is defined as the number of linearly independent row vectors of the matrix and the **column rank** is the number of independent column vectors of the matrix. ■

The column rank is none other than the rank of the linear transformation defined by $\mathbf{T}$, and the row rank is the rank of the transformation defined by the transpose. We have the following important result.

Proposition 1.1.2 *For any matrix, the row and column ranks are equal and the common value is called the* **rank** *of the matrix.* ■

Definition 1.1.13 The **nullity** of a matrix is the dimension of the null space of the linear transformation defined by the matrix. ■

Proposition 1.1.3 *Let* $\mathbf{T}$ *be an* $m \times n$ *matrix. The sum of the rank of* $\mathbf{T}$ *and the nullity of* $\mathbf{T}$ *is equal to* n.

Corollary 1.1.1 *An* $n \times n$ *matrix is invertible if, and only if, its nullity is zero or, equivalently, its rank is* n. *Equivalently, a linear operator on a finite dimensional space is one-to-one if, and only if, it is onto.* ■

Definition 1.1.14 Let $\mathbf{T}$ be an $n \times n$ matrix with complex entries and $\mathbf{T}^*$ its adjoint. The matrix is said to be **normal** if

$$\mathbf{T}\mathbf{T}^* = \mathbf{T}^*\mathbf{T}.$$

The matrix is said to be **unitary** if

$$\mathbf{T}\mathbf{T}^* = \mathbf{T}^*\mathbf{T} = \mathbf{I}.$$

The matrix is said to be **self-adjoint** or **hermitian** if $\mathbf{T} = \mathbf{T}^*$. ■

Definition 1.1.15 Let $\mathbf{T}$ be an $n \times n$ matrix with complex entries. It is said to be **positive semidefinite** if for every $n \times 1$ matrix with complex entries, i.e. a column vector, $\mathbf{x}$, we have

$$\mathbf{x}^*\mathbf{T}\mathbf{x} \geq 0.$$

The matrix $\mathbf{T}$ is said to be **positive definite** if, in addition, the above inequality is strict if $\mathbf{x} \neq \mathbf{0}$. ■

Remark 1.1.2 A hermitian matrix is equal to its adjoint and the inverse of a unitary matrix is its adjoint. A matrix $\mathbf{T}$, with real entries, which is equal to its transpose is called **symmetric** and one whose inverse is its own transpose is called **orthogonal**. In case the matrix $\mathbf{T}$ has real entries, we can still define positive semidefiniteness (or positive definiteness) by considering real column vectors $\mathbf{x}$ in the above definition. ■

We now introduce an important notion, *viz.* that of the determinant. Before we do this, we need some notation. Let $\mathcal{S}_n$ denote the set of all permutations of n symbols. A *transposition* is a permutation wherein two symbols exchange places with each other and all other symbols are left invariant. It is known that every permutation is the product (in the sense of composition of mappings) of transpositions. A permutation is *even* if it is the product of an even number of transpositions and *odd* if it is the product of an odd number of transpositions. The *signature* of a permutation σ, denoted sgn σ, is +1 if the permutation is even and -1 if it is odd.

Definition 1.1.16 Let $\mathbf{T} = (t_{ij})$ be an $n \times n$ matrix. The **determinant** of $\mathbf{T}$, denoted $\det(\mathbf{T})$, is given by the formula

$$\det(\mathbf{T}) = \sum_{\sigma \in S_n} (\mathrm{sgn}\ \sigma) t_{1,\sigma(1)} \ldots t_{n,\sigma(n)}.$$

■

We list below the important properties of the determinant.

Proposition 1.1.4 *(i) If* $\mathbf{I}$ *is the identity matrix of order* n*, then* $\det(\mathbf{I}) = 1$*;*

(ii) if $\mathbf{T}$ *and* $\mathbf{S}$ *are two* $n \times n$ *matrices, then*

$$\det(\mathbf{ST}) = \det(\mathbf{S}).\det(\mathbf{T}).$$

In particular, $\mathbf{T}$ *is invertible if, and only if,* $\det(\mathbf{T}) \neq 0$ *and*

$$\det(\mathbf{T}^{-1}) = (\det(\mathbf{T}))^{-1}.$$

(iii) if $\mathbf{T}$ *is an* $n \times n$ *matrix, then*

$$\det(\mathbf{T}') = \det(\mathbf{T}).$$

■

Definition 1.1.17 An invertible matrix is also said to be a **non-singular** matrix. Otherwise, it is said to be **singular**. ■

If $\mathbf{T}$ is a non-singular matrix of order n, then, given any $n \times 1$ column vector $\mathbf{b}$, there exists a unique $n \times 1$ column vector $\mathbf{x}$ such that

$$\mathbf{Tx} = \mathbf{b}.$$

This is because the corresponding linear transformation is invertible and hence onto (which gives the existence of the solution) and one-to-one (which gives the uniqueness of the solution). If $\mathbf{T}$ is singular, this is no longer the case and we have the following result.

Proposition 1.1.5 (Fredhölm Alternative) *Let* $\mathbf{T}$ *be a singular matrix of order* n *and let* $\mathbf{b}$ *be an* $n \times 1$ *column vector. Then, either the system of* n *linear equations in* n *unknowns (written in matrix notation)*

$$\mathbf{Tx} = \mathbf{b}$$

has no solution or has an infinite number of solutions. The latter possibility occurs if, and only if

$$\mathbf{b}'\mathbf{u} = 0$$

for all (column) vectors $\mathbf{u}$ *such that* $\mathbf{T}'\mathbf{u} = \mathbf{0}$. ■

We now come to a very important notion in linear algebra and functional analysis.

Definition 1.1.18 Let $\mathbf{T}$ be a square matrix of order n with complex entries. A complex number λ is said to be an **eigenvalue** of $\mathbf{T}$ if there exists an $n \times 1$ vector $\mathbf{u} \neq \mathbf{0}$ such that

$$\mathbf{Tu} = \lambda \mathbf{u}.$$

Such a vector $\mathbf{u}$ is called an **eigenvector** of $\mathbf{T}$ associated to the eigenvalue λ. The set of all eigenvectors associated to an eigenvalue λ is a subspace of $\mathbb{C}^n$ and is called the **eigenspace** associated to λ. ■

From the above definition, we see that $\lambda \in \mathbb{C}$ is an eigenvalue of a square matrix $\mathbf{T}$ if, and only if the matrix $\mathbf{T} - \lambda\mathbf{I}$ is *not* invertible. Thus, λ will be an eigenvalue of $\mathbf{T}$ if, and only if,

$$\det(\mathbf{T} - \lambda\mathbf{I}) = 0. \qquad (1.1.2)$$

The expression on the left-hand side of (1.1.2) is a polynomial of degree equal to the order of $\mathbf{T}$ and is called the **characteristic polynomial** of $\mathbf{T}$. Thus, every eigenvalue is a root of the characteristic polynomial and so every matrix of order n has a non-empty set of at most n distinct eigenvalues. Counting multiplicity, there are exactly n eigenvalues. The equation (1.1.2) is called the **characteristic equation** of the matrix $\mathbf{T}$.

Remark 1.1.3 If a matrix has only real entries, the eigenvalues can still be all purely complex. In that case, considered as a linear transformation on a real vector space, there will be no eigenvalues in the sense of the preceding definition. ■

Definition 1.1.19 Let λ be an eigenvalue of a matrix $\mathbf{T}$. Its **algebraic multiplicity** is its multiplicity as a root of the characteristic polynomial. Its **geometric multiplicity** is the dimension of the eigenspace associated to it. ■

Proposition 1.1.6 *The geometric multiplicity of an eigenvalue does not exceed its algebraic multiplicity.* ■

Definition 1.1.20 The set of all eigenvalues of a matrix is called its **spectrum**. The maximum of the absolute values of the eigenvalues is called its **spectral radius**. ■

The sum of the diagonal entries of a square matrix $\mathbf{T}$ is called its **trace** and is denoted by $\mathrm{tr}(\mathbf{T})$. It is easy to see from the characteristic equation that the trace is the sum of all the eigenvalues taking into account their multiplicities. Similarly the determinant of a matrix is the product of all its eigenvalues (again counting multiplicity).

Proposition 1.1.7 *(i) The eigenvalues of the adjoint of a matrix are the complex conjugates of the eigenvalues of the original matrix;*
(ii) the eigenvalues of a hermitian matrix are all real;
(iii) the eigenvalues of a hermitian and positive definite matrix are positive;
(iv) the eigenvalues of a unitary matrix lie on the unit circle of the complex plane. ■

The eigenvalues of a hermitian matrix admit a 'variational characterization'. Let $\mathbf{T}$ be a hermitian matrix of order n. Its eigenvalues are all real and let them be numbered in increasing order as follows:

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n.$$

Let v_i, $1 \le i \le n$ be a collection of eigenvectors, where v_i is associated to λ_i. Set $V_0 = \{\mathbf{0}\}$ and

$$V_i = \mathrm{span}\{v_1, \ldots, v_i\}$$

for $1 \le i \le n$.

We define the *Rayleigh quotient* associated to the matrix $\mathbf{T}$ as follows:

$$R_{\mathbf{T}}(x) = \frac{x^*\mathbf{T}x}{x^*x}, x \neq \mathbf{0},$$

which is real valued, since $\mathbf{T}$ is hermitian.

Proposition 1.1.8 *(i) The eigenvectors v_i can be chosen such that*

$$v_i^* v_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j; \end{cases}$$

(ii) for each $1 \leq i \leq n$*, we have*

$$\begin{aligned}\lambda_i &= R_{\mathbf{T}}(v_i) \\ &= \max_{x \in V_i} R_{\mathbf{T}}(x) \\ &= \min_{x \perp V_{i-1}} R_{\mathbf{T}}(x) \\ &= \min_{W \subset \mathbb{C}^n, \dim(W)=i} \max_{x \in W} R_{\mathbf{T}}(x).\end{aligned}$$

In particular,

$$\lambda_1 = \min_{x \in \mathbb{C}^n} R_{\mathbf{T}}(x), \text{ and } \lambda_n = \max_{x \in \mathbb{C}^n} R_{\mathbf{T}}(x).$$ ■

By analogy, if $\mathbf{T}$ is a real symmetric matrix, a similar result holds with the adjoint being replaced by the transpose in the definition of the Rayleigh quotient and $\mathbb{C}^n$ being replaced by $\mathbb{R}^n$ in the formulae above.

Remark 1.1.4 A matrix is diagonalizable if, and only if, it admits a basis of eigenvectors. In particular, if all the eigenvalues of a matrix are distinct, it is diagonalizable. All normal matrices are diagonalizable. In particular, hermitian matrices are diagonalizable. ■

For more details on linear spaces, the reader is referred to, for instance, Artin [1].

1.2 Topological Spaces

In this section, we recall the important definitions and results of topology which will be used in the sequel.

Definition 1.2.1 A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that

(i) X and $\emptyset$ are in $\mathcal{T}$;
(ii) the union of any collection of members in $\mathcal{T}$ is a member of $\mathcal{T}$;
(iii) the intersection of any finite collection of members of $\mathcal{T}$ is a member of $\mathcal{T}$. The pair $\{X, \mathcal{T}\}$ is said to be a **topological space**, and the members of the topology $\mathcal{T}$ are called **open sets**. The complements of open sets are called **closed sets**. If $x \in X$, then a **neighbourhood** of x is any open set containing x. ■

It is clear from the above definition that X and $\emptyset$ are both open and closed. Further, a finite union and an arbitrary intersection of closed sets are closed. It then follows that given any set $A \subset X$, there is a smallest closed set containing it. This is called the **closure** of the set A and is usually denoted by $\overline{A}$. If $\overline{A} = X$, we say that A is **dense** in X. Similarly, given any set $A \subset X$, there is a largest open set contained in A. This set is called the **interior** of A and is denoted by A°. A set $A \subset X$ is said to be **nowhere dense** if $(\overline{A})^\circ = \emptyset$.

If $\{X, \mathcal{T}\}$ is a topological space and if $Y \subset X$, then Y inherits a natural topology from that of X. The open sets are those of the form $U \cap Y$, where U is open in X.

Definition 1.2.2 A topological space $\{X, \mathcal{T}\}$ is said to be **Hausdorff** if for every pair of distinct elements x and y in X, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$. ■

Definition 1.2.3 A **metric** on a set X is a function $d : X \times X \to [0, \infty)$ such that

(i) $d(x, y) = 0$ if, and only if, $x = y$;
(ii) for all x and y in X, we have

$$d(x, y) = d(y, x);$$

(iii) for all x, y and z in X, we have

$$d(x, z) \leq d(x, y) + d(y, z). \tag{1.2.1}$$

The pair $\{X, d\}$ is called a **metric space**. ■

Remark 1.2.1 The notion of a metric generalizes that of a distance, as we know it, in the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. The inequality (1.2.1) is just an abstract version of a familiar theorem from Euclidean plane geometry which states that the sum of the lengths of two sides of a triangle is greater than the length of the third side. For this reason, it is called the *triangle inequality*. Of all the conditions on a metric, this one will need the most non-trivial verification. ■

If $\{X, d\}$ is a metric space, then it is easy to see that we have a topology (called the metric topology) induced on X by the metric d which is defined as follows. A non-empty set $U \subset X$ is open if, and only if, for every $x \in U$, there exists $r > 0$ such that

$$B(x; r) \stackrel{\text{def}}{=} \{y \in X | d(x, y) < r\} \subset U.$$

The set $B(x; r)$ described above is called the (open) ball centred at x and of radius r. It is a simple exercise to check that open balls themselves are open sets. It is also immediate to see that this topology is Hausdorff.

On $\mathbb{R}$ or $\mathbb{C}$, we have the 'usual' metric defined by

$$d(x, y) = |x - y|.$$

The topology induced by this metric will be called the 'usual' topology on $\mathbb{R}$ or $\mathbb{C}$, as the case may be. Similarly, on $\mathbb{R}^N$ (or $\mathbb{C}^N$), we have the 'usual' Euclidean distance which defines a metric on that space: if $x = (x, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ are vectors in $\mathbb{R}^N$ (respectively, $\mathbb{C}^N$), then

$$d(x, y) = \left(\sum_{i=1}^{N} |x_i - y_i|^2 \right)^{\frac{1}{2}}.$$

The topology induced by this metric will be referred to as the 'usual' topology on $\mathbb{R}^N$ (respectively, $\mathbb{C}^N$).

In the metric topology defined above, we see that every open set is the union of open balls. In a general topological space a collection $\mathcal{B}$ of open sets is called a **base** for the topology if every open set can be expressed as the union of members of $\mathcal{B}$.

Definition 1.2.4 Let $\{X, \mathcal{T}\}$ be a topological space and let $\mathcal{S}$ be a collection of open sets in X. We say that $\mathcal{S}$ is a **subbase** for the topology $\mathcal{T}$ if every open set can be expressed as unions of finite intersections of members of $\mathcal{S}$. ■

Clearly any topology containing $\mathcal{S}$ will have to contain $\mathcal{T}$. Thus $\mathcal{T}$ is the smallest topology containing $\mathcal{S}$. The set of finite intersections of members of $\mathcal{S}$ form a base for the topology.

Definition 1.2.5 Let $\{X, \mathcal{T}\}$ be a topological space and let A be an arbitrary subset of X. A point $x \in X$ is said to be a **limit point** of A if every neighbourhood of x contains a point of A (different from x, in case $x \in A$). ■

Definition 1.2.6 Let $\{X, \mathcal{T}\}$ be a topological space and let $\{x_n\}$ be a sequence of elements in X. We say that the sequence **converges** to a point $x \in X$ if for every neighbourhood U of x, we can find a positive integer N (depending on U) such that $x_k \in U$ for all $k \geq N$. In this case, we write $x_n \to x$ in X. ■

Definition 1.2.7 Let $\{X_i, \mathcal{T}_i\}$, $i = 1, 2$, be two topological spaces and let $f : X_1 \to X_2$ be a given function. We say that f is **continuous** if $f^{-1}(U)$ is an open set in X_1 for every open set U in X_2. If f is a bijection such that both f and f^{-1} are continuous, then f is said to be a **homeomorphism** and the two topological spaces are said to be homeomorphic to each other. ■

The following propositions are easy to prove.

Proposition 1.2.1 *Let $\{X, d\}$ be a metric space and let $\{x_n\}$ be a sequence in X. Then $x_n \to x$ in X if, and only if, for every $\varepsilon > 0$, there exists a positive integer N such that*

$$d(x_k, x) < \varepsilon \text{for every } k \geq N.$$ ■

In particular, every convergent sequence is **bounded**, *i.e.*, it can be contained in a (sufficiently large) ball.

Proposition 1.2.2 *Let $\{X_i, d_i\}$, $i = 1, 2$, be metric spaces and let $f : X_1 \to X_2$ be a given function. The following are equivalent:*

(i) f *is continuous;*
(ii) for every $x \in X$ *and for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that whenever* $d_1(x, y) < \delta$*, we have* $d_2(f(x), f(y)) < \varepsilon$*;*
(iii) if $x_n \to x$ *in* X_1*, then* $f(x_n) \to f(x)$ *in* X_2*.* ■

Remark 1.2.2 When property (ii) (or, equivalently, (iii)) holds for a particular point $x \in X$, we say that f is *continuous at* x. Thus, f is continuous if, and only if, it is continuous at each point of X. If f is continuous and, for a given $\varepsilon > 0$, the $\delta > 0$ described in statement (ii) above does not depend on the point x, then the function is said to be *uniformly continuous* on X. ■

Let $\{X, d\}$ be a metric space and let E be an arbitrary subset of X. Let $x \in X$. Define

$$d(x, E) = \inf_{y \in E} d(x, y).$$

This is called the distance of the point x from the set E. The following proposition is easy to prove.

Proposition 1.2.3 *Let* $\{X, d\}$ *be a metric space and let* $E \subset X$*. Then*
(i) *for all* x *and* $y \in X$*, we have*

$$|d(x, E) - d(y, E)| \leq d(x, y).$$

Thus, the function $x \mapsto d(x, E)$ *is a uniformly continuous function on* X*;*
(ii) *if* E *is a closed set, then,* $d(x, E) = 0$ *if, and only if,* $x \in E$*; more generally, if* E *is any subset of* X*, we have*

$$\overline{E} = \{x \in X | d(x, E) = 0\}.$$

Definition 1.2.8 Let $\{X, d\}$ be a metric space. A sequence $\{x_n\}$ in X is said to be **Cauchy** if, for every $\varepsilon > 0$, there exists a positive integer N such that

$$d(x_k, x_l) < \varepsilon$$

for every $k \geq N, l \geq N$. ■

It is simple to verify that every Cauchy sequence is bounded. It is also easy to see that every convergent sequence in a metric space is Cauchy. The converse is not true, and this leads to the following important definition.

Definition 1.2.9 A metric space is said to be **complete** if every Cauchy sequence is convergent. ■

With their usual metric, the spaces $\mathbb{R}$ and $\mathbb{C}$ are complete.

We now introduce an important notion which we will study in detail in later chapters.

Definition 1.2.10 Let $\mathcal{I}$ be an arbitrary indexing set. Let X be a set and $\{X_i, \mathcal{T}_i\}_{i \in \mathcal{I}}$ be topological spaces. Let $f_i : X \to X_i$ be given functions. The **weak topology** generated by the functions f_i, $i \in \mathcal{I}$, is the smallest topology on X such that all the f_i are continuous. ■

From the above definition it follows that a subbase for the weak topology generated by the f_i is the collection of all sets of the form $f_i^{-1}(U)$ where U is an arbitrary open set in X_i and the index i ranges over the indexing set $\mathcal{I}$. A typical neighbourhood of a point $x \in X$ will therefore be a *finite* intersection of sets of the form $f_i^{-1}(U_i)$ where U_i is a neighbourhood of $f_i(x)$ in X_i.

Definition 1.2.11 Let $\mathcal{I}$ be an indexing set and let $\{X_i, \mathcal{T}_i\}$, $i \in \mathcal{I}$ be topological spaces. Set $X = \Pi_{i \in \mathcal{I}} X_i$. Let $x = (x_i)_{i \in \mathcal{I}}$. Let $p_i : x \in X \mapsto x_i \in X_i$ be the i-th coordinate projection. The **product topology** on X is the weak topology generated by the coordinate projections, i.e. it is the smallest topology such that the projections are all continuous. ■

Thus, sets of the form $\Pi_{i \in \mathcal{I}} U_i$, where $U_i = X_i$ for all $i \neq i_0$ (an arbitrary element of $\mathcal{I}$) and U_{i_0} is open in X_{i_0}, form a subbase for the product topology. A base for the topology is the collection of all sets of the form $\Pi_{i \in \mathcal{I}} U_i$ where $U_i = X_i$ for all but a finite number of indices and, for those indices, U_i is an open set in X_i.

Definition 1.2.12 Let $\{X, \mathcal{T}\}$ be a topological space and let $\emptyset \neq K \subset X$. A collection of open sets $\mathcal{F}$ is said to be an **open cover** of K if the union of the members of $\mathcal{F}$ contains K. A **subcover** of $\mathcal{F}$ is a subcollection of members of $\mathcal{F}$ which is also an open cover of K. ■

Definition 1.2.13 Let $\{X, \mathcal{T}\}$ be a topological space and let $\emptyset \neq K \subset X$. The set K is said to be a **compact set** if every open cover of K admits a finite subcover. ■

If X is itself a compact set, we say that it is a compact space. We can also describe compactness via closed sets.

Definition 1.2.14 A collection $\mathcal{A}$ of subsets of a set X is said to have **finite intersection property** if every finite subcollection has non-empty intersection. ■

The following proposition is easily proved.

Proposition 1.2.4 *A non-empty subset K of a topological space is compact if, and only if, every collection of closed sets in K having finite intersection property has non-empty intersection.* ■

Definition 1.2.15 Let $\{X, \mathcal{T}\}$ be a topological space and let $\emptyset \neq K \subset X$. The set K is said to be **sequentially compact** if every sequence in K has a convergent subsequence. ■

We list important facts about compact sets in the following proposition.

Proposition 1.2.5 *(i) Every continuous image of a compact set is compact;*
(ii) the product of compact sets is compact;

(iii) *compact subsets of Hausdorff spaces are closed;*
(iv) *a compact subset of a metric space is closed and bounded;*
(v) *for the usual topology on* $\mathbb{R}^N$, *a subset is compact if, and only if, it is closed and bounded;*
(vi) *every continuous real-valued function on a compact space is bounded and attains its maximum and minimum values in that set.*
(vii) *Every continuous real-valued function on a compact metric space is uniformly continuous.* ■

Compact metric spaces are very special. In order to characterize them, we need the following notion.

Definition 1.2.16 A metric space $\{X, d\}$ is said to be **totally bounded** if, for every $\varepsilon > 0$, there exists finite set of points $\{x_i\}_{i=1}^{k(\varepsilon)}$ such that

$$X \subset \cup_{i=1}^{k(\varepsilon)} B(x_i; \varepsilon).$$ ■

Proposition 1.2.6 *Let* $\{X, d\}$ *be a metric space. The following statements are equivalent:*

(i) X *is compact;*
(ii) X *is sequentially compact;*
(iii) *every infinite subset of* X *has a limit point;*
(iv) X *is complete and totally bounded.* ■

We conclude this section with one final important topological notion.

Definition 1.2.17 Let $\{X, \mathcal{T}\}$ be a topological space. We say that X is **connected** if there do not exist non-empty open sets U and V such that $X = U \cup V$ and $U \cap V = \emptyset$. A subset $A \subset X$ is said to be connected if there do not exist disjoint open sets U and V such that $A = A \cap (U \cup V)$, $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$. ■

Definition 1.2.18 A non-empty subset A of a topological space is said to be **path connected** if given any pair of points x and y in A, there exists a continuous function $\gamma : [0, 1] \to A$ such that $\gamma(0) = x$ and $\gamma(1) = y$. ■

Proposition 1.2.7 *(i)* *Every continuous image of a connected set is connected;*
(ii) *the product of connected sets is connected;*
(iii) *every path connected set is connected.* ■

In particular, every ball in a metric space is path connected, and hence, connected. The only connected sets in $\mathbb{R}$ are intervals.

For a detailed study of topological spaces, the reader is referred to, for instance, Simmons [2].

1.3 Measure and Integration

In this section, we will recall basic facts and results of Lebesgue's theory of measure and integration.

Definition 1.3.1 Let X be a set. A σ-**algebra** is a collection $\mathcal{S}$ of subsets of X such that

(i) $X \in \mathcal{S}$;
(ii) if $A \in \mathcal{S}$, then $A^c \in \mathcal{S}$, where A^c denotes the complement of A in X;
(iii) if $A_i \in \mathcal{S}$ for $i \in \mathbb{N}$, then
$$\cup_{i=1}^{\infty} A_i \in \mathcal{S}.$$

The pair $(X, \mathcal{S})$ is then called a **measurable space**. The members of $\mathcal{S}$ are called **measurable sets**. ■

In other words, a σ-algebra on a set X is collection of subsets of Xcontaining X and which is closed under the set theoretic operations of complementation and countable unions.

Definition 1.3.2 Let $(X, \mathcal{S})$ be a measurable space. Let $f : X \to \mathbb{R}$ be a given function. It is said to be a **measurable function** if, for every $\alpha \in \mathbb{R}$, we have

$$f^{-1}((\alpha, \infty)) = \{x \in X \,|\, f(x) > \alpha\} \in \mathcal{S}.$$

A complex valued function is said to be measurable if its real and imaginary parts are measurable. ■

Proposition 1.3.1 *Let $(X, \mathcal{S})$ be a measurable space and let $f : X \to \mathbb{R}$ be a given function. Then, the following are equivalent:*

(i) $f^{-1}((\alpha, \infty)) \in \mathcal{S}$, *for all* $\alpha \in \mathbb{R}$;
(ii) $f^{-1}([\alpha, \infty)) \in \mathcal{S}$, *for all* $\alpha \in \mathbb{R}$;
(iii) $f^{-1}((-\infty, \alpha)) \in \mathcal{S}$, *for all* $\alpha \in \mathbb{R}$;
(iv) $f^{-1}((-\infty, \alpha]) \in \mathcal{S}$, *for all* $\alpha \in \mathbb{R}$. ■

Given any collection of subsets of a set X, there is a smallest σ-algebra containing the collection and it is called the σ-algebra generated by the given collection of sets. If $(X, \mathcal{T})$ is a topological space , then the σ-algebra generated by the open sets is called the **Borel** σ-algebra and its members are called Borel sets.

Definition 1.3.3 Let $(X, \mathcal{S})$ be a measurable space. A **measure** on X is a function $\mu : \mathcal{S} \to [0, \infty]$ such that, if A_i, $i \in \mathbb{N}$ are members of $\mathcal{S}$ which are pairwise disjoint, i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$, then

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

The triple $(X, \mathcal{S}, \mu)$ is called a **measure space**. ■

Definition 1.3.4 Let $(X, \mathcal{S}, \mu)$ be a measure space. If $\mu(X) < \infty$, then μ is said to be a finite measure. If X can be covered by a countable union of measurable sets, each with finite measure, then μ is said to be a σ-**finite** measure. ■

Example 1.3.1 Let X be any non-empty set and let $\mathcal{S}$ be the collection of all subsets of X, which is obviously a σ-algebra. If $A \subset X$, define $\mu(A)$ to be the number of elements in A, if A is a finite set and to be ∞, otherwise. This defines a measure and is called the **counting measure** on X. ■

Example 1.3.2 Let $(X, \mathcal{S})$ be as in the preceding example and let $x_0 \in X$ be a given point. let $A \subset X$. Define

$$\mu(A) = \begin{cases} 1, & \text{if } x_0 \in A \\ 0, & \text{otherwise.} \end{cases}$$

This defines a measure on X which is called the **Dirac measure** concentrated at the point x_0. ■

The most important example of a measure is the **Lebesgue measure** defined on $\mathbb{R}^N$.

Consider the real line $\mathbb{R}$ endowed with its usual topology. Then, it is possible to construct a σ-algebra $\mathcal{M}$ on $\mathbb{R}$ which contains all the Borel sets and to define a measure μ such that for any interval of the form $[a, b)$ (or $[a, b]$, $(a, b]$, (a, b)), where $a, b \in \mathbb{R}$, we have

$$\mu([a, b)) = b - a.$$

This measure also has the following additional properties:

(i) (**Completeness**) If $E \in \mathcal{M}$ and $\mu(E) = 0$, then, for any $F \subset E$, we have $F \in \mathcal{M}$ and, *a fortiori*, $\mu(F) = 0$;

(ii) (**Translation Invariance**) If $E \in \mathcal{M}$ and if $a \in \mathbb{R}$, then

$$a + E = \{a + x | x \in E\} \in \mathcal{M}$$

and, further, $\mu(a + E) = \mu(E)$;

(iii) (**Regularity**) If $E \in \mathcal{M}$, then

$$\begin{aligned} \mu(E) &= \inf\{\mu(W) | E \subset W, W \text{ open}\} \\ &= \sup\{\mu(F) | F \subset E, F \text{ compact}\}. \end{aligned}$$

Properties (i)–(iii) above determine the measure μ up to a multiplicative constant. The fact that the measure of an interval is its length determines the measure uniquely and it is called the **Lebesgue measure** on $\mathbb{R}$ and the members of $\mathcal{M}$ are called **Lebesgue measurable** sets.

In the same way, it is possible to construct the Lebesgue measure on $\mathbb{R}^N$, $N \geq 1$, with the properties (i)–(iii) and such that for any cell of the form $E = \Pi_{i=1}^N [a_i, b_i)$, we have

$$\mu(E) = \Pi_{i=1}^N (b_i - a_i).$$

Definition 1.3.5 Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $E \in \mathcal{S}$. The **characteristic function** of E, denoted χ_E is defined by

$$\chi_E(x) = \begin{cases} 1, \text{ if} x \in E, \\ 0, \text{ if} x \notin E. \end{cases}$$

A **simple function** is a function $S : X \to \mathbb{R}$ such that

$$s(x) = \sum_{i=1}^{m} \alpha_i \chi_{A_i}(x),$$

where $\alpha_i \in \mathbb{R}$ and $A_i \in \mathcal{S}$ for $1 \leq i \leq m$. ■

We are now in a position to define the Lebesgue integral. Let $(X, \mathcal{S}, \mu)$ be a measure space. The integral of a non-negative simple function $s = \sum_{i=1}^m \alpha_i \chi_i$ is defined by

$$\int_X s \, \mathrm{d}\mu \stackrel{\text{def}}{=} \sum_{i=1}^{m} \alpha_i \mu(E_i).$$

(Since the A_i may have infinite measure, we insist on the non-negativity of the function (so that $\alpha_i \geq 0$ for all $1 \leq i \leq m$) in order to avoid anomalous situations where infinite quantities may need to be subtracted from each other.)

Proposition 1.3.2 *Let $(X, \mathcal{S})$ be a measurable space and let $f : X \to \mathbb{R}$ be a non-negative measurable function. Then there exists a sequence $\{s_n\}$ of non-negative simple functions such that, for all n,*

$$0 \leq s_n \leq s_{n+1} \leq f$$

and

$$\lim_{n\to\infty} s_n(x) = f(x)$$

for all $x \in X$. ■

In view of the above proposition, we may now define the integral of any non-negative measurable function as follows:

$$\int_X f \, \mathrm{d}\mu \overset{\text{def}}{=} \sup \left\{ \int_X s \, \mathrm{d}\mu | s \text{ simple}, 0 \leq s \leq f \right\}.$$

In the case of a simple function, it is easy to see that both these definitions coincide. Now, let $f : X \to \mathbb{R}$ be any measurable function. Set

$$f^+ = \max\{f, 0\}, \text{ and } f^- = -\min\{f, 0\}.$$

Then, f^+ and f^- are non-negative measurable functions and

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.$$

Definition 1.3.6 Let $(X, \mathcal{S}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function. Then f is said to be **integrable** if

$$\int_X |f| \, \mathrm{d}\mu < \infty. \qquad \blacksquare$$

It is easy to see that

$$\int_X |f| \, \mathrm{d}\mu = \int_X f^+ \, \mathrm{d}\mu + \int_X f^- \, \mathrm{d}\mu$$

so that, if f is integrable, then both the integrals on the right-hand side of the above relation are finite. Then, we can unambiguously define the integral of f as follows:

$$\int_X f \, \mathrm{d}\mu \overset{\text{def}}{=} \int_X f^+ \, \mathrm{d}\mu - \int_X f^- \, \mathrm{d}\mu.$$

Integration is a linear operation: if f and g are integrable functions and if α and $\beta \in \mathbb{R}$, we have

$$\int_X (\alpha f + \beta g) \, \mathrm{d}\mu = \alpha \int_X f \, \mathrm{d}\mu + \beta \int_X g \, \mathrm{d}\mu.$$

Notation: If we wish to specify the variable of integration as well, we will write

$$\int_X f(x) \, \mathrm{d}\mu(x)$$

in place of $\int_X f \, \mathrm{d}\mu$.

Example 1.3.3 If we define the counting measure on the set of natural numbers $\mathbb{N}$, then any real-valued function f on $\mathbb{N}$ is measurable and can be identified with a sequence $\{a_n\}$, where $f(n) = a_n$. It can be seen that

$$\int_{\mathbb{N}} f \, \mathrm{d}\mu = \sum_{n=1}^{\infty} a_n.$$

An integrable function corresponds to an absolutely convergent series. ■

Example 1.3.4 If X is any non-empty set and if δ_{x_0} is the Dirac measure concentrated at $x_0 \in X$, then

$$\int_X f \, \mathrm{d}\delta_{x_0} = f(x_0)$$

for any function $f : X \to \mathbb{R}$. ■

We now list some of the main results from the theory of the Lebesgue integral, which behaves very nicely with respect to limit processes.

Theorem 1.3.1 (Monotone Convergence Theorem) *Let $(X, \mathcal{S}, \mu)$ be a measure space. let $\{f_n\}$ be an increasing sequence of non-negative measurable functions on X. Let $f_n(x) \to f(x)$ for all $x \in X$. Then f is measurable and*

$$\lim_{n\to\infty} \int_X f_n \, d\mu \to \int_X f \, d\mu. \tag{1.3.1}$$

■

Theorem 1.3.2 (Fatou's Lemma) *Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of non-negative measurable functions on X. Then*

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

■

Theorem 1.3.3 (Dominated Convergence Theorem) *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions such that $f_n(x) \to f(x)$ for all $x \in X$. Assume that $|f_n| \le g$ for all n, where g is integrable. Then, f is also integrable and*

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0.$$

In particular, (1.3.1) holds. ■

A property is said to hold almost everywhere (abbreviated as a.e.) in a measure space if it holds on the complement of a set of measure zero. The above results are valid even if we replace convergence for all $x \in X$ by convergence a.e. for the given sequences of functions.

Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be two measure spaces. Consider the product set $X \times Y$. A measurable rectangle is a subset of the product set of the form $A \times B$, where $A \in \mathcal{S}$ and $B \in \mathcal{T}$. An elementary set is a finite disjoint union of measurable rectangles. The product σ-algebra $\mathcal{S} \times \mathcal{T}$ on $X \times Y$ is the smallest σ-algebra generated by elementary sets.

Definition 1.3.7 Let $Q \subset X \times Y$. Then the x-**section** of Q is the subset of Y defined by

$$Q_x = \{y \in Y | (x, y) \in Q\}.$$

Similarly, the y-**section** of Q is a subset of X and is given by

$$Q^y = \{x \in X | (x, y) \in Q\}.$$

If $f : X \times Y \to \mathbb{R}$ is a function, then its x- and y-**sections** are functions $f_x : Y \to \mathbb{R}$ and $f^y : X \to \mathbb{R}$, respectively, given by

$$f_x(y) = f(x, y) = f^y(x)$$

for all $x \in X$ and $y \in Y$. ■

Proposition 1.3.3 *(i) If $Q \in \mathcal{S} \times \mathcal{T}$, then, for every $x \in X$ and $y \in Y$, we have $Q_x \in \mathcal{T}$ and $Q^y \in \mathcal{S}$;*

(ii) if $f : X \times Y \to Y$ is a $\mathcal{S} \times \mathcal{T}$-measurable function, then, for every $x \in X$ and $y \in Y$, we have that f_x is $\mathcal{T}$-measurable on Y and f^y is $\mathcal{S}$-measurable on X.

Theorem 1.3.4 *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be σ-finite measure spaces. Let $Q \in \mathcal{S} \times \mathcal{T}$. For $x \in X$ and $y \in Y$, define*

$$\varphi(x) = \lambda(Q_x), \text{ and } \psi(y) = \mu(Q^y).$$

Then φ is $\mathcal{S}$-measurable, ψ is $\mathcal{T}$-measurable and, further,

$$\int_X \varphi \, d\mu = \int_Y \psi \, d\lambda. \tag{1.3.2}$$

■

We can now use the above theorem to define the measure on the product of two σ-finite measure spaces.

Definition 1.3.8 Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be two σ-finite measure spaces. The **product measure** $\mu \times \lambda$ on $X \times Y$ is defined by

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x)\, d\mu(x) = \int_Y \mu(Q^y)\, d\lambda(y)$$

for every $Q \in \mathcal{S} \times \mathcal{T}$. ■

The following result helps us to evaluate integrals over the product space via iterated integrals and is very useful.

Theorem 1.3.5 (Fubini's Theorem) *Let* $(X, \mathcal{S}, \mu)$ *and* $(Y, \mathcal{T}, \lambda)$ *be* σ*-finite measure spaces. Let* $f : X \times Y \to \mathbb{R}$ *be* $\mathcal{S} \times \mathcal{T}$*-measurable.*

(i) Let f *be non-negative. Set*

$$\varphi(x) = \int_Y f_x\, d\lambda \quad \text{and} \quad \psi(y) = \int_X f^y\, d\mu \tag{1.3.3}$$

for $x \in X$ *and* $y \in Y$*. Then* φ *is* $\mathcal{S}$*-measurable,* ψ *is* $\mathcal{T}$*-measurable and*

$$\int_X \varphi\, d\mu = \int_Y \psi\, d\lambda; \tag{1.3.4}$$

(ii) if f *is an arbitrary measurable function such that*

$$\varphi(x) \stackrel{\text{def}}{=} \int_Y |f|_x\, d\lambda$$

verifies

$$\int_X \varphi\, d\mu < \infty,$$

then f *is integrable with respect to* $\mu \times \lambda$*;*

(iii) let f *be integrable with respect to* $\mu \times \lambda$*. Then* f_x *is integrable with respect to* λ *for almost every* $x \in X$*; similarly* f^y *is integrable with respect to* μ *for almost every* $y \in Y$*; the functions* φ *and* ψ *defined by (1.3.3) are integrable with respect to* μ *and* λ*, respectively, and (1.3.4) holds.*

Remark 1.3.1 The σ-finiteness of the measures is essential. ■

Remark 1.3.2 The relation (1.3.4) may be written in the form of iterated integrals as follows:

$$\int_X \left(\int_Y f(x, y)\, \mathrm{d}\lambda(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x, y)\, \mathrm{d}\mu(x) \right) \mathrm{d}\lambda(y)$$

and the common value is equal to $\int_{X\times Y} f\, d(\mu \times \lambda)$ whenever Fubini's theorem is applicable. ■

Remark 1.3.3 In (ii), f is integrable if the analogous result for the y-section of $|f|$ holds. Thus, if either of the iterated integrals for $|f|$ is finite, then f is integrable over the product space and (1.3.4) will hold. ■

For further details, the reader is referred to Halmos [3], Royden [4] or Rudin [5].

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Chapter 2
Normed Linear Spaces

2.1 The Norm Topology

In order to do analysis on vector spaces, we need to endow these spaces with a topological structure which is compatible with the linear structure. This is made precise in the following definition.

Definition 2.1.1 A **topological vector space** is a vector space V that is endowed with a Hausdorff topology such that the maps

$$(x, y) \in V \times V \mapsto x + y \in V \text{ and } (\alpha, x) \in \mathbb{F} \times V \mapsto \alpha x \in V$$

are continuous, each product space being endowed with the appropriate product topology using the given topology of V and the usual topology on the scalar field $\mathbb{F}(= \mathbb{R} \text{ or } \mathbb{C})$. ■

We will restrict our attention to a class of topological vector spaces called normed linear spaces, which we now proceed to define. Recall that the base field $\mathbb{F}$ will always be $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.2 A **norm** on a vector space V is a function $\|\cdot\| : V \to [0, \infty)$ such that

(i) $\|x\| = 0$ if, and only if, $x = \mathbf{0}$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for every $\alpha \in \mathbb{F}$ and every $x \in V$;
(iii) (Triangle Inequality) for every x and y in V, we have

$$\|x + y\| \leq \|x\| + \|y\|. \tag{2.1.1}$$

■

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Associated to a norm on a vector space V, we have a metric defined by

$$d(x, y) = \|x - y\|.$$

It is immediate to verify that this defines a metric. The triangle inequality (2.1.1) yields the inequality (1.2.1) (which is also called by the same name) via the relation

$$x - z = (x - y) + (y - z).$$

(Thus, the norm of a vector is its distance from the origin and is a generalization of the notion of the length of a vector, as we know it, in Euclidean space.)

Thus, V is endowed with a metric topology. In this topology, a sequence $\{x_n\}$ converges to x in V if, and only if

$$\|x_n - x\| \to 0.$$

Now if $\{x_n\}$ and $\{y_n\}$ are sequences in V and $\{\alpha_n\}$ a sequence in $\mathbb{F}$, we have, for $x, y \in V$ and $\alpha \in \mathbb{F}$,

$$\begin{aligned} \|(x_n + y_n) - (x + y)\| &\le \|x_n - x\| + \|y_n - y\|, \\ \|\alpha_n x_n - \alpha x\| &\le |\alpha_n| \|x_n - x\| + |\alpha_n - \alpha| \|x\|. \end{aligned}$$

Thus, if $x_n \to x$, $y_n \to y$ in V and if $\alpha_n \to \alpha$ in $\mathbb{F}$, it immediately follows that $x_n + y_n \to x + y$ and that $\alpha_n x_n \to \alpha x$ in V. Thus addition and scalar multiplication are continuous and so V becomes a topological vector space with this metric topology.

Definition 2.1.3 A **normed linear space** is a vector space V endowed with a norm. The metric topology induced by the norm is called its **norm topology**. ■

The norm itself is a continuous function with respect to this topology. Indeed, if x and y are in V, then, since $x = (x - y) + y$, the triangle inequality yields

$$\|x\| \le \|x - y\| + \|y\|$$

which we rewrite as

$$\|x\| - \|y\| \le \|x - y\|.$$

Interchanging the roles of x and y we finally obtain

$$|\ \|x\| - \|y\|\ | \le \|x - y\|$$

from which the continuity of the function $x \in V \mapsto \|x\| \in \mathbb{R}$ follows.

Definition 2.1.4 A normed linear space is said to be a **Banach space** if it is complete under the norm topology. ■

2.2 Examples

We will now look at several examples of normed linear spaces. Essentially, they can be classified into three groups - finite dimensional spaces, sequence spaces and function spaces. In the examples that follow, we will set $\mathbb{F} = \mathbb{R}$. The reader can easily make the necessary changes to cover the case when $\mathbb{R}$ is replaced by $\mathbb{C}$.

Example 2.2.1 We can consider $\mathbb{R}$ as a vector space over itself. The map $x \in \mathbb{R} \mapsto |x|$ is easily seen to define a norm which generates the usual topology on $\mathbb{R}$. Since $\mathbb{R}$ is complete, it thus becomes a Banach space. ■

Example 2.2.2 Let $1 \leq p < \infty$. For $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, define

$$\|x\|_p = \left(\sum_{i=1}^{N} |x_i|^p\right)^{\frac{1}{p}}.$$

It is easy to see that conditions (i)–(ii) of Definition 2.1.2 are verified. We will presently prove the triangle inequality, and thus $\mathbb{R}^N$ with the norm $\|\cdot\|_p$ will become a normed linear space. It is, again, immediate to see that a sequence $\{x^{(n)}\}$ in $\mathbb{R}^N$ converges in this norm to $x \in \mathbb{R}^N$ if, and only if, for every $1 \leq i \leq N$, we have $x_i^{(n)} \to x_i$. Similarly, $\{x^{(n)}\}$ is Cauchy in this norm if, and only if, for every $1 \leq i \leq N$, the sequences $\{x_i^{(n)}\}$ are Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, it now follows that $\mathbb{R}^N$ is also complete with respect to each of the norms $\|\cdot\|_p$ defined above. Thus for each of these norms, $\mathbb{R}^N$ is a Banach space. ■

We now proceed to prove the triangle inequality for each of the norms $\|.\|_p$ for $1 \leq p < \infty$.

Definition 2.2.1 Let $1 \leq p \leq \infty$. If $p = 1$, set $p^* = \infty$ and vice versa. Otherwise, let $1 < p^* < \infty$ be such that

$$\frac{1}{p} + \frac{1}{p^*} = 1 \tag{2.2.1}$$

The number p^* defined thus is called the **conjugate exponent** of p. ■

Lemma 2.2.1 *Let $1 < p < \infty$. Let p^* be its conjugate exponent. Then, if a and b are non-negative real numbers, we have*

$$a^{1/p}b^{1/p^*} \leq \frac{a}{p} + \frac{b}{p^*}. \tag{2.2.2}$$

Proof Let $t \geq 1$ and consider the function

$$f(t) = k(t-1) - t^k + 1$$

for some $k \in (0, 1)$. Then $f'(t) = k(1 - t^{k-1}) \geq 0$ since $k < 1$. Thus, f is an increasing function on $[1, \infty)$ and, since $f(1) = 0$, we immediately deduce that

$$t^k \leq k(t-1) + 1 \tag{2.2.3}$$

for $t \geq 1$ and $0 < k < 1$.

Now, if a or b is zero, then (2.2.2) is obviously true. So let us assume that $a \geq b > 0$.

The inequality (2.2.2) now follows by setting $t = a/b$ and $k = 1/p$ in (2.2.3) and using the definition of p^*. ■

Lemma 2.2.2 *(**Hölder's inequality**) Let $1 < p < \infty$. Let p^* be its conjugate exponent. Then, for $x, y \in \mathbb{R}^N$,*

$$\sum_{i=1}^{N} |x_i y_i| \leq \|x\|_p \|y\|_{p^*}. \tag{2.2.4}$$

Proof Since the result is trivially true for $x = \mathbf{0}$ or $y = \mathbf{0}$, we can assume, without loss of generality, that both x and y are non-zero vectors. Then, set

$$a = \frac{|x_i|^p}{\|x\|_p^p} \quad \text{and} \quad b = \frac{|y_i|^{p^*}}{\|y\|_{p^*}^{p^*}}$$

for a fixed $1 \leq i \leq N$. Then (2.2.2) yields

$$\frac{|x_i y_i|}{\|x\|_p \|y\|_{p^*}} \leq \frac{1}{p} \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{p^*} \frac{|y_i|^{p^*}}{\|y\|_{p^*}^{p^*}}.$$

Summing over the range of the index i, we get

$$\frac{\sum_{i=1}^{N} |x_i y_i|}{\|x\|_p \|y\|_{p^*}} \leq \frac{1}{p} + \frac{1}{p^*} = 1$$

which proves (2.2.4). ■

Lemma 2.2.3 *(**Minkowski's Inequality**) Let $1 \leq p < \infty$. Let $x, y \in \mathbb{R}^N$. Then*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p. \tag{2.2.5}$$

Proof The proof is obvious if $p = 1$. Let us, therefore, assume that $1 < p < \infty$. Let p^* be the conjugate exponent. Then,

$$\sum_{i=1}^N |x_i + y_i|^p \le \sum_{i=1}^N |x_i + y_i|^{p-1}|x_i| + \sum_{i=1}^N |x_i + y_i|^{p-1}|y_i|$$
$$\le (\|x\|_p + \|y\|_p)\left(\sum_{i=1}^N |x_i + y_i|^{(p-1)p^*}\right)^{\frac{1}{p^*}}$$

by a simple application of Hölder's inequality (2.2.4). But $(p-1)p^* = p$ by definition and so,

$$\|x + y\|_p^p \le (\|x\|_p + \|y\|_p)\|x + y\|_p^{p/p^*}.$$

Since the result is obviously true when $x + y = \mathbf{0}$, we can assume, without loss of generality, that $x + y \ne \mathbf{0}$ and so, dividing both sides of the above inequality by $\|x + y\|^{p/p^*}$ and using, once again, the definition of p^*, we get (2.2.5). ■

Since Minkowski's inequality is exactly the triangle inequality for the norm $\|\cdot\|_p$, our proof that $\mathbb{R}^N$ is a Banach space for each of these norms is complete.

Remark 2.2.1 The inequalities of Hölder and Minkowski are clearly true when x and $y \in \mathbb{C}^N$ and so $\mathbb{C}^N$ is also a Banach space for each of the norms $\|\cdot\|_p$, $1 \le p < \infty$. ■

Remark 2.2.2 When $p = 2$, we have $p^* = 2$ as well. In this case Hölder's inequality is known as the **Cauchy-Schwarz inequality**. The inequality (2.2.2), in this case, turns out to be the familiar inequality relating the arithmetic and geometric means of two positive real numbers. The norm $\|\cdot\|_2$ is also called the *Euclidean norm* since it corresponds to the usual Euclidean distance in $\mathbb{R}^N$. ■

Example 2.2.3 For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, define

$$\|x\|_\infty = \max_{1\le i\le N} |x_i|.$$

It is easy to verify that this also defines a norm on $\mathbb{R}^N$. Again convergence and the Cauchy criterion hold if and only if they hold componentwise and so $\mathbb{R}^N$ is a Banach space for this norm as well. Again all these assertions hold for $\mathbb{C}^N$ as well. It is immediate to see that Hölder's inequality is true when $p = 1$ as well. ■

Remark 2.2.3 The spaces $\mathbb{R}^N$ (or $\mathbb{C}^N$ when the base field is $\mathbb{C}$) with the norm $\|\cdot\|_p$, where $1 \le p \le \infty$ are usually denoted by ℓ_p^N in the literature. ■

Example 2.2.4 The notation $\|\cdot\|_\infty$ for the norm defined in Example 2.2.3 can be 'justified' as follows. Let $x \in \mathbb{R}^N$. Assume that the maximum for $|x_i|$ is attained for a *single* index, say, i_0. Then,

$$\|x\|_p = |x_{i_0}|\left[1 + \sum_{i\ne i_0}\left(\frac{|x_i|}{|x_{i_0}|}\right)^p\right]^{\frac{1}{p}}.$$

Thus, since $|x_i|/|x_{i_0}| < 1$ for $i \neq i_0$, we get that

$$\|x\|_p \to \|x\|_\infty$$

when $p \to \infty$. ■

Remark 2.2.4 We now consider sets of real (or complex) sequences

$$x = (x_1, x_2, \ldots, x_i, \ldots).$$

Let $1 \leq p < \infty$. We define the space

$$\ell_p = \left\{ x | \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}.$$

We define vector addition and scalar multiplication (over the corresponding field) componentwise, i.e. if $x = (x_i)$ and $y = (y_i)$ are sequences in ℓ_p and if α is a scalar, we set

$$x + y = (x_i + y_i) \text{ and } \alpha x = (\alpha x_i).$$

We also define

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}.$$

We will prove the triangle inequality for $\|.\|_p$ (which will also simultaneously show that ℓ_p is closed under vector addition and hence that it is a vector space). Since properties (i) and (ii) of Definition 2.1.2 are obvious, it will follow that $\|\cdot\|_p$ defines a norm on ℓ_p.

Let $x, y \in \ell_p$. Then, for any positive integer N, we have

$$\begin{aligned} \sum_{i=1}^{N} |x_i + y_i|^p &\leq \left[\left(\sum_{i=1}^{N} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{N} |y_i|^p \right)^{\frac{1}{p}} \right]^p \\ &\leq \left[\|x\|_p + \|y\|_p \right]^p \end{aligned}$$

using Minkowski's inequality (2.2.5) for the integer N. Thus, since N was arbitrary, we deduce that

$$\sum_{i=1}^{\infty} |x_i + y_i|^p \leq \left[\|x\|_p + \|y\|_p \right]^p < \infty$$

which shows that $x + y \in \ell_p$ and also proves the triangle inequality for $\|\cdot\|_p$. ■

Remark 2.2.5 The triangle inequality

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

for x and y in ℓ_p is again referred to as Minkowski's inequality. We can also prove Hölder's inequality: if $x \in \ell_p$ and $y \in \ell_{p^*}$ where p^* is the conjugate exponent, then

$$\sum_{i=1}^{\infty} |x_i y_i| \;\le\; \|x\|_p \|y\|_{p^*}. \qquad \blacksquare$$

Proposition 2.2.1 *Let $1 \le p < \infty$. Then ℓ_p is a Banach space.*

Proof We just need to prove the completeness of the space. Let $\{x^{(n)}\}$ be a Cauchy sequence in ℓ_p, i.e. given $\varepsilon > 0$, there exists N such that

$$\sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(l)}|^p \;<\; \varepsilon \tag{2.2.6}$$

for all $m \ge N, l \ge N$. Thus, it is clear that for each fixed subscript i, the sequence $\{x_i^{(n)}\}$ is Cauchy in $\mathbb{R}$ (or $\mathbb{C}$, as the case may be). Thus, there exists x_i such that $x_i^{(n)} \to x_i$ for each i. Set

$$x = (x_1, x_2, \ldots, x_i, \ldots).$$

We will first show that $x \in \ell_p$. Since $\{x^{(n)}\}$ is a Cauchy sequence, it is bounded. Thus, there exists a $C > 0$ such that

$$\|x^{(n)}\|_p^p \le C, \text{ for all } n.$$

Let k be any fixed positive integer. Then,

$$\sum_{i=1}^{k} |x_i^{(n)}|^p \le C$$

which implies that

$$\sum_{i=1}^{k} |x_i|^p \le C.$$

Since k is arbitrary, this shows that

$$\sum_{i=1}^{\infty} |x_i|^p \le C < \infty.$$

This shows that $x \in \ell_p$.

Now, for any positive integer k and all $m, l \geq N$, it follows from (2.2.6) that

$$\sum_{i=1}^{k} |x_i^{(m)} - x_i^{(l)}|^p < \varepsilon.$$

Passing to the limit as $l \to \infty$, we get that for any $m \geq N$ and for any k,

$$\sum_{i=1}^{k} |x_i^{(m)} - x_i|^p \leq \varepsilon.$$

Since k is arbitrary, we deduce that for $m \geq N$,

$$\|x^{(m)} - x\|_p^p \leq \varepsilon,$$

i.e. $x^{(n)} \to x$ in ℓ_p. This completes the proof. ■

Example 2.2.5 Set

$$\ell_\infty = \left\{ x = (x_i) | \sup_{1 \leq i < \infty} |x_i| < +\infty \right\},$$

i.e. the space of all bounded real (or complex) sequences. This is clearly a vector space under componentwise addition and scalar multiplication. Define

$$\|x\|_\infty = \sup_{1 \leq i < \infty} |x_i|.$$

This makes ℓ_∞ a Banach space (check!). ■

Remark 2.2.6 Once again, Hölder's inequality holds for $p = 1$ as well. ■

Remark 2.2.7 For those readers who are acquainted with measure theory, the spaces ℓ_p^N and ℓ_p, for $1 \leq p \leq \infty$ are particular cases of the Lebesgue spaces $L^p(\mu)$ where $\{X, \mathcal{S}, \mu\}$ is a measure space. In the case of ℓ_p^N, we have $X = \{1, 2, \ldots, N\}$ and in the case of ℓ_p we have $X = \mathbb{N}$, the set of all natural numbers. In either case, the σ-algebra is the collection of all subsets of X and the measure μ is the counting measure. We will study L^p spaces in detail in Chap. 6. ■

Our final example is that of a function space.

Example 2.2.6 Let $\mathcal{C}[0, 1]$ denote the set of all continuous real-valued functions on the closed interval $[0, 1]$. This becomes a vector space under the operations of addition and scalar multiplications defined pointwise, i.e.

$$(f + g)(x) = f(x) + g(x) \text{ and } (\alpha f)(x) = \alpha f(x)$$

for f and g in $\mathcal{C}[0,1]$, $\alpha \in \mathbb{R}$ and for $x \in [0,1]$. Define

$$\|f\| = \sup_{x\in[0,1]} |f(x)| \left(= \max_{x\in[0,1]} |f(x)| \right).$$

This is well defined since $[0,1]$ is compact and so every continuous function is bounded and attains its maximum. The verification that this defines a norm on $\mathcal{C}[0,1]$ is routine and is left to the reader.

Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}[0,1]$. This implies that for every $\varepsilon > 0$, there exists a positive integer N such that, for all $x \in [0,1]$ and for all $n \geq N$ and $m \geq N$, we have

$$|f_n(x) - f_m(x)| < \varepsilon. \tag{2.2.7}$$

Thus the pointwise sequences $\{f_n(x)\}$ are all Cauchy and hence convergent. Define

$$f(x) = \lim_{n\to\infty} f_n(x).$$

We will show that the function f thus defined is in $\mathcal{C}[0,1]$ and that $\|f_n - f\| \to 0$. This will show that $\mathcal{C}[0,1]$, with the given norm, is a Banach space.

Let ε and N be as above. Then, keeping $n \geq N$ fixed and passing to the limit as $m \to \infty$ in (2.2.7), we get

$$|f_n(x) - f(x)| \leq \varepsilon \tag{2.2.8}$$

for all $n \geq N$ and for all $x \in [0,1]$. Fix a point $x_0 \in [0,1]$. Since f_N is continuous, there exists $\delta > 0$ such that for all $|x_0 - y| < \delta$, we have

$$|f_N(x_0) - f_N(y)| < \varepsilon.$$

Thus, if $|x_0 - y| < \delta$, we get, using the above inequality and also the inequality (2.2.8),

$$|f(x_0) - f(y)| \leq |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(y)| + |f_N(y) - f(y)| \leq 3\varepsilon.$$

This proves that f is continuous and from (2.2.8), we see that $\|f_n - f\| \to 0$. ■

Remark 2.2.8 The convergence described in the preceding example is what is known in the literature as *uniform convergence*. The norm is often referred to as the 'sup-norm'. ■

We conclude this section by showing a standard method of producing new normed linear spaces from existing ones. Let V be a normed linear space and let W be a closed subspace of V, i.e. W is a linear subspace of V and is closed under the norm topology. We define an equivalence relation on V by

$$x \sim y \Leftrightarrow x - y \in W.$$

The equivalence class containing a vector $x \in V$ is called a coset and is denoted as $x + W$. It consists of all elements of the form $x + w$ where $w \in W$. The set of all cosets is called the **quotient space** and is denoted V/W. Addition and scalar multiplication on V/W are defined by

$$(x + W) + (y + W) = (x + y) + W \text{ and } \alpha(x + W) = \alpha x + W.$$

If $x \sim x'$ and $y \sim y'$, then, clearly, $x + y \sim x' + y'$ and $\alpha x \sim \alpha x'$, since W is a linear subspace of V. Thus, addition and scalar multiplication are well defined. Thus the quotient space becomes a vector space. On this, we define

$$\|x + W\|_{V/W} = \inf_{w \in W} \|x + w\|.$$

In other words, the 'norm' defined above is the infimum of the norms of all the elements in the coset and so, clearly, it is well defined.

Proposition 2.2.2 *Let V be a normed linear space and let W be a* closed *subspace. Then, $\|\cdot\|_{V/W}$ defined above is a norm on the quotient space V/W. Further, if V is a Banach space, so is V/W.*

Proof Clearly $\|x + W\|_{V/W} \geq 0$ for all $x \in V$. If $x + W = \mathbf{0} + W$ in V/W, we have $x \in W$; then $-x \in W$ and so $0 \leq \|x + W\|_{V/W} \leq \|x + (-x)\| = 0$ and so $\|x + W\|_{V/W} = 0$. Conversely, if $\|x + W\|_{V/W} = 0$, then, by definition, there exists a sequence $\{w_n\}$ in W such that $\|x + w_n\| \to 0$. This means that $w_n \to -x$ in V and, since W is closed, it follows that $-x \in W$ and so $x \in W$ as well. This means that $x \sim \mathbf{0}$, i.e. $x + W$ is the zero element of V/W.

If $\alpha \neq 0$, then $\alpha x + w = \alpha(x + w')$ where $w' = \alpha^{-1}w \in W$. From this it is easy to see that $\|\alpha x + W\|_{V/W} = |\alpha|\|x + W\|_{V/W}$. The case $\alpha = 0$ is obvious.

Finally, we prove the triangle inequality.

$$\begin{aligned}
\|x + y + W\|_{V/W} &= \inf\{\|x + y + w\| | w \in W\} \\
&= \inf\{\|x + y + w + w'\| | w, w' \in W\} \\
&\leq \inf\{\|x + w\| + \|y + w'\| | w, w' \in W\} \\
&= \inf\{\|x + w\| | w \in W\} + \inf\{\|y + w'\| | w' \in W\} \\
&= \|x + W\|_{V/W} + \|y + W\|_{V/W}.
\end{aligned}$$

Thus, V/W is a normed linear space. Now assume that V is complete. Let $\{x_n + W\}$ be a Cauchy sequence in V/W. Then, we can find a subsequence such that

$$\|(x_{n_k} + W) - (x_{n_{k+1}} + W)\|_{V/W} < \frac{1}{2^k}.\text{(why?)}$$

Now choose $y_k \in x_{n_k} + W$ such that $\|y_k - y_{k+1}\| < 1/2^k$. Then the sequence $\{y_k\}$ is Cauchy (why?) and so, since V is complete, $y_k \to y$ in V. Thus

$$\|(x_{n_k} + W) - (y + W)\|_{V/W} \leq \|y_k - y\| \to 0.$$

Thus, the Cauchy sequence $\{x_n + W\}$ has a convergent subsequence $\{x_{n_k} + W\}$ and so the Cauchy sequence itself must be convergent and converge to the same limit (why?). Hence V/W is complete. ■

2.3 Continuous Linear Transformations

An important aspect of functional analysis is to study mappings between normed linear spaces which 'respect' the linear and topological structures. We make this notion precise in the following definition.

Definition 2.3.1 Let V and W be normed linear spaces. A linear transformation $T : V \to W$ is said to be a **continuous linear transformation** or, a **continuous linear operator**, if it is continuous as a map between the topological spaces V and W (endowed with their norm topologies). If W is the base field, then a continuous linear transformation is called a **continuous linear functional**. ■

Definition 2.3.2 A subset of a normed linear space is **bounded** if it can be contained in a ball. ■

The following proposition gives an important characterization of continuous linear transformations.

Proposition 2.3.1 *Let V and W be normed linear spaces and let $T : V \to W$ be a linear transformation. The following are equivalent:*

(i) T is continuous;
(ii) T is continuous at $\mathbf{0}$;
(iii) there exists a constant $K > 0$ such that, for all $x \in V$,

$$\|T(x)\|_W \leq K\|x\|_V \tag{2.3.1}$$

where $\|\cdot\|_V$ and $\|\cdot\|_W$ denote the respective norms in the spaces V and W;
(iv) if $B = \{x \in V \mid \|x\|_V \leq 1\}$ is the (closed) unit ball in V, then $T(B)$ is a bounded set in W.

Proof (i) $\Leftrightarrow$ (ii) If T is continuous, then, clearly, it is continuous at $\mathbf{0} \in V$. Conversely, let T be continuous at $\mathbf{0} \in V$. Let $x \in V$ be arbitrary and let $x_n \to x$ in V. Then $x_n - x \to \mathbf{0}$ in V and so, $T(x_n - x) \to \mathbf{0}$ in W, i.e. $T(x_n) \to T(x)$ in W. Thus, T is continuous.

(ii) $\Leftrightarrow$ (iii) If T is continuous at $\mathbf{0} \in V$, there exists a $\delta > 0$ such that $\|x\|_V < \delta$ implies that $\|T(x)\|_W < 1$. For any $x \in X$, set $y = \frac{\delta}{2\|x\|_V}x$ so that $\|y\|_V = \delta/2 < \delta$ and so $\|T(y)\|_W < 1$. By linearity, it follows that

$$\|T(x)\|_V \leq \frac{2}{\delta}\|x\|_V$$

which proves (2.3.1) with $K = 2/\delta$. Conversely, if (2.3.1) is true, then whenever $x_n \to \mathbf{0}$ in V, it follows that $T(x_n) \to \mathbf{0}$ in W, i.e. T is continuous at $\mathbf{0}$.

(iii) $\Leftrightarrow$ (iv) By virtue of (2.3.1), it follows that $\|T(x)\|_W \leq K$ for all $x \in B$. Thus, $T(B)$ is bounded in W. Conversely, if $T(B)$ is bounded in W, there exists a $K > 0$ such that $\|T(x)\|_W \leq K$ for all $x \in B$. Now, if $\mathbf{0} \neq x \in V$ is arbitrary, set $y = x/\|x\|$. Then $\|T(y)\|_W \leq K$ from which (2.3.1) follows, by linearity. ■

Remark 2.3.1 Continuous linear transformations are also known as **bounded linear transformations** since they map bounded sets into bounded sets. ■

The above proposition inspires the following definition.

Definition 2.3.3 Let V and W be normed linear spaces and let $T : V \to W$ be a continuous linear transformation. Let B be the closed unit ball in V. The **norm** of T, denoted $\|T\|$, is given by

$$\|T\| = \sup_{x \in B} \|T(x)\|_W. \tag{2.3.2}$$

■

The following proposition gives alternative characterizations of the norm of a continuous linear transformation.

Proposition 2.3.2 *Let V and W be normed linear spaces and let $T : V \to W$ be a continuous linear transformation. Then*

$$\begin{aligned}\|T\| &= \sup\{\|T(x)\|_W | \|x\|_V = 1\}\\ &= \sup\{\|T(x)\|_W/\|x\|_V | \mathbf{0} \neq x \in V\}\\ &= \inf\{K > 0 | \|T(x)\|_W \leq K\|x\|_V \text{for all } x \in V\}.\end{aligned}$$

Proof Let us set

$$\begin{aligned}\alpha &= \sup\{\|T(x)\|_W | \|x\|_V = 1\},\\ \beta &= \sup\{\|T(x)\|_W/\|x\|_V | \mathbf{0} \neq x \in V\} \text{and}\\ \gamma &= \inf\{K > 0 | \|T(x)\|_W \leq K\|x\|_V \text{ for all } x \in V\}.\end{aligned}$$

Clearly, $\alpha \leq \beta$. If x is a non-zero vector in V, then $x/\|x\|_V$ has unit norm and $\|T(x)\|_W/\|x\|_V = \|T(x/\|x\|_V)\|_W$. This shows that we also have $\beta \leq \alpha$. If $K > 0$ is any number in the set defining γ, then it follows immediately that $\beta \leq K$ and so, *a fortiori*, we have $\beta \leq \gamma$. Now, we also have that $\|T(x)\|_W \leq \beta\|x\|_V$ for all $x \in V$ and so, by definition, $\gamma \leq \beta$. Thus, we have

$$\alpha = \beta = \gamma.$$

Clearly $\|T\| \geq \alpha$ by definition. If K is in the set defining γ, then, for all $x \in V$ such that $\|x\|_V \leq 1$, we have $\|T(x)\|_W \leq K$ and so $\|T\| \leq K$. Thus, we get that $\|T\| \leq \gamma = \alpha$. Thus we get that

$$\|T\| = \alpha = \beta = \gamma. \qquad \blacksquare$$

Corollary 2.3.1 *If V and W are normed linear spaces and if $T : V \to W$ is a continuous linear transformation, then*

$$\|T(x)\|_W \leq \|T\|\|x\|_V \tag{2.3.3}$$

for all $x \in V$. $\blacksquare$

Let V and W be normed linear spaces. Let us denote by $\mathcal{L}(V, W)$, the set of all continuous linear maps from V into W. If T_1 and T_2 are such maps, let us define $T_1 + T_2$ by

$$(T_1 + T_2)(x) = T_1(x) + T_2(x)$$

for all $x \in V$. Clearly, $T_1 + T_2$ is also a linear transformation. Now,

$$\|(T_1 + T_2)(x)\|_W \leq \|T_1(x)\|_W + \|T_2(x)\|_W \leq (\|T_1\| + \|T_2\|)\|x\|_V$$

by virtue of the triangle inequality and the above corollary. Thus, it follows that $T_1 + T_2$ is also a continuous linear transformation and that

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|.$$

Similarly, if T is a continuous linear transformation and if α is a scalar, we define

$$(\alpha T)(x) = \alpha T(x)$$

for all $x \in V$. It is then easy to see that αT is also continuous and that

$$\|\alpha T\| = |\alpha|\|T\|.$$

The zero element of $\mathcal{L}(V, W)$ is the trivial map which maps every element of V into the null vector of W. The element $-T$ is defined by $(-T)(x) = -T(x)$. Thus, $\mathcal{L}(V, W)$ is a vector space; in fact, it is a normed linear space for the norm of a continuous linear transformation defined above.

Proposition 2.3.3 *Let V and W be normed linear spaces. If W is complete, then $\mathcal{L}(V, W)$ is also complete.*

Proof Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(V, W)$. Then, given $\varepsilon > 0$, we can find a positive integer N such that, for all m and $n \geq N$, we have

$$\|T_n - T_m\| < \varepsilon.$$

Let $x \in V$. Then,

$$\|T_n(x) - T_m(x)\|_W \leq \|T_n - T_m\|\|x\|_V$$

and so it follows that the sequence $\{T_n(x)\}$ is Cauchy in W. Since W is complete, this sequence is convergent. Let us define

$$T(x) = \lim_{n\to\infty} T_n(x).$$

Clearly, the map $x \mapsto T(x)$ is linear. We will show that it is continuous and that $\|T_n - T\| \to 0$. This will complete the proof.

Since the sequence $\{T_n\}$ is Cauchy, it is bounded, i.e. there exists $M > 0$ such that, for all positive integers n, we have $\|T_n\| \leq M$. Now, since for any $x \in V$, we have $\|T_n(x)\|_W \leq \|T_n\|\|x\|_V \leq M\|x\|_V$, it follows, on passing to the limit as $n \to \infty$, that, for all $x \in V$,

$$\|T(x)\|_W \leq M\|x\|_V.$$

Thus, T is continuous and so $T \in \mathcal{L}(V, W)$.

Let $\varepsilon > 0$ and let N be as defined earlier by the Cauchy property of the given sequence. Let B be the closed unit ball in V. For all $x \in B$, we have

$$\|T_n(x) - T_m(x)\|_W \leq \|T_n - T_m\| < \varepsilon.$$

Keeping n fixed and letting m tend to infinity, we get

$$\|T_n(x) - T(x)\|_W \leq \varepsilon$$

for all $x \in B$. This shows that, for $n \geq N$, we have $\|T_n - T\| \leq \varepsilon$, i.e. $T_n \to T$ in $\mathcal{L}(V, W)$. This completes the proof. ■

In particular, since the scalar field is a Banach space over itself (cf. Example 2.2.1), the set of all continuous linear functionals $\mathcal{L}(V, \mathbb{R})$ (or, $\mathcal{L}(V, \mathbb{C})$, as the case may be) is always a Banach space.

Definition 2.3.4 Let V be a normed linear space. The space of all continuous linear functionals on V is a Banach space and is called the **dual space** of V. It is denoted by V^*. ■

Another particular case is when $W = V$. In this case we write $\mathcal{L}(V)$ for the space of all continuous linear operators instead of $\mathcal{L}(V, V)$. This space is Banach if V is Banach. On this space, we have a third operation (after addition and scalar multiplication), namely composition of operators: if T_1 and T_2 are continuous linear operators, we define T_1T_2 by

$$(T_1T_2)(x) = T_1(T_2(x)).$$

Now, for any $x \in V$, we have

$$\|(T_1T_2)(x)\|_V \leq \|T_1\|\|T_2(x)\|_V \leq \|T_1\|\|T_2\|\|x\|_V.$$

Thus, T_1T_2 is also a continuous linear operator and, further,

$$\|T_1T_2\| \leq \|T_1\|\|T_2\|. \tag{2.3.4}$$

Further, multiplication is a continuous operation. Indeed, if $T_n \to T$ and $T_n' \to T'$ in $\mathcal{L}(V)$, we have

$$\|T_nT_n' - TT'\| \leq \|T_n\|\|T_n' - T'\| + \|T'\|\|T_n - T\|.$$

Since $\|T_n\|$ is bounded independent of n, it follows that $T_nT_n' \to TT'$. Finally, if I is the identity mapping, i.e. $I(x) = x$ for all $x \in V$, we have

$$\|I\| = 1.$$

Definition 2.3.5 A Banach space V on which we have a multiplication operation $(x, y) \in V \times V \mapsto xy \in V$ such that addition and multiplication make it a ring and such that

$$\|xy\| \leq \|x\|\|y\| \text{ and } \|\mathbf{1}\| = 1$$

where $\mathbf{1}$ is the multiplicative identity in V, is called a **Banach algebra**. ■

Thus, $\mathcal{L}(V)$, where V is a Banach space, is a Banach algebra.

Let us now study various examples of continuous linear transformations.

Example 2.3.1 Any linear transformation $T : \mathbb{R}^N \to \mathbb{R}^M$ is given by an $M \times N$ matrix. Assume that $\mathbb{R}^N$ is the space ℓ_1^N. Then, for any norm on $\mathbb{R}^M$, any linear transformation is continuous. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ be the standard basis of $\mathbb{R}^N$ (cf. Example 1.1.2). If $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, then $x = \sum_{i=1}^N x_i\mathbf{e}_i$. Then $T(x) = \sum_{i=1}^N x_iT(\mathbf{e}_i)$. Thus,

$$\|T(x)\|_{\mathbb{R}^M} \leq K \sum_{i=1}^{N} |x_i| = K\|x\|_1$$

where

$$K = \max_{1 \leq i \leq N} \|T(\mathbf{e}_i)\|_{\mathbb{R}^M}.$$

■

Example 2.3.2 Let $a_1, ..., a_N$ be scalars. For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, define

$$f(x) = \sum_{i=1}^{N} a_i x_i.$$

Then f is a linear functional on $\mathbb{R}^N$ which is continuous if $\mathbb{R}^N = \ell_1^N$. ■

We will see later that these transformations and functionals are continuous for *any* norm defined on $\mathbb{R}^N$.

Example 2.3.3 Let $x = (x_i) \in \ell_2$. Define

$$T(x) = \left(\frac{x_1}{1}, \frac{x_2}{2}, \ldots, \frac{x_i}{i}, \ldots\right).$$

Then, since

$$\sum_{i=1}^{\infty} \left|\frac{x_i}{i}\right|^2 \leq \sum_{i=1}^{\infty} |x_i|^2 < \infty,$$

we have that T is a continuous linear operator on ℓ_2 and that $\|T\| \leq 1$. The map T is not onto. In fact, the range of T consists of all square summable sequences (y_i) such that

$$\sum_{i=1}^{\infty} i^2 |y_i|^2 < \infty.$$

■

Example 2.3.4 Let $1 \leq p \leq \infty$ and let p^* be its conjugate exponent. For $x \in \ell_p$ and $y \in \ell_{p^*}$, define

$$f_y(x) = \sum_{i=1}^{\infty} x_i y_i.$$

Then, by Hölder's inequality, we have

$$|f_y(x)| \leq \sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_{p^*}.$$

Thus f_y defines a continuous linear functional on ℓ_p and

$$\|f_y\| \leq \|y\|_{p^*}.$$

We will see later that, when $1 \leq p < \infty$, all continuous linear functionals on ℓ_p occur only in this way and that the last inequality is, in fact, an equality. ■

Example 2.3.5 Consider an infinite matrix $(a_{ij})_{i,j=1}^{\infty}$. This can be used to define a linear mapping on ℓ_p as follows. Let $x = (x_i) \in \ell_p$. Define a sequence $A(x)$ by

$$A(x)_i = \sum_{j=1}^{\infty} a_{ij} x_j.$$

Showing that A defines a continuous linear map of ℓ_p into itself is usually a non-trivial problem and some examples are given in the exercises at the end of this chapter. We now give an example (due to Schur).

Assume that $a_{ij} \geq 0$ for all i and j. Assume further there exists a sequence $\{p_i\}$ of positive real numbers and $\beta > 0$ and $\gamma > 0$ such that

$$\sum_{i=1}^{\infty} a_{ij} p_i \leq \beta p_j$$

for all $j \in \mathbb{N}$ and also such that

$$\sum_{j=1}^{\infty} a_{ij} p_j \leq \gamma p_i$$

for all $i \in \mathbb{N}$. Then $A \in \mathcal{L}(\ell_2)$ and $\|A\|^2 \leq \beta\gamma$.

To see this, let $x = (x_i) \in \ell_2$. We write

$$\sum_{j=1}^{\infty} a_{ij} x_j = \sum_{j=1}^{\infty} \sqrt{a_{ij}} \sqrt{p_j} \frac{\sqrt{a_{ij}} x_j}{\sqrt{p_j}}.$$

Applying the Cauchy-Schwarz inequality, this yields

$$|A(x)_i|^2 \leq \left(\sum_{j=1}^{\infty} a_{ij} p_j \right) \left(\sum_{j=1}^{\infty} \frac{a_{ij} |x_j|^2}{p_j} \right).$$

It follows from the hypotheses that

$$\begin{aligned}\sum_{i=1}^{\infty} |A(x)_i|^2 &\leq \sum_{i=1}^{\infty} \gamma p_i \sum_{j=1}^{\infty} \tfrac{a_{ij}|x_j|^2}{p_j} \\ &= \gamma \sum_{j=1}^{\infty} \tfrac{|x_j|^2}{p_j} \sum_{i=1}^{\infty} a_{ij} p_i \\ &\leq \gamma \sum_{j=1}^{\infty} \tfrac{|x_j|^2}{p_j} \beta p_j \\ &= \beta\gamma \|x\|_2^2\end{aligned}$$

which establishes the claim.

An interesting particular case is that of the *Hilbert matrix*. Set

$$a_{ij} = \frac{1}{i + j + 1}$$

for $0 \leq i, j \leq \infty$. Set $p_i = 1/\sqrt{i + \frac{1}{2}}$. Since the matrix is symmetric, it suffices to check one of the two conditions. Now,

$$\begin{aligned}\sum_{i=0}^{\infty} a_{ij} p_i &= \sum_{i=0}^{\infty} \tfrac{1}{(i+\frac{1}{2}+j+\frac{1}{2})\sqrt{i+\frac{1}{2}}} \\ &< \int_0^{\infty} \frac{\mathrm{d}x}{(x + j + \frac{1}{2})\sqrt{x}} \\ &= 2\int_0^{\infty} \frac{\mathrm{d}t}{t^2 + j + \frac{1}{2}} \\ &= \tfrac{\pi}{\sqrt{j+\frac{1}{2}}}.\end{aligned}$$

Thus, by Schur's test, the matrix defines a continuous linear operator A on ℓ_2 whose norm is less than, or equal to π. (In fact, it has been shown by Hardy, Littlewood and Polya that the norm is exactly π). ■

Example 2.3.6 (Cesàro Operator) Let $x = (x_i) \in \ell_p$ where $1 < p < \infty$. Define

$$(T(x))_n = \frac{x_1 + \cdots + x_n}{n}.$$

We show that $T \in \mathcal{L}(\ell_p)$ and that

$$\|T\| \leq \frac{p}{p-1}.$$

Indeed,

$$|T(x)_n| \leq \frac{|x_1| + \cdots + |x_n|}{n}.$$

Set $A_n = |x_1| + \cdots + |x_n|$ and $\alpha_n = A_n/n$, for $n \geq 1$ and set $x_0 = A_0 = 0$. Then

$$\begin{aligned}
\alpha_n^p - \tfrac{p}{p-1}\alpha_n^{p-1}|x_n| &= \alpha_n^p - \tfrac{p}{p-1}\alpha_n^{p-1}(n\alpha_n - (n-1)\alpha_{n-1}) \\
&= \left(1 - \tfrac{np}{p-1}\right)\alpha_n^p + \tfrac{(n-1)p}{p-1}\alpha_n^{p-1}\alpha_{n-1} \\
&= \left(1 - \tfrac{np}{p-1}\right)\alpha_n^p + \tfrac{(n-1)p}{p-1}(\alpha_n^p)^{\frac{p-1}{p}}(\alpha_{n-1}^p)^{\frac{1}{p}}.
\end{aligned}$$

Recall that $p/(p-1) = p^*$, the conjugate exponent of p. Thus by Lemma 2.2.1, we get

$$\begin{aligned}
\alpha_n^p - \tfrac{p}{p-1}\alpha_n^{p-1}|x_n| &\leq \left(1 - \tfrac{np}{p-1}\right)\alpha_n^p + \tfrac{(n-1)p}{p-1}\left(\tfrac{p-1}{p}\alpha_n^p + \tfrac{1}{p}\alpha_{n-1}^p\right) \\
&= \tfrac{1}{p-1}[(n-1)\alpha_{n-1}^p - n\alpha_n^p].
\end{aligned}$$

Fix a positive integer N. Then summing both sides over n running between 1 and N, and noticing that the right-hand side is a telescoping sum, we get

$$\sum_{n=1}^{N}\alpha_n^p - \frac{p}{p-1}\sum_{n=1}^{N}\alpha_n^{p-1}|x_n| \leq -\frac{N}{p-1}\alpha_N^p \leq 0.$$

By an application of Hölder's inequality, we now deduce that

$$\sum_{n=1}^{N}\alpha_n^p \leq \frac{p}{p-1}\sum_{n=1}^{N}\alpha_n^{p-1}|x_n| \leq \frac{p}{p-1}\left(\sum_{n=1}^{N}\alpha_n^p\right)^{\frac{p-1}{p}}\left(\sum_{n=1}^{N}|x_n|^p\right)^{\frac{1}{p}}.$$

Dividing both sides by $(\sum_{n=1}^{N}\alpha_n^p)^{((p-1)/p)}$, which is strictly positive for non-zero x, we get

$$\left(\sum_{n=1}^{N}\alpha_n^p\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\sum_{n=1}^{N}|x_n|^p\right)^{\frac{1}{p}}.$$

Since N was arbitrarily chosen, we deduce that

$$\|T(x)\|_p \leq \frac{p}{p-1}\|x\|_p$$

which establishes our claim. (In fact, Hardy, Littlewood and Polya also show that $\|T\| = p/(p-1)$.) ■

Example 2.3.7 (Volterra integral operator) Let $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function. For $f \in \mathcal{C}[0, 1]$ and $s \in [0, 1]$, define

$$T(f)(s) = \int_0^s K(s,t)f(t)\mathrm{d}t.$$

Since K is continuous on the compact set $[0, 1] \times [0, 1]$, it is bounded and uniformly continuous. Assume that, for all s and t in $[0, 1]$, we have

$$|K(s,t)| \le \kappa.$$

Further, given $\varepsilon > 0$, there exists $\delta > 0$ such that, whenever $|s_1 - s_2| < \delta$, we have

$$|K(s_1,t) - K(s_2,t)| < \varepsilon$$

for all $t \in [0, 1]$, by virtue of the uniform continuity. Without loss of generality, we can assume that $\delta \le \varepsilon$. Thus,

$$T(f)(s_1) - T(f)(s_2) = \int_0^{s_1} (K(s_1,t) - K(s_2,t))f(t)\mathrm{d}t + \int_{s_2}^{s_1} K(s_2,t)f(t)\mathrm{d}t.$$

If $\|f\|$ denotes the norm of $f \in \mathcal{C}[0, 1]$ as defined in Example 2.2.6, we have

$$|T(f)(s_1) - T(f)(s_2)| \le \varepsilon\|f\|s_1 + \delta\kappa\|f\| < (1+\kappa)\|f\|\varepsilon$$

whenever $|s_1 - s_2| < \delta$. This shows that $T(f)$ is a continuous function. As the mapping T being clearly linear, it thus defines a linear operator on $\mathcal{C}[0, 1]$. Further,

$$|T(f)(s)| \le \kappa\|f\|s \le \kappa\|f\|.$$

Thus T is a continuous linear operator on $\mathcal{C}[0, 1]$ and $\|T\| \le \kappa$. ■

So far, we have been seeing examples of continuous linear transformations. We now give an example of a linear transformation which is *not* continuous.

Example 2.3.8 Consider the space $\mathcal{C}^1[0, 1]$ of continuous functions on $[0, 1]$ which are continuously differentiable on $(0, 1)$ and whose derivatives can be extended continuously to $[0, 1]$. This is a subspace of $\mathcal{C}[0, 1]$. Let both these spaces be endowed with the 'sup-norm' (cf. Example 2.2.6). Then, the map $T : \mathcal{C}^1[0, 1] \to \mathcal{C}[0, 1]$ defined by $T(f) = f'$, where f' denotes the derivative of f, is not continuous. To see this, consider the sequence of functions $\{f_n\}$ defined by $f_n(t) = t^n$ for $n \ge 1$. Then, it is easy to see that $\|f_n'\| = n$ while $\|f_n\| = 1$. Hence there can be no constant $C > 0$ such that $\|T(f)\| \le C\|f\|$ for all $f \in \mathcal{C}^1[0, 1]$. Thus, T is not continuous. ■

Definition 2.3.6 Let V be a normed linear space and let $T \in \mathcal{L}(V)$ be a bijection. If T^{-1} is also continuous, then T is said to **invertible** or an **isomorphism**. ■

Note: When dealing with normed linear spaces, the word isomorphism is understood in the topological sense: not only is it an isomorphism in the usual algebraic sense, i.e. it is linear and is a bijection, but it also implies that both the mapping and its inverse are continuous.

Definition 2.3.7 Two norms defined on the same vector space are said to be **equivalent** if the topologies induced by these two norms coincide. ■

Proposition 2.3.4 *Let V be a vector space and let $\|\cdot\|_{(1)}$ and $\|\cdot\|_{(2)}$ be two norms defined on it. The two norms are equivalent if, and only if, there exist two constants $C_1 > 0$ and $C_2 > 0$ such that, for all $x \in V$, we have*

$$C_1\|x\|_{(1)} \leq \|x\|_{(2)} \leq C_2\|x\|_{(1)}.$$

Proof The topologies induced by the two norms coincide if, and only if, the identity mapping

$$I : \{V, \|\cdot\|_{(1)}\} \to \{V, \|\cdot\|_{(2)}\}$$

is an isomorphism. This is equivalent to saying that there exist two constants $K_1 > 0$ and $K_2 > 0$ such that

$$\|x\|_{(2)} \leq K_2\|x\|_{(1)} \text{ and } \|x\|_{(1)} \leq K_1\|x\|_{(2)}$$

for all $x \in V$. This proves the proposition on setting $C_1 = K_1^{-1}$ and $C_2 = K_2$. ■

Example 2.3.9 Let $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ (or $\mathbb{C}^N$). Then, clearly,

$$\|x\|_\infty \leq \|x\|_1 \leq N\|x\|_\infty.$$

Thus these two norms are equivalent and the topologies induced on $\mathbb{R}^N$ (respectively, $\mathbb{C}^N$) by the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ coincide. It is a simple matter to check that the topology induced by $\|\cdot\|_\infty$ is none other than the product topology on $\mathbb{R}^N$ (respectively $\mathbb{C}^N$) when $\mathbb{R}$ (respectively $\mathbb{C}$) is given its usual topology for every component.

Since $(|x_1|^2 + \cdots + |x_N|^2) \leq (|x_1| + \cdots + |x_N|)^2$, we have that $\|x\|_2 \leq \|x\|_1$. An application of the Cauchy-Schwarz inequality shows that, on the other hand, $\|x\|_1 \leq \sqrt{N}\|x\|_2$. Thus, for every $x \in \mathbb{R}^N$, we have that

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{N}\|x\|_2.$$

It is also very easy to see that

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{N}\|x\|_\infty.$$

Thus all the three norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent and the corresponding topology on $\mathbb{R}^N$, in each case, is the usual topology. ■

We will, in fact, now prove a much stronger result.

Proposition 2.3.5 *Any two norms on a finite dimensional vector space are equivalent.*

Proof Let V be a finite dimensional normed linear space with dimension N. We will show that V is isomorphic to the space ℓ_1^N. Thus, given two norms on V, it will be isomorphic to ℓ_1^N for each of those norms, and from this we will deduce the equivalence of the norms.

Step 1. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ be the standard basis of ℓ_1^N. Fix a basis $\{v_1, \ldots, v_N\}$ for V. Define $T : \ell_1^N \to V$ by setting $T(\mathbf{e}_i) = v_i$ for all $1 \leq i \leq N$ and then extending T linearly to all of ℓ_1^N. Clearly T is a bijection and an identical argument as in Example 2.3.1 shows that T is continuous.

Step 2. Assume, if possible, that T^{-1} is not continuous. Then the continuity must fail at $\mathbf{0}$ and so we can find a sequence $\{y_n\}$ in V and a real number $\varepsilon > 0$ such that $\|T^{-1}(y_n)\|_1 \geq \varepsilon > 0$ while $y_n \to \mathbf{0}$. Set $z_n = y_n/\|T^{-1}(y_n)\|_1$. Then, $z_n \to \mathbf{0}$ and $\|T^{-1}(z_n)\|_1 = 1$. Now, the set

$$B = \{x \in \ell_1^N \mid \|x\|_1 \leq 1\}$$

is compact. To see this, observe that B is a closed and bounded set and, as we saw above, the topology on ℓ_1^N is the same as the usual topology on $\mathbb{R}^N$. Consequently, it is also sequentially compact, and so, there exists a subsequence $\{z_{n_k}\}$ such that $\{T^{-1}(z_{n_k})\}$ is convergent. Let $T^{-1}(z_{n_k}) \to x$, where $\|x\|_1 = 1$. Since T is continuous, we then deduce that $z_{n_k} \to T(x)$ which then implies that $T(x) = \mathbf{0}$. But T is a one-one map and $\|x\|_1 = 1$ implies that $x \neq \mathbf{0}$ which shows that $T(x) \neq \mathbf{0}$ as well. This gives us a contradiction. Hence T^{-1} must also be continuous.

Step 3. Thus, whatever be the norm on V, the same map T is always an isomorphism between ℓ_1^N and V which implies in turn that the identity map on V, considered as a map of normed linear spaces when V is provided with two different norms, must be an isomorphism as well. Hence any two norms on V are equivalent. ■

Remark 2.3.2 We mentioned in Proposition 1.2.5 that a set in $\mathbb{R}^N$ (or $\mathbb{C}^N$) is compact if, and only if, it is closed and bounded. However, the topology given there is the 'usual' topology, *viz.* that of ℓ_2^N. However, thanks to the preceding proposition, we now know that the topology is the same for all the spaces ℓ_p^N or, for any other norm on $\mathbb{R}^N$ (respectively, $\mathbb{C}^N$) and so a subset thereof will be compact if, and only if, it is bounded and closed. ■

Corollary 2.3.2 *Any finite dimensional normed linear space is complete. In particular, any finite dimensional subspace of a normed linear space is closed.* ■

Example 2.3.10 Let $f \in \mathcal{C}[0, 1]$. Define

$$\|f\|_1 = \int_0^1 |f(t)|\, \mathrm{d}t.$$

It is simple to check that this defines a norm on $\mathcal{C}[0, 1]$. Consider the sequence $\{f_n\}$ defined by

$$f_n(x) = \begin{cases} 1 - nx, & \text{for } 0 \le x \le \frac{1}{n} \\ 0, & \text{for } \frac{1}{n} \le x \le 1. \end{cases}$$

Clearly, $\|f_n\| = 1$ for all n (where $\|\cdot\|$ denotes the usual 'sup' norm) while $\|f_n\|_1 = \int_0^1 f_n(t)\, \mathrm{d}t = 1/2n$ which tends to zero as n tends to infinity. Thus it is clear that these two norms cannot be equivalent. Thus in infinite dimensional spaces, two norms are not, in general, equivalent. ■

Since all norms on $\mathbb{R}^N$ (respectively, $\mathbb{C}^N$) generate the same topology, it is now clear that any matrix generates a continuous linear transformation, whatever the norm on that space may be. Thus, if $\mathbf{T}$ is an $N \times N$ matrix and if $\|\cdot\|$ is a norm on $\mathbb{R}^N$ (respectively, $\mathbb{C}^N$), we can define

$$\|\mathbf{T}\| = \sup_{\|\mathbf{x}\| \le 1, \mathbf{x} \ne \mathbf{0}} \frac{\|\mathbf{T}\mathbf{x}\|}{\|\mathbf{x}\|} \tag{2.3.5}$$

or, via any of the other equivalent formulations as in Proposition 2.3.2. Since the unit ball (and hence the unit sphere) is compact, the 'sup' above is, in fact, a 'max'. If $\mathbf{T}$ and $\mathbf{S}$ are matrices of order N, then $\mathbf{TS}$ represents the composition of the corresponding linear transformations and so we also have

$$\|\mathbf{TS}\| \le \|\mathbf{T}\|.\|\mathbf{S}\|. \tag{2.3.6}$$

Let $\mathcal{M}_N$ denote the set of all matrices of order N with entries from the corresponding field. This itself (under the operations of matrix addition and scalar multiplication) is a vector space (of dimension N^2). Any norm on this space which satisfies (2.3.6) is called a **matrix norm**. If such a norm were induced by a vector norm on $\mathbb{R}^N$ (respectively, $\mathbb{C}^N$) via (2.3.5), then we always have

$$\|\mathbf{I}\| = 1$$

for the identity matrix $\mathbf{I}$.

In particular, for the vector norms $\|\cdot\|_p$ defining the spaces ℓ_p^N for $1 \le p \le \infty$, we denote the induced matrix norms by $\|\cdot\|_{p,N}$.

Example 2.3.11 Since $\mathcal{M}_N$ is a vector space of dimension N^2 over the corresponding field, we can string out its rows to form a vector of that dimension and define the usual Euclidean norm. Thus, if $\mathbf{T} = (t_{ij})$, then define

$$\|\mathbf{T}\|_E = \left(\sum_{i,j=1}^{N} |t_{ij}|^2\right)^{\frac{1}{2}} = \sqrt{\mathrm{tr}(\mathbf{T}^*\mathbf{T})}.$$

This obviously defines a norm on $\mathcal{M}_N$. It is also a matrix norm. For this, we only need to check the validity of (2.3.6). It $\mathbf{T} = (t_{ij})$ and $\mathbf{S} = (s_{ij})$, then

$$\begin{aligned}\|\mathbf{TS}\|_E^2 &= \sum_{i,j=1}^{N} \left|\sum_{k=1}^{N} t_{ik}s_{kj}\right|^2 \\ &\leq \sum_{i,j=1}^{N} \left(\sum_{k=1}^{N} |t_{ik}|^2\right)\left(\sum_{k=1}^{N} \left|s_{kj}\right|^2\right)\end{aligned}$$

by the Cauchy-Schwarz inequality. Thus,we get

$$\|\mathbf{TS}\|_E^2 \leq \sum_{i,k=1}^{N} |t_{ik}|^2 \sum_{k,j=1}^{N} \left|s_{kj}\right|^2 = \|\mathbf{T}\|_E^2\|\mathbf{S}\|_E^2$$

which shows that $\|\cdot\|_E$ is indeed a matrix norm. This is also known as the *Hilbert-Schmidt norm*.

However, notice that this norm is not induced by any vector norm when $N \geq 2$. Indeed to see this, observe that $\|\mathbf{I}\|_E = \sqrt{N} \neq 1$. ■

The proof of the Proposition 2.3.5 depended crucially on the fact that the unit ball in a finite dimensional space is compact. In fact this property characterizes finite dimensional spaces, which we now proceed to show. We begin with a very useful technical result.

Let V be a normed linear space. If $E \subset V$, we define the distance of a vector $x \in V$ from E as

$$d(x, E) = \inf_{y\in E} \|x - y\|.$$

This is the same notion of the distance of a point from a subset in a metric space if we look at V as a metric space for the metric $d(x, y) = \|x - y\|$.

Lemma 2.3.1 *(**Riesz**) Let V be a normed linear space and let $W \subset V$ be a closed and proper subspace. Then, for every $\varepsilon > 0$, we can find a vector $u \in V$ (depending on ε) such that*

$$\|u\| = 1 \quad \text{and} \quad d(u, W) \geq 1 - \varepsilon.$$

Proof Since W is a proper subspace, there exists $v \in V\backslash W$, so that $\delta = d(v, W) > 0$. Now, choose $w \in W$ such that

$$\delta \leq \|v - w\| \leq \frac{\delta}{1 - \varepsilon}.$$

Set $u = (v - w)/\|v - w\|$ so that $\|u\| = 1$. Let $z \in W$ be an arbitrary element. Then

$$\|u - z\| = \|v - (w + \|v - w\|z)\|/\|v - w\| \geq \delta/(\delta/(1 - \varepsilon)) = 1 - \varepsilon$$

by the definition of δ (since $w + \|v - w\|z \in W$) and the choice of w. This completes the proof. ■

Proposition 2.3.6 *A normed linear space V is finite dimensional if, and only if, the closed unit ball in V, i.e. the set*

$$B = \{x \in V \mid \|x\| \leq 1\},$$

is compact.

Proof Assume that V is finite dimensional, with dimension N. Let $T : \ell_1^N \to V$ be the canonical mapping as defined in Proposition 2.3.5. We have seen that T is an isomorphism. It then follows that $T^{-1}(B)$ is bounded and closed in ℓ_1^N and so it is compact (cf. Remark 2.3.2). Consequently $B = T(T^{-1}(B))$ is compact as well.

Conversely, let us suppose that B is compact. Then, there exists a positive integer n and points $x_i \in B$, $1 \leq i \leq n$, such that

$$B \subset \cup_{i=1}^n B(x_i, 1/2), \tag{2.3.7}$$

where $B(x_i, 1/2) = \{x \in V \mid \|x - x_i\| < 1/2\}$ is the open ball centred at x_i and of radius $1/2$. Set $W = \text{span}\{x_1, ..., x_n\}$. We claim that $W = V$ and this will prove that the dimension of V is less than, or equal to, n and so V has to be finite dimensional. Assume the contrary. Then W will be a proper and closed (since it is finite dimensional) subspace of V. Now, by the preceding lemma of Riesz, we have the existence of $u \in V$ such that $\|u\| = 1$ (and so $u \in B$) and such that $d(u, W) \geq 2/3$. In particular, it follows that $u \in B$ is such that $\|u - x_i\| \geq 2/3$ for all $1 \leq i \leq n$ which contradicts (2.3.7). This completes the proof. ■

Example 2.3.12 Consider the space ℓ_2 of all square summable sequences. Consider the sequence $\mathsf{e}_i \in \ell_2$ which has its ith component equal to unity and all other components equal to zero. Then, since $\|\mathsf{e}_i\|_2 = 1$, it belongs to the closed unit ball in that space. Now, if $i \neq j$, we have

$$\|\mathsf{e}_i - \mathsf{e}_j\|_2 = \sqrt{2}$$

and so the the sequence $\{\mathsf{e}_i\}$ can never have a convergent subsequence. Thus, in the infinite dimensional space ℓ_2, we directly see that the unit ball is not sequentially compact and hence it is not compact. ■

2.4 Applications to Differential Equations

One of the famous results in analysis is Banach's contraction mapping theorem (also known as Banach's fixed point theorem), which is stated as follows.

Theorem 2.4.1 *(***Contraction Mapping Theorem***) Let (X, d) be a complete metric space and let $F : X \to X$ be a* contraction*, i.e. there exists a constant $0 < c < 1$ such that, for all x and $y \in X$, we have*

$$d(F(x), F(y)) \leq c\, d(x, y).$$

Then F has a unique fixed point, i.e. there exists a unique point $x^* \in X$ such that

$$F(x^*) = x^*.$$

Further, given any $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = F(x_n), n \geq 0,$$

converges to x^*. ■

Proof Obviously, F is continuous. Let $x_0 \in X$ and let the x_n be as defined in the statement of the theorem. Then, by hypothesis,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(F(x_n), F(x_{n-1})) \\ &\leq cd(x_n, x_{n-1}) \end{aligned}$$

and, proceeding recursively, we deduce that

$$d(x_{n+1}, x_n) \leq c^n d(x_1, x_0).$$

Thus, if $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (c^n + c^{n+1} + \cdots + c^{m-1}) d(x_1, x_0) \end{aligned}$$

which can be made arbitrarily small for large n and m since the geometric series $\sum_{k=1}^{\infty} c^k$ is convergent for $0 < c < 1$. Thus $\{x_n\}$ is a Cauchy sequence and, since X is complete, it converges to some $x^* \in X$. The continuity of F and the definition of the x_n now imply that $x^* = F(x^*)$.

If there were two distinct fixed points of F, say, x and y, then, we have that

$$0 < d(x, y) = d(F(x), F(y)) \leq cd(x, y)$$

which is a contradiction, since $0 < c < 1$. This completes the proof. ∎

We now give a well known application of this result.

Theorem 2.4.2 (**Picard's Theorem**) *Let R be a closed rectangle in the plane $\mathbb{R}^2$ whose sides are parallel to the coordinate axes. Let $f : R \to \mathbb{R}$ be a function which is continuous and which is such that $\frac{\partial f}{\partial y}$ exists and is continuous on R. Let (x_0, y_0) be a point in the interior of R. Then, there exists $h > 0$ such that the initial value problem*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$
$$y(x_0) = y_0$$

has a unique solution in the interval $(x_0 - h, x_0 + h)$.

Proof Since R is compact and f and $\frac{\partial f}{\partial y}$ are continuous on R, there exist $K > 0$ and $M > 0$ such that

$$|f(x, y)| \leq K \quad \text{and} \quad \left|\frac{\partial f}{\partial y}(x, y)\right| \leq M$$

for all $(x, y) \in R$. Then, by the mean value theorem, it follows that, for all (x, y_1) and (x, y_2) in R, we have

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|.$$

It is easy to see that a function $y = y(x)$ is a solution of the initial value problem above if, and only if, it satisfies the following:

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,\mathrm{d}t$$

for all x.

Now, choose $h > 0$ such that $Mh < 1$. Consider the rectangle R' defined as follows:

$$R' = \{(x, y) | |x - x_0| \leq h, |y - y_0| \leq Kh\}.$$

By choosing h small enough, we can ensure that $R' \subset R$. Now consider

$$X = \{g \in \mathcal{C}[x_0 - h, x_0 + h] \,|\, |g(x) - y_0| \leq Kh \text{ for all } x\}.$$

Then, we see that X is a closed subspace of $\mathcal{C}[x_0 - h, x_0 + h]$ and so is a complete metric space (with the distance induced by the 'sup-norm'). Let $g \in X$. Define

$$F(g)(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\,\mathrm{d}t$$

for all $x \in [x_0 - h, x_0 + h]$. Then, clearly, $F(g)$ is continuous on that interval and

$$|F(g)(x) - y_0| \leq K|x - x_0| \leq Kh.$$

Further, if g_1 and g_2 are in X, we have

$$\begin{aligned} |F(g_1)(x) - F(g_2)(x)| &\leq \int_{x_0}^{x} |f(t, g_1(t)) - f(t, g_2(t))| \, \mathrm{d}t \\ &\leq M\|g_1 - g_2\||x - x_0| \\ &\leq Mh\|g_1 - g_2\| \end{aligned}$$

from which it follows that $\|F(g_1) - F(g_2)\| \leq Mh\|g_1 - g_2\|$ which shows that F maps the complete metric space X into itself and is a contraction. Thus F has a unique fixed point $y \in X$ which solves the initial value problem. This completes the proof. ■

We now prove a corollary of the contraction mapping theorem which will also be useful in proving the existence of solutions to higher order initial value problems.

Corollary 2.4.1 *Let (X, d) be a complete metric space and let $F : X \to X$ be a mapping such that, for some positive integer n, the map $F^n : X \to X$ is a contraction. Then F has a unique fixed point.*

Proof Since $F^n : X \to X$ is a contraction, this mapping has a unique fixed point x^* by the contraction mapping theorem. Now,

$$F(x^*) = F(F^n(x^*)) = F^{n+1}(x^*) = F^n(F(x^*))$$

and thus, $F(x^*)$ is also a fixed point for F^n. By the uniqueness of the fixed point, it follows that

$$F(x^*) = x^*$$

and so F has a fixed point, *viz.* x^*.

On the other hand, any fixed point y of F is also a fixed point of F^n since

$$F^n(y) = F^{n-1}(F(y)) = F^{n-1}(y) = \cdots = F(y) = y.$$

Thus F also must have a unique fixed point. ■

Example 2.4.1 Consider the Volterra integral operator $T : \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]$ defined in Example 2.3.7, *viz.*

$$T(f)(s) = \int_0^s K(s, t) f(t) \, \mathrm{d}t$$

where $K : [0, 1] \times [0, 1] \to \mathbb{R}$ is a continuous function. Consider the following problem: find $u \in \mathcal{C}[0, 1]$ such that

$$u(s) = w(s) + \lambda \int_0^s K(s, t)u(t)\,\mathrm{d}t \tag{2.4.1}$$

where $w \in \mathcal{C}[0, 1]$ is a given function and $\lambda \in \mathbb{R}$. The equation (2.4.1) is called a *Volterra integral equation*. It is clear that a solution u of (2.4.1) is a fixed point of the (affine linear) mapping $F : \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]$ defined by

$$F(u) = w + \lambda T(u).$$

Now, for any $s \in [0, 1]$ and for any u_1 and $u_2 \in \mathcal{C}[0, 1]$,

$$\begin{aligned} |F(u_1)(s) - F(u_2)(s)| &= |\lambda \int_0^s K(s, t)((u_1(t) - u_2(t))\,\mathrm{d}t| \\ &\leq |\lambda|\kappa\|u_1 - u_2\|s \end{aligned}$$

where

$$\kappa = \max_{[0,1]\times[0,1]} |K(s, t)|.$$

Hence, we get

$$\|F(u_1) - F(u_2)\| \leq |\lambda|\kappa\|u_1 - u_2\|.$$

Again,

$$\begin{aligned} |F^2(u_1)(s) - F^2(u_2)(s)| &= |\lambda \int_0^s K(s, t)(F(u_1)(t) - F(u_2)(t))\,\mathrm{d}t| \\ &\leq |\lambda|\kappa \int_0^s |F(u_1)(t) - F(u_2)(t)|\,\mathrm{d}t \\ &\leq |\lambda|^2\kappa^2\|u_1 - u_2\| \int_0^s t\,\mathrm{d}t \\ &= |\lambda|^2\kappa^2\|u_1 - u_2\|\tfrac{t^2}{2} \end{aligned}$$

whence we deduce that

$$\|F^2(u_1) - F^2(u_2)\| \leq \frac{|\lambda|^2\kappa^2}{2}\|u_1 - u_2\|.$$

Proceeding in this way, we get, for any positive integer n,

$$\|F^n(u_1) - F^n(u_2)\| \leq \frac{|\lambda|^n \kappa^n}{n!} \|u_1 - u_2\|.$$

Since $|\lambda|^n \kappa^n / n!$ is the general term of the convergent exponential series $\exp(|\lambda|\kappa)$, it tends to zero as n tends to infinity and so, for sufficiently large n, we have

$$\frac{|\lambda|^n \kappa^n}{n!} < 1$$

and hence F^n is a contraction. Thus, by the preceding corollary, F has a unique fixed point. In other words, the Volterra integral equation (2.4.1) has a unique solution. ■

We now show that the study of certain differential equations can be reduced to the study of the Volterra integral equation.

Consider the equation

$$x''(s) + p(s)x'(s) + q(s)x(s) = f(s) \tag{2.4.2}$$

for $s \in (0, 1)$, where p, q and f are given continuous functions. Consider the initial conditions

$$x(0) = \alpha \text{ and } x'(0) = \beta \tag{2.4.3}$$

where α and β are two given real numbers. Set $u(s) = x''(s)$. Then,

$$x'(s) = \beta + \int_0^s u(t)\, \mathrm{d}t.$$

Again,

$$\begin{aligned} x(s) &= \alpha + \int_0^s x'(t)\, \mathrm{d}t \\ &= \alpha + \beta s + \int_0^s \int_0^t u(\tau)\, \mathrm{d}\tau\, \mathrm{d}t \\ &= \alpha + \beta s + \int_0^s \int_\tau^s u(\tau)\, \mathrm{d}t\, \mathrm{d}\tau \\ &= \alpha + \beta s + \int_0^s u(\tau)(s - \tau)\, \mathrm{d}\tau. \end{aligned}$$

Thus, (2.4.2) can be written in the form (2.4.1) with

$$w(s) = f(s) - [\beta p(s) + \alpha q(s) + \beta s q(s)],$$

$\lambda = 1$ and

$$K(s,t) = -\left[p(s) + (s-t)q(s)\right].$$

Hence, the initial value problem (2.4.2)–(2.4.3) has a unique solution.

2.5 Exercises

Note: In all the function spaces which occur below, it is assumed that vector addition and scalar multiplication are defined pointwise.

2.1 Let c denote the space of all (real, or complex) sequences which are convergent, equipped with the norm $\|\cdot\|_\infty$. Let c_0 denote the subspace of sequences converging to zero. Show that these spaces are complete.

2.2 Let c_{00} denote the space of all sequences such that, except for a finite number of terms, all other terms are zero. Show that c_{00} is dense in c_0.

2.3 Show that ℓ_1 is a dense subspace of c_0.

2.4 Let $f \in \mathcal{C}[0,1]$. Define

$$\|f\|_p = \left(\int_0^1 |f(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}}$$

where $1 \leq p < \infty$. Show that this defines a norm on $\mathcal{C}[0,1]$.

2.5 Show that the space $\mathcal{C}[0,1]$ with the norm $\|\cdot\|_1$ defined in the previous exercise is not complete by producing a Cauchy sequence which is not convergent.

2.6 Let $f \in \mathcal{C}(\mathbb{R})$. The *support* of f is the closure of the set of points where f does not vanish. Let $\mathcal{C}_c(\mathbb{R})$ denote the space of all continuous real-valued functions on $\mathbb{R}$ whose support is a compact subset of $\mathbb{R}$. Show that it is a normed linear space with the 'sup-norm' and that it is not complete.

2.7 Let $\mathcal{C}_0(\mathbb{R})$ denote the space of all continuous real-valued functions on $\mathbb{R}$ which *vanish at infinity, i.e.* if $f \in \mathcal{C}_0(\mathbb{R})$, then, given any $\varepsilon > 0$, there exists a compact subset $K \subset \mathbb{R}$ such that

$$|f(x)| < \varepsilon$$

for all $x \in \mathbb{R}\backslash K$. Show that $\mathcal{C}_0(\mathbb{R})$ is a Banach space with the 'sup-norm'. Show also that the space $\mathcal{C}_c(\mathbb{R})$ defined in the previous exercise is dense in $\mathcal{C}_0(\mathbb{R})$.

2.8 Show that the space $BUC(\mathbb{R})$ of bounded and uniformly continuous real-valued functions, defined on $\mathbb{R}$, is a Banach space when equipped with the sup-norm.

2.9 Let $\mathcal{C}^1[0, 1]$ denote the space of all continuous real-valued functions on $[0, 1]$ which are continuously differentiable on $(0, 1)$ and whose derivatives can be continuously extended to $[0, 1]$. For $f \in \mathcal{C}^1[0, 1]$, define

$$\|f\| = \max_{t\in[0,1]}\{|f(t)|, |f'(t)|\}$$

where f' denotes the derivative of f. Show that $\mathcal{C}^1[0, 1]$ is a Banach space for this norm. State and prove an analogous result for $\mathcal{C}^k[0, 1]$, the space of all continuous real-valued functions on $[0, 1]$ which are k times continuously differentiable on $(0, 1)$ and all those derivatives possessing continuous extensions to $[0, 1]$.

2.10 Let $f \in \mathcal{C}^1[0, 1]$ and let f' denote its derivative. Define

$$\|f\|_1 = \left(\int_0^1 (|f(t)|^2 + |f'(t)|^2)\mathrm{d}t\right)^{\frac{1}{2}}.$$

Show that $\|\cdot\|_1$ defines a norm on $\mathcal{C}^1[0, 1]$. If we define

$$|f|_1 = \left(\int_0^1 |f'(t)|^2\mathrm{d}t\right)^{\frac{1}{2}},$$

does $|\cdot|_1$ define a norm on $\mathcal{C}^1[0, 1]$?

2.11 Let

$$V = \{f \in \mathcal{C}^1[0, 1] | f(0) = 0\}.$$

Show that $|\cdot|_1$ defines a norm on V.

2.12 Let V be a Banach space with norm $\|\cdot\|_V$. Set

$$X = \mathcal{C}([0, 1]; V)$$

to be the space of all continuous functions from $[0, 1]$ into the space V. Define, for $f \in X$,

$$\|f\|_X = \sup_{t\in[0,1]} \|f(t)\|_V = \left(\max_{t\in[0,1]} \|f(t)\|_V\right)$$

Show that $\|\cdot\|_X$ is well defined and that it defines a norm on X. Show also that, under this norm, X is a Banach space.

2.13 Let V be a normed linear space and let $W \subset V$ be a closed subspace. Let $\pi : V \to V/W$ denote the canonical mapping given by $\pi(x) = x + W$, for every $x \in V$. If $U \subset V$ is an open set, show that $\pi(U)$ is open in V/W.

2.14 Let V and W be normed linear spaces and let $T : V \to W$ be a linear transformation. Show that T is continuous if, and only if, T maps Cauchy sequences in V into Cauchy sequences in W.

2.15 Let $\mathcal{C}^1[0, 1]$ be endowed with the norm as in Exercise 2.9 above. Let $\mathcal{C}[0, 1]$ be endowed with the usual 'sup-norm'. Show that $T : \mathcal{C}^1[0, 1] \to \mathcal{C}[0, 1]$ defined by $T(f) = f'$ is a continuous linear transformation and that $\|T\| = 1$.

2.16 Let $\mathcal{C}[0, 1]$ be endowed with its usual norm. For $f \in \mathcal{C}[0, 1]$, define

$$T(f)(t) = \int_0^t f(s)\, \mathrm{d}s,\ t \in [0, 1].$$

For every positive integer n, show that, in $\mathcal{L}(\mathcal{C}[0, 1])$,

$$\|T^n\| = 1/n!$$

2.17 Let $V = \mathcal{C}[0, 1]$ be equipped with its usual norm. Let W denote the same space, but equipped with the norm $\|\cdot\|_2$ (cf. Exercise 2.4 above). If T is defined as in Exercise 2.16 above, compute $\|T\|$ in $\mathcal{L}(V, W)$.

2.18 Consider the space $\mathcal{C}_c(\mathbb{R})$ defined in Exercise 2.6 above. For $f \in \mathcal{C}_c(\mathbb{R})$, define

$$\varphi(f) = \int_{-\infty}^{\infty} f(t)\, \mathrm{d}t.$$

Show that φ is well defined and that it is a linear functional on this space. Is it continuous?

2.19 Let $\{t_i\}_{i=1}^n$ be given points in the closed interval $[0, 1]$. Let $\{\omega_i\}_{i=1}^n$ be given real numbers. Let $f \in \mathcal{C}[0, 1]$. Define

$$\varphi(f) = \sum_{i=1}^n \omega_i f(t_i).$$

Show that φ defines a continuous linear functional on $\mathcal{C}[0, 1]$ and that

$$\|\varphi\| = \sum_{i=1}^n |\omega_i|.$$

2.20 Let $\mathcal{M}_n$ denote the set of all $n \times n$ matrices with complex entries. Let $\|\cdot\|_{p,n}$ denote the matrix norm induced by the vector norm $\|\cdot\|_p$ on $\mathbb{C}^n$, for $1 \leq p \leq \infty$. If $\mathbf{A} = (a_{ij}) \in \mathcal{M}_n$, show that

$$\|\mathbf{A}\|_{1,n} = \max_{1 \le j \le n} \left\{ \sum_{i=1}^{n} |a_{ij}| \right\}.$$

State and prove an analogous result for $\|\mathbf{A}\|_{\infty,n}$.

2.21 With the notations introduced in the preceding exercise, show that

$$\|\mathbf{A}\|_{2,n} = \sqrt{\rho(\mathbf{A}^*\mathbf{A})}$$

where $\rho(\mathbf{T})$ denotes the spectral radius of a matrix $\mathbf{T}$. (Hint: Use Proposition 1.1.8). If $\mathbf{A}$ is a normal matrix, show that $\|\mathbf{A}\|_{2,n} = \rho(\mathbf{A})$.

2.22 With the notations introduced above, show that, for any matrix $\mathbf{A} \in \mathcal{M}_n$, we have

$$\|\mathbf{A}\|_{2,n} \le \|\mathbf{A}\|_E \le \sqrt{n}\|\mathbf{A}\|_{2,n}$$

where $\|\cdot\|_E$ is the norm introduced in Example 2.3.11.

2.23 Let D be an $n \times n$ diagonal matrix with diagonal entries given by $d_{ii} = \beta_i$, $1 \le i \le n$. Compute $\|D\|_{p,n}$ for $1 \le p \le \infty$.

2.24 If $\|\cdot\|$ defines a matrix norm on $\mathcal{M}_n$, show that $\rho(\mathbf{A}) \le \|\mathbf{A}\|$ for all $\mathbf{A} \in \mathcal{M}_n$.

2.25 Let $\mathbf{A} \in \mathcal{M}_n$ be invertible and let $\|\cdot\|$ be a matrix norm. The *condition number* of $\mathbf{A}$ is defined as

$$\mathrm{cond}(\mathbf{A}) = \|\mathbf{A}\|.\|\mathbf{A}^{-1}\|.$$

Show that

(a) $\mathrm{cond}(\mathbf{A}) \ge 1$ for any invertible matrix $\mathbf{A} \in \mathcal{M}_n$;
(b) $\mathrm{cond}(\alpha\mathbf{A}) = \mathrm{cond}(\mathbf{A})$ for any invertible matrix $\mathbf{A}$ and for any scalar $\alpha \ne 0$;
(c) for any invertible and normal matrix $\mathbf{A}$,

$$\mathrm{cond}_{2,n}(\mathbf{A}) = \frac{\max_{1 \le i \le n} |\lambda_i(\mathbf{A})|}{\min_{1 \le i \le n} |\lambda_i(\mathbf{A})|}$$

where $\{\lambda_i(\mathbf{A})\}_{i=1}^n$ are the eigenvalues of $\mathbf{A}$ and $\mathrm{cond}_{2,n}(\mathbf{A})$ denotes the condition number of $\mathbf{A}$ with respect to the norm $\|\cdot\|_{2,n}$.

2.26 For what class of matrices does $\mathrm{cond}_{2,n}$ attain its minimum value?

2.27 Let $\mathbf{A} = (a_{ij})$ be a 2×2 matrix which is invertible. Show that

$$\mathrm{cond}_{2,2}(\mathbf{A}) = \sigma + (\sigma^2 - 1)^{\frac{1}{2}}$$

where

$$\sigma = \frac{\sum_{i,j=1}^2 |a_{ij}|^2}{2|\det(\mathbf{A})|}.$$

2.28 Let $\mathcal{M}_n$ be endowed with the topology generated by any matrix norm. Let $GL_n(\mathbb{C})$ denote the set of all invertible matrices in $\mathcal{M}_n$. Show that $GL_n(\mathbb{C})$ is an open and dense set. Is it connected?

2.29 (a) Let $\mathcal{D}_n$ be the subset of all $n \times n$ matrices with distinct eigenvalues. Show that $\mathcal{D}_n$ is dense in $\mathcal{M}_n$ (endowed with any matrix norm);
(b) prove the Cayley-Hamilton theorem for any diagonalizable matrix: 'Every $n \times n$ matrix satisfies its characteristic equation';
(c) deduce the Cayley-Hamilton theorem for all $n \times n$ matrices.

2.30 Let $\mathbf{A} \in \mathcal{M}_n$ be an invertible matrix. Show that

$$\inf_{\mathbf{B} \text{ is singular}} \|\mathbf{A} - \mathbf{B}\|_{2,n} = \frac{1}{\|\mathbf{A}^{-1}\|_{2,n}}.$$

2.31 Show that the set of all orthogonal matrices in the space of all $n \times n$ real matrices (endowed with any norm topology) is compact.

2.32 Let $1 \le p < q \le \infty$. Show that $\ell_p \subset \ell_q$ and that, for all $x \in \ell_p$,

$$\|x\|_q \le \|x\|_p.$$

2.33 Let $\omega = \{\omega_k\}_{k=1}^{\infty}$ be a bounded sequence. Define $D_\omega : \ell_2 \to \ell_2$ by

$$D_\omega(x) = (\omega_1 x_1, \ldots, \omega_k x_k, \ldots),$$

where $x = (x_1, \ldots, x_k, \ldots) \in \ell_2$. Show that D_ω is continuous and that it is invertible if, and only if, $\inf_k\{|\omega_k|\} > 0$.

2.34 Consider an infinite matrix $(a_{ij}),\ i, j \in \mathbb{N}$ of scalars. Let $x = (x_i) \in \ell_p$, $1 \le p \le \infty$. Define a sequence $A(x)$ whose ith component is given by

$$\sum_{j=1}^{\infty} a_{ij} x_j.$$

(a) Assume that

$$\alpha = \sup_j \sum_{i=1}^{\infty} |a_{ij}| < \infty.$$

Show that $A \in \mathcal{L}(\ell_1)$ and that $\|A\| = \alpha$;

(b) assume that

$$\alpha = \sup_i \sum_{j=1}^{\infty} |a_{ij}| < \infty.$$

Show that $A \in \mathcal{L}(\ell_\infty)$ and that $\|A\| = \alpha$;

(c) assume that

$$\alpha = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|^2 < \infty.$$

Show that $A \in \mathcal{L}(\ell_2)$ and that $\|A\|^2 \leq \alpha$;

(d) assume that $A \in \mathcal{L}(\ell_p, \ell_q)$ and that $A \in \mathcal{L}(\ell_p, \ell_r)$ where $1 \leq p, q, r < \infty$. Let $\theta \in (0, 1)$. Let $\frac{1}{s} = \frac{\theta}{q} + \frac{1-\theta}{r}$. Show that $A \in \mathcal{L}(\ell_p, \ell_s)$ and that

$$\|A\|_{\mathcal{L}(\ell_p,\ell_s)} \leq \|A\|^{\theta}_{\mathcal{L}(\ell_p,\ell_q)} \|A\|^{1-\theta}_{\mathcal{L}(\ell_p,\ell_r)}.$$

2.35 Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers such that $\sum_{k=0}^{\infty} |a_k| < \infty$. Consider the infinite lower triangular matrix

$$\begin{bmatrix} a_0 & 0 & 0 & \cdots\cdots \\ a_1 & a_0 & 0 & \cdots\cdots \\ a_2 & a_1 & a_0 & \cdots\cdots \\ a_3 & a_2 & a_1 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots\cdots \end{bmatrix}.$$

Let A be the linear map defined on ℓ_2 by this matrix (as in the preceding exercise). Show that $A \in \mathcal{L}(\ell_2)$ and that

$$\|A\| \leq \sum_{k=0}^{\infty} |a_k|.$$

2.36 Let V be a Banach space. Let $\{A_n\}$ be a sequence of continuous linear operators on V. Let

$$S_n = \sum_{k=1}^{n} A_k.$$

If $\{S_n\}$ is a convergent sequence in $\mathcal{L}(V)$, we say that the *series*

$$\sum_{k=1}^{\infty} A_k$$

is *convergent* and the limit of the sequence $\{S_n\}$ is called the sum of the series. If $\sum_{k=1}^{\infty} \|A_k\| < \infty$, we say that the series $\sum_{k=1}^{\infty} A_k$ is *absolutely convergent*. Show that any absolutely convergent series in convergent.

2.37 Let V be a Banach space. If $A \in \mathcal{L}(V)$ is such that $\|A\| < 1$, show that the series

$$I + \sum_{k=1}^{\infty} A^k$$

is convergent and that its sum is $(I - A)^{-1}$.

2.38 (a) Let V be a Banach space and let $A \in \mathcal{L}(V)$. Show that the series

$$I + \sum_{k=1}^{\infty} \frac{A^k}{k!}$$

is convergent. The sum is denoted $\exp(A)$;

(b) if $A, B \in \mathcal{L}(V)$ are such that $AB = BA$, show that

$$\exp(A + B) = \exp(A)\exp(B);$$

(c) deduce that $\exp(A)$ is invertible for any $A \in \mathcal{L}(V)$;

(d) let

$$A = \begin{bmatrix} \alpha & -\omega \\ \omega & \alpha \end{bmatrix},$$

where α and ω are real numbers. Show that, for any $t \in \mathbb{R}$,

$$\exp(tA) = e^{\alpha t} \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix}.$$

2.39 Let V be a Banach space. Show that $\mathcal{G}$, the set of invertible linear operators in $\mathcal{L}(V)$ is an open subset of $\mathcal{L}(V)$ (endowed with its usual norm topology).

2.40 (a) Define $T : \ell_2 \to \ell_2$ and $S : \ell_2 \to \ell_2$ by

$$\begin{aligned} T(x) &= (0, x_1, x_2, \dots) \\ S(x) &= (x_2, x_3, \dots) \end{aligned}$$

where $x = (x_1, x_2, \dots) \in \ell_2$. Show that T and S define continuous linear operators on ℓ_2 and that $ST = I$ while $TS \neq I$. (Thus, T and S, which are called the *right* and *left shift operators* respectively, are not invertible);

(b) if A is a continuous linear operator on ℓ_2 such that $\|A - T\| < 1$, show that A is also not invertible. Deduce that, in general, $\mathcal{G}$, defined in Exercise 2.39 above, is not dense in $\mathcal{L}(V)$, if V is infinite dimensional. (Compare this with the finite dimensional case, cf. Exercise 2.28).

2.41 Let $\mathcal{P}$ denote the space of all polynomials in one variable with real coefficients. Let $\mathsf{p} \in \mathcal{P}$ and let $\mathsf{p} = \sum_{i=1}^{n} a_i x^i$, where $a_i \in \mathbb{R}$ for $1 \leq i \leq n$. Define

$$\|\mathsf{p}\|_1 = \sum_{i=1}^{n} |a_i| \text{ and } \|\mathsf{p}\|_\infty = \max_{1\leq i\leq n} |a_i|.$$

Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ define norms on $\mathcal{P}$ and that they are not equivalent.

2.42 Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on $\mathcal{C}[0,1]$ are not equivalent.

2.43 Let V be a normed linear space and let W be a finite dimensional (and hence, closed) subspace of V. Let $x \in V$. Show that there exists $w \in W$ such that

$$\|x + W\| = \|x + w\|.$$

2.44 Let V be a normed linear space and let $f : V \to \mathbb{R}$ be a non-zero linear map. Assume that its null space (or, kernel)

$$Z = \{x \in V | f(x) = 0\},$$

is closed in V. Let $x_0 \in V$ such that $f(x_0) \neq 0$.

(a) Show that, for any $x \in V$,

$$\|x + Z\|_{V/Z} = \frac{|f(x)|}{|f(x_0)|} \|x_0 + Z\|_{V/Z};$$

(b) deduce that f is continuous;
(c) show that

$$\|f\| = \frac{|f(x_0)|}{\|x_0 + Z\|_{V/Z}}.$$

2.45 Let E_i be Banach spaces for $1 \leq i \leq 3$. Let $A \in \mathcal{L}(E_1, E_2)$ and $B \in \mathcal{L}(E_1, E_3)$. Assume, further, if K is any bounded set in E_1, then $\overline{B(K)}$ is compact in E_3. Finally, assume that $x \in E_1 \mapsto \|A(x)\|_{E_2} + \|B(x)\|_{E_3}$ defines a norm on E_1 which is equivalent to the norm $\|.\|_{E_1}$.

(a) Show that $\mathrm{Ker}(A)$, the kernel of A, is finite dimensional;
(b) let $\mathcal{R}(A)$ denote the range of A. Show that the canonical mapping $\overline{A} : E_1/(\mathrm{Ker}(A)) \to \mathcal{R}(A)$ defined by $\overline{A}(x + \mathrm{Ker}(A)) = A(x)$ for $x \in E_1$, is an isomorphism;
(c) deduce that $\mathcal{R}(A)$ is a closed subspace of E_3.

2.46 Let V and Y be Banach spaces and let W be a dense subspace of V. Let $A \in \mathcal{L}(W, Y)$. Show that there exists a unique continuous extension $\widetilde{A} \in \mathcal{L}(V, Y)$ (i.e. $\widetilde{A}|_W = A$) and that $\|\widetilde{A}\|_{\mathcal{L}(V,Y)} = \|A\|_{\mathcal{L}(W,Y)}$.

2.47 (Completion of a normed linear space) Let V be a normed linear space. We say that two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in V are equivalent if $\|x_n - y_n\| \to 0$ as $n \to \infty$.

(a) Show that this defines an equivalence relation. Let the set of all equivalence classes be denoted by $\overline{V}$.
(b) let $\overline{x}$ and $\overline{y}$ denote the equivalence classes of the Cauchy sequences $\{x_n\}$ and $\{y_n\}$ respectively. Let $\mathbf{0}$ denote the equivalence class of the sequence all of whose terms are zero. Let α be a scalar. Define $\overline{x} + \overline{y}$ to be the equivalence class of the sequence $\{x_n + y_n\}$ and $\alpha\overline{x}$ to be that of the sequence $\{\alpha x_n\}$. Show that these operations are well defined and make $\overline{V}$ a vector space;
(c) with the above notations, define $\|\overline{x}\|_{\overline{V}} = \lim_{n\to\infty} \|x_n\|_V$. Show that this is well defined and that it defines a norm on $\overline{V}$;
(d) define $\mathsf{i} : V \to \overline{V}$ by setting $\mathsf{i}(x)$ to be the equivalence class of the sequence all of whose terms are equal to x, for any $x \in V$. Show that $\mathsf{i} \in \mathcal{L}(V, \overline{V})$ and that it is an injection. Show also that the image $\mathsf{i}(V)$ is dense in $\overline{V}$;
(e) show that $\overline{V}$ is complete (and this space is called the *completion* of V). (Hint: Given a Cauchy sequence $\{\overline{x}^{(n)}\}$ in $\overline{V}$, choose $x_n \in V$ such that $\|\overline{x}^{(n)} - \mathsf{i}(x_n)\|_{\overline{V}} < 1/n$. Show that $\{x_n\}$ is a Cauchy sequence in V and if $\overline{x}$ denotes its equivalence class, show that $\overline{x}^{(n)} \to \overline{x}$ in $\overline{V}$.)

2.48 Let V and W be normed linear spaces and let $U \subset V$ be an open subset. Let $J : U \to W$ be a mapping. We say that J is *(Fréchet) differentiable* at $u \in U$ if there exists $T \in \mathcal{L}(V, W)$ such that

$$\lim_{h\to\mathbf{0}} \frac{\|J(u+h) - J(u) - T(h)\|}{\|h\|} = 0.$$

(Equivalently,

$$J(u+h) - J(u) - T(h) = \varepsilon(h), \lim_{h\to\mathbf{0}} \frac{\|\varepsilon(h)\|}{\|h\|} = 0.)$$

(a) If such a T exists, show that it is unique. (We say that T is the *(Fréchet) derivative* of J at $u \in U$ and write $T = J'(u)$.)
(b) If J is differentiable at $u \in U$, show that J is continuous at $u \in U$.

2.49 Let V and W be normed linear spaces and let $U \subset V$ be an open subset. Let $J : U \to W$ be a mapping. We say that J is *Gâteau differentiable* at $u \in U$ along a vector $w \in V$ if

$$\lim_{t\to 0} \frac{1}{t}(J(u+tw) - J(u))$$

exists. (We call the limit the *Gâteau derivative* of J at u along w).

If J is Fréchet differentiable at $u \in U$, show that it is Gâteau differentiable at u along any vector $w \in V$ and that the corresponding Gâteau derivative is given by $J'(u)w$.

2.50 Let V and W be normed linear spaces and let $A \in \mathcal{L}(V, W)$. Let $w_0 \in W$ be given. define $J : V \to W$ by $J(u) = A(u) + w_0$. Show that J is differentiable at every $u \in V$ and that $J'(u) = A$.

2.51 Let $U = GL_n(\mathbb{C}) \subset \mathcal{M}_n$ (cf. Exercise 2.28). Define $J(\mathbf{A}) = \mathbf{A}^{-1}$ for $\mathbf{A} \in U$. Show that J is differentiable at every $\mathbf{A} \in U$ and that, if $\mathbf{H} \in \mathcal{M}_n$, we have

$$J'(\mathbf{A})(\mathbf{H}) = -\mathbf{A}^{-1}\mathbf{H}\mathbf{A}^{-1}.$$

2.52 (a) Let $\mathbf{A} \in GL_n(\mathbb{C})$. Show that

$$\det(\mathbf{I} + \mathbf{A}) = 1 + \mathrm{tr}(\mathbf{A}) + \varepsilon(A)$$

where

$$\lim_{\mathbf{A} \to \mathbf{0}} \frac{|\varepsilon(\mathbf{A})|}{\|\mathbf{A}\|} = 0$$

(for any matrix norm);

(b) deduce that if we define $J(\mathbf{A}) = \det(\mathbf{A})$ for $\mathbf{A} \in GL_n(\mathbb{C})$, then J is differentiable at all such $\mathbf{A}$ and that, if $\mathbf{H} \in \mathcal{M}_n$, then

$$J'(\mathbf{A})(\mathbf{H}) = \det(\mathbf{A})\mathrm{tr}(\mathbf{A}^{-1}\mathbf{H}).$$

2.53 (Chain Rule) Let V, W and Z be normed linear spaces and let $f : V \to W$ and $g : W \to Z$ be mappings such that f is differentiable at a point $v \in V$ and g is differentiable at $f(v) = w \in W$. Show that the map $g \circ f : V \to Z$ is differentiable at $v \in V$ and that

$$(g \circ f)'(v) = g'(f(v)) \circ f'(v).$$

2.54 (a) Let V be a real normed linear space and let $J : V \to \mathbb{R}$ be a given mapping. A subset K of V is said to be *convex* if, for every u and $v \in K$ and for every $t \in [0, 1]$, we have that

$$tu + (1 - t)v \in K.$$

Let $K \subset V$ be a closed convex set. Assume that J attains its minimum over K at $u \in K$. If J is differentiable at u, then show that

$$J'(u)(v - u) \geq 0$$

for every $v \in K$;

(b) let $K = V$. If J attains its minimum at $u \in V$ and if J is differentiable at u, show that $J'(u) = \mathbf{0}$.

2.55 Let V be a real normed linear space. A mapping $J : V \to \mathbb{R}$ is said to be *convex* if, for every u and $v \in V$ and for every $t \in [0, 1]$, we have

$$J(tu + (1-t)v) \leq tJ(u) + (1-t)J(v).$$

(a) If $J : V \to \mathbb{R}$ is convex and differentiable at every point, show that

$$J(v) - J(u) \geq J'(u)(v-u)$$

for every u and $v \in V$;

(b) let $J : V \to \mathbb{R}$ be convex and differentiable at every point of V. Let $K \subset V$ be a closed convex set. Let $u \in K$ be such that

$$J'(u)(v-u) \geq 0$$

for every $v \in K$. Show that

$$J(u) = \min_{v \in K} J(v);$$

(c) if $J : V \to \mathbb{R}$ is convex and differentiable at every point of V, and if $u \in V$ is such that $J'(u) = \mathbf{0}$, show that J attains its minimum (over all of V) at u.

Remark 2.5.1 Exercise 2.54 gave necessary conditions for a differentiable function J to attain a minimum at a point u. The preceding exercise shows that these conditions are also sufficient in the case of convex functions. ■

2.56 Let $m > n$. Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$. Consider the linear system of equations

$$\mathbf{Ax} = \mathbf{b}.$$

This system may not have a solution since the number of equations exceeds the number of unknowns. A *least squares approximate* solution is a vector $\mathbf{x}_0 \in \mathbb{R}^n$ such that

$$\|\mathbf{Ax}_0 - \mathbf{b}\|_2 = \min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{Ax} - \mathbf{b}\|_2.$$

Show that such a solution must satisfy the linear system

$$\mathbf{A}^*\mathbf{Ax}_0 = \mathbf{A}^*\mathbf{b}$$

and that this system has a unique solution if the rank of $\mathbf{A}$ is n.

Chapter 3
Hahn-Banach Theorems

3.1 Analytic Versions

The analytic form of the Hahn-Banach theorem concerns the extension of linear functionals defined on a subspace of a normed linear space to the entire space, preserving the norm of the functional. We will prove a slightly more general result in this direction.

Theorem 3.1.1 (Hahn-Banach) *Let V be a vector space over $\mathbb{R}$. Let $p : V \to \mathbb{R}$ be a mapping such that*

$$\left.\begin{array}{r} p(\alpha x) = \alpha p(x) \\ p(x+y) \leq p(x) + p(y) \end{array}\right\} \tag{3.1.1}$$

for all x and y in V and for all $\alpha > 0$ in $\mathbb{R}$. Let W be a subspace of V and let $g : W \to \mathbb{R}$ be a linear map such that

$$g(x) \ \leq \ p(x)$$

for all $x \in W$. Then, there exists a linear extension $f : V \to \mathbb{R}$ of g (i.e. $f(x) = g(x)$ for all $x \in W$) which is such that

$$f(x) \ \leq \ p(x)$$

for all $x \in V$.

Proof Step 1. Let $\mathcal{P}$ denote the collection of all pairs (Y, h), where Y is a subspace of V containing W and $h : Y \to \mathbb{R}$ a linear map which is an extension of g and which is also such that

$$h(x) \ \leq \ p(x)$$

S. Kesavan, *Functional Analysis*, Texts and Readings in Mathematics,
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for all $x \in Y$. Clearly $\mathcal{P}$ is non-empty, since $(W, g) \in \mathcal{P}$. Consider the partial order defined on $\mathcal{P}$ by

$$(Y, h) \preceq (\widetilde{Y}, \widetilde{h})$$

if $Y \subset \widetilde{Y}$ and $\widetilde{h}$ is a linear extension of h. Let $\mathcal{Q} = \{(Y_i, h_i) \mid i \in I\}$ be a chain in $\mathcal{P}$. Define

$$Y = \cup_{i\in I} Y_i$$

and let $h : Y \to \mathbb{R}$ be defined by $h(x) = h_i(x)$ if $x \in Y_i$. Since $\mathcal{Q}$ is a chain, it is immediate to see that h is well defined and also that it is a linear extension of each of the h_i. Also $h(x) \leq p(x)$ for all $x \in Y$. Thus, $(Y, h) \in \mathcal{P}$ and $(Y_i, h_i) \preceq (Y, h)$ for each $i \in I$, and thus every chain has an upper bound. Hence, by Zorn's lemma, $\mathcal{P}$ has a maximal element (Z, f).

Step 2. We will show that $Z = V$, which will complete the proof. Assuming the contrary, let $x_0 \notin Z$. Consider the linear subspace of V given by

$$Y = \{x + tx_0 \mid x \in Z, \ \ t \in \mathbb{R}\}.$$

We will define a linear extension $h : Y \to \mathbb{R}$ of f such that $(Y, h) \in \mathcal{P}$, thus contradicting the maximality of f. Define

$$h(x + tx_0) = f(x) + \alpha t$$

where α will be suitably determined. In order that $(Y, h) \in \mathcal{P}$, we need that

$$f(x) + \alpha t \ \leq \ p(x + tx_0)$$

for all $x \in Z$ and for all $t \in \mathbb{R}$. If $t > 0$, this reduces to (in view of (3.1.1))

$$f\left(\frac{1}{t}x\right) + \alpha \ \leq \ p\left(\frac{1}{t}x + x_0\right)$$

or, equivalently,

$$f(x) + \alpha \ \leq \ p(x + x_0) \tag{3.1.2}$$

for all $x \in Z$. Similarly, by considering $t < 0$, we deduce that

$$f(x) - \alpha \ \leq \ p(x - x_0) \tag{3.1.3}$$

for all $x \in Z$. In other words, it is necessary that α be chosen such that

$$\sup_{x\in Z}[f(x) - p(x - x_0)] \ \leq \ \alpha \ \leq \ \inf_{x\in Z}[p(x + x_0) - f(x)]. \tag{3.1.4}$$

But, for all x and y in Z, we have

$$f(x) + f(y) = f(x + y) \ \leq \ p(x + y) \ \leq \ p(x + x_0) + p(y - x_0)$$

or, equivalently,

$$f(y) - p(y - x_0) \ \leq \ p(x + x_0) - f(x).$$

Hence, it is possible to choose α such that (3.1.4) is true, giving the desired contradiction, which completes the proof. ■

Theorem 3.1.2 (Hahn-Banach) *Let V be a normed linear space over $\mathbb{R}$. Let W be a subspace of V and let $g : W \to \mathbb{R}$ be a continuous linear functional on W. Then there exists a continuous linear extension $f : V \to \mathbb{R}$ of g such that*

$$\|f\|_{V^*} = \|g\|_{W^*}.$$

Proof Set $p(x) = \|g\|_{W^*}\|x\|$. Then p verifies (3.1.1). Also $g(x) \leq p(x)$ for all $x \in W$. Thus, there exists a linear extension of g, *viz.* $f : V \to \mathbb{R}$ such that, for all $x \in V$, $f(x) \leq \|g\|_{W^*}\|x\|$. This implies that f is continuous and that $\|f\|_{V^*} \leq \|g\|_{W^*}$. But, since $f = g$ on W, it follows that we do have equality of the norms of f and g. This completes the proof. ■

We will now prove the same result for complex vector spaces.

If V is a normed linear space over $\mathbb{C}$, and if $f \in V^*$, then for every $x \in V$, we can write $f(x) = g(x) + ih(x)$, where $g(x)$ and $h(x)$ are the real and imaginary parts of $f(x)$, respectively. Then g and h will be real-valued continuous linear functionals on V, now considered as a real vector space by restricting scalar multiplication to real scalars only.

Proposition 3.1.1 *Let V be a normed linear space over $\mathbb{C}$. Let $f : V \to \mathbb{C}$ be a continuous linear functional. Let $f = g + ih$ where g and h are real-valued linear functionals as described above. Then*

$$f(x) = g(x) - ig(ix)$$

for all $x \in V$ and, further, $\|f\| = \|g\|$.

Proof Let $x \in V$. Then $f(ix) = if(x)$. Expressing this in terms of the real and imaginary parts of f, we get

$$g(ix) + ih(ix) = ig(x) - h(x)$$

which shows that $h(x) = -g(ix)$. Now, let $f(x) = e^{i\theta}|f(x)|$ where $\theta \in [0, 2\pi)$. Then,

$$|f(x)| = e^{-i\theta}f(x) = f(e^{-i\theta}x) = g(e^{-i\theta}x)$$

since the left extreme of the above relation is real. Thus $|f(x)| \leq \|g\| \; \|x\|$ which implies that $\|f\| \leq \|g\|$. On the other hand,

$$|f(x)|^2 = |g(x)|^2 + |h(x)|^2$$

which yields $|g(x)| \leq |f(x)| \leq \|f\| \; \|x\|$ whence we get $\|g\| \leq \|f\|$. This completes the proof. ■

Theorem 3.1.3 (Hahn-Banach) *Let V be a normed linear space over $\mathbb{C}$. Let W be a subspace of V and let $g : W \to \mathbb{C}$ be a continuous linear functional on W. Then there exists a continuous linear extension $f : V \to \mathbb{C}$ of g such that*

$$\|f\|_{V^*} = \|g\|_{W^*}.$$

Proof Let $g = h(x) - ih(ix)$ where h is the real part of g. We consider V as a real normed linear space by restricting ourselves to scalar multiplication by reals only. Then, there exists $\widetilde{h} : V \to \mathbb{R}$ which is a linear extension of h and such that $\|\widetilde{h}\|_{V^*} = \|h\|_{W^*}$. Now set

$$f(x) = \widetilde{h}(x) - i\widetilde{h}(ix)$$

for all $x \in V$. Then, clearly, $f(x + y) = f(x) + f(y)$ and, for real scalars α, $f(\alpha x) = \alpha f(x)$. Now,

$$f(ix) = \widetilde{h}(ix) - i\widetilde{h}(-x) = i(\widetilde{h}(x) - i\widetilde{h}(ix)) = if(x)$$

and thus f is complex linear as well. Further, by the preceding proposition,

$$\|f\|_{V^*} = \|\widetilde{h}\|_{V^*} = \|h\|_{W^*} = \|g\|_{W^*}.$$

This completes the proof. ■

Corollary 3.1.1 *Let V be a normed linear space and $x_0 \in V$ a non-zero vector. Then, there exists $f \in V^*$ such that $\|f\| = 1$ and $f(x_0) = \|x_0\|$.*

Proof Let W be the one-dimensional space spanned by x_0. Define $g(\alpha x_0) = \alpha\|x_0\|$. Then $\|g\|_{W^*} = 1$. Hence, there exists $f \in V^*$ such that $\|f\|_{V^*} = 1$ and which extends g. Hence $f(x_0) = g(x_0) = \|x_0\|$. ■

Remark 3.1.1 If V is a normed linear space and if x and y are distinct points in V, then, clearly, there exists $f \in V^*$ such that $f(x) \neq f(y)$ (consider $x_0 = x - y \neq \mathbf{0}$). We say that V^* separates points of V. ■

Corollary 3.1.2 *Let V be a normed linear space. Let $x \in V$. then*

$$\|x\| = \sup_{f \in V^*, \; \|f\| \leq 1} |f(x)| = \max_{f \in V^*, \; \|f\| \leq 1} |f(x)|. \tag{3.1.5}$$

Proof Clearly, $|f(x)| \leq \|f\| \, \|x\| \leq \|x\|$ when $\|f\| \leq 1$. On the other hand, by the preceding corollary, there exists $f \in V^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$ when x is non-zero. Thus the result is established for non-zero vectors and is trivially true for the null vector. ■

3.2 Reflexivity

Compare the relation

$$\|f\| = \sup_{x \in V, \ \|x\| \leq 1} |f(x)|, \tag{3.2.1}$$

which is a *definition*, with the relation (3.1.5), which is a *result* of the theory. In the former, the supremum need not be attained, while in the latter the supremum is always attained and hence is a maximum.

This is the starting point for the investigation of a very nice property of Banach spaces called reflexivity.

Let $x \in V$ and define

$$J_x(f) = f(x)$$

for $f \in V^*$. Then, by virtue of (3.1.5), it follows that $J_x \in (V^*)^* = V^{**}$ and that, in fact,

$$\|J_x\|_{V^{**}} = \|x\|_V.$$

Thus $J : V \to V^{**}$ given by $x \mapsto J_x$ is a norm preserving linear transformation. Such a map is called an isometry. The map J is clearly injective and maps V isometrically onto a subspace of V^{**}.

Definition 3.2.1 A Banach space V is said to be **reflexive** if the canonical imbedding $J : V \to V^{**}$, given above, is surjective. ■

Example 3.2.1 Since the canonical map $J : V \to V^{**}$ is an isometry, it is injective. Thus, if V is finite dimensional, then $\dim(V^{**}) = \dim(V^*) = \dim(V)$ and so, by dimension considerations, J has to be surjective as well. Thus, every finite dimensional space is automatically reflexive. ■

Since V^{**}, being a dual space, is always complete, the notion of reflexivity makes sense only for Banach spaces.

By applying Corollary 3.1.2 to V^*, it is readily seen that the supremum in (3.2.1) is attained for reflexive Banach spaces. Consequently, if there exists $f \in V^*$ such that the supremum in (3.2.1) is not attained, then the space is not reflexive.

Remark 3.2.1 A deep result due to James is that the converse is also true: *if V is a Banach space such that the supremum is attained in (3.2.1) for all $f \in V^*$, then V is reflexive.* ■

We will study reflexive spaces in greater detail in Chap. 5. In the remainder of this section, we will look at several examples.

Example 3.2.2 Let $V = \mathcal{C}[0, 1]$, the space of continuous real-valued functions defined on the interval $[0, 1]$, equipped with the usual 'sup-norm'. Consider the linear functional φ defined on V by

$$\varphi(f) = \int_0^{\frac{1}{2}} f(t)\, \mathrm{d}t - \int_{\frac{1}{2}}^{1} f(t)\, \mathrm{d}t,$$

for every $f \in V$. Clearly $|\varphi(f)| \leq \|f\|_\infty$ and so $\varphi \in V^*$ and $\|\varphi\| \leq 1$. Now consider the sequence of functions $\{f_n\}$ in V, where, for each sufficiently large positive integer n, we have

$$f_n(t) = \begin{cases} +1, & \text{if } t \in \left[0, \frac{1}{2} - \frac{1}{n}\right], \\ 1 + n\left(\frac{1}{2} - \frac{1}{n} - x\right), & \text{if} t \in \left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right], \\ -1, & \text{if, } t \in \left[\frac{1}{2} + \frac{1}{n}, 1\right]. \end{cases}$$

The graph of f_n is given in Fig. 3.1.

We see from this picture (by computing the relevant areas) that $\varphi(f_n) = 1 - \frac{1}{n}$. Since $\|f_n\| = 1$ for all n, it follows that $\|\varphi\| = 1$.

We now show that there is no function $f \in V$ such that $\|f\| = 1$ and such that $\varphi(f) = \|\varphi\| = 1$. Indeed if there were such a function, then consider the function g defined on $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ by

$$g(t) = +1, \text{ if } t \in \left(0, \frac{1}{2}\right), \text{ and } g(t) = -1 \text{ if } t \in \left(\frac{1}{2}, 1\right).$$

Then

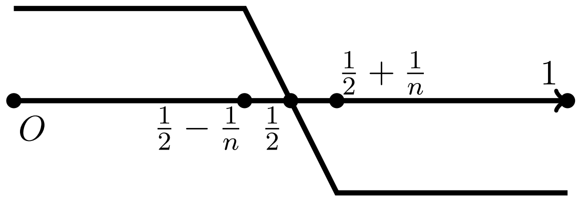
$$1 = \int_0^{\frac{1}{2}} g(t)\, \mathrm{d}t - \int_{\frac{1}{2}}^{1} g(t)\, \mathrm{d}t = \int_0^{\frac{1}{2}} f(t)\, \mathrm{d}t - \int_{\frac{1}{2}}^{1} f(t)\, \mathrm{d}t.$$

Then

$$\int_0^{\frac{1}{2}} (g(t) - f(t))\, \mathrm{d}t = \int_{\frac{1}{2}}^{1} (g(t) - f(t))\, \mathrm{d}t. \tag{3.2.2}$$

But $|f(t)| \leq 1$, i.e. $-1 \leq f(t) \leq +1$ for all t and so the integrand on the left-hand side of (3.2.2) is non-negative, while that on the right-hand side is non-positive. Thus each of the integrals in (3.2.2) is zero. But then, again, since those integrands are of

Fig. 3.1 The function f_n



constant sign, it follows that $f \equiv +1$ on $(0, \frac{1}{2})$ and that $f \equiv -1$ on $(\frac{1}{2}, 1)$, which contradicts the continuity of f.

Thus, φ is a continuous linear functional on $\mathcal{C}[0, 1]$ for which $\|\varphi\|$ is not attained on the unit sphere and we conclude that $\mathcal{C}[0, 1]$ is not reflexive. ■

Example 3.2.3 Let $1 < p < \infty$. Let p^* be the conjugate exponent of p (cf. Definition 2.2.1). Let $y \in \ell_{p^*}$. We already saw that (cf. Example 2.3.4) the linear functional f_y defined on ℓ_p by

$$f_y(x) = \sum_{i=1}^{\infty} x_i y_i$$

for $x = (x_i) \in \ell_p$, is continuous and that, in fact,

$$\|f_y\| \ \leq \ \|y\|_{p^*}.$$

Now, let $f \in \ell_p^*$. Define

$$f_i = f(\mathsf{e}_i)$$

where e_i is the sequence whose ith entry is unity and all other entries are zero. Set $\mathsf{f} = (f_i)$.

Let n be any positive integer. Define

$$x_i = \begin{cases} 0, & \text{if } 1 \leq i \leq n \text{ and } f_i = 0, \\ |f_i|^{p^*}/f_i, & \text{if } 1 \leq i \leq n \text{ and } f_i \neq 0, \\ 0, & \text{if } i > n. \end{cases}$$

Then, since it is a finite sequence, $x = (x_i) \in \ell_p$ and $x = \sum_{i=1}^{n} x_i \mathsf{e}_i$. Thus,

$$f(x) = \sum_{i=1}^{n} x_i f_i = \sum_{i=1}^{n} |f_i|^{p^*}.$$

Consequently

$$\sum_{i=1}^{n} |f_i|^{p^*} \leq \|f\| \|x\|_p = \|f\| \sum_{i=1}^{n} \left(|f_i|^{p^*}\right)^{\frac{1}{p}}$$

using the definition of the x_i and that of p^*. This yields

$$\left(\sum_{i=1}^{n}|f_i|^{p^*}\right)^{\frac{1}{p^*}} \leq \|f\|.$$

Since n was arbitrary, we deduce that $\mathsf{f} \in \ell_{p^*}$ and that $\|\mathsf{f}\|_{p^*} \leq \|f\|$.

For any $x = (x_i) \in \ell_p$, we have $\sum_{i=1}^{n} x_i \mathsf{e}_i \to x$ in ℓ_p and so, by the continuity of f, it follows that

$$f(x) = \sum_{i=1}^{\infty} x_i f_i$$

or, in other words, $f = f_{\mathsf{f}}$. Hence,

$$\|\mathsf{f}\|_{p^*} \leq \|f\| \leq \|\mathsf{f}\|_{p^*}$$

as already observed.

Thus every element of the dual space of ℓ_p occurs in this fashion and the map $y \mapsto f_y$ is an isometry of ℓ_{p^*} onto ℓ_p^*. Thus, we can write

$$\ell_p^* = \ell_{p^*}$$

using this isometry.

Similarly, we can write

$$\ell_{p^*}^* = \ell_p.$$

It is easy to see that using these identifications of the dual spaces, the canonical isometry from ℓ_p into $\ell_p^{**} = \ell_p$ is nothing but the identity map, which is onto. Thus, the spaces ℓ_p, for $1 < p < \infty$ are all reflexive. ■

Remark 3.2.2 Though in the above example, we have not dealt with the case of real and complex sequence spaces separately, it is customary to identify the dual of the complex sequence space ℓ_p with ℓ_{p^*} via the following relation: if $y \in \ell_{p^*}$, then we define $f_y \in \ell_p^*$ by

$$f_y(x) = \sum_{i=1}^{\infty} x_i \overline{y_i}.$$

The mapping $y \in \ell_{p^*} \mapsto f_y \in \ell_p^*$ satisfies $f_{y+z} = f_y + f_z$ and $f_{\alpha y} = \overline{\alpha} f_y$. Such a mapping is called *conjugate linear*. This identification will be especially useful in the case of ℓ_2 which will be made clear when we study *Hilbert spaces* (cf. Chap. 7). ■

Example 3.2.4 Proceeding in a manner similar to that in the preceding example, we can show that $\ell_1^* = \ell_\infty$. In other words, given any continuous linear functional f on ℓ_1, we have

$$f(x) = \sum_{i=1}^{\infty} x_i f_i$$

for all $x = (x_i) \in \ell_1$, where $f_i = f(\mathsf{e}_\mathsf{i})$. Further, setting $\mathsf{f} = (f_i)$, we have

$$\|\mathsf{f}\|_\infty = \|f\|.$$

We now show that there is no similar identification of ℓ_∞^* with ℓ_1. Of course, if $y \in \ell_1$, the functional f_y defined by

$$f_y(x) = \sum_{i=1}^{\infty} x_i y_i \tag{3.2.3}$$

for all $x = (x_i) \in \ell_\infty$ is continuous and $\|f_y\| = \|y\|_1$. But there exist continuous linear functionals on ℓ_∞ which do not arise this way. So the identity mapping of ℓ_1 which is still the canonical imbedding of ℓ_1 into ℓ_1^{**}, is not surjective. Thus the space ℓ_1 is not reflexive.

To see this, let c be the subspace of all convergent sequences in ℓ_∞ (cf. Exercise 2.1). For $x = (x_i) \in c$, define

$$g(x) = \lim_{i\to\infty} x_i.$$

Then $g : c \to \mathbb{R}$ is linear and $|g(x)| \leq \|x\|_\infty$. Thus, g is continuous as well and so, by the Hahn-Banach theorem, can be extended to a continuous linear functional f on ℓ_∞, preserving the norm. We claim that this continuous linear functional cannot be obtained from an element of ℓ_1 by the above outlined procedure.

Assume the contrary and let $y = (y_i) \in \ell_1$ be such that $f = f_y$. Consider the sequence $\{x^{(n)}\}$ in ℓ_∞ given by

$$x^{(n)} = \{0, 0, \ldots, 0, 1, 1, 1, \ldots\}$$

where the 1's start from the n-th entry. Then $\|x^{(n)}\|_\infty = 1$ and $x^{(n)} \in c$. We have

$$1 = f(x^{(n)}) = \sum_{i=n}^{\infty} y_i$$

which is impossible since $y \in \ell_1$ implies that

$$\sum_{i=n}^{\infty} |y_i| \ \to \ 0$$

as $n \to \infty$. ■

Example 3.2.5 Consider the space c of all real sequences which are convergent. This is a closed subspace of ℓ_∞ (cf. Exercise 2.1). Let $y = (y_1, y_2, \ldots, y_k, \ldots) \in \ell_1$. Then, if $x = (x_1, x_2, \ldots, x_k, \ldots) \in c$, we have that y defines a continuous linear functional on c, via the action defined by (3.2.3) and it is easy to see that the norm of this functional is given by $\|y\|_1$. Assume that the supremum in (3.2.1) is attained on the unit sphere of c. Without loss of generality, we may assume that there exists $x \in c$, with $\|x\|_\infty = 1$, such that $< y, x >= \|y\|_1$ (why?). Let $\|y\|_1 = 1$. Thus,

$$1 = \|y\|_1 = \sum_{k=1}^{\infty} y_k x_k.$$

Since, $\|x\|_\infty = 1$, it follows that, for each k, $y_k x_k \leq |y_k x_k| \leq |y_k|$. Then it follows from the preceding equation that, for each k, we have

$$|y_k| = y_k x_k.$$

Now assume that, for each k, $y_k \neq 0$ and that $y_k = (-1)^k |y_k|$. (Example: $y_k = (-1)^k (\frac{1}{2})^k$.) Then it follows that $x_k = (-1)^k$, which is a contradiction since $x \notin c$ in this case. Thus, for all such $y \in \ell_1$, the supremum is not attained in (3.2.1) and so c is not reflexive. ■

Example 3.2.6 Consider the space c_0 of all real sequences which converge to zero. This is a closed subspace of c (cf. Exercise 2.1). One can prove that $c_0^* = \ell_1$. If $y \in \ell_1$ and if $x \in c_0$, again, the action of the functional defined by y on x is given by (3.2.3), where, as in the preceding example, x_k and y_k are the components of x and y, respectively. If $\|y\|_1 = 1$ and if $\|x\|_\infty = 1$, we have that $|x_k| < 1$ for all $k \geq N$, for some positive integer N. Then it is clear that $| < y, x > | < \|y\|_1 = 1$. Thus, for no continuous linear functional on c_0 we have that the supremum in (3.2.1) is attained. Thus, c_0 is not reflexive. ■

3.3 Geometric Versions

In this section we will study the geometric versions of the Hahn-Banach theorem which concerns the separation of convex sets by means of hyperplanes.

Definition 3.3.1 Let V be a real normed linear space. An **affine hyperplane** is a set of the form

$$H = \{x \in V \mid f(x) = \alpha\},$$

denoted by $[f = \alpha]$, where f is a non-zero linear functional on V. ■

Proposition 3.3.1 *A hyperplane $[f = \alpha]$ is closed if, and only if, f is a continuous linear functional.*

Proof Clearly, if f is continuous, then $[f = \alpha]$ is closed. Conversely, assume that the hyperplane H, given by $[f = \alpha]$, is closed. Then, its complement H^c is open and, since $f \not\equiv 0$, it is non-empty.(For, if $\alpha \neq 0$, then $\mathbf{0} \in H^c$; if $\alpha = 0$, and $f \not\equiv 0$, there exists $x \in V$ such that $f(x) \neq 0$ and so $x \in H^c$.)

Without loss of generality, assume that $x_0 \in H^c$ is such that $f(x_0) < \alpha$. Since H^c is open, there exists $r > 0$ such that the open ball centred at x_0 and of radius $2r$, denoted $B(x_0; 2r)$, is contained in H^c. Now, for all $x \in B(x_0; 2r)$, we have $f(x) < \alpha$. (If not, there exists $x_1 \in B(x_0; 2r)$ such that $f(x_1) > \alpha$. Let

$$t = \frac{f(x_0) - \alpha}{f(x_0) - f(x_1)}.$$

Then $0 < t < 1$ and, if $x_t = tx_1 + (1-t)x_0$, then $f(x_t) = \alpha$. But $B(x_0; 2r)$ is a convex set and so $x_t \in B(x_0; 2r)$, which is a contradiction.)

Thus, for any $z \in V$ such that $\|z\| \leq 1$, we get $f(x_0 + rz) \leq \alpha$ or, equivalently,

$$f(z) \leq \frac{\alpha - f(x_0)}{r}.$$

Thus the image of the unit ball is bounded and so f is continuous. ∎

Proposition 3.3.2 *Let C be an open and convex set in a real normed linear space V such that $\mathbf{0} \in C$. For $x \in V$, set*

$$p(x) = \inf\{\alpha > 0 \mid \alpha^{-1}x \in C\}.$$

(The function p is called the Minkowski functional *of C.) Then, there exists $M > 0$ such that*

$$0 \leq p(x) \leq M\|x\| \tag{3.3.1}$$

for all $x \in V$. We also have

$$C = \{x \in V \mid p(x) < 1\}. \tag{3.3.2}$$

Further, p satisfies (3.1.1).

Proof Since $\mathbf{0} \in C$ and C is open, there exists an open ball $B(\mathbf{0}; 2r)$, centred at $\mathbf{0}$ and of radius $2r$, contained in C. Now, if $x \in V$, we have $rx/\|x\| \in C$ and so, by definition, $p(x) \leq \frac{1}{r}\|x\|$ which proves (3.3.1).

Let $x \in C$. Since C is open, and since $\mathbf{0} \in C$, there exists $\varepsilon > 0$ such that $(1 + \varepsilon)x \in C$. Thus, $p(x) \leq (1+\varepsilon)^{-1} < 1$. Conversely, let $x \in V$ such that $p(x) < 1$. Then, there exists $0 < t < 1$ such that $\frac{1}{t}x \in C$. Then, as C is convex, we also have $t\frac{1}{t}x + (1-t)\mathbf{0} \in C$, i.e. $x \in C$. This proves (3.3.2).

If $\alpha > 0$, it is easy to see that $p(\alpha x) = \alpha p(x)$. This is the first relation in (3.1.1). Now, let x and $y \in V$. Let $\varepsilon > 0$. Then

$$\frac{1}{p(x)+\varepsilon}x \in C \quad \text{and} \quad \frac{1}{p(y)+\varepsilon}y \in C.$$

Set

$$t = \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}$$

so that $0 < t < 1$. Then, as C is convex,

$$t\frac{1}{p(x)+\varepsilon}x + (1-t)\frac{1}{p(y)+\varepsilon}y = \frac{1}{p(x)+p(y)+2\varepsilon}(x+y) \in C$$

which implies that

$$p(x+y) \le p(x)+p(y)+2\varepsilon$$

from which the second relation in (3.1.1) follows since ε was chosen arbitrarily. ■

Proposition 3.3.3 *Let C be a non-empty open convex set in a real normed linear space V and assume that $x_0 \notin C$. Then, there exists $f \in V^*$ such that $f(x) < f(x_0)$ for all $x \in C$.*

Proof Without loss of generality, we can assume that $\mathbf{0} \in C$. (If $\mathbf{0} \notin C$, let $x_1 \in C$. Then we consider the convex set $C - \{x_1\}$ which contains the origin and does not contain $x_0 - x_1$; if f is as in the proposition, we have $f(x - x_1) < f(x_0 - x_1)$ for all $x \in C$ which yields $f(x) < f(x_0)$ for all $x \in C$.)

Let W be the one-dimensional space spanned by x_0. Define $g : W \to \mathbb{R}$ by

$$g(tx_0) = t.$$

By definition of the Minkowski functional, since $\frac{1}{t}tx_0 = x_0 \notin C$, we have that

$$g(tx_0) = t \ \le p(tx_0)$$

for $t > 0$. Since the Minkowski functional is non-negative, this inequality holds trivially for $t \le 0$ as well. Thus, by the Hahn-Banach theorem (cf. Theorem 3.1.1), there exists a linear extension f of g to the whole of V such that, for all $x \in V$,

$$f(x) \le p(x) \le M\|x\|$$

(cf. (3.3.1)) which yields $|f(x)| \le M\|x\|$, and so f is continuous as well. Now, if $x \in C$,

$$f(x) \le p(x) \ < \ 1 = g(x_0) = f(x_0)$$

by (3.3.2) and this completes the proof. ■

Theorem 3.3.1 (Hahn-Banach) *Let A and B be two non-empty and disjoint convex subsets of a real normed linear space V. Assume that A is open. Then, there exists* *a closed hyperplane which separates A and B, i.e. there exists $f \in V^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \leq \alpha \leq f(y)$$

for all $x \in A$ and $y \in B$.

Proof Let $C = A - B = \{x - y \mid x \in A, y \in B\}$. Since

$$C = \cup_{y \in B}(A - \{y\}),$$

we see immediately that C is both open and convex. Since A and B are disjoint, it also follows that $\mathbf{0} \notin C$. Hence, by the preceding proposition, there exists $f \in V^*$ such that $f(z) < 0$ for all $z \in C = A - B$. In other words, $f(x) < f(y)$ for all $x \in A$ and $y \in B$. Choose $\alpha \in \mathbb{R}$ such that

$$\sup_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y).$$

This completes the proof. ■

Theorem 3.3.2 (Hahn-Banach) *Let A and B be non-empty and disjoint convex sets in a real normed linear space V. Assume that A is closed and that B is compact. Then A and B can be separated* strictly *by a closed hyperplane, i.e. there exists $f \in V^*$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ such that*

$$f(x) \leq \alpha - \varepsilon \text{ and } f(y) \geq \alpha + \varepsilon$$

for all $x \in A$ and $y \in B$.

Proof Let $\eta > 0$. Then, if $B(\mathbf{0}; \eta)$ is the open ball of radius $\eta > 0$ centred at $\mathbf{0}$, then $A + B(\mathbf{0}; \eta)$ and $B + B(\mathbf{0}; \eta)$ are non-empty, open and convex. Further, if $\eta > 0$ is sufficiently small, the two sets are disjoint as well. If not, there exists a sequence $\eta_n \to 0$ and $x_n \in A$, $y_n \in B$ such that $\|x_n - y_n\| \leq 2\eta_n$. Since B is compact, there exists a subsequence y_{n_k} which converges to $y \in B$. This implies then that $x_{n_k} \to y$ and, since A is closed, $y \in A$, i.e. $y \in A \cap B$, which is a contradiction.

Thus, we can choose $\eta > 0$ such that $A + B(\mathbf{0}; \eta)$ and $B + B(\mathbf{0}; \eta)$ are disjoint. Then, by the preceding theorem, there exists $f \in V^*$ and $\alpha \in \mathbb{R}$ such that for all $x \in A$, $y \in B$ and z_1 and z_2 in the closed unit ball, we have

$$f\left(x + \frac{\eta}{2}z_1\right) \leq \alpha \leq f\left(y + \frac{\eta}{2}z_2\right).$$

This implies that

$$f(x) + \frac{\eta}{2}\|f\| \leq \alpha \leq f(y) - \frac{\eta}{2}\|f\|.$$

This proves the result if we set $\varepsilon = \frac{\eta}{2}\|f\|$. ■

The following corollary is very useful in testing whether a given subspace of a normed linear space is dense or not.

Corollary 3.3.1 *Let W be a subspace of a real normed linear space V. Assume that $\overline{W} \neq V$. Then, there exists $f \in V^*$ such that $f \not\equiv 0$ and such that $f(x) = 0$ for all $x \in W$.*

Proof Let $x_0 \in V \setminus \overline{W}$. Let $A = \overline{W}$ and $B = \{x_0\}$. Then A is closed, B is compact and they are non-empty and disjoint convex sets. Thus, there exists $f \in V^*$ and $\alpha \in \mathbb{R}$ such that for all $x \in \overline{W}$,

$$f(x) \ < \ \alpha \ < \ f(x_0).$$

Since W is a linear subspace, it follows that for all $\lambda \in \mathbb{R}$, we have $\lambda f(x) < \alpha$ for all $x \in W$. Now, since $\mathbf{0} \in W$, we have $\alpha > 0$. On the other hand, setting $\lambda = n$, we get that, for any $x \in W$,

$$f(x) \ < \ \frac{\alpha}{n}$$

whence we see that $f(x) \leq 0$ for all $x \in W$. Again, if $x \in W$, we also have $-x \in W$ and so $f(-x) \leq 0$ as well and so $f(x) = 0$ for all $x \in W$ and $f(x_0) > \alpha > 0$. ■

Remark 3.3.1 In case of normed linear spaces over $\mathbb{C}$, the conclusions of Theorems 3.3.1 and 3.3.2 hold with f being replaced by $\mathrm{Re}(f)$, the real part of f. This follows from Proposition 3.1.1. It is now easy to see that Corollary 3.3.1 is also valid for complex spaces. For another proof of this result, see the exercises at the end of this chapter. ■

Remark 3.3.2 A topological vector space is said to be *locally convex* if every point has a local basis made up of convex sets, i.e. every open neighbourhood of each point contains a convex open neighbourhood of that point.

The proofs of the geometric versions of the Hahn-Banach theorems go through *mutatis mutandis* in the case of locally convex spaces. In particular, Corollary 3.3.1 is also true for such spaces. For details, see Rudin [1]. ■

3.4 Vector-Valued Integration

In this section we will apply the Hahn-Banach theorem to give a meaning to integration of vector-valued functions.

Let us consider the unit interval [0, 1] endowed with the Lebesgue measure. Let V be a normed linear space over $\mathbb{R}$. Let $\varphi : [0, 1] \to V$ be a continuous mapping. We would like to give a meaning to the integral

$$\int_0^1 \varphi(t) \, \mathrm{d}t$$

as a vector in V in a manner that the familiar properties of integrals are preserved.

Using our experience with the integral of a continuous real-valued function, one could introduce a partition

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

and form Riemann sums of the form

$$\sum_{i=1}^{n}(x_i - x_{i-1})\varphi(\xi_i)$$

where $\xi_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$, and define the integral (if it exists) as a suitable limit of such sums. Assume that such a limit exists and denote it by $y \in V$. Let $f \in V^*$. Then, by the continuity and linearity of f, it will follow that $f(y)$ will be the limit of the Riemann sums of the form

$$\sum_{i=1}^{n}(x_i - x_{i-1})f(\varphi(\xi_i)).$$

But since $f \circ \varphi : [0, 1] \to \mathbb{R}$ is continuous, the above limit of Riemann sums is none other than

$$\int_0^1 f(\varphi(t))\, \mathrm{d}t.$$

Thus the integral of φ must satisfy the relation

$$f\left(\int_0^1 \varphi(t)\, \mathrm{d}t\right) = \int_0^1 f(\varphi(t))\, \mathrm{d}t \tag{3.4.1}$$

for all $f \in V^*$.

Notice that since V^* separates points of V (cf. Corollary 3.1.1 and Remark 3.1.1), such a vector, if it exists, must be unique. We use this to define the integral of a vector-valued function.

Definition 3.4.1 Let V be a real normed linear space and let $\varphi : [0, 1] \to \mathbb{R}$ be a continuous mapping. The integral of φ over $[0, 1]$, denoted

$$\int_0^1 \varphi(t)\, \mathrm{d}t,$$

is that vector in V which satisfies (3.4.1) for all $f \in V^*$. ■

Proposition 3.4.1 *Let $\varphi : [0, 1] \to V$ be a continuous mapping into a real Banach space V. Then the integral of φ over* [0, 1] *exists.*

Proof Since [0, 1] is compact, the set $\overline{H}$ which is the closure (in V) of the set H which is the convex hull of $\varphi([0, 1])$ (i.e. the smallest convex set containing $\varphi([0, 1])$), is compact, by the completeness of V.

Let L be an arbitrary finite collection of continuous linear functionals on V. Define

$$E_L = \left\{ y \in \overline{H} \mid f(y) = \int_0^1 f(\varphi(t))\, \mathrm{d}t \quad \text{for all} \quad f \in L \right\}.$$

It is immediate to see that E_L is a closed set.
Step 1: For any such finite collection L of continuous linear functionals, $E_L \neq \emptyset$. To see this, let $L = \{f_1, \ldots, f_k\}$. Define $\mathcal{A} : V \to \mathbb{R}^k$ by

$$\mathcal{A}(x) = (f_1(x), \ldots, f_k(x)).$$

Then $\mathcal{A}$ is a continuous linear transformation and so $K = \mathcal{A}(\overline{H})$ is a compact and convex set. If $(t_1, \ldots, t_k) \notin K$, then, by the Hahn-Banach theorem (cf. Theorem 3.3.2), we can find constants $c_1, \ldots, c_k$ such that

$$\sum_{i=1}^{k} c_i u_i \ < \ \sum_{i=1}^{k} c_i t_i$$

for all $(u_1, \ldots, u_k) \in K$. In particular, for all $t \in [0, 1]$, we have

$$\sum_{i=1}^{k} c_i f_i(\varphi(t)) \ < \ \sum_{i=1}^{k} c_i t_i.$$

Integrating this inequality over [0, 1], we get

$$\sum_{i=1}^{k} c_i m_i \ < \ \sum_{i=1}^{k} c_i t_i$$

where

$$m_i = \int_0^1 f_i(\varphi(t))\, \mathrm{d}t.$$

In other words, if $(t_1, \ldots, t_k) \notin K$, then $(t_1, \ldots, t_k) \neq (m_1, \ldots, m_k)$. Thus, $(m_1, \ldots, m_k) \in K$. Thus, there exists $y \in \overline{H}$ such that, for $1 \leq i \leq k$, we have

$$m_i = f_i(y).$$

This means that $y \in E_L$, i.e. E_L is non-empty.
Step 2. Let I be a finite indexing set and let L_i be finite collections of elements in V^* for each $i \in I$. Then $L = \cup_{i\in I} L_i$ is still finite and further, since it is easy to see that

$$\cap_{i\in I} E_{L_i} = E_L,$$

it follows from the previous step that the class of closed sets

$$\{E_L \mid L \text{ a finite subset of } V^*\}$$

has finite intersection property. Since $\overline{H}$ is compact, it now follows that

$$\cap_{L,\ \text{finite subset of } V^*} E_L \neq \emptyset.$$

In particular, there exists y such that $y \in E_{\{f\}}$ for every $f \in V^*$, i.e. y satisfies

$$f(y) = \int_0^1 f(\varphi(t))\, \mathrm{d}t$$

for every $f \in V^*$. Thus $y = \int_0^1 \varphi(t)\, \mathrm{d}t$. This completes the proof. ■

Proposition 3.4.2 *Let V be a real normed linear space and let $\varphi : [0, 1] \to V$ be continuous. Then*

$$\left\| \int_0^1 \varphi(t)\, dt \right\| \leq \int_0^1 \|\varphi(t)\|\, dt. \tag{3.4.2}$$

Proof By Corollary 3.1.1, there exists $f \in V^*$ such that $\|f\| = 1$ and $f(y) = \|y\|$ where

$$y = \int_0^1 \varphi(t)\, \mathrm{d}t.$$

Thus,

$$\begin{aligned} \left\| \int_0^1 \varphi(t)\, \mathrm{d}t \right\| &= f\left(\int_0^1 \varphi(t)\, \mathrm{d}t \right) = \int_0^1 f(\varphi(t))\, \mathrm{d}t \\ &\leq \int_0^1 |f(\varphi(t))|\, \mathrm{d}t \leq \int_0^1 \|\varphi(t)\|\, \mathrm{d}t. \end{aligned}$$

This completes the proof. ■

Remark 3.4.1 Let $\varphi : [a, b] \to V$ be a continuous mapping. Define $\psi : [0, 1] \to V$ by $\psi(t) = \varphi(a + t(b - a))$. Then we can define

$$\int_a^b \varphi(t)\, \mathrm{d}t = (b - a) \int_0^1 \psi(t)\, \mathrm{d}t.$$

It is easy to verify that for all $f \in V^*$, we have

$$f\left(\int_a^b \varphi(t)\, \mathrm{d}t\right) = \int_a^b f(\varphi(t))\, \mathrm{d}t.$$

Again the result of Proposition 3.4.2 remains valid in this case as well.

Assume now that $\varphi : [0, \infty) \to V$ is continuous. Assume further that the limit

$$\lim_{\lambda \to \infty} \int_0^\lambda \varphi(t)\, \mathrm{d}t$$

exists, i.e. for any sequence $\lambda_n \to \infty$ (as $n \to \infty$), we have that the limit

$$\lim_{n \to \infty} \int_0^{\lambda_n} \varphi(t)\, \mathrm{d}t$$

exists and is independent of the sequence chosen. Then we define

$$\int_0^\infty \varphi(t)\, \mathrm{d}t = \lim_{\lambda \to \infty} \int_0^\lambda \varphi(t)\, \mathrm{d}t$$

and again it follows that, for all $f \in V^*$, we have

$$f\left(\int_0^\infty \varphi(t)\, \mathrm{d}t\right) = \int_0^\infty f(\varphi(t))\, \mathrm{d}t.$$

The result of Proposition 3.4.2 continues to hold. We can define integrals over other infinite intervals, if they exist, in a similar manner. ■

3.5 An Application to Optimization Theory

We conclude this chapter with an application of the Hahn-Banach theorem to optimization theory.

Definition 3.5.1 A **cone** in a real vector space V is a set C such that:

(i) $\mathbf{0} \in C$;

(ii) if $x \in C$ and $\lambda \geq 0$, then $\lambda x \in C$. ■

Lemma 3.5.1 *Let v_i, $1 \leq i \leq n$ be elements in a normed linear space V. Define*

$$C = \left\{ \sum_{i=1}^{n} \lambda_i v_i \mid \lambda_i \geq 0,\ 1 \leq i \leq n \right\}.$$

Then C is a closed convex cone.

Proof Step 1. Clearly, C is convex since, for any $0 < t < 1$, we have

$$t \sum_{i=1}^{n} \lambda_i v_i + (1-t) \sum_{i=1}^{n} \mu_i v_i = \sum_{I=1}^{n} (t\lambda_i + (1-t)\mu_i) v_i$$

and so, if $\lambda_i \geq 0$ and $\mu_i \geq 0$ for all i, we also have $t\lambda_i + (1-t)\mu_i \geq 0$. Also if $x \in C$, it is obvious that $\lambda x \in C$ for any $\lambda \geq 0$. Thus, C is a convex cone.
Step 2. Assume that the v_i are linearly independent. Let $\{w_n\}$ be a sequence in C and assume that $w_m \to w$ in V. If

$$W = \text{span}\{v_1, \ldots, v_n\},$$

then W is a finite dimensional subspace of V and is hence closed as well. Thus $w \in W$. If $w_m = \sum_{i=1}^{n} \lambda_i^m v_i$ with $\lambda_i^m \geq 0$ for all $1 \leq i \leq n$ and for all m, then $\lambda_i^m \to \lambda_i \geq 0$ for each $1 \leq i \leq n$ and $w = \sum_{i=1}^{n} \lambda_i v_i$. Thus $w \in C$ and so C is closed.

Step 3. Assume now that the v_i are linearly dependent. Then there exists a linear relation between them and so we can find scalars α_i such that $\sum_{i=1}^{n} \alpha_i v_i = \mathbf{0}$ and such that the set

$$J = \{i \mid 1 \leq i \leq n, \alpha_i < 0\}$$

is non-empty.

Let $v \in C$ be such that $v = \sum_{i=1}^{n} \lambda_i v_i$ with $\lambda_i \geq 0$ for all $1 \leq i \leq n$. Then $v = \sum_{i=1}^{n} (\lambda_i + t\alpha_i) v_i$ for any $t \in \mathbb{R}$. Define

$$t = \min_{i \in J} \left\{ -\frac{\lambda_i}{\alpha_i} \right\} \geq 0.$$

Then, for all $1 \leq i \leq n$, we have $\lambda_i + t\alpha_i \geq 0$ and at least one of them must vanish. Then

$$C = \cup_{j\in I}\left\{v = \sum_{i\in I\backslash\{j\}} \lambda_i v_i \mid \lambda_i \geq 0 \text{ for all } i \in I\backslash\{j\}\right\}$$

where $I = \{1, 2, \ldots, n\}$. Each set in the union described on the right-hand side is a cone but generated by fewer elements from V. Iterating this procedure, we can ultimately write C as the finite union of cones each generated by a linearly independent set of vectors and hence, by the preceding step, each of these cones will be closed as well. Hence C, being the finite union of closed sets, is closed. This completes the proof. ■

Theorem 3.5.1 (Farkas-Minkowski Lemma) *Let V be a real reflexive Banach space and let $\{f_0, f_1, \ldots, f_n\}$ be elements of V^* such that if for some $x \in V$ we have $f_i(x) \geq 0$ for all $1 \leq i \leq n$, then $f_0(x) \geq 0$ as well. Then, there exists scalars $\lambda_i \geq 0,\ 1 \leq i \leq n$ such that*

$$f_0 = \sum_{i=1}^{n} \lambda_i f_i.$$

Proof Let

$$C = \left\{\sum_{i=1}^{n} \lambda_i f_i \mid \lambda_i \geq 0,\ 1 \leq i \leq n\right\}$$

which is a closed convex cone in V^* by the preceding lemma. Assume that $f_0 \notin C$. Then, by the Hahn-Banach Theorem (cf. Theorem 3.3.2) there exist $\varphi \in V^{**}$ and $\alpha \in \mathbb{R}$ such that

$$\varphi(f_0) < \alpha < \varphi(f)$$

for all $f \in C$. Since $\mathbf{0} \in C$, it follows that $\alpha < 0$. Thus $\varphi(f_0) < 0$ as well.

Now, since V is reflexive, there exists $x \in V$ such that $\varphi = J_x$ and so $f_0(x) < 0$. On the other hand, since C is a cone, for all $\lambda > 0$, and for all $f \in C$, we have $\lambda f \in C$ and so $\varphi(\lambda f) > \alpha$ or, $\varphi(f) > \alpha/\lambda$ whence we deduce, on letting λ tend to infinity, that $\varphi(f) \geq 0$, i.e. $f(x) \geq 0$ for all $f \in C$. In particular $f_i(x) \geq 0$ for all $1 \leq i \leq n$ while $f_0(x) < 0$, which is a contradiction. Thus $f_0 \in C$ and the proof is complete. ■

The Farkas-Minkowski lemma is a key step in the proof of the *Kuhn-Tucker conditions* which play the same role in characterizing minima in the presence of constraints in the form of *inequalities* as that played by *Lagrange multipliers* in characterizing minima in the presence of constraints in the form of *equalities*. While the Kuhn-Tucker conditions are necessary in general 'nonlinear programming', they are necessary and sufficient in 'convex programming', i.e. when the functional to be minimized and the constraints are all given by convex functions.

Let V be a real normed linear space and let $J : V \to \mathbb{R}$ be a given functional. Let $K \subset V$ be a closed and convex subset. Then, if J attains a minimum over K at $u \in K$ and if J is differentiable at u, a necessary condition is that

$$J'(u)(v-u) \ \geq \ 0$$

for all $v \in K$ (cf. Exercise 2.54). We would like to generalize this to sets K which are not necessarily convex. To this end, we introduce the following definition.

Definition 3.5.2 Let V be a real normed linear space and let $U \subset V$ be a non-empty subset. Let $u \in U$. Then, the **tangent cone**, denoted $C(u)$, at u is the union of the origin and the set of all vectors $w \in V$ such that

(i) there exists a sequence $\{u_k\}$ in U, $u_k \neq u$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} u_k = u$;

(ii)
$$\lim_{k\to\infty} \frac{u_k - u}{\|u_k - u\|} = \frac{w}{\|w\|}.$$ ■

Remark 3.5.1 The second condition in the above definition may be written, in an equivalent fashion, as follows:

$$u_k = u + \|u_k - u\| \left(\frac{w}{\|w\|} + \delta_k \right)$$

where $\delta_k \to \mathbf{0}$ as $k \to \infty$. ■

Remark 3.5.2 It is clear that $C(u)$ is a cone (cf. Definition 3.5.1) since $w \in C(u)$ implies that $\lambda w \in C(u)$ as well, for any $\lambda > 0$ (with the same associated sequence $\{u_k\}$). This is a cone (which is not necessarily convex) with its vertex at the origin. Its translate

$$u + C(u) = \{u + w \mid w \in C(u)\},$$

is a cone with vertex at u. This translated cone contains the (half) tangents at u of all curves in U which pass through u, as it is easy to see from the Taylor expansion (at u) of the function describing the curve. ■

Proposition 3.5.1 *Let U be a non-empty subset of a real normed linear space V and let $u \in U$. Then, $C(u)$ is a closed cone.*

Proof Let $w_n \in C(u)$ and let $w_n \to w$ in V. Without loss of generality, we can assume that $w \neq \mathbf{0}$ (since the origin is always in $C(u)$, by definition) and hence that $w_n \neq \mathbf{0}$ for all n. There exist $u_k^n \in U$ such that $u_k^n \neq u$ for all n and such that $u_k^n \to u$ as $k \to \infty$. Further,

$$u_k^n = u + \|u_k^n - u\| \left(\frac{w_n}{\|w_n\|} + \delta_k^n \right)$$

where $\delta_k^n \to \mathbf{0}$ as $k \to \infty$.

Choose a sequence $\{\varepsilon_n\}$ of positive reals which converges to zero as $n \to \infty$. Then, we can find positive integers $k(n)$ such that

$$\|u_{k(n)}^n - u\| \ < \ \varepsilon_n, \ \text{and } \|\delta_{k(n)}^n\| \le \varepsilon_n.$$

Consider the sequence $\{u_{k(n)}^n\}$. Clearly, $u_{k(n)}^n \to u$ and $u_{k(n)}^n \neq u$ for all n. Further,

$$u_{k(n)}^n = u + \|u_{k(n)}^n - u\| \left[\frac{w}{\|w\|} + \left(\delta_{k(n)}^n + \frac{w_n}{\|w_n\|} - \frac{w}{\|w\|} \right) \right]$$

which shows that $w \in C(u)$ since

$$\eta_n = \delta_{k(n)}^n + \left(\frac{w_n}{\|w_n\|} - \frac{w}{\|w\|} \right) \ \to \ \mathbf{0}$$

given that $w_n \to w$. This completes the proof. ■

Proposition 3.5.2 *Let $J : U \subset V \to \mathbb{R}$ be a functional defined on a set U of a real normed linear space V. Assume that J attains a relative minimum at $u \in U$ and that J is differentiable at u. Then*

$$J'(u)(v - u) \ \ge \ 0$$

for all $v \in u + C(u)$.

Proof Let $v \in u + C(u)$. Then $w = v - u \in C(u)$. Let $\{u_k\}$ be a sequence in U associated to w as in the definition of $C(u)$. Then, since J is differentiable at u, we have

$$\begin{aligned} J(u_k) - J(u) &= J'(u)(u_k - u) + \|u_k - u\|\varepsilon_k \\ &= \|u_k - u\| \left(\tfrac{1}{\|w\|} J'(u)w + J'(u)\delta_k + \varepsilon_k \right) \end{aligned}$$

where $\delta_k \to \mathbf{0}$ and $\varepsilon_k \to 0$ as $k \to \infty$. Setting $\eta_k = \|w\|(J'(u)\delta_k + \varepsilon_k)$, we see that $\eta_k \to 0$ as well. Since J attains a relative minimum at u, it follows that

$$0 \le J(u_k) - J(u) = \frac{\|u_k - u\|}{\|w\|}(J'(u)w + \eta_k)$$

from which it immediately follows that $J'(u)w \ge 0$, which completes the proof. ■

The above result can be used to derive optimality conditions when a functional J is being minimized under constraints. We consider a finite set of functionals φ_i :

$V \to \mathbb{R}$ for $1 \leq i \leq m$ and set

$$U = \{v \in V \mid \varphi_i(u) \leq 0,\ \ 1 \leq i \leq m\}. \tag{3.5.1}$$

Of particular interest is the case when the functionals φ_i are affine linear, i.e. there exist $f_i \in V^*$ and $d_i \in \mathbb{R}$ for $1 \leq i \leq m$ such that

$$\varphi_i(u) = f_i(u) + d_i \tag{3.5.2}$$

for $1 \leq i \leq m$. In this case, we have a simple characterization of the tangent cone.

Proposition 3.5.3 *Let U be as given by (3.5.1) and let the constraints φ_i be affine linear, given by (3.5.2). Then, for any $u \in U$,*

$$C(u) = \{w \in V \mid f_i(w) \leq 0,\ \ i \in I(u)\} \tag{3.5.3}$$

where

$$I(u) = \{i \mid 1 \leq i \leq m,\ \ \varphi_i(u) = 0\}.$$

Proof Notice that (cf. Exercise 2.50) $\varphi_i'(u) = f_i$. If $i \in I(u)$, then φ_i attains its maximum over U at u. Then, by Proposition 3.5.2, $\varphi_i'(u)(w) \leq 0$ for all $w \in C(u)$.

Conversely, if $w \neq \mathbf{0}$ satisfies $f_i(w) \leq 0$ for all $i \in I(u)$, set $u_k = u + \varepsilon_k w$ where $\{\varepsilon_k\}$ is a sequence of positive reals converging to zero. then $u_k \neq u$ and $u_k \to u$. If $i \notin I(u)$, then $\varphi_i(u) < 0$, and so, by continuity, $\varphi_i(u_k) < 0$ for large enough k. If $i \in I(u)$, then $\varphi_i(u) = 0$ and so

$$\varphi_i(u_k) = f_i(u_k - u) = \varepsilon_k f_i(w) \leq 0.$$

Thus, for k large enough, we have that $u_k \in U$. Finally we see immediately that

$$\frac{u_k - u}{\|u_k - u\|} = \frac{w}{\|w\|}.$$

Thus, it follows that $w \in C(u)$. This completes the proof. ■

Theorem 3.5.2 (Kuhn-Tucker Conditions) *Let V be a real, reflexive Banach space. Let φ_i, $1 \leq i \leq m$ be as in (3.5.2) and let U be as in (3.5.1). Let $J : V \to \mathbb{R}$ be a functional which attains a relative minimum at $u \in U$. Assume that J is differentiable at u. Then, there exist constants $\lambda_i(u)$ such that*

$$\left.\begin{aligned} J'(u) + \sum_{i=1}^{m} \lambda_i(u)\varphi_i'(u) &= \mathbf{0} \\ \sum_{i=1}^{m} \lambda_i(u)\varphi_i(u) &= 0 \\ \lambda_i(u) \geq 0,\ \ 1 \leq i &\leq m. \end{aligned}\right\} \tag{3.5.4}$$

Proof By Propositions 3.5.2 and 3.5.3, we have that for all w such that $\varphi_i'(u)w \leq 0,\ \ i \in I(u)$, we have $J'(u)w \geq 0$. Thus, by the Farkas-Minkowski lemma, there exist $\lambda_i(u) \geq 0$ for $i \in I(u)$ such that

$$J'(u) = -\sum_{i \in I(u)} \lambda_i(u)\varphi_i'(u).$$

Setting $\lambda_i(u) = 0$ for all $i \notin I(u)$, we get (3.5.4). This completes the proof. ■

The above theorem can be generalized to cases when the φ_i are not affine. In this situation, in addition to differentiability at u, we need to assume another technical condition of 'admissibility' on the constraints at u. In particular, when the constraints $\varphi_i,\ \ 1 \leq i \leq m$ are all convex, the admissibility condition reads as follows:

- either, all the φ_i are affine and the set U given by (3.5.1) is non-empty;
- or, there exists an element $v^* \in V$ such that $\varphi_i(v^*) \leq 0$ for all $1 \leq i \leq m$ and $\varphi_i(v^*) < 0$ whenever φ_i is not affine linear.

If J is differentiable at u and the constraints are differentiable and admissible (at u), then (3.5.4) is a necessary condition for u to be a relative minimum of J at u. In addition, if J and the constraints φ_i are all convex, then (3.5.4) is both necessary and sufficient. Interested readers can find further details in the book *Introduction à l'analyse numérique matricielle et à l'optimisation* by P.G. Ciarlet (Masson, Paris, France, 1982; English translation, Cambridge University Press, Cambridge, UK, 1989).

3.6 Exercises

3.1 Give an example to show that the functional whose existence is guaranteed by Corollary 3.1.1, is not unique.

3.2 A normed linear space V is said to be *strictly convex* if for x and $y \in V$ such that $x \neq y,\ \ \|x\| = \|y\| = 1$, we have

$$\left\| \frac{1}{2}(x+y) \right\| \ < \ 1.$$

Show that the spaces ℓ_1 and ℓ_∞ are not strictly convex.

3.3 For x and y in ℓ_2, show that

$$\left\|\frac{1}{2}(x+y)\right\|_2^2 + \left\|\frac{1}{2}(x-y)\right\|_2^2 = \frac{1}{2}\left[\|x\|_2^2 + \|y\|_2^2\right].$$

Deduce that the space ℓ_2 is strictly convex.

3.4 Show that the space $\mathcal{C}[0, 1]$ is not strictly convex.

3.5 If V is a normed linear space such that V^* is strictly convex, show that given a subspace W of V and a continuous linear functional f on W, its Hahn-Banach extension to all of V is unique. In particular, show also that the functional whose existence is guaranteed by Corollary 3.1.1, is unique.

3.6 Use Corollary 3.1.1 to prove Corollary 3.2.1. (Hint: consider the quotient space $V/\overline{W}$.)

3.7 Let V be a normed linear space and let W be a subspace of V. Let X be a finite dimensional space and let $T : W \to X$ be a continuous linear transformation. Show that there exists a continuous linear transformation $\widetilde{T} : V \to X$ which extends T.

3.8 Let V be a normed linear space and let W be a finite dimensional subspace. Show that there exists a closed subspace Z of V such that

$$V = W \oplus Z.$$

3.9 (a) Let c_0 denote the space of all real sequences which converge to zero, equipped with the norm $\|.\|_\infty$. Prove that (cf. Example 3.2.6)

$$c_0^* = \ell_1.$$

(b) Show that (cf. Example 3.2.4)

$$\ell_1^* = \ell_\infty.$$

(c) For any positive integer N, show that

$$\left(\ell_\infty^N\right)^* = \ell_1^N.$$

3.10 Let $y \in \ell_1$. Given a sequence $x = (x_1, \ldots, x_k, \ldots)$, define $f_y(x)$ using (3.2.3). Show that $f_y \in \ell_\infty^*$ and that $f_y \in c^*$. In both cases, show that $\|f_y\| = \|y\|_1$.

3.11 Let

$$y = \left(0, 1 - \frac{1}{2}, 1 - \frac{1}{3}, \ldots, 1 - \frac{1}{n}, \ldots\right) \in \ell_\infty.$$

Show that the functional $f_y \in \ell_1^*$ does not attain the supremum in (3.2.1). (This gives another proof of the non-reflexivity of the space ℓ_1.)

3.12 Let V be a normed linear space and let C denote the open ball with center at the origin and of radius $r > 0$. Compute the Minkowski functional of C.

3.13 Let V be a normed linear space and let W be a subspace of V. Define

$$W^\perp = \{g \in V^* \mid g(x) = 0 \text{ for all } x \in W\}.$$

(a) Show that $W^\perp$ is a closed subspace of V^*.

(b) Let $f \in V^*$. Show that
$$d(f, W^\perp) = \left\| f|_W \right\|_{W^*}$$
where $f|_W$ is the restriction of f to W and
$$d(f, W^\perp) = \inf_{g \in W^\perp} \|f - g\|_{V^*}.$$

(c) Let $f \in W^*$ and let $\widetilde{f} \in V^*$ be an extension of f preserving its norm. Define $\sigma : W^* \to V^*/W^\perp$ by
$$\sigma(f) = \widetilde{f} + W^\perp.$$
Show that σ is well defined and that it is an isometric isomorphism of W^* onto $V^*/W^\perp$.

(d) Let W be a closed subspace. Let $\pi : V \to V/W$ be the canonical quotient map $x \mapsto x + W$. For $f \in (V/W)^*$, define $\tau(f) = f \circ \pi \in V^*$. Show that the range of τ is equal to $W^\perp$ and that the map $\tau : (V/W)^* \to W^\perp$ is an isometric isomorphism.

3.14 Let V be a normed linear space and let Z be a subspace of V^*. Define

$$Z^\perp = \{x \in V \mid g(x) = 0 \text{ for all } g \in Z\}.$$

(a) Show that $Z^\perp$ is a closed subspace of V.

(b) Show that, if W is a subspace of V, then
$$\left(W^\perp\right)^\perp = \overline{W}.$$

(c) If Z is a subspace of V^*, show that
$$\left(Z^\perp\right)^\perp \supset \overline{Z}.$$

(d) If V is reflexive and if Z is a subspace of V^*, show that

$$\left(Z^{\perp}\right)^{\perp} = \overline{Z}.$$

3.15 Let φ and ψ be continuous mappings from [0, 1] into a real normed linear space V. For arbitrary scalars α and β, show that

$$\int_0^1 (\alpha\varphi(t) + \beta\psi(t))\,\mathrm{d}t = \alpha\int_0^1 \varphi(t)\,\mathrm{d}t + \beta\int_0^1 \psi(t)\,\mathrm{d}t.$$

3.16 Let V and W be normed linear spaces and let $A \in \mathcal{L}(V, W)$. Let $\varphi : [0, 1] \to V$ be a continuous mapping. Show that

$$A\left(\int_0^1 \varphi(t)\,\mathrm{d}t\right) = \int_0^1 A(\varphi(t))\,\mathrm{d}t.$$

3.17 Let $\varphi : [a, b] \to V$ be a continuous mapping and let $c \in (a, b)$. Show that

$$\int_a^b \varphi(t)\,\mathrm{d}t = \int_a^c \varphi(t)\,\mathrm{d}t + \int_c^b \varphi(t)\,\mathrm{d}t.$$

3.18 Let $\varphi : \mathbb{R} \to V$ be a continuous mapping. Let $a, b, h \in \mathbb{R}$. Show that

$$\int_a^b \varphi(t)\,\mathrm{d}t = \int_{a+h}^{b+h} \psi(t)\,\mathrm{d}t$$

where $\psi(t) = \varphi(t - h)$.

3.19 Let $\varphi : [0, 1] \to \mathcal{C}[0, 1]$ be a continuous mapping and let $\psi \in \mathcal{C}[0, 1]$ be given by

$$\psi = \int_0^1 \varphi(t)\,\mathrm{d}t.$$

(a) Show that for any $s \in [0, 1]$, we have

$$\psi(s) = \int_0^1 \varphi(t)(s)\,\mathrm{d}t.$$

(b) Let $t_0 \in (0, 1)$. Show that

$$\int_0^{t_0}\int_0^1 \varphi(t)(s)\,\mathrm{d}t\,\mathrm{d}s = \int_0^1\int_0^{t_0} \varphi(t)(s)\,\mathrm{d}s\,\mathrm{d}t.$$

3.20 Let $\varphi : [0,1] \to V$ be a continuous mapping and let $t \in [0,1)$. Show that

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\varphi(s)\,\mathrm{d}s = \varphi(t).$$

3.21 Let V be a normed linear space and let $T \in \mathcal{L}(V)$ such that $\|T\| \le 1$. Let $\lambda > 0$ and let $x \in V$. Define

$$R(x) = \int_0^\infty e^{-\lambda t}T(x)\,\mathrm{d}t.$$

Show that $R \in \mathcal{L}(V)$ and that

$$\|R\| \le \frac{1}{\lambda}.$$

3.22 (a) Let V be a real Banach space. Let $k > 0$. Define

$$X = \left\{u \in \mathcal{C}([0,\infty); V) \mid \sup_{t\ge 0} e^{-kt}\|u(t)\|_V < \infty\right\}.$$

Define

$$\|u\|_X = \sup_{t\ge 0} e^{-kt}\|u(t)\|_V.$$

Show that X is a Banach space with this norm.

(b) Let $f : V \to V$ be a mapping. Assume that there exists $L > 0$ such that

$$\|f(u) - f(v)\|_V \le L\|u - v\|_V$$

for all u and $v \in V$. For $u \in X$, define

$$F(u)(t) = u_0 + \int_0^t f(u(s))\,\mathrm{d}s$$

where $u_0 \in V$ is a fixed vector. Show that $F(u) \in X$ and that, for any u and $v \in X$, we have

$$\|F(u) - F(v)\|_X \le \frac{L}{k}\|u - v\|_X.$$

(c) Deduce that, for any $u_0 \in V$, there exists a unique $u \in \mathcal{C}([0, \infty); V)$ such that

$$u(t) = u_0 + \int_0^t f(u(s))\,\mathrm{d}s$$

which is also the solution of the initial value problem

$$\begin{array}{l} \frac{du}{\mathrm{d}t}(t) = f(u(t)),\ \ t > 0 \\ u(0) = u_0. \end{array}$$

3.23 Let V be any vector space and let $f_0, f_1, \ldots, f_n$ be linear functionals on V. Let $\mathrm{Ker}(f_i)$ denote the kernel of $f_i,\ \ 0 \leq i \leq n$. Assume that

$$\cap_{i=1}^n \mathrm{Ker}(f_i) \ \subset \ \mathrm{Ker}(f_0).$$

Show that there exist scalars $\alpha_i,\ \ 1 \leq i \leq n$ such that

$$f_0 = \sum_{i=1}^N \alpha_i f_i.$$

(Hint: Consider the image of the map $\mathcal{A}$ as defined in Sect. 3.4 and apply Corollary 3.3.1 to its image.)

3.24 Let V be a real normed linear space and let $K \subset V$ be a compact and convex subset. Let $C \subset V^*$ be a convex cone. Assume that for each $f \in C$, there exists a vector $x \in K$ (depending on f), such that $f(x) \geq 0$. Show that there exists $x \in K$ such that $f(x) \geq 0$ for all $f \in C$. (Hint: For $f \in C$, consider

$$K_f = \{x \in K \mid f(x) \geq 0\}$$

which is a non-empty closed set. Show that this collection of closed sets has finite intersection property.)

Reference

1. W. Rudin, *Functional Analysis* (McGraw-Hill, 1973)

Chapter 4
Baire's Theorem and Applications

4.1 Baire's Theorem

Baire's theorem is a result on complete metric spaces which will be used in this chapter to prove some very important results on Banach spaces.

Theorem 4.1.1 (Baire) *Let (X, d) be a complete metric space. Let $\{V_n\}_{n=1}^{\infty}$ be a collection of open dense sets. Then*

$$\cap_{n=1}^{\infty} V_n$$

is also dense.

Proof Let W be any non-empty open set in X. We need to show that the intersection $\cap_n V_n$ has a point in W.

Since V_1 is dense, it follows that $W \cap V_1 \neq \emptyset$. Thus, we can find a point x_1 in this intersection (which is also an open set). Hence, there exists $r_1 > 0$ such that the open ball $B(x_1; r_1)$, with centre at x_1 and radius r_1, is contained in $W \cap V_1$. By shrinking r if necessary, we may also assume that the closure of this ball $\overline{B(x_1; r_1)}$ is also a subset of $W \cap V_1$ and that $0 < r_1 < 1$.

If $n \geq 2$, assume that we have chosen x_{n-1} and r_{n-1} suitably. The denseness of V_n shows that $V_n \cap B(x_{n-1}; r_{n-1}) \neq \emptyset$ and so we can choose x_n and r_n such that $0 < r_n < 1/n$ and

$$\overline{B(x_n; r_n)} \subset V_n \cap B(x_{n-1}; r_{n-1}).$$

Thus we now have a sequence of points $\{x_n\}$ in X. If $i > n$ and $j > n$, it is clear that both x_i and x_j both lie in $B(x_n; r_n)$ and so $d(x_i, x_j) < 2r_n < 2/n$ and thus, the sequence $\{x_n\}$ is Cauchy. Since X is complete, it follows that there exists $x \in X$ such that $x_n \to x$.

Now, for $i > n$, it follows that $x_i \in B(x_n; r_n)$ and so $x \in \overline{B(x_n; r_n)}$ for all n and so $x \in V_n$ for all n. Also $x \in \overline{B(x_1; r_1)}$ and so $x \in W$ as well. This completes the proof. ■

S. Kesavan, *Functional Analysis*, Texts and Readings in Mathematics,
https://doi.org/10.1007/978-981-19-7633-9_4

Remark 4.1.1 The main significance of this theorem is that, in particular, the intersection of a countable collection of non-empty open dense sets in a (non-empty) complete metric space is, again, non-empty. ■

Remark 4.1.2 An equivalent way of stating this theorem is that a complete metric space cannot be the countable union of nowhere dense sets. In the literature, countable unions of nowhere dense sets are said to be of the *first category* and all other sets are said to be of the *second category*. Thus, Baire's theorem states that every complete metric space is of second category and so, often in the literature, it is referred to as the *Baire category theorem.* ■

A countable intersection of open sets in a topological space is called a G_δ set. Since the countable union of countable sets is again countable, the following corallary is an immediate consequence of Baire's theorem.

Corollary 4.1.1 *In a complete metric space, the intersection of any countable collection of dense G_δ sets is again a dense G_δ set.* ■

Corollary 4.1.2 *In a complete metric space which has no isolated points, a countable dense set can never be a G_δ set.* ■

Proof Let $E = \{x_k\}_{k=1}^{\infty}$ be a countable dense set in a complete metric space X. If it is a G_δ set, then, there exist open sets V_n such that

$$E = \cap_{n=1}^{\infty} V_n.$$

Clearly, since $E \subset V_n$ for each n, each set V_n is dense as well. Set

$$W_n = V_n \backslash \cup_{k=1}^{n} \{x_k\}.$$

Then each W_n is also dense and open, since X has no isolated points. But $\cap_{n=1}^{\infty} W_n = \emptyset$, which contradicts Baire's theorem. Hence the result. ■

Let V be a normed linear space and let $W \subset V$ be a closed and proper subspace. Then, by a simple scaling argument, it is immediate to see that W cannot contain a ball of V with centre at the origin and of positive radius. By translation, it now follows that W cannot contain any open ball of the form $B_V(x; r)$, where B_V indicates that it is a ball in V. Thus, it follows that W has empty interior and so W is nowhere dense.

Example 4.1.1 Consider the space c_{00} of all sequences which have at most a finite number of non-zero terms. In other words, if e_n is the sequence with one in the n-th place and zero elsewhere, we have that

$$c_{00} = \text{span}\{\mathsf{e}_n \mid n \in \mathbb{N}\}.$$

Let $W_n = \text{span}\{\mathsf{e}_k \mid 1 \leq k \leq n\}$. Then, each W_n is finite dimensional and,

$$c_{00} = \cup_{n=1}^{\infty} W_n.$$

Whatever be the norm on c_{00}, we have that W_n is a closed and proper subspace of c_{00} for each n, and hence, by our earlier observation, each W_n is nowhere dense. By Baire's theorem it now follows that c_{00} cannot be complete for *any* norm. ■

4.2 Principle of Uniform Boundedness

As a first application of Baire's theorem we prove the following result.

Theorem 4.2.1 (Banach-Steinhaus) *Let V be a Banach space and let W be a normed linear space. Let I be an arbitrary indexing set and, for each $i \in I$, let $T_i \in \mathcal{L}(V, W)$. Then, either there exists $M > 0$ such that*

$$\|T_i\| \leq M, \text{ for all } i \in I,$$

or,

$$\sup_{i \in I} \|T_i(x)\| = \infty$$

for all x belonging to some dense G_δ set in V.

Proof For each $x \in V$, set

$$\varphi(x) = \sup_{i \in I} \|T_i(x)\|.$$

Let

$$V_n = \{x \in V \mid \varphi(x) > n\}.$$

Since each T_i is continuous and since the norm is a continuous function, it is easy to see that V_n is open for each n.

Assume now that there exists N such that V_N fails to be dense in V. Then, there exists $x_0 \in V$ and $r > 0$ such that $x + x_0 \notin V_N$ if $\|x\| < r$. (In other words, there is an open ball $B(x_0; r)$, centred at x_0 and of radius r, which does not intersect V_N.) This implies that $\varphi(x + x_0) \leq N$ for all such x and so, for all $i \in I$,

$$\|T_i(x + x_0)\| \leq N.$$

Thus, if $\|x\| \leq r/2$, we have, for all $i \in I$,

$$\|T_i(x)\| \leq \|T_i(x + x_0)\| + \|T_i(x_0)\| \leq 2N.$$

It follows from this that, for all $i \in I$,

$$\|T_i\| \le \frac{4N}{r}$$

and so the first alternative holds with $M = 4N/r$.

The other possibility is that each V_n is dense, and so, V being complete, by Baire's theorem, $\cap_n V_n$ is a dense G_δ and for each $x \in \cap_n V_n$, we have that $\varphi(x) = \infty$. This completes the proof. ■

An immediate consequence of the above result is the following.

Corollary 4.2.1 *If V is a Banach space and W is a normed linear space and if $T_i \in \mathcal{L}(V, W)$ for an indexing set I such that*

$$\sup_{i \in I} \|T_i(x)\| < \infty$$

for every $x \in V$, then there exists $M > 0$ such that

$$\|T_i\| \le M \text{ for each } i \in I. \qquad ■$$

In other words, if the T_i are all pointwise bounded, then they are uniformly bounded in norm. For this reason, the Banach-Steinhaus theorem is also referred to as the *principle of uniform boundedness*.

Corollary 4.2.2 *Let V be a Banach space and let W be a normed linear space and let $\{T_n\}$ be a sequence of continuous linear transformations from V into W such that, for each $x \in V$, the sequence $\{T_n x\}$ is convergent in W. Define*

$$T(x) = \lim_{n\to\infty} T_n(x).$$

Then $T \in \mathcal{L}(V, W)$ and

$$\|T\| \le \liminf_{n\to\infty} \|T_n\|. \tag{4.2.1}$$

Proof It is clear that T is linear. By the Banach-Steinhaus theorem, it follows that $\{\|T_n\|\}$ is a bounded sequence. Let $\|T_n\| \le C$ for all n. Then, for each $x \in V$ and for all n, we have

$$\|T_n(x)\| \le C\|x\|.$$

Passing to the limit as $n \to \infty$, we deduce that

$$\|T(x)\| \le C\|x\|$$

for each $x \in V$ and so $T \in \mathcal{L}(V, W)$. The relation (4.2.1) follows from the inequality

$$\|T_n(x)\| \le \|T_n\| \, \|x\|$$

for each $x \in V$. ■

Corollary 4.2.3 *Let V be a Banach space and let $B \subset V$ be a subset. Assume that*

$$f(B) = \{f(x) \mid x \in B\}$$

is a bounded subset of the scalar field for each $f \in V^*$. Then B is a bounded subset of V.

Proof For $x \in B$, consider the functional $J_x \in V^{**}$ defined by $J_x(f) = f(x)$ for $f \in V^*$. Then, we know that (cf. Corollary 3.1.2)

$$\|J_x\|_{V^{**}} = \|x\|_V.$$

Taking B as the indexing set and V^* as the Banach space, it follows from the Banach-Steinhaus theorem that $\|J_x\|$ is uniformly bounded in V^{**}, which is the same as saying that B is bounded in V. ■

Remark 4.2.1 To check the boundedness of a set V in a Banach space, it thus suffices to verify that its image under each continuous linear functional is bounded. In finite dimensional spaces, this is what we precisely do. We check that the image under each coordinate projection is bounded and these form a basis for the dual space. In the language of weak topologies (to be studied later), the conclusion of the preceding corollary is read as 'weakly bounded implies bounded'. ■

4.3 Application to Fourier Series

Let $f : [-\pi, \pi] \to \mathbb{R}$ be an integrable function. We can write its formal *Fourier series* (in exponential form) as follows:

$$f(t) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) \exp(\text{int})$$

where

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \exp(-ins) \, \mathrm{d}s \tag{4.3.1}$$

are the *Fourier coefficients* of f. The first question that springs to the mind is 'in what sense does the Fourier series of a function represent the function?' In particular, does the Fourier series of a continuous 2π-periodic function f converge to $f(t)$ at every point $t \in [-\pi, \pi]$? This is relevant since each term in the Fourier series is a continuous 2π-periodic function. Unfortunately, the answer is 'No!'. It was Dirichlet who first established (around 1829, nearly seven decades after a lengthy controversy began in Europe—about the validity of representing a function in terms of sines and cosines—and raged through the latter half of the eighteenth century) the sufficient conditions for the Fourier series of a function to converge to its value at a point. This was later strengthened by Jordan. In fact the study of the validity of Fourier expansions led to a lot of mathematical development such as making precise the notion of a function, Cantor's theory of infinite sets, the theories of integration by Riemann and by Lebesgue and theories of summability of series.

In this section, we will use the Banach-Steinhaus theorem to show that there exists a very large class of continuous 2π-periodic functions whose Fourier series fail to converge on a very large set of points.

To study the convergence of Fourier series, we need to study its partial sums:

$$s_m(f)(t) = \sum_{n=-m}^{m} \widehat{f}(n) \exp(\text{int}).$$

Using the formula for the Fourier coefficients (4.3.1), this becomes

$$s_m(f)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) D_m(t-s)\, \mathrm{d}s \tag{4.3.2}$$

where

$$D_m(t) = \sum_{n=-m}^{m} \exp(\text{int}).$$

The function D_m is called the *Dirichlet kernel*. If we multiply $D_m(t)$ successively by $\exp(it/2)$ and $\exp(-it/2)$ and subtract, we see immediately that

$$D_m(t) = \begin{cases} \frac{\sin(m+\frac{1}{2})t}{\sin\frac{t}{2}}, & \text{if } t \neq 2k\pi, \text{ for some non-negative integer } k \\ 2m+1, & \text{if } t = 2k\pi \text{ for some non-negative integer } k. \end{cases}$$

Proposition 4.3.1 *We have*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |D_n(t)|\, dt = +\infty. \tag{4.3.3}$$

Proof For $t \in \mathbb{R}$, we have $|\sin t| \leq |t|$ and so

$$\int_{-\pi}^{\pi} |D_n(t)|\, \mathrm{d}t \geq 4 \int_{0}^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{t}\, \mathrm{d}t$$

$$
\begin{aligned}
&= 4 \int_{0}^{(n+\frac{1}{2})\pi} \frac{|\sin t|}{t}\, \mathrm{d}t \\
&> 4 \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t}\, \mathrm{d}t \\
&> 4 \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{k\pi}\, \mathrm{d}t \\
&= \frac{8}{\pi} \sum_{k=1}^{n} \frac{1}{k}
\end{aligned}
$$

from which (4.3.3) follows immediately. ■

Proposition 4.3.2 *Let $V = \mathcal{C}_{\text{per}}[-\pi, \pi]$, the space of continuous 2π-periodic functions with the usual sup-norm (denoted $\|\cdot\|_\infty$) and define $\phi_n \colon V \to \mathbb{R}$ by*

$$\phi_n(f) = s_n(f)(0)$$

where $s_n(f)$ is the n-th partial sum of the Fourier series of f. Then ϕ_n is a continuous linear functional on V and

$$\|\phi_n\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt. \tag{4.3.4}$$

Proof On one hand,

$$\phi_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(t)\, \mathrm{d}t$$

(cf. (4.3.2); D_n is an even function). Thus,

$$|\phi_n(f)| \ \le \ \|f\|_\infty \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, \mathrm{d}t$$

and so

$$\|\phi_n\| \ \le \ \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, \mathrm{d}t.$$

Now, let $E_n = \{t \in [-\pi, \pi] \mid D_n(t) \geq 0\}$. Define

$$f_m(t) = \frac{1 - md(t, E_n)}{1 + md(t, E_n)}$$

where $d(t, A) = \inf\{|t - s| \mid s \in A\}$ is the distance of t from a set A. Since $d(t, A)$ is a continuous function (cf. Proposition 1.2.3), $f_m \in \mathcal{C}_{\text{per}}[-\pi, \pi]$, (it is periodic since D_n is even and so E_n is a symmetric set about the origin). Also $\|f_m\|_\infty \leq 1$ and $f_m(t) \to 1$ if $t \in E_n$ while $f_m(t) \to -1$ if $t \in E_n^c$. By the dominated convergence theorem, it now follows that

$$\phi_n(f_m) \to \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| \, \mathrm{d}t$$

from which (4.3.4) follows. ■

We now apply the Banach-Steinhaus theorem to the Banach space $V = \mathcal{C}_{\text{per}}[-\pi, \pi]$ and the collection of continuous linear functionals $\{\phi_n\}$. Since, by Propositions 4.3.1 and 4.3.2, we have $\|\phi_n\| \to \infty$ as $n \to \infty$, it follows there exists a dense G_δ-set (of continuous 2π-periodic functions) in V such that the Fourier series of all these functions diverge at $t = 0$. We could have very well dealt with any other point in the interval $[-\pi, \pi]$ in the same manner.

By another application of Baire's theorem, we can strengthen this further.

Let E_x be the dense G_δ-set of continuous 2π-periodic functions in V such that the Fourier series of these functions diverge at x. Let $\{x_i\}$ be a countable set of points in $[-\pi, \pi]$ and let

$$E = \cap_{i=1}^{n} E_{x_i} \subset V.$$

Then, by Baire's theorem, E is also a dense G_δ-set (cf. Corollary 4.1.1). Thus for each $f \in E$, the Fourier series of f diverges at x_i for all $1 \leq i \leq \infty$. Define

$$s^*(f; x) = \sup_n |s_n(f)(x)|.$$

Hence $\{x \mid s^*(f; x) = \infty\}$ is a G_δ-set in $(-\pi, \pi)$ for each f. If we choose the x_i above so that $\{x_i\}$ is dense (take all rationals, for instance in $(-\pi, \pi)$) then we have the following result.

Proposition 4.3.3 *The set $E \subset V$ is a dense G_δ-set such that for all $f \in E$, the set $Q_f \subset (-\pi, \pi)$ where its Fourier series diverges, is a dense G_δ-set in $(-\pi, \pi)$.* ■

By virtue of Corollary 4.1.2, it follows that there exist uncountably many 2π-periodic continuous functions on $[-\pi, \pi]$ whose Fourier series diverge on an uncountable subset of $(-\pi, \pi)$.

4.4 The Open Mapping and Closed Graph Theorems

In this section, we will study two more important consequences of Baire's theorem which will have a lot of applications.

We begin by setting up some notation. If A and B are subsets in a vector space V, we set

$$A + B = \{x + y \mid x \in A, y \in B\}.$$

Similarly, if λ is a scalar, we set

$$\lambda A = \{\lambda x \mid x \in A\}.$$

If A is a convex set, then we have $A + A = 2A$, for if $x \in A$, then $2x = x + x$ and so $2A \subset A + A$. On the other hand, if x and $y \in A$, by convexity, we have $\frac{1}{2}(x + y) \in A$ and so $x + y \in 2A$. Thus, $A + A \subset 2A$.

Proposition 4.4.1 *Let V and W be Banach spaces. Let $T \in \mathcal{L}(V, W)$ be* surjective. *Then, there exists a constant $c > 0$ such that*

$$B_W(\mathbf{0}; c) \subset T(B_V(\mathbf{0}; 1)). \tag{4.4.1}$$

where B_V and B_W denote open balls in the spaces V and W, respectively.

Proof Step 1. We will first show that there exists a constant $c > 0$ such that

$$B_W(\mathbf{0}; 2c) \subset \overline{T(B_V(\mathbf{0}; 1))}. \tag{4.4.2}$$

Set $X_n = n\overline{T(B_V(\mathbf{0}; 1))}$. Then each X_n is a closed set. Since T is linear and surjective, it follows immediately that $W = \cup_{n=1}^{\infty} X_n$. Hence, W being complete, it follows from Baire's theorem (cf. Remark 4.1.2) that there exists n such that X_n has non-empty interior. Hence by change of scale, it follows that

$$(\overline{T(B_V(\mathbf{0}; 1))})^\circ \neq \emptyset.$$

Thus, there exists $y_0 \in W$ and $c > 0$ such that

$$B_W(y_0; 4c) \subset \overline{T(B_V(\mathbf{0}; 1))}.$$

In particular $y_0 \in \overline{T(B_V(\mathbf{0}; 1))}$ and, by symmetry, so does $-y_0$. Since any element of $B_W(y_0; 4c)$ may be written as $y_0 + z$ where $z \in B_W(\mathbf{0}; 4c)$, any such z can be written as $z = (y_0 + z) + (-y_0)$ and so it follows from the above that

$$B_W(\mathbf{0}; 4c) \subset \overline{T(B_V(\mathbf{0}; 1))} + \overline{T(B_V(\mathbf{0}; 1))}.$$

But $\overline{T(B_V(\mathbf{0}; 1))}$ is convex and so the set on the right-hand side in the above inclusion is, in fact, $2\overline{T(B_V(\mathbf{0}; 1))}$ and so we have (4.4.2).

Step 2. We now prove (4.4.1). Let $y \in B_W(\mathbf{0}; c)$. We need to find $x \in B_V(\mathbf{0}; 1)$ such that $T(x) = y$. Let $\varepsilon > 0$. There exists $z \in V$ such that $\|z\|_V < 1/2$ and $\|y - T(z)\|_W < \varepsilon$ by virtue of (4.4.2) (applied to $2y$). Set $\varepsilon = c/2$ and let $z_1 \in V$ be such that $\|z_1\|_V < 1/2$ and $\|T(z_1) - y\|_W < c/2$.

We can iterate this procedure. By another application of (4.4.2) (to $4(T(z_1) - y)$) we can find $z_2 \in V$ such that

$$\|z_2\|_V < \frac{1}{4}, \ \|T(z_1 + z_2) - y\|_W < \frac{c}{4}.$$

Thus, we can find, by repeated use of (4.4.2), a sequence $\{z_n\}$ in V such that

$$\|z_n\|_V < \frac{1}{2^n}, \ \|T(z_1 + \cdots + z_n) - y\|_W < \frac{c}{2^n}.$$

Then, it follows that the sequence $\{z_1 + \cdots + z_n\}$ is Cauchy in V, and, since V is complete, it will converge to an element $z \in V$ such that $\|z\|_V < 1$ and we will also have $T(z) = y$. This completes the proof. ■

Remark 4.4.1 Step 1 of the above proof used the completeness of W to apply Baire's theorem. Step 2 used the completeness of the space V. ■

Remark 4.4.2 If $r > 0$, the same arguments can be used to show that there exists $s > 0$ such that $B_W(\mathbf{0}; s) \subset T(B_V(\mathbf{0}; r))$. ■

Theorem 4.4.1 (Open Mapping Theorem) *Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$ be* surjective. *Then T is an open map.*

Proof We need to show that T maps open sets in V onto open sets in W. Let G be an open set in V. Let $y \in T(G)$. Then, there exists $x \in G$ such that $y = T(x)$. Since G is open, there exists $r > 0$ such that $x + B_V(\mathbf{0}; r) \subset G$. Hence, $y + T(B_V(\mathbf{0}; r)) \subset T(G)$. But by the previous proposition, there exists $s > 0$ such that $B_W(\mathbf{0}; s) \subset T(B_V(\mathbf{0}; r))$ and so $y + B_W(\mathbf{0}; s) \subset T(G)$ which means that $T(G)$ is open. ■

Corollary 4.4.1 *Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$ be a bijection. Then T is an isomorphism.*

Proof Since T is a continuous bijection, in particular it is onto and so, by the open mapping theorem, it is an open map, which means that T^{-1} is also continuous. Thus T is an isomorphism. ■

Corollary 4.4.2 *Let V be a Banach space with respect to two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume that there exists $c > 0$ such that*

$$\|x\|_1 \leq c\|x\|_2$$

for all $x \in V$. Then, the two norms are equivalent.

Proof The identity map $I: \{V, \|\cdot\|_2\} \to \{V, \|\cdot\|_1\}$ is a linear bijection which is also continuous by the given hypothesis. Hence the inverse map is also continuous, and this means that there exists a constant $C > 0$ such that

$$\|x\|_2 \le C\|x\|_1$$

for all $x \in V$. Hence the norms are equivalent. ■

If V and W are normed linear spaces and $T: V \to W$ is any mapping define the **graph** of T, denoted $G(T)$ as follows:

$$G(T) = \{(x, y) \mid y = T(x)\} \subset V \times W$$

If V and W are normed linear spaces and $T: V \to W$ is continuous, then $G(T)$ is closed in $V \times W$. Indeed, if $(x_n, T(x_n)) \to (x, y)$ in $V \times W$, then $x_n \to x$ in V and $T(x_n) \to y$ in W. But, by continuity, $T(x_n) \to T(x)$ and so $y = T(x)$. Thus $(x, y) \in G(T)$ and so $G(T)$ is closed. In fact, we have the following general topological result.

Lemma 4.4.1 *Let X and Y be topological spaces, with Y being Hausdorff. Let $f: X \to Y$ be a given mapping. Then, if f is continuous, the graph*

$$G(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$$

is closed in the product space $X \times Y$.

Proof Let $(x, y) \in (X \times Y)\backslash G(f)$. Thus, $y \ne f(x)$ and so, since Y is Hausdorff, there exist open sets $\mathcal{U}$ and $\mathcal{V}$ in Y such that $f(x) \in \mathcal{U}$, $y \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$. Now, $f^{-1}(\mathcal{U}) \times \mathcal{V}$ is open in $X \times Y$ and contains the point (x, y). Further, this set does not intersect $G(f)$. (For, if (u, v) were in the intersection, we would have $v = f(u)$, $u \in f^{-1}(\mathcal{U})$ and $v \in \mathcal{V}$. Thus, $f(u) = v \in \mathcal{U} \cap \mathcal{V}$, which is a contradiction.) Thus, the complement of $G(f)$ contains an open neighbourhood of each of its points and is hence open; *i.e.* $G(f)$ is closed in $X \times Y$. ■

If V and W are Banach spaces and T is linear, then the converse is also true as the following theorem shows.

Theorem 4.4.2 (Closed Graph Theorem) *Let V and W be Banach spaces and let $T: V \to W$ be a linear mapping. Assume that $G(T)$, the graph of T, is closed in $V \times W$. Then T is continuous.*

Proof For $x \in V$, define

$$\|x\|_1 = \|x\|_V + \|T(x)\|_W.$$

Then $\|\cdot\|_1$ defines a norm on V. If $\{x_n\}$ is a sequence in V which is Cauchy with respect to this norm, then evidently it is Cauchy with respect to the norm $\|\cdot\|_V$

as well. Then, since V is complete for the norm $\|\cdot\|_V$, we have that $x_n \to x$ in V (in the topology of the norm $\|\cdot\|_V$). Again, since $\{x_n\}$ is Cauchy with respect to the norm $\|\cdot\|_1$, it also follows that $\{T(x_n)\}$ is Cauchy in W and so, since W is complete, $T(x_n) \to y$ in W. Since $G(T)$ is closed, it follows that $y = T(x)$. Thus, it follows that $x_n \to x$ in the topology of the norm $\|\cdot\|_1$ as well. Thus $\{V, \|\cdot\|_1\}$ is also complete. Hence, by the preceding corollary, since $\|x\|_V \leq \|x\|_1$ for all $x \in V$, it follows that $\|\cdot\|_V$ and $\|\cdot\|_1$ are equivalent. Hence, there exists $C > 0$ such that $\|x\|_1 \leq C\|x\|_V$ for all $x \in V$. In particular, $\|T(x)\|_W \leq C\|x\|_V$ for all $x \in V$ and so T is continuous. ■

Remark 4.4.3 The closed graph theorem gives a convenient way for verifying the continuity of linear maps between Banach spaces V and W. We just have to verify that if $x_n \to x$ in V and if $T(x_n) \to y$ in W, then $y = T(x)$. ■

Remark 4.4.4 In the Banach-Steinhaus theorem, it was sufficient that the domain space V was complete. The range space W could be any normed linear space. In the case of the open mapping and closed graph theorems, however, it is essential that both the spaces V and W are complete. ■

Remark 4.4.5 One can first prove the closed graph theorem and deduce the open mapping theorem from it. In fact one can prove that the three theorems—the Banach-Steinhaus theorem, the open mapping theorem, and the closed graph theorem—are all equivalent to each other; *i.e.* each of these statements implies the other two. Further, one can prove the Banach-Steinhaus theorem without using Baire's theorem. Thus, all these theorems can be proved without using Baire's theorem. (However, the proofs using Baire's theorem are simpler and more natural.) The important point is that we work with complete spaces. For more details on the points mentioned above, the interested reader is referred to an expository article by the author, entitled *The grand theorems of functional analysis revisited: a Baire-free approach*, which appeared in the *Mathematics Newsletter* of the Ramanujan Mathematical Society, Vol. 31, No. 3, December 2020–March 2021, pp. 89–93. The article can also be found in the author's home page: www.imsc.res.in/~kesh. ■

4.5 Annihilators

In the remaining sections of this chapter, we will apply the open mapping theorem to obtain results about the range and the kernel of linear maps. In order to do this, we need to introduce an important notion.

Definition 4.5.1 Let V be a Banach space and let W be a subspace of V. Let Z be a subspace of V^*. The **annihilator** of W is the subspace of V^* given by

$$W^{\perp} = \{f \in V^* \mid f(x) = 0 \text{ for all } x \in W\}.$$

The annihilator of Z is the subspace of V given by

$$Z^\perp = \{x \in V \mid f(x) = 0 \text{ for all } f \in Z\}. \qquad \blacksquare$$

It is easy to see that $W^\perp$ is a closed subspace of V^* and that $Z^\perp$ is a closed subspace of V. Further,

$$\left(W^\perp\right)^\perp = \overline{W} \tag{4.5.1}$$

and

$$\left(Z^\perp\right)^\perp \supset \overline{Z} \tag{4.5.2}$$

(cf. Exercise 3.14). Also, if G and H are subspaces of V such that $G \subset H$, then $H^\perp \subset G^\perp$.

Example 4.5.1 Let $V = \ell_1$ so that $V^* = \ell_\infty$. Consider c_{00} (cf. Example 4.1.1) as a subspace of ℓ_1. Then if $y \in c_{00}^\perp$, we have that $y \in \ell_\infty$ and that, for all $x \in c_{00}$, $\sum_{k=1}^\infty y_k x_k = 0$. In particular, if we take $x = \mathsf{e}_n$, which is the sequence with unity in the n-th place and zero elsewhere, we have that $y_n = 0$ for every positive integer n. Thus $c_{00}^\perp = \{\mathbf{0}\}$ and so $c_{00}^{\perp\perp} = \ell_1 = \overline{c_{00}}$, since c_{00} is dense in ℓ_1.

On the other hand, we can also consider c_{00} as a subspace of $V^* = \ell_\infty$. Again, it is immdeiate to see that $c_{00}^\perp = \{\mathbf{0}\} \subset \ell_1$ and so $c_{00}^{\perp\perp} = \ell_\infty$ which strictly contains $\overline{c_{00}} = c_0$, the space of all sequences converging to zero. $\blacksquare$

Proposition 4.5.1 *Let G and H be closed subspaces of a Banach space V. Then*

$$G \cap H = \left(G^\perp + H^\perp\right)^\perp \tag{4.5.3}$$

and

$$G^\perp \cap H^\perp = (G + H)^\perp. \tag{4.5.4}$$

Proof Clearly, $G \cap H \subset \left(G^\perp + H^\perp\right)^\perp$. Also, $G^\perp \subset G^\perp + H^\perp$ and $H^\perp \subset G^\perp + H^\perp$. Hence,

$$\left(G^\perp + H^\perp\right)^\perp \subset \left(G^\perp\right)^\perp = \overline{G} = G$$

using (4.5.1) and the fact that G is closed. Similarly, we have

$$\left(G^\perp + H^\perp\right)^\perp \subset H$$

and (4.5.3) follows immediately. The relation (4.5.4) is obvious. $\blacksquare$

Combining the result of the preceding proposition with the relations (4.5.1)–(4.5.2), we deduce the following corollary.

Corollary 4.5.1 *Let G and H be closed subspaces of a Banach space V. Then*

$$(G \cap H)^{\perp} \supset \overline{G^{\perp} + H^{\perp}} \tag{4.5.5}$$

and

$$\left(G^{\perp} \cap H^{\perp}\right)^{\perp} = \overline{G + H}. \tag{4.5.6}$$

■

Proposition 4.5.2 *Let G and H be two closed subspaces of a Banach space V such that $G + H$ is also closed. Then, there exists a constant $C > 0$ such that, for every $z \in G + H$, there exist $x \in G$ and $y \in H$ satisfying*

$$z = x + y,\ \|x\| \le C\|z\|,\ \|y\| \le C\|z\|. \tag{4.5.7}$$

Proof Consider the product space $G \times H$ with the norm

$$\|(x, y)\|_{G\times H} = \|x\| + \|y\|.$$

This is a Banach space since G and H are closed. Consider the map $G \times H \to G + H$ given by $(x, y) \mapsto x + y$. This map is clearly continuous, linear and onto. Since $G + H$ is a closed subspace of the Banach space V, it is also a Banach space and so, by the open mapping theorem, there exists a constant $c > 0$ such that if $z \in G + H$ with $\|z\| < c$, then there exists $(x, y) \in G \times H$ such that $z = x + y$ and such that $\|x\| + \|y\| < 1$. Given any $z \in G + H$, by considering, for instance, the vector $\frac{c}{2\|z\|}z$ we see immediately that z can be written as $z = x + y$ with $x \in G,\ y \in H$ such that

$$\|x\| \le \frac{2}{c}\|z\|,\ \|y\| \le \frac{2}{c}\|z\|$$

which proves the result with $C = 2/c$. ■

Corollary 4.5.2 *Let G and H be closed subspaces of a Banach space V such that $G + H$ is also closed.Then, there exists a constant $C > 0$ such that*

$$d(x, G \cap H) \le C\,[d(x, G) + d(x, H)] \tag{4.5.8}$$

for all $x \in V$.

Proof Let $x \in V$ and let $\varepsilon > 0$ be arbitrary. Then, there exist $a \in G$ and $b \in H$ such that

$$\|x - a\| \le d(x, G) + \varepsilon \text{ and } \|x - b\| \le d(x, H) + \varepsilon.$$

set $z = a - b \in G + H$. Then, there exists $c > 0$ (which depends only on G and H), $a' \in G$ and $b' \in H$ such that

$$a - b = a' + b',\ \|a'\| \le c\|a - b\|,\ \|b'\| \le c\|a - b\|.$$

Now, $a - a' = b' + b \in G \cap H$. Thus,

$$\begin{aligned} d(x, G \cap H) &\leq \|x - a\| + \|a'\| \\ &\leq \|x - a\| + c\|a - b\| \\ &\leq (1 + c)\|x - a\| + c\|x - b\| \\ &\leq (1 + c)d(x, G) + cd(x, H) + (1 + 2c)\varepsilon \\ &\leq (1 + c)(d(x, G) + d(x, H)) + (1 + 2c)\varepsilon \end{aligned}$$

which completes the proof (with $C = (1 + c)$), since ε is an arbitrarily small quantity. ■

We are now in a position to prove a deeper result on annihilators.

Theorem 4.5.1 *Let G and H be closed subspaces of a Banach space V. The following are equivalent:*
(i) *$G + H$ is closed in V.*
(ii) *$G^\perp + H^\perp$ is closed in V^*.*
(iii)

$$G + H = \left(G^\perp \cap H^\perp\right)^\perp.$$

(iv)

$$G^\perp + H^\perp = (G \cap H)^\perp.$$

Proof Since an annihilator is always closed, the implication (iv) $\Rightarrow$ (ii) is trivial. The equivalence (i) $\Leftrightarrow$ (iii) is an immediate consequence of (4.5.6). To complete the proof, we need to show that (i) $\Rightarrow$ (iv) and that (ii) $\Rightarrow$ (i), which we now proceed to do.

Step 1. (i) $\Rightarrow$ (iv). Assume that $G + H$ is closed. Now, by (4.5.5), we already know that $(G \cap H)^\perp \supset G^\perp + H^\perp$. Hence, to prove (iv), it suffices to prove the reverse inclusion. Let $f \in (G \cap H)^\perp$. Define a linear functional φ on $G + H$ as follows: let $x = a + b \in G + H$ with $a \in G$ and $b \in H$; set

$$\varphi(x) = f(a).$$

If $x = a' + b'$ is another decomposition of x with $a' \in G$ and $b' \in H$, it follows that $a - a' = b' - b \in G \cap H$ and so $f(a) = f(a')$. Thus the definition of $\varphi(x)$ is independent of the decomposition chosen, and so φ is a well-defined linear functional. Further, since $G + H$ is closed, we can choose a decomposition $x = a + b$ such that $\|a\| \leq C\|x\|$ (where C depends only on G and H). Consequently, $\|\varphi(x)\| \leq C\|f\|\ \|x\|$ and it follows that φ is a continuous linear functional on $G + H$. Hence, we can extend it to a continuous linear functional $\widetilde{\varphi}$ on V, by the Hahn-Banach theorem. Now, $f = \varphi = \widetilde{\varphi}$ on G and so $f - \widetilde{\varphi} \in G^\perp$. Also if $x \in H$, then since $\varphi(x) = 0$, it follows that $\widetilde{\varphi} \in H^\perp$. Hence we have that

$$f = (f - \widetilde{\varphi}) + \widetilde{\varphi} \in G^\perp + H^\perp$$

which proves the reverse inclusion that we sought to establish.
Step 2. (ii) $\Rightarrow$ (i). For any $f \in V^*$, we have that (cf. Exercise 3.8b)

$$d(f, G^\perp) = \|f|_G\| = \sup_{x\in G,\ \|x\|\le 1} |f(x)|,$$

$$d(f, H^\perp) = \|f|_H\| = \sup_{x\in H,\ \|x\|\le 1} |f(x)|,$$

and (in view of (4.5.4)) that

$$d(f, G^\perp \cap H^\perp) = d(f, (G+H)^\perp) = \|f|_{\overline{G+H}}\| = \sup_{x\in\overline{G+H},\ \|x\|\le 1} |f(x)|.$$

By Corollary 4.5.2, there exists a constant $C > 0$ such that, for every $f \in V^*$,

$$d(f, G^\perp \cap H^\perp) \le C\left[d(f, G^\perp) + d(f, H^\perp)\right]$$

since we are assuming that $G^\perp + H^\perp$ is closed in V^*. In other words, we have, for every $f \in V^*$,

$$\sup_{x\in\overline{G+H},\ \|x\|\le 1} |f(x)| \le C\left[\sup_{x\in G,\ \|x\|\le 1} |f(x)| + \sup_{x\in H,\ \|x\|\le 1} |f(x)|\right]. \tag{4.5.9}$$

Step 3. We now claim that the above relation implies that

$$\overline{B_G(\mathbf{0}; 1) + B_H(\mathbf{0}; 1)} \supset \frac{1}{C} B_{\overline{G+H}}(\mathbf{0}; 1) \tag{4.5.10}$$

where $B_G(\mathbf{0}; 1)$ denotes the (open) unit ball in G and so on. If not, let $x_0 \in \overline{G+H}$ with $\|x_0\| < 1/C$ such that

$$x_0 \notin \overline{B_G(\mathbf{0}; 1) + B_H(\mathbf{0}; 1)}.$$

Assume that V is a real Banach space. Then, by the Hahn-Banach theorem, there exists $f \in V^*$ and a real number α such that, for all $z \in B_G(\mathbf{0}; 1) + B_H(\mathbf{0}; 1)$, we have

$$f(z) < \alpha < f(x_0). \tag{4.5.11}$$

In particular, $f(x_0) > \alpha > 0$. Further, if $z = x + y \in B_G(\mathbf{0}; 1) + B_H(\mathbf{0}; 1)$, then so does $-z = -x - y$. Thus

$$
\begin{aligned}
\sup_{x\in G,\ \|x\|\le 1}|f(x)| + \sup_{x\in H,\ \|x\|\le 1}|f(x)| &\le \alpha \\
&< f(x_0) \\
&= \|x_0\| f\left(\tfrac{1}{\|x_0\|}x_0\right) \\
&< \tfrac{1}{C} f\left(\tfrac{1}{\|x_0\|}x_0\right) \\
&\le \tfrac{1}{C} \sup_{\substack{x\in\overline{G+H} \\ \|x\|\le 1}} |f(x)|
\end{aligned}
$$

which contradicts (4.5.9). This establishes (4.5.10).

In case V is a complex Banach space, then (4.5.11) holds for the real part of a continuous linear functional f and the preceding sequence of inequalities hold with $|f(x)|$ being replaced by $|\mathrm{Re}(f)(x)|$. However (cf. Proposition 3.1.1) since the norm $\|f\|$ is the same as the norm $\|\mathrm{Re}(f)\|$ (the latter being considered as a real linear functional), and since the supremum over the unit ball gives the norm of the functional concerned, the same conclusion holds with $|f(x)|$ as well and so we again get a contradiction to (4.5.9).

Step 4. Now consider the spaces $E = G \times H$ with the norm $\|(x, y)\|_E = \max\{\|x\|, \|y\|\}$ and $F = \overline{G + H}$ with the norm from V. Both are Banach spaces. Define $T\colon E \to F$ by $T((x, y)) = x + y$. Then T is continuous and linear. Further, in view of (4.5.10), we also have that

$$\overline{T\,(B_E(\mathbf{0}; 1))} \supset \frac{1}{C} B_F(\mathbf{0}; 1).$$

Now, this implies that (cf. Step 2 of the proof of Proposition 4.4.1)

$$T\,(B_E(\mathbf{0}; 1)) \supset \frac{1}{2C} B_F(\mathbf{0}; 1).$$

But then, it is now immediate to see that T must be onto. In other words, $\overline{G + H} = G + H$; *i.e.* $G + H$ is closed. This completes the proof. ■

4.6 Complemented Subspaces

Let V be a vector space. If W is a subspace of V, then by completing a basis of W to get a basis of V, we can easily produce a subspace Z of V such that $V = W \oplus Z$. The question which we wish to address now is that if W is a *closed* subspace of a Banach space V, whether there exists a *closed* subspace Z as above.

Definition 4.6.1 Let G be a closed subspace of a Banach space V. A closed subspace H of V is said to be a **complement** of G if $V = G \oplus H$, i.e. $G \cap H = \{\mathbf{0}\}$ and $V = G + H$. ■

Remark 4.6.1 If G has a complement H in V, then every $x \in V$ has a unique decomposition $z = x + y$ with $x \in G$ and $y \in H$. By Proposition 4.5.2, it follows that the maps $z \mapsto x$ and $z \mapsto y$ are continuous. Thus, we have continuous projections from V onto G and H. If G and H were not closed, these projections need not be continuous. ■

Example 4.6.1 Let G be a finite dimensional subspace of a Banach space V. Then G has a complement (cf. Exercises 3.7 and 3.8). ■

Example 4.6.2 Every closed subspace G with finite codimension has a complement. This is trivial since any algebraic complement, *i.e.* any subspace H such that $V = G \oplus H$, is finite dimensional and is hence closed. ■

Remark 4.6.2 Subspaces of finite codimension typically occur in the following way. Let Z be a subspace of V^* of dimension d. Then its annihilator $Z^\perp$ will be a subspace of V with codimension d. To see this, let $\{f_1, \ldots, f_d\}$ be a basis for Z. Define $\varphi\colon V \to \mathbb{R}^d$ by $\varphi(x) = (f_1(x), \ldots, f_d(x))$. This map is onto. If not, by the Hahn-Banach theorem, there exist scalars $\{\alpha_1, \ldots, \alpha_d\}$, not all zero, such that

$$\sum_{i=1}^{d} \alpha_i f_i(x) = 0$$

for every $x \in V$. But this implies that $\sum_{i=1}^{d} \alpha_i f_i = \mathbf{0}$ in V^*, which contradicts the linear independence of the $\{f_i\}$. Thus, we can find vectors $\{e_1, \ldots, e_d\}$ in V such that

$$f_i(e_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

It is now easy to verify that $\{e_1, \ldots, e_d\}$ are linearly independent and that their span (which has dimension d) is a complement to $Z^\perp$. ■

Remark 4.6.3 We will see later (cf. Chap. 7) that in a *Hilbert space*, every closed subspace is complemented. In fact, a deep result of Lindenstrauss and Tzafriri states that any Banach space which is not isomorphic to a Hilbert space will always have uncomplemented closed subspaces. Since Hilbert spaces are always reflexive, it follows that, in particular, non-reflexive spaces will always have uncomplemented closed subspaces. For example de Vito has shown that the space c_0 is uncomplemented in ℓ_∞. ■

Definition 4.6.2 Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$ be surjective. A linear transformation $S \in \mathcal{L}(W, V)$ is said to be a **right inverse** of T if $T \circ S = I_W$, the identity operator on W. ■

Proposition 4.6.1 *Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$ be surjective. The following are equivalent:*
(i) *T has a right inverse.*
(ii) *The subspace $N = \mathrm{Ker}(T)$ is complemented in V.*

Proof Step 1. (i) $\Rightarrow$ (ii). Let $S \in \mathcal{L}(W, V)$ be a right inverse for T. Consider the image $S(W) \subset V$ of S. Clearly, $S(W) \cap N = \{\mathbf{0}\}$. Also, if $x \in V$, then $x - S(T(x)) \in N$. Thus $V = S(W) \oplus N$. We now show that $S(W)$ is closed. Indeed, if we have a sequence $\{y_n\}$ in W such that $S(y_n) \to x$ in V, then $y_n = T(S(y_n)) \to T(x)$ in W. Then, it follows that

$$x = \lim_{n\to\infty} S(y_n) = S(T(x))$$

i.e. $x \in S(W)$, which proves that $S(W)$ is closed. Thus N is complemented in V.

Step 2. (ii) $\Rightarrow$ (i). Assume that a closed subspace M of V is a complement to N in V. Then, we have a continuous projection $P\colon V \to M$. Let $y \in W$. Let $x \in V$ such that $T(x) = y$ (T is surjective). If $x' \in V$ also satisfies $T(x') = y$, then $x - x' \in N$ and so $P(x) = P(x')$. Thus, the map $y \mapsto P(x)$ is well-defined. Define $S(y) = P(x)$. Then $T(S(y)) = T(P(x)) = T(x) = y$ (since $x - P(x) \in N$ and so $T(x) = T(P(x))$). Thus $T \circ S = I_W$. We now show that S is continuous as well. Since T is onto, there exists a constant $C > 0$ such that $B_W(\mathbf{0}; C) \subset T(B_V(\mathbf{0}; 1))$. Thus, for any $y \in W$, $\frac{C}{2\|y\|_W} y \in B_W(\mathbf{0}; C)$ and so there exists $x' \in V$ with $\|x'\|_V < 1$ such that

$$T(x') = \frac{C}{2\|y\|_W} y.$$

Hence, if we set $x = \frac{2}{C}\|y\|_W x'$, we get that $T(x) = y$ and that

$$\|x\|_V < \frac{2}{C}\|y\|_W.$$

Thus,

$$\|S(y)\|_V = \|P(x)\|_V \le \|P\|\|x\|_V \le \frac{2}{C}\|P\|\ \|y\|_W$$

which proves the continuity of S. Thus S is a right inverse for T. ■

Example 4.6.3 Consider the Banach spaces $V = \mathcal{C}^1[0, 1]$ and $W = \mathcal{C}[0, 1]$ with their usual norms. Consider the map $T\colon V \to W$ given by $T(f) = f'$, the derivative of f. By the fundamental theorem of calculus, this is clearly a surjective map and its kernel is the set of all constant functions, which is a one-dimensional subspace, and so it is complemented (cf. Example 4.6.1). Hence, T admits a right inverse. Indeed, we can explicitly define a right-inverse:

$$S(f)(t) = \int_0^t f(s)\,\mathrm{d}s,\ t \in [0, 1].$$ ■

4.7 Unbounded Operators, Adjoints

In this section, we will look at linear transformations between Banach spaces which are not necessarily defined on all of the space, but only on a subspace. Further, they may not map bounded sets into bounded sets. Such transformations are said to be unbounded and bounded linear transformations form a special subclass.

Definition 4.7.1 Let V and W be Banach spaces. An **unbounded linear operator** (or, **transformation**) from V into W is any linear map defined on a subspace of V taking values in W. The domain of definition of the transformation A is called the **domain** of A and is denoted $D(A)$. Thus,

$$A\colon D(A) \subset V \to W.$$

The image of A is a subspace of W and is called the **range** of A and is denoted $\mathcal{R}(A)$. The operator A is said to be **bounded** if there exists a constant $C > 0$ such that

$$\|A(x)\|_W \ \leq \ C\|x\|_V \tag{4.7.1}$$

for all $x \in D(A)$. The operator A is said to be **densely defined** if $\overline{D(A)} = V$. The **graph** of a linear operator A is denoted $G(A)$ and is given by

$$G(A) = \{(x, A(x)) \in V \times W \mid x \in D(A)\}.$$

The operator A is said to be **closed** if the graph $G(A)$ is closed in $V \times W$. ■

To define a linear operator, we thus need to specify its domain and then its action on vectors in the domain. Given a linear operator $A\colon D(A) \subset V \to W$, we denote its kernel by $\mathcal{N}(A)$. Thus,

$$\mathcal{N}(A) = \{x \in D(A) \mid A(x) = \mathbf{0}\}.$$

Remark 4.7.1 If $A\colon D(A) \subset V \to W$ is closed, then $\mathcal{N}(A)$ is a closed subspace in V. ■

Example 4.7.1 Let $V = W = \mathcal{C}[0, 1]$. Let $D(A) = \mathcal{C}^1[0, 1]$. Define $A\colon D(A) \subset V \to V$ by $A(u) = u'$ where u' stands for the derivative of the function u. Clearly, A is densely defined and $\mathcal{N}(A)$ is the subspace of all constant functions. It follows from the fundamental theorem of calculus that A is surjective. If $u_n \to u$ uniformly and if $u_n' \to v$ uniformly, we know that u is differentiable and that $u' = v$. Thus, A is a closed operator. Finally, A is unbounded. This, in fact, is the content of Example 2.3.8. ■

Notation: Let V be a Banach space. Let $x \in V$ and $f \in V^*$. We introduce the **duality bracket** $< f, x >$, which will also be denoted $< f, x >_{V^*,V}$ in case we need to specify the spaces involved, via the relation

$$< f, x > = f(x).$$

Let V and W be Banach spaces and let $A: D(A) \subset V \to W$ be a *densely defined* linear operator. Let

$$Z = \left\{ v \in W^* \mid \begin{array}{c} \text{there exists } C > 0 \text{ such that for all } x \in D(A) \\ | < v, A(x) >_{W^*,W} | \leq C\|x\|_V \end{array} \right\}. \quad (4.7.2)$$

Note that the constant C mentioned above depends on v. Clearly, Z is a subspace of W^*. If $v \in Z$, define, for $x \in D(A)$,

$$g(x) = < v, A(x) >_{W^*,W} .$$

By the definition of Z, it follows that g defines a continuous linear functional on $D(A)$. Hence, by the Hahn-Banach theorem, we can extend it to a continuous linear functional g_v on all of V. Since $D(A)$ is dense in V, such an extension is unique.

To summarize, we have a map $v \mapsto g_v$ from Z into V^*. It is also easy to see that this map is linear.

Definition 4.7.2 Let V and W be Banach spaces and let $A: D(A) \subset V \to W$ be a densely defined linear operator. Let Z be as defined above and for each $v \in Z$, let $g_v \in V^*$ be as defined above. We set $D(A^*) = Z$ and define $A^*(v) = g_v$ for $v \in D(A^*)$. The linear operator $A^*: D(A^*) \subset W^* \to V^*$ is called the **adjoint** of the operator A. ■

Thus, the adjoint is defined for densely defined linear operators. Notice that there is no reason for A^* to be densely defined. We have the following important duality relationship: for all $u \in D(A)$ and all $v \in D(A^*)$, we have

$$< A^*(v), u >_{V^*,V} = < v, A(u) >_{W^*,W} . \quad (4.7.3)$$

Example 4.7.2 In a finite dimensional space, every subspace is closed. Thus, any linear transformation is closed; it is densely defined if, and only if, it is defined over the entire space. Hence, every linear transformation defined on the entire space has an adjoint and, since every linear transformation is bounded, we also have that the adjoint is defined on the entire dual space of the range. The dual of $\mathbb{C}^n$ is identified with $\mathbb{C}^n$ with the duality product being given by

$$< y, x > = \sum_{i=1}^{n} x_i \overline{y_i}$$

where $x = (x_1, \ldots, x_n)$ is in the base space and $y = (y_1, \ldots, y_n)$ is in the dual space. If $A: \mathbb{C}^n \to \mathbb{C}^m$ is a linear transformation which is represented by the $m \times n$ matrix $\mathbf{A}$, then we have, for all $\mathbf{y} \in \mathbb{C}^m$ and for all $\mathbf{x} \in \mathbb{C}^n$,

$$< A^*y, x > = < y, Ax > = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j\right)\overline{y_i} = \sum_{j=1}^{n} x_j \overline{\left(\sum_{i=1}^{m}\overline{a_{ij}}y_i\right)}$$

from which it follows that the matrix representing A^* is none other than the adjoint matrix $\mathbf{A}^*$ (cf. Definition 1.1.11).

In the same way, if the matrix $\mathbf{A}$ represents a linear transformation A from $\mathbb{R}^n$ into $\mathbb{R}^m$, the adjoint of this transformation will be represented by the transpose matrix $\mathbf{A}'$. ■

Example 4.7.3 Consider the space ℓ_2 of square summable real sequences. Then its dual space can be identified with itself (cf. Example 3.2.3). Consider the linear operators on ℓ_2 given by $T(x) = (0, x_1, x_2, \ldots)$ and $S(x) = (x_2, x_3, \ldots)$ where $x = (x_1, x_2, \ldots) \in \ell_2$. Then it is easy to check that $T^* = S$ and that $S^* = T$. ■

Proposition 4.7.1 *Let V and W be Banach spaces and let $A: D(A) \subset V \to W$ be a densely defined linear operator. Then A^* is closed.*

Proof We need to show that $G(A^*)$, the graph of A^*, is closed in $W^* \times V^*$. Let $v_n \to v$ in W^* and let $A^*(v_n) \to f$ in V^*. Let $u \in D(A)$. Then

$$< A^*(v_n), u >_{V^*,V} = < v_n, A(u) >_{W^*,W} .$$

Passing to the limit as $n \to \infty$, we get

$$< f, u >_{V^*,V} = < v, A(u) >_{W^*,W} \tag{4.7.4}$$

for all $u \in D(A)$. Thus,

$$| < v, A(u) >_{W^*,W} | \leq \|f\|_{V^*} \|u\|_V$$

which shows that $v \in D(A^*)$. It also follows from (4.7.4) that $f = A^*(v)$, thus proving that $G(A^*)$ is closed. ■

The graphs of A and A^* are connected by a simple relation. Let V and W be Banach spaces. Define

$$\mathcal{J}: W^* \times V^* \to V^* \times W^*$$

by

$$\mathcal{J}(v, f) = (-f, v).$$

Proposition 4.7.2 *Let V and W be Banach spaces and let $A: D(A) \subset V \to W$ be a densely defined linear operator. Let $\mathcal{J}$ be as defined above. Then*

$$\mathcal{J}(G(A^*)) = (G(A))^{\perp}.$$

Proof Let $u \in D(A)$ be an arbitrary element so that $(u, A(u))$ is an arbitrary element of $G(A)$.

$$
\begin{aligned}
(v, f) \in G(A^*) &\Leftrightarrow < f, u >_{V^*,V} = < v, A(u) >_{W^*,W} \\
&\Leftrightarrow < -f, u >_{V^*,V} + < v, A(u) >_{W^*,W} = 0 \\
&\Leftrightarrow \mathcal{J}(v, f) \in (G(A))^{\perp}
\end{aligned}
$$

which completes the proof. ■

The following result characterizes densely defined and closed operators that are bounded.

Proposition 4.7.3 *Let V and W be Banach spaces and let $A: D(A) \subset V \to W$ be a densely defined and closed linear operator. The following are equivalent:*
(i) $D(A) = V$.
(ii) *A is bounded.*
(iii) $D(A^*) = W^*$.
(iv) *A^* is bounded.*
In this case, we also have

$$\|A\| = \|A^*\|. \tag{4.7.5}$$

Proof (i) $\Rightarrow$ (ii). If $D(A) = V$ and A is closed, then it is continuous by the closed graph theorem, and hence is bounded.
(ii) $\Rightarrow$ (iii). If A is bounded, then it follows from the definition of A^* that $D(A^*) = W^*$.
(iii) $\Rightarrow$ (iv). Since $G(A^*)$ is always closed, the result again follows from the closed graph theorem.
(iv) $\Rightarrow$ (i). First of all, we show that $D(A^*)$ is closed. Indeed, if $\{v_n\}$ is a sequence in $D(A^*)$ converging to $v \in W^*$, then,

$$\|A^*(v_n - v_m)\| \le C\|v_n - v_m\|$$

since A^* is bounded. Thus, it follows that $\{A^*(v_n)\}$ is Cauchy in V^*. Let $A^*(v_n) \to f$ in V^*. Since $G(A^*)$ is always closed, it follows then that $v \in D(A^*)$ and that $A^*(v) = f$. Thus $D(A^*)$ is closed.

Now set $G = G(A) \subset V \times W$ and $H = \{\mathbf{0}\} \times W \subset V \times W$. Both are closed subspaces. Further, $G + H = D(A) \times W$. On the other hand, since $(G(A))^{\perp} = \mathcal{J}(G(A^*))$, we have $G^{\perp} + H^{\perp} = V^* \times D(A^*)$, which is closed. Thus, by Theorem 4.5.1, it follows that $G + H$ is closed as well, which implies that $D(A)$ is closed. Thus, $D(A) = \overline{D(A)} = V$.

This proves the equivalence of all the four statements. Under these conditions, we now prove (4.7.5). For all $u \in V$ and for all $v \in W^*$, we have

$$< v, A(u) >_{W^*,W} = < A^*(v), u >_{V^*,V} \tag{4.7.6}$$

which yields

$$| < v, A(u) >_{W^*,W} | \;\leq\; \|A^*\| \|v\|_{W^*} \|u\|_V$$

whence we deduce that (cf. Corollary 3.1.2)

$$\|A(u)\|_W \;\leq\; \|A^*\| \; \|u\|_V$$

which implies that $\|A\| \leq \|A^*\|$.

Again, by virtue of (4.7.6), it follows that

$$\|A^*(v)\|_{V^*} = \sup_{u\in V,\ \|u\|\leq 1} | < A^*(v), u >_{V^*,V} | \;\leq\; \|A\| \; \|v\|_{W^*}$$

which implies that $\|A^*\| \leq \|A\|$. This completes the proof. ■

Proposition 4.7.4 *Let V and W be Banach spaces and let $A \colon D(A) \subset V \to W$ be a densely defined linear operator. Let $G = G(A) \subset V \times W$ and let $H = V \times \{\mathbf{0}\} \subset V \times W$. Then*
(i) $\mathcal{N}(A) \times \{\mathbf{0}\} = G \cap H$.
(ii) $V \times \mathcal{R}(A) = G + H$.
(iii) $\{\mathbf{0}\} \times \mathcal{N}(A^*) = G^\perp \cap H^\perp$.
(iv) $\mathcal{R}(A^*) \times W^* = G^\perp + H^\perp$.

Proof The proof is an immediate consequence of the definitions and the relation $G(A)^\perp = \mathcal{J}(G(A^*))$. The details are left as an exercise. ■

Corollary 4.7.1 *Let V and W be Banach spaces and let $A \colon D(A) \subset V \to W$ be a* closed *and* densely defined *linear operator. Then*

$$\mathcal{N}(A) = \mathcal{R}(A^*)^\perp, \tag{4.7.7}$$

$$\mathcal{N}(A^*) = \mathcal{R}(A)^\perp, \tag{4.7.8}$$

$$\mathcal{N}(A)^\perp \supset \overline{\mathcal{R}(A^*)} \tag{4.7.9}$$

and

$$\mathcal{N}(A^*)^\perp = \overline{\mathcal{R}(A)}. \tag{4.7.10}$$

Proof Let G and H be as in the preceding proposition. We know that (cf. Proposition 4.5.1) $G \cap H = (G^\perp + H^\perp)^\perp$. Thus, we get that

$$\mathcal{N}(A) \times \{\mathbf{0}\} = \mathcal{R}(A^*)^\perp \times \{\mathbf{0}\}.$$

This proves (4.7.7). Again, by Proposition 4.5.1, we know that $G^\perp \cap H^\perp = (G + H)^\perp$ which yields

$$\{\mathbf{0}\} \times \mathcal{N}(A^*) = \{\mathbf{0}\} \times \mathcal{R}(A)^\perp$$

which proves (4.7.8). The remaining two relations are proved from these two and applying (4.5.1) and (4.5.2). ■

The following example illustrates the above corollary in all its aspects.

Example 4.7.4 Consider the map $A \in \mathcal{L}(\ell_1)$ defined by

$$A(x) = \left(x_1, \frac{x_2}{2}, \ldots, \frac{x_k}{k}, \ldots\right),$$

where $x = (x_1, x_2 \ldots x_k, \ldots) \in \ell_1$. Then $A^* \in \mathcal{L}(\ell_\infty)$ and, by definition of the adjoint, it is immediate to see that, for $y = (y_1, y_2, \ldots, y_k, \ldots) \in \ell_\infty$, we have

$$A^*(y) = \left(y_1, \frac{y_2}{2}, \ldots, \frac{y_k}{k}, \ldots\right).$$

We now see that $\mathcal{N}(A) = \{\mathbf{0}\} \subset \ell_1$ and that $\mathcal{N}(A^*) = \{\mathbf{0}\} \subset \ell_\infty$.

Now clearly $c_{00} \subset \mathcal{R}(A)$ and so $\ell_1 = \overline{c_{00}} \subset \overline{\mathcal{R}(A)} \subset \ell_1$ and so $\overline{\mathcal{R}(A)} = \ell_1$. Since for any subspace W we have that $W^\perp = \overline{W}^\perp$, we see that (4.7.8) and (4.7.10) are verified.

Again, $c_{00} \subset \mathcal{R}(A^*)$. If $y \in \mathcal{R}(A^*)$, then $y \in \ell_\infty$ and, there exists $x \in \ell_\infty$ such that $x_k = ky_k$ for every positive integer k. This implies that $|y_k| \leq \|x\|/k \to 0$ as $k \to \infty$. Consequently, $\mathcal{R}(A^*) \subset c_0$. Thus,

$$c_{00} \subset \mathcal{R}(A^*) \subset c_0,$$

from which we deduce that $\overline{\mathcal{R}(A^*)} = c_0$, which is strictly contained in $\ell_\infty = \mathcal{N}(A)^\perp$. Thus, the inclusion in (4.7.9) can be strict. Since $c_{00} \subset c_0$, we have that $c_0^\perp \subset c_{00}^\perp = \{\mathbf{0}\}$ (cf Example 4.5.1). Thus $\mathcal{R}(A^*)^\perp = \{\mathbf{0}\}$ as well and this verifies (4.7.7). ■

Theorem 4.7.1 *Let V and W be Banach spaces and let $A: D(A) \subset V \to W$ be a* closed *and* densely defined *linear operator. Then, the following are equivalent:*

(i) $\mathcal{R}(A)$ *is closed in* W.
(ii) $\mathcal{R}(A^*)$ *is closed in* V^*.
(iii) $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$.
(iv) $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp$.

Proof Using the same notations as in the preceding proposiition and its corollary, we have:

(i) $\Leftrightarrow$ $G + H$ is closed in $V \times W$.
(ii) $\Leftrightarrow$ $G^\perp + H^\perp$ is closed in $V^* \times W^*$.
(iii) $\Leftrightarrow$ $G + H = (G^\perp \cap H^\perp)^\perp$.
(iv) $\Leftrightarrow$ $G^\perp + H^\perp = (G \cap H)^\perp$.

The equivalence of these statements follows from Theorem 4.5.1. ■

Remark 4.7.2 As already remarked (cf. Example 4.7.2), all linear transformations on finite dimensional spaces are closed and densely defined means defined on all the space. The statement that $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$ is none other than the well-known Fredhölm alternative (cf. Proposition 1.1.5). ■

We conclude with a useful characterization of surjective maps.

Theorem 4.7.2 *Let V and W be Banach spaces and let $A: D(A) \subset V \to W$ be a* closed *and* densely defined *linear operator. The following are equivalent:*

(i) *A is onto, i.e. $\mathcal{R}(A) = W$.*
(ii) *There exists a constant $C > 0$ such that, for all $v \in D(A^*)$,*

$$\|v\|_{W^*} \leq C\|A^*(v)\|_{V^*}. \tag{4.7.11}$$

(iii) *$\mathcal{N}(A^*) = \{\mathbf{0}\}$ and $\mathcal{R}(A^*)$ is closed in V^*.*

Proof (i) $\Rightarrow$ (iii). If A is onto, then $\mathcal{R}(A) = W$ and is hence closed. Thus by the preceding theorem, $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp$, which is also closed. Further (cf. Corollary 4.7.1), $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp = \{\mathbf{0}\}$.

(iii) $\Rightarrow$ (i). If $\mathcal{N}(A^*) = \{\mathbf{0}\}$, and $\mathcal{R}(A^*)$ is closed, it follows from the preceding theorem that $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp = W$. Thus A is onto.

(ii) $\Rightarrow$ (iii). By virtue of (4.7.11), it follows that $\mathcal{N}(A^*) = \{\mathbf{0}\}$. Let $\{v_n\}$ be a sequence in $D(A^*)$ such that $A^*v_n \to f$ in V^*. Then, again, by (4.7.11), it follows that

$$\|v_n - v_m\|_{W^*} \leq C\|A^*(v_n) - A^*(v_m)\|_{W^*}$$

and so $\{v_n\}$ is a Cauchy sequence. Let $v_n \to v$ in W^*. Since $G(A^*)$ is closed, it follows then that $v \in D(A^*)$ and that $A^*(v) = f$. Thus $\mathcal{R}(A^*)$ is closed.

(iii) $\Rightarrow$ (ii). Using the notation established in Proposition 4.7.4, $\mathcal{N}(A^*) = \{\mathbf{0}\}$ implies that $G^\perp \cap H^\perp = \{\mathbf{0}\}$ and $\mathcal{R}(A^*)$ closed implies that $G^\perp + H^\perp$ is closed. Hence, by Proposition 4.5.2, there exists a constant $C > 0$ such that for every $z \in G^\perp + H^\perp = \mathcal{R}(A^*) \times W^*$, there exist $a \in G^\perp$ and $b \in H^\perp$ such that $z = a + b$, and $\|a\|_{V^*\times W^*} \leq C\|z\|_{V^*\times W^*}$ and $\|b\|_{V^*\times W^*} \leq C\|z\|_{V^*\times W^*}$. Further, this decomposition is unique, since $G^\perp \cap H^\perp = \{\mathbf{0}\}$. Let $v \in D(A^*)$. Set $z = (A^*(v), \mathbf{0}) \in \mathcal{R}(A^*) \times W^* = G^\perp + H^\perp$. Then $a = (A^*(v), -v) \in G^\perp$ and $b = (\mathbf{0}, v) \in H^\perp$ and $a + b = z$. The inequality(4.7.11) now follows immediately. ■

Remark 4.7.3 A similar result can be stated and proved for the surjectivity of A^*. ■

Remark 4.7.4 If V and W are finite dimensional, then the ranges of A and A^* are automatically closed. In this case we have:

$$A \text{ onto} \Leftrightarrow A^* \text{ one} - \text{one},$$
$$A^* \text{ onto} \Leftrightarrow A \text{ one} - \text{one}.$$

However, in infinite dimensional Banach spaces, if we do not have information on the range being closed, we can only say:

$$\begin{array}{c} A \text{ onto} \Rightarrow A^* \text{ one} - \text{one}, \\ A^* \text{ onto} \Rightarrow A \text{ one} - \text{one}. \end{array}$$

For instance, if $V = W = \ell_2$, and if

$$A(x) = \left(x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots\right),$$

then $A = A^*$ (check!) and A is clearly one-one, but not onto (cf. Example 2.3.3). ∎

Remark 4.7.5 To show that A is onto, we usually use the relation (4.7.11). We assume that $A^*(v) = f$ and show that $\|v\|_{W^*} \leq C\|f\|_{V^*}$. This is called the method of *a priori* estimates. We do not worry about the existence of solutions to the equation $A^*(v) = f$, for a given f; if a solution exists, we obtain estimates for its norm. ∎

Example 4.7.5 Let V and W be real Banach spaces and assume that W is reflexive. Let

$$a(\cdot, \cdot): V \times W \to \mathbb{R}$$

be a bilinear form such that:
(i) $a(\cdot, \cdot)$ is continuous, *i.e.* there exists $M > 0$ such that, for all $v \in V$ and $w \in W$, we have

$$|a(v, w)| \leq M\|v\|_V\|w\|_W;$$

(ii) there exists $\alpha > 0$ such that, for all $w \in W$,

$$\sup_{v \in V,\ v \neq \mathbf{0}} \frac{|a(v, w)|}{\|v\|_V} \geq \alpha\|w\|_W;$$

(iii) there exists a constant $\beta > 0$ such that, for all $v \in V$, we have

$$\sup_{w \in W,\ w \neq \mathbf{0}} \frac{|a(v, w)|}{\|w\|_W} \geq \beta\|v\|_V.$$

Then, given $f \in V^*$ and $g \in W^*$, there exist unique elements $v_0 \in V$ and $w_0 \in W$ such that

$$\begin{array}{ll} a(v_0, w) = \langle g, w \rangle_{W^*,W} & \text{for all } w \in W; \\ a(v, w_0) = \langle f, v \rangle_{V^*,V} & \text{for all } v \in V. \end{array}$$

To see this, define $A: V \to W^*$ by $< A(v), w >_{W^*,W} = a(v, w)$ for all $w \in W$. By the continuity of the bilinear form, A is a well-defined and bounded linear operator. Since W is reflexive, we have $A^*: W \to V^*$, which is also a bounded linear operator (cf. Proposition 4.7.3) and it is easy to see that $< A^*(w), v >_{V^*,V} = a(v, w)$. By (ii), we have

$$\|A^*(w)\|_{V^*} \geq \alpha \|w\|_W.$$

Thus, by Theorem 4.7.2, we deduce that A is onto. By (iii), we have

$$\|A(v)\|_{W^*} \geq \beta \|v\|_V$$

whence A is one-one as well. In the same way, A^* is one-one and onto as well. Thus, there exist unique solutions to the equations

$$A(v_0) = g, \text{ and } A^*(w_0) = f$$

which proves the result. ■

4.8 Exercises

4.1 Show that a Banach space cannot have a basis whose elements form a countable set. Deduce that the space $\mathcal{P}$ of all polynomials in one variable cannot be complete for any norm.

4.2 Let $\mathcal{M}_n$ denote the space of all $n \times n$ matrices with complex entries. Show that the following subsets are nowhere dense:

(a) the set of all singular matrices in $\mathcal{M}_n$;
(b) the set of all matrices in $\mathcal{M}_n$ whose trace is zero.

4.3 (a) Let $\{a_n\}$ be a sequence of real numbers such that for any given real sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$, the series

$$\sum_{n=1}^{\infty} a_n x_n$$

converges. Show that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (Hint: Use Exercise 3.9(a)).
(b) Let $1 < p < \infty$. Let $\{a_n\}$ be a sequence of real numbers such that for all $x = (x_n) \in \ell_p$, the series $\sum_{n=1}^{\infty} a_n x_n$ is convergent. Show that $a = (a_n) \in \ell_{p^*}$.

4.4 (Numerical quadrature) Let $V = \mathcal{C}[0, 1]$. For each positive integer n, define

$$\varphi_n(f) = \sum_{m=0}^{p_n} \omega_m^n f(x_m^n)$$

for all $f \in V$, where $\{x_m^n\}_{m=0}^{p_n}$ are given points in $[0, 1]$ and $\{\omega_m^n\}_{m=0}^{p_n}$ are real numbers (called *weights*). Let

$$\varphi(f) = \int_0^1 f(t)\mathrm{d}t$$

for $f \in V$.

(a) Show that $\varphi_n(f) \to \varphi(f)$ for every $f \in V$, as $n \to \infty$, if, and only if, the following conditions are verified:
(i) $\varphi_n(f_j) \to \varphi(f_j)$ as $n \to \infty$, for every integer $j \geq 0$, where $f_j(t) = t^j$;
(ii)

$$\sup_n \left\{ \sum_{m=0}^{p_n} |\omega_m^n| \right\} < \infty.$$

(cf. Exercise 2.19.)
(b) If $\omega_m^n \geq 0$ for all n and for all $0 \leq m \leq p_n$, show that the condition (ii) above is redundant.
(c) (Trapezoidal rule) Set $p_n = n$ and $x_m^n = m/n$ for $0 \leq m \leq n$. Let

$$\omega_m^n = \begin{cases} \frac{1}{n} & \text{if } m \neq 0, n \\ \frac{1}{2n} & \text{if } m = 0 \text{ or } n. \end{cases}$$

Show that $\varphi_n(f) \to \varphi(f)$ for all $f \in V$, as $n \to \infty$.

4.5 Let V be a Banach space and let $\{S(t)\}_{t \geq 0}$ be a family of continuous linear operators on V. Assume that the following conditions hold:

(i) $S(0) = I$, the identity operator on V.
(ii) For all $t_1 \geq 0$ and $t_2 \geq 0$, we have

$$S(t_1 + t_2) = S(t_1) \circ S(t_2).$$

(iii) For all $x \in V$,

$$\lim_{t \downarrow 0} S(t)(x) = x.$$

Then, we say that $\{S(t)\}_{t \geq 0}$ is a c_0-*semigroup of operators* on V.

(a) Let $A \in \mathcal{L}(V)$. Define $S(t) = \exp(tA)$ (cf. Exercise 2.38) for $t \geq 0$. Show that $\{S(t)\}_{t \geq 0}$ forms a c_0-semigroup of operators on V.
(b) Let V denote the space of all bounded and uniformly continuous real-valued functions on $\mathbb{R}$ provided with the usual 'sup-norm'. For $t \geq 0$, define $S(t)$ by

$$S(t)(f)(\tau) = f(t + \tau)$$

for $\tau \in \mathbb{R}$. Show that $S(t) \in \mathcal{L}(V)$ for each $t \geq 0$ and that $\{S(t)\}_{t \geq 0}$ is a c_0-semigroup of operators on V.

4.6 Let V be a Banach space and let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup of operators on V.

(a) Show that there exists $M > 0$ (which, without loss of generality, can be chosen to be greater than, or equal to, unity) and $\eta > 0$ such that, for all $0 \leq t \leq \eta$, we have

$$\|S(t)\| \leq M.$$

(b) Deduce that if $\omega = \eta^{-1} \log M \geq 0$, then

$$\|S(t)\| \leq Me^{\omega t}$$

for all $t \geq 0$.
(c) The semigroup is said to be *exponentially stable* if we can find $M > 0$ and $\omega < 0$ such that the preceding inequality is true. Show that a c_0-semigroup $\{S(t)\}_{t\geq 0}$ is exponentially stable if, and only if, there exists a $t_0 > 0$ such that $\|S(t_0)\| < 1$.
(d) Prove that for every $x \in V$, fixed, the mapping $t \mapsto S(t)x$ is continuous from the interval $[0, \infty)$ into V.
(e) Prove that

$$\lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} S(\tau)(x)\, \mathrm{d}\tau = S(t)(x)$$

for all $t \geq 0$ and for every $x \in V$.

4.7 Let V be a real Banach space and let $a(\cdot,\cdot)\colon V \times V \to \mathbb{R}$ be a continuous bilinear form (cf. Example 4.7.5). Assume that for every $x \in V$, $x \neq \mathbf{0}$,

$$a(x, x) > 0.$$

Let $T\colon V \to V$ be a linear map such that, for all $x \in V$ and $y \in V$, we have

$$a(T(x), y) = a(x, T(y)).$$

Show that $T \in \mathcal{L}(V)$.

4.8 Let V and W be Banach spaces and let $\{f_i\}_{i\in I}$ (where I is an indexing set) be a collection of continuous linear functionals on W which separates points in W. Let $T\colon V \to W$ be a linear map. If $f_i \circ T$ is continuous for each $i \in I$, show that $T \in \mathcal{L}(V, W)$.

4.9 Let X, Y and Z be Banach spaces. Let $T \in \mathcal{L}(X, Z)$ and $A \in \mathcal{L}(Y, Z)$. Assume that for every $x \in X$, there exists a unique $y \in Y$ such that $A(y) = T(x)$. Define $B\colon X \to Y$ by $Bx = y$. Show that $B \in \mathcal{L}(X, Y)$.

4.10 Let V be a Banach space and let $W \subset V$ be a subspace. Show that $W^\perp = \overline{W}^\perp$.

4.11 Let V be a Banach space and let $f \in V^*$. Let Z denote the null space (*i.e.* kernel) of f. Compute $Z^\perp$.

4.12 Show that c_0 is a complemented subspace of c.

4.13 Let V and W be Banach spaces. let $T \in \mathcal{L}(V, W)$. We say that $S \in \mathcal{L}(W, V)$ is a *left inverse* of T if $S \circ T = I_V$, where I_V is the identity operator on V. Show that T has a left inverse if, and only if, T is injective and $\mathcal{R}(A)$ is closed and complemented in W.

4.14 (a) Let W be a Banach space and let $T: D(T) \subset W \to W$ be a closed and densely defined linear operator. Set $V = D(T)$ and define, for $x \in V$,

$$\|x\| = \|x\|_W + \|T(x)\|_W.$$

Show that V is a Banach space for this norm.
(b) If V is also a Banach space for some other norm $\|.\|_V$, and if this norm is such that both the inclusion map of V into W and the map T are in $\mathcal{L}(V, W)$, show that there exists a constant $C > 0$ such that, for all $x \in V$,

$$\|x\|_V \leq C\left(\|x\|_W + \|T(x)\|_W\right).$$

4.15 Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$ be surjective. Show that T is injective if, and only if, there exists a constant $c > 0$ such that, for all $v \in V$, we have

$$\|T(v)\|_W \geq c\|v\|_V.$$

4.16 Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$. Assume that there exists a constant $c > 0$ such that, for all $v \in V$,

$$\|T(v)\|_W \geq c\|v\|_V.$$

Let Z denote the null space of T. For $v \in V$, define $\overline{T}(v + Z) = T(v)$. Show that the mapping $\overline{T}: V/Z \to \mathcal{R}(T)$ is well-defined and that it is an isomorphism.

4.17 Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$. Show that T is invertible if, and only if, T^* is invertible.

4.18 Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$. If T and T^* are injective, and if $\mathcal{R}(T)$ is closed, show that T and T^* are invertible.

4.19 Consider c_{00} with the norm $\|\cdot\|_\infty$. Define $T: c_{00} \to c_{00}$ by

$$T(x) = \left(x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \ldots\right),$$

where $x = (x_n) \in c_{00}$. Show that T is a bijection and that $T \in \mathcal{L}(c_{00})$. Show, however, that T is not an isomorphism. Why does this not contradict Corollary 4.4.1?

4.20 Let V be a Banach space and let $\{S(t)\}_{t \geq 0}$ be a c_0-semigroup of operators on V. Define

$$D(A) = \left\{x \in V \mid \lim_{h\downarrow 0} \frac{1}{h}(S(h)(x) - x) \text{ exists}\right\}$$

and, for $x \in D(A)$,

$$A(x) = \lim_{h\downarrow 0} \frac{1}{h}(S(h)(x) - x).$$

The operator $A\colon D(A) \subset V \to V$ is called the *infinitesimal generator* of the semigroup $\{S(t)\}_{t\geq 0}$. If $A \in \mathcal{L}(V)$, show that it is the infinitesimal generator of the semigroup $\{\exp(tA)\}_{t\geq 0}$.

4.21 Let V and $\{S(t)\}_{t\geq 0}$ be as in Exercise 4.5(b). Show that the infinitesimal generator of the semigroup is given by:

$$D(A) = \{f \in V \mid f' \text{ exists and } f' \in V\},$$

and

$$A(f) = f'$$

where f' denotes the derivative of f.

4.22 Let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup of operators on a Banach space V. Let $A\colon D(A) \subset V \to V$ be its infinitesimal generator. For any $x \in V$, show that

$$\int_0^t S(\tau)(x)\, \mathrm{d}\tau \in D(A)$$

and that

$$A\left(\int_0^t S(\tau)(x)\, \mathrm{d}\tau\right) = S(t)(x) - x.$$

4.23 Let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup of operators on a Banach space V. Let $A\colon D(A) \subset V \to V$ be its infinitesimal generator.

(a) Let $x \in D(A)$. Show that $S(t)(x) \in D(A)$ for all $t > 0$ and that

$$\frac{\mathrm{d}}{\mathrm{d}t}(S(t)(x)) = A(S(t)(x)) = S(t)(A(x)).$$

(b) If $x \in D(A)$ and if $0 \leq t_2 < t_1$, show that

$$S(t_1)(x) - S(t_2)(x) = \int_{t_2}^{t_1} S(\tau)(A(x))\, \mathrm{d}\tau = \int_{t_2}^{t_1} A(S(\tau)(x))\, \mathrm{d}\tau.$$

4.24 Let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup of operators on a Banach space V. Let $A: D(A) \subset V \to V$ be its infinitesimal generator. Show that A is densely defined and closed.

4.25 Let V be a Banach space and let $A: D(A) \subset V \to V$ be a linear operator. We say that a map $t \mapsto u(t)$ from $[0, \infty)$ into V is a solution to the initial value problem:

$$\begin{aligned} \tfrac{\mathrm{d}u(t)}{\mathrm{d}t} &= A(u(t)),\ t > 0, \\ u(0) &= x \end{aligned}$$

if $u(t) \in D(A)$ for all $t > 0$ and if it verifies the above equations.
(a) If $x \in D(A)$, and if A is the infinitesimal generator of a c_0-semigroup of operators $\{S(t)\}_{t\geq 0}$ on V, show that the only solution to the above initial value problem is given by $u(t) = S(t)(x)$ for $t \geq 0$ (Hint: Clearly $u(t)$ defined thus is a solution by the Exercise 4.23; to show uniqueness, differentiate the map $\tau \mapsto S(t-\tau)(u(\tau))$, $\tau \in (0, \infty)$.)
(b) If V is a Banach space and if $\{S_1(t)\}_{t\geq 0}$ and $\{S_2(t)\}_{t\geq 0}$ are two c_0-semigroups of operators on V which have the same infinitesimal generator, show that $S_1(t) = S_2(t)$ for all $t \geq 0$.
(c) Deduce that the only semigroups on V whose infinitesimal generators are in $\mathcal{L}(V)$ are of the form $\{\exp(tA)\}_{t\geq 0}$ with $A \in \mathcal{L}(V)$.

4.26 Let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup of operators on a Banach space V. Let $A: D(A) \subset V \to V$ be its infinitesimal generator. Assume further that, for all $t \geq 0$, $\|S(t)\| \leq 1$. (Such a semigroup is called a *semigroup of contractions*.)
(a) Let $\lambda > 0$. For $x \in V$, define

$$R(\lambda)(x) = \int_0^\infty e^{-\lambda t} S(t)(x)\, \mathrm{d}t$$

(cf. Exercise 3.21). If $x \in D(A)$, show that $R(\lambda)(x) \in D(A)$ for all $\lambda > 0$ and that

$$R(\lambda)(A(x)) = A(R(\lambda)(x)).$$

(b) Show that for all $\lambda > 0$,

$$(\lambda I - A)(R(\lambda)(x)) = x$$

for all $x \in V$ and that

$$R(\lambda)((\lambda I - A)(x)) = x$$

for all $x \in D(A)$, where I is the identity map on V.

Remark 4.8.1 Exercise 4.26 shows that if A is the infinitesimal generator of a semigroup of contractions, then the (unbounded) linear operator $\lambda I - A$ is invertible and that its inverse is $R(\lambda)$ which is a bounded linear operator defined on V. Thus, together with Exercise 4.24, we see that in order that an (unbounded) linear operator A be the infinitesimal generator of a semigroup of contractions, it has to be densely defined, closed and for all $\lambda > 0$, $\|(\lambda I - A)^{-1}\| \leq 1/\lambda$. In fact these conditions are also sufficient for A to be the infinitesimal generator of a semigroup of contractions. This is the content of the famous *Hille-Yosida theorem*. Generalizations to other c_0-semigroups also exist. The usefulness of this result stems from the fact that many partial differential equations of the evolution type (for instance, the heat, wave and Schrödinger equations) can be cast in the form of an initial value problem as stated in Exercise 4.25 involving an unbounded linear operator and so the existence of uniqueness of solutions will follow from the fact that the operator is the infinitesimal generator of a c_0-semigroup. For more details see, for instance Kesavan [1]. ■

4.27 Let V and W be Banach spaces and let $A: D(A) \subset V \to W$ be a closed operator. Let $B \in \mathcal{L}(V, W)$. Show that $(A + B): D(A) \subset V \to W$ is also closed.

Reference

1. Kesavan, S., *Topics in Functional Analysis and Applications* (New Age International formerly Wiley-Eastern, 1989)

Chapter 5
Weak and Weak* Topologies

5.1 The Weak Topology

In this chapter, we will study topologies on Banach spaces which are weaker (i.e., coarser) than the norm topology.

Definition 5.1.1 Let V be a Banach space. The **weak topology** on V is the coarsest (i.e. smallest) topology such that every element of V^* is continuous. Open (respectively, closed) sets in the weak topology will be called weakly open (respectively, weakly closed) sets. ■

We have already encountered the notion of the weak topology on a given set such that a family of functions is continuous (cf. Definition 1.2.10). The weakly open sets are precisely the class of all arbitrary unions of finite intersections of sets of the form $f^{-1}(U)$ where $f \in V^*$ and U is an open set in $\mathbb{R}$ (or $\mathbb{C}$, in the case of complex Banach spaces).

A basic neighbourhood system for the weak topology is, therefore, the collection of sets of the form

$$U = \{x \in V \mid |f_i(x - x_0)| < \varepsilon \text{ for all } i \in I\}$$

where $x_0 \in V$, $\varepsilon > 0$, I is a finite indexing set and $f_i \in V^*$ for all $i \in I$. The set U described above forms a weakly open neighbourhood of the point $x_0 \in V$ (cf. the discussion following Definition 1.2.10).

Proposition 5.1.1 *The weak topology is Hausdorff.*

Proof Let x and y be distinct points in V. Then, since V^* separates points in V (cf. Remark 3.1.1), there exists $f \in V^*$ such that $f(x) \neq f(y)$. Choose disjoint open neighbourhoods U of $f(x)$ and V of $f(y)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint weakly open neighbourhoods of x and y, respectively. This completes the proof. ■

Notation: Given a sequence $\{x_n\}$ in V, we write $x_n \to x$ if the sequence converges to $x \in V$ in the norm topology, i.e., if $\|x_n - x\| \to 0$ as $n \to \infty$. If the sequence converges to x in the weak topology, we write $x_n \rightharpoonup x$.

S. Kesavan, *Functional Analysis*, Texts and Readings in Mathematics,
https://doi.org/10.1007/978-981-19-7633-9_5

Proposition 5.1.2 *Let V be a Banach space and let $\{x_n\}$ be a sequence in V.*

(i) *$x_n \rightharpoonup x$ in V if, and only if, $f(x_n) \to f(x)$ for all $f \in V^*$.*
(ii) *If $x_n \to x$ in V, then $x_n \rightharpoonup x$.*
(iii) *If $x_n \rightharpoonup x$ in V, then $\{\|x_n\|\}$ is bounded and*

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

(iv) *If $x_n \rightharpoonup x$ in V and $f_n \to f$ in V^*, then $f_n(x_n) \to f(x)$.*

Proof (i) This is a direct consequence of the definition of the weak topology.
(ii) Let $f \in V^*$ be an arbitrary element. Then

$$|f(x_n) - f(x)| \le \|f\| \, \|x_n - x\| \to 0.$$

The result now follows from (i).

(iii) This follows from (i) and the Banach-Steinhaus theorem (cf. Corollary 4.2.2 applied to the sequence $\{J_{x_n}\}$ in V^{**}).

(iv) We have

$$\begin{aligned} |f_n(x_n) - f(x)| &\le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &\le \|f_n - f\| \, \|x_n\| + |f(x_n) - f(x)|. \end{aligned}$$

The first term on the right-hand side tends to zero since $\|x_n\|$ is bounded (by (iii)) and $\|f_n - f\| \to 0$. The second term also tends to zero (by (i)). This completes the proof. ∎

Example 5.1.1 Consider ℓ_2 the space of all square summable real sequences. We can identify ℓ_2^* with ℓ_2 (cf. Example 3.2.3). Consider the sequence $\{\mathsf{e}_n\} \in \ell_2$. If $x \in \ell_2 = \ell_2^*$, with $x = (x_i)$, we have that

$$< x, \mathsf{e}_n >_{\ell_2^*, \ell_2} = x_n$$

which tends to zero since $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. Thus, the sequence $\{\mathsf{e}_n\}$ converges weakly to $\mathbf{0}$ in ℓ_2. Notice that this sequence has no subsequence which converges in the norm topology since

$$\|\mathsf{e}_n - \mathsf{e}_m\|_2 = \sqrt{2}$$

for all $n \ne m$. Thus, while norm convergence implies weak convergence, nothing can be said about the reverse implication.

We can similarly prove that for all $1 < p < \infty$, the sequence $\{\mathsf{e}_n\}$ converges weakly to zero in ℓ_p. ∎

Proposition 5.1.3 *If V is a finite dimensional space, then the norm and weak topologies coincide.*

Proof Since the weak topology is coarser than the norm topology, every weakly open set is also open in the norm topology. We thus have to prove the converse. Let U be open in the norm topology and let $x_0 \in U$. There exists $r > 0$ such that the open ball $B(x_0; r) \subset U$. Let $\dim(V) = n$ and let $\{v_1, \ldots, v_n\}$ be a basis for V such that, without loss of generality, $\|v_i\| = 1$ for all $1 \leq i \leq n$. If $x \in V$, then $x = \sum_{i=1}^n x_i v_i$ and define f_i to be the i-th coordinate projection, i.e., $f_i(x) = x_i$. Then

$$\|x - x_0\| = \left\| \sum_{i=1}^n f_i(x - x_0) v_i \right\| \leq \sum_{i=1}^n |f_i(x - x_0)|.$$

Define

$$W = \left\{ x \in V \mid |f_i(x - x_0)| < \frac{r}{n},\ 1 \leq i \leq n \right\}.$$

Then W is a weakly open neighbourhood of x_0, and it is clear from the above computations that $W \subset B(x_0; r) \subset U$. Thus U is open in the weak topology as well, and this completes the proof. ■

Thus, in a finite dimensional space, the weak and norm open (respectively, closed) sets are the same. However, in infinite dimensional spaces, the weak topology is strictly coarser than the norm topology. We will presently see examples of norm closed (respectively, open) sets which are not closed (respectively, open) in the weak topology (cf. Examples 5.1.2 and 5.1.3). However, for convex sets, the situation is different.

Proposition 5.1.4 *Let C be a convex and (norm) closed subset of a Banach space V. Then C is also weakly closed. (The converse is always true, even without the convexity hypothesis.)*

Proof We will assume that V is a real Banach space, for simplicity. Let C be a closed and convex set in V and let $x_0 \notin C$. Then, by the Hahn-Banach theorem, there exists $f \in V^*$ and $\alpha \in \mathbb{R}$ such that $f(x_0) < \alpha < f(x)$ for all $x \in C$. Then the set

$$U = \{x \in V \mid f(x) < \alpha\}$$

is a weakly open neighbourhood of x_0 which does not meet C. Thus the complement of C is weakly open and so C is weakly closed. ■

Definition 5.1.2 Let X be a topological space and let $f: X \to \mathbb{R}$ be a given function. We say that f is **lower semicontinuous** if, for every $\alpha \in \mathbb{R}$, the set

$$f^{-1}((-\infty, \alpha]) = \{x \in X \mid f(x) \leq \alpha\}$$

is closed in X. ■

Clearly, every continuous map is lower semicontinuous. If $x_n \to x$ in X, and if $f: X \to \mathbb{R}$ is lower semicontinuous, then

$$f(x) \;\le\; \liminf_{n\to\infty} f(x_n). \tag{5.1.1}$$

For, if $\alpha = \liminf_{n\to\infty} f(x_n)$, then, given any $\varepsilon > 0$, there exists a subsequence x_{n_k} such that $f(x_{n_k}) \le \alpha + \varepsilon$ for all k. Since $f^{-1}((-\infty, \alpha+\varepsilon])$ is closed, it follows that $f(x) \le \alpha + \varepsilon$ and since $\varepsilon > 0$ was arbitrarily chosen, (5.1.1) follows.

Corollary 5.1.1 *Let V be a Banach space and let $\varphi\colon V \to \mathbb{R}$ be* convex *and lower semicontinuous (with respect to the norm topology). Then φ is also lower semicontinuous with respect to the weak topology. In particular, the map $x \mapsto \|x\|$, being continuous, is also lower semicontinuous with respect to the weak topology and, if $x_n \rightharpoonup x$ in V, we have*

$$\|x\| \;\le\; \liminf_{n\to\infty} \|x_n\|. \tag{5.1.2}$$

Proof For every $\alpha \in \mathbb{R}$, the set $\varphi^{-1}((-\infty, \alpha])$ is closed (in the norm topology) and is convex. Hence it is weakly closed. This completes the proof. ∎

Notation: Let V be Banach space. We will use the following notations.

$$\begin{aligned} D &= \{x \in V \mid \|x\| < 1\} \text{ (open unit ball).} \\ B &= \{x \in V \mid \|x\| \le 1\} \text{ (closed unit ball).} \\ S &= \{x \in V \mid \|x\| = 1\} \text{ (unit sphere).} \end{aligned}$$

Example 5.1.2 Let V be an infinite dimensional Banach space. Let S be the unit sphere in V. Then S is never weakly closed, though it is closed in the norm topology. To see this, let $x_0 \in V$ such that $\|x_0\| < 1$. Consider any weakly open neighbourhood U of x_0 of the form

$$U = \{x \in V \mid |f_i(x - x_0)| < \varepsilon,\ 1 \le i \le n\}$$

where $\varepsilon > 0$ and $f_i \in V^*$ for $1 \le i \le n$. Consider the map $\mathcal{A}\colon V \to \mathbb{R}^n$ defined by

$$\mathcal{A}(x) = (f_1(x), \ldots, f_n(x)).$$

This map cannot be injective (otherwise, we will have $\dim(V) \le n$, which is a contradiction). Thus, there exists $y_0 \neq \mathbf{0}$, such that $f_i(y_0) = 0$ for all $1 \le i \le n$. Then $x_0 + ty_0 \in U$ for all $t \in \mathbb{R}$. Set $g(t) = \|x_0 + ty_0\|$. Then $g(0) < 1$ while $g(t) \to +\infty$ as $t \to +\infty$. Hence, there exists t_0 such that $g(t_0) = 1$. Thus, $x_0 + t_0 y_0 \in U \cap S$. We have thus proved that every weakly open neighbourhood of every point in the open unit ball, D, intersects the unit sphere, S. Hence the closed unit ball, B, must lie in the weak closure of S. But B being closed (in the norm topology) and convex, is itself weakly closed. Thus the weak closure of the unit sphere, S, is the closed unit ball, B. Thus, S is not weakly closed. ∎

Example 5.1.3 Let V be an infinite dimensional Banach space. Then the open unit ball D is not weakly open. As seen in the preceding example, every weakly open neighbourhood of a point $x_0 \in D$ contains an affine subspace of the form $\{x_0 + ty_0 \,|t \in \mathbb{R}\}$ where y_0 is chosen as before. Thus D cannot contain a weakly open neighbourhood of any of its points, and hence D cannot be weakly open. ■

Thus, in an infinite dimensional Banach space, the weak topology is strictly coarser than the norm topology.

Proposition 5.1.5 *(***Schur's lemma***) In the space ℓ_1, a sequence is convergent in the weak topology if, and only if, it converges in the norm topology.*

Proof By Proposition 5.1.2, every sequence which converges in norm, also converges weakly. Conversely, let $\{x_n\}$ be a weakly convergent sequence. Without loss of generality, assume that $x_n \rightharpoonup \mathbf{0}$. Let

$$x_n = (x_n^1, x_n^2, \ldots, x_n^k, \ldots).$$

Consider the functional f_i which is the projection to the i-th coordinate. Then, since the sequence weakly converges to zero, it follows that $f_i(x_n) \to 0$, i.e.

$$\lim_{n\to\infty} x_n^i = 0$$

for very positive integer i. Assume, if possible, that $\{x_n\}$ does not converge to zero in norm. Then, there exist $\varepsilon > 0$ such that, for infinitely many n,

$$\sum_{k=1}^{\infty} |x_n^k| \ \geq \ \varepsilon.$$

Thus, working with a suitable subsequence if necessary, we may assume that this is true for all n.

Set $n_0 = m_0 = 1$. Define, for $k \geq 1$, n_k and m_k inductively as follows.

- n_k is the smallest integer greater than n_{k-1} such that

$$\sum_{j=1}^{m_{k-1}} |x_{n_k}^j| \ < \ \frac{\varepsilon}{5}.$$

(This is possible since we know that each coordinate sequence tends to zero.)

- Now choose m_k to be the smallest integer greater than m_{k-1} such that

$$\sum_{j=m_k+1}^{\infty} |x_{n_k}^j| \ < \ \frac{\varepsilon}{5}.$$

(This is possible since the sequence $(x_{n_k}) \in \ell_1$.)

Now define $y = (y^j) \in \ell_\infty = \ell_1^*$ as follows:

For $m_{k-1} + 1 \leq j \leq m_k$,

$$y^j = \begin{cases} 0 & \text{if } x_{n_k}^j = 0, \\ \frac{|x_{n_k}^j|}{x_{n_k}^j} & \text{otherwise.} \end{cases}$$

By varying k over all positive integers, y^j will be defined for all positive integers j. Clearly $\|y\|_\infty = 1$. Also

$$\left| \sum_{j=1}^{\infty} \left(x_{n_k}^j y^j - |x_{n_k}^j| \right) \right| \leq 2 \sum_{j=1}^{m_{k-1}} |x_{n_k}^j| + 2 \sum_{j=m_k+1}^{\infty} |x_{n_k}^j| \leq \frac{4\varepsilon}{5}.$$

Thus,

$$\left| \sum_{j=1}^{\infty} x_{n_k}^j y^j \right| \geq \varepsilon - \frac{4\varepsilon}{5} = \frac{\varepsilon}{5}$$

which contradicts the weak convergence of $\{x_{n_k}\}$ to zero. Hence the result. ■

Remark 5.1.1 The space ℓ_1 being infinite dimensional, the weak and norm topologies are different. Nevertheless, we see from the preceding proposition that the convergent sequences for these topologies are the same. While two metric spaces which have the same convergent sequences are equivalent (i.e. their topologies are the same), two topological spaces with the same convergent sequences need not be the same. This illustrates the inadequacy of considering just sequences in a general topological space. We also conclude that the weak topology on ℓ_1 is not metrizable. ■

Example 5.1.4 Consider the sequence $\{\mathsf{e}_n\}$ in ℓ_1. Since $\|\mathsf{e}_n - \mathsf{e}_m\|_1 = 2$ for all $n \neq m$, we see that this sequence is not convergent in the norm topology. In fact it does not even have a convergent sequence. It cannot have a weakly convergent subsequence either, since such a subsequence will have to converge in the norm topology as well, by Schur's lemma. ■

Definition 5.1.3 Let V and W be Banach spaces and let $T : V \to W$ be a linear mapping. We say that T is **weakly continuous** if T is continuous as a mapping from V into W, each space being endowed with its weak topology. ■

Lemma 5.1.1 *Let $T : V \to W$ be a linear mapping. Then T is weakly continuous if, and only if, for every $f \in W^*$, the map $x \mapsto f(T(x))$ is a weakly continuous map from V into $\mathbb{R}$ (or $\mathbb{C}$, in case of complex Banach spaces).*

Proof If T is weakly continuous, then clearly its composition with any $f \in W^*$ will also be weakly continuous.

Now let $f \in W^*$. Let U be open in $\mathbb{R}$ (or $\mathbb{C}$). Then $f^{-1}(U)$ is weakly open in W, by definition. If $f \circ T$ is weakly continuous, we also have that $T^{-1}(f^{-1}(U)$ is weakly open in V. But by the definition of the weak topology, every weakly open set in W is the union of finite intersections of sets of the form $f^{-1}(U)$, where U is open in $\mathbb{R}$ (or $\mathbb{C}$) and $f \in W^*$. Thus, it follows from the above that the inverse image of every weakly open set in W is weakly open in V; i.e. T is weakly continuous. ■

Our final result in this section shows that as far as continuity of linear maps is concerned, the topology really does not matter.

Theorem 5.1.1 *Let V and W be Banach spaces and let $T: V \to W$ be a linear map. then $T \in \mathcal{L}(V, W)$ if, and only if, T is weakly continuous.*

Proof Let $T \in \mathcal{L}(V, W)$. If $f \in W^*$, then the map $f \circ T \in V^*$ and so is weakly continuous as well. Thus by lemma 5.1.1, it follows that T is weakly continuous.

Conversely, if T is weakly continuous, since the weak topology is Hausdorff, it follows that the graph $G(T)$ is closed when $V \times W$ is given the product topology induced by the weak topologies (cf. Lemma 4.4.1). But this is clearly the weak topology of $V \times W$ (why?) and so $G(T)$ is weakly closed in $V \times W$ and so is closed for its norm topology as well. The continuity of T (between the norm topologies of V and W) is now a consequence of the closed graph theorem. ■

5.2 The Weak* Topology

Let V be a Banach space. Then its dual space, V^*, has its natural norm topology. It also is endowed with its weak topology, *viz.* the coarsest topology such that all the elements of V^{**} are continuous. We now define an even coarser topology on V^*.

Definition 5.2.1 The **weak* topology** on V^* is the coarsest topology such that the functionals $\{J_x \mid x \in V\}$ are all continuous, where $J: x \mapsto J_x$ is the canonical imbedding of V into V^{**}. ■

Clearly, the weak* topology is coarser than the weak topology on V^*. Thus if $\mathcal{S}$, $\mathcal{W}$ and $\mathcal{W}^*$ denote the norm, weak and weak* topologies, respectively, on V^*, we have

$$\mathcal{W}^* \subset \mathcal{W} \subset \mathcal{S}.$$

Remark 5.2.1 It is clear that if V is a reflexive Banach space, then the weak and weak* topologies on V^* coincide. ■

Proposition 5.2.1 *Let V be a Banach space. The weak* topology on V^* is Hausdorff.*

Proof Let f_1 and f_2 be distinct elements of V^*. Then, there exists $x \in V$ such that $f_1(x) \neq f_2(x)$. Choose disjoint neighbourhoods U_1 of $f_1(x)$ and U_2 of $f_2(x)$ in $\mathbb{R}$ (or $\mathbb{C}$, as the case may be). Then, by definition, the sets

$$J_x^{-1}(U_1) = \{f \in V^* \mid f(x) \in U_1\} \text{ and } J_x^{-1}(U_2) = \{f \in V^* \mid f(x) \in U_2\}$$

are both weak* open sets and are clearly disjoint and contain f_1 and f_2, respectively. This completes the proof. ■

As in the case of the weak topology, we can describe the weak* open neighbourhoods of elements of V^* as follows. Let I be a finite indexing set and let $x_i \in V$ for $i \in I$. Let $\varepsilon > 0$. Then, a weak* open neighbourhood of $f_0 \in V^*$ can be written as

$$\{f \in V^* \mid |(f - f_0)(x_i)| < \varepsilon,\ i \in I\}.$$

Notation: Let $\{f_n\}$ be a sequence in V^*. If f_n converges to f in V^* in the norm topology, we write $f_n \to f$. If it converges to f in the weak topology of V^*, we will write, as before, $f_n \rightharpoonup f$. If the sequence converges in the weak* topology of V^*, we will write $f_n \overset{*}{\rightharpoonup} f$.

The proof of the following proposition is easy and is left to the reader as an exercise.

Proposition 5.2.2 *Let V be a Banach space and let $\{f_n\}$ be a sequence in V^*.*

(i) $f_n \overset{*}{\rightharpoonup} f$ *in V^* if, and only if, $f_n(x) \to f(x)$ for every $x \in V$.*
(ii) $f_n \to f \Rightarrow f_n \rightharpoonup f \Rightarrow f_n \overset{*}{\rightharpoonup} f$.
(iii) *If $f_n \overset{*}{\rightharpoonup} f$ in V^* and $x_n \to x$ in V, then $f_n(x_n) \to f(x)$.* ■

Example 5.2.1 Consider the sequence $\{\mathsf{e}_n\}$ in ℓ_1. We saw that (cf. Example 5.1.4) that it is not convergent in the norm and weak topologies. Since $c_0^* = \ell_1$, if $x = (x_1, \ldots, x_k, \ldots) \in c_0$, we have that

$$< \mathsf{e}_n, x >_{\ell_1, c_0} = x_n \to 0,$$

as $n \to \infty$. Thus $\mathsf{e}_n \overset{*}{\rightharpoonup} \mathbf{0}$ in ℓ_1. ■

The next proposition shows that the functionals $\{J_x \mid x \in V\}$ are the only ones which are continuous with respect to the weak* topology.

Proposition 5.2.3 *Let φ be a linear functional on V^* which is continuous with respect to the weak* topology. Then, there exists $x \in V$ such that $\varphi = J_x$.*

Proof Let $\widetilde{D}$ be the open unit ball in $\mathbb{R}$ (or $\mathbb{C}$, in case of complex Banach spaces). Since φ is weak* continuous, there exists a weak* neighbourhood of the origin in V^*, say, U, such that $\varphi(U) \subset \widetilde{D}$. Assume that

$$U = \{f \in V^* \mid |f(x_i)| < \varepsilon,\ 1 \leq i \leq n\}$$

where $\varepsilon > 0$ and $x_i \in V$ for $1 \leq i \leq n$. Thus, for every $f \in U$, we have that

$$|\varphi(f)| < 1.$$

Assume that for some $f \in V^*$, we have $f(x_i) = 0$ for all $1 \leq i \leq n$. Then $f \in U$. Further, for any real number k, we have that $kf(x_i) = 0$ for all $1 \leq i \leq n$ and so $kf \in U$ as well. Thus, for all positive integers k,

$$|\varphi(f)| < \frac{1}{k}$$

and so $\varphi(f) = 0$. It then follows (cf. Exercise 3.23) that there exist scalars α_i for $1 \leq i \leq n$ such that

$$\varphi = \sum_{i=1}^{n} \alpha_i J_{x_i}.$$

This proves the result with $x = \sum_{i=1}^{n} \alpha_i x_i$. ∎

Corollary 5.2.1 *A weak* closed hyperplane must be of the form*

$$H = \{f \in V^* \mid f(x) = \alpha\}$$

where $x \in V$ and α is a scalar.

Proof For simplicity, we will assume that the base field is $\mathbb{R}$. Since H is a weak* closed hyperplane, it is closed in the norm topology as well and so (cf. Proposition 3.3.1) there exists $\varphi \in V^{**}$ such that

$$H = \{f \in V^* \mid \varphi(f) = \alpha\}$$

for some real number α. Let $f_0 \in H^c$, the complement of H. Since H is weak* closed, there exists a weak* open neighbourhood of the form

$$U = \{f \in V^* \mid |(f - f_0)(x_i)| < \varepsilon,\ 1 \leq i \leq n\}$$

(where $\varepsilon > 0$ and $x_i \in V$ for $1 \leq i \leq n$) of f_0 contained in H^c. Now, U is a convex set. Thus, it is easy to see that either $\varphi(f) < \alpha$ for all $f \in U$ or $\varphi(f) > \alpha$ for all $f \in U$. Assume the former (the proof in the latter case will be similar). Let $W = U - \{f_0\} = \{f - f_0 \mid f \in U\}$. Then,

$$W = \{g \in V^* \mid g + f_0 \in U\}$$

and so $g \in W$ if, and only if, $-g \in W$. Thus $W = -W$. Now, if $\varphi(f) < \alpha$ for all $f \in U$, it follows that

$$\varphi(g) < \alpha - \varphi(f_0)$$

for all $g \in W$. Since $W = -W$, it then follows that

$$|\varphi(g)| < |\alpha - \varphi(f_0)|$$

for all $g \in W$. Since we can always find $f_0 \in H^c$ such that $|\alpha - \varphi(f_0)| < \eta$, for any $\eta > 0$, and since W is weak* open, it follows that φ is weak* continuous at the origin, and so, by linearity, weak* continuous everywhere. Then, by the preceding proposition, $\varphi = J_x$ for some $x \in V$. This completes the proof. ■

Since every finite dimensional space V is reflexive, the weak and weak* topologies on V^* coincide. We have already seen, in Sect. 5.1, that the norm and weak topologies coincide. Thus, in finite dimensional spaces, all the three topologies are the same.

However, the above corollary shows that, in infinite dimensional and non-reflexive spaces, the weak* topology is strictly coarser than the weak topology. If $\varphi \in V^{**}\backslash J(V)$, then the hyperplane $[\varphi = \alpha]$ is a convex and (norm) closed set and hence is weakly closed but it is not weak* closed.

One might wonder the purpose of impoverishing the norm topologies on Banach spaces and their duals to produce the weak and weak* topologies. One important off shoot of this is process is that by decreasing the number of open sets, we increase the chances of a set being compact, which is a very useful topological property. We saw, in Chap. 2, that in infinite dimensional spaces, the closed unit ball cannot be compact. The ball becomes compact in the weak* topology.

Theorem 5.2.1 *(**Banach-Alaoglu**) Let V be a Banach space. Then, B^*, the closed unit ball in V^*, is weak* compact.*

Proof Consider the product space

$$X = \Pi_{x \in V}[-\|x\|, \|x\|]$$

with the usual product topology inherited from $\mathbb{R}$. This space is clearly compact since each bounded and closed interval in $\mathbb{R}$ is compact. Let $f \in B^*$. Then, for each $x \in V$, we have $f(x) \in [-\|x\|, \|x\|]$. Thus, the map $f \mapsto \varphi(f) = (f(x))_{x \in V}$ is a bijection from B^* onto its image in X. If B is endowed with the topology induced by the weak* topology of V^*, then the definitions of this topology and the product topology on X tell us that φ is a homeomorphism. We thus just need to show that $\varphi(B^*)$ is closed in X, which will prove $\varphi(B^*)$, and hence B^*, to be compact.

Let $(f_x)_{x \in V} \in \overline{\varphi(B^*)}$. Define, for $x \in V$, $f(x) = f_x$. The proof will be complete if we can show that f is linear; since $|f(x)| \leq \|x\|$ for all $x \in V$, it will then follow that $f \in B^*$, i.e. $\overline{\varphi(B^*)} = \varphi(B^*)$.

Let $\varepsilon > 0$. Then, given x and $y \in V$, we can find $g \in B^*$ such that

$$|g(x) - f(x)| < \frac{\varepsilon}{3}, \ |g(y) - f(y)| < \frac{\varepsilon}{3}, \ |g(x+y) - f(x+y)| < \frac{\varepsilon}{3}.$$

Thus,

$$|f(x+y) - f(x) - f(y)| < \varepsilon$$

and, since ε was arbitrarily chosen, we deduce that

$$f(x+y) = f(x) + f(y)$$

for every x and $y \in V$. Similarly, we can show that if α is a scalar and if $x \in V$,

$$f(\alpha x) = \alpha f(x).$$

Thus f is linear and the proof is complete. ■

Lemma 5.2.1 *Let V be a Banach space and let $f_i \in V^*$, $1 \leq i \leq n$. Let α_i, $1 \leq i \leq n$ be scalars. Then, the following are equivalent:*

(i) *For every $\varepsilon > 0$, there exists $x_\varepsilon \in V$ with $\|x_\varepsilon\| \leq 1$ and such that*

$$|f_i(x_\varepsilon) - \alpha_i| < \varepsilon$$

for all $1 \leq i \leq n$.
(ii) *For all scalars β_i, $1 \leq i \leq n$, we have*

$$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| \leq \left\|\sum_{i=1}^{n} \beta_i f_i\right\|.$$

Proof (i) $\Rightarrow$ (ii). Let $s = \sum_{i=1}^{n} |\beta_i|$. By (i),

$$\left|\sum_{i=1}^{n} (\beta_i f_i(x_\varepsilon) - \beta_i \alpha_i)\right| < \varepsilon s$$

which implies that

$$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| \leq \varepsilon s + \left\|\sum_{i=1}^{n} \beta_i f_i\right\|$$

from which (ii) follows, since we can choose $\varepsilon > 0$ to be arbitrarily small.

(ii) $\Rightarrow$ (i). Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ (the proof when the scalar field is $\mathbb{C}$ is similar). Define $\mathcal{A}$: $V \to \mathbb{R}^n$ by $\mathcal{A}(x) = (f_1(x), \ldots, f_n(x))$. We then need to show that $\overline{\alpha} \in \overline{\mathcal{A}(B)}$ where B is the closed unit ball in V. If not, by the Hahn-Banach theorem, we can find scalars λ and $\beta_1, \ldots, \beta_n$ such that, for every $x \in B$,

$$\sum_{i=1}^{n} \alpha_i \beta_i > \lambda > \sum_{i=1}^{n} \beta_i f_i(x)$$

which implies that

$$\left\| \sum_{i=1}^{n} \beta_i f_i \right\| \leq \lambda < \sum_{i=1}^{n} \alpha_i \beta_i$$

which contradicts (ii). This completes the proof. ■

Proposition 5.2.4 *Let V be a Banach space. Let B be the closed unit ball in V and B^{**} the closed unit ball in V^{**}. Let $J: V \to V^{**}$ be the canonical imbedding. Then, B^{**} is the weak* closure of $J(B)$ in V^{**}.*

Proof Since B^{**} is weak* compact, it is weak* closed. Let $\varphi_0 \in B^{**}$. Consider a weak* open neighbourhood of φ_0 of the form

$$U = \{\varphi \in V^{**} \mid |(\varphi - \varphi_0)(f_i)| < \varepsilon,\ 1 \leq i \leq n\}$$

where $\varepsilon > 0$ and $f_i \in V^*,\ 1 \leq i \leq n$. Let $\alpha_i = \varphi_0(f_i),\ 1 \leq i \leq n$. Then, for scalars $\beta_i,\ 1 \leq i \leq n$, we have,

$$\left| \sum_{i=1}^{n} \beta_i \alpha_i \right| = \left| \varphi_0 \left(\sum_{i=1}^{n} \beta_i f_i \right) \right| \leq \left\| \sum_{i=1}^{n} \beta_i f_i \right\|.$$

Then, by the preceding lemma,there exists $x \in B$ such that $J_x \in U$. Thus U intersects $J(B)$ and this shows that $J(B)$ is weak* dense in B^{**} which completes the proof. ■

Remark 5.2.2 Let V be a Banach space. Since the map $J: V \to V^{**}$ is an isometry, it follows that $J(B)$ is closed in the norm topology of V^{**}. Thus, either $J(B) = B^{**}$, which is true if, and only if, V is reflexive, or $J(B)$ is a closed and proper subset of B^{**}. Thus, in the non-reflexive case, $J(B)$ is not dense in B^{**} for the norm topology. ■

Example 5.2.2 Let B_0 be the closed unit ball in c_0. With the usual identifications, since $c_0^* = \ell_1$ and $\ell_1^* = \ell_\infty$, we can easily see that the canonical mapping $J: c_0 \to \ell_\infty$ is just the inclusion mapping. Thus B_0 is closed and weakly closed in ℓ_∞. However, its weak* closure in ℓ_∞ is B_∞, the closed unit ball in ℓ_∞. ■

5.3 Reflexive Spaces

Let us recall that a Banach space V is said to be reflexive if the canonical imbedding $J: V \to V^{**}$ is surjective (cf. Definition 3.1.1). We also saw that the spaces ℓ_p for $1 < p < \infty$ are examples of reflexive spaces while ℓ_1 is not reflexive. In Chap. 7, we will see that every *Hilbert space* is reflexive.

In this section, we will study some important properties of reflexive spaces.

Notation: Given a Banach space V, we will denote the closed unit balls in V, V^* and V^{**} by B, B^* and B^{**}, respectively.

Theorem 5.3.1 *A Banach space V is reflexive if, and only if, B is weakly compact.*

Proof Assume that B is weakly compact. Since $J: V \to V^{**}$ is an isometry, it is continuous and hence weakly continuous as well (cf. Theorem 5.1.1) and so $J(B)$ is weakly compact. Hence it is weak* compact as well. The weak* topology being Hausdorff, it follows that $J(B)$ is weak* closed. But then (cf. Proposition 5.2.4) it follows that $J(B) = B^{**}$. This immediately implies that J is surjective; i.e. V is reflexive.

Conversely, let V be reflexive. Then the weak and weak* topologies on V^* coincide. Hence, by the Banach-Alaoglu theorem, B^* is weakly compact. Then, by the preceding arguments, it follows that V^* is reflexive. Then, just as we saw earlier, it follows that B^{**} is weakly compact. Since V is reflexive, we have $B = J^{-1}(B^{**})$. Also, since $J^{-1}: V^{**} \to V$ is continuous, it is weakly continuous as well and so B is weakly compact. ■

Corollary 5.3.1 *Let V and W be Banach spaces and let $T: V \to W$ be an isometric isomorphism. Then, if V is reflexive, so is W.*

Proof Let B_V and B_W be the closed unit balls in V and W, respectively. Since T is an isometric isomorphism, we have that $T(B_V) = B_W$. Now, T being continuous, it is weakly continuous as well. Since V is reflexive, we have that B_V is weakly compact and so $B_W = T(B_V)$ is also weakly compact, which implies that W is reflexive. ■

Corollary 5.3.2 *Let V be a reflexive Banach space and let W be a* closed *subspace of V. The W is also reflexive.*

Proof It is easy to see that the weak topology on W is none other than the topology induced on W by the weak topology of V (cf. Exercise 5.1). Since V is reflexive, it follows that B is weakly compact. The unit ball in W is none other than $W \cap B$. But W being a closed subspace, it is weakly closed and since B is weakly compact, it follows that $W \cap B$ is weakly compact as well. Thus, it follows that W is reflexive. ■

Corollary 5.3.3 *Let V be a Banach space. Then, V is reflexive if, and only if, V^* is reflexive.*

Proof We already saw in the proof of Theorem 5.3.1 that if V is reflexive, then V^* is reflexive.

Conversely, let V^* be reflexive. Then, as before, V^{**} is reflexive. Now, $J(V)$ is a closed subspace of V^{**} and so, by the preceding corollary, it follows that $J(V)$ is reflexive. But then $J^{-1}: J(V) \to V$ is an isometric isomorphism and so V is reflexive by Corollary 5.3.1. ■

Example 5.3.1 We can recover many of the results we proved in Sect. 3.2, using those above. Since we know that ℓ_1 is not reflexive, it follows that c_0 and ℓ_∞ are not reflexive either, since $c_0^* = \ell_1$ and $\ell_1^* = \ell_\infty$. Since c_0 is a closed subspace of c (the space of convergent sequences, equipped with the norm $\|\cdot\|_\infty$), it follows that c is not reflexive. ■

Corollary 5.3.4 *Let V be a reflexive Banach space. Let $K \subset V$ be a closed, bounded and* convex *subset. Then, K is weakly compact.*

Proof Since K is bounded, there exists a positive integer m such that $K \subset mB$. Then, since K is convex and closed, it is weakly closed and since mB is weakly compact, it follows that K is weakly compact. ■

Example 5.3.2 Consider the spaces $V = \mathcal{C}^1[0, 1]$ and $W = \mathcal{C}[0, 1]$ with their usual norms. Then we saw (cf. Example 4.6.3) that the map $T: V \to W$ defined by $T(f) = f'$, for every $f \in V$, admits a right inverse S. Let B be the closed unit ball in W. Then $S(B)$ is a bounded and convex set in V. It is also closed. To see this, let $\{f_n\}$ be a sequence in B such that $S(f_n) \to g$ in V. Then, $f_n = T(S(f_n)) \to T(g)$ in W and so $T(g) \in B$. But then $S(f_n) \to S(T(g))$, i.e. $g = S(T(g)) \in S(B)$. If V were reflexive, it would then follow from the above corollary that $S(B)$ is weakly compact and, since continuous maps are also weakly continuous, we could then deduce that $B = T(S(B))$ is weakly compact. But that would imply that $W = \mathcal{C}[0, 1]$ is reflexive, which we know to be false. Thus $V = \mathcal{C}^1[0, 1]$ is not reflexive.

The same maps can be used between $V = \mathcal{C}^k[0, 1]$ and $\mathcal{C}^{k-1}[0, 1]$, where $k \geq 2$, and we can inductively prove that $\mathcal{C}^k[0, 1]$ is not reflexive for any positive integer k. ■

Proposition 5.3.1 *Let V and W be Banach spaces, with W being reflexive, and let $A: D(A) \subset V \to W$ be a linear transformation which is* closed *and* densely defined. *Then A^* is also densely defined.*

Proof Let $\varphi \in W^{**}$ which vanishes on $D(A^*)$. It suffices to show that $\varphi = \mathbf{0}$. Since W is reflexive, there exists $y \in W$ such that

$$< \varphi, v >_{W^{**}, W^*} = < v, y >_{W^*, W}$$

for all $v \in W^*$. Thus, we need to show that $y = \mathbf{0}$ given that $< w, y >_{W^*, W} = 0$ for all $w \in D(A^*)$. If not, then $(\mathbf{0}, y) \notin G(A)$, the graph of A. Since $G(A)$ is closed, by hypothesis, there exists $(f, v) \in V^* \times W^*$ such that

$$< f, u >_{V^*, V} + < v, A(u) >_{W^*, W} = 0 \tag{5.3.1}$$

for all $u \in D(A)$ and such that $< v, y >_{W^*, W} \neq 0$, by virtue of the Hahn-Banach theorem (cf. Corollary 3.3.1). It follows from (5.3.1) that $v \in D(A^*)$ which then implies that $< v, y >_{W^*, W} = 0$ which is a contradiction. ■

Thus, in the circumstances of the above proposition, we can define the second adjoint $A^{**} = (A^*)^*$ from V^{**} into W^{**}. If we now assume that both V and W are reflexive, then we can identify V with V^{**} and W with W^{**} via their respective canonical imbeddings. In this case, we will then have $A^{**}: D(A^{**}) \subset V \to W$.

Theorem 5.3.2 *Let V and W be* reflexive *Banach spaces and let $A: D(A) \subset V \to W$ be a* closed *and* densely defined *linear transformation. Then, $A^{**} = A$.*

Proof It suffices to show that the graphs $G(A)$ and $G(A^{**})$ are the same. Recall that if we define $\mathcal{J}: W^* \times V^* \to V^* \times W^*$ by $\mathcal{J}(v, f) = (-f, v)$, we then have $\mathcal{J}(G(A^*)) = G(A)^{\perp}$ (cf. Proposition 4.7.2). Then $(G(A)^{\perp})^{\perp} = (\mathcal{J}(G(A^*))^{\perp} \subset V \times W$. Thus, $(G(A)^{\perp})^{\perp}$ consists of all $(v, w) \in V \times W$ such that

$$< -A^*(\varphi), v >_{V^*,V} + < \varphi, w >_{W^*,W} = 0$$

for all $\varphi \in D(A^*)$. This is equivalent to saying that $v \in D(A^{**})$ and that $w = A^{**}(v)$. Thus,

$$G(A^{**}) = (G(A)^{\perp})^{\perp} = \overline{G(A)} = G(A)$$

since $G(A)$ is closed and this completes the proof. ■

5.4 Separable Spaces

In this section, we will study the relationship between separable spaces and weak topologies.

Definition 5.4.1 A topological space is said to be **separable** if it contains a countable dense set. ■

Proposition 5.4.1 *Let V be a Banach space. If V^* is separable, then so is V.*

Proof Let $\{f_n\}$ be a countable dense set in V^*. Choose $\{x_n\}$ in V such that

$$\|x_n\| = 1 \text{ and } f_n(x_n) > \frac{1}{2}\|f_n\|.$$

Assume, for simplicity, that the base field is $\mathbb{R}$. Let W be the linear subspace generated by the sequence $\{x_n\}$ and let W_0 be the set of all finite linear combinations of the $\{x_n\}$ with *rational* coefficients. Then W_0 is countable, and it is dense in W. So it suffices to show that W is dense in V. Let $f \in V^*$ which vanishes on W. We need to show then that f vanishes on all of V (i.e. f is identically zero). Let $\varepsilon > 0$. Then, there exists f_m such that $\|f - f_m\| < \varepsilon$, by the density of the $\{f_n\}$. Since $f(x_n) = 0$ for all n, we have

$$\frac{1}{2}\|f_m\| < f_m(x_m) = (f_m - f)(x_m) \leq \|f_m - f\|.$$

Thus,

$$\|f\| \le \|f_m - f\| + \|f_m\| < 3\varepsilon$$

from which it follows that $f = \mathbf{0}$ since ε was arbitrarily chosen. ■

Example 5.4.1 The converse of the above proposition is not true. We know that $\ell_1^* = \ell_\infty$. While ℓ_1 is separable (the set of all sequences with only a finite number of non-zero components, all of which are rational, forms a countable dense set of ℓ_1), ℓ_∞ is not separable. To see this, we prove that no countable set in ℓ_∞ can be dense. Indeed, let $\{f_n\}$ be a countable set in ℓ_∞ where $f_n = (f_n^i)$. Define $f = (f^i)$ by

$$f^i = \begin{cases} 0, \text{ if } |f_i^i| \ge 1, \\ 2, \text{ if } |f_i^i| < 1. \end{cases}$$

Then $f \in \ell_\infty$ and $\|f - f_n\|_\infty \ge 1$ for all n. Thus $\{f_n\}$ cannot be dense in ℓ_∞. ■

Corollary 5.4.1 *Let V be a Banach space. Then V is both separable and reflexive if, and only if, V^* is both separable and reflexive.*

Proof If V^* is both separable and reflexive, then so is V. Conversely, if V is separable and reflexive, so is $J(V) = V^{**}$, where J is the canonical imbedding of V into V^{**}. Thus, it now follows that V^* is separable and reflexive. ■

Example 5.4.2 It follows, from the above corollary, that if a Banach space V is separable, and if V^* is not separable, then V is not reflexive. Consider the space $V = \mathcal{C}[0, 1]$, with the usual sup-norm. As a consequence of the Weierstrass approximation theorem, it follows that V is separable (why?). Now, for every $x \in [0, 1]$, define $\delta_x \in V^*$ by $\delta_x(f) = f(x)$, for every $f \in V$. Then, it follows that if $x \neq y$, we have $\|\delta_x - \delta_y\|_{V^*} = 2$ (why?). Thus if $\{\varphi_n\}$ is any countable set in V^*, it can intersect at most a countable number of the open balls of the form $B_{V^*}(\delta_x; 1/2)$, since all such balls are mutually disjoint. Thus, no countable set can be dense in V^*; i.e. the space V^* is not separable. Consequently, we now have a different proof (cf. Example 3.2.2) of the non-reflexivity of $\mathcal{C}[0, 1]$. ■

Theorem 5.4.1 *Let V be a Banach space. Then, V is separable if, and only if, the weak* topology on B^*, the closed unit ball in V^*, is metrisable.*

Proof Assume that V is separable. Let $\{x_n\}$ be a countable dense set in V. We may assume, without loss of generality, that $x_n \neq \mathbf{0}$ for all n (why?). For f and $g \in B^*$, define

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n \|x_n\|} |(f - g)(x_n)|. \tag{5.4.1}$$

It is easy to check that $d(., .)$ is well-defined and that it defines a metric on B^*. Let U be a weak* open neighbourhood of $f_0 \in B^*$ of the form

$$U = \{f \in B^* \mid |(f - f_0)(y_i)| < \varepsilon,\ 1 \le i \le k\}$$

where $\varepsilon > 0$ and $y_i \in V$ for $1 \leq i \leq k$. Since $\{x_n\}$ is dense, there exists x_{n_i} such that $\|y_i - x_{n_i}\| < \varepsilon/4$ for each $1 \leq i \leq k$. Now choose $r > 0$ such that

$$r 2^{n_i} \|x_{n_i}\| < \frac{\varepsilon}{2} \text{ for all } 1 \leq i \leq k.$$

Consider the ball $B_d(f_0; r)$ in B^* provided with the metric defined in (5.4.1). If f belongs to this ball, i.e. $d(f, f_0) < r$, then for each $1 \leq i \leq k$, we have

$$\begin{aligned}|(f - f_0)(y_i)| &\leq |(f - f_0)(y_i - x_{n_i})| + |(f - f_0)(x_{n_i})| \\ &\leq 2.\tfrac{\varepsilon}{4} + 2^{n_i} \|x_{n_i}\| r \\ &< \varepsilon.\end{aligned}$$

Thus $B_d(f_0; r) \subset U$ and so every weak* open set is also open in the metric topology.

On the other hand, consider a ball $B_d(f_0; r)$. Consider the weak* open neighbourhood of f_0 given by

$$U_k^\varepsilon = \left\{ f \in B^* \mid \left| (f - f_0) \left(\frac{1}{\|x_i\|} x_i \right) \right| < \varepsilon,\ 1 \leq i \leq k \right\}.$$

Choose $\varepsilon < r/2$ and k such that

$$\sum_{n=k+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^k} < \frac{r}{4}.$$

If $f \in U_k^\varepsilon$, then

$$\begin{aligned}d(f, f_0) &= \textstyle\sum_{n=1}^{k} \frac{1}{2^n \|x_n\|} |(f - f_0)(x_n)| + \sum_{n=k+1}^{\infty} \frac{1}{2^n \|x_n\|} |(f - f_0)(x_n)| \\ &< \varepsilon \textstyle\sum_{n=1}^{k} \frac{1}{2^n} + 2 \sum_{n=k+1}^{\infty} \frac{1}{2^n} \\ &< \tfrac{r}{2} + 2 \tfrac{r}{4} \\ &= r.\end{aligned}$$

Thus, $U_k^\varepsilon \subset B_d(f_0; r)$ which shows that every open set in the metric topology is also weak* open. Thus, the weak* and the metric topologies on B^* are the same.

Conversely, assume that the weak* topology on B^* is metrisable. Consider, for each positive integer n, the ball $B_d(\mathbf{0}; \frac{1}{n})$, where d is the metric defined on B^*. This ball then contains a weak* open neighbourhood of zero, say U_n which can be written in the form

$$U_n = \{f \in B^* \mid |f(x)| < \varepsilon_n, \text{ for all } x \in \Phi_n\}$$

where $\varepsilon_n > 0$ and Φ_n is a finite set in V. The set

$$D = \cup_{n=1}^{\infty} \Phi_n$$

is then countable and the set E of all finite rational linear combinations of the elements of D is a countable set which will be dense in the subspace generated by D. If $f \in V^*$ is such that $f(x) = 0$ for all x in the subspace generated by D, then clearly, $f \in U_n$ for each n. Thus,

$$f \in \cap_{n=1}^{\infty} U_n \subset \cap_{n=1}^{\infty} B_d(\mathbf{0}; 1/n) = \{\mathbf{0}\}.$$

Thus the subspace generated by D is itself dense in V and so the countable set E is also dense in V and hence V is separable.

This completes the proof. ■

Corollary 5.4.2 *Let V be a separable Banach space. Then, every bounded sequence in V^* has a weak* convergent subsequence.*

Proof A bounded sequence in V^* is contained in some ball, which is weak* compact. Since V is separable, the weak* topology on this ball is metrisable and so the ball is weak* sequentially compact as well. ■

In a metric space compactness and sequential compactness are equivalent (cf. Proposition 1.2.6); this is not true in a general topological space. Thus a sequence in a compact topological space may fail to have a convergent subsequence, as the following example shows.

Example 5.4.3 Consider the space ℓ_∞. Define

$$f_n(x) = x_n$$

for $x = (x_n) \in \ell_\infty$. Then clearly $f_n \in \ell_\infty^*$ and, further, $\|f_n\|_{\ell_\infty^*} = 1$, for all n. The unit ball in ℓ_∞^* is weak* compact, by the Banach-Alaoglu theorem. Assume, if possible, that there exists a weak* convergent subsequence $\{f_{n_k}\}$ for this sequence. This, in view of Proposition 5.2.2 (i), imples that $\{x_{n_k}\}$ is convergent for every $x = (x_n) \in \ell_\infty$, which is clearly absurd. Thus $\{f_n\}$ cannot have any weak* convergent subsequence, eventhough it lies in a weak* compact set. ■

Example 5.4.4 We saw that (cf. Example 5.2.1) $\{\mathsf{e}_n\}$ has no weakly convergent subsequence in ℓ_1. But since ℓ_1 is separable and so c_0 is also separable (cf. Proposition 5.4.1) and, by the preceding corollary, $\{\mathsf{e}_n\}$ must have a weak* convergent subsequence. In fact, we have seen, in the above-mentioned example, that $\{\mathsf{e}_n\}$ weak* converges to zero. Thus the weak and weak* convergent sequences in ℓ_1 are not the same (while its norm and weakly convergent sequences are the same). ■

Theorem 5.4.2 *Let V be a reflexive Banach space. Then every bounded sequence has a weakly convergent subsequence.*

Proof Let $\{x_n\}$ be a bounded sequence in V. Let $W = \overline{\text{span}\{\{x_n\}\}}$, i.e. the smallest closed subspace containing the sequence in question. Then, W is also reflexive and, by construction, it is separable (why?). So W^* is also reflexive and separable. Then, every bounded sequence in W^{**} has a weak* convergent subsequence and since W^{**} is also reflexive, the weak and weak* topologies are the same. In particular, $\{J(x_n)\}$ has a weakly convergent subsequence in W^{**}, where $J: W \to W^{**}$ is the canonical imbedding, and since $J^{-1}: W^{**} \to W$ is an isometry and hence weakly continuous, $\{x_n\}$ has a weakly convergent subsequence in W. Since the weak topology in W is the topology induced on W by the weak topology on V (cf. Exercise 5.1), it follows that this subsequence converges weakly in V as well. ■

Remark 5.4.1 The converse of the above theorem is also true, and it is a deep result due to Eberlein and Šmulian: If every bounded sequence admits a weakly convergent subsequence in a Banach space, then the space is reflexive. ■

5.5 Uniformly Convex Spaces

We know that the unit ball in a normed linear space is convex. However, the nature of the boundary of this ball depends on the norm. For instance, in $\mathbb{R}^2$, with the euclidean metric (i.e. $\mathbb{R}^2 = \ell_2^2$), the unit ball is a very symmetric object which 'bulges uniformly' in all directions. On the other hand, if we consider $\mathbb{R}^2$ as ℓ_1^2 or as ℓ_∞^2, then the unit ball will be, the rhombus bounded by the lines $(\pm x_1) + (\pm x_2) = 1$ or the unit square, respectively. In both these cases, the boundary has a lot of 'flat' portions. Uniform convexity makes precise the notion of the boundary 'bulging uniformly' in all directions. This is a condition describing the 'geometry' of the norm, but has an important 'analytic' consequence, which will be the main theorem of this section. It also has important consequences in the calculus of variations, which we will see in the next section.

Definition 5.5.1 A normed linear space is said to be **uniformly convex** if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever we have x and $y \in V$ satisfying

$$\|x\| \le 1, \ \|y\| \le 1 \text{ and } \|x - y\| > \varepsilon,$$

it follows that

$$\left\| \frac{1}{2}(x + y) \right\| < 1 - \delta.$$

■

In other words, given two points on the boundary which are at a distance of ε from each other, then, irrespective of the position of these points, the midpoint of the chord joining them should lie in the interior, at a minimum distance away from the boundary, the minimum distance being prescribed uniformly.

Uniform convexity is stronger than the notion of strict convexity (cf. Exercise 3.2).

Example 5.5.1 The spaces ℓ_1^N and ℓ_∞^N are not uniformly convex. In fact, they are not even strictly convex. ■

Example 5.5.2 The space ℓ_2^N is uniformly convex. Let x and $y \in \ell_2^N$. Then it is easy to verify that

$$\left\|\frac{1}{2}(x+y)\right\|_2^2 + \left\|\frac{1}{2}(x-y)\right\|_2^2 = \frac{1}{2}(\|x\|_2^2 + \|y\|_2^2). \tag{5.5.1}$$

If $\|x\|_2 \leq 1,\ \|y\|_2 \leq 1$ and $\|x-y\|_2 > \varepsilon$, with ε sufficiently small, we see that

$$\left\|\frac{1}{2}(x+y)\right\|_2^2 < 1 - \frac{\varepsilon^2}{4} = (1-\delta)^2$$

where

$$\delta = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

■

Remark 5.5.1 When $N = 2$, the relation (5.5.1) is the familiar parallellogram law or Apollonius' theorem in plane geometry. The relation (5.5.1) is also valid for the space ℓ_2 and so ℓ_2 is also uniformly convex. In fact, we will see, in Chap. 7, that this relation is valid in any *Hilbert space* and so every Hilbert space will be uniformly convex. ■

Remark 5.5.2 We will see in Chap. 6, that a relation similar to (5.5.1) is also valid for the spaces ℓ_p^N and ℓ_p whenever $2 \leq p < \infty$ (cf. Proposition 6.2.1) and so all these spaces are uniformly convex. A similar inequality also holds for $1 < p < 2$ but the proof is more difficult. Thus, all these spaces are uniformly convex. ■

Theorem 5.5.1 *Let V be a uniformly convex Banach space. Then V is reflexive.*

Proof If $J\colon V \to V^{**}$ is the canonical mapping, it is enough to show that the image of the closed unit ball B in V under J is the closed unit ball B^{**} in V^{**}. Since V is a Banach space, $J(B)$ is a closed set in V^{**} and so it suffices to show that it is dense in B^{**} (cf. Remark 5.2.2).

Let $\varphi \in B^{**}$. Assume that $\|\varphi\|_{V^{**}} = 1$. Let $\varepsilon > 0$. We will show that there exists $x \in B$ such that

$$\|\varphi - J(x)\|_{V^{**}} < \varepsilon.$$

The same will then be true for all elements of B^{**} as well (why?).

Let $\delta > 0$ correspond to ε in the definition of uniform convexity. Choose $f \in V^*$, with $\|f\|_{V^*} = 1$ and such that

$$\varphi(f) > 1 - \frac{\delta}{2}. \tag{5.5.2}$$

Define
$$U = \{\xi \in V^{**} \mid |(\xi - \varphi)(f)| < \delta/2\}.$$

Then U is a weak* open neighbourhood of φ in V^{**}. Since $J(B)$ is weak* dense in B^{**} (cf. Proposition 5.2.4), it follows that there exists $x \in B$ such that $J(x) \in U$.

Assume now that $\|J(x) - \varphi\|_{V^{**}} > \varepsilon$. In other words, $\varphi \notin J(x) + \varepsilon B^{**}$. Since εB^{**} is weak* compact (by the Banach-Alaoglu theorem), it is weak* closed and so is its translation by $J(x)$. Thus, there exists a weak* open neighbourhood U_1 of φ such that for all $\xi \in U_1$, we still have $\|\xi - J(x)\|_{V^{**}} > \varepsilon$. Again, as before, there exists $x_1 \in B$ such that $J(x_1) \in U \cap U_1$ by the weak* density of $J(B)$ in B^{**}. Thus,

$$\begin{aligned} |\varphi(f) - f(x)| &< \tfrac{\delta}{2} \\ |\varphi(f) - f(x_1)| &< \tfrac{\delta}{2} \end{aligned}$$

and so
$$2\varphi(f) \ < \ \delta + |f(x + x_1)| \ < \ \delta + \|x + x_1\|_V.$$

By virtue of (5.5.2), it follows from the above that

$$\left\| \frac{1}{2}(x + x_1) \right\|_V \ > \ 1 - \delta$$

which contradicts the uniform convexity since $J(x_1) \in U_1$ and so

$$\|x - x_1\|_V = \|J(x) - J(x_1)\|_{V^{**}} \ > \ \varepsilon$$

while $\|x\|_V = \|x_1\|_V \leq 1$. Thus, it follows that $\|J(x) - \varphi\|_{V^{**}} \leq \varepsilon$ which shows that $J(B)$ is dense in B^{**} as already observed. ■

Remark 5.5.3 The converse of this theorem is not true. A reflexive space need not be uniformly convex. For instance, ℓ_1^N is not uniformly convex, but since it is finite dimensional, it is reflexive. ■

Proposition 5.5.1 *Let V be a uniformly convex Banach space. Let $x_n \rightharpoonup x$ in V. Assume that*
$$\limsup_{n\to\infty} \|x_n\| \ \leq \ \|x\|. \tag{5.5.3}$$

Then $x_n \to x$ in V.

Proof We already know that (cf. Proposition 5.1.2 (iii))

$$\liminf_{n\to\infty} \|x_n\| \ \geq \ \|x\|. \tag{5.5.4}$$

Thus, by (5.5.3) and (5.5.4), we deduce that $\|x_n\| \to \|x\|$. If $x = \mathbf{0}$, this completes the proof. Assume now that $x \neq \mathbf{0}$. Then, from the convergence of the norms, we deduce that (for large n), $x_n \neq \mathbf{0}$. Set $y_n = x_n/\|x_n\|$ and $y = x/\|x\|$. Observe then that, by hypothesis and the convergence of the norms of x_n, it follows that $y_n \rightharpoonup y$ in V. The proof will be complete if we show that $y_n \to y$.

Since $y_n \rightharpoonup y$, we have

$$1 = \|y\| \leq \liminf_{n\to\infty} \left\| \frac{1}{2}(y_n + y) \right\| \leq \limsup_{n\to\infty} \left\| \frac{1}{2}(y_n + y) \right\| \leq 1.$$

Thus we have

$$\|y_n\| = \|y\| = 1 \text{ and } \left\| \frac{1}{2}(y_n + y) \right\| \to 1.$$

Hence, by uniform convexity, if $\varepsilon > 0$ is an arbitrary number, we must have $\|y_n - y\| \leq \varepsilon$ for n sufficiently large. This proves that $y_n \to y$ and hence that $x_n \to x$ in V as already observed. ■

5.6 Application: Calculus of Variations

In this section, we will apply the results of the preceding sections to obtain some important results in the calculus of variations, which can be described as the theory of optimization in infinite dimensional spaces.

Proposition 5.6.1 *Let V be a reflexive Banach space and let $K \subset V$ be a non-empty, closed and convex subset. Let $\varphi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Assume further that*

$$\lim_{\|x\|\to\infty} \varphi(x) = +\infty. \tag{5.6.1}$$

Then, φ attains a minimum in K.

Proof Since K is convex and closed, it is weakly closed. Let $\{x_n\}$ be a minimizing sequence in K for φ, i.e., $\varphi(x_n) \to \inf_{x\in K} \varphi(x)$. Then, since (5.6.1) implies that the sequence is bounded and since V is reflexive, it has a weakly convergent subsequence, say, $\{x_{n_k}\}$, converging weakly to some $x \in V$. But since K is weakly closed, we have $x \in K$. Further, by the lower semicontinuity and convexity of φ, it follows that φ is also weakly lower semicontinuous (cf. Corollary 5.1.1) and so

$$\inf_{y\in K} \varphi(y) \leq \varphi(x) \leq \liminf_{k\to\infty} \varphi(x_{n_k}) = \inf_{y\in K} \varphi(y).$$

Thus,

$$\varphi(x) = \min_{y\in K} \varphi(y).$$

This completes the proof. ■

Remark 5.6.1 The condition (5.6.1) is usually called the condition of *coercivity* of the function(al) φ. Thus, a coercive, convex and lower semicontinuous functional defined on a non-empty, closed and convex subset of a reflexive Banch space always attains a minimum. The method of proof used above is usually known as the *direct method of the calculus of variations*. A minimizing sequence is shown to have a convergent subsequence (in a suitable topology), and the limit is shown to be the desired minimum. ■

Remark 5.6.2 In the proof of the preceding proposition, the coercivity condition was really needed only when K was not bounded.

In a metric space, the lower semicontinuity of a function φ is equivalent to the condition that if $x_n \to x$, then

$$\varphi(x) \le \liminf_{n\to\infty} \varphi(x_n). \tag{5.6.2}$$

However, in a general topological space, the lower semicontinuity implies the above relation but the converse is not true. A function which satisfies the above relation for all convergent sequences is called *sequentially lower semicontinuous*. In the context of the weak topology, we can thus say that $\varphi : V \to \mathbb{R}$ is *weakly sequentially lower semicontinuous* if whenever $x_n \rightharpoonup x$ in V, we have that (5.6.2) is true. Thus, the preceding proposition holds even when φ is only weakly sequentially lower semicontinuous, since the coercivity condition would imply that every minimizing sequence is bounded and the rest of the proof follows as before. ■

The following result is an immediate consequence of the preceding proposition.

Theorem 5.6.1 *Let V be a reflexive Banach space and let K be a closed convex subset of V. Then, for any $x \in V$, there exists $y \in K$ such that*

$$\|x - y\| = \min_{z\in K} \|x - z\|. \tag{5.6.3}$$

Further, if V is also uniformly convex, then such a y is unique.

Proof The functional $z \mapsto \|x - z\|$ is clearly coercive, convex and weakly lower semicontinuous. Thus the existence of y follows from the preceding proposition. Assume that V is uniformly convex. Assume that there exist $y_i \in K$, $i = 1, 2$ such that

$$\alpha = \|x - y_1\| = \|x - y_2\| = \min_{z\in K} \|x - z\|.$$

Let us assume that $\|y_1 - y_2\| > \varepsilon > 0$. Then

$$\|(x - y_1) - (x - y_2)\| > \varepsilon.$$

Then, by the uniform convexity, we have that

$$\left\| x - \frac{1}{2}(y_1 + y_2) \right\| = \left\| \frac{1}{2}[(x - y_1) + (x - y_2)] \right\| < \alpha(1 - \delta) < \alpha$$

for some $\delta > 0$. Since K is convex, we have $\frac{1}{2}(y_1 + y_2) \in K$ and so the above relation contradicts the minimality of y_1 and y_2. Thus it follows that $y_1 = y_2$, and the proof of the uniqueness is complete. ■

Example 5.6.1 In general we have non-uniqueness of the minimizer for the problem (5.6.3) if the space is not uniformly convex. Consider the space ℓ_1^2 (which is $\mathbb{R}^2$ with the norm $\|.\|_1$). It is reflexive since it is finite dimensional, but it is not uniformly convex. Consider the set $K = B$, the closed unit ball in ℓ_1^2, which is a closed and convex set. Let $x = (1, 1)$. Let $y = (a, b) \in K$. Then,

$$\begin{aligned} \|x - y\|_1 &= |1 - a| + |1 - b| \\ &\geq 1 - |a| + 1 - |b| \\ &= 2 - (|a| + |b|) \\ &\geq 1. \end{aligned}$$

However, if $a + b = 1,\ a \geq 0,\ b \geq 0$, then, for all such points $y = (a, b)$, we have

$$\|x - y\|_1 = 1 - a + 1 - b = 1.$$

Thus we have uncountably many y which satisfy (5.6.3). ■

If V were not reflexive, then we can guarantee neither the existence, nor the uniqueness.

Example 5.6.2 Let V be a non-reflexive Banach space. Let $f \in V^*$, $f \neq 0$ such that $\|f\|$ is not attained on the unit sphere (cf. Examples 3.2.2, 3.2.5, 3.2.6 and Exercise 3.11). Without loss of generality, we can assume that $\|f\| = 1$. Now define

$$K = \{z \in V \mid f(z) = 1\}.$$

By the continuity and linearity of f, it follows immediately that K is a closed convex set. Let us take $x = 0$. If $z \in K$, then

$$1 = f(z) \leq \|f\| \, \|z\|.$$

Thus $\|z\| \geq 1$ for all $z \in K$. Now, there exists a sequence $\{x_n\}$ such that $\|x_n\| = 1$ for all n and such that $f(x_n) \to \|f\| = 1$. Thus $f(x_n)$ is non-zero for large enough n. Define $z_n = x_n/f(x_n)$ so that $z_n \in K$. Further $\|z_n\| \to 1$. Thus

$$\inf_{z \in K} \|z\| = 1.$$

But there is no $z \in K$ such that $\|z\| = 1$ since $\|f\|$ is not attained on the unit sphere. ■

Remark 5.6.3 We have already quoted (cf. Remark 3.2.1) a theorem of James which assures us that a space is reflexive if, and only if, every continuous linear functional attains its norm on the unit sphere. Hence, if V is non-reflexive, there always exist continuous linear functionals which fail to attain their norm on the unit sphere. Thus, in view of James' theorem and the preceding example, we deduce that a Banach space V is reflexive if, and only if, for every closed convex set $K \subset V$, and for every point $x \in V$, there exists $y \in K$ such that (5.6.3) is true. ■

A particular case of this optimization problem is that when $K = W$, a closed subspace of V. In that case, given $x \in V$ and $y \in W$ such that (5.6.3) is true, it follows that $\|x + W\|_{V/W} = \|x - y\|_V$. We have seen earlier (cf. Exercise 2.43) that such a y exists for every $x \in V$ if W is finite dimensional. It follows from our discussion above that such a y will exist for every $x \in V$ when V is reflexive and W is any closed subspace thereof. Our next example shows that there exist closed subspaces of non-reflexive spaces for which this fails to happen.

Example 5.6.3 Let V be a non-reflexive Banach space and let f be a non-zero continuous linear functional on V. Let W be the null space of f. Assume that for some $x_0 \in V \backslash W$, there exists $w \in W$ such that

$$\|x_0 - w\| = \min_{z \in W} \|x_0 - z\| = \|x_0 + W\|_{V/W}.$$

Then (cf. Exercise 2.44)

$$\|f\| = \frac{|f(x_0)|}{\|x_0 + W\|_{V/W}} = \frac{|f(x_0 - w)|}{\|x_0 - w\|},$$

which shows that f attaines its norm for the unit vector $(x_0 - w)/\|x_0 - w\|$. Thus, if W is the null space of a continuous linear functional whose norm is not attained on the unit sphere, and if x_0 is point not in that subspace, such a w cannot exist. ■

5.7 Exercises

5.1 Let V be a Banach space and let W be a closed subspace of V. Show that the weak topology on W is the topology induced on W by the weak topology on V.

5.2 Let V be a Banach space and let W be a subspace of V. Show that the closure of W under the weak topology coincides with $\overline{W}$, the closure of W in the norm topology.

5.3 Let V be a Banach space and let W be a subspace of V. Show that $W^\perp \subset V^*$ is weak* closed.

5.4 Let V be a Banach space. Show that V with its weak topology and V^* with its weak* topology are both locally convex topological vector spaces (cf. Remark 3.3.2).

5.5 Use the preceding exercise and Remark 3.3.2 to show that if V is a Banach space and if W is a subspace of V^*, then the weak* closure of W is $(W^\perp)^\perp$. (Compare this with Exercise 3.9.)

5.6 Let V be a Banach space.
(a) Show that $x_n \rightharpoonup x$ in V if, and only if,
(i) $\{\|x_n\|\}$ is bounded and (ii) $f(x_n) \to f(x)$ for all $f \in \mathcal{S}$, where $\mathcal{S}$ is a subset of V^* whose span is dense in V^*.
(b) Show that $f_n \overset{*}{\rightharpoonup} f$ in V^* if, and only if,
(i) $\{\|f_n\|\}$ is bounded and (ii) $f_n(x) \to f(x)$ for all $x \in \mathcal{S}$, where $\mathcal{S}$ is a subset of V whose span is dense in V.

5.7 Let $1 < p < \infty$. Let $x_n = (x_n^j)$, $1 \le n < \infty$, and $x = (x^j)$ be elements of ℓ_p. Show that $x_n \rightharpoonup x$ in ℓ_p if, and only if, the sequence $\{x_n\}$ is bounded and $x_n^j \to x^j$ for every positive integer j.

5.8 Let V be a Banach space and let $W \subset V$ be a closed subspace. Let $\{x_n\}$ be a sequence in W. Show that the sequence converges weakly in W if, and only if, it converges weakly in V.

5.9 Is the sequence $\{\mathsf{e}_n\}$ weakly convergent in c_0? Is it weakly convergent in ℓ_∞?

5.10 Let $x_n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, 0, \ldots)$. In which of the spaces ℓ_p, $1 \le p \le \infty$, does this sequence converge weakly?

5.11 Let $x_n = (0, \ldots, 0, 1, 1, 1, \ldots)$, where the first entry equal to unity is in the n-th place. Does the sequence $\{x_n\}$ converge weakly in c?

5.12 Let V be a Banach space and let $x_n \rightharpoonup x$ in V. If $\|x_n\| \le \|x\|$ for every n, show that $\|x_n\| \to \|x\|$.

5.13 Let $x_n = \mathsf{e}_n - \mathsf{e}_1 \in c_0$. Show that $\{x_n\}$ converges weakly in c_0 and that $\{\|x_n\|\}$ converges to the norm of the weak limit. Show also that $\{x_n\}$ does not converge in norm.

5.14 Show that if $x_n \rightharpoonup x$ in ℓ_2 and if $\|x_n\|_2 \to \|x\|_2$, then $x_n \to x$ in ℓ_2.

5.15 Let V and W be Banach spaces and let $T : V \to W$ be a linear map. Show that the following are equivalent:
(i) If $x_n \to x$ in V, then $T(x_n) \to T(x)$ in W.
(ii) If $x_n \rightharpoonup x$ in V, then $T(x_n) \rightharpoonup T(x)$ in W.
(iii) If $x_n \to x$ in V, then $T(x_n) \rightharpoonup T(x)$ in W.

5.16 Let V and W be Banach spaces. Let V be provided with the norm topology and W with the weak topology. Let $T : V \to W$ be a linear map. Show that T is continuous if, and only if, $T \in \mathcal{L}(V, W)$. What happens when the topologies are interchanged?

5.17 Let V be a Banach space. Show that every weak* open neighbourhood of the origin in V^* is unbounded.

5.18 Let $\{x_n\}$ be the sequence defined in Exercise 5.10. Is it weak* convergent in ℓ_1?

5.19 Show that $\mathsf{e}_n \overset{*}{\rightharpoonup} \mathbf{0}$ in all the spaces ℓ_p, $1 \leq p \leq \infty$.

5.20 Let $\{x_n\}$ be the sequence defined in Exercise 5.11. Is it weak* convergent in ℓ_∞?

5.21 Let V be a reflexive space and let $T \in \mathcal{L}(V, \ell_1)$ be surjective. Show that it does not admit a right inverse.

5.22 Define $T : c_0 \to \ell_1$ as follows: let $x = (x_n) \in c_0$ and

$$T(x) = \left(\frac{x_n}{n^2}\right).$$

Show that $T \in \mathcal{L}(c_0, \ell_1)$. If B is the closed unit ball in c_0, show that $T(B)$ is not closed in ℓ_1.

5.23 Let V be a reflexive Banach space. If W is a Banach space and if $T \in \mathcal{L}(V, W)$, show that $T(B)$ is closed in W, where B is the closed unit ball in V.

5.24 Show that ℓ_1 does not contain an infinite dimensional subspace that is reflexive.

5.25 Let $1 \leq p < \infty$. Show that ℓ_p is separable.

5.26 Show that c_0 is separable.

5.27 Show that $\mathcal{C}[0, 1]$ is separable.

5.28 Let V be a reflexive real Banach space and let $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a continuous bilinear form (cf. Example 4.7.5). Assume that $a(\cdot, \cdot)$ is V-*elliptic* (or, *coercive*); i.e. there exists $\alpha > 0$ such that, for all $x \in V$,

$$a(x, x) \geq \alpha \|x\|^2.$$

Let $x \in V$. Define $A(x) : V \to \mathbb{R}$ by

$$A(x)(y) = a(x, y).$$

(a) Show that $A(x) \in V^*$ for every $x \in V$.
(b) Show that $A \in \mathcal{L}(V, V^*)$.
(c) Show that for every $x \in V$,

$$\|A(x)\| \geq \alpha\|x\|.$$

(d) Show that $A : V \to V^*$ is surjective.
(e) Deduce that, for every $f \in V^*$, there exists a unique $x \in V$ such that

$$a(x, y) = f(y) \tag{5.7.1}$$

for all $y \in V$.

5.29 In the preceding exercise, assume further that $a(\cdot, \cdot)$ is symmetric i.e. $a(x, y) = a(y, x)$ for all x and $y \in V$. Let $f \in V^*$. Define, for $x \in V$,

$$J(x) = \frac{1}{2}a(x, x) - f(x).$$

(a) For any closed convex subset $K \subset V$, show that there exists $x \in K$ such that

$$J(x) = \min_{y \in K} J(y). \tag{5.7.2}$$

(b) Show that $x \in K$ satisfies (5.7.2) if, and only if,

$$a(x, y - x) \geq f(y - x) \tag{5.7.3}$$

for every $y \in K$.
(c) Show that the solution $x \in K$ of (5.7.3) (and hence, that of (5.7.2)) is unique.
(d) If K is a closed convex cone (cf. Definition 3.5.1), show that the solution $x \in K$ of (5.7.3) is characterized by

$$a(x, x) = f(x) \text{ and } a(x, y) \geq f(y)$$

for all $y \in K$.
(e) If $K = V$, show that the solution x of (5.7.3) is the solution of (5.7.1).

Chapter 6
L^p Spaces

6.1 Basic Properties

The Lebesgue spaces, also known as the L^p spaces, constitute a rich source of examples and counterexamples in functional analysis. They also form an important class of function spaces when studying the applications of mathematical analysis. In this chapter, we will study the important properties of these spaces from the functional analytic point of view. Let $(X, \mathcal{S}, \mu)$ be a measure space (cf. Sect. 1.3).

Let $f: X \to \mathbb{R}$ be a real-valued measurable function defined on X. Let $1 \leq p < \infty$. We define

$$\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \tag{6.1.1}$$

and we say that f is p-**integrable** (integrable, if $p = 1$ and square integrable, if $p = 2$) if $\|f\|_p < \infty$. Next, let $M > 0$. We set

$$\{|f| > M\} = \{x \in X \mid |f(x)| > M\}.$$

We now define

$$\|f\|_\infty = \inf\{M > 0 \mid \mu(\{|f| > M\}) = 0\} \tag{6.1.2}$$

and we say that f is *essentially bounded* if $\|f\|_\infty < \infty$.

Proposition 6.1.1 (Hölder's Inequality) *Let $1 \leq p < \infty$ and let p^* be the conjugate exponent. If f is p-integrable and g is p^*-integrable (essentially bounded, if $p = 1$), then*

$$\int_X |fg| \, d\mu \leq \|f\|_p \|g\|_{p^*}. \tag{6.1.3}$$

S. Kesavan, *Functional Analysis*, Texts and Readings in Mathematics,
https://doi.org/10.1007/978-981-19-7633-9_6

Proof If $p = 1$, then $p^* = \infty$. Then

$$|f(x)g(x)| \leq |f(x)|.\|g\|_\infty$$

for almost every $x \in X$ and then (6.1.3) follows on integrating this inequality over X.

Let us now assume that $1 < p < \infty$ so that $1 < p^* < \infty$ as well. The relation (6.1.3) is trivially true if $\|f\|_p$ (respectively, $\|g\|_{p^*}$) equals zero, for then f (respectively, g) will be equal to zero almost everywhere. So we assume further that $\|f\|_p \neq 0$ and that $\|g\|_{p^*} \neq 0$. Then (cf. Lemma 2.2.1)

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{p^*}|g(x)|^{p^*}$$

for all $x \in X$. Assume now that $\|f\|_p = \|g\|_{p^*} = 1$. Then, integrating the above inequality over X, we get

$$\int_X |fg|\, d\mu \leq \frac{1}{p} + \frac{1}{p^*} = 1.$$

For the general case, apply the preceding result to the functions $f/\|f\|_p$ and $g/\|g\|_{p^*}$ to get (6.1.3). ■

Remark 6.1.1 When $p = p^* = 2$, once again (6.1.3) is known as the **Cauchy-Schwarz inequality**. ■

Proposition 6.1.2 (Minkowski's Inequality) *Let* $1 \leq p \leq \infty$. *Let* f *and* g *be* p-*integrable. Then* $f + g$ *is also* p-*integrable and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{6.1.4}$$

Proof We assume that $\|f + g\|_p \neq 0$, since, otherwise, the result is trivially true. Since the function $t \mapsto |t|^p$ is convex for $1 \leq p < \infty$, we have that

$$|f(x) + g(x)|^p \leq 2^{p-1}(|f(x)|^p + |g(x)|^p)$$

from which it follows that $f + g$ is also p-integrable. Thus, if $1 < p < \infty$, we have

$$\int_X |f + g|^p\, d\mu \leq \int_X |f + g|^{p-1}|f|\, d\mu + \int_X |f + g|^{p-1}|g|\, d\mu.$$

We apply Hölder's inequality to each of the terms on the right-hand side. Notice that $|f(x) + g(x)|^{(p-1)p^*} = |f(x) + g(x)|^p$ by the definition of p^*. Thus $|f + g|^{p-1}$ is p^*-integrable and

$$\|\,|f + g|^{p-1}\|_{p^*} = \|f + g\|_p^{\frac{p}{p^*}}.$$

Thus,

$$\|f+g\|_p^p \le \|f+g\|_p^{\frac{p}{p^*}}(\|f\|_p + \|g\|_p).$$

Dividing both sides by $\|f+g\|_p^{\frac{p}{p^*}}$ and using, once again, the definition of p^*, we get (6.1.4). The cases where $p = 1$ and $p = \infty$ follow trivially from the inequality

$$|f(x)+g(x)| \le |f(x)|+|g(x)|.$$

This completes the proof. ■

It is now easy to see that the space of all p-integrable functions ($1 \le p < \infty$) and that of all essentially bounded functions are vector spaces and that the map $f \mapsto \|f\|_p$ for $1 \le p \le \infty$ verifies all the properties of the norm, except that $\|f\|_p = 0$ *does not* imply that $f = 0$, but that $f = 0$ *almost everywhere* (a.e.; cf. Sect. 1.3).

Given two measurable functions f and g, we say that $f \sim g$ if $f = g$ almost everywhere, *i.e.* $f(x) = g(x)$ everywhere, except over a subset of measure zero. This defines an equivalence relation. If $f \sim g$, then for $1 \le p \le \infty$, we have that $\|f\|_p = \|g\|_p$. Further the set of all equivalence classes forms a vector space with respect to pointwise addition and scalar multiplication defined via arbitrary representatives of equivalence classes. In other words, if $f_1 \sim f_2$ and if $g_1 \sim g_2$, then $f_1 + g_1 \sim f_2 + g_2$ and, for any scalar α, we also have $\alpha f_1 \sim \alpha f_2$ and so on. Since $\|\cdot\|_p$ is also constant on any equivalence class, we can define the 'norm' of an equivalence class via any representative function of that class. Further, if $\|f\|_p = 0$, then f will belong to the equivalence class of the function which is identically zero. Thus the set of all equivalence classes, with $\|\cdot\|_p$, becomes a normed linear space.

Definition 6.1.1 Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $1 \le p < \infty$. The space of all equivalence classes, under the equivalence relation defined by equality of functions almost everywhere, of all p-integrable functions is a normed linear space with the norm of an equivalence class being the $\|\cdot\|_p$-'norm' of any representative of that class. This space is denoted $L^p(\mu)$. The space of all equivalence classes of all essentially bounded functions with the norm of an equivalence class being defined as the $\|\cdot\|_\infty$-'norm' of any representative of that class is denoted $L^\infty(\mu)$. ■

While we may often talk of 'L^p-functions' we must keep in mind that we are really talking about equivalence classes of functions and that we carry out computations via representatives of those equivalence classes.

Notation We will denote elements of $L^p(\mu)$ by lowercase Roman letters in sanserif font and a generic representative of the equivalence class it represents by the same lowercase Roman letter in italicized font. Thus if we have $\mathsf{f} \in L^p(\mu)$, a generic representative will be f and so, for instance,

$$\|\mathsf{f}\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$.
Similarly, the equivalence class of a function f will be denoted by f.

Notation If $X = \Omega$, an open set of $\mathbb{R}^n$ provided with the Lebesgue measure, then the corresponding spaces $L^p(\mu)$ will be denoted $L^p(\Omega)$. In particular, if $\mathbb{R}$ is provided with the Lebesgue measure and if (a, b) is an interval, where $-\infty \leq a < b \leq +\infty$, then the L^p spaces on (a, b) will be denoted $L^p(a, b)$.

Example 6.1.1 Let $X = \{1, 2, \ldots, n\}$. Let $\mathcal{S}$ be the collection of all subsets of X and let μ be the counting measure (cf. Example 1.3.1). Then a measurable function can be identified with an n-tuple $(a_1, a_2, \ldots, a_n)$. In this case $L^p(\mu) = \ell_p^n$. Notice that in this example, equality almost everywhere is the same as equality everywhere and so every equivalence class is a singleton. ■

Example 6.1.2 Let $X = \mathbb{N}$, the set of all natural numbers and let $\mathcal{S}$ be the collection of all subsets of X. Let μ be the counting measure. In this case, functions are identified with real sequences and $L^p(\mu) = \ell_p$. Again, in this example, equivalence classes are just singletons. ■

Proposition 6.1.3 *Let $(X, \mathcal{S}, \mu)$ be a finite measure space, i.e. $\mu(X) < \infty$. Then*

$$L^p(\mu) \subset L^q(\mu)$$

with the inclusion being continuous, whenever $1 \leq q \leq p$.

Proof The result is trivial if $p = \infty$. Let $1 \leq q < p < \infty$ and let $\mathsf{f} \in L^p(\mu)$. Then, by Hölder's inequality, we have

$$\begin{aligned}
\int |f|^q \, d\mu &\leq \left(\int_X (|f|^q)^{\frac{p}{q}} \, d\mu \right)^{\frac{q}{p}} \left(\int_X d\mu \right)^{1-\frac{q}{p}} \\
&= \left(\int_X |f|^p \, d\mu \right)^{\frac{q}{p}} (\mu(X))^{1-\frac{q}{p}} \\
&= \|\mathsf{f}\|_p^q . (\mu(X))^{1-\frac{q}{p}}
\end{aligned}$$

which yields

$$\|\mathsf{f}\|_q \leq C \; \|\mathsf{f}\|_p$$

where

$$C = (\mu(X))^{\frac{1}{q}-\frac{1}{p}}.$$

This completes the proof. ■

Example 6.1.3 No such inclusions hold in infinite measure spaces. For instance, the sequence $(\frac{1}{n})$ belongs to ℓ_2 but not to ℓ_1. ■

Example 6.1.4 Nothing can be said about the reverse inclusions. For example, if $f(x) = 1/\sqrt{x}$, then $\mathsf{f} \in L^1(0, 1)$ but $\mathsf{f} \notin L^2(0, 1)$. However, we know that (cf. Exercise 2.32) $\ell_p \subset \ell_q$ for all $1 \leq p < q \leq \infty$. ■

Theorem 6.1.1 *Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $1 \leq p \leq \infty$. Then $L^p(\mu)$ is a Banach space.*

Proof Case 1. Let $1 \leq p < \infty$. Let $\{\mathsf{f}_n\}$ be a Cauchy sequence in $L^p(\mu)$. It is enough to show that there exists a convergent subsequence (why?). Choose a subsequence such that

$$\|f_{n_k} - f_{n_{k+1}}\|_p \leq \frac{1}{2^k}.$$

Set

$$g_n(x) = \sum_{k=1}^{n} |f_{n_{k+1}}(x) - f_{n_k}(x)|$$

and

$$g(x) = \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

Then

$$\|g_n\|_p \leq 1.$$

It follows that $g_n(x) \to g(x)$ and, by the monotone convergence theorem (cf. Theorem 1.3.1), we see that $\|g\|_p \leq 1$. In particular, $g(x) < \infty$ almost everywhere. Further, if $k \geq l \geq 2$, we have

$$\begin{aligned} |f_{n_k}(x) - f_{n_l}(x)| &\leq |f_{n_k}(x) - f_{n_{k-1}}(x)| + \ldots + |f_{n_{l+1}}(x) - f_{n_l}(x)| \\ &\leq g(x) - g_{l-1}(x). \end{aligned}$$

Thus, it follows that, for almost every $x \in X$, $\{f_{n_k}(x)\}$ is a Cauchy sequence and converges almost everywhere to a finite limit $f(x)$ and that, for such x,

$$|f(x) - f_{n_k}(x)| \leq g(x)$$

for $k \geq 2$. Set $f = 0$ elsewhere, which is a set of measure zero. It then follows that f is p-integrable. Further, $|f_{n_k}(x) - f(x)|^p \to 0$ almost everywhere and is bounded by $|g(x)|^p$ which is integrable. Hence, by the dominated convergence theorem (cf. Theorem 1.3.3), we deduce that $\|f_{n_k} - f\|_p \to 0$. Thus we have that

$$\mathsf{f}_{n_k} \to \mathsf{f}$$

in $L^p(\mu)$.

Case 2. $p = \infty$. Let $\{\mathsf{f}_n\}$ be Cauchy in $L^\infty(\mu)$. Then, for each k, there exists a positive integer N_k such that

$$\|\mathsf{f}_m - \mathsf{f}_n\|_\infty < \frac{1}{k}$$

for all $m, n \geq N_k$. Thus, there exists a set E_k of measure zero, such that

$$|f_m(x) - f_n(x)| \leq \frac{1}{k}$$

for all $m, n \geq N_k$ and for all $x \in X \backslash E_k$. Setting $E = \cup_{k=1}^\infty E_k$, we see that E is of measure zero and for all $x \in X \backslash E$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Thus, for all such x, $f_n(x) \to f(x)$. Passing to the limit as $m \to \infty$, we see that, for all $x \in X \backslash E$, and for all $n \geq N_k$,

$$|f(x) - f_n(x)| \leq \frac{1}{k}.$$

Hence, it follows that f is essentially bounded and that $\mathsf{f_n} \to \mathsf{f}$ in $L^\infty(\mu)$.

This completes the proof. ■

Corollary 6.1.1 *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $\mathsf{f}_n \to \mathsf{f}$ in $L^p(\mu)$ for some $1 \leq p \leq \infty$. Then, there exists a subsequence f_{n_k} such that*
(i) *$f_{n_k}(x) \to f(x)$ almost everywhere.*
(ii) *$|f_{n_k}(x)| \leq h(x)$ almost everywhere for some $\mathsf{h} \in L^p(\mu)$.*

Proof The result is obvious in the case $p = \infty$. Let $1 \leq p < \infty$. Then, as in the case of the preceding theorem, we have a subsequence $\{\mathsf{f}_{n_k}\}$ which converges to a function $\widetilde{\mathsf{f}}$ in $L^p(\mu)$and also such that $f_{n_k}(x) \to \widetilde{f}(x)$ almost everywhere. It is then clear that $\widetilde{\mathsf{f}} = \mathsf{f}$, *i.e.* $\widetilde{f} = f$ almost everywhere, and this proves (i). To see (ii), take $h = \widetilde{f} + g$, where g is as in the proof of the preceding theorem. ■

6.2 Duals of L^p Spaces

In Chap. 3, we identified the dual of the space ℓ_p with ℓ_{p^*} where $1 \leq p < \infty$ and p^* is the conjugate exponent of p. Similar results are true in more general L^p spaces.

Proposition 6.2.1 (Clarkson's Inequality) *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $2 \le p < \infty$. Then if f and $\mathsf{g} \in L^p(\mu)$,*

$$\left\| \frac{1}{2}(\mathsf{f}+\mathsf{g}) \right\|_p^p + \left\| \frac{1}{2}(\mathsf{f}-\mathsf{g}) \right\|_p^p \le \frac{1}{2}\left(\|\mathsf{f}\|_p^p + \|\mathsf{g}\|_p^p \right). \tag{6.2.1}$$

Proof Consider the function

$$\varphi(x) = (x^2+1)^{\frac{p}{2}} - x^p - 1$$

for $x \ge 0$. Then it is simple to check that $\varphi(0) = 0$ and that $\varphi'(x) > 0$ for $x > 0$ when $p \ge 2$. Thus, it follows that for all $x \ge 0$,

$$(x^2+1)^{\frac{p}{2}} \ge x^p + 1,$$

when $p \ge 2$. Hence, if α and β are positive real numbers, we have

$$(\alpha^2+\beta^2)^{\frac{p}{2}} \ge \alpha^p + \beta^p.$$

Combining this with the fact that the function $t \mapsto t^{\frac{p}{2}}$ is convex on the set $\{t \in \mathbb{R} \mid t \ge 0\}$, we get, for any $x \in X$ and for any f and $\mathsf{g} \in L^p(\mu)$,

$$\begin{aligned} \left|\tfrac{f(x)+g(x)}{2}\right|^p + \left|\tfrac{f(x)-g(x)}{2}\right|^p &\le \left(\left|\tfrac{f(x)+g(x)}{2}\right|^2 + \left|\tfrac{f(x)-g(x)}{2}\right|^2 \right)^{\frac{p}{2}} \\ &= \left(\tfrac{|f(x)|^2+|g(x)|^2}{2} \right)^{\frac{p}{2}} \\ &\le \tfrac{1}{2}(|f(x)|^p + |g(x)|^p) \end{aligned}$$

which yields (6.2.1) on integration over X. ■

Corollary 6.2.1 *Let $(X, \mathcal{S}, \mu)$ be a measure space. Then, the spaces $L^p(\mu)$ are reflexive when $2 \le p < \infty$.*

Proof Arguing as in Example 5.5.2, it is easy to see that (6.2.1) implies that $L^p(\mu)$ is uniformly convex when $2 \le p < \infty$. The reflexivity now follows from Theorem 5.5.1. ■

Theorem 6.2.1 (Riesz Representation Theorem) *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $1 < p < \infty$. Let p^* be the conjugate exponent. Then the dual of $L^p(\mu)$ is isometrically isomorphic to $L^{p^*}(\mu)$. In particular, the spaces $L^p(\mu)$ are reflexive for all $1 < p < \infty$.*

Proof Step 1. Let $\mathsf{g} \in L^{p^*}(\mu)$. Define $T_{\mathsf{g}}: L^p(\mu) \to \mathbb{R}$ by

$$T_{\mathsf{g}}(\mathsf{f}) = \int_X fg \, \mathrm{d}\mu$$

for $\mathsf{f} \in L^p(\mu)$. Clearly, T_{g} is a linear functional, and, by Hölder's inequality, it is continuous as well. In fact, we have

$$\|T_{\mathsf{g}}\| \leq \|\mathsf{g}\|_{p^*}.$$

Now, consider the function

$$f(x) = \begin{cases} |g(x)|^{p^*-2}g(x), & \text{if } g(x) \neq 0 \\ 0, & \text{if } g(x) = 0. \end{cases}$$

Then $|f|^p = |g|^{(p^*-1)p} = |g|^{p^*}$ so that f is p-integrable. Also

$$T_{\mathsf{g}}(f) = \int_X |g|^{p^*} \, \mathrm{d}\mu$$

from which we deduce that

$$\|T_{\mathsf{g}}\| = \|\mathsf{g}\|_{p^*}.$$

Thus, the map $\mathsf{g} \mapsto T_{\mathsf{g}}$ is an isometry from $L^{p^*}(\mu)$ into $L^p(\mu)^*$. Hence its image is closed. It is enough now to show that the image is dense.

Step 2. We now show that $L^p(\mu)$ is reflexive for all $1 < p < \infty$. This has already been proved for $2 \leq p < \infty$. Thus $L^p(\mu)^*$ is also reflexive for such p and so is every closed subspace of this dual space. Thus, by the preceding step, $L^{p^*}(\mu)$ which is isometrically isomorphic to a closed subspace of the dual of $L^p(\mu)$ is also reflexive for $2 \leq p < \infty$. But then $1 < p^* \leq 2$. This proves that $L^p(\mu)$ is also reflexive when $1 < p \leq 2$. This establishes the claim.

Step 3. We are now in a position to show that the isometry $\mathsf{g} \mapsto T_{\mathsf{g}}$ from $L^{p^*}(\mu)$ to the dual space $L^p(\mu)^*$ is onto. As already observed, the image is a closed subspace and we now show that it is dense. Indeed, let $\varphi \in L^p(\mu)^{**}$ vanish on the image. We need to show that φ is the zero functional. Since all the $L^p(\mu)$ are reflexive, this means that there exists $\mathsf{f} \in L^p(\mu)$ such that, for all $\mathsf{g} \in L^{p^*}(\mu)$, we have

$$\int_X fg \, \mathrm{d}\mu = 0.$$

Once again, choosing $g = |f|^{p-2}f$ (and equal to zero where f vanishes) we deduce that $\mathsf{f} = \mathsf{0}$. This completes the proof. ■

Remark 6.2.1 We have seen earlier that $\ell_1^* = \ell_\infty$. In the same way, it is true that for σ-finite measure spaces, we have $L^1(\mu)^* = L^\infty(\mu)$. However, the proof of this result relies on very measure theoretic arguments and we shall omit it. Nevertheless, in the next section, we will prove it for a very important class of L^1 spaces. ■

Remark 6.2.2 The measure theoretic proof mentioned in the preceding remark covers completely the cases $1 \leq p < \infty$. However, it works only for σ-finite measure spaces. The proof that we have presented here shows that the hypothesis of σ-finiteness is not necessary for the cases $1 < p < \infty$, for the Riesz representation theorem to hold. ∎

6.3 The Spaces $L^p(\Omega)$

In this section, we will study the properties of a very important class of L^p spaces defined on open sets in the Euclidean spaces $\mathbb{R}^N$.

Let $\Omega \subset \mathbb{R}^N$ be an open set. Consider the Lebesgue measure on this set. Then, as mentioned in Sect. 6.1, we will denote the corresponding L^p spaces by $L^p(\Omega)$.

In the sequel, if we say that a certain function space is contained in (respectively, is dense in), $L^p(\Omega)$, we will understand that we are talking about the set of all equivalence classes of functions in that space being contained in (respectively, being dense in) $L^p(\Omega)$.

Proposition 6.3.1 *Let S be the set of all simple functions which vanish outside a set of finite measure. Then S is dense in $L^p(\Omega)$ for $1 \leq p < \infty$.*

Proof Let $\varphi \in S$. Since φ vanishes outside a set of finite measure, it is automatically p-integrable for $1 \leq p < \infty$. Let $f \geq 0$ be a p-integrable function. Then, there exists a sequence $\{\varphi_n\}$ of non-negative simple functions which increase to f (cf. Proposition 1.3.2). Since f is p-integrable, so is φ_n and so φ_n will also vanish outside a set of finite measure. Further

$$|\varphi_n(x) - f(x)|^p \leq 2^p |f(x)|^p$$

for $x \in \Omega$ and, since $|f|^p$ is integrable, it follows from the dominated convergence theorem that

$$\int_\Omega |\varphi_n - f|^p \, \mathrm{d}x \ \to \ 0$$

as $n \to \infty$. If f is an arbitrary p-integrable function, then we have sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of simple functions vanishing outside sets of finite measure and such that

$$\int_\Omega |\varphi_n - f^+|^p \, \mathrm{d}x \ \to \ 0 \text{ and } \int_\Omega |\psi_n - f^-|^p \, \mathrm{d}x \ \to \ 0.$$

Thus $\chi_n = \varphi_n - \psi_n$ is a simple function which vanishes outside a set of finite measure and

$$\int_\Omega |\chi_n - f|^p \, \mathrm{d}x \ \to \ 0$$

as $n \to \infty$. This proves the result. ■

Theorem 6.3.1 *Let* $1 \leq p < \infty$. *Let* $\Omega \subset \mathbb{R}^N$ *be open. Then,* $\mathcal{C}_c(\Omega)$*, the space of all continuous functions with compact support contained in* Ω*, is dense in* $L^p(\Omega)$.

Proof By the preceding proposition, we know that S is dense in $L^p(\Omega)$. Thus, given $\varphi \in S$, it is enough to show that it can be approximated (in the L^p-norm) as closely as we wish by a continuous function with compact support. Indeed, let $\varepsilon > 0$. By Lusin's theorem (cf. Royden [1]), we can find a continuous function g, with compact support, such that $g = \varphi$ except possibly on a set whose measure is less than ε and also such that $|g(x)| \leq \|\varphi\|_\infty$. Then

$$\int_\Omega |g - \varphi|^p \, \mathrm{d}x \ \leq \ 2^p \|\varphi\|_\infty^p \varepsilon.$$

This shows that $\mathcal{C}_c(\Omega)$ is dense (with respect to the norm $\|.\|_p$) in S which, in turn, is dense in $L^p(\Omega)$. This proves the result. ■

Remark 6.3.1 In fact it can be shown that the space of infinitely differentiable functions with compact support contained in Ω is dense in $L^p(\Omega)$ for $1 \leq p < \infty$. For this we need to develop the theory of convolution of functions (cf. Theorem 6.3.3 below). For details, see Kesavan [2]. ■

Corollary 6.3.1 *Let* $\Omega \subset \mathbb{R}^N$ *be an open set. Let* $1 \leq p < \infty$. *Then,* $L^p(\Omega)$ *is separable.*

Proof Recall that, by the Weierstrass approximation theorem, a continuous functiuon on a compact set can be uniformly approximated by means of a polynomial and hence, by a polynomial with *rational* coefficients and such polynomials form a countable set.

We can write

$$\Omega = \cup_{n=1}^\infty \Omega_n$$

where $\Omega_n = \Omega \cap B(\mathbf{0}; n)$; here $B(\mathbf{0}; n)$ is the ball centred at the origin and with radius n in $\mathbb{R}^N$. Notice that Ω_n is bounded and is hence relatively compact; *i.e.* $\overline{\Omega_n}$ is compact.

Let $\varepsilon > 0$ and let f be p-integrable over Ω. Then, by the preceding theorem, we can find a continuous function g, with compact support such that $\|f - g\|_p < \varepsilon$. Since the support of g is compact, its support will lie in some Ω_n. Thus, we can find a polynomial p with rational coefficients such that, for all $x \in \Omega_n$,

$$|g(x) - \mathsf{p}(x)| \ < \ \frac{\varepsilon}{|\Omega_n|^{\frac{1}{p}}}$$

where $|\Omega_n|$ denotes the (Lebesgue) measure of Ω_n. Setting $\mathsf{p} = 0$ outside Ω_n, we then see that $\|g - \mathsf{p}\|_p < \varepsilon$ so that $\|f - \mathsf{p}\|_p < 2\varepsilon$. Thus any p-integrable function can be approximated in the norm $\|\cdot\|_p$ by means of a function which vanishes outside some Ω_n and is equal to a polynomial with rational coefficients inside Ω_n. The collection of all such functions being countable, we deduce that $L^p(\Omega)$ is separable for $1 \leq p < \infty$. ■

Proposition 6.3.2 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, $L^\infty(\Omega)$ is not separable.*

Proof Let $x \in \Omega$. Let $r = r(x) > 0$ be chosen such that the ball $B(x; r) \subset \Omega$. Define

$$\chi_x(y) = \begin{cases} 1, & \text{if } y \in B(x;r) \\ 0, & \text{otherwise.} \end{cases}$$

Set

$$U_x = \{\mathsf{f} \in L^\infty(\Omega) \mid \|\mathsf{f} - \chi_x\|_\infty < 1/4\}.$$

Then, for each $x \in \Omega$, U_x is a non-empty open subset of $L^\infty(\Omega)$. If $x \neq y$, then $\|\chi_x - \chi_y\|_\infty = 1$ (why?). Hence $U_x \cap U_y = \emptyset$.

Now let $E = \{\mathsf{f}_n\}$ be any countable set in $L^\infty(\Omega)$. If such a set were dense, then $E \cap U_x \neq \emptyset$ for each $x \in \Omega$. However, any f_n can belong to at most one such open set U_x since the sets U_x are pairwise disjoint. This is a contradiction since the number of open sets U_x is uncountable. Thus, no countable set in $L^\infty(\Omega)$ can be dense. ■

Remark 6.3.2 In general, if a normed linear space contains an uncountable number of disjoint open balls, then the space cannot be separable. This idea was already used in Example 5.4.2. ■

Definition 6.3.1 Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f\colon \Omega \to \mathbb{R}$ be a measurable function. We say that f is **locally integrable** if $\int_K |f|\, \mathrm{d}x < \infty$ for every compact set $K \subset \Omega$. ■

We denote the set of all locally integrable functions on Ω by $L^1_{\mathrm{loc}}(\Omega)$.

Proposition 6.3.3 *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ be such that*

$$\int_\Omega fg\, dx = 0$$

for all $g \in \mathcal{C}_c(\Omega)$. Then $f = 0$ almost everywhere in Ω.

Proof Step 1. We first assume that f is integrable on Ω and that $|\Omega|$, the measure of Ω, is finite. Let $\varepsilon > 0$. Then, there exists a continuous function f_1, with compact support, such that $\|f - f_1\|_1 < \varepsilon$ (cf. Theorem 6.3.1). Thus, if $g \in \mathcal{C}_c(\Omega)$, we have

$$\left| \int_\Omega f_1 g\, \mathrm{d}x \right| = \left| \int_\Omega (f_1 - f) g\, \mathrm{d}x \right| \leq \varepsilon \|g\|_\infty. \tag{6.3.1}$$

Let

$$K_1 = \{x \in \Omega \mid f_1(x) \geq \varepsilon\}$$
$$K_2 = \{x \in \Omega \mid f_1(x) \leq -\varepsilon\}.$$

Then, K_1 and K_2 are disjoint and compact sets (since f_1 is a continuous function with compact support) and by Urysohn's lemma, we can construct a continuous function h, also with compact support such that $h \equiv 1$ on K_1 and $h \equiv -1$ on K_2. Further, we can also have $|h(x)| \leq 1$ for all $x \in \Omega$. Set $K = K_1 \cup K_2$. Then

$$\int_\Omega f_1 h \,\mathrm{d}x = \int_{\Omega\setminus K} f_1 h \,\mathrm{d}x + \int_K f_1 h \,\mathrm{d}x,$$

whence, in view of (6.3.1), we have

$$\int_K |f_1| \,\mathrm{d}x = \int_K f_1 h \,\mathrm{d}x \leq \varepsilon + \int_{\Omega\setminus K} |f_1 h| \,\mathrm{d}x \leq \varepsilon + \int_{\Omega\setminus K} |f_1| \,\mathrm{d}x.$$

Since $|f_1(x)| \leq \varepsilon$ on $\Omega\setminus K$, we deduce that

$$\begin{aligned}\int_\Omega |f_1| \,\mathrm{d}x &= \int_K |f_1| \,\mathrm{d}x + \int_{\Omega\setminus K} |f_1| \,\mathrm{d}x \\ &\leq \varepsilon + 2\int_{\Omega\setminus K} |f_1| \,\mathrm{d}x \\ &\leq \varepsilon + 2\varepsilon|\Omega|.\end{aligned}$$

Thus,

$$\|f\|_1 \leq \|f - f_1\|_1 + \|f_1\|_1 \leq 2\varepsilon + 2\varepsilon|\Omega|.$$

Since ε is arbitrary, it follows that $f(x) = 0$ almost everywhere in Ω.

Step 2. In the general case, we again write $\Omega = \cup_{n=1}^\infty \Omega_n$ where $\Omega_n = \Omega \cap B(\mathbf{0}; n)$. Then applying the result of Step 1 to the restriction of f to Ω_n, denoted $f|_{\Omega_n}$, we get that $f|_{\Omega_n} = 0$ almost everywhere in Ω_n from which it immediately follows that $f = 0$ almost everywhere in Ω. ■

Let us now turn our attention to the space $L^1(\Omega)$.

Theorem 6.3.2 (Riesz Representation Theorem) *Let $\Omega \subset \mathbb{R}^N$ be an open set. The dual of the space $L^1(\Omega)$ is isometrically isomorphic to $L^\infty(\Omega)$.*

Proof Step 1. There exists $\mathsf{w} \in L^2(\Omega)$ such that $w(x) \geq \varepsilon_K > 0$ for all $x \in K$ for every compact subset K of Ω. Indeed, define $w(x) = \alpha_n > 0$ on the set

$$E_n = \{x \in \Omega \mid n \leq |x| < n+1\},$$

where $|x|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^N$. Now choose the constants α_n such that

$$\sum_{n=0}^{\infty} \alpha_n^2 |E_n| \ < \ \infty,$$

where $|E_n|$ denotes the (Lebesgue) measure of the set E_n. Then w has the required properties.

Step 2. Let $\varphi \in L^1(\Omega)^*$. Consider the mapping $\mathsf{f} \mapsto \varphi(\mathsf{wf})$ from $L^2(\Omega)$ into $\mathbb{R}$. Clearly, this defines a linear functional which, by Hölder's inequality, is also continuous. Thus, by the Riesz representation theorem (cf. Theorem 6.2.1) applied to the case $p = 2$, there exists $\mathsf{v} \in L^2(\Omega)$ such that

$$\varphi(\mathsf{wf}) = \int_\Omega f v \, \mathrm{d}x$$

for all $\mathsf{f} \in L^2(\Omega)$. Thus, we have

$$\left| \int_\Omega f v \, \mathrm{d}x \right| \ \leq \ \|\varphi\| . \|wf\|_1. \tag{6.3.2}$$

Step 3. Set $u(x) = v(x)/w(x)$ for $x \in \Omega$. Since w never vanishes, this is well-defined and u is measurable. We claim that $\mathsf{u} \in L^\infty(\Omega)$ and that $\|\mathsf{u}\|_\infty \leq \|\varphi\|$. To see this it is sufficient to show that, for any constant $C > \|\varphi\|$, we have that the set

$$A = \{x \in \Omega \mid |u(x)| > C\}$$

is of measure zero.

Assume the contrary for some such $C > \|\varphi\|$.Then, there exists a subset B of A of finite and positive measure. Consider the function

$$f(x) = \begin{cases} +1, \text{ if } x \in B \text{ and } u(x) > 0, \\ -1, \text{ if } x \in B \text{ and } u(x) < 0, \\ 0, \quad \text{if} x \in \Omega \backslash B. \end{cases}$$

Clearly, f is square integrable (since the measure of B is finite) and we can use it in (6.3.2). We then get

$$\int_B |u| w \, \mathrm{d}x \ \leq \ \|\varphi\| \int_B w \, \mathrm{d}x$$

and, using the definition of A, which contains B, we get

$$C\int_B w\, \mathrm{d}x \le \|\varphi\| \int_B w\, \mathrm{d}x$$

which is a contradiction to the choice of C, since $\int_B w\, \mathrm{d}x > 0$.

Step 4. Thus we now have $u \in L^\infty(\Omega)$ with $\|u\|_\infty \le \|\varphi\|$ such that, for all $f \in L^2(\Omega)$,

$$\varphi(wf) = \int_\Omega fuw\, \mathrm{d}x.$$

Let $g \in \mathcal{C}_c(\Omega)$. Then, by choice of w, $f = g/w$ is square integrable and so, we get, for all g continuous with compact support in Ω,

$$\varphi(g) = \int_\Omega ug\, \mathrm{d}x. \tag{6.3.3}$$

Since $\mathcal{C}_c(\Omega)$ is dense in $L^1(\Omega)$, the above relation also holds for all $g \in L^1(\Omega)$. Further, it follows that

$$|\varphi(g)| \le \|g\|_1 \|u\|_\infty$$

for all $g \in L^1(\Omega)$ from which we deduce that $\|\varphi\| \le \|u\|_\infty$.

Step 5. Thus, for every $\varphi \in L^1(\Omega)^*$, we have $u \in L^\infty(\Omega)$ such that $\|\varphi\| = \|u\|_\infty$ and such that (6.3.3) holds for all $g \in L^1(\Omega)$. Such a u is unique as well. Indeed if we have two essentially bounded functions u_1 and u_2 such that

$$\int_\Omega g(u_1 - u_2)\, \mathrm{d}x = 0$$

for all $g \in L^1(\Omega)$, then it is in particular true for all $g \in \mathcal{C}_c(\Omega)$ and, since essentially bounded functions are locally integrable, it follows that (cf. Proposition 6.3.3) $u_1 - u_2 = 0$ almost everywhere, *i.e.* $u_1 = u_2$ in $L^\infty(\Omega)$.

Step 6. If $u \in L^\infty(\Omega)$, then if we define T_u as a linear functional on $L^1(\Omega)$ via the right-hand side of (6.3.3), then we have just seen that $u \mapsto T_u$ is surjective and that it is an isometry between $L^\infty(\Omega)$ and $L^1(\Omega)^*$. This completes the proof. ■

Proposition 6.3.4 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, $L^1(\Omega)$ is not reflexive.*

Proof Without loss of generality, assume that Ω contains the origin. Let n be sufficiently large so that the ball centred at the origin and of radius $1/n$, denoted B_n, is contained in Ω. Let $\alpha_n = |B_n|^{-1}$, where, as usual, $|B_n|$ denotes the (Lebesgue) measure of B_n. Let $f_n(x) = \alpha_n$ for all $x \in B_n$ and let it vanish on $\Omega \backslash B_n$. Then $f_n \in L^1(\Omega)$ and $\|f_n\|_1 = 1$ for all n. If $L^1(\Omega)$ were reflexive, then the sequence $\{f_n\}$ would contain a weakly convergent subsequence (cf. Theorem 5.4.2), say $\{f_{n_k}\}$. Let f be its weak limit. Then, for every $h \in L^\infty(\Omega)$ we must have

$$\int_\Omega f_{n_k} h \,\mathrm{d}x \;\to\; \int_\Omega f h \,\mathrm{d}x. \tag{6.3.4}$$

Now choose $h \in \mathcal{C}_c(\Omega \backslash \{\mathbf{0}\})$. Then, for sufficiently large k, we have that

$$\int_\Omega f_{n_k} h \,\mathrm{d}x = 0$$

(since the two functions in the integrand will then have disjoint supports) and so, it follows that, for all such h, we have $\int_\Omega f h \,\mathrm{d}x = 0$. By Proposition 6.3.3, it then follows that $f(x) = 0$ almost everywhere in $\Omega \backslash \{\mathbf{0}\}$ and so $f(x) = 0$ almost everywhere on Ω as well. On the other hand, if we choose $h(x) = 1$ for all $x \in \Omega$ in (6.3.4), we get $\int_\Omega f \,\mathrm{d}x = 1$, which is a contradiction. Thus, $L^1(\Omega)$ is not reflexive. ■

Corollary 6.3.2 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then, $L^\infty(\Omega)$ is not reflexive.*

Proof Since $L^1(\Omega)^* \cong L^\infty(\Omega)$, the result follows immediately from the preceding proposition (cf. Corollary 5.3.3). ■

To sum up, we have that $L^p(\Omega)^* \cong L^{p^*}(\Omega)$ for $1 \le p < \infty$. The spaces $L^p(\Omega)$ are separable for $1 \le p < \infty$ and reflexive for $1 < p < \infty$. The space $L^\infty(\Omega)$ is neither separable nor reflexive.

We conclude by proving an important inequality.

Theorem 6.3.3 (Young's Inequality) *Let $1 < p < \infty$. Let $f \in L^1(\mathbb{R}^N)$ and let $g \in L^p(\mathbb{R}^N)$. Then the map*

$$x \;\mapsto\; \int_{\mathbb{R}^N} f(y) g(x-y)\, dy$$

is well-defined almost everywhere in $\mathbb{R}^N$. The function thus defined is denoted $f * g$ and is called the convolution *of f and g. Further, $f * g \in L^p(\mathbb{R}^N)$ and we also have*

$$\|f * g\|_p \;\le\; \|f\|_1 \|g\|_p. \tag{6.3.5}$$

Proof Let $h \in L^{p^*}(\mathbb{R}^N)$, where p^* is the conjugate exponent of p. The function $(x, y) \mapsto f(y)g(x-y)h(x)$ is measurable on $\mathbb{R}^N \times \mathbb{R}^N$; consider the iterated integral

$$I = \int_{\mathbb{R}^N_x} \int_{\mathbb{R}^N_y} |f(y)g(x-y)h(x)| \,\mathrm{d}y \,\mathrm{d}x.$$

Since the Lebesgue measure is translation invariant, we get that

$$\begin{aligned} I &= \int_{\mathbb{R}^N_y} |f(y)| \left(\int_{\mathbb{R}^N_x} |g(x-y)h(x)| \, \mathrm{d}x \right) \mathrm{d}y \\ &\leq \|g\|_p \|h\|_{p^*} \int_{\mathbb{R}^N} |f(y)| \, \mathrm{d}y \\ &= \|f\|_1 \|g\|_p \|h\|_{p^*} \ < \ \infty. \end{aligned}$$

Thus by Fubini's theorem, the integral

$$\int_{\mathbb{R}^N} f(y)g(x-y)h(x) \, \mathrm{d}y$$

exists for almost all $x \in \mathbb{R}^N$. Let us choose $h \in L^{p^*}(\mathbb{R}^N)$ such that $h(x) \neq 0$ for all $x \in \mathbb{R}^N$. For example, we can choose $h(x) = \exp(-|x|^2)$. Thus, it follows that the integral

$$\int_{\mathbb{R}^N} f(y)g(x-y) \, \mathrm{d}y$$

exists for almost all $x \in \mathbb{R}^N$ and so the convolution $f * g$ is well-defined. Further, by the above computation it follows that the map

$$h \ \mapsto \int_{\mathbb{R}^N} h(x)(f * g)(x) \, \mathrm{d}x$$

is a continuous linear functional on $L^{p^*}(\mathbb{R}^N)$ whose norm is bounded by $\|f\|_1 \|g\|_p$. It follows from the Riesz representation theorem that $f * g \in L^p(\mathbb{R}^N)$ and that (6.3.5) holds. ■

Remark 6.3.3 By a simple change of variable it is easy to see that we can also write the convolution of f and g as

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y) \, \mathrm{d}y.$$

Remark 6.3.4 The result of Theorem 6.3.3 is valid for the case $p = 1$ as well. The proof of this fact is left as an exercise. ■

6.4 The Spaces $W^{1,p}(a, b)$

In this section, we will study a very special case of a class of function spaces called *Sobolev spaces*.

Throughout this section, we assume that (a,b) is a finite interval in $\mathbb{R}$ and that $1 \leq p < \infty$. We will denote by $\mathcal{D}(a,b)$ the space of infinitely differentiable functions with compact support contained in the interval (a,b). Recall that (cf. Remark 6.3.1) $\mathcal{D}(a,b)$ is dense in $L^p(a,b)$ for $1 \leq p < \infty$.

Lemma 6.4.1 *Let* $\mathsf{f} \in L^p(a,b)$. *Assume that there exists* $\mathsf{g} \in L^p(a,b)$ *such that, for all* $\varphi \in \mathcal{D}(a,b)$, *we have*

$$\int_a^b f\varphi'\,dx = -\int_a^b g\varphi\,dx. \tag{6.4.1}$$

Then such a g *is unique.*

Proof If there were two functions g_1 and g_2 satisfying (6.4.1) for a given f, then

$$\int_a^b (g_1 - g_2)\varphi\,\mathrm{d}x = 0$$

for all $\varphi \in \mathcal{D}(a,b)$. Since $g_1 - g_2$ is locally integrable, it now follows that $g_1(x) = g_2(x)$ almost everywhere (cf. Proposition 6.3.3). ∎

Definition 6.4.1 Let $(a,b) \subset \mathbb{R}$ be a finite interval and let $1 \leq p < \infty$. The Sobolev space $W^{1,p}(a,b)$ is given by

$$W^{1,p}(a,b) = \{\mathsf{f} \in L^p(a,b) \mid \text{there exists } \mathsf{g} \in L^p(a,b) \text{ satisfying (6.4.1)}\}.$$

Further, we define

$$\|\mathsf{f}\|_{1,p} = (\|\mathsf{f}\|_p^p + \|\mathsf{g}\|_p^p)^{\frac{1}{p}}. \blacksquare$$

It is a routine verification to see that $\|\cdot\|_{1,p}$ defines a norm on $W^{1,p}(a,b)$ and this is left to the reader. Thus, $W^{1,p}(a,b)$ is a normed linear space.

Example 6.4.1 Let $f \in \mathcal{C}^1[a,b]$. Clearly $\mathsf{f} \in L^p(a,b)$. If f' denotes the derivative of f, then $f' \in \mathcal{C}[a,b]$ and so $\mathsf{f}' \in L^p(a,b)$ as well. Further, if $\varphi \in \mathcal{D}(a,b)$, then since $\varphi(a) = \varphi(b) = 0$, we have, by integration by parts,

$$\int_a^b f\varphi'\,\mathrm{d}x = -\int_a^b f'\varphi\,\mathrm{d}x.$$

Thus, $\mathsf{f} \in W^{1,p}(a,b)$ and it satisfies (6.4.1) with $g = f'$. ∎

By analogy with the preceding example, if $\mathsf{f} \in W^{1,p}(a,b)$, and if g is the associated function as in (6.4.1), then we denote g by f'. In particular, we have

$$\|\mathsf{f}\|_{1,p} = (\|\mathsf{f}\|_p^p + \|\mathsf{f}'\|_p^p)^{\frac{1}{p}}.$$

In the literature, f' is known as the *generalized* or *distributional derivative* of f.

Proposition 6.4.1 *Let $1 \le p < \infty$ and let $(a, b) \subset \mathbb{R}^n$ be a finite interval. Then, $W^{1,p}(a, b)$ is a Banach space.*

Proof We just need to prove the completeness. Let $\{\mathsf{f}_n\}$ be a Cauchy sequence in $W^{1,p}(a, b)$. Then $\{\mathsf{f}_n\}$ and $\{\mathsf{f}_n'\}$ are both Cauchy sequences in $L^p(a, b)$. Let $\mathsf{f}_n \to \mathsf{f}$ and $\mathsf{f}_n' \to g$ in $L^p(a, b)$. Now, if $\varphi \in \mathcal{D}(a, b)$, we have

$$\int_a^b f_n \varphi' \, \mathrm{d}x = -\int_a^b f_n' \varphi \, \mathrm{d}x$$

for all n. Passing to the limit as $n \to \infty$, we deduce that the pair (f, g) satisfies (6.4.1). Thus $\mathsf{f} \in W^{1,p}(a, b)$ and $\mathsf{f}' = \mathsf{g}$. Further, it follows that $\mathsf{f}_n \to \mathsf{f}$ in $W^{1,p}(a, b)$. This completes the proof. ■

Proposition 6.4.2 *The space $W^{1,p}(a, b)$ is reflexive if $1 < p < \infty$ and separable if $1 \le p < \infty$.*

Proof Since the space $L^p(a, b)$ is reflexive if $1 < p < \infty$, so is the space $(L^p(a, b))^2$ (why?). Similarly, $(L^p(a, b))^2$ is separable if $1 \le p < \infty$. Now, the space $W^{1,p}(a, b)$ is isometric to a subspace of $(L^p(a, b))^2$ via the mapping $\mathsf{f} \mapsto (\mathsf{f}, \mathsf{f}')$. Since $W^{1,p}(a, b)$ is complete, the image is a closed subspace of $(L^p(a, b))^2$ and so it inherits the reflexivity and separability properties from that space. This completes the proof. ■

We will now study some finer properties of these Sobolev spaces.

Lemma 6.4.2 *Let $\varphi \in \mathcal{D}(a, b)$. Then, there exists $\psi \in \mathcal{D}(a, b)$ such that $\psi' = \varphi$ if, and only if,*

$$\int_a^b \varphi(t) \, dt = 0.$$

Proof Assume that $\varphi = \psi'$ for some $\psi \in \mathcal{D}(a, b)$. Then, since $\psi(a) = \psi(b) = 0$, it follows that

$$\int_a^b \varphi(t) \, \mathrm{d}t = \int_a^b \psi'(t) \, \mathrm{d}t = \psi(b) - \psi(a) = 0.$$

Conversely, let $\int_a^b \varphi(t) \, \mathrm{d}t = 0$. Let the support of φ be contained in $[c, d] \subset (a, b)$. Now, define

$$\psi(t) = \int_a^t \varphi(s) \, \mathrm{d}s.$$

Clearly ψ is infinitely differentiable since $\psi' = \varphi$. Further, ψ vanishes on the interval (a, c) and, by hypothesis, on the interval (d, b) as well. Thus the support of ψ is also contained in $[c, d]$ and so $\psi \in \mathcal{D}(a, b)$. This completes the proof. ■

Corollary 6.4.1 *Let* f $\in L^p(a, b)$ *where* $1 \le p < \infty$. *Assume that*

$$\int_a^b f\varphi'\,dx = 0$$

for all $\varphi \in \mathcal{D}(a, b)$. *Then* f *is equal to a constant almost everywhere in* (a, b).

Proof Choose $\varphi_0 \in \mathcal{D}(a, b)$ such that $\int_a^b \varphi_0(t)\,\mathrm{d}t = 1$. Let $\varphi \in \mathcal{D}(a, b)$ be an arbitrary element. Set

$$\phi = \varphi - \left(\int_a^b \varphi(t)\,\mathrm{d}t\right)\varphi_0.$$

Then $\int_a^b \phi(t)\,\mathrm{d}t = 0$ and so $\phi = \psi'$ for some $\psi \in \mathcal{D}(a, b)$. Thus, $\int_a^b f\phi\,\mathrm{d}t = 0$ which yields

$$\int_a^b f\varphi = \int_a^b \varphi\,\mathrm{d}t.\int_a^b f\varphi_0\,\mathrm{d}t.$$

Setting $c = \int_a^b f\varphi_0\,\mathrm{d}t$, we get

$$\int_a^b (f - c)\varphi\,\mathrm{d}t = 0$$

for all $\varphi \in \mathcal{D}(a, b)$, whence, by Proposition 6.3.3, it follows that $f(x) = c$ almost everywhere in (a, b). This completes the proof. ■

Remark 6.4.1 If φ_1 were another function in $\mathcal{D}(a, b)$ such that $\int_a^b \varphi_1(t)\mathrm{d}t = 1$, then since $\int_a^b (\varphi_0 - \varphi_1)\,\mathrm{d}t = 0$, it follows that $\varphi_0 - \varphi_1 = \psi'$ for some $\psi \in \mathcal{D}(a, b)$. Therefore, by hypothesis, $\int_a^b f(\varphi_0 - \varphi_1)\,\mathrm{d}t = 0$. Thus, the constant c defined in the above proof does not depend on the choice of the function φ_0 whose integral is unity. ■

Let us denote by $\mathcal{C}^\infty[a, b]$ the space of all functions which are infinitely differentiable in the open interval (a, b) and such that the functions and all their derivatives possess continuous extensions to $[a, b]$.

Proposition 6.4.3 *Let* $1 \le p < \infty$. *Then* $\mathcal{C}^\infty[a, b]$ *is dense in* $W^{1,p}(a, b)$.

Proof It is clear that if $f \in \mathcal{C}^\infty[a,b]$, then $\mathsf{f} \in W^{1,p}(a,b)$ and its distributional derivative is just its classical derivative. Now, let $\mathsf{f} \in W^{1,p}(a,b)$. Since $\mathsf{f}' \in L^p(a,b)$, choose $\varphi_n \in \mathcal{D}(a,b)$ such that $\varphi_n \to \mathsf{f}'$ in $L^p(a,b)$. Define

$$\psi_n(x) = \int_a^x \varphi_n(t)\,\mathrm{d}t.$$

Then $\psi_n \in \mathcal{C}^\infty[a,b]$. Further, for $x \in [a,b]$,

$$|\psi_n(x) - \psi_m(x)| \le \int_a^b |\varphi_n(t) - \varphi_m(t)|\,\mathrm{d}t \le (b-a)^{\frac{1}{p^*}} \|\varphi_n - \varphi_m\|_p$$

by Hölder's inequality. Thus,

$$\|\psi_n - \psi_m\|_p \le (b-a)\|\varphi_n - \varphi_m\|_p.$$

It then follows that $\{\psi_n\}$ is Cauchy in $L^p(a,b)$ (since $\{\varphi_n\}$is Cauchy) and so let $\psi_n \to \mathsf{h}$ in $L^p(a,b)$. Since $\psi_n' = \varphi_n$, it is now easy to verify that $\mathsf{h} \in W^{1,p}(a,b)$ and that $\mathsf{h}' = \mathsf{f}'$. By the preceding corollary, it follows that $\mathsf{f} - \mathsf{h}$ is equal to a constant, say c. Thus, if we set $\chi_n = \psi_n + c$, then $\chi_n \in \mathcal{C}^\infty[a,b]$, $\chi_n \to \mathsf{f}$ in $L^p(a,b)$ and $\chi_n' \to \mathsf{f}'$ in $L^p(a,b)$. This completes the proof. ∎

We now briefly digress to recall some facts about absolutely continuous functions.

Definition 6.4.2 A function $f: [a,b] \to \mathbb{R}$ is said to be **absolutely continuous** on $[a,b]$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever we have a finite collection of disjoint intervals $\{(x_i, x_i')\}_{i=1}^n$ contained in (a,b) satisfying

$$\sum_{i=1}^n (x_i' - x_i) < \delta,$$

we have

$$\sum_{i=1}^n |f(x_i') - f(x_i)| < \varepsilon.$$

∎

Clearly, any absolutely continuous function is uniformly continuous. It can also be shown (cf. Royden [1]) that an absolutely continuous function is differentiable almost everywhere and that its derivative is an integrable function. The following two results are very important (cf. Royden [1]).

Theorem 6.4.1 *A function $f\colon [a,b] \to \mathbb{R}$ can be expressed as an indefinite integral of an integrable function if, and only if, it is absolutely continuous. In this case we have*

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

■

Theorem 6.4.2 (Integration by parts) *Let f and g be absolutely continuous functions on $[a,b]$. Then*

$$\int_a^b f(t)g'(t)\, dt = f(b)g(b) - f(a)g(a) - \int_a^b f'(t)g(t)\, dt.$$

■

If $f \in \mathcal{C}^1[a,b]$, then it is absolutely continuous. In particular, if $\varphi \in \mathcal{D}(a,b)$, it is absolutely continuous. Consequently, by virtue of intergation by parts, it follows that if f is absolutely continuous on $[a,b]$ and if $\varphi \in \mathcal{D}(a,b)$, then

$$\int_a^b f\varphi'\, \mathrm{d}t = -\int_a^b f'\varphi\, \mathrm{d}t$$

so that the distributional derivative of f is f'.

Proposition 6.4.4 *Let $1 \le p < \infty$. Let $\mathsf{f} \in W^{1,p}(a,b)$. Then f is absolutely continuous, i.e. f is equal, almost everywhere, to an absolutely continuous function.*

Proof Let us define

$$u(x) = \int_a^x f'(t)\, \mathrm{d}t.$$

Since $\mathsf{f}' \in L^p(a,b)$ and since $p \ge 1$, it follows that f' is integrable on (a,b) and so u is an absolutely continuous function. Further, since integration by parts is valid for absolutely continuous functions, it follows that for all $\varphi \in \mathcal{D}(a,b)$, we have

$$\int_a^b u\varphi'\, \mathrm{d}x = -\int_a^b f'\varphi\, \mathrm{d}x.$$

Thus $\mathsf{u} \in W^{1,p}(a,b)$ and $\mathsf{u}' = \mathsf{f}'$. Then, as before, it follows that $f - u$ is equal to a constant almost everywhere. Thus $f(x) = u(x) + c$ almost everywhere in x and the latter function is absolutely continuous. ■

The above proposition states that $W^{1,p}(a,b)$ consists of absolutely continuous functions (upto equality almost everywhere). In particular, we can say that $W^{1,p}(a,b)$ is contained in $\mathcal{C}[a,b]$; *i.e.* every element of $W^{1,p}(a,b)$ is represented by means of an (absolutely) continuous function. Such a representative must be unique, for, if two continuous functions are equal almost everywhere, then they are equal everywhere (why?).

Theorem 6.4.3 (Sobolev) *The inclusion map from $W^{1,p}(a,b)$ into $\mathcal{C}[a,b]$ is continuous.*

Proof Let f_n, f be in $W^{1,p}(a,b)$ with absolutely continuous representatives f_n, f. Assume that $\mathsf{f}_n \to \mathsf{f}$ in $W^{1,p}(a,b)$. Then $\|f_n - f\|_p \to 0$ and $\|f_n' - f'\|_p \to 0$. Now, by absolute continuity, we have

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)\,\mathrm{d}t \tag{6.4.2}$$

and

$$f(x) = f(a) + \int_a^x f'(t)\,\mathrm{d}t. \tag{6.4.3}$$

We claim that $\{f_n(a)\}$ is Cauchy. If not, there exists $\varepsilon > 0$ such that, for every N, there exist $m, n \geq N$ satisfying $|f_m(a) - f_n(a)| \geq \varepsilon$. Then, it follows from (6.4.2) that

$$|f_m(x) - f_n(x)| \;\geq\; \varepsilon - \|f_m' - f_n'\|_p (x-a)^{\frac{1}{p^*}}$$

by an application of Hölder's inequality. Choose N large enough such that, for all $n, m \geq N$, we have

$$\|f_m' - f_n'\|_p (b-a)^{\frac{1}{p^*}} < \frac{\varepsilon}{2}.$$

Then, for all $x \in (a,b)$ we have

$$|f_m(x) - f_n(x)| \;\geq\; \frac{\varepsilon}{2}$$

whence it would follow that

$$\|f_m - f_n\|_p \geq (b-a)^{\frac{1}{p}} \frac{\varepsilon}{2} \;>\; 0$$

whch contradicts the fact the $\{\mathsf{f}_n\}$ is cauchy in $L^p(a,b)$.

Thus $\{f_n(a)\}$ is Cauchy and now, for any $x \in [a,b]$,

$$|f_m(x) - f_n(x)| \;\leq\; |f_m(a) - f_n(a)| + \|f_m' - f_n'\|_p (b-a)^{\frac{1}{p^*}}$$

by another application of (6.4.2) and Hölder's inequality. This shows that $\{f_n\}$ is uniformly Cauchy, and so it converges to a continuous function $\widetilde{f}$ on $[a,b]$. But since $\|f_n - f\|_p \to 0$, it follows that (cf. Corollary 6.1.1), at least for a subsequence, we have $f_{n_k}(x) \to f(x)$ almost everywhere, from which we deduce that $f \equiv \widetilde{f}$. Thus, $f_n \to f$ in $\mathcal{C}[a,b]$ which completes the proof. ■

Theorem 6.4.4 (Rellich) *The unit ball in $W^{1,p}(a,b)$ is relatively compact in $L^p(a,b)$.*

Proof The inclusion map $W^{1,p}(a,b) \subset L^p(a,b)$ is the composition of the following inclusion maps:

$$W^{1,p}(a,b) \ \subset \ \mathcal{C}[a,b] \ \subset \ L^p(a,b).$$

The first inclusion above is continuous by the preceding theorem. The space $\mathcal{C}[a,b]$ is a subspace of $L^\infty(a,b)$, and the 'sup-norm' is the same as $\|.\|_\infty$. Now it follows that the second inclusion is also continuous by Proposition 6.1.3.

Let B be the unit ball in $W^{1,p}(a,b)$. Thus, if $\mathsf{f} \in B$, then

$$\|\mathsf{f}\|_p^p + \|\mathsf{f}'\|_p^p \ \leq \ 1.$$

Then, by the preceding theorem, it follows that B is bounded in $\mathcal{C}[a,b]$ as well, since the inclusion map is continuous (cf. Proposition 2.3.1 (iv)). Further, let $x, y \in [a,b]$. Assume, without loss of generality, that $x \leq y$. Then

$$|f(x) - f(y)| \ \leq \ \left| \int_x^y f'(t)\,\mathrm{d}t \right| \ \leq \ \|f'\|_p |y-x|^{\frac{1}{p^*}} \ \leq \ |y-x|^{\frac{1}{p^*}}.$$

It now follows immediately that B is equicontinuous as well since for $\varepsilon > 0$, if we choose $\delta < \varepsilon^{p^*}$, then $|x-y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$ for all $\mathsf{f} \in B$. Thus, by the theorem of Ascoli, it follows that B is relatively compact in $\mathcal{C}[a,b]$.

Thus, any sequence in B will have a subsequence which is convergent in $\mathcal{C}[a,b]$, which will also, *a fortiori*, converge in $L^p(a,b)$. This proves that B is relatively compact in $L^p(a,b)$. ■

Definition 6.4.3 Let $(a,b) \subset \mathbb{R}$ be a finite interval and let $1 \leq p < \infty$. The closure of $\mathcal{D}(a,b)$ in $W^{1,p}(a,b)$ is denoted $W_0^{1,p}(a,b)$. ■

Theorem 6.4.5 *Let* $\mathsf{f} \in W^{1,p}(a,b)$ *with f absolutely continuous. Then* $\mathsf{f} \in W_0^{1,p}(a,b)$ *if, and only if, $f(a) = f(b) = 0$.*

Proof Let $\mathsf{f} \in W_0^{1,p}(a,b)$. Then there exists a sequence $\{\varphi_n\}$ in $\mathcal{D}(a,b)$ such that $\varphi_n \to \mathsf{f}$ in $W^{1,p}(a,b)$. Then $\varphi_n \to f$ uniformly on $[a,b]$ and so it follows immediately that $f(a) = f(b) = 0$.

Conversely, let $f(a) = f(b) = 0$. Then (cf. Theorem 6.4.1) we have

$$f(x) = \int_a^x f'(t)\,\mathrm{d}t$$

and so it follows that $\int_a^b f'(t)\,\mathrm{d}t = 0$. Let $\varphi_n \in \mathcal{D}(a,b)$ such that $\|\varphi_n - f'\|_p \to 0$. Then

$$\left| \int_a^b \varphi_n\,\mathrm{d}t - \int_a^b f'\,\mathrm{d}t \right| \leq \|\varphi_n - f'\|_p (b-a)^{\frac{1}{p^*}} \;\to\; 0$$

and so

$$\int_a^b \varphi_n\,\mathrm{d}t \;\to\; 0.$$

Let $\varphi_0 \in \mathcal{D}(a,b)$ such that $\int_a^b \varphi_0\,\mathrm{d}t = 1$. Then if

$$\psi_n = \varphi_n - \left(\int_a^b \varphi_n\,\mathrm{d}t \right) \varphi_0,$$

we also have that $\|\psi_n - f'\|_p \to 0$ and $\int_a^b \psi_n\,\mathrm{d}t = 0$. Thus $\psi_n = \chi_n'$ where $\chi_n \in \mathcal{D}(a,b)$ as well (cf. Lemma 6.4.2). Since

$$\chi_n(x) = \int_a^x \psi_n\,\mathrm{d}t,$$

it follows that χ_n converges to f uniformly and so $\|\chi_n - f\|_p \to 0$ as well. Thus $\chi_n \in \mathcal{D}(a,b)$ and $\chi_n \to \mathsf{f}$ in $W^{1,p}(a,b)$. This proves that $\mathsf{f} \in W_0^{1,p}(a,b)$. ■

Notice that if $f \equiv 1$, then $f' \equiv 0$ so that, in $W^{1,p}(a,b)$, the map $\mathsf{f} \mapsto \|\mathsf{f}'\|_p$ does not define a norm, but only a *seminorm*; *i.e.* while $\|\mathsf{f}'\|_p = 0$ does not imply that $\mathsf{f} = \mathbf{0}$, all other properties of a norm are satisfied. However, in the space $W_0^{1,p}(a,b)$, we have the following result.

Theorem 6.4.6 (Poincaré's Inequality) *Let* $\mathsf{f} \in W_0^{1,p}(a,b)$. *Then*

$$\|\mathsf{f}\|_p \;\leq\; (b-a)\|\mathsf{f}'\|_p. \tag{6.4.4}$$

Thus the function $\mathsf{f} \mapsto \|\mathsf{f}'\|_p$ *defines a norm on* $W_0^{1,p}(a,b)$ *equivalent to the usual norm on this space.*

Proof Let f be absolutely continuous and represent f. If $\mathsf{f} \in W_0^{1,p}(a,b)$, then, since $f(a) = 0$, we have

$$f(x) = \int_a^x f'(t)\, \mathrm{d}t.$$

Then, by Hölder's inequality, we have

$$|f(x)| \ \leq \ \|f'\|_p (b-a)^{\frac{1}{p^*}}.$$

Thus,

$$\|f||_p \ \leq \ \|f'\|_p (b-a)^{\frac{1}{p}+\frac{1}{p^*}} = (b-a)\|f'\|_p$$

which proves (6.4.4).

In particular, if $\|\mathsf{f}'\|_p = 0$, it follows that $\|\mathsf{f}\|_p = 0$ and so $\mathsf{f} = \mathbf{0}$ in $W_0^{1,p}(a,b)$. The other properties of a norm are easily verified. Thus we have two norms on $W_0^{1,p}(a,b)$:

$$\|\mathsf{f}\|_{1,p} \text{ and } |\mathsf{f}|_{1,p} \ \stackrel{\text{def}}{=} \ \|\mathsf{f}'\|_p.$$

Clearly,

$$|\mathsf{f}|_{1,p} \ \leq \ \|\mathsf{f}\|_{1,p} \ \leq \ [(b-a)^p + 1]^{\frac{1}{p}} |\mathsf{f}|_{1,p}.$$

Thus the two norms are equivalent. ■

Sobolev spaces can also be defined when $p = \infty$. The definition can also be extended to cover functions defined on arbitrary open sets $\Omega \subset \mathbb{R}^N$. It is also possible to define 'higher-order distributional derivatives' and define Sobolev spaces $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, based on these derivatives. All these spaces have properties similar to those proved in this section, with or without additional hypotheses. For a detailed study of Sobolev spaces, see Kesavan [2]. See also the Exercises 6.25–6.27.

6.5 Exercises

6.1 Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $1 \leq p, q, r < \infty$ be such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

If $\mathsf{f} \in L^p(\mu)$ and $\mathsf{g} \in L^q(\mu)$, show that $\mathsf{fg} \in L^r(\mu)$ and that

$$\|\mathsf{fg}\|_r \ \leq \ \|\mathsf{f}\|_p \|\mathsf{g}\|_q.$$

6.2 Let $(X, \mathcal{S}, \mu)$ be a measure space and let $1 \leq p < \infty$. Define, for $t > 0$,

$$h_f(t) = \mu(\{|f| > t\}).$$

Show that

$$\|\mathsf{f}\|_p^p = p \int_0^\infty t^{p-1} h_f(t)\, \mathrm{d}t.$$

(Hint: Write h_f as an integral over a subset of X and apply Fubini's theorem (cf. Theorem 1.3.5)).

6.3 (a) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let f_n, g_n, f, g be measurable functions such that $f_n \to f$ and $g_n \to g$ almost everywhere in X. Assume further that $|f_n(x)| \leq g_n(x)$ for all $x \in X$ and that

$$\int_X g_n \,\mathrm{d}\mu \ \to \int_X g \,\mathrm{d}\mu \ < \infty$$

as $n \to \infty$. Show that

$$\int_X f_n \,\mathrm{d}\mu \ \to \int_X f \,\mathrm{d}\mu$$

as $n \to \infty$. (Hint: Apply Fatou's lemma (cf. Theorem 1.3.2) to $g_n + f_n \geq 0$ and to $g_n - f_n \geq 0$.)
(b) Let $1 \leq p < \infty$. Let f_n and $\mathsf{f} \in L^p(\mu)$ and assume that $f_n(x) \to f(x)$ almost everywhere in X. Show that $\mathsf{f}_n \to \mathsf{f}$ in $L^p(\mu)$ if, and only if, $\|\mathsf{f}_n\|_p \to \|\mathsf{f}\|_p$.

6.4 Let $(X, \mathcal{S}, \mu)$ be a measure space and let $1 \leq p < \infty$. Let $\mathsf{f}_\mathsf{n} \to \mathsf{f}$ in $L^p(\mu)$. Let g_n be a sequence of measurable functions converging to a measurable function g almost everywhere in X. Assume further that g_n and g are all uniformly bounded by a constant $M > 0$ in X. Show that $\mathsf{f}_n\mathsf{g}_n \to \mathsf{fg}$ in $L^p(\mu)$.

6.5 Let $(X, \mathcal{S}, \mu)$ be a measure space. A sequence of measurable functions f_n is said to *converge in measure* in X to a measurable function f if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{|f_n - f| \geq \varepsilon\}) = 0.$$

In this case, we write $f_n \xrightarrow{\mu} f$. If $1 \leq p < \infty$ and if $\mathsf{f}_n \to \mathsf{f}$ in $L^p(\mu)$, show that $f_n \xrightarrow{\mu} f$.

6.6 Let $(X, \mathcal{S}, \mu)$ be a measure space and let $1 < p < \infty$. Let $f\colon X \times X \to \mathbb{R}$ be such that for almost every $y \in X$, the section f^y (cf. Definition 1.3.7) is p-integrable. Define, for $x \in X$,

$$g(x) = \int_X f(x, y)\, \mathrm{d}\mu(y).$$

If $\int_X \|f^y\|_p \mathrm{d}\mu(y) < \infty$, show that $\mathsf{g} \in L^p(\mu)$ and that

$$\|\mathsf{g}\|_p \ \leq \ \int\limits_X \|f^y\|_p \mathrm{d}\mu(y).$$

6.7 Show that the inclusion $\mathcal{C}_c(\mathbb{R}) \subset L^1(\mathbb{R})$ is not continuous.

6.8 Let $g \in \mathcal{C}_c(\mathbb{R})$. Define $\varphi(g) = g(0)$. Then φ can be extended to a continuous linear functional on $L^\infty(\mathbb{R})$. Show that there does not exist $\mathsf{f} \in L^1(\mathbb{R})$ such that

$$\varphi(g) = \int\limits_{\mathbb{R}} gf \ \mathrm{d}x$$

for all $g \in L^\infty(\mathbb{R})$. (This gives another proof that $L^\infty(\mathbb{R})$ is not reflexive.)

6.9 Let $h \in \mathbb{R}^N$. For a (Lebesgue) measurable function f defined on $\mathbb{R}^N$, define its translation by h by

$$f_h(x) = f(x+h).$$

If $\mathsf{f} \in L^p(\mathbb{R}^N)$, show that $\mathsf{f}_h \in L^p(\mathbb{R}^N)$ and that

$$\|\mathsf{f} - \mathsf{f}_h\|_p \ \to \ 0$$

as $h \to \mathbf{0}$ in $\mathbb{R}^N$ for any $1 \leq p < \infty$.

6.10 Let

$$f_n = \chi_{[n,n+1]},$$

the characteristic function of the closed interval $[n, n+1]$ for $n \in \mathbb{N}$ (cf. Definition 1.3.5).

(a) Then $\{\mathsf{f}_n\}$ is a bounded sequence in $L^1(0,\infty)$. Show that it does not have a weakly convergent subsequence. (In view of Theorem 5.4.2, this gives another proof that $L^1(\mathbb{R})$ is not reflexive.)

(b) Show that $\{\mathsf{f}_n\}$ converges weakly in $L^p(0,\infty)$, for all $1 < p < \infty$.

(c) Show that $\{\mathsf{f}_n\}$ is weak* convergent in $L^\infty(0,\infty)$.

6.11 Let $f \in \mathcal{C}[0,1]$ be such that, for all $n \geq 0$,

$$\int\limits_0^1 x^n f(x) \ \mathrm{d}x = 0.$$

Show that $f \equiv 0$.

6.12 (Hardy's inequality) Let $\mathsf{f} \in L^p(0,\infty)$, where $1 < p < \infty$. Define

$$g(x) = \frac{1}{x} \int\limits_0^x f(t) \ \mathrm{d}t$$

for $x \in (0, \infty)$. Show that $\mathsf{g} \in L^p(0, \infty)$ and that

$$\|\mathsf{g}\|_p \leq \frac{p}{p-1}\|\mathsf{f}\|_p.$$

(Hint: Prove it first for $f \in \mathcal{C}_c(0, \infty)$, $f \geq 0$.)

6.13 A function φ: $(0, \infty) \to \mathbb{R}$ is said to be a *step function* if

$$\varphi(x) = \sum_{j=1}^{n} \alpha_j \chi_{I_j}(x)$$

where I_j, $1 \leq j \leq n$ are intervals contained in $(0, \infty)$ and, as usual, χ_E denotes the characteristic function of a set E. Show that step functions in $(0, \infty)$ are dense in $L^1(0, \infty)$.

6.14 (Riemann-Lebesgue lemma) Let h be a bounded and measurable function on $(0, \infty)$ such that

$$\lim_{c\to\infty} \frac{1}{c} \int_0^c h(t)\, \mathrm{d}t = 0.$$

(a) Let $f = \chi_{[c,d]}$, where $[c, d] \subset (0, \infty)$. Show that

$$\lim_{\omega\to\infty} \int_0^\infty f(t)h(\omega t)\, \mathrm{d}t = 0. \tag{6.5.5}$$

(b) Deduce that (6.5.5) is true for all $\mathsf{f} \in L^1(0, \infty)$.
(c) If $\mathsf{f} \in L^1(a, b)$ where $(a, b) \subset (0, \infty)$, show that

$$\lim_{n\to\infty} \int_a^b f(t) \cos nt\, \mathrm{d}t = \lim_{n\to\infty} \int_a^b f(t) \sin nt\, \mathrm{d}t = 0.$$

6.15 (a) Let $(a, b) \subset (0, \infty)$ be any finite interval. Let $f_n(t) = \cos nt$ and let $g_n(t) = \sin nt$. Show that $\mathsf{f}_n \rightharpoonup \mathbf{0}$ and $\mathsf{g}_n \rightharpoonup \mathbf{0}$ in $L^p(a, b)$ for any $1 \leq p < \infty$.
(b) What is the weak limit of h_n in $L^p(a, b)$ for $1 \leq p < \infty$ where $h_n(t) = \cos^2 nt$?

6.16 (a) Consider the trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt).$$

Show that it can written in the *amplitude-phase* form

$$\frac{a_0}{2} + d_n \cos(nt - \phi_n).$$

Write down the relations between a_n, b_n and d_n, ϕ_n.
(b) (Cantor-Lebesgue theorem) Show that if a trigonometric series as in (a) above converges over a set E whose measure is strictly positive, then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$. (Hint: Use the amplitude-phase form of the series.)

6.17 If f and $\mathsf{g} \in L^1(\mathbb{R}^N)$, show that

$$(\mathsf{f} * \mathsf{g})(x) = \int_{\mathbb{R}^N} f(y)g(x-y)\,\mathrm{d}y$$

is well-defined for almost all $x \in \mathbb{R}^N$. Show also that $\mathsf{f} * \mathsf{g} \in L^1(\mathbb{R}^N)$ and that

$$\|\mathsf{f} * \mathsf{g}\|_1 \le \|\mathsf{f}\|_1 \|\mathsf{g}\|_1.$$

6.18 Let $\{\rho_\varepsilon\}_{\varepsilon>0}$ be a family of $\mathcal{C}^\infty$ functions in $\mathbb{R}^N$ such that for each $\varepsilon > 0$, we have that $\rho_\varepsilon(x) \ge 0$ for all $x \in \mathbb{R}^N$, the support of ρ_ε is contained in the closed ball with centre at the origin and radius ε, and

$$\int_{\mathbb{R}^N} \rho_\varepsilon(x)\,\mathrm{d}x = 1.$$

(a) If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, show that $\rho_\varepsilon * f \to f$, pointwise, as $\varepsilon \to 0$.
(b) If $f\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous, show that $\rho_\varepsilon * f \to f$, uniformly, as $\varepsilon \to 0$.
(c) Let $\varphi \in \mathcal{C}_c(\mathbb{R}^N)$. Show that $\rho_\varepsilon * \varphi$ converges uniformly to φ on $\mathbb{R}^N$ as $\varepsilon \to 0$.
(d) Show also that the support of $\rho_\varepsilon * \varphi$ is contained in the set

$$\mathrm{supp}(\varphi) + \overline{B}(0;\varepsilon),$$

where $\mathrm{supp}(\varphi)$ denotes the support of φ and $\overline{B}(0;\varepsilon)$ is the closed ball in $\mathbb{R}^N$ with centre at the origin and of radius ε.
(e) Deduce that, if $\mathsf{u} \in L^p(\mathbb{R}^N)$, $1 \le p < \infty$, then $\rho_\varepsilon * \mathsf{u}$ converges to u in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$.

6.19 Let $\{\mathsf{f}_n\}$ be a bounded sequence in $L^p(a,b)$, where (a,b) is an open interval in $\mathbb{R}$ and $1 \le p \le \infty$. Show that $\mathsf{f}_n \rightharpoonup \mathsf{f}$ in $L^p(a,b)$ when $1 < p < \infty$ and $\mathsf{f}_n \overset{*}{\rightharpoonup} \mathsf{f}$ in $L^\infty(a,b)$ if, and only if, for every $\varphi \in \mathcal{D}(a,b)$, we have

$$\int_a^b f_n\varphi\,\mathrm{d}x \to \int_a^b f\varphi\,\mathrm{d}x.$$

6.20 Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Define the sequence $\{f_n\}$ as follows. Let $f_n(x) = f(nx)$ on $[0, \frac{1}{n}]$ and extend this periodically to each subinterval $[\frac{k-1}{n}, \frac{k}{n}]$ for $2 \le k \le n$. Let $m = \int\limits_0^1 f(t)\,\mathrm{d}t$. Show that $\mathsf{f}_n \rightharpoonup \mathsf{f}$ in $L^p(0,1)$ for $1 < p < \infty$ and that $\mathsf{f}_n \overset{*}{\rightharpoonup} \mathsf{f}$ in $L^\infty(0,1)$, where $f(t) = m$ for all $t \in [0,1]$.

6.21 Let $(a,b) \subset \mathbb{R}$ be a finite interval and let $f\colon [a,b] \to \mathbb{R}$ be a Lipschitz continuous function *i.e.* there exists $K > 0$ such that for all $x, y \in [a,b]$, we have

$$|f(x) - f(y)| \le K|x - y|.$$

Show that $\mathsf{f} \in W^{1,p}(a,b)$ for all $1 \le p < \infty$.

6.22 Let $a < c < b$ in $\mathbb{R}$. Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Assume that $\mathsf{f} \in W^{1,p}(a,c)$ and that $\mathsf{f} \in W^{1,p}(c,b)$. Show that $\mathsf{f} \in W^{1,p}(a,b)$.

6.23 Let $f\colon [-1,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0, \text{ if } x \in [-1,0) \\ 1, \text{ if } x \in [0,1] \end{cases}$$

Show that $\mathsf{f} \notin W^{1,p}(-1,1)$ for $1 \le p < \infty$. (Thus, continuity is essential in the previous exercise.)

6.24 (a) Let $f\colon [a,b] \to \mathbb{R}$ be absolutely continuous and assume that

$$\int\limits_a^b f(t)\,\mathrm{d}t = 0.$$

Let $1 \le p < \infty$. Show that

$$|f(x)| \le (b-a)^{\frac{1}{p^*}} \|f'\|_p$$

for all $x \in [a,b]$.
(b) (*Poincaré-Wirtinger Inequality*) Deduce that, for all $\mathsf{f} \in W^{1,p}(a,b)$ such that $\int_a^b f(t)\,\mathrm{d}t = 0$, we have

$$\|\mathsf{f}\|_p \le (b-a)\|\mathsf{f}'\|_p.$$

6.25 (a) Define $W^{1,p}(\mathbb{R})$ exactly as in Definition 6.4.1, with $\mathbb{R}$ replacing the interval (a,b) in that definition as well as in relation (6.4.1). Let ζ be a $\mathcal{C}^\infty$ function on $\mathbb{R}$ with compact support. Show that, if $\mathsf{f} \in W^{1,p}(\mathbb{R})$, then $\zeta\mathsf{f} \in W^{1,p}(\mathbb{R})$ as well, where $(\zeta f)(x) = \zeta(x) f(x)$.
(b) Let ζ be a $\mathcal{C}^\infty$ function on $\mathbb{R}$ with compact support contained in $[-2,2]$ such that

$0 \leq \zeta(x) \leq 1$ for all $x \in \mathbb{R}$ and such that $\zeta \equiv 1$ on $[-1, 1]$. Define $\zeta_m(x) = \zeta(x/m)$ for all $x \in \mathbb{R}$. Show that if $u \in W^{1,p}(\mathbb{R})$ for $1 \leq p < \infty$, then $\zeta_m u \to u$ in $W^{1,p}(\mathbb{R})$ as $m \to \infty$.

6.26 Let $(a, b) \subset \mathbb{R}$ be a finite open interval. Let $m > 1$ be a positive integer. Define, for $1 \leq m < \infty$,

$$W^{m,p}(a,b) = \left\{ \mathsf{f} \in L^p(a,b) \mid \begin{array}{c} \text{there exist } \mathsf{g}_i \in L^p(a,b),\ 1 \leq i \leq m \\ \text{such that} \\ \int_a^b f \frac{d^i\varphi}{\mathrm{d}x^i}\, \mathrm{d}x = (-1)^i \int_a^b g_i \varphi \, \mathrm{d}x \\ \text{for all } \varphi \in \mathcal{D}(a,b) \end{array} \right\}.$$

The functions g_i are called the generalized successive derivatives of f and we denote $\mathsf{f}^{(i)} = g_i$. Define

$$\|\mathsf{f}\|_{m,p} = \left(\|f\|_p^p + \sum_{i=1}^m \|\mathsf{f}^{(i)}\|_p^p \right)^{\frac{1}{p}}$$

for $\mathsf{f} \in W^{m,p}(a,b)$.
(a) Show that $\|.\|_{m,p}$ defines a norm on $W^{m,p}(a,b)$ which makes it into a Banach space which is separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$.
(b) Show that if $\mathsf{f} \in W^{m,p}(a,b)$, then $f \in \mathcal{C}^{m-1}[a,b]$.

6.27 Let $W_0^{m,p}(a,b)$ denote the closure of $\mathcal{D}(a,b)$ in $W^{m,p}(a,b)$.
(a) Show that $\mathsf{f} \in W^{m,p}(a,b)$ belongs to $W_0^{m,p}(a,b)$ if, and only if, $f(a) = f(b) = 0$ and $f^{(i)}(a) = f^{(i)}(b) = 0$ for all $1 \leq i \leq m-1$.
(b) Show that

$$\mathsf{f} \ \mapsto \ |\mathsf{f}|_{m,p} \stackrel{\text{def}}{=} \|\mathsf{f}^{(m)}\|_p$$

defines a norm on $W_0^{m,p}(a,b)$ which is equivalent to the usual norm $\|.\|_{m,p}$.

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Chapter 7
Hilbert Spaces

7.1 Basic Properties

Hilbert spaces form a special class of Banach spaces with the geometric notion of orthogonality of vectors, or more generally, the notion of an angle between vectors, built into them.

Consider the space $\mathbb{R}^2$. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are vectors in $\mathbb{R}^2$, then we define the scalar product of these vectors by

$$x.y \ = \ x_1y_1 + x_2y_2 \ = \ |x|.|y| \cos\theta$$

where $|x| = \|x\|_2, \ |y| = \|y\|_2$ and θ is the angle between the two vectors. The scalar product is linear in each of the two variables. It is symmetric in these variables and $x.x = \|x\|_2^2$. It turns out that these properties are crucial and we generalize these to other vector spaces.

Definition 7.1.1 Let V be a real normed linear space. An **inner product** on V is a form $(\cdot, \cdot) : V \times V \to \mathbb{R}$ such that

(i) it is symmetric, *i.e.* for all x and $y \in V$,

$$(x, y) \ = \ (y, x);$$

(ii) it is bilinear: in particular, if x, y and $z \in V$ and if α and $\beta \in \mathbb{R}$, then

$$(\alpha x + \beta y, z) \ = \ \alpha(x, z) + \beta(y, z);$$

(iii) for all $x \in V$,

$$(x, x) \ = \ \|x\|^2. \qquad \blacksquare$$

S. Kesavan, *Functional Analysis*, Texts and Readings in Mathematics,
https://doi.org/10.1007/978-981-19-7633-9_7

Remark 7.1.1 The linearity with respect to the second variable is, clearly, a consequence of conditions (i) and (ii) above. ■

Remark 7.1.2 In case the base field is $\mathbb{C}$, then the inner product is a *sesquilinear form*. If x and $y \in V$, we have

$$(y, x) = \overline{(x, y)}.$$

Thus, we have conjugate linearity with respect to the second variable, *i.e.* if x, y and $z \in V$ and if α and $\beta \in \mathbb{C}$, then

$$(x, \alpha y + \beta z) = \overline{\alpha}(x, y) + \overline{\beta}(x, z).$$

■

In view of condition (iii), we say that the norm comes from the inner product.

Definition 7.1.2 A **Hilbert space** is a Banach space whose norm comes from an inner product. ■

Example 7.1.1 Consider the space $\mathbb{R}^n$. For $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$, we define

$$(x, y) = \sum_{i=1}^{n} x_i y_i.$$

This defines an inner product and the norm associated to it is the norm $\|.\|_2$. Thus ℓ_2^n is a Hilbert space. In the case of $\mathbb{C}^n$, the inner product is given by

$$(x, y) = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Again the norm is $\|.\|_2$. ■

Example 7.1.2 Consider the space ℓ_2. For x and $y \in \ell_2$, define

$$(x, y) = \sum_{i=1}^{\infty} x_i y_i$$

where $x = (x_i)$ and $y = (y_i)$ are real sequences. Again, if the base field is $\mathbb{C}$, then we define

$$(x, y) = \sum_{i=1}^{\infty} x_i \overline{y_i}.$$

This makes ℓ_2 into a Hilbert space. ■

As we have seen in the previous chapter, these are particular cases of the Lebesgue spaces L^2.

Example 7.1.3 Let $(X, \mathcal{S}, \mu)$ be a measure space. If f and $\mathsf{g} \in L^2(\mu)$, and if f and g represent these classes, respectively, then

$$(\mathsf{f}, \mathsf{g}) = \int_X fg \, \mathrm{d}\mu$$

defines an inner product which makes $L^2(\mu)$ as a Hilbert space, in the real case. If we are in the complex case, then the inner product should read

$$(\mathsf{f}, \mathsf{g}) = \int_X f\overline{g} \, \mathrm{d}\mu.$$

■

Example 7.1.4 Let $(a, b) \subset \mathbb{R}$ be a finite interval. We denote by $H^1(a, b)$ the space $W^{1,2}(a, b)$ and by $H_0^1(a, b)$ the space $W_0^{1,2}(a, b)$. Then both these spaces are Hilbert spaces with the inner product given by

$$(\mathsf{f}, \mathsf{g}) = \int_a^b (fg + f'g') \, \mathrm{d}x.$$

By virtue of the Poincaré inequality (cf. Theorem 6.4.6), the space $H_0^1(a, b)$ is also a Hilbert space with the inner product

$$(\mathsf{f}, \mathsf{g})_1 = \int_a^b f'g' \, \mathrm{d}x.$$

■

Let H be a Hilbert space and let x and $y \in H$. Then

$$\|x + y\|^2 = (x + y, x + y) = \|x\|^2 + 2(x, y) + \|y\|^2 \tag{7.1.1}$$

in the real case; if the field is $\mathbb{C}$, then the middle term on the right will be replaced by $2\mathrm{Re}(x, y)$, where $\mathrm{Re}\, z$ denotes the real part of a complex number z. Writing a similar expression for $\|x - y\|^2$ and adding the two, we get

$$\left\|\frac{1}{2}(x + y)\right\|^2 + \left\|\frac{1}{2}(x - y)\right\|^2 = \frac{1}{2}(\|x\|^2 + \|y\|^2). \tag{7.1.2}$$

This is known as the *parallelogram identity*. In case of $\mathbb{R}^2 = \ell_2^2$, this is the familiar result from plane geometry which relates the sum of the squares of the lengths of the diagonals of a parallellogram to that of the sides. It is also known as *Apollonius' theorem*.

Remark 7.1.3 A theorem of Fréchet, Jordan and von Neumann states that a Banach space whose norm satisfies the parallelogram identity (7.1.2) is a Hilbert space, *i.e.* the norm comes from an inner product. ■

Example 7.1.5 The space $\mathcal{C}[-1, 1]$ cannot be made into a Hilbert space. To see this, consider the functions

$$u(x) = \min\{x, 0\}, \text{ and } v(x) = x$$

defined on $[-1, 1]$. Then $\|u\| = \|v\| = 1$ while we have

$$\left\| \frac{1}{2}(u + v) \right\| = 1 and \left\| \frac{1}{2}(u - v) \right\| = \frac{1}{2}$$

and the parallelogram identity is not satisfied by this pair of functions. ■

Proposition 7.1.1 *Every Hilbert space is uniformly convex and hence is reflexive.*

Proof The proof is of the uniform convexity follows from the parallelogram identity (7.1.2) exactly as described in Example 5.5.2; the reflexivity now follows from Theorem 5.5.1. ■

We now prove a fundamental inequality for Hilbert spaces.

Theorem 7.1.1 *(**Cauchy-Schwarz Inequality**) Let H be a Hilbert space and let x and $y \in H$. Then*

$$|(x, y)| \leq \|x\| \, \|y\|. \tag{7.1.3}$$

Equality occurs in this inequality if, and only if, x and y are scalar multiples of each other.

Proof Let θ be a complex number such that $|\theta| = 1$ and $\theta(x, y) = |(x, y)|$. Let $t \in \mathbb{R}$. We have

$$\begin{aligned} 0 &\leq \|\theta x - ty\|^2 \\ &= \|x\|^2 - 2t\mathrm{Re}(\theta x, y) + t^2\|y\|^2 \\ &= \|x\|^2 - 2t|(x, y)| + t^2\|y\|^2. \end{aligned}$$

Since we have a quadratic polynomial which is always of constant sign, the roots of this polynomial must be coincident or complex. Thus, we deduce that

$$4|(x, y)|^2 \leq 4\|x\|^2\|y\|^2$$

which yields (7.1.3).

Equality occurs in (7.1.3) if, and only if, the polynomial has two coincident roots. Thus, there exists t_0 such that $\theta x = t_0 y$, or, in other words $x = \alpha y$ where $\alpha = \theta^{-1} t_0$. ■

Corollary 7.1.1 *Let H be a Hilbert space. Let $y \in H$. Define*

$$f_y(x) = (x, y)$$

for all $x \in H$. Then $f_y \in H^*$ and $\|f_y\| = \|y\|$.

Proof Clearly f_y is a linear functional. By the Cauchy-Schwarz inequality, we have

$$|f_y(x)| \leq \|x\|.\|y\|$$

which shows that $f_y \in H^*$ and that $\|f_y\| \leq \|y\|$. If $y \neq \mathbf{0}$, then set $x = y/\|y\|$. Then $f_y(x) = \|y\|$, which shows that $\|f_y\| = \|y\|$. ■

Remark 7.1.4 We will see in the next section that all continuous linear functionals on a Hilbert space occur in this manner. ■

Corollary 7.1.2 *Let H be a Hilbert space and let $x_n \rightharpoonup x$ and $y_n \to y$ in H. Then*

$$(x_n, y_n) \to (x, y).$$

Proof Observe that

$$\begin{aligned} |(x_n, y_n) - (x, y)| &\leq |(x_n, y_n - y)| + |(x_n - x, y)| \\ &\leq \|x_n\|.\|y_n - y\| + |f_y(x_n - x)| \end{aligned}$$

by the Cauchy-Schwarz inequality and the preceding corollary. Now, since any weakly converging sequence is bounded and since $y_n \to y$, the first term on the right-hand side tends to zero. The second term also tends to zero by virtue of the preceding corollary, since $x_n \rightharpoonup x$ in H. ■

Remark 7.1.5 Since norm convergence implies weak convergence, it follows *a fortiori* that if $x_n \to x$ and $y_n \to y$ in H, then $(x_n, y_n) \to (x, y)$. ■

Theorem 7.1.2 *Let H be a Hilbert space and let $K \subset H$ be a closed and convex subset of H. Then, for every $x \in H$, there exists a unique element $P_K(x) \in K$ such that*

$$\|x - P_K(x)\| = \min_{y \in K} \|x - y\|. \tag{7.1.4}$$

Further, if H is a real Hilbert space, then $P_K(x) \in K$ is characterized by the following relations:

$$(x - P_K(x), y - P_K(x)) \leq 0 \tag{7.1.5}$$

for every $y \in K$.

Proof Since H is uniformly convex, the existence and uniqueness of $P_K(x)$ has been proved in Theorem 5.6.1. Let $y \in K$. For any $0 < t < 1$, set $z = (1-t)P_K(x) + ty$ which belongs to K by convexity. Then, by virtue of (7.1.4),

$$\|x - P_K(x)\| \leq \|x - z\| = \|(x - P_K(x)) - t(y - P_K(x))\|.$$

Squaring both sides, we get

$$\|x - P_K(x)\|^2 \leq \|x - P_K(x)\|^2 - 2t(x - P_K(x), y - P_K(x)) + t^2\|y - P_K(x)\|^2.$$

Cancelling the common term *viz.* $\|x - P_K(x)\|^2$, dividing throughout by t and letting $t \to 0$, we get (7.1.5).

Conversely, if $P_K(x) \in K$ is an element satisfying (7.1.5), then, for any $y \in K$, we have

$$\begin{aligned}\|x - P_K(x)\|^2 &= \|(x-y) + (y - P_K(x))\|^2 \\ &= \|x-y\|^2 + 2(x-y, y - P_K(x)) + \|y - P_K(x)\|^2 \\ &= \|x-y\|^2 + 2(x - P_K(x), y - P_K(x)) - \|y - P_K(x)\|^2 \\ &\leq \|x-y\|^2.\end{aligned}$$

Thus $P_K(x)$ also satisfies (7.1.4). ■

Remark 7.1.6 If H is a complex Hilbert space, then $(x - P_K(x), y - P_K(x))$ is replaced by its real part in (7.1.5). ■

Remark 7.1.7 The element $P_K(x)$, which is closest to x in K, is called the *projection of x onto K*. In general, the mapping $x \mapsto P_K(x)$ is *not* linear. In $\mathbb{R}^2$, the condition (7.1.5) means that, for all $y \in K$, the lines joining x to $P_K(x)$ and y to $P_K(x)$ will always make an obtuse angle. ■

We now study some properties of the mapping $P_K : H \to K$.

Proposition 7.1.2 *Let H be a Hilbert space and let K be a closed and convex subset of H. Let $P_K : H \to K$ be as defined by the preceding theorem. Then, for all x and $y \in H$, we have*

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|.$$

Proof Assume, for simplicity, that H is a real Hilbert space. By virtue of (7.1.5), we have

$$(x - P_K(x), P_K(y) - P_K(x)) \leq 0$$

and

$$(y - P_K(y), P_K(x) - P_K(y)) \leq 0.$$

Adding these two inequalities, we get

$$(x - y, P_K(y) - P_K(x)) + \|P_K(y) - P_K(x)\|^2 \leq 0.$$

Thus,

$$\|P_K(y) - P_K(x)\|^2 \leq (y - x, P_K(y) - P_K(x))$$

and the result now follows from the Cauchy-Schwarz inequality being applied to the term on the right-hand side. ■

Corollary 7.1.3 *Let M be a closed subspace of a Hilbert space H. Then the projection P_M is a continuous linear mapping. Further, for $x \in H$, the element $P_M(x) \in M$ is characterized by*

$$(P_M(x), y) = (x, y) \tag{7.1.6}$$

for every $y \in M$.

Proof If (7.1.6) holds, then (7.1.5) holds trivially. Conversely, if $P_M(x) \in M$ is the projection of x onto M, then $P_M(x)$ satisfies (7.1.5). Let $y \in M$. Set $z = y + P_M(x) \in M$, since M is a subspace. Then (7.1.5) yields

$$(x - P_M(x), y) \leq 0$$

for all $y \in M$. Since we also have $-y \in M$, we get the reverse inequality as well and this proves (7.1.6). It now follows from (7.1.6) that P_M is a linear map and it is continuous by the preceding proposition. This completes the proof. ■

Remark 7.1.8 If M is a closed subspace of a Hilbert space H, the vector $x - P_M(x)$ is orthogonal to every vector in M. Thus, P_M is called the **orthogonal projection** of H onto M. ■

Theorem 7.1.3 *Let H be a Hilbert space and let M be a closed subspace. then M is complemented in H.*

Proof Set

$$M^\perp = \{y \in H | (x, y) = 0 \text{ for all } x \in M\}.$$

It is immediate to check that $M^\perp$ is a subspace. It is also closed. For, let $\{y_n\}$ be a sequence in $M^\perp$ and let $y_n \to y$ in H. If $x \in M$ is arbitrary, then since $(x, y_n) = 0$ for all n, we get that $(x, y) = 0$ as well and so $y \in M^\perp$ which establishes our claim. If $x \in H$, then $P_M(x) \in M$ and $x - P_M(x) \in M^\perp$ by the preceding corollary. Thus $H = M + M^\perp$. Further, if $x \in M \cap M^\perp$, we then have that $\|x\|^2 = (x, x) = 0$ and so $M \cap M^\perp = \{\mathbf{0}\}$. Thus $H = M \oplus M^\perp$ and the proof is complete. ■

Remark 7.1.9 The subspace $M^\perp$ consisting of all vectors orthogonal to all elements of M is called the **orthogonal complement** of M. The notation is not accidental. We will see in the next section that (at least in the case of real Hilbert spaces) the orthogonal complement can be identified with the annihilator of M. ■

Remark 7.1.10 If M is a closed subspace of a Hilbert space H, and if P_M is the orthogonal projection on to M, then, clearly, $P_M(x) = x$ for every $x \in M$. If $x \in M^\perp$, since $x - P_M(x) \in M^\perp$ always, we have that $P_M(x) \in M \cap M^\perp$ and so $P_M(x) = 0$. Conversely, if $P_M(x) = 0$, for some $x \in H$, then $x = x - P_M(x) \in M^\perp$. Thus $M^\perp$ is the null space (or kernel) of P_M. ■

7.2 The Dual of a Hilbert Space

Earlier, we saw that every vector in a Hilbert space gave rise to a continuous linear functional. The main result of this section is to show that all continuous linear functionals arise in this way.

Theorem 7.2.1 *(***Riesz Representation Theorem***) Let H be a Hilbert space. Let $\varphi \in H^*$. Then, there exists a unique vector $y \in H$ such that*

$$\varphi(x) \ = \ (x, y) \tag{7.2.1}$$

for all $x \in H$. Further, $\|\varphi\| = \|y\|$.

Proof We saw that (cf. Corollary 7.1.1), given $y \in H$, the functional f_y defined by

$$f_y(x) \ = \ (x, y)$$

is in H^* and that $\|f_y\| = \|y\|$. Thus, the mapping $\Phi : H \to H^*$ defined by $\Phi(y) = f_y$ is an isometry of H into H^* and so its image is closed in H^*. If we show that the image is dense in H^*, then it will follow that $H^* = \Phi(H)$, or, in other words, that Φ is onto and this will complete the proof.

Consider a linear functional φ on H^* which vanishes on $\Phi(H)$. Since every Hilbert space is uniformly convex and hence, reflexive, this means that there exists $x \in H$ such that $f_y(x) = 0$ for all $y \in H$. This implies that $(x, y) = 0$ for all $y \in H$. In particular,

$$\|x\|^2 \ = \ (x, x) \ = \ 0$$

which shows that $x = \mathbf{0}$, *i.e.* φ is zero. This shows that $\Phi(H)$ is dense in H^* and the proof is complete. ■

Remark 7.2.1 It is also possible to directly prove this theorem without using the reflexivity of H. This will be outlined in the exercises at the end of this chapter. ■

Remark 7.2.2 Let H be a Hilbert space. Then every element of the dual, H^*, can be represented as f_x, where $x \in H$. We can then define an inner product on H^* by

$$(f_x, f_y)_* = (y, x).$$

It is easy to see that this defines an inner product which gives rise to the usual norm on H^*. Thus, H^* also becomes a Hilbert space in its own right. In the same way, H^{**} also becomes a Hilbert space. Now, we have two natural mappings from H into H^{**}. The first is the usual canonical imbedding $x \mapsto J(x)$. The second is the mapping $x \mapsto f_{f_x}$, *i.e.* the composition of the Riesz map $H \to H^*$ and that of $H^* \to H^{**}$. We will show that these are the same. The latter map, by the Riesz representation theorem, is onto and so J will be onto, giving another proof of the reflexivity of a Hilbert space, provided we prove the Riesz representation theorem independently, as suggested in Remark 7.2.1. To see that the maps are the same, observe that if $f = f_y \in H^*$, then

$$\begin{array}{rcccl} f_{f_x}(f) = (f, f_x)_* & = & (f_y, f_x)_* & = & (x, y) \\ J(x)(f) = \ f(x) & = & f_y(x) & = & (x, y). \end{array}$$

This establishes the claim. ■

Remark 7.2.3 As a consequence of the Riesz representation theorem, the map $y \mapsto f_y$ is an isometry of H onto H^*. It is linear if H is real and conjugate linear if H is complex. Thus, at least in the real case, we can identify a Hilbert space with its own dual via the Riesz isometry. ■

Remark 7.2.4 In the case of real Hilbert spaces, while we can identify a Hilbert space with its dual, we have to be careful in doing so and we cannot do it to *every* space under consideration at a time. A typical example of such a situation is the following. Let V and H be real Hilbert spaces. Let $V \subset H$ and let V be dense in H. Let us assume further that there exists a constant $C > 0$ such that

$$\|v\|_H \leq C\|v\|_V$$

for every $v \in V$.

Let us now identify H^* with H via the Riesz representation theorem. Let $f \in H$. Then the map $v \mapsto (v, f)_H$ defines a continuous linear functional on V since

$$|(v, f)_H| \leq \|v\|_H\|f\|_H \leq C\|v\|_V\|f\|_H$$

for all $v \in V$. Let us denote this linear functional by $T(f)$. Thus $T \in \mathcal{L}(H, V^*)$ and

$$\|T(f)\|_{\mathcal{L}(H,V^*)} \leq C\|f\|_H.$$

If $T(f) = \mathbf{0}$, then $(v, f) = 0$ for all $v \in V$ and so, by density, we have $f = \mathbf{0}$. Thus T is one-one as well. Finally, we claim that the image of T is dense in V^*. Indeed, if $\varphi \in V^{**}$ vanishes on $T(H)$, then, by reflexivity, there exists $v \in V$ such that $T(f)(v) = 0$ for all $f \in H$ *i.e.* $(v, f) = 0$ for all $f \in H$. Since $V \subset H$, it follows that $(v, v) = 0$, *i.e.* $v = \mathbf{0}$, which means that φ is identically zero, which establishes the claim.

Thus we have the following scheme:

$$V \subset H \cong H^* \subset V^*$$

where both the inclusions are dense. It would now be clearly absurd for us to identify V with V^* as well. Thus we cannot simultaneously identify V and H with their respective duals. The space H in this case is called the *pivot space* and is identified with its dual, whereas the other spaces, though they are also Hilbert spaces, will not be identified with their respective duals. This situation typically arises when we have a parametrized family of Hilbert spaces as in the case of the *Sobolev spaces* (cf. Kesavan [1]). In particular, we can set $V = H_0^1(a, b)$ (cf. Example 7.1.4) and $H = L^2(a, b)$. We identify $L^2(a, b)$ with its dual while we denote the dual of $H_0^1(a, b)$ by $H^{-1}(a, b)$ and we have the inclusions

$$H_0^1(a, b) \subset L^2(a, b) \cong (L^2(a, b))^* \subset H^{-1}(a, b).$$

■

Let H be a Hilbert space and let $A \in \mathcal{L}(H)$. For a fixed $y \in H$, the map $x \mapsto (A(x), y)$ clearly defines a continuous linear functional on H and so, by the Riesz representation theorem, this functional can be written as the inner product of x with a vector (which depends on y). This leads us to the following definition.

Definition 7.2.1 Let H be a Hilbert space and let $A \in \mathcal{L}(H)$. We define the **adjoint** of A as the mapping $A^* : H \to H$ given by

$$(x, A^*(y)) = (A(x), y) \tag{7.2.2}$$

for all x and $y \in H$. ■

Remark 7.2.5 In the case of real Hilbert spaces, since H and H^* can be identified via the Riesz isometry, the map A^* is just the adjoint in the sense of Definition 4.7.2. ■

The following proposition lists the properties of the adjoint map.

Proposition 7.2.1 *Let H be a Hilbert space. Let A_i, $i = 1, 2$ and A be continuous linear operators on H. Let α be a scalar. Then*

(i) $\|A\| = \|A^*\|$*;*
(ii) $\|A^*A\| = \|A\|^2$*;*
(iii) $A^{**} = A$*;*
(iv) $(A_1 + A_2)^* = A_1^* + A_2^*$*;*
(v) $(A_1A_2)^* = A_2^*A_1^*$*;*
(vi) $(\alpha A)^* = \overline{\alpha}A^*$ *(to be interpreted as* αA^* *in the real case).*

Proof By the Cauchy-Schwarz inequality, we have for any $x \in H$,

$$\|x\| = \sup_{\|y\|\leq 1} |(x, y)|.$$

It then follows that if $A \in \mathcal{L}(H)$, then

$$\|A\| = \sup_{\|x\|\leq 1} \|A(x)\| = \sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} |(A(x), y)|.$$

Then, using (7.2.2) and the Cauchy-Schwarz inequality, we get

$$\|A\| = \sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} |(A(x), y)| = \sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} |(x, A^*(y))| \leq \|A^*\|.$$

Similarly,

$$\|A^*\| = \sup_{\|y\|\leq 1} \sup_{\|x\|\leq 1} |(x, A^*(y))| = \sup_{\|y\|\leq 1} \sup_{\|x\|\leq 1} |(A(x), y)| \leq \|A\|.$$

This proves (i). Again, if x and $y \in H$, then

$$|(A^*A(x), y)| = |(A(x), A(y))| \leq \|A\|^2\|x\|.\|y\|$$

from which, we deduce that

$$\|A^*A\| \leq \|A\|^2.$$

On the other hand,

$$\|A(x)\|^2 = (A(x), A(x)) = (A^*A(x), x) \leq \|A^*A\|.\|x\|^2$$

by the Cauchy-Schwarz inequality and we deduce that

$$\|A\|^2 \leq \|A^*A\|.$$

This proves (ii). The other relations follow trivially from (7.2.2). ■

Remark 7.2.6 A Banach algebra, B, is said to be a *-*algebra* if there exists a mapping $x \mapsto x^*$ from B into itself satisfying the properties analogous to (iii)–(vi) of the above proposition. Such a mapping is said to be an *involution*. If, in addition, properties (i) and (ii) are also true, it is said to be a *B^*-algebra*. Thus, if H is a Hilbert space, the $\mathcal{L}(H)$ is a B^*-algebra with the involution being given by the adjoint mapping. ■

Definition 7.2.2 Let H be a Hilbert space and let $A \in \mathcal{L}(H)$. A is said to be **self-adjoint** if $A^* = A$. It is said to be **normal** if $AA^* = A^*A$. It is said to be **unitary** if $AA^* = A^*A = I$, where I is the identity operator on H. ■

Example 7.2.1 Any orthogonal projection in a Hilbert space is self-adjoint. If $P : H \to M$ is the orthogonal projection of a Hilbert space H onto a closed subspace M, then, for any x and $y \in H$, we have

$$(P^*(x), y) = (x, P(y)) = (P(x), P(y)) = (P(x), y)$$

by repeated application of Corollary 7.1.3. Since x and y are arbitrary elements of H, it follows that $P = P^*$. ■

Example 7.2.2 In ℓ_2^n, the operator defined by a hermetian matrix is self-adjoint, that defined by a normal matrix is normal and that defined by a unitary matrix (orthogonal matrix, if the base field is $\mathbb{R}$) is unitary (cf. Definition 1.1.14). ■

Remark 7.2.7 If $A : D(A) \subset H \to H$ is a densely defined linear transformation in a Hilbert space H, it is easy to see how to define the adjoint $A^* : D(A^*) \subset H \to H$. Again, we have for $u \in D(A)$ and $v \in D(A^*)$,

$$(A(u), v) = (u, A^*(v)).$$

All the results of Sect. 4.7, in particular, Proposition 4.7.3 and Theorem 4.7.1, are true. ■

Proposition 7.2.2 *Let H be a Hilbert space and let $P \in \mathcal{L}(H)$. Then, P is an orthogonal projection if, and only if, $P = P^2 = P^*$.*

Proof Let P be an orthogonal projection on to a closed subspace M of H. Then, it is clear that $P = P^2$ (cf. Remark 7.1.10). Now, let $x, y \in H$ be arbitrary elements. Then, since P is an orthogonal projection, we have

$$(P^*(x), y) = (x, P(y)) = (P(x), P(y)) = (P(x), y),$$

by repeated application of Corollary 7.1.3. This proves that $P = P^*$. Thus, $P = P^2 = P^*$. ■

Conversely, assume that $P \in \mathcal{L}(H)$ is such that $P = P^2 = P^*$. Let N denote the null space of P, which is closed. Let $M = N^\perp$, so that $M^\perp = N$, and M is closed as well. Let $x \in H$. If $y \in N$, then $(P(x), y) = (x, P^*(y)) = (x, P(y)) = 0$, since $P = P^*$ and $P(y) = \mathbf{0}$. Thus, $P(x) \in M$. Further, if $x \in H$, then $x - P(x) \in N$, since $P = P^2$. In particular, if $x \in M$, then $x - P(x) \in M \cap N$ and so $x = P(x)$. Thus, the range of P is M. Now, for any $x \in H$, and for any $y \in M$, we have $(x - P(x), y) = 0$, *i.e.* $(x, y) = (Px, y)$ and so P is the orthogonal projection on to M (cf. Corollary 7.1.3). ■

We conclude this section with a result which is special to *complex* Hilbert spaces.

Proposition 7.2.3 *Let H be a* complex *Hilbert space and let $T \in \mathcal{L}(H)$.*

(a) If $(T(x), x) = 0$ for all $x \in H$, then $T = \mathbf{0}$.
(b) If $(T(x), x) \geq 0$ for all $x \in H$, then $T = T^*$.

Proof Let $x, y \in H$. Then, $(T(x + y), x + y) = (T(x), x) = (T(y), y) = 0$. We deduce from this that

$$(T(x), y) + (T(y), x) = 0.$$

Further, $(T(x + iy), x + iy) = 0$, which yields

$$i(T(y), x) - i(T(x), y) \;=\; 0.$$

From these two relations we deduce that, for all $x, y \in H$, we have $(T(x), y) = 0$ which implies that $T(x) = 0$ for all $x \in H$. This proves (a). ■

If $(T(x), x) \geq 0$ for all $x \in H$, then

$$0 \leq (T(x), x) \;=\; \overline{(T(x), x)} \;=\; (x, T(x)) \;=\; (T^*(x), x).$$

Thus, we get $((T - T^*)(x), x) = 0$ for all $x \in H$ and so, by (a), we deduce that $T - T^* = \mathbf{0}$. This proves (b). ■

Example 7.2.3 The above results are not valid on real Hilbert spaces. For instance, let $H = \ell_2^2$, *i.e.* $\mathbb{R}^2$ equipped with the usual euclidean norm $\|\cdot\|_2$. Consider the linear transformation T given by the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $(T(\mathbf{x}), \mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbb{R}^2$, but T is neither zero, nor is it self-adjoint. ■

7.3 Application: Variational Inequalities

Let H be a real Hilbert space and let $a(\cdot, \cdot) : H \times H \to \mathbb{R}$ be a continuous bilinear form (cf. Example 4.7.5). Let $M > 0$ such that

$$|a(x,y)| \leq M\|x\|.\|y\| \tag{7.3.1}$$

for all x and $y \in H$. Assume further that $a(\cdot,\cdot)$ is H-elliptic (or, coercive; cf. Exercise 5.28). Let $\alpha > 0$ such that

$$a(x,x) \geq \alpha\|x\|^2 \tag{7.3.2}$$

for all $x \in H$.

Example 7.3.1 The inner product of a real Hilbert space is a symmetric, continuous and coercive bilinear form. Conversely, if $a(\cdot,\cdot)$ is a symmetric, continuous and coercive bilinear form, then

$$(x,y)_a \stackrel{\text{def}}{=} a(x,y)$$

defines a new inner product on H. The associated norm (cf. Exercise 7.3) is

$$\|x\|_a = \sqrt{a(x,x)}.$$

Thanks to the continuity and coercivity of the bilinear form, we have

$$\sqrt{\alpha}\|x\| \leq \|x\|_a \leq \sqrt{M}\|x\|.$$

Thus the two norms on H are equivalent. ∎

Example 7.3.2 Let $H = \ell_2^n$. Let $\mathbf{A}$ be an $n \times n$ matrix. If $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n = \ell_2^N$ are vectors, define

$$a(x,y) = \mathbf{y}'\mathbf{A}\mathbf{x}$$

where $\mathbf{y}'$ is the transpose of the column vector $\mathbf{y}$. Then $a(.,.)$ defines a continuous bilinear form on ℓ_2^n. If $\mathbf{A}$ is symmetric, then the bilinear form is symmetric as well. If $\mathbf{A}$ is positive definite, then the bilinear form is coercive. ∎

Theorem 7.3.1 *(**Stampacchia's Theorem**) Let H be a real Hilbert space and let $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ be a continuous and coercive bilinear form on H (satisfying (7.3.1) and (7.3.2)). Let K be a closed and convex subset of H. Let $f \in H$. Then, there exists a unique $x \in K$ such that, for all $y \in K$,*

$$a(x, y-x) \geq (f, y-x). \tag{7.3.3}$$

Proof Let $u \in H$ be fixed. The map $v \mapsto a(u,v)$ is a continuous linear functional on H, by the continuity of the bilinear form. Thus, by the Riesz representation theorem, there exists $A(u) \in H$ such that

$$(A(u), v) = a(u,v)$$

for all $v \in H$. Clearly, the map $u \mapsto A(u)$ is linear. Further, by (7.3.1) and (7.3.2) we have

$$\|A(u)\| \leq M\|u\| \text{and} (A(u), u) \geq \alpha\|u\|^2$$

for all $u \in H$. Thus $A \in \mathcal{L}(H)$. Now, (7.3.3) is equivalent to finding $x \in K$ such that

$$(A(x), y - x) \geq (f, y - x)$$

for all $y \in K$. If $\rho > 0$ is any constant (to be determined suitably), this is equivalent to finding $x \in K$ such that

$$(-\rho A(x) + \rho f + x - x, y - x) \leq 0$$

for all $y \in K$. In other words (cf. Theorem 7.1.2),

$$x = P_K(x - \rho A(x) + \rho f) \stackrel{\text{def}}{=} S(x).$$

Thus, we seek a fixed point of the mapping $S : K \to K$. Let x_1 and $x_2 \in K$. Then, by Proposition 7.1.2, we have

$$\|S(x_1) - S(x_2)\| \leq \|x_1 - x_2 - \rho(A(x_1) - A(x_2))\|.$$

Squaring both sides, we get

$$\begin{aligned} \|S(x_1) - S(x_2)\|^2 &\leq \|x_1 - x_2\|^2 - 2\rho(x_1 - x_2, A(x_1) - A(x_2)) \\ &\quad +\rho^2\|A(x_1) - A(x_2)\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2 M^2)\|x_1 - x_2\|^2 \end{aligned}$$

using (7.3.1) and (7.3.2). Now, choosing ρ such that

$$0 < \rho < \frac{2\alpha}{M^2},$$

we have $1 - 2\rho\alpha + \rho^2 M^2 < 1$ so that $S : K \to K$ is a contraction. Since K is closed, by the contraction mapping theorem (cf. Theorem 2.4.1) we deduce that there exists a unique fixed point $x \in K$ for S which completes the proof. ■

Corollary 7.3.1 *(**Lax-Milgram Lemma**) Let H be a Hilbert space and let $a(\cdot, \cdot)$: $H \times H \to \mathbb{R}$ be a continuous and coercive bilinear form. Let $f \in H$. Then, there exists a unique $x \in H$ such that*

$$a(x, y) = (f, y)$$

for every $y \in H$.

Proof Applying the preceding theorem with $K = H$, there exists a unique $x \in H$ satisfying (7.3.3). Replacing y by $y + x$, we get

$$a(x, y) \geq (f, y)$$

for every $y \in H$. Since $-y \in H$ as well, we also get the reverse inequality. Hence the result. ■

Remark 7.3.1 The Lax-Milgram lemma was already proved in Exercise 5.28. If, in addition $a(\cdot, \cdot)$ is symmetric, then the preceding results have been proved via Exercise 5.29. In that case, the solution x has a variational characterization, *viz.* $x \in K$ is the minimizer of the functional

$$J(y) = \frac{1}{2}a(y, y) - (f, y)$$

over K. For this reason, (7.3.3) is called a *variational inequality*. In the terminology of the calculus of variations, (7.3.3) is the equivalent of the *Euler-Lagrange* condition for the minimization of a functional. In the case of *unconstrained minimization i.e.* $K = H$, this becomes an *equation* instead of an inequality, as seen in the Lax-Milgram Lemma, and corresponds to the vanishing of the 'first variation' of J (cf. Kesavan [2]).

Indeed, it is easy to see that J is Fréchet differentiable (cf. Exercise 2.48) and that

$$J'(x)(y) = a(x, y) - (f, y)$$

for any x and $y \in H$. Thus (7.3.3) and the Lax-Milgram lemma are just the results of Exercise 2.54 when $a(\cdot, \cdot)$ is symmetric.

The Lax-Milgram lemma forms the basis of a wide class of numerical methods, known as *finite element methods*, to solve boundary value problems for elliptic partial differential equations (cf. Kesavan [1]). ■

Remark 7.3.2 In the symmetric case, as explained in Example 7.3.1, $a(\cdot, \cdot)$ defines a new inner product whose norm is equivalent to the usual norm. Thus the dual space remains the same and so the Lax-Milgram lemma is just a restatement of the Riesz representation theorem. ■

Remark 7.3.3 If M is a closed subspace of a Hilbert space H, then we can use $K = M$ in the proof of Corollary 7.3.1 to prove the existence of a unique $u \in M$ such that

$$a(u, y) = (f, y), \text{ for every } y \in M.$$ ■

7.4 Orthonormal Sets

As mentioned earlier, orthogonality is a very important notion special to Hilbert spaces. In this section, we will take a closer look at this property.

Definition 7.4.1 Let H be a Hilbert space and let $\mathcal{I}$ be an indexing set. A subset $S = \{u_i \in H | i \in \mathcal{I}\}$ is said to be **orthonormal** if

$$\|u_i\| = 1 \text{ for all } i \in \mathcal{I}$$

and

$$(u_i, u_j) = 0 \text{ for all } i, j \in \mathcal{I}, i \neq j. \qquad \blacksquare$$

Remark 7.4.1 If we use the *Kronecker symbol, viz.* δ_{ij} which equals unity if $i = j$ and equals zero if $i \neq j$, then the above relations can be written as

$$(u_i, u_j) = \delta_{ij}$$

for all i and $j \in \mathcal{I}$. ■

Remark 7.4.2 An orthonormal set of vectors is automatically linearly independent. For, if we have a linear relation of the form

$$\sum_{k=1}^{n} \alpha_k u_{i_k} = \mathbf{0},$$

then, taking the inner product with u_{i_j} and using the orthonormality of the vectors, we get $\alpha_j = 0$ for any $1 \leq j \leq n$. ■

Example 7.4.1 The sequence $\{\mathsf{e}_n\}$ in ℓ_2 (cf. Example 2.3.12) forms an orthonormal set. Similarly, the standard basis in ℓ_2^n (cf. Example 1.1.2) forms an orthonormal set. ■

Example 7.4.2 Consider the interval $X = [0, 1]$ endowed with the Lebesgue measure. The corresponding space $L^2(\mu)$ is denoted $L^2(0, 1)$ (cf. Sect. 6.3). The sequence $\{\mathsf{f}_n\}$ where f_n is the equivalence class represented by the function $f_n(t) = \sqrt{2} \sin n\pi t$, forms an orthonormal set. ■

Proposition 7.4.1 *(***Gram-Schmidt Orthogonalization***) Let H be a Hilbert space and let $\{x_1, \cdots, x_n\}$ be a set of linearly independent vectors in H. Then there exists an orthonormal set of vectors $\{e_1, \cdots, e_n\}$ in H such that, for each $1 \leq i \leq n$, the vector e_i is a linear combination of the vectors $x_1, \cdots, x_i$.*

Proof Clearly, none of the x_i can be the null vector. Define

$$e_1 = \frac{1}{\|x_1\|} x_1.$$

Next, consider the vector $x_2 - (x_2, e_1)e_1$. This vector cannot vanish since x_1 and x_2 are linearly independent and e_1 is a scalar multiple of x_1. Thus, we can define

$$e_2 = \frac{1}{\|x_2 - (x_2, e_1)e_1\|} [x_2 - (x_2, e_1)e_1].$$

It is now immediate to check that $\|e_1\| = \|e_2\| = 1$ and that $(e_1, e_2) = 0$. Further, e_2 is a linear combination of x_1 and x_2, since e_1 is just a scalar multiple of x_1.

We can now proceed inductively. Assume that we have constructed the vectors $e_1, \cdots, e_k$, for $1 \leq k \leq n-1$. We then define

$$e_{k+1} = \frac{1}{\|x_{k+1} - \sum_{i=1}^{k} (x_{k+1}, e_i)e_i\|} \left[x_{k+1} - \sum_{i=1}^{k} (x_{k+1}, e_i)e_i \right].$$

It is now easy to verify that the set $\{e_1, \cdots, e_n\}$ verifies the conditions mentioned in the statement of the proposition. ■

In the exercises at the end of this chapter, we will see important examples of the Gram-Schmidt orthogonalization process leading to various well known special functions of mathematical physics.

Remark 7.4.3 Consider the space $\mathbb{R}^n = \ell_2^n$. For any $1 \leq j \leq n$, the sets $\{x_1, \cdots, x_j\}$ and $\{e_1, \cdots, e_j\}$ span the same subspace. Thus, we can write

$$x_j = \sum_{i=1}^{j} r_{ij} e_i.$$

Let $\mathbf{A}$ be the matrix whose columns are the x_j, and $\mathbf{Q}$ the matrix whose columns are the e_j. Let $\mathbf{R}$ be the matrix whose entries are the r_{ij}. For any j, we have that $r_{ij} = 0$ if $i > j$. Thus, $\mathbf{R}$ is an upper triangular matrix. Further, we see that

$$\mathbf{A} = \mathbf{QR}.$$

Since the columns of $\mathbf{A}$ are linearly independent, the matrix $\mathbf{A}$ is invertible. Since the columns of $\mathbf{Q}$ are orthonormal, the matrix $\mathbf{Q}$ is orthogonal. Thus, the Gram-Schmidt orthogonalization process proves the following result from matrix theory: *every invertible matrix can be decomposed into the product of an orthogonal matrix and an upper triangular matrix.* ■

Remark 7.4.4 The process of producing orthonormal vectors from linearly independent ones is quite useful in several contexts. For instance, let us consider a continuous function on the interval [0, 1]. We wish to approximate it by a polynomial. Amongst several ways of doing this, one is the *least squares approximation*. We look for a polynomial p of degree at most n such that

$$\int_0^1 |f(t) - \mathsf{p}(t)|^2 \, \mathrm{d}t = \min_{\mathsf{q} \in \mathcal{P}_n} \int_0^1 |f(t) - \mathsf{q}(t)|^2 \, \mathrm{d}t$$

where $\mathcal{P}_n$ is the space of all polynomials (in one variable) of degree less than or equal to n. In other words, we are looking for the projection of f onto the subspace of polynomials of degree less than , or equal to, n in the space $L^2(0, 1)$. We know that such a p exists uniquely and that it is characterized by (cf. Corollary 7.1.3)

$$(\mathsf{p}, \mathsf{q}) = (f, \mathsf{q})$$

for all $\mathsf{q} \in \mathcal{P}_n$. By linearity, it is sufficient to check the above relation for just the basis elements of $\mathcal{P}_n$. The standard basis of $\mathcal{P}_n$ consists of the functions $\mathsf{p}_0, \mathsf{p}_1, \cdots, \mathsf{p}_n$ where

$$\mathsf{p}_0(t) \equiv 1 \text{ and } \mathsf{p}_k(t) = t^k$$

for $t \in [0, 1]$ and for all $1 \leq k \leq n$. Writing

$$\mathsf{p}(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_n t^n,$$

we then derive the following linear system:

$$\mathbf{Ax} = \mathbf{f}$$

where, $\mathbf{A} = (a_{ij})$ is the $(n + 1) \times (n + 1)$ matrix given by

$$a_{ij} = (\mathsf{p}_j, \mathsf{p}_i) = \int_0^1 t^{i+j} \, \mathrm{d}t = \frac{1}{i + j + 1};$$

$\mathbf{x}$ is the $(n + 1) \times 1$ column vector whose components are the unknown coefficients of p, $\alpha_0, \alpha_1, \cdots, \alpha_n$; $\mathbf{f}$ is the $(n + 1) \times 1$ column vector whose i-th component is

$$f_i = (f, \mathsf{p}_i) = \int_0^1 f(t) t^i \, \mathrm{d}t.$$

Solving this linear system yields p. However, especially when n is large, the matrix $\mathbf{A}$ is known to be very difficult to invert numerically; it is an example of what is known as a *highly ill-conditioned matrix, i.e.* even small errors in the data can lead to very large errors in the solution of any linear system involving this matrix.

On the other hand, if we replace the standard basis by a basis consisting of orthonormal polynomials $\mathsf{q}_0, \mathsf{q}_1, \cdots, \mathsf{q}_n$, then we can write

$$\mathsf{p} = \sum_{j=0}^{n} \beta_j \mathsf{q}_j.$$

Now,

$$(\mathsf{p}, \mathsf{q}_i) = \sum_{j=0}^{n} \beta_j (\mathsf{q}_j, \mathsf{q}_i) = \sum_{j=0}^{n} \beta_j \delta_{ji} = \beta_i$$

and so

$$\beta_i = \int_0^1 f(t)\mathsf{q}_i(t)\, \mathrm{d}t.$$

Thus, without solving any linear system, we can directly compute the least squares approximation. ∎

Example 7.4.3 Let us compute some of the elements of the orthonormal set obtained from the standard basis of the space of polynomials of degree at most n in $L^2(-1, 1)$. Recall that $\mathsf{p}_i(t) = t^i$ for $0 \le i \le n$.

$$\|\mathsf{p}_0\|_2 = \left(\int_{-1}^{1} \mathrm{d}t \right)^{\frac{1}{2}} = \sqrt{2}.$$

Thus $\mathsf{q}_0(t) = 1/\sqrt{2}$ for all $t \in [-1, 1]$. Now consider the function

$$q_1(t) = t - \frac{1}{\sqrt{2}} \left(\int_{-1}^{1} t\, \mathrm{d}t \right) \mathsf{q}_0(t) = t.$$

Then $\|q_1\|_2 = \sqrt{2}/\sqrt{3}$. Thus,

$$\mathsf{q}_1(t) = \frac{\sqrt{3}}{\sqrt{2}} t$$

for all $t \in [-1, 1]$. Next, we consider

$$q_2(t) = t^2 - \frac{\sqrt{3}}{\sqrt{2}}\left(\int_{-1}^{1} t^3\,\mathrm{d}t\right)\frac{\sqrt{3}}{\sqrt{2}}t - \frac{1}{\sqrt{2}}\left(\int_{-1}^{1} t^2\,\mathrm{d}t\right)\frac{1}{\sqrt{2}} = t^2 - \frac{1}{3}.$$

Then, a straight forward computation yields that $\|q_2\|_2 = 2\sqrt{2}/3\sqrt{5}$. Thus, for all $t \in [-1, 1]$, we get

$$\mathsf{q}_2(t) = \frac{\sqrt{5}}{2\sqrt{2}}(3t^2 - 1).$$

Similarly, we can show that

$$\mathsf{q}_3(t) = \frac{\sqrt{7}}{2\sqrt{2}}(5t^3 - 3t)$$

and so on.

An easier way of computing these polynomials will be seen in the exercises at the end of this chapter. ■

Proposition 7.4.2 *Let $\{e_1, \cdots, e_n\}$ be a finite orthonormal set in a Hilbert space H. Then, for any $x \in H$,*

$$\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2. \tag{7.4.1}$$

Further, $x - \sum_{i=1}^{n}(x, e_i)e_i$ is orthogonal to e_j for all $1 \le j \le n$.

Proof We know that $\|x - \sum_{i=1}^{n}(x, e_i)e_i\|^2 \ge 0$. Expanding this, and using the orthonormality of the set, we get (7.4.1) immediately. Further,

$$\left(x - \sum_{i=1}^{n}(x, e_i)e_i, e_j\right) = (x, e_j) - \sum_{i=1}^{n}(x, e_i)\delta_{ij} = (x, e_j) - (x, e_j) = 0.$$

This completes the proof. ■

Proposition 7.4.3 *Let H be a Hilbert space. Let $\mathcal{I}$ be an indexing set and let $\{e_i | i \in \mathcal{I}\}$ be an orthonormal set in H. Let $x \in H$. Define*

$$\mathcal{S} = \{i \in \mathcal{I} | (x, e_i) \ne 0\}. \tag{7.4.2}$$

Then, $\mathcal{S}$ is atmost countable.

Proof Define

$$\mathcal{S}_n = \left\{ i \in \mathcal{I} | |(x, e_i)|^2 > \frac{\|x\|^2}{n} \right\}.$$

By (7.4.1), it follows that $\mathcal{S}_n$ has at most $n - 1$ elements for any positive integer n. Since

$$\mathcal{S} = \cup_{n=1}^{\infty}\mathcal{S}_n,$$

it follows that $\mathcal{S}$ is at most countable. ∎

The preceding proposition helps us to define (infinite) sums over arbitrary orthonormal sets. Let $\{e_i | i \in \mathcal{I}\}$ be an orthonormal set in H for an indexing set $\mathcal{I}$. Let $x \in H$. We wish to define the sum

$$\sum_{i\in\mathcal{I}} |(x, e_i)|^2.$$

Let $\mathcal{S}$ be the set defined by (7.4.2). If $\mathcal{S} = \emptyset$, we define the above sum to be zero. If it is a finite set, then the above sum is just the finite sum of the corresponding non-zero terms. If it is countably infinite, then we choose a numbering $e_1, e_2, \cdots, e_n, \cdots$ for the elements in the orthonormal set whose inner product with x is non-zero. Then we define the above sum to be

$$\sum_{n=1}^{\infty} |(x, e_n)|^2.$$

The sum is independent of the numbering chosen since this is a series of positive terms and so any rearrangement thereof will yield the same sum.

We are now in a position to generalize (7.4.1).

Theorem 7.4.1 *(***Bessel's Inequality***) Let H be a Hilbert space and let $\{e_i | i \in \mathcal{I}\}$ be an orthonormal set in H, for some indexing set $\mathcal{I}$. Let $x \in H$. Then*

$$\sum_{i\in\mathcal{I}} |(x, e_i)|^2 \leq \|x\|^2. \tag{7.4.3}$$

Proof Let $\mathcal{S}$ be defined by (7.4.2). If $\mathcal{S}$ is empty, there is nothing to prove. If it is finite, the result is the same as (7.4.1), which has already been proved. If $\mathcal{S}$ is countably infinite, then, since (7.4.1) establishes the result for all partial sums, (7.4.3) follows. ∎

Let $\{e_i | i \in \mathcal{I}\}$ be an orthonormal set in a Hilbert space H. Given a vector $x \in H$, we now try to give a meaning to the sum

$$\sum_{i\in\mathcal{I}} (x, e_i)e_i$$

as a vector in H. Once again, let $\mathcal{S}$ be the set defined by (7.4.2). If it is empty, we define the above sum to be the null vector. If it is finite, then we define it to be the (finite) sum of the corresponding terms. Let us, therefore, assume now that $\mathcal{S}$ is a countably infinite set. Let us number the elements $E = \{e_i | i \in \mathcal{S}\}$ as $\{e_1, e_2, \cdots, e_n, \cdots\}$. Define

$$y_n = \sum_{i=1}^{n} (x, e_i)e_i.$$

If $m > n$, then

$$\|y_m - y_n\|^2 = \sum_{i=n+1}^{m} |(x, e_i)|^2$$

using the orthonormality of the set. But the sum on the right-hand side can be made arbitrarily small for large n and m since it is part of the tail of a convergent series (cf. (7.4.3)). Thus, the sequence $\{y_n\}$ is Cauchy and hence converges to a limit, say, y in H.

Assume now that the elements of the set E above are rearranged so that

$$E = \{e'_1, e'_2, \cdots, e'_n, \cdots\}$$

where each e'_i is equal to a unique e_j. Once again, we define

$$y'_n = \sum_{i=1}^{n} (x, e'_i)e'_i.$$

As before $\{y'_n\}$ is Cauchy and will converge to an element $y' \in H$.

We claim that $y = y'$ so that, whatever the manner in which we number the elements of E, we get the same limit vector, which we will unambiguously define as the required infinite vector sum.

Let $\varepsilon > 0$. Choose N sufficiently large such that, for all $n \geq N$, we have

$$\|y_n - y\| < \varepsilon,\ \|y'_n - y'\| < \varepsilon \text{ and } \sum_{i=N+1}^{\infty} |(x, e_i)|^2 < \varepsilon^2.$$

Fix $n \geq N$. Then we can find $m \geq N$ such that

$$\{e_1, \cdots, e_n\} \subset \{e'_1, \cdots, e'_m\}.$$

Then, the difference $y'_m - y_n$ will consist of a finite number of terms of the form $(x, e_i)e_i$ where all the i concerned are greater than $n (\geq N)$. Hence, it follows that

$$\|y'_m - y_n\|^2 \leq \sum_{i=N+1}^{\infty} |(x, e_i)|^2 < \varepsilon^2.$$

Thus,

$$\|y - y'\| \leq \|y - y_n\| + \|y_n - y'_m\| + \|y'_m - y'\| < 3\varepsilon$$

which proves that $y = y'$ since $\varepsilon > 0$ can be chosen arbitrarily small.

To sum up, we choose an arbitrary numbering of E and write $E = \{e_1, e_2, \cdots, e_n, \cdots\}$ and define

$$\sum_{i\in\mathcal{I}}(x,e_i)e_i \;=\; \lim_{n\to\infty}\sum_{j=1}^{n}(x,e_j)e_j.$$

The following result is now an immediate consequence of this definition and of Proposition 7.4.2.

Proposition 7.4.4 *Let H be a Hilbert space and let $\{e_i|i\in\mathcal{I}\}$ be an orthonormal set in H. let $x\in H$. Then*

$$x-\sum_{i\in\mathcal{I}}(x,e_i)e_i$$

is orthogonal to every $e_j,\ \ j\in\mathcal{I}$. ■

Definition 7.4.2 An orthonormal set in a Hilbert space is said to be **complete** if it is maximal with respect to the partial ordering on orthonormal sets induced by set inclusion. A complete orthonormal set is also called an **orthonormal basis**. ■

Proposition 7.4.5 *Every Hilbert space admits an orthonormal basis.*

Proof Given any chain (with respect to the partial ordering induced by set inclusion on orthonormal sets), the union of its members gives an upper bound. Hence, by Zorn's lemma, there exists a maximal orthonormal set. ■

Theorem 7.4.2 *Let H be a Hilbert space and let $\{e_i|i\in\mathcal{I}\}$ be an orthonormal set in H. The following are equivalent:*

(i) *The orthonormal set is complete.*
(ii) *If $x\in H$ is such that $(x,e_i)=0$ for all $i\in\mathcal{I}$, then $x=\mathbf{0}$.*
(iii) *If $x\in H$, then*

$$x \;=\; \sum_{i\in\mathcal{I}}(x,e_i)e_i. \tag{7.4.4}$$

(iv) *If $x\in H$, then*

$$\|x\|^2 \;=\; \sum_{i\in\mathcal{I}}|(x,e_i)|^2 \tag{7.4.5}$$

(This is known as **Parseval's identity**.*)*

Proof (i) $\Rightarrow$ (ii). Assume that the orthonormal set is complete and that $(x,e_i)=0$ for all $i\in\mathcal{I}$. If $x\neq\mathbf{0}$, then the set

$$\{e_i|i\in\mathcal{I}\}\cup\left\{\frac{1}{\|x\|}x\right\}$$

is also an orthonormal set strictly larger than the given set which contradicts the maximality of the given set. ■

(ii) $\Rightarrow$ (iii). We know that $x - \sum_{i\in\mathcal{I}}(x, e_i)e_i$ is orthogonal to every $e_j,\ j \in \mathcal{I}$. This immediately gives (7.4.4).
(iii) $\Rightarrow$ (iv). Set $y_n = \sum_{i=1}^{n}(x, e_i)e_i$ where $\{e_1, e_2, \cdots, e_n \cdots\}$ is a numbering of the elements of the set E explained earlier when defining the sum $\sum_{i\in\mathcal{I}}(x, e_i)e_i$. Then a straight forward computation yields

$$\|y_n\|^2 = \sum_{i=1}^{n} |(x, e_i)|^2.$$

This immediately yields (7.4.5) on passing to the limit as $n \to \infty$.
(iv) $\Rightarrow$ (i). If the given set was not complete, then there exists $e \in H$ such that $\|e\| = 1$ and $(e, e_i) = 0$ for every $i \in \mathcal{I}$. But then, this will contradict (7.4.5) (applied to the vector e). ■

Corollary 7.4.1 *Let H be a Hilbert space and let $\{e_i | i \in \mathcal{I}\}$ be an orthonormal set. It is complete if, and only if, the subspace of all (finite) linear combinations of the e_i is dense in H.*

Proof If the orthonormal set is complete, then by the preceding theorem, every element of H is the limit of finite linear combinations of the e_i by (7.4.4) and so the subspace spanned by the e_i is dense in H.

Conversely, if the subspace spanned by the e_i is dense in H, then if $x \in H$ is such that it is orthogonal to all the elements of this subspace, then $x = \mathbf{0}$. In particular, if $(x, e_i) = 0$ for all $i \in \mathcal{I}$, then, clearly, x is orthogonal to the subspace spanned by the e_i and so it must vanish. Thus, statement (ii) of the preceding theorem is satisfied and so the orthonormal set is complete. ■

Corollary 7.4.2 *Let H be a Hilbert space and let $\{e_1, e_2, \cdots, e_n \cdots\}$ be a sequence in H which is also an orthonormal basis for H. Then $e_n \rightharpoonup \mathbf{0}$.*

Proof Let $x \in H$. Then, by (7.4.5), it follows that $(x, e_n) \to 0$ as $n \to \infty$. Then, by the Riesz representation theorem, it follows that $e_n \rightharpoonup \mathbf{0}$. ■

Remark 7.4.5 Notice that if $\{e_n\}$ is an orthonormal sequence which is also complete in a Hilbert space H, then it weakly converges to the null vector while it does not have a norm convergent subsequence, since

$$\|e_n - e_m\| = \sqrt{2}$$

for all $n \neq m$. ■

Theorem 7.4.3 *An infinite dimensional Hilbert space has a countable orthonormal basis if, and only if, it is separable.*

Proof For simplicity, assume that the space is a real Hilbert space. If the space has a countable orthonormal basis $\{e_n\}$, then the set of all finite linear combinations of the $\{e_n\}$ is dense in H, by (7.4.4). The set of all finite linear combinations of the $\{e_n\}$ with rational coefficients then forms a countable dense subset.

Conversely, assume that the space is separable. If $\{e_i | i \in \mathcal{I}\}$ is an orthonormal set, then, since $\|e_i - e_j\| = \sqrt{2}$ for $i \neq j$, it follows that the balls $B(e_i; \sqrt{2}/4)$, $i \in \mathcal{I}$, are all mutually disjoint. If $\mathcal{I}$ were uncountable, then the space cannot be separable (cf. Remark 6.3.2). Thus, any orthonormal set can be at most countable. In particular, any orthonormal basis must be countable. ■

Example 7.4.4 By virtue of (7.4.5), the orthonormal sets in ℓ_2 and ℓ_2^n described in Example 7.4.1 are orthonormal bases of those spaces. ■

Example 7.4.5 (**Fourier series**) Consider the space $L^2(-\pi, \pi)$. The set

$$\{\mathsf{f}_0\} \cup \{\mathsf{f}_n, \mathsf{g}_n | n \in \mathbb{N}\}$$

where

$$f_0(t) = \frac{1}{\sqrt{2\pi}},\ f_n(t) = \frac{\cos nt}{\sqrt{\pi}} \text{ and } g_n(t) = \frac{\sin nt}{\sqrt{\pi}}$$

for $t \in (-\pi, \pi)$, forms an orthonormal set. By Theorem 6.3.1, the space of continuous functions with compact support contained in $(-\pi, \pi)$ is dense in $L^2(-\pi, \pi)$. Such functions vanish at the end points of the interval $[-\pi, \pi]$ and so they are 2π-periodic on the interval $[-\pi, \pi]$. Consider the space spanned by the orthonormal set mentioned above. By an application of the Stone-Weierstrass theorem (cf. Rudin [3]), it follows that this space is dense in the space of all 2π-periodic continuous functions with respect to the sup norm, which is nothing but the norm $\|.\|_\infty$. Since the interval $(-\pi, \pi)$ has finite measure, this implies that this space is dense with respect to the norm $\|.\|_2$ as well (cf. Proposition 6.1.3). This shows that the space spanned by this orthonormal set is dense in $L^2(-\pi, \pi)$ as well and so, by Corollary 7.4.1, it follows that it is a complete orthonormal set in $L^2(-\pi, \pi)$.

Thus if $\mathsf{f} \in L^2(-\pi, \pi)$, we have that

$$\begin{aligned} f(t) = &\left(\int_{-\pi}^{\pi} f(t)\frac{1}{\sqrt{2\pi}}\,\mathrm{d}t\right)\frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(t)\frac{\cos nt}{\sqrt{\pi}}\,\mathrm{d}t\right)\frac{\cos nt}{\sqrt{\pi}} \\ &+ \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(t)\frac{\sin nt}{\sqrt{\pi}}\,\mathrm{d}t\right)\frac{\sin nt}{\sqrt{\pi}} \end{aligned}$$

by virtue of (7.4.4). This can be rewritten as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,\mathrm{d}t,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,\mathrm{d}t.$$

This is nothing but the classical Fourier series of a function f and the $a_n,\ n \geq 0$ and $b_n,\ n \geq 1$ are the usual Fourier coefficients of f. The above series expansion means that the partial sums of the Fourier series converge to f in the $\|.\|_2$ norm. In other words if

$$f_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos nt + b_n \sin nt),$$

for $t \in (-\pi, \pi)$, then

$$\int_{-\pi}^{\pi} |f_N(t) - f(t)|^2\,\mathrm{d}t \to 0$$

as $N \to \infty$. The analogue of the Parseval identity (7.4.5) reads as follows:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(t)|^2\,\mathrm{d}t = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2).$$

■

Example 7.4.6 (**Fourier sine series**) Consider the space $L^2(0, \pi)$. Consider the set

$$\left\{\sqrt{\frac{2}{\pi}}\sin nt | n \in \mathbb{N}\right\}.$$

This is an orthonormal set in $L^2(0, \pi)$ as one can easily verify. Let f $\in L^2(0, \pi)$ be orthogonal to every element of this set. Extend f as an odd function to all the interval $(-\pi, \pi)$. Thus, we set $f(x) = -f(-x)$ if $x \in [-\pi, 0)$. Since f is an odd function, it follows that

$$\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \int_{-\pi}^{\pi} f(t)\cos nt\,\mathrm{d}t = 0$$

for all $n \in \mathbb{N}$. Since we also have that

$$\int_{-\pi}^{\pi} f(t)\sin nt\,\mathrm{d}t = 2\int_{0}^{\pi} f(t)\sin nt = 0$$

for all $n \in \mathbb{N}$, it follows that $\mathsf{f} = \mathbf{0}$ in $L^2(-\pi, \pi)$ and so $\mathsf{f} = \mathbf{0}$ in $L^2(0, \pi)$ as well. Thus, by Theorem 7.4.2 (ii), it follows that the given set is complete in $L^2(0, \pi)$. In particular, if $\mathsf{f} \in L^2(0, \pi)$, we can write the series expansion

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

where

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(t)\sin nt\,\mathrm{d}t.$$

This is called the Fourier sine series of the function f. ∎

By analogy, if H is a separable Hilbert space with an orthonormal basis $\{e_n | n \in \mathbb{N}\}$ and if $x \in H$, we call

$$x = \sum_{n=1}^{\infty} (x, e_n)e_n$$

as its Fourier expansion and the coefficients (x, e_n) are called its Fourier coefficients.

Remark 7.4.6 Let V be a Banach space and let $\{e_n\}$ be a sequence of vectors in V such that every vector $x \in V$ can be written as

$$x = \sum_{n=1}^{\infty} \alpha_n(x)e_n = \lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n(x)e_n.$$

Then $\{e_n\}$ is called a *Schauder basis* for V. Thus, in every separable Hilbert space, an orthonormal basis forms a Schauder basis. The set $\{\mathsf{e}_n\}$ in ℓ_p (cf. Example 3.1.1) is a Schauder basis for ℓ_p for $1 \leq p < \infty$.

In the literature, a usual basis of the vector space, *i.e.* a set of linearly independent elements such that every vector is a finite linear combination of vectors from the set, is called a *Hamel basis*. Notice that, by Baire's theorem (cf. Exercise 4.1), a

Banach space cannot have a countable Hamel basis, while it may have a (countable) Schauder basis. ■

In the next chapter we will see how orthonormal bases occur very naturally in Hilbert spaces.

We conclude with the following important result.

Theorem 7.4.4 *(***Riesz-Fischer***) Let H be an infinite dimensional separable Hilbert space. Then H is isometrically isomorphic to ℓ_2.*

Proof Since H is separable, it admites a countably infinite orthonormal basis $\{e_k\}_{k=1}^{\infty}$. If $x \in H$, then we have

$$\|x\|^2 = \sum_{k=1}^{\infty} |(x, e_k)|^2.$$

Thus we can define an isometry $T : H \to \ell_2$ by $(T(x))_k = (x, e_k)$. We only need to show that this map is surjective.

Let $c = (c_1, c_2, \cdots, c_k, \cdots) \in \ell_2$. Define

$$x_n = \sum_{k=1}^{n} c_k e_k \in H.$$

If $n > m$, then

$$\|x_n - x_m\|^2 = \sum_{k=m+1}^{n} |c_k|^2.$$

Since $c \in \ell_2$, it then follows that the sequence $\{x_n\}$ is Cauchy and hence will converge to some $x \in H$. For any fixed $k \in \mathbb{N}$, we have, for all $n \geq k$, $(x_n, e_k) = c_k$, by construction. Passing to the limit, we have $(x, e_k) = c_k$. Thus

$$x = \sum_{k=1}^{\infty} c_k e_k,$$

which shows that $T(x) = c$. This completes the proof. ■

7.5 Exercises

7.1 Let V be a real Banach space and assume that the parallelogram identity holds in V. Define

$$(u, v) = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2).$$

Show that this defines an inner product which induces the given norm and hence that V is a Hilbert space.

7.2 Let V be a complex Banach space and assume that the parallelogram identity holds in V. Define

$$(u, v) = \frac{1}{4}[\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2].$$

Show that this defines an inner product which induces the given norm and hence that V is a Hilbert space.

7.3 Let H be a real Hilbert space and let $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ be a symmetric and continuous bilinear form such that $a(x,x) > 0$ for all $x \in H, x \neq \mathbf{0}$. Define $\|x\|_a = \sqrt{a(x,x)}$, for all $x \in H$.

(a) Show that, for every $x, y \in H$,

$$|a(x,y)| \leq \|x\|_a \|y\|_a.$$

(b) Show that $\|\cdot\|_a$ defines a norm on H.

7.4 Let H be a Hilbert space and let M be a non-zero and proper closed subspace of H. Let $P : H \to M$ be the orthogonal projection of H onto M. Show that $\|P\| = 1$.

7.5 Let $H = \ell_2^n$. Let $\mathbf{J}$ be the $n \times n$ matrix all of whose entries are equal to $1/n$. Show that

$$\|\mathbf{J}\|_{2,n} = \|\mathbf{I} - \mathbf{J}\|_{2,n} = 1$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.

7.6 Show that the following matrix defines an orthogonal projection in ℓ_2^3. Find the range of the projection.

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

7.7 Let

$$K = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}.$$

If $a < 0$ and $b < 0$, compute $P_K(\mathbf{z})$, where $\mathbf{z} = (a, b)$.

7.8 Let H be a real Hilbert space and let K be a closed convex cone in H with vertex at the origin (cf. Definition 3.5.1).

(a) If $x, y \in K$ and if α and β are non-negative scalars, show that $\alpha x + \beta y \in K$.
(b) If $x \in H$, show that $P_K(x)$ is characterized by the following relations:

$$(x, P_K(x)) = \|P_K x\|^2, \text{and} (x - P_K(x), y) = 0 \text{for every} y \in K.$$

7.9 Let $H = L^2(-\pi, \pi)$. Write down, explicitly, the orthogonal projection onto each of the following closed subspaces:

(a) $M = \{f \in H | f(t) = f(-t), \text{for every} t \in (-\pi, \pi)\}$.
(b) $M = \{f \in H | \int_{-\pi}^{\pi} f(t)\, \mathrm{d}t = 0\}$.
(c) $M = \{f \in H | f \equiv 0 \,\text{on}(-\pi, 0)\}$.

7.10 (a) Let H be a Hilbert space and let φ be a non-zero continuous linear functional on H. Let $M = \mathrm{Ker}(\varphi)$. Show that M has codimension one.
(b) Let $g \in M^\perp$ be a unit vector such that any $y \in H$ can be written as

$$y = \lambda g + z$$

where $z \in M$. Define $x = \varphi(g)g$. Show that x is such that

$$\varphi(y) = (y, x)$$

for all $y \in H$. (This gives a direct proof of the Riesz representation theorem.)

7.11 Let H be a Hilbert space and let $W \subset H$ be a closed subspace. Let $u \in H$. Define $\varphi(w) = (w, u)$, for every $w \in W$. Write down the Hahn-Banach (*i.e.* norm preserving) extension of φ to all of H.

7.12 Let H be a Hilbert space and let $x_n \rightharpoonup x$ in H. If $\|x_n\| \to \|x\|$, show that $x_n \to x$ in H.

7.13 Let $x = (x_1, x_2, \cdots, x_k, \cdots) \in \ell_2$. Define

$$T(x) = \left(x_1, \frac{x_2}{2}, \cdots, \frac{x_k}{k}, \cdots\right).$$

Show that $T \in \mathcal{L}(\ell_2)$ and that its range is not closed.

7.14 Let H be a Hilbert space and let $A \in \mathcal{L}(H)$ be such that $A = A^*$. Let $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space (or kernel) and range of A, respectively. Which of the following statements are true?

(a) $H = \mathcal{N}(A) \oplus \mathcal{R}(A)$.
(b) $H = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$.
(c) Neither of the above need hold.

7.15 Let H be a Hilbert space. Show that if $T \in \mathcal{L}(H, \ell_1)$ has closed range, then the range of T is finite-dimensional.

7.16 Let H be a Hilbert space and let $U : H \to H$ be a unitary operator. Show that U is an isometry, *i.e.* $\|Ux\| = \|x\|$ for all $x \in H$.

7.17 Let H be a real Hilbert space and let $a(\cdot, \cdot) : H \times H \to \mathbb{R}$ be a continuous and H-elliptic bilinear form (cf. Sect. 7.3) with constants $M > 0$ (for continuity) and $\alpha > 0$ (for ellipticity). Let $f \in H$.

(a) Let $W \subset H$ be a closed subspace. Show that there exists a unique $w \in W$ such that
$$a(w, v) = (f, v) \tag{7.5.1}$$
for all $v \in W$.

(b) Show that, if $w \in W$ is as above, then
$$\|w\| \leq \frac{1}{\alpha}\|f\|.$$

(c) Let $u \in H$ be the unique vector such that
$$a(u, v) = (f, v)$$
for all $v \in H$. Show that
$$\|u - w\| \leq \frac{M}{\alpha} \inf_{v \in W} \|u - v\|.$$

(d) Let H be separable and let $\{u_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. Let $W_n = \text{span}\{u_1, \cdots, u_n\}$. Let w_n be the solution of (7.5.1) when $W = W_n$. Show that $w_n \to u$ as $n \to \infty$.

7.18 Let H be an infinite dimensional separable Hilbert space. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Let $V_k = \text{span}\{e_j | 1 \leq j \leq k\}$. Let $x \in H$. Compute the distance of x from V_k.

7.19 Consider the space $L^2(0, 1)$. Define $r_0(t) \equiv 1$ and
$$r_n(t) = \sum_{i=1}^{2^n} (-1)^{i-1} \chi_{[\frac{i-1}{2^n}, \frac{i}{2^n}]}(t)$$
where χ_E denotes the characteristic function of a set E.

(a) Show that
$$r_n(t) = \text{sgn}(\sin 2^n \pi t), 0 \leq t \leq 1$$
where $\text{sgn}(t)$ equals 1 when $t \geq 0$ and equals -1 when $t < 0$.

(b) Show that $\{\mathsf{r}_n(t)\}_{n=0}^{\infty}$ is orthonormal in $L^2(0, 1)$ but that it is not complete.

7.20 Let $(a, b) \subset \mathbb{R}$ be a finite interval and let $\{\phi_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $L^2(a, b)$. Define
$$\Phi_{ij}(t, s) = \phi_i(t)\phi_j(s)$$

for $(t, s) \in (a, b) \times (a, b)$. Show that $\{\Phi_{ij}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ forms an orthonormal basis for $L^2((a, b) \times (a, b))$.

7.21 Show that the sets

$$\left\{\frac{1}{\sqrt{\pi}}\right\} \cup \left\{\sqrt{\frac{2}{\pi}}\cos nt | n \in \mathbb{N}\right\}$$

is a complete orthonormal set in $L^2(0, \pi)$. (Thus, a function in $L^2(0, \pi)$ can be expanded as a series of cosines and this is called its *Fourier cosine series*.)

7.22 Consider $L^2(-1, 1)$ and the linearly independent set of functions p_n where $\mathsf{p}_n(t) = t^n$. Applying the Gram-Schmidt orthogonalization procedure, we obtain an orthonormal sequence $\{\mathsf{q}_n\}$ of polynomials (cf. Example 7.4.3).

(a) Define

$$P_n(t) = \sqrt{\frac{2}{2n+1}}\mathsf{q}_n(t).$$

These are the *Legendre polynomials*. Show that $P_n(t)$ consists only of even powers of t when n is even, and of only odd powers of t, when n is odd.

(b) Show that $P_0(t) \equiv 1$, $P_1(t) = t$ and that, for $n \geq 1$,

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t)$$

for $t \in [-1, 1]$, given that $P_n(1) = 1$ for all $n \geq 0$. (This gives a simple recursive formula to compute the Legendre polynomials.)

(c) Prove Rodrigues' Formula:

$$P_n(t) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}(t^2 - 1)^n.$$

7.23 (a) Consider the space

$$\widetilde{H} = \left\{f : \mathbb{R} \to \mathbb{R} | \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}|f(x)|^2\,\mathrm{d}x < \infty\right\}$$

and let H be the space of all equivalence classes (with respect to equality almost everywhere) of functions in $\widetilde{H}$. Define the inner product

$$(\mathsf{f}, \mathsf{g}) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} f(x)g(x)\,\mathrm{d}x.$$

Show that H is a Hilbert space.

(b) Show that if $f_n(x) = x^n$, then f_n belongs to H for every $n \in \{0\} \cup \mathbb{N}$.
(c) Apply the Gram-Schmidt process to the linearly independent set $\{\mathsf{f}_n\}$ to obtain an orthonormal set h_n. Define

$$H_n(x) = \sqrt{2^n n!}\,h_n(x).$$

These are the *Hermite polynomials*. Compute H_0 and H_1.
(d) Prove Rodrigues' Formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2}.$$

7.24 Let $\mathsf{f}, \mathsf{g} \in L^2(-\pi, \pi)$ and let their Fourier series be given by

$$f(t) = \tfrac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$
$$g(t) = \tfrac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos nt + d_n \sin nt).$$

Show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} (a_n c_n + b_n d_n).$$

7.25 Compute the Fourier series of the function:

$$f(t) = \begin{cases} -1 & -\pi \le t < 0 \\ 1 & 0 < t \le \pi. \end{cases}$$

7.26 Compute the Fourier cosine series of the function $f(t) = \sin t$ on $[0, \pi]$.

7.27 (a) Compute the Fourier sine series and the Fourier cosine series of the function $f(t) = t$ on $[0, \pi]$.
(b) Evaluate:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^4}$$

using Parseval's identity.

7.28 (a) Let $\mathsf{f} \in L^2(-\pi, \pi)$. Let its Fourier series be given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Extend the function f to all of $\mathbb{R}$ by periodicity, *i.e.* such that $f(t + 2\pi) = f(t)$ for all $t \in \mathbb{R}$. Define

$$F(t) = \int_0^t \left(f(s) - \frac{a_0}{2}\right)\, \mathrm{d}s.$$

Show that $F : \mathbb{R} \to \mathbb{R}$ is also 2π-periodic.

(b) Show that its Fourier series is given by

$$F(t) \;=\; \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin nt - \frac{b_n}{n} \cos nt \right).$$

(c) Show that the above series converges to F uniformly on $\mathbb{R}$.

7.29 Let $\mathsf{f} \in H_0^1(-\pi, \pi)$. Show that if its Fourier series expansion is given by

$$f(t) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

then the Fourier series expansion of f' is given by

$$f'(t) \;=\; \sum_{n=1}^{\infty} (nb_n \cos nt - na_n \sin nt).$$

7.30 Let H be an infinite dimensional, seprable Hilbert space. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a bounded sequence of scalars. For $x \in H$, define

$$A(x) \;=\; \sum_{k=1}^{\infty} \lambda_k (x, e_k) e_k.$$

Show that $A(x)$ is well-defined for each $x \in H$ and that $A \in \mathcal{L}(H)$.

7.31 Let H be a Hilbert space and let $A : D(A) \subset H \to H$ be a linear operator. We say that it is *dissipative* if $(A(u), u) \leq 0$ for all $u \in D(A)$. We say that it is *maximal dissipative* if, in addition $\mathcal{R}(I - A) = H$, where I denotes the identity operator on H. Let A be the infinitesimal generator of a c_0-semigroup of contractions (cf. Exercises 4.5, 4.20 and 4.26) on H. Show that it is maximal dissipative.

7.32 Let H be a Hilbert space and let $A : D(A) \subset H \to H$ be a maximal dissipative operator. Show that if $B : D(B) \subset H \to H$ is dissipative and is an extension of A, *i.e.* $D(A) \subset D(B)$ and $B|_{D(A)} = A$, then $D(B) = D(A)$. (This justifies the adjective 'maximal').

7.33 Let H be a Hilbert space and let $A : D(A) \subset H \to H$ be a maximal dissipative operator. Show that it is closed and densely defined.

7.34 (a) Let H be a Hilbert space and let $A : D(A) \subset H \to H$ be a dissipative operator. Let $\lambda > 0$. If $\mathcal{R}(\lambda I - A) = H$, show that $(\lambda I - A)^{-1}$ exists in $\mathcal{L}(H)$ and that

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}.$$

(b) If A is a dissipative operator and if $\mathcal{R}(\lambda_0 I - A) = H$ for some $\lambda_0 > 0$, show that $\mathcal{R}(\lambda I - A) = H$ for all $0 < \lambda < 2\lambda_0$.

(c) Deduce that if A is a maximal dissipative operator, then $\mathcal{R}(\lambda I - A) = H$ for all $\lambda > 0$.

Remark 7.5.1 Comparing the results of the above exercises with the comments made in Remark 4.8.1, we deduce that an operator $A : D(A) \subset H \to H$ will be the infinitesimal generator of a c_0-semigroup of contractions if, and only if, it is maximal dissipative. Unlike Banach spaces, where the Hille-Yosida theorem involves verification of infinitely many conditions, one for each $\lambda > 0$, this is much easier to verify in Hilbert spaces. The dissipativity is usually very easy to check. Further, it is enough to verify that the equation $(\lambda I - A)x = f$ has a solution $x \in D(A)$ for every $f \in H$ just for *one* fixed $\lambda > 0$. ■

7.35 Let $H_i,\ i = 1, 2, 3$ be Hilbert spaces with norms $\|\cdot\|_i,\ i = 1, 2, 3$, respectively. Let $T : D(T) \subset H_1 \to H_2$ and $S : D(S) \subset H_2 \to H_3$ be closed and densely defined linear transformations. Assume that $\mathcal{R}(T) \subset \mathcal{N}(S)$. Assume further, that there exists a constant $C > 0$ such that, for all $x \in D(S) \cap D(T^*)$, we have

$$\|T^*x\|_1^2 + \|Sx\|_3^2 \geq C\|x\|_2^2.$$

(a) Let $\widetilde{H}_2 = \mathcal{N}(S)$ and let $\widetilde{T}^*$ denote the adjoint of $T : D(T) \subset H_1 \to \widetilde{H}_2$. Show that $\widetilde{T}^*$ has closed range.

(b) If $P : H_2 \to \mathcal{N}(S)$ is the orthogonal projection, show that

$$T^*(x) = \widetilde{T}^*(Px)$$

for all $x \in D(T^*)$.

(c) Deduce that T has closed range.

(d) Show that $\mathcal{R}(T) = \mathcal{N}(S)$.

7.36 Let H be a Hilbert space and let $GL(H)$ be the set of all invertible continuous linear operators on H. Then $GL(H)$ is a group with respect to the binary operation defined via composition of operators. Consider the unit circle $S^1 \subset \mathbb{R}^2$ with its usual topology inherited from $\mathbb{R}^2$. Representing a point $g \in S^1$ as $(\cos\theta, \sin\theta)$ where $\theta \in [0, 2\pi)$, we have that S^1 is a group under the operation defined via $(\theta_1, \theta_2) \mapsto (\theta_1 + \theta_2)\mathrm{mod}(2\pi)$. A *representation* of S^1 is a group homomorphism $\widehat{\pi} : S^1 \to GL(H)$ which is also continuous. For simplicity, we will denote the image of $g = (\cos\theta, \sin\theta)$ by $\widehat{\pi}(\theta)$.

(a) Show that every representation is uniformly bounded, *i.e.* there exists a constant $C > 0$ such that

$$\|\widehat{\pi}(\theta)\| \leq C$$

for all $\theta \in [0, 2\pi)$.

(b) Define

$$< u, v > = \frac{1}{2\pi} \int_0^{2\pi} (\widehat{\pi}(2\pi - \theta)(u), \widehat{\pi}(2\pi - \theta)(v)) \, \mathrm{d}\theta.$$

Show that $< \cdot, \cdot >$ defines a new inner product on H whose induced norm is equivalent to the original norm on H.

(c) With respect to the inner product $< \cdot, \cdot >$ on H, show that $\widehat{\pi}(\theta)$ is a unitary operator on H for every $\theta \in [0, 2\pi)$. (We say that every representation of S^1 is equivalent to a *unitary representation*.)

7.37 Let $V = \ell_2^N$ and let $\{\mathbf{A}_n\}$ be a sequence of $N \times N$ matrices such that $\mathbf{A}_n = \mathbf{A}_n^*$ for each n. Assume further that, for each $\mathbf{v} \in V$, we have that the sequence $\{(\mathbf{A}_n\mathbf{v}, \mathbf{v})\}$ is non-negative and decreasing as $n \to \infty$. Show that there exists a matrix $\mathbf{A} = \mathbf{A}^*$ such that $(\mathbf{A}\mathbf{v}, \mathbf{v}) \geq 0$ for all $\mathbf{v} \in V$ and such that $\mathbf{A}_n \to \mathbf{A}$ in $\mathcal{L}(V)$.

References

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Chapter 8
Compact Operators

8.1 Basic Properties

In this chapter we will study a special class of linear transformations between Banach spaces which generalize several properties of linear transformations between finite dimensional spaces.

Definition 8.1.1 Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$. If $\dim(\mathcal{R}(T))$ is finite, then we say that T is of **finite rank**. If the image of every bounded set in V is relatively compact in W, we say that T is **compact**. ■

Remark 8.1.1 For compactness, it is sufficient to verify that $T(B)$ is relatively compact in W when B is the (closed) unit ball in V. Equivalently, $T \in \mathcal{L}(V, W)$ is compact if, and only if, given any bounded sequence $\{x_n\}$ in V, the sequence $\{T(x_n)\}$ admits a convergent subsequence in W. ■

Example 8.1.1 Since any bounded set in a finite dimensional space is relatively compact, it follows that every continuous linear transformation of finite rank is compact. ■

Example 8.1.2 Let V be finite dimensional. Then any $T \in \mathcal{L}(V, W)$, where W is an arbitrary Banach space, is of finite rank and hence is compact. ■

Example 8.1.3 Let V be an infinite dimensional Banach space and let $I : V \to V$ be the identity map. Since the unit ball in V is not compact (cf. Propositon 2.3.6), it follows that the identity map is not compact. ■

Example 8.1.4 Let X, Y and Z be Banach spaces and let $S \in \mathcal{L}(X, Y)$ and let $T \in \mathcal{L}(Y, Z)$. If one of these is compact, it is easy to see that their composition is also compact. In particular, since the identity map in an infinite dimensional Banach space cannot be compact, it follows that compact maps on infinite dimensional spaces are not invertible. ■

S. Kesavan, *Functional Analysis*, Texts and Readings in Mathematics,
https://doi.org/10.1007/978-981-19-7633-9_8

Example 8.1.5 Let $\widehat{i} : \mathcal{C}^1[0,1] \to \mathcal{C}[0,1]$ be the canonical inclusion map. Let us denote the usual norm in $\mathcal{C}^1[0,1]$ (cf. Exercise 2.9) by $\|\cdot\|_{1,\infty}$ and the usual norm in $\mathcal{C}[0,1]$ by $\|\cdot\|_\infty$. Consider the set of all $f \in \mathcal{C}^1[0,1]$ such that $\|f\|_{1,\infty} \leq c$. Then, clearly, $\|f\|_\infty \leq c$ as well. Further, if x and y are points in $[0,1]$, we have, by the mean value theorem,

$$|f(x) - f(y)| \;\leq\; (\sup_{t\in[0,1]} |f'(t)|)|x-y| \;\leq\; c|x-y|.$$

It follows from this that the set of all functions considered here is uniformly bounded and equicontinuous in $\mathcal{C}[0,1]$. Thus, by Ascoli's theorem, the set is relatively compact and so the inclusion map is compact. ■

Example 8.1.6 Let $K \in \mathcal{C}([0,1]\times[0,1])$. Let

$$\kappa = \sup_{(x,y)\in[0,1]\times[0,1]} |K(x,y)|.$$

Let $f \in \mathcal{C}[0,1]$. Define

$$T(f)(x) = \int_0^1 K(x,y)f(y)\,dy.$$

Then, it is easy to see that $T(f)$ is continuous and that $\|T(f)\|_\infty \leq \kappa\|f\|_\infty$ for all $f \in \mathcal{C}[0,1]$ and so T is a continuous linear operator on $\mathcal{C}[0,1]$. Thus, if we look at the set of all $f \in \mathcal{C}[0,1]$ such that $\|f\|_\infty \leq c$, it follows that the set of all $T(f)$ is uniformly bounded. Further, if x and y are points in $[0,1]$, we have

$$\begin{aligned}|T(f)(x) - T(f)(y)| &\leq \int_0^1 K(x,t) - K(y,t)|.|f(t)|\,dt \\ &\leq c\sup_{t\in[0,1]} |K(x,t) - K(y,t)|.\end{aligned}$$

Now, since K is uniformly continuous (since $[0,1]\times[0,1]$ is compact), given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t \in [0,1]$, we have

$$|K(x,t) - K(y,t)| \;<\; \varepsilon$$

whenever $|x-y| < \delta$. This shows that the set of all $T(f)$ where $\|f\|_\infty \leq c$ is equicontinuous and so T is a compact operator. ■

Example 8.1.7 By virtue of Rellich's theorem (cf. Theorem 6.4.4), it follows that the canonical inclusion map $\widehat{i} : W^{1,p}(a,b) \to L^p(a,b)$ is compact, where $(a,b) \subset \mathbb{R}$ is a finite interval and $1 \leq p < \infty$. ■

Example 8.1.8 Let $\mathsf{f} \in L^2(a,b)$ be given. The map $\mathsf{v} \mapsto \int_0^1 vf\, \mathrm{d}x$ is obviously a continuous linear functional on $H_0^1(0,1)$. Since, by Poincaré's inequality,

$$(\mathsf{v},\mathsf{w})_1 = \int_0^1 v'w'\, \mathrm{d}x$$

defines an inner product on this space (cf. Example 7.1.4), it follows from the Riesz representation theorem that there exists a unique $\mathsf{u} \in H_0^1(a,b)$ such that

$$\int_0^1 u'v'\, \mathrm{d}x = \int_0^1 fv\, \mathrm{d}x$$

for all $\mathsf{v} \in H_0^1(0,1)$. Further, setting $\mathsf{v} = \mathsf{u}$ in the above relation and using Poincaré's inequality (cf. Theorem 6.4.6), we get

$$|\mathsf{u}|_1^2 \stackrel{\text{def}}{=} \|\mathsf{u}'\|_2^2 \le \|\mathsf{f}\|_2\|\mathsf{u}\|_2 \le \|\mathsf{f}\|_2|\mathsf{u}|_1$$

from which we get that

$$|\mathsf{u}|_1 \le \|\mathsf{f}\|_2.$$

Thus, the map

$$\mathsf{f} \mapsto \mathsf{u} \stackrel{\text{def}}{=} G\mathsf{f}$$

is continuous. Composing this with the canonical inclusion of $H_0^1(0,1)$ into $L^2(0,1)$, we get that $\mathsf{f} \mapsto G\mathsf{f} = \mathsf{u}$ is a compact operator on $L^2(0,1)$. ■

Example 8.1.9 This is a generalization of the preceding example. Let $\alpha : (0,1) \to \mathbb{R}$ be a function such that $0 < c \le \alpha(x) \le C$ for all $x \in (0,1)$, where c and C are fixed positive constants. Define $a(\cdot,\cdot) : H_0^1(0,1) \times H_0^1(0,1) \to \mathbb{R}$ by

$$a(\mathsf{v},\mathsf{w}) = \int_0^1 \alpha(x)v'(x)w'(x)\, \mathrm{d}x$$

for v and w in $H_0^1(0,1)$. Then, it is easy to check that $a(\cdot,\cdot)$ defines a continuous, symmetric and elliptic bilinear form on $H_0^1(0,1)$. Thus, if $\mathsf{f} \in L^2(0,1)$ is given, there exists a unique $\mathsf{u} \in H_0^1(0,1)$ such that

$$a(\mathsf{u},\mathsf{v}) = \int_0^1 fv\, \mathrm{d}x$$

for all $\mathsf{v} \in H_0^1(0,1)$, by virtue of the Lax-Milgram lemma (cf. Corollary 7.3.1). Set $u = G(f)$. Again, it is easy to check that

$$|\mathsf{u}|_1 \le \frac{1}{c}\|\mathsf{f}\|_2$$

and so the mapping $G : L^2(0,1) \to H_0^1(0,1)$ is continuous. Composing this with the canonical inclusion of $H_0^1(0,1)$ into $L^2(0,1)$, we get that G defines a compact operator on $L^2(0,1)$. ■

Remark 8.1.2 In the preceding example, since $\mathsf{u} \in H_0^1(0,1)$, we see that $u(0) = u(1) = 0$ (cf. Theorem 6.4.5). Assuming that u and α are sufficiently smooth, we get, on integrating by parts, that

$$\int_0^1 -(\alpha(x)u'(x))'v(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x$$

for all $\mathsf{v} \in H_0^1(0,1)$. In particular, this is true for all v represented by $\mathcal{C}^\infty$ functions with compact support in $(0,1)$. For this reason, we say that

$$-(\alpha u')' = f$$

in the sense of distributions (cf. Kesavan [1]) and u is called a *weak* (or generalized) solution of the boundary value problem:

$$\begin{aligned} -\tfrac{\mathrm{d}}{\mathrm{d}x}\left(\alpha\tfrac{\mathrm{d}u}{\mathrm{d}x}\right) &= f \text{ in } (0,1) \\ u(0) &= u(1) = 0. \end{aligned}$$

The case of Example 8.1.8 corresponds to the case when $\alpha \equiv 1$ and so the differential operator is $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in the above boundary value problem. The operator G mapping the data f to the solution u is sometimes called the *Green's operator*. ■

Example 8.1.10 Let $a : [0,1] \to \mathbb{R}$ be a continuous function which is not identically zero. Define $A : L^2(0,1) \to L^2(0,1)$ by

$$A(\mathsf{f})(t) = a(t)f(t).$$

It is easy to verify that $A \in \mathcal{L}(L^2(0,1))$. Let $a(t_0) \neq 0$ for some $t_0 \in (0,1)$. Then, there exists a compact interval $J = [t_0 - \alpha, t_0 + \alpha]$ such that

$$|a(t)| \ge \frac{1}{2}|a(t_0)| > 0$$

for all $t \in J$. Let $\{\widetilde{\mathsf{f}}_n\}$ be an orthonormal basis for $L^2(J)$. Define

$$f_n(t) = \begin{cases} \widetilde{f_n}(t), & \text{if } t \in J \\ 0, & \text{if } t \in [0,1]\backslash J. \end{cases}$$

Then $\|\mathsf{f}_n\|_2 = 1$ for all n and, if $n \neq m$, we have

$$\begin{aligned} \|A(\mathsf{f}_n) - A(\mathsf{f}_m)\|_2^2 &= \int_0^1 |a(t)|^2 |f_n(t) - f_m(t)|^2 \, \mathrm{d}t \\ &\geq \frac{|a(t_0)|^2}{4} \int_J |\widetilde{f_n}(t) - \widetilde{f_m}(t)|^2 \, \mathrm{d}t \\ &= \frac{|a(t_0)|^2}{2}. \end{aligned}$$

Thus the sequence $\{A(\mathsf{f}_n)\}$ does not have a convergent subsequence and so A is not compact. ■

Proposition 8.1.1 *Let V and W be Banach spaces and let $\mathcal{K}(V, W)$ be the set of all compact linear operators from V into W. Then, $\mathcal{K}(V, W)$ is a closed subspace of $\mathcal{L}(V, W)$.*

Proof It is immediate to check that a linear combination of compact operators is compact. Thus $\mathcal{K}(V, W)$ is a subspace of $\mathcal{L}(V, W)$. We thus have to check that if $T_n \in \mathcal{K}(V, W)$ and if $T_n \to T$ in $\mathcal{L}(V, W)$, then $T \in \mathcal{K}(V, W)$. Let B be a ball in V. We need to show that its image $T(B)$ in W is relatively compact. Since W is a complete metric space, it suffices to show that given $\varepsilon > 0$, we can cover $T(B)$ by a finite number of balls of radius ε (cf. Proposition 1.2.6). Since $\|T_n - T\| \to 0$, choose n sufficiently large such that $\|T_n - T\| < \varepsilon/2$. Since T_n is compact, we have

$$T_n(B) \subset \cup_{i \in \mathcal{I}} B_W(f_i; \varepsilon/2)$$

where $\mathcal{I}$ is a finite indexing set and $f_i \in W$. (Here B_W denotes a ball in W). We then have

$$T(B) \subset \cup_{i \in \mathcal{I}} B_W(f_i; \varepsilon).$$

Thus, T is compact. ■

Corollary 8.1.1 *If $T_n : V \to W$ are all of finite rank and if $\|T_n - T\| \to 0$, then T is compact.* ■

Thus, a continuous linear transformation which can be approximated by transformations of finite rank is compact. An interesting question is the converse. Can every compact linear transformation be approximated by those of finite rank. In general, this is not true. However, we have the following result (see also Exercise 8.15).

Proposition 8.1.2 *Let V be an infinite dimensional Banach space and let W be an infinite dimensional Hilbert space. Let $T \in \mathcal{K}(V, W)$. Then T is the limit of transformations of finite rank.*

Proof Let B be the closed unit ball in V. Let $K = \overline{T(B)}$, which is compact, by hypothesis. Let $\varepsilon > 0$ and let $f_i \in W$ for $i \in \mathcal{I}$, a finite indexing set, such that

$$K \subset \cup_{i \in \mathcal{I}} B_W(f_i; \varepsilon).$$

Let $G = \text{span}\{f_i : i \in \mathcal{I}\}$ which is a finite dimensional subspace of W. Let $P : W \to G$ be the orthogonal projection onto G, which exists since W is closed. Then $P \circ T$ is of finite rank.

Now, let $x \in B$. Then, there exists f_{i_0} such that $\|T(x) - f_{i_0}\| < \varepsilon$. Since the norm of a projection is unity (cf. Exercise 7.4), we have

$$\|(P \circ T)(x) - f_{i_0}\| = \|(P \circ T)(x) - P(f_{i_0})\| < \varepsilon$$

which implies that, for all $x \in B$,

$$\|(P \circ T)(x) - T(x)\| < 2\varepsilon$$

i.e. $\|(P \circ T) - T\| \le 2\varepsilon$. This completes the proof. ■

Example 8.1.11 Let $\mathsf{k} \in L^2((0, 1) \times (0, 1))$. For $\mathsf{f} \in L^2(0, 1)$, define

$$K(f)(t) = \int_0^1 k(t, s) f(s) \, \mathrm{d}s.$$

Then, by an application of the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_0^1 K(f)(t)|^2 \, \mathrm{d}t &= \int_0^1 \left| \int_0^1 k(t, s) f(s) \, \mathrm{d}s \right|^2 \mathrm{d}t \\ &\le \int_0^1 \left(\int_0^1 |k(t, s)|^2 \, \mathrm{d}s \right) \left(\int_0^1 |f(s)|^2 \, \mathrm{d}s \right) \mathrm{d}t \\ &= \|\mathsf{f}\|_2^2 \int_0^1 \int_0^1 |k(t, s)|^2 \, \mathrm{d}s \, \mathrm{d}t \end{aligned}$$

which shows that $K \in \mathcal{L}(L^2(0, 1))$ and that

$$\|K\| \le \|\mathsf{k}\|_{L^2((0,1)\times(0,1))}.$$

Let $\{\phi_n\}$ be an orthonormal basis for $L^2(0, 1)$. Then $\{\Phi_{ij}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ is an orthonormal basis for $L^2((0, 1) \times (0, 1))$ where

$$\Phi_{ij}(t, s) = \phi_i(t)\phi_j(s)$$

(cf. Exercise 7.20). Thus

$$\mathsf{k} = \sum_{i,j=1}^{\infty} (k, \Phi_{ij})\Phi_{ij}.$$

If we set

$$\mathsf{k}_n(t, s) = \sum_{i,j=1}^{n} (\mathsf{k}, \Phi_{ij})\Phi_{ij}(t, s),$$

then $\|\mathsf{k} - \mathsf{k}_n\|_2 \to 0$. Consequently, if we define

$$K_n(\mathsf{f})(t) = \int_0^1 k_n(t, s) f(s)\, \mathrm{d}s$$

for $\mathsf{f} \in L^2(0, 1)$, we have that $\|K_n - K\| \leq \|\mathsf{k}_n - \mathsf{k}\|_2 \to 0$. Clearly, the image of K_n is contained in the span of $\{\phi_1, \cdots, \phi_n\}$ and so K_n is of finite rank. Hence K is a compact operator on $L^2(0, 1)$. (The operator K is called the *Hilbert-Schmidt operator* induced by k.) ■

Proposition 8.1.3 *Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$. Then T is compact if, and only if, $T^* : W^* \to V^*$ is compact.*

Proof Assume that T is compact. Set $K = \overline{T(B_V)}$, where B_V is the closed unit ball in V. Then, K is compact and $\|w\| \leq \|T\|$ for all $w \in K$. Now, let $\{v_n\}$ be a sequence in B_{W^*}. Consider the sequence $\{\varphi_n\}$ defined by $\varphi_n(w) = v_n(w)$ for $w \in K$. Then, this is a sequence of continuous functions on the compact metric space K. We have$|\varphi_n(w)| \leq \|T\|$ for all $w \in K$. Further $|\varphi_n(w_1) - \varphi_n(w_2)| \leq \|w_1 - w_2\|$ and so the sequence $\{\varphi_n\}$ is uniformly bounded and equicontinuous in $\mathcal{C}(K)$. By Ascoli's theorem, it has a uniformly convergent subsequence$\{\varphi_{n_k}\}$. Thus, in particular

$$\sup_{x \in B_V} |\varphi_{n_k}(T(x)) - \varphi_{n_l}(T(x))|$$

can be made arbitrarily small for large k and l. In other words,

$$\sup_{x \in B_V} |T^*(v_{n_k})(x) - T^*(v_{n_l})(x)|$$

can be made arbitrarily small for large k and l, i.e. the sequence $\{T^*(v_{n_k})\}$ is a Cauchy sequence and is hence convergent. This shows that T^* is compact.

Now assume that T^* is compact. Then, by the preceding arguments, it follows that $T^{**} : V^{**} \to W^{**}$ is compact. Let $J_V : V \to V^{**}$ and $J_W : W \to W^{**}$ be the canonical imbeddings. Let $x \in B_V$. Then, for any $v \in W^*$, we have $T^{**}(J_V(x))(v) = J_V(x)(T^*(v)) = T^*(v)(x) = v(T(x)) = J_W(T(x))(v)$. Thus $T^{**}(J_V(B_V)) = J_W$

$(T(B_V))$. Consequently, $J_W(T(B_V))$ is relatively compact in W^{**} and since J_W is an isometric isomorphism of W onto $J_W(W)$, it follows that $T(B_V)$ is relatively compact in W. Thus, T is compact and the proof is complete. ■

8.2 Riesz-Fredhölm Theory

In this section we will be interested in operators of the form $I - T$ on a Banach space V, where I is the identity operator on V and $T \in \mathcal{L}(V)$ is a compact operator. Such operators are called *compact perturbations of the identity*. Such operators have properties very similar to those of linear operators on finite dimensional spaces.

Given a continuous linear operator A on a Banach space V, we will denote its range by $\mathcal{R}(A)$ and its kernel (or null space) by $\mathcal{N}(A)$.

Theorem 8.2.1 *(***Fredhölm Alternative***) Let V be a Banach space and let $T : V \to V$ be compact. Let $T^* : V^* \to V^*$ denote its adjoint. Then:*

*(*a*) $\mathcal{N}(I - T)$ is finite dimensional;*
*(*b*) $\mathcal{R}(I - T)$ is closed and*

$$\mathcal{R}(I - T) = \mathcal{N}(I - T^*)^{\perp};$$

*(*c*) $\mathcal{N}(I - T) = \{\mathbf{0}\}$ if, and only if, $\mathcal{R}(I - T) = V$;*
*(*d*) $\dim(\mathcal{N}(I - T)) = \dim(\mathcal{N}(I - T^*))$.*

Proof (a) Let B_1 be the closed unit ball in $\mathcal{N}(I - T)$. If $x \in B_1$, then $\|x\| \leq 1$ and, further, $x = T(x)$. Thus, $B_1 \subset T(B)$, where B is the closed unit ball in V. Since T is compact, it follows that B_1 is compact and so $\mathcal{N}(I - T)$ is finite dimensional (cf. Proposition 2.3.6).

(b) Let $f_n \in \mathcal{R}(I - T)$ and assume that $f_n \to f$ in V. Let $f_n = u_n - T(u_n)$. Since $\mathcal{N}(I - T)$ is finite dimensional, it follows that there exists $v_n \in \mathcal{N}(I - T)$ such that

$$\|u_n - v_n\| = d(u_n, \mathcal{N}(I - T))$$

(cf. Exercise 2.43). We can now write

$$f_n = (u_n - v_n) - T(u_n - v_n).$$

We now claim that $\{u_n - v_n\}$ is a bounded sequence in V. If not, for a subsequence, we have $\|u_{n_k} - v_{n_k}\| \to \infty$. Set

$$w_{n_k} = \frac{1}{\|u_{n_k} - v_{n_k}\|}(u_{n_k} - v_{n_k}).$$

Then $\|w_{n_k}\| = 1$. Further,

$$w_{n_k} - T(w_{n_k}) = \frac{1}{\|u_{n_k} - v_{n_k}\|} f_{n_k}$$

and so it follows that $w_{n_k} - T(w_{n_k}) \to \mathbf{0}$. Since T is compact, $\{w_{n_k}\}$ has a subsequence $\{w_{n_{k_l}}\}$ such that $T(w_{n_{k_l}}) \to z \in V$. Then, it follows from the preceding arguments that $w_{n_{k_l}} \to z$ as well. Then $T(w_{n_{k_l}}) \to T(z)$ and so we deduce that $z = T(z)$, i.e. $z \in \mathcal{N}(I - T)$.

Now, since $v_{n_{k_l}} \in \mathcal{N}(I - T)$, it follows that

$$d(w_{n_{k_l}}, \mathcal{N}(I - T)) = \frac{d(u_{n_{k_l}}, \mathcal{N}(I - T))}{\|u_{n_{k_l}} - v_{n_{k_l}}\|} = 1.$$

Thus, on one hand $z \in \mathcal{N}(I - T)$, while on the other hand, the above considerations imply that $d(z, \mathcal{N}(I - T)) = 1$, which is a contradiction. This establishes the claim.

Since $\{u_n - v_n\}$ is a bounded sequence, and since T is compact, we have that, for a subsequence, $T(u_{n_k} - v_{n_k}) \to g \in V$. Thus $u_{n_k} - v_{n_k} \to f + g = h$, say. It then follows that $T(h) = T(f) + T(g) = g$. This implies that

$$h - T(h) = f$$

i.e. $\mathcal{R}(I - T)$ is closed. Then, by virtue of Theorem 4.7.1, we deduce that

$$\mathcal{R}(I - T) = \mathcal{N}(I - T^*)^{\perp}.$$

(c) Assume that $\mathcal{N}(I - T) = \{\mathbf{0}\}$. Assume further that $V_1 = \mathcal{R}(I - T) \neq V$. Since V_1 is a closed subspace of V, it is also a Banach space. Further, if $x \in V$, we have

$$T(x - T(x)) = (I - T)(T(x))$$

and so $T(V_1) \subset V_1$. Thus, $T : V_1 \to V_1$ is also compact and so $I - T$ restricted to V_1 also has a closed range. Let $V_2 = (I - T)(V_1)$. Again, it follows that V_2 is a proper subspace of V_1. (If not, for every $x \in V$, it will follow that there exists $y \in V$ such that

$$(I - T)(x) = (I - T)^2(y)$$

from which it will follow that $x = (I - T)y$, since $I - T$ is assumed to be injective. This will then imply that $V = V_1$, contrary to our assumption.) Inductively, we thus get a decreasing sequence of closed subspaces $\{V_n\}$ such that $V_n = (I - T)^n(V)$ and V_{n+1} is a proper subspace of V_n for all n. Then, by Riesz' lemma (cf. Lemma 2.3.1), there exists $u_n \in V_n$ such that $\|u_n\| = 1$ and $d(u_n, V_{n+1}) \geq 1/2$.

Now,

$$T(u_m - u_n) = (u_n - T(u_n)) - (u_m - T(u_m)) + (u_m - u_n).$$

Assume that $n > m$. Then $u_n - T(u_n) \in V_{n+1}$, $u_m - T(u_m) \in V_{m+1}$, $u_n \in V_n$ and

$$V_{n+1} \subset V_n \subset V_{m+1} \subset V_m.$$

Thus,

$$T(u_m - u_n) = u_m - w$$

where $w \in V_{m+1}$. Thus, if $m \neq n$, by construction,

$$\|T(u_m) - T(u_n)\| \geq \frac{1}{2}.$$

Hence, $\{T(u_n)\}$ cannot have a convergent subsequence while $\{u_n\}$ is bounded, contradicting the fact that T is compact. Thus, it follows that $\mathcal{R}(I - T) = V$.

Conversely, if $\mathcal{R}(I - T) = V$, then $\mathcal{N}(I - T^*) = \mathcal{R}(I - T)^\perp = \{\mathbf{0}\}$. Since T^* is also compact, it now follows that $\mathcal{R}(I - T^*) = V^*$. Then, again, $\mathcal{N}(I - T) = \mathcal{R}(I - T^*)^\perp = \{\mathbf{0}\}$.

(d) Let $d = \dim(\mathcal{N}(I - T))$ and let $d^* = \dim(\mathcal{N}(I - T^*))$. Assume that $d < d^*$. Since $\mathcal{N}(I - T)$ is finite dimensional, it is complemented and so there exists a continuous projection $P : V \to \mathcal{N}(I - T)$. Also $\mathcal{R}(I - T) = \mathcal{N}(I - T^*)^\perp$ and so the range of $(I - T)$ has finite codimension d^*. Let W be a d^*-dimensional subspace of V which is a complement to $\mathcal{R}(I - T)$. Since we have assumed that $d < d^*$, there exists a linear map $\Lambda : \mathcal{N}(I - T) \to W$ which is injective but *not* surjective. Define

$$S = T + \Lambda \circ P.$$

Since T is compact and $\Lambda \circ P$ is of finite rank, it follows that S is also compact. Let $u - S(u) = \mathbf{0}$. Then

$$\mathbf{0} = (u - T(u)) + \Lambda(P(u))$$

where the first term belongs to $\mathcal{R}(I - T)$ while the second belongs to its complement, W. Thus $u - T(u) = \mathbf{0}$ and so $u \in \mathcal{N}(I - T)$. Thus, $P(u) = u$. Further, $\mathbf{0} = \Lambda(P(u)) = \Lambda(u)$ and so $u = \mathbf{0}$ since Λ is injective. Thus $\mathcal{N}(I - S) = \{\mathbf{0}\}$ and so $\mathcal{R}(I - S) = V$.

Now choose $f \in W \backslash \mathcal{R}(\Lambda)$. Let $u \in V$ be such that $u - S(u) = f$, i.e. $(u - T(u)) - \Lambda(P(u)) = f$. Since both f and $\Lambda(P(u))$ belong to W, which is the complement of $\mathcal{R}(I - T)$, it follows that $u - T(u) = \mathbf{0}$ which implies that $f = -\Lambda(P(u)) \in \mathcal{R}(\Lambda)$, a contradiction. Thus,

$$d^* \leq d.$$

Similarly,

$$\dim(\mathcal{N}(I - T^{**})) \leq \dim(\mathcal{N}(I - T^{*})) = d^{*}.$$

But, if $J : V \to V^{**}$ is the canonical imbedding, then it is easy to see that

$$J(\mathcal{N}(I - T)) \subset \mathcal{N}(I - T^{**})$$

and so $d \leq \dim(\mathcal{N}(I - T^{**})) \leq d^{*}$. Thus, we deduce that $d = d^{*}$ and this completes the proof. ■

Remark 8.2.1 The conclusion (c) holds for all operators on finite dimensional spaces and is referred to in short as 'one-one if, and only if, onto'. This property carries over to compact perturbations of the identity in infinite dimensional Banach spaces. In general it is not true for a general operator on a Banach space. If $x = (x_i) \in \ell_2$ and we define $T \in \mathcal{L}(\ell_2)$ by

$$T(x) = (0, x_1, x_2, \cdots),$$

then, clearly T is one-one but not onto. ■

Remark 8.2.2 The Fredhölm alternative is stated as follows: let V be a Banach space and let $T \in \mathcal{L}(V)$ be a compact operator. Let $f \in V$. Then

- Either, the equation $u - T(u) = \mathbf{0}$ has only the zero solution and so the equation $w - T(w) = f$ has a unique solution for all $f \in V$,
- Or, the equation $u - T(u) = \mathbf{0}$ has d linearly independent solutions and the equation $w - T(w) = f$ will have a solution if, and only if, f satisfies d orthogonality relations i.e. $f \in \mathcal{N}(I - T^{*})^{\perp}$. (In the case of finite dimensional spaces, these are the familiar consistency conditions to be satisfied by the right-hand side vector in case of singular matrices.) ■

Remark 8.2.3 Theorem 8.2.1 is the starting point of the theory of *Fredhölm operators*. A Fredhölm operator $A : V \to W$ between Banach spaces V and W is one which is such that $\mathcal{N}(A)$ is finite dimensional and $\mathcal{R}(A)$ is closed with finite codimension. The (Fredhölm) *index* of A is given by

$$i(A) = \dim(\mathcal{N}(A)) - \operatorname{codim}(\mathcal{R}(A)).$$

Compact perturbations of the identity on a Banach space are thus Fredhölm operators with index zero. It can be shown that the class of Fredhölm operators is open in $\mathcal{L}(V)$ and that the index is a continuous function in this space. Since it is integer valued, the index will be constant on connected components of this class. ■

We conclude this section by proving a useful result which could be considered as a generalization of the Lax-Milgram lemma (cf. Corollary 7.3.1).

Proposition 8.2.1 *Let V and H be Hilbert spaces such that $V \subset H$, the inclusion being dense and compact. Let $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a continuous bilinear form such that $a(v, v) = 0$ if, and only if, $v = \mathbf{0}$. Assume, further, that there exist constants $\alpha > 0$ and $\beta > 0$ such that, for every $v \in V$,*

$$a(v, v) \ \geq \ \alpha\|v\|_V^2 - \beta\|v\|_H^2. \tag{8.2.1}$$

Let $f \in H$ be given. Then, there exists a unique $u \in V$ such that

$$a(u, v) \ = \ (f, v)_H \tag{8.2.2}$$

for every $v \in V$.

Proof Consider the bilinear form $A(\cdot, \cdot) : V \times V \to \mathbb{R}$ given by

$$A(v, w) \ = \ a(v, w) + \beta(v, w)_H.$$

Then, $A(\cdot, \cdot)$ is clearly continuous and, by virtue of (8.2.1), we have

$$A(v, v) \ \geq \ \alpha\|v\|_V^2.$$

Further, the map $v \mapsto (f, v)_H$ is clearly a continuous linear functional on V. Hence, by the Lax-Milgram lemma, there exists a unique $u \in V$ such that

$$A(u, v) \ = \ (f, v)_H$$

for every $v \in V$. Set $u = G(f)$. Thus, $G \in \mathcal{L}(H, V)$. Since the inclusion of V in H is compact, we have that $G \in \mathcal{L}(H)$ is compact.

Now, if $u \in V$ is a solution of (8.2.2), then

$$a(u, v) + \beta(u, v)_H \ = \ (f + \beta u, v)_H$$

for every $v \in V$, and *vice-versa*. Thus, $u \in V$ satisfies (8.2.2) if, and only if, $u = G(f + \beta u)$. This can be rewritten as

$$z - \beta G(z) \ = \ f, \tag{8.2.3}$$

where $z = f + \beta u$. Thus, (8.2.2) has a unique solution for given f if, and only if, the same is true for (8.2.3).

Now, $I - \beta G$ is a compact perturbation of the identity and so, it is surjective if, and only if, it is injective. Assume that $(I - \beta G)w = \mathbf{0}$. Thus, $w = G(\beta w)$, and thus, $w \in V$ and, for every $v \in V$, we have

$$a(w, v) + \beta(w, v)_H \ = \ (\beta w, v)_H.$$

We hence deduce that $a(w, v) = 0$ for every $v \in V$ and so, in particular, we have $a(w, w) = 0$, whence, we deduce that $w = \mathbf{0}$. Thus, $I - \beta G$ is injective and, therefore, surjective as well. This completes the proof. ■

Example 8.2.1 Let $V = H^1(0, 1)$ and $H = L^2(0, 1)$. Let $k > 0$. Set

$$a(u, v) = \int_0^1 u'(x)v'(x)\,\mathrm{d}x + k[u(1)v(1) + u(0)v(0)],$$

for u and $v \in H^1(0, 1)$. If $a(v, v) = 0$, we see that $v' = 0$ and $v(1) = v(0) = 0$. This implies that $v = 0$. Further,

$$a(v, v) \geq \int_0^1 |v'(x)|^2\,\mathrm{d}x = \|v\|_{1,2}^2 - \|v\|_2^2$$

where

$$\|v\|_{1,2} = \left(\int_0^1 (|v'(x)|^2 + |v(x)|^2)\,\mathrm{d}x\right)^{\frac{1}{2}}$$

is the norm in $H^1(0, 1)$. Thus, all the hypotheses of the preceding proposition are satisfied and so, given $f \in L^2(0, 1)$, we have a unique solution $u \in H^1(0, 1)$ satisfying

$$a(u, v) = \int_0^1 f(x)v(x)\,\mathrm{d}x \tag{8.2.4}$$

for every $v \in H^1(0, 1)$.

It can be shown that (8.2.4) is the weak formulation (cf. Remark 8.1.2) of the following boundary value problem:

$$\begin{aligned} -\tfrac{\mathrm{d}^2u}{\mathrm{d}x^2} &= f \text{ in } (0, 1) \\ ku(0) - u'(0) = ku(1) + u'(1) &= 0. \end{aligned} \tag{8.2.5}$$

The boundary condition in (8.2.5) is called a *Robin boundary condition*. ■

8.3 Spectrum of an Operator

In this section, we will assume that all Banach spaces are complex.

Definition 8.3.1 Let V be a Banach space and let $T \in \mathcal{L}(V)$. The **spectrum** of T is the set $\sigma(T)$ of scalars defined by

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible}\}.$$

The **resolvent** of T is the set $\rho(T)$ defined by

$$\rho(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is invertible}\}. \qquad \blacksquare$$

If $T \in \mathcal{L}(V)$ is such that $\|T\| < 1$, then (cf. Exercise 2.37) $I - T$ is invertible and its inverse is given by the *Neumann series*

$$(I - T)^{-1} = I + T + T^2 + T^3 + \cdots$$

and, it is easy to see that

$$\|(I - T)^{-1}\| \leq \frac{1}{1 - \|T\|}.$$

In particular, if T is any invertible operator on V and if $S \in \mathcal{L}(V)$ such that

$$\|S\| < \frac{1}{\|T^{-1}\|},$$

then it follows that $T - S$ is also invertible. We deduce from this that the set of all invertible operators is open in $\mathcal{L}(V)$. In particular, if $\lambda \in \rho(T)$, then $\lambda + \delta \in \rho(T)$ as well, if δ is sufficiently small. Thus, it follows that $\rho(T)$ is an open subset of $\mathbb{C}$ and so the spectrum is a closed subset of $\mathbb{C}$.

Now assume that $|\lambda| > \|T\|$. Then

$$T - \lambda I = -\lambda(I - \lambda^{-1}T)$$

and, since $\|\lambda^{-1}T\| = |\lambda|^{-1}\|T\| < 1$, it follows that $T - \lambda I$ is invertible and so $\lambda \in \rho(T)$. Thus,

$$\sigma(T) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}.$$

Thus, the spectrum is a compact subset of the complex plane.

Definition 8.3.2 Let $B(0; r)$ denote the closed ball in the complex plane with centre at the origin and radius r. Let V be a Banach space and let $T \in \mathcal{L}(V)$. The **spectral radius** of T, denoted $r(T)$, is defined by

$$r(T) = \inf\{r > 0 \mid \sigma(T) \subset B(0; r)\}. \qquad \blacksquare$$

We now show that the spectrum of a continuous linear operator on a (complex) Banach space is always a non-empty set. Let $\lambda \in \rho(T)$. Set $T(\lambda) = (T - \lambda I)^{-1}$. Since we can write $T(\lambda) = \lambda^{-1}((1/\lambda)T - I)^{-1}$, it follows that, for $|\lambda| > \|T\|$, we have

$$\|T(\lambda)\| \leq \frac{1}{|\lambda|}\frac{1}{\left(1 - \frac{\|T\|}{|\lambda|}\right)}$$

and so $\|T(\lambda)\|$ is bounded and tends to zero as $|\lambda| \to \infty$.

Now let λ and μ belong to $\rho(T)$. Then

$$\begin{aligned} T(\lambda) &= T(\lambda)(T - \mu I)T(\mu) \\ &= T(\lambda)(T - \lambda I + (\lambda - \mu)I)T(\mu) \\ &= (I + (\lambda - \mu)T(\lambda))T(\mu) \end{aligned}$$

which implies that

$$T(\lambda) - T(\mu) = (\lambda - \mu)T(\lambda)T(\mu). \tag{8.3.1}$$

This is usually known as the *resolvent equation*.

Now, let $f \in \mathcal{L}(V)^*$. Define

$$\varphi(\lambda) = f(T(\lambda))$$

for $\lambda \in \rho(T)$. Then,

$$|\varphi(\lambda)| \leq \|f\|.\|T(\lambda)\|. \tag{8.3.2}$$

Further, by (8.3.1), it follows that

$$\lim_{\lambda \to \mu} \frac{\varphi(\lambda) - \varphi(\mu)}{\lambda - \mu} = f(T(\mu)^2). \tag{8.3.3}$$

We conclude from (8.3.2) that $\varphi(\lambda)$ is bounded and vanishes as λ approaches infinity, and from (8.3.3) that φ is differentiable at every point of $\rho(T)$. Thus, if $\rho(T) = \mathbb{C}$, then φ will be a bounded and entire function and hence, by Liouville's theorem, a constant. Since it vanishes at infinity, it will follow that $\varphi \equiv 0$. Since f can be arbitrarily chosen, it follows then that $T(\lambda) = 0$ for all $\lambda \in \mathbb{C} = \rho(T)$ which is absurd since it is the inverse of the map $T - \lambda I$. Thus $\rho(T)$ cannot be equal to the entire complex plane. In other words, $\sigma(T)$ is non-empty.

Thus, the spectrum of a continuous linear operator, T, is a non-empty compact set of the complex plane lying inside the ball centered at the origin and having a radius equal to $\|T\|$.

Remark 8.3.1 The non-emptiness of the spectrum is crucially dependent on the fact that the scalar field is the field of complex numbers. Consider the real Banach space $\mathbb{R}^2$ (with any norm). Consider the linear operator,T, defined via the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $T - \lambda I$ is not invertible if, and only if $\lambda^2 + 1 = 0$ which has no real solution. Thus the spectrum of T is empty, if the space is $\mathbb{R}^2$, while it is the set $\{i, -i\}$ when the space is $\mathbb{C}^2$. ■

Definition 8.3.3 An element λ of the spectrum $\sigma(T)$ of a continuous linear operator T on a Banach space V is called an **eigenvalue** of T if $\mathcal{N}(T - \lambda I) \neq \{\mathbf{0}\}$. In this case $\dim(\mathcal{N}(T - \lambda I))$ is called the **geometric multiplicity** of the eigenvalue λ. The non-zero elements of the space $\mathcal{N}(T - \lambda I)$ are called the **eigenvectors** of T associated to the eigenvalue λ. ■

Remark 8.3.2 In finite dimensions, the spectrum coincides with the set of all eigenvalues. This need not be the case in infinite dimensions. For example, consider the (complex) space ℓ_2 and the map

$$x = (x_i) \mapsto T(x) = (0, x_1, x_2, \cdots).$$

Then, since T is not onto, it is not invertible. Thus $0 \in \sigma(T)$. However, since T is injective, $\lambda = 0$ is not an eigenvalue. ■

Proposition 8.3.1 *Let H be a Hilbert space and let $T \in \mathcal{L}(V)$. Then $\lambda \in \sigma(T)$ if, and only if, $\overline{\lambda} \in \sigma(T^*)$.*

Proof Observe that the adjoint of $T - \lambda I$ is $T^* - \overline{\lambda} I$. The result is now an immediate consequence of Theorem 4.7.1. ■

Example 8.3.1 Let $V = \ell_2$. Consider the map

$$x \mapsto S(x) = (x_2, x_3, \cdots)$$

where $x = (x_1, x_2, x_3, \cdots) \in \ell_2$. Then, clearly, $\|S(x)\|_2 \leq \|x\|_2$ and so $\|S\| \leq 1$. (In fact $\|S\|_2 = 1$; why?) The operator S is called the *left shift operator*. Thus,

$$\sigma(S) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

Let $\lambda \neq 0$ and assume that $S(x) = \lambda x$. Then, for all $i \geq 1$, we have $x_{i+1} = \lambda x_i$ which yields $x_i = \lambda^{i-1} x_1$. Since $x \in \ell_2$, we have that $x_i \to 0$ as $i \to \infty$, and so it follows that $|\lambda| < 1$. On the other hand, if $0 < |\lambda| < 1$, the vector

$$x = (1, \lambda, \lambda^2, \lambda^3, \cdots) \in \ell_2$$

and, indeed, $S(x) = \lambda x$. Further, $S(\mathsf{e}_1) = \mathbf{0}$. Thus, it follows that every λ such that $|\lambda| < 1$ is an eigenvalue of S and so is in the spectrum of S. Since the spectrum is closed, it now follows that

$$\sigma(S) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

If $|\lambda| = 1$, then though it is in the spectrum, it is not an eigenvalue.

It is now immediate to see that the adjoint of S is the mapping T on ℓ_2 given by

$$T(x) = (0, x_1, x_2, x_3, \cdots).$$

This is called the *right shift operator*. By Proposition 8.3.1, it follows that the spectrum of T is also the closed unit disc in the complex plane. Now, if $T(x) = \lambda x$, then $\lambda x_1 = 0$ and, for all $i \geq 2$, we have $\lambda x_i = x_{i-1}$. If $\lambda = 0$, then $x_i = 0$ for all i and so $\mathcal{N}(T) = \{\mathbf{0}\}$. If $\lambda \neq 0$, then $x_1 = 0$ and again, it follows that $x_i = 0$ for all i. Thus, there are no eigenvalues for T. ■

Example 8.3.2 Let H be a Hilbert space and let $P \in \mathcal{L}(H)$ be a non-trivial orthogonal projection, i.e. $P \neq \mathbf{0}$, $P \neq I$. For every x in the range of P, we have $P(x) = x$ and for every y in the null space, we have $P(x) = 0$. Thus 0 and 1 are eigenvalues of P. If $\lambda \neq 0, 1$, then it is a simple matter to check that

$$\frac{1}{\lambda}\left[I + \frac{1}{\lambda - 1}P\right]$$

is an inverse for $\lambda I - P$. Thus $\sigma(P) = \{0, 1\}$. ■

We conclude this section by characterising the spectra of some special classes of operators on a Hilbert space.

Theorem 8.3.1 *Let H be a Hilbert space. Let $T \in \mathcal{L}(H)$.*

(a) If T is self-adjoint, then $\sigma(T) \subset \mathbb{R}$.
(b) If T is such that $(T(x), x) \geq 0$ for every $x \in H$, then $\sigma(T) \subset [0, \infty) \subset \mathbb{R}$.
(c) If T is unitary, then $\sigma(T) \subset \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

Proof (a) Let $T = T^*$ and let $\lambda \in \mathbb{C}$, such that $\mathrm{Im}(\lambda) \neq 0$. Now, if $x \in H$, then

$$(T(x), x) = (x, T^*(x)) = (x, T(x)) = \overline{(T(x), x)}.$$

Thus $(T(x), x)$ is always a real number. Then

$$\mathrm{Im}((T - \lambda I)(x), x)) = -\mathrm{Im}(\lambda)\|x\|^2.$$

Consequently,

$$|\mathrm{Im}(\lambda)|\ \|x\|^2 \leq \|(T - \lambda I)(x)\|\ \|x\|,$$

whence we deduce that, for every $x \in H$,

$$\|(T - \lambda I)(x)\| \geq |\mathrm{Im}(\lambda)|\ \|x\|.$$

In particular, this implies that $T - \lambda I$ is injective. In the same way, we have, for every $x \in H$,

$$\|(T - \overline{\lambda} I)(x)\| \geq |\mathrm{Im}(\lambda)| \, \|x\|.$$

Since $(T - \lambda I)^* = T - \overline{\lambda} I$, we deduce that $T - \lambda I$ is surjective (cf. Theorem 4.7.2) and hence, by the open mapping theorem, it is invertible. Thus $\lambda \in \rho(T)$ and so $\sigma(T) \subset \mathbb{R}$.

(b) If $(T(x), x) \geq 0$ for every $x \in H$, it follows that $T = T^*$ (cf. Proposition 7.2.3). Thus, $\sigma(T) \subset \mathbb{R}$. Let $\lambda < 0$. Then

$$|\lambda| \, \|x\|^2 = -\lambda \|x\|^2 \leq ((T - \lambda I)(x), x) \leq \|(T - \lambda I)(x)\| \, \|x\|,$$

from which we deduce that, for all $x \in H$,

$$\|(T - \lambda I)(x)\| \geq |\lambda| \, \|x\|.$$

Since $T - \lambda I$ is now self-adjoint, it again follows, from Theorem 4.7.2, that it is invertible. This proves that $\sigma(T) \subset [0, \infty)$.

(c) If T is unitary, then $\|T(x)\| = \|T^*(x)\| = \|x\|$. If $|\lambda| \neq 1$, then

$$\|T(x) - \lambda x\| \geq | \, \|Tx\| - |\lambda| \|x\| \, | = |1 - |\lambda| \, | \|x\|,$$

and

$$\|T^*(x) - \lambda x\| \geq | \, \|T^* x\| - |\lambda| \|x\| \, | = |1 - |\lambda| \, | \|x\|,$$

Again, from Theorem 4.7.2, it follows that $T - \lambda I$ is invertible. This completes the proof. ■

8.4 Spectrum of a Compact Operator

In the previous section, we saw from the examples that the spectrum of an operator, in infinite dimensionl spaces, can be of various kinds. In the case of the left shift operator, there is a continuum of eigenvalues and a continuum of elements in the spectrum which are not eigenvalues, while the right shift operator has a spectrum which is a continuum of complex numbers but without any eigenvalues. The spectrum of an orthogonal projection consists of only two elements, both being eigenvalues. We will now see that compact operators behave more or less like operators on finite dimensional spaces.

Theorem 8.4.1 *Let V be an infinite dimensional Banach space and let $T \in \mathcal{L}(V)$ be a compact operator. Then,*

(a) $0 \in \sigma(T)$;
(b) *Every non-zero element of the spectrum is an eigenvalue of* T *with finite (geometric) multiplicity;*
(c) *If* $\{\lambda_n\}$ *is a sequence of distinct non-zero eigenvalues of* T*, and if* $\lambda_n \to \lambda$*, then* $\lambda = 0$ *(in particular, the non-zero elements of the spectrum are all isolated);*
(d) *One of the following alternatives holds:*

- $\sigma(T) = \{0\}$,
- $\sigma(T)\backslash\{0\}$ *is finite,*
- $\sigma(T)\backslash\{0\}$ *consists of a sequence of eigenvalues which converges to zero.*

Proof (a) We have already seen (cf. Example 8.1.4) that a compact operator is not invertible. Thus $0 \in \sigma(T)$.
(b) Let $\lambda \neq 0$. Now $T - \lambda I = -\lambda(I - \lambda^{-1}T)$ and the Riesz-Fredhölm theory applies. Thus, if $\mathcal{N}(I - \lambda^{-1}T) = \{\mathbf{0}\}$, then $T - \lambda^{-1}I$ is onto as well. Then, by the open mapping theorem, it is also invertible. Thus, if $\lambda \neq 0$ is in the spectrum of T, it follows that $\mathcal{N}(T - \lambda I) = \mathcal{N}(I - \lambda^{-1}T) \neq \{\mathbf{0}\}$ and that it is finite dimensional (cf. Theorem 8.2.1).
(c) Since λ_n is an eigenvalue for each n, let $u_n \neq \mathbf{0}$ be such that $T(u_n) = \lambda_n u_n$. The set $\{u_n\}$ is a linearly independent set. Indeed, assume that the set $\{u_1, \cdots, u_n\}$ is linearly independent (this is true for $n = 1$ since $u_1 \neq \mathbf{0}$) and that we can write

$$u_{n+1} = \sum_{j=1}^{n} \alpha_j u_j.$$

Then,

$$\sum_{j=1}^{n} \alpha_j \lambda_{n+1} u_j = \lambda_{n+1} u_{n+1} = T(u_{n+1}) = \sum_{j=1}^{n} \alpha_j \lambda_j u_j.$$

Since the set $\{u_1, \cdots, u_n\}$ is linearly independent and since $\lambda_j \neq \lambda_{n+1}$ for all such j, we deduce that $\alpha_j = 0$ for $1 \leq j \leq n$, whence $u_{n+1} = \mathbf{0}$, which is a contradiction. This completes the induction and proves the linear independence of the set of eigenvectors $\{u_n\}$.

Set $V_n = \text{span}\{u_1, \cdots, u_n\}$. Then we have an increasing sequence of subspaces such that V_n is a proper subspace of V_{n+1} for all n. These subspaces are all finite dimensional and hence, closed. By Riesz' lemma we can find $v_n \in V_n$ such that $\|v_n\| = 1$ and $d(v_n, V_{n-1}) \geq 1/2$ for all $n \geq 2$.

Let $\lambda_n \to \lambda \neq 0$. Then $\{\frac{1}{\lambda_n} v_n\}$ is a bounded sequence and so, since T is compact, the sequence

$$\left\{\frac{1}{\lambda_n} T(v_n)\right\}$$

must have a convergent subsequence. Now, if $2 \leq m < n$, we have

$$V_{m-1} \subset V_m \subset V_{n-1} \subset V_n.$$

Thus,

$$\frac{1}{\lambda_n}T(v_n) - \frac{1}{\lambda_m}T(v_m) = \frac{1}{\lambda_n}(T(v_n) - \lambda_n v_n) - \frac{1}{\lambda_m}(T(v_m) - \lambda_m v_m) + (v_n - v_m).$$

Now, $T(v_n) - \lambda_n v_n \in V_{n-1}$ (why?) and, similarly, $T(u_m) - \lambda_m v_m \in V_{m-1}$. Thus

$$\frac{1}{\lambda_n}T(v_n) - \frac{1}{\lambda_m}T(v_m) = v_n - w$$

where $w \in V_{n-1}$ and so, by construction,

$$\left\| \frac{1}{\lambda_n}T(v_n) - \frac{1}{\lambda_m}T(v_m) \right\| \geq \frac{1}{2}$$

which contradicts the fact that the sequence $\{(1/\lambda_n)T(v_n)\}$ has a convergent subsequence. This proves that $\lambda = 0$.

(d) In view of (c) we deduce that, for all positive integrs n, the set

$$\sigma(T) \cap \left\{ \lambda \in \mathbb{C} \mid |\lambda| \geq \frac{1}{n} \right\}$$

is either empty, or finite (since $\sigma(T)$ is compact). Thus $\sigma(T)\backslash\{\mathbf{0}\}$ is empty, finite or countably infinite. In the last case, by the compactness of $\sigma(T)$, every subsequence of this set has a convergent subsequence which must converge to zero. Hence the entire sequence in $\sigma(T)\backslash\{\mathbf{0}\}$ converges to zero. ■

Example 8.4.1 If $T : V \to V$ is an operator of finite rank, then obviously it can only have a finite number of eigenvalues. Now, consider the map $T : \ell_2 \to \ell_2$ defined by

$$T(x) = (0, x_1, x_2, \cdots, x_n, 0, 0, \cdots)$$

where $x = (x_1, x_2, \cdots)$. Then, $T^{n+1}(x) = \mathbf{0}$ for all $x \in \ell_2$. Then it is clear that the only eigenvalue of T is zero. ■

Example 8.4.2 Let $\{\alpha_n\}$ be a sequence of complex numbers such that $\alpha_n \to 0$. Define $T \in \mathcal{L}(\ell_2)$ by

$$T(x) = (\alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_n x_n, \cdots)$$

where $x = (x_1, x_2, \cdots)$. Define

$$T_n(x) = (\alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_n x_n, 0, 0, \cdots).$$

Then T_n is of finite rank. Further, for any $x = (x_i) \in \ell_2$, we have

$$\|T(x) - T_n(x)\|_2^2 = \sum_{k=n+1}^{\infty} |\alpha_k|^2 |x_k|^2 \le \left(\sup_{k \ge n+1} |\alpha_k|^2\right) \|x\|^2$$

from which we immediately see that $\|T - T_n\| \to 0$. Thus T is compact. Clearly each α_n is an eigenvalue with associated eigenvector e_n. If $\alpha_n \neq 0$ for any n, then zero is not an eigenvalue of T (since T will then be injective). If a finite number of the α_n are zero, then zero will be an eigenvalue of finite multiplicity. If infinitely many α_n are zero, then zero will be an eigenvalue with infinite multiplicity. Thus while zero is always in the spectrum of a compact operator, it need not be an eigenvalue or it could be an eigenvalue of finite or infinite multiplicity, unlike non-zero elements of the spectrum which are always eigenvalues of finite multiplicity. If all but a finite number of the α_i are zero, $\sigma(T)\backslash\{0\}$ is finite. If all the α_i are non-zero, then $\sigma(T)\backslash\{0\}$ consists of an infinite sequence of eigenvalues converging to zero. Thus, all the possibilitis mentioned in the statement of the theorem above can occur. ■

Example 8.4.3 Let $T : \ell_2 \to \ell_2$ be given by

$$T(x) = (\alpha_1 x_1, \cdots, \alpha_n x_n, \cdots)$$

where $x = (x_1, x_2, \cdots)$, and where $\{\alpha_n\}$ is a sequence of complex numbers. We saw that if $\alpha_n \to 0$, then T is compact. Conversely, if T is compact, it follows that $\alpha_n \to 0$. Indeed, since α_n are all in the spectrum, by Theorem 8.4.1 (c), we deduce that $\alpha_n \to 0$. ■

8.5 Compact Self-adjoint Operators

In this section, we will study the spectrum of a compact self-adjoint operator on a Hilbert space. If H is a Hilbert space and $T \in \mathcal{L}(H)$ is self-adjoint, recall that

$$(T(x), y) = (x, T(y))$$

for all x and $y \in H$.

Example 8.5.1 Consider the operator $G : L^2(0, 1) \to L^2(0, 1)$ defined in Example 8.1.9. Then G is self-adjoint. To see this, let f and $\mathsf{g} \in L^2(0, 1)$. Set $\mathsf{u} = G(\mathsf{f})$ and $\mathsf{v} = G(\mathsf{g})$. Then (cf. Example 8.1.9)

$$(G(\mathsf{f}), \mathsf{g}) = \int_0^1 gu \, \mathrm{d}x = \int_0^1 \alpha(x) v'(x) u'(x) \, \mathrm{d}x = \int_0^1 f v \, \mathrm{d}x = (\mathsf{f}, G(\mathsf{g}))$$

which establishes our claim. In particular, the operator G considered in Example 8.1.8 is also self-adjoint. Both these are thus examples of compact and self-adjoint operators on $L^2(0, 1)$. ■

Let H be a Hilbert space. We saw, in Sect. 8.3, that if $T \in \mathcal{L}(H)$ is a self-adjoint operator, then $\sigma(T) \subset \mathbb{R}$. Thus, if T is a compact self-adjoint operator, all it eigenvalues are real. Further, if $T(u) = \lambda u$, and $T(v) = \mu v$, where $\lambda \neq \mu$ and u and v are non-zero, we have

$$\lambda(u, v) = (T(u), v) = (u, T(v)) = \mu(u, v).$$

Hence, it follows that $(u, v) = 0$. Thus, we have the following result.

Proposition 8.5.1 *Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be a compact self-adjoint operator. Then, eigenvectors corrsponding to distinct eigenvalues are orthogonal to each other.* ■

Proposition 8.5.2 *Let $T : H \to H$ be a self-adjoint compact operator. Set*

$$m = \inf_{u \in H; \|u\|=1} (T(u), u) \text{ and } M = \sup_{u \in H; \|u\|=1} (T(u), u).$$

Then, $\sigma(T) \subset [m, M]$ and, further, both m and M belong to $\sigma(T)$.

Proof Let $\lambda > M$. Then, for any $u \in H$, we have

$$(T(u), u) \le M\|u\|^2 < \lambda\|u\|^2.$$

Thus, if we set $a(u, v) = (\lambda u - T(u), v)$, we have

$$a(u, v) = (u, \lambda v - T(v)) = \overline{a(v, u)}, \text{ and } a(u, u) \ge (\lambda - M)\|u\|^2$$

since λ is real and T is self-adjoint. Thus, $a(\cdot, \cdot)$ defines a new inner product whose norm is equivalent to the original one. Hence, by the Riesz representation theorem, it follows that for any $f \in H$, there exists a unique $u \in H$ such that

$$(\lambda u - T(u), v) = (f, v)$$

for all $v \in H$, i.e. $\lambda I - T$ is onto; since it is also one-one (why?) and continuous, it is invertible, by the open mapping theorem. Thus $\lambda \in \rho(T)$.

Again, consider $a(u, v) = (Mu - T(u), v)$. This almost defines an inner product on H; observe that $a(Mu - T(u), u) \ge 0$ for all $u \in H$ but this quantity being zero does not imply that $u = \mathbf{0}$. Nevertheless, we can still prove the Cauchy-Schwarz inequality for this 'inner product' using the same arguments as in the proof of Theorem 7.1.1. Thus, we deduce that

$$|(Mu - T(u), v)| \le (Mu - T(u), u)^{\frac{1}{2}}(Mv - T(v), v)^{\frac{1}{2}}$$

for all u and $v \in H$. Thus,

$$|(Mu - T(u), v)| \leq \|MI - T\|^{\frac{1}{2}}\|v\|(Mu - T(u), u)^{\frac{1}{2}}.$$

This implies that

$$\|Mu - T(u)\| \leq C(Mu - T(u), u)^{\frac{1}{2}}$$

where $C = \|MI - T\|^{\frac{1}{2}}$.

Let $\{u_n\}$ be a sequence in H such that $\|u_n\| = 1$ for all n and such that $(T(u_n), u_n) \to M$ as $n \to \infty$. Then, it follows that $Mu_n - T(u_n) \to \mathbf{0}$. This proves that $M \in \sigma(T)$. (If not, then $MI - T$ would be invertible and we would have

$$u_n = (MI - T)^{-1}(Mu_n - T(u_n)) \to \mathbf{0}$$

while $\|u_n\| = 1$, which is a contradiction.)

The results for m will now follow from the above by considering $-T$ instead of T. ■

Corollary 8.5.1 *Let $T \in \mathcal{L}(H)$ be a compact and self-adjoint operator such that $\sigma(T) = \{0\}$. Then $T = \mathbf{0}$.*

Proof Let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ stand for the real and imaginary parts of a complex number z. By the preceding proposition, it follows that $(T(u), u) = 0$ for all $u \in H$. Then (cf. Proposition 7.2.3) it follows that $T = \mathbf{0}$. ■

Remark 8.5.1 The above corollary remains true even if H is a real Hilbert space, thanks to the self-adjointness of T. For, if $T(u), u) = 0$ for all $u \in H$, then,

$$0 = (T(x+y), x+y) = (T(x), y) + (T(y), x) = 2(T(x), y),$$

for arbitrary x and y in H. From this, we deduce that $T = \mathbf{0}$. ■

Theorem 8.5.1 *Let H be an infinite dimensional separable Hilbert space and let $T \in \mathcal{L}(H)$ be a compact self-adjoint operator on H. Then H admits an orthonormal basis consisting of eigenvectors of T.*

Proof Let $\sigma(T) = \{0\} \cup \{\lambda_n \mid n \in \mathbb{N}\}$. Set $\lambda_0 = 0$. The numbers λ_n for $n \in \mathbb{N}$ are all distinct eigenvalues of T. Thus, if we set $E_0 = \mathcal{N}(T)$ and $E_n = \mathcal{N}(T - \lambda_n I)$ for $n \in \mathbb{N}$, it follows that

$$0 \leq \dim(E_0) \leq \infty \text{ and } 0 < \dim(E_n) < \infty$$

for $n \in \mathbb{N}$.

The spaces E_n, $n \geq 0$ are all mutually orthogonal (cf. Proposition 8.5.1). Let F be the linear subspace generated by the E_n, $n \geq 0$. We will show that F is dense in H. In other words, we will show that $F^\perp = \{\mathbf{0}\}$.

Clearly $T(F) \subset F$. If $u \in F^\perp$ and $v \in F$, we have

$$(T(u), v) = (u, T(v)) = 0.$$

Thus $T(F^\perp) \subset F^\perp$ as well. Let T_0 be the restriction of T to $F^\perp$. Then $T_0 \in \mathcal{L}(F^\perp)$ is compact and self-adjoint. Hence, if T_0 has a non-zero element $\widetilde{\lambda}$ in its spectrum, then it must be an eigenvalue. But then, it will mean that there exists $u \neq \mathbf{0}$ in $F^\perp$ such that $\widetilde{\lambda}u = T_0 u = Tu$ which implies that $u \in F$, which is impossible. Thus $\sigma(T_0) = \{0\}$, whence, by the preceding corollary, $T_0 = \mathbf{0}$. Thus

$$F^\perp \subset \mathcal{N}(T) \subset F$$

i.e. $F^\perp = \{\mathbf{0}\}$.

Now, if E_0 is a non-zero subspace, it is either finite dimensional or is an infinite dimensional separable subspace of H and so it has an orthonormal basis. Each E_n, $n \geq 1$, is finite dimensional and admits an orthonormal basis. The union of all these bases forms an orthonormal basis of H, since F is dense in H (cf. Corollary 7.4.1). ■

Remark 8.5.2 In view of Remark 8.5.1, the above result is true for in the case of infinite dimensional separable *real* Hilbert spaces as well. ■

The above result is known as the *spectral theorem for compact self-adjoint operators on a Hilbert space*. We will now present the same result in another form, completely in terms of operators. Let us denote by P_n, the orthogonal projection of H onto E_n, $n \geq 0$. Since the eigenvectors of distinct eigenvalues are orthogonal to each other, it follows that $P_n P_m = \mathbf{0}$, when $n \neq m$. Let $\{u_k\}_{k=1}^\infty$ denote an orthonormal basis of H consisting of eigenvectors of T. Let $T(u_k) = \mu_k u_k$, where each μ_k is equal to one of the λ_n. Then, if $x \in H$, we have that

$$x = \sum_{k=1}^\infty (x, u_k)u_k.$$

Then it follows that

$$T(x) = \sum_{k=1}^\infty \mu_k (x, u_k)u_k.$$

Let $d(n) < +\infty$ denote the dimension of E_n, $n \geq 1$. Let $\{u_{k(1)}, \cdots, u_{k(d(n))}\}$ be those vectors in the collection $\{u_k\}_{k=1}^\infty$ which are in E_n. Then,

$$P_n(x) = \sum_{j=1}^{d(n)} (P_n(x), u_{k(j)})u_{k(j)} = \sum_{j=1}^{d(n)} (x, u_{k(j)})u_{k(j)},$$

since P_n is an orthogonal projection. Similarly we can write the expansion when $n = 0$, to get

$$x = P_0(x) + \sum_{n=1}^{\infty} P_n(x).$$

Then, we get

$$T(x) = \sum_{k=1}^{\infty} \lambda_n P_n(x).$$

Now, for any n, we have

$$\begin{aligned}
\left\| T - \sum_{j=1}^{n} \lambda_j P_j \right\|^2 &= \sup_{\|x\|=1} \left\| T(x) - \sum_{j=1}^{n} \lambda_j P_j(x) \right\|^2 \\
&= \sup_{\|x\|=1} \sum_{j=n+1}^{\infty} \lambda_j^2 \|P_j x\|^2 \\
&\leq \sup_{j>n} \lambda_j^2 \to 0.
\end{aligned}$$

Thus we can write

$$T = \sum_{n=1}^{\infty} \lambda_n P_n.$$

This is the second form of the spectral theorem.

We now give a variational characterization of the non-zero eigenvalues of a compact self-adjoint operator, T, on a separable Hilbert space, H.

Let us number the non-negative eigenvalues of T in decreasing order of magnitude and the non-positive eigenvalues in increasing order of magnitude. Each eigenvalue is repeated as many times as its (geometric) multiplicity. Thus, we have

$$\begin{aligned}
\lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq 0 \\
\lambda_1^- \leq \lambda_2^- \leq \cdots \leq 0.
\end{aligned}$$

Let u_m^+ be a normalized eigenvalue of λ_m^+ and u_n^- a normalized eigenvalue of λ_n^- such that the set of all the u_m^+ and u_n^- form an orthonormal basis of H. Let V_n^+ be the finite dimensional space generated by the vectors $\{u_1^+, \cdots, u_n^+\}$ and let $V_0 = \{\mathbf{0}\}$. Similarly we can define V_n^- for $n \geq 0$. With these notations, we can now prove the following result.

Theorem 8.5.2 *For each $m \geq 1$, we have*

$$\lambda_m^+ = (T(u_m^+), u_m^+)$$

$$= \max_{v \neq \mathbf{0};\ v \perp V_{m-1}^+} \frac{(T(v), v)}{\|v\|^2}$$

$$= \min_{V \subset H;\ \dim(V) = m-1}\ \max_{v \neq \mathbf{0};\ v \perp V} \frac{(T(v), v)}{\|v\|^2}.$$

Proof Since u_m^+ is an eigenvector corresponding to the eigenvalue λ_m^+, and since $\|u_m^+\| = 1$, it follows immediately that

$$\lambda_m^+ = (T(u_m^+), u_m^+) \tag{8.5.1}$$

On the other hand, if $v \in H$, we have (cf. Theorem 7.4.2)

$$v = \sum_k (v, u_k^+) u_k^+ + \sum_n (v, u_n^-) u_n^-$$

and so

$$T(v) = \sum_k \lambda_k^+ (v, u_k^+) u_k^+ + \sum_n \lambda_n^- (v, u_n^-) u_n^-.$$

We then deduce that

$$(T(v), v) = \sum_k \lambda_k^+ |(v, u_k^+)|^2 + \sum_n \lambda_n^- |(v, u_n^-)|^2 \le \sum_k \lambda_k^+ |(v, u_k^+)|^2.$$

If, in addition, $v \perp V_{m-1}$, then,

$$(T(v), v) \le \sum_{k \ge m} \lambda_k^+ |(v, u_k^+)|^2 \le \lambda_m^+ \sum_{k \ge m} |(v, u_k^+)|^2 \le \lambda_m^+ \|v\|^2$$

by repeated use of Parseval identity (cf. Theorem 7.4.2). Since $u_m^+ \perp V_{m-1}$, the preceding computation and (8.5.1) imply that

$$\lambda_m^+ = \max_{v \neq \mathbf{0}; v \perp V_{m-1}} \frac{(T(v), v)}{\|v\|^2}. \tag{8.5.2}$$

Finally, let V be a $(m-1)$-dimensional subspace of H. Then, there exists a nonzero vector u in the m-dimensional space V_m^+ such that $u \perp V$ (why?). Thus

$$\sup_{v \neq \mathbf{0}; v \perp V} \frac{(T(v), v)}{\|v\|^2} \ge \frac{(T(u), u)}{\|u\|^2}.$$

But $u = \sum_{i=1}^{m} \alpha_i u_i^+$ and so $\|u\|^2 = \sum_{i=1}^{m} \alpha_i^2$; thus,

$$(T(u), u) \;=\; \sum_{i=1}^{m} \alpha_i^2 \lambda_i^+ \;\geq\; \lambda_m^+ \|u\|^2.$$

Combining these inequalities, we get

$$\sup_{v \neq \mathbf{0};\, v \perp V} \frac{(T(v), v)}{\|v\|^2} \;\geq\; \lambda_m^+.$$

In fact, since T is compact, the 'sup' above is a 'max' (why?). Further, in view of (8.5.2), we deduce that

$$\lambda_m^+ \;=\; \min_{V \subset H;\, \dim(V) = m-1} \; \max_{v \neq \mathbf{0};\, v \perp V} \frac{(T(v), v)}{\|v\|^2}. \tag{8.5.3}$$

This completes the proof. ∎

Remark 8.5.3 We can prove analogous relations for the eigenvalues λ_n^-. ∎

Corollary 8.5.2 *With the preceding notations, we have*

$$\lambda_1^+ \;=\; \max_{v \neq \mathbf{0};\, v \in H} \frac{(T(v), v)}{\|v\|^2}.$$

∎

Example 8.5.2 Let us consider, once again, the operator G defined on $L^2(0, 1)$ as in Example 8.1.9 (and also Example 8.5.1). Consider the following 'eigenvalue problem': find $(\mathsf{u}, \lambda) \in (H_0^1(0, 1)\backslash\{\mathbf{0}\}) \times \mathbb{R}$ such that

$$a(\mathsf{u}, \mathsf{v}) \;=\; \lambda(\mathsf{u}, \mathsf{v}) \tag{8.5.4}$$

for all $\mathsf{v} \in H_0^1(0, 1)$. In view of the definition of G, it follows that

$$G(\lambda \mathsf{u}) \;=\; \mathsf{u}.$$

Since $a(.,.)$ is coercive, there is no solution corresponding to $\lambda = 0$. If $\lambda \neq 0$, we can write

$$G(\mathsf{u}) \;=\; \lambda^{-1}\mathsf{u}.$$

Thus the problem reduces to finding the spectrum of G which is a self-adjoint compact operator. Since $(G(\mathsf{u}), \mathsf{u}) \geq 0$ for all $\mathsf{u} \in L^2(0, 1)$, it follows that all eigenvalues λ are strictly positive. Since G has a sequence of eigenvalues $\mu_n \to 0$, the problem (8.5.4) admits a sequence of eigenvalues $\lambda_n = \mu_n^{-1}$ which tend to infinity. In view of the preceding corollary, we get that

$$\lambda_1^{-1} \;=\; \mu_1 \;=\; \max_{\mathsf{u} \neq \mathsf{0};\, \mathsf{u} \in L^2(0,1)} \frac{(G(\mathsf{u}), \mathsf{u})}{\|\mathsf{u}\|_2^2}.$$

Notice that the equation (8.5.4) is the 'weak formulation' (cf. Remark 8.1.2) of the following two-point boundary value problem:

$$\begin{array}{rcl} -\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha\frac{\mathrm{d}u}{\mathrm{d}x}\right) & = & \lambda u \text{ in}(0,1) \\ u(0) & = & u(1) = 0. \end{array}$$

Thus, eigenvectors of such boundary value problems generate, in a natural way, orthonormal bases for Hilbert spaces like $L^2(0, 1)$. Several special functions of mathematical physics are, in fact, eigenvectors of certain boundary value problems for second order differential equations. For instance, we cite the cases of the Bessel functions, Legendre polynomials (cf. Exercise 7.22) and the Hermite polynomials (cf. Exercise 7.23). For details, the reader is referred to Simmons [2]. ■

Remark 8.5.4 Notice that, in the preceding example, all the eigenvectors will, in fact, belong to $H_0^1(0, 1)$. Using arguments very similar to those in the proof of Theorem 8.5.2, it can also be shown that if the eigenvalues $\{\lambda_n\}$ are numbered in increasing order of magnitude (taking into account the geometric multiplicities), then they can be given the following variational characterization in $H_0^1(0, 1)$. Let $\{\mathsf{w}_n\}$ be the orthonormal basis of eigenvectors (in $L^2(0, 1)$) with w_n corresponding to λ_n for each n. Let V_n be the finite dimensional space spanned by $\{\mathsf{w}_1, \cdots, \mathsf{w}_n\}$. Set

$$R(\mathsf{v}) = \frac{a(\mathsf{v}, \mathsf{v})}{\|\mathsf{v}\|_2^2}$$

for $\mathsf{v} \in H_0^1(0, 1), \ \mathsf{v} \neq \mathsf{0}$. Then

$$\begin{aligned} \lambda_n &= R(\mathsf{w}_n) \\ &= \max_{\mathsf{v}\neq\mathsf{0};\ \mathsf{v}\in V_n} R(\mathsf{v}) \\ &= \min_{\mathsf{v}\neq\mathsf{0};\ \mathsf{v}\perp V_{n-1}} R(\mathsf{v}) \\ &= \min_{V\subset H_0^1(0,1);\ \dim(V)=n} \ \max_{\mathsf{v}\neq\mathsf{0};\ \mathsf{v}\in V} R(\mathsf{v}). \end{aligned}$$

In particular, we have

$$\lambda_1 = \min_{\mathsf{v}\neq\mathsf{0};\ \mathsf{v}\in H_0^1(0,1)} R(\mathsf{v}).$$

For details, see Kesavan [1]. (A proof is outlined in Exercise 8.30 in the next section.) This result is an infinite dimensional version of the characterization of eigenvalues of a hermetian positive definite matrix (cf. Proposition 1.1.8). ■

Example 8.5.3 A particular case of the preceding example is the case where $\alpha(x) \equiv 1$. The corresponding differential equation is

$$\begin{aligned} -\frac{d^2u}{dx^2} &= \lambda u \ \text{ in } (0,1) \\ u(0) &= u(1) = 0. \end{aligned}$$

A simple computation yields that the only possible solutions to this equation are given by

$$\lambda_n \ = \ n^2\pi^2; \ u_n \ = \ A_n \sin n\pi x$$

where A_n is an arbitrary constant. Choosing $A_n = \sqrt{2}$, we get an orthonormal sequence in $L^2(0,1)$. We have already seen that it is complete (cf. Example 7.4.6). In particular, we have

$$\lambda_1 \ = \ \pi^2 \ = \ \min_{\mathsf{v}\neq 0;\ \mathsf{v}\in H_0^1(0,1)} \frac{\int_0^1 |\nabla v|^2 \, dx}{\int_0^1 |v|^2 \, dx}.$$

Thus, for every $\mathsf{v} \in H_0^1(0,1)$, we deduce that

$$\|\mathsf{v}\|_2 \ \leq \ \frac{1}{\pi}\|\mathsf{v}'\|_2.$$

This is just Poincaré's inequality (cf. Theorem 6.4.6), except that we have a better constant in the estimate compared to the inequality (6.4.4). In fact, we have shown in this case that $1/\pi$ is the best possible constant and equality is attained in the above inequality for the function $x \mapsto \sin \pi x$. ■

8.6 Exercises

8.1 Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$ be compact. Show that if $x_n \rightharpoonup x$ in V, then $T(x_n) \to T(x)$ in W.

8.2 (a) Let V and W be Banach spaces and let V be reflexive. Let $T \in \mathcal{L}(V, W)$. Assume that whenever $x_n \rightharpoonup x$ in V, we have $T(x_n) \to T(x)$ in W. Show that T is compact.
(b) Show, by means of an example, that the conclusion is not true, in general, if we drop the assumption that V is reflexive.

8.3 Let H be an infinite dimensional Hilbert space. Let A and B belong to $\mathcal{L}(H)$. If AB is compact, is it necessary that at least one of A or B is compact?

8.4 (a) Let V be a reflexive Banach space and let $T \in \mathcal{L}(V, \ell_1)$. Show that T is compact.
(b) Let W be a reflexive Banach space and let $T \in \mathcal{L}(c_0, W)$. Show that T is compact.

8.5 Let H be an infinite dimensional Hilbert space and let $T \in \mathcal{L}(c_0, H)$. If $\mathcal{R}(T)$ is closed, show that T is not injective.

8.6 Let H be an infinite dimensional Hilbert space. Let $T \in \mathcal{L}(H)$ be compact. If the range of T is closed, show that T is of finite rank.

8.7 Let V and W be infinite dimensional Hilbert spaces. Let $\{v_1, \cdots, v_n\} \subset V$ and $\{w_1, \cdots, w_n\} \subset W$. For every $x \in V$, define

$$T(x) = \sum_{k=1}^{n} (x, v_k) w_k.$$

Show that $T \in \mathcal{L}(V, W)$ and that it is compact.

8.8 Let V and W be infinite dimensional Hilbert spaces and let $T \in \mathcal{L}(V, W)$ be an operator of finite rank, say, n. If $\{w_1, \cdots, w_n\}$ is an orthonormal basis for the range of T, show that there exist $v_1, \cdots, v_n$ in V such that, for every $x \in V$,

$$T(x) = \sum_{k=1}^{n} (x, v_k) w_k.$$

8.9 Let H be an infinite dimensional Hilbert space and let $\{v_1, \cdots, v_n\} \subset H$. Let M be the subspace spanned by the set $\{v_k\}_{k=1}^{n}$. Let $\{w_1, \cdots, w_n\}$ be a linearly independent set of vectors in $M^{\perp}$. For $x \in H$, define

$$T(x) = \sum_{k=1}^{n} (x, v_k) w_k.$$

For every scalar α, show that $I + \alpha T$ is invertible.

8.10 Let H be a Hilbert space and let $A \in \mathcal{L}(H)$ be compact. Assume that $(A(x), x) \geq 0$ for all $x \in H$. Show that there does not exist a constant $\alpha > 0$ such that

$$(A(x), x) \geq \alpha \|x\|^2$$

for all $x \in H$.

8.11 Let V be a Banach space and let $T \in \mathcal{L}(H)$ be compact. Show that if $S \in \mathcal{L}(V)$, then $S(I - T) = I$ if, and only if, $(I - T)S = I$. In this case, deduce that $I - (I - T)^{-1}$ is compact.

8.12 Let H be a Hilbert space. Let $T \in \mathcal{L}(H)$ be compact. Show that

$$\|T\| = \max_{x \in H;\ \|x\|=1} \|T(x)\|.$$

8.13 Let $H_i,\ i = 1, 2, 3$ be Hilbert spaces such that

$$H_1 \subset H_2 \subset H_3$$

with continuous inclusions. Assume, further, that the inclusion of H_1 in H_2 is compact. Show that for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|u\|_{H_2} \leq \varepsilon\|u\|_{H_1} + C_\varepsilon\|u\|_{H_3}$$

for all $u \in H_1$.

8.14 Let $\{a_j\}_{j=1}^\infty$ be a sequence of numbers such that $\sum_{j=1}^\infty |a_j| < \infty$. Consider the infinite matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ a_3 & a_4 & a_5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Let A be the linear mapping defined on ℓ_2 by this matrix (cf. Exercise 2.34). Show that $A \in \mathcal{L}(\ell_2)$ and that it is compact.

8.15 (a) Let W be a Banach space. Assume that there exists a sequence of operators of finite rank, $\{P_n\}$, in $\mathcal{L}(W)$ such that $P_n(y) \to y$ for all $y \in W$. Show that, if V is any Banach space, and if $T \in \mathcal{L}(V, W)$ is compact, then T is the limit (in $\mathcal{L}(V, W)$) of operators of finite rank.
(b) Deduce that, if V is any Banach space, and if $T \in \mathcal{L}(V, \ell_p)$, where $1 \leq p < \infty$, is compact, then T is the limit of operators of finite rank.

8.16 Let $x = (x_1, x_2, \cdots) \in \ell_2$. Define

$$\begin{aligned} S(x) &= (0, x_1, x_2, \cdots) \\ M(x) &= (x_1, x_2/2, x_3/3, \cdots). \end{aligned}$$

Set $T = MS$. Show that $T \in \mathcal{L}(\ell_2)$ is compact and compute its spectrum.

8.17 Consider the complex spaace ℓ_p, $1 \leq p < \infty$. For $x = (x_1, \cdots, x_k, \cdots) \in \ell_p$, define

$$T(x) = (\beta_1 x_1, \cdots, \beta_k x_k, \cdots),$$

where $\{\beta_k\}_{k=1}^\infty$ is a bounded sequence of scalars. Show that $T \in \mathcal{L}(\ell_p)$ and compute its spectrum.

8.18 Let $V = \mathcal{C}[0, 1]$ equipped with its usual norm. For $f \in V$, define $T(f) \in V$ by

$$T(f)(t) = tf(t),\ t \in [0, 1].$$

(a) Show that $T \in \mathcal{L}(V)$ and that it is not compact.

(b) Show that T has no eigenvalues.
(c) Show that $\sigma(T) = [0, 1]$.

8.19 Let $H = L^2(0, 1)$, considered as a real Hilbert space (of real-valued square integrable functions on $(0, 1)$). For $f \in H$, define $T(f)$ by

$$T(f)(t) = tf(t), \ t \in (0, 1).$$

Show that $T \in \mathcal{L}(H)$ is self-adjoint and that it has no eigenvalues.

8.20 Let $L^2(0, 1)$ be the space of complex valued square integrable functions. Let K be the Hilbert-Schmidt operator (cf. Example 8.1.11) defined on this space by the function $i\mathsf{k}$ where

$$\mathsf{k}(t, s) = \begin{cases} 1 \text{ if } s \leq t \\ -1 \text{ if } s > t. \end{cases}$$

Show that all the eigenvalues of K are given by

$$\lambda_k = \frac{2}{(2k+1)\pi}, \ k \in \mathbb{Z}$$

with corresponding eigenvector $t \mapsto \exp(i(2k+1)\pi t)$.

8.21 Consider the operator $K \in \mathcal{L}(L^2(0, 1))$ defined by

$$K(\mathsf{f})(t) = \int_0^t f(s) \, \mathrm{d}s.$$

Show that it is compact and compute its spectrum.

8.22 Let V be a Banach space and let $T \in \mathcal{L}(V)$ be compact. Let L and M be closed subspaces of V such that M is strictly contained in L. Assume that $(I - T)L \subset M$. Show that there exists $x \in L$ such that $\|x\| = 1$ and $\|T(x) - T(y)\| \geq 1/2$ for all $y \in M$.

8.23 Let V be a Banach space and let $T \in \mathcal{L}(V)$ be compact.

(a) Show that for all $n \in \mathbb{N}$, the operator $(I - T)^n$ is a compact perturbation of the identity.
(b) Let $N_k = \mathcal{N}((I - T)^k)$ for $k \in \mathbb{N}$. Show that N_k is an increasing sequence of finite dimensional spaces which is stationary, i.e. $N_k \subset N_{k+1}$ for all $k \in \mathbb{N}$ and there exists $l \in \mathbb{N}$ such that $N_k = N_l$ for all $k > l$ and l is the least positive integer with that property. (The dimension of N_l is called the *algebraic multiplicity* of T.) (Hint: use Exercise 8.13.)
(c) If V is a Hilbert space and if T is self-adjoint, show that $l = 1$, i.e. the algebraic and geometric multiplicities coincide.

(d) Let $F_k = \mathcal{R}((I - T)^k)$. Show that $\{F_k\}$ is a decreasing sequence of closed subspaces of V and that $F_k = F_l$ for all $k > l$.
(e) Show that $V = N_l \oplus F_l$.
(f) Show that $T(N_l) \subset N_l$ and that $T(F_l) \subset F_l$.
(g) Show that $I - T$ is an isomorphism of F_l onto itself.

8.24 (a) Let V be a Banach space and let $T \in \mathcal{L}(V)$ be compact. Let $\lambda \neq 0$. Show that the results of the previous exercise are valid when $I - T$ is replaced by $\lambda I - T$. (If λ is an eigenvalue of T, then $\mathcal{N}(\lambda I - T)$ is called the *eigenspace* associated to λ while the space N_l obtained in section (b) of the preceding exercise is called the *generalized eigenspace* associated to λ and is denoted $N(\lambda)$. The corresponding complement F_l is denoted $F(\lambda)$.
(b) Show that if λ and μ are disctinct non-zero eigenvalues of T, then $N(\mu) \subset F(\lambda)$.

8.25 (a) Let H be a Hilbert space and let S and $T \in \mathcal{L}(H)$. Let $\lambda \neq 0$ be an eigenvalue of ST with eigenvector u. Show that $T(u)$ is an eigenvector of TS.
(b) Show that the non-zero eigenvalues of ST and TS are the same and that

$$T(\mathcal{N}(ST - \lambda I)) = \mathcal{N}((TS - \lambda I))$$

for each such eigenvalue λ.

8.26 Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be self-adjoint. Show that

$$\|T\| = \sup_{x \in H;\ \|x\|=1} |(T(x), x)|.$$

8.27 Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be compact and self-adjoint. Show that either $\|T\|$ or $-\|T\|$ is an eigenvalue of T.

8.28 Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be self-adjoint. If T^2 is compact, show that T is compact as well. Generalize this result to the case when T^n is compact for some positive integer n.

8.29 Let H be an infinite dimensional Hilbert space and let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. For $x \in H$, define

$$T(x) = \sum_{k=1}^{\infty} (x, e_{2k+1}) e_{2k}.$$

Show that $T \in \mathcal{L}(H)$. SHow that T is not compact, while T^2 is compact.

8.30 Let V and H be real Hilbert spaces with inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_H$, respectively; let the corresponding norms be denoted by $\|\cdot\|_V$ and $\|\cdot\|_H$. Assume that $V \subset H$, the inclusion being continuous and compact. Assume, further, that V is a dense subspace of H. Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a continuous, symmetric and coercive bilinear form

(a) Let $f \in H$. Show that there exists a unique vector $u \in V$ such that

$$a(u, v) = (f, v)_H$$

for every $v \in V$.

(b) Define $G(f) = u$. Show that $G \in \mathcal{L}(H)$ and that it is self-adjoint and compact. Show, further, that

$$(G(u), u)_H > 0$$

for every $u \in H,\ u \neq \mathbf{0}$.

(c) Show that the image of G is dense in V.

(d) Show that there exists a sequence of positive real numbers $\{\lambda_n\}$ and an orthonormal basis $\{u_n\}$ of H such that each $u_n \in V$ and

$$a(u_n, v) = \lambda_n (u_n, v)_H$$

for every $v \in V$.

(e) Set $v_n = (1/\sqrt{\lambda_n})u_n$. Show that $\{v_n\}$ forms an orthonormal basis of V (for the inner product induced on V by the bilinear form $a(.,.)$).

(f) Show that

$$\lambda_1 = \min_{v \in V;\ v \neq \mathbf{0}} \frac{a(v, v)}{\|v\|_H^2}.$$

(g) If $u \in V$ is such that $\|u\|_H^2 = 1$ and $a(u, u) = \lambda_1$, then show that

$$a(u, v) = \lambda_1 (u, v)_H$$

for every $v \in V$.

(h) Let V_n be the space spanned by $\{u_1, u_2, \cdots, u_n\}$. Set

$$R(v) = \frac{a(v, v)}{\|v\|_H^2}$$

for $v \in V,\ v \neq \mathbf{0}$. Let

$$V_n^\perp = \{v \in V \mid (v, w)_H = 0 \text{ for all } w \in V_n\}.$$

Show that

$$\begin{aligned} \lambda_n &= R(u_n) \\ &= \max\nolimits_{v \in V_n;\ v \neq \mathbf{0}} R(v) \\ &= \min\nolimits_{v \in V_{n-1}^\perp; v \neq \mathbf{0}} R(v) \\ &= \min\nolimits_{W \subset V;\ \dim(W) = n}\ \max\nolimits_{v \in W;\ v \neq \mathbf{0}} R(v). \end{aligned}$$

Remark 8.6.1 Exercise 8.30 above is an abstract version of the situation described in Remark 8.5.4. In that case we have $V = H_0^1(0, 1)$ and $H = L^2(0, 1)$. ■

8.31 Let V and H be as in the preceding exercise. Let $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a continuous and coercive bilinear form. For $g \in H$, denote by $u_g \in V$, the unique solution such that

$$a(u_g, v) = (g, v)_H$$

for every $v \in V$.

(a) Show that there exists a constant $C > 0$ such that

$$\|u_g\|_V \leq \|g\|_H.$$

(b) Let $K \subset H$ be a closed convex subset. Let $f \in H$ be given. For $\theta \in K$, let $u(\theta) \in V$ be given by $u(\theta) = U_{f+\theta} = u_f + u_\theta$. Show that if $\theta_n \rightharpoonup \theta$ weakly in H, then $u(\theta_n) \to u(\theta)$ in H (i.e. in norm).

(c) Let, further, $u_0 \in H$ be given. Define

$$J(\theta) = \frac{1}{2}\|u(\theta) - u_0\|_H^2 + \frac{1}{2}\|\theta\|_H^2.$$

Show that there exists a unique $\theta^* \in K$ such that

$$J(\theta^*) = \min_{\theta \in K} J(\theta).$$

(d) Show that J is differentiable; if $h \in H$, show that

$$J'(\theta)(h) = (p + \theta, h)_H$$

where $p \in V$ satisfies

$$a(v, p) = (u(\theta) - u_0, v)_H$$

for every $v \in V$.

(e) Deduce that if θ^* minimizes J over K, then

$$(\theta^* + p^*, \theta - \theta^*)_H \geq 0$$

where $p^* \in V$ is the unique solution of the problem

$$a(v, p^*) = (u(\theta^*) - u_0, v)_H$$

for every $v \in V$.

(f) Conversely, if $(\theta^*, u^*, p^*) \in K \times V \times V$ satisfies the system:

$$\begin{aligned} a(u^*, v) &= (f + \theta^*, v)_H \text{ for every } v \in V \\ a(v, p^*) &= (u^* - u_0, v)_H \text{ for every } v \in V \\ (\theta^* + p^*, \theta - \theta^*)_H &\geq 0 \text{ for every } \theta \in K, \end{aligned}$$

show that θ^* minimizes J over K.

Remark 8.6.2 The preceding exercise is an example of an abstract *optimal control problem*. The set K is the set of *admissible controls* and H is the *control space*. The equation defining $u(\theta)$ is called the *state equation* and $u(\theta)$ is the *state* corresponding to the *control* θ. The state u_0 is the desirable state and so we seek to find a control such that the corresponding state is as close to the desired state as possible. We thus seek to minimize J which also takes into account the 'cost' of exercising the control. The minimizer θ^* is called the *optimal control*. The corresponding state u^* is the *optimal state*. The element p defined in (d) above is called the *adjoint state* and the equation defining it is the *adjoint state equation*. The system in (f) above, characterizing the optimal solution, is called the *optimality system*. ■

8.32 Let H be a complex Hilbert space and let $u, v \in H$. Let $T \in \mathcal{L}(H)$ be defined by $T(x) = (x, u)v$, for every $x \in H$. Show that T is self-adjoint if, and only if, $u = \alpha v$, where $\alpha \in \mathbb{R}$.

8.33 Let H be an infinite dimensionl separable Hilbert space. Let $A \in \mathcal{L}(H)$ be a compact and self-adjoint operator all of whose eigenvalues are non-negative. Let $\{w_k\}_{k=1}^{\infty}$ be an orthonormal basis of eigenvectors for H and let $A(w_k) = \lambda_k w_k$, for every k. For every $x \in H$, define

$$B(x) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}(x, w_k)w_k.$$

Show that $B \in \mathcal{L}(H)$ and that it is compact and self-adjoint. Show that $B^2 = A$.

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