Lemma If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Let $n \in \mathbb{N}$, A be a set.

Proof: If $n = 0$, $A = \emptyset$ (the only set of size 0).

Then $\mathcal{P}(\emptyset) = \{\emptyset\}$ has size $1 = 2^0$.

So $P(0)$ is true.

Assume $P(k)$ is true for some $k \geq 0$.

Let A be an arbitrary set of size $k+1$

Then $A = \{a_1, a_2, \ldots a_{k+1}\}$.

Every subset of A contains a_{k+1} or not.

So $\mathcal{P}(A) = S_1 \cup S_2$

where $S_1 \cap S_2 = \emptyset$,

$S_1 = \mathcal{P}(\{a_1, a_2, \ldots a_k\})$

and

S_2 = the set of all subsets of A which contain a_{k+1}.

Then S_2 has the same size as S_1 because there is a bijective map (Week 6) from S_1 to S_2 which simply inserts a_{k+1} into the set:

$f: S_1 \to S_2$ defined by

$f(B) = B \cup \{a_{k+1}\}$.

So $|S_1| = |S_2|$.

$|S_1| = 2^k$ ~~since~~ by inductive assumption

$\therefore |\mathcal{P}(A)| = |S_1| + |S_2|$

(because $S_1 \cap S_2 = \emptyset$)

$= 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$

So $P(k) \to P(k+1)$.

Then by PMI $P(n)$ is true for all $n \geq 0$.

$\blacksquare$