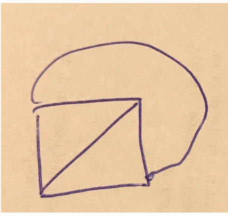
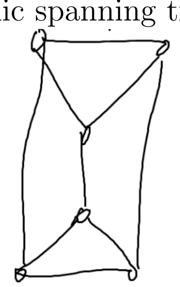
DISCRETE MATH 37181 HOMEWORK SHEET 11

Instructions. Try these sometime after your tutorial and before the next lecture. Set aside some time each week to keep up with the homework. Partial solutions at the end of the PDF.

1. Draw two non-isomorphic spanning trees for each of these graphs.

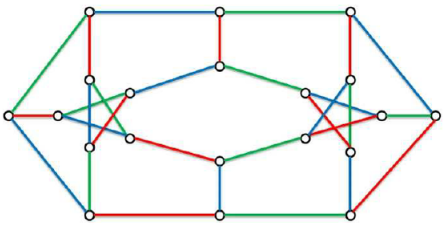




2. Prove that for a tree with n vertices with some vertex of degree $k < n$, the longest simple path has length at most $n - k + 1$. Give examples to show this bound is achieved sometimes, but not all of the time.

3. What is the negation of this statement: every tree has at least two vertices of degree 1. Is it true or false? Prove you are right.

4. Consider the graph $G =$



   (a) Does G have a Hamilton cycle?

   (b) Does G have an Euler path?

   (c) How many edges in a spanning tree for G?

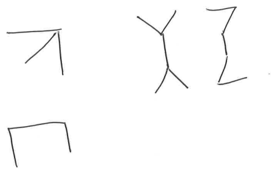
5. Consider the statement: for all $n \in \mathbb{N}$, $2^{2^n} + 1$ is prime. Is it true or false? If true, prove it. If false, prove that by giving a counterexample.

END OF HOMEWORK SHEET 11

Date: Week 11.

Brief solutions:

1.



2. $n = 1$: single vertex of degree 0, bound is $n - k + 1 = 1 + 1 = 2$ and the longest simple path has length 0. For $n = 2$, only tree is a single edge, both vertices degree 1, so $n-k+1 = 2-1+1 = 2$ and longest simple path has length 1. To prove the bound always holds, induction.

   For $n > 2$ we always have a tree which is a path of length $n - 1$ (degree sequence $1122\dots2$) so the bound is sharp for these trees, and there is always a tree with degree sequence $11\dots1(n-1)$ which looks like a star, so the bound is not sharp for these trees.

3. there exists a tree that does not have at least two degree 1 vertices.

   Negation is true because the tree with a single vertex and no edges exists.

   If G is a tree with at least two vertices, then G has at least two vertices of degree 1 is true though, and a useful fact (prove it - induction!)

5. False: $n = 5$. See
   https://en.wikipedia.org/wiki/Fermat_number#Primality_of_Fermat_numbers