DISCRETE MATH 37181 HOMEWORK SHEET 2

INSTRUCTIONS. Try these sometime after your tutorial and before the next lecture. Set aside some time each week to keep up with the homework. Partial solutions at the end of the PDF.

1. Recall the definition of $a|b$ (divides). For each of the following statements, either prove using either a *direct*, *contrapositive* or *contradiction* proof, or show that it is false by giving a *counterexample*.

   (a) For all $x \in \mathbb{Z}$, $4|x$ implies $4|x^2$.

   (b) For all $x \in \mathbb{Z}$, $4|x^2$ implies $4|x$.

> **Definition 1.** Let $x, y, d \in \mathbb{Z}$ with $d \geqslant 1$. We say $x \equiv y \mod d$ if $d|(x-y)$ ($(x-y)$ is divisible by d).
>
> For example, $12 \equiv 2 \mod 5$ because $12 - 2 = 10$ is divisible by 5, and $-15 \equiv 0 \mod 5$.

2. Which of the following statements is true?

   A. $3 \equiv 7 \mod 5$

   B. $7 \equiv 3 \mod 5$

   C. $15 \equiv 7 \mod 5$

   D. $5 \equiv 15 \mod 5$

   E. none of A–D

   F. all of A–D

3. Complete the proof of the following statement: if $x^2 \equiv 0 \mod 5$ then $x \equiv 0 \mod 5$.

   Proof: Suppose $x = 5d + i$ where $i \in \{1, 2, 3, 4\}$. [1] Then

   A. if $x \equiv 0 \mod 5$ then $i = 0$ and $x^2 \equiv i^2 = 0$.

   B. $x^2 = 25d^2 + 10di + i^2 \equiv i^2 \mod 5$. If $i = 0$ then $i^2 = 0$ and so $x \equiv 0 \mod 5$.

   C. $x^2 = 25d^2 + 10di + i^2 \equiv i^2 \mod 5$. Since $i \in \{1, 2, 3, 4\}$ then $i^2 \in \{1, 4, 9 \equiv 4, 16 \equiv 1\}$ so $x^2 \not\equiv 0 \mod 5$.

   D. x^2 is a multiple of 5 so $x^2 \equiv i^2 = 0$.

   E. none of the above are a correct proof.

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[1](in other words, suppose $x \not\equiv 0 \mod 5$).

4. The proof style used in Question 3 was:

   A. direct
   B. contrapositive
   C. contradiciton
   D. none of the above.

5. Let $d \in \mathbb{Z}$. Prove or disprove:

   (a) $\forall a \in \mathbb{Z},\ a \equiv a \mod d$

   (b) $\forall a, b \in \mathbb{Z},\ a \equiv b \mod d$ implies $b \equiv a \mod d$

   (c) $\forall a, b \in \mathbb{Z},\ a \equiv b \mod d$ and $b \equiv a \mod d$ implies $a = b$

   (d) $\forall a, b, c \in \mathbb{Z},\ a \equiv b \mod d$ and $b \equiv c \mod d$ implies $a \equiv c \mod d$

   [2]

Recall from Worksheet 2,
we proved $\sqrt{2}$ is not a rational number (cannot write as $\frac{a}{b}$ for $a, b \in \mathbb{Z}$).

Let $\mathbb{R}$ denote all real numbers (expressible as decimals) and $\mathbb{Q}$ the subset of all rational numbers (expressible as $\frac{a}{b}$ with $a, b \in \mathbb{Z}$).

6. Let the universe of discourse be $\mathbb{R}$. Prove that

$$\forall x \forall y\ [(x < y) \rightarrow \exists z (x < z < y \wedge z \in \mathbb{Q})$$

[3]

[2]These four properties of a *relation* are called reflexive, symmetric, antisymmetric and transitive.
[3]Hint: cases: Case 1: $x, y \in \mathbb{Q}$. Case 2: $x \in Q, y \in \mathbb{R} \setminus Q$ (this means reals minus rational. In other words, y is real but not rational.)

Brief solutions:

1 (a) Direct (b) False, counterexample $x = 2$.

2 D.

3 C.

4 B. We start by assuming x is not divisible by 5, and concluding that x^2 is not divisible by 5.

5 (c) is false: $6 \equiv 4 \mod 2$ and $4 \equiv 6 \mod 2$ but $6 \neq 4$. The others are true. Here is (a): Direct proof. Let $a \in \mathbb{Z}$. Then $(a - a) = 0 = 0.d$ for any $d \in \mathbb{Z}$ so by definition $a \equiv a \mod d$.

and here is (b): Direct proof. Let $a, b \in \mathbb{Z}$. If $a \equiv b \mod d$ then (by Definition 1) $d|(a - b)$ so $a - b = dk$ for some $k \in \mathbb{Z}$. Then $b - a = d(-k)$ so $d|(b - a)$ so $b \equiv a \mod d$.

(d) Direct proof: $a \equiv b \mod d$ means $a - b = dk$ for some $k \in \mathbb{Z}$, and $b \equiv c \mod d$ means $c - a = ds$ for some $s \in \mathbb{Z}$. So $c - a = c - b + b - a = dk + ds = d(k + s)$ is divisible by d, so $a \equiv c \mod d$.

More on this when we do *relations* in the set theory/functions section.

6 This question is saying that between any two real numbers, no matter how close they are, you can always see a rational number. So there are lots of rational numbers!

Proof (incomplete, try to write a complete proof based on this): Case 1: If x, y both rational, so $x = \frac{a}{b}, y = \frac{c}{d}$ with $a, b, c, d \in \mathbb{Q}$, then $y - x = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}$ is the distance from x to y on the number line, so $x + \frac{bc-ad}{2bd}$ is half way. This is a rational number since

$$x + \frac{bc - ad}{2bd} = \frac{a}{b} + \frac{bc - ad}{2bd} = \frac{2da + bc - ad}{2bd}$$

and numerator, denominator are both integers (whole numbers).

Case 2: x is rational but y is irrational. If there is some integer $q \in \mathbb{Z}$ such that $x < q$ and $y > q$ then just take q to be the rational number in between x and y. Otherwise, consider the decimal expansions of x and y. They are not identical since $y > x$, but they both have the same number in front of the decimal point, so write them on top of each other like this:

$$\begin{array}{ccccccc} x & = & q & . & a_1 & a_2 & \cdots \\ y & = & q & . & b_1 & b_2 & \cdots \end{array}$$

where a_i, b_i are digits between 0 and 9 and $q \in \mathbb{Z}$. Since $y \neq x$ then there is some smallest i where $a_i \neq b_i$. Then let $r = q.a_1a_2 \dots a_{i-1}b_i$. Since y is irrational, we know the b_j's must continue infinitely (they cannot all be 0 after b_i) so $x < r < y$ and r is rational because it is

$$q + \frac{a_1 \dots a_{i-1}b_i}{10^i}.$$

Note: in any analysis textbook or online you should be able to find a much better proof than mine. Using decimal expansions is dangerous, for example what is the difference between $0.99999999\dots$ and 1?

Case 3: similar to case 2

Case 4: if they are both irrational (the final possibility), maybe their difference is irrational or not. For example $\pi - e$ versus $(1 + \sqrt{2}) - \sqrt{2}$. So your final proof needs to consider two possibilities: the difference is rational, or not.