DISCRETE MATH 37181 HOMEWORK SHEET 5

INSTRUCTIONS. Try these sometime after your tutorial and before the next lecture. Set aside some time each week to keep up with the homework. Partial solutions at the end of the PDF.

1. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by the *recursive* definition

$$\begin{array}{rcll} f(0) & = & 1 & \\ f(n) & = & nf(n-1) & n > 0. \end{array}$$

   The value of $f(7)$ is

   A. 2520 **C**. 5040 **E**. -420

   B. 28 **D**. 420 **F**. none of (A)–(E).

2. What is another name for the function defined in Question 1?

3. Define a function $A : \mathbb{N}^2 \to \mathbb{N}$ using the following *recursive* definition.

$$\begin{array}{rcll} A(0,n) & = & n+1 & n \geqslant 0, \\ A(m,0) & = & A(m-1,1) & m > 0, \\ A(m,n) & = & A(m-1, A(m,n-1)) & m,n > 0. \end{array}$$

   Then $A(2,1)$ is equal to

   A. 5 **C**. 100 **E**. 13

   B. 4 **D**. 6 **F**. none of the above.

4. Let $A = \{1,2,3,4\}$ and $B = \{a,b,c\}$.
   (a) Give an example of a one-to-one function from A to B.[1]
   (b) Give an example of an onto function from A to B.
   (c) How many different functions are there from A to B?
   (d) Give an example of a relation from A to B that is not a function.

Date: Week 5.

[1]Hint: give an example means write $f = \{(1,a),(2,b),(2,c),\dots$ and explain why it satisfies the required properties

5. Let $A = \mathbb{N}$ and $\mathscr{R}$ be the relation defined by "$a\mathscr{R}b$ if $a < b$ or $5 \mid (b - a)$".

   So for example $(1, 6) \in \mathscr{R}$ and $(1, -6) \notin \mathscr{R}$.

   (a) Is $\mathscr{R}$ reflexive?

   (b) Is $\mathscr{R}$ symmetric?

   (c) Is $\mathscr{R}$ antisymmetric?

   (d) Is $\mathscr{R}$ transitive?

6. If A is a set, the notation $|A|$ means the number of elements in A. This is also called the *size* of A, or the *cardinality* of A. Let $|A| = 4$ and $|B| = 3$.

   (a) What is $|A \times B|$?

   (b) How many functions are there from A to B?

   (c) How many relations are there from A to B?

   (d) How many one-to-one functions are there from A to B?

   (e) How many one-to-one functions are there from B to A?

7. Read this code

```
int max(int n,const int a[]) {
    int m = a[0];
    // m equals the maximum value in a[0...0]
    int i = 1;
    while (i != n) {
        // m equals the maximum value in a[0...i-1]
        if (m < a[i])
            m = a[i];
        // m equals the maximum value in a[0...i]
        ++i;
        // m equals the maximum value in a[0...i-1]
    }
    // m equals the maximum value in a[0...i-1], and i==n
    return m;
}
```

   (a) decide what you think it does

   (b) prove termination

   (c) find a loop invariant, and prove it.

Brief solutions:

1. **C** 5040

2. $n!$ "n factorial"

3. **A** 5
$$A(1,1) = A(0, A(1,0)) = A(0, A(0,1)) = A(0,2) = 3$$
$$A(1,2) = A(0, A(1,1)) = A(1,1) + 1 = 4$$
$$A(1,3) = A(0, A(1,2)) = A(1,2) + 1 = 5$$
$$A(2,1) = A(1, A(2,0)) = A(1, A(1,1)) = A(1,3) = 5$$

4. (a) Does not exist. Not enough elements in B. Need $|A| \leqslant |B|$.
   (b) $f = \{(1,a),(2,b),(3,c)(4,a)\}$.
   (c) $3^4 = 81$. 3 choices for $f(1)$, 3 choices for $f(2)$, etc.
   (d) $\mathscr{R} = \{(1,a),(1,b)\}$.

5. (a) Yes since 5 divides $(a - a) = 0$ for all $a \in \mathbb{N}$.
   (b) Not symmetric, $(1,2) \in \mathscr{R}$ because $1 < 2$, but $(1-2) = -1$ is not divisible by 5 and $2 \not< 1$ so $(2,1) \notin \mathscr{R}$.
   (c) Not antisymmetric. $(0,5)$ and $(5,0)$ are both in $\mathscr{R}$ but $0 \neq 5$.
   (d) Not transitive. $(4,7)$ and $(7,2)$ are in $\mathscr{R}$ but $(4,2)$ is not.

6. (a) 12.
   (b) $3^4 = 81$ (same as Question 4(c)).
   (c) A relation can be any subset of $\mathscr{P}(A \times B)$, and the size of the power set of a set of size 12 is $2^{12} = 4096$ (induction problem Lecture 4).
   (d) There are none.
   (e) For $f(a)$ there are 4 possible choices. Then for $f(b)$ we only have 3 numbers to choose from, and then for $f(c)$ only 2 numbers left, so $4 \times 3 \times 2 = 24$.