37181 DISCRETE MATHEMATICS

Lecture 11: Big O, complexity of algorithms

PLAN

- Big O
- comparing speed of algorithms

COMPARING SPEEDS OF ALGORITHMS

for i in [1, n]

```
power_slow(a real; n positive integer)
x = 1.0
for i = 1 to n do
    x = x*a
return x
```

i	x
	1.0
1	a
2	$a * a = a^2$
3	$a^2 * a = a^3$
4	$a^3 * a = a^4$
$\vdots$	
n	a^n

~~$L(i, x)$:~~

~~$x = a^i$~~

COMPARING SPEEDS OF ALGORITHMS

```
power_slow(a real; n positive integer)
x = 1.0
for i = 1 to n do
    x = x*a
return x
```

one iteration

power_slow(3, 6):

i	x
1	3
2	3^2
3	
4	
5	
6	3^6

Question

how many steps?

iterations of for loop $= 6$.

```
power_slow(a real; n positive integer)
x = 1.0
for i = 1 to n do
    x = x*a
return x
```

m

power_slow(3, 100):

i	x
	1.0
1	
2	
100	

100 steps.

COMPARING SPEEDS OF ALGORITHMS

```
power_fast(a real; n positive integer)
x = 1.0
i = n
while i > 0 do
    if i is odd then
        x = x*a
    i = floor(i/2)
    if i > 0 then
        a = a*a
return x
```

i	x	a
n	1.0	a
$\lfloor \frac{n}{2} \rfloor$		

A "step" of this algorithm is one iteration of the while loop.

COMPARING SPEEDS OF ALGORITHMS

```
power_fast(a real; n positive integer)
x = 1.0
i = n
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    if i is odd then
        x = x*a
    i = floor(i/2)
    if i > 0 then
        a = a*a
return x
```

One step

power_fast(3, 6):

i	x	a
6	1.0	3
3	1.0	9
1	9	81
0	9·81 = 3^6	81

: 3^4

3^6

If input $n = 2^k$

```
power_fast(a real; n positive integer)
x = 1.0
i = n
while i > 0 do
    if i is odd then
        x = x*a
    i = floor(i/2)
    if i > 0 then
        a = a*a
return x
```

i	x	a
100	1.0	3
50		
25		
12		
6		
3		
1		
0		

power_fast(3, 100):

$100 = 2^6 + 2^5 + 2^2$

64

36

If $n = 2^k$ $\quad k > 0$

$\frac{2^6}{2} = 2^5$

i	x	a
2^k		
2^{k-1}		
2^{k-2}		
$\vdots$		
2^1		
2^0		
0		

2^{k-k}

1

\# steps?

k

$\log_2 n = \log_2(2^k)$
$= k$

$k = \log_2 n$

ignoring size of a.

How can we *formally* capture the idea that the second algorithm is *faster*? Can we estimate that on input of size n, the algorithm will take roughly some function of n steps?

eg $\log_2 n$

Look back at the two algorithms and try to roughly guess a function for each one.

$\text{power\_fast}(a, 2^k) \to \log_2 k$

$\text{power\_fast}(a, 1{,}000{,}000)$

$2^{\square} + 2^{\square} + 2^{\square}$

$f_1(n) = n$
$f_2(n) = \log_2 n$

Luckily, someone already thought of how to capture the idea of algorithm speed/complexity, using this definition.

time

# of steps

stronger.

Definition

Let $f, g : \mathbb{N}_+ \to \mathbb{R}$. We say that g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that

$m = 1000$

$$|f(n)| \leqslant m|g(n)|$$

absolute value

for all $n \in \mathbb{N}, n \geqslant k$.

$(k=6)$

$n =$ 1 2 3 4 5 6 7

$f(n)$:

could be bigger

$|f(6)|$ $|f(7)|$

$\leq 1000|g(6)|$ $1000|g(7)|$

$g(n)$:

f is dominated by g.

if

$\exists\ m \in \mathbb{R}_+,\ k \in \mathbb{N}_+$

such that

$\forall\ n \geq k$

$$|f(n)| \leq m\,|g(n)|$$

COMPARING SPEEDS OF ALGORITHMS

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Let $f, g : \mathbb{N}_+ \to \mathbb{R}$. We say that g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that

$$|f(n)| \leqslant m|g(n)|$$

for all $n \in \mathbb{N}, n \geqslant k$. "in Big O of g".

We use the notation $f \in O(g)$ and read this as "f is in Big O of g".

in

$O(g) = \{ f : \mathbb{N}_+ \to \mathbb{R} \mid \ldots$

Luckily, someone already thought of how to capture the idea of algorithm speed/complexity, using this definition.

Definition

Let $f, g : \mathbb{N}_+ \to \mathbb{R}$. We say that g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that

$$|f(n)| \leqslant m|g(n)|$$

for all $n \in \mathbb{N}, n \geqslant k$.

We use the notation $f \in O(g)$ and read this as "f is in Big O of g".

$O(g)$ is the set of all functions from $\mathbb{N}_+$ to $\mathbb{R}$ which are dominated by g.

whom

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}, n \geqslant k$.

Ex: who dominates who: $f(n) = \log_2 n + 5$ versus $g(n) = n$:

small ?

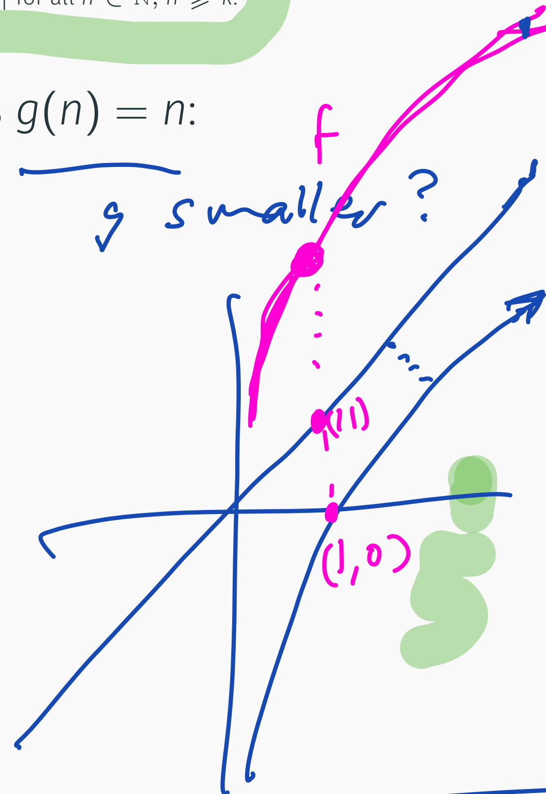
g smaller ?

Guess: $f \in O(g)$?

$\exists\, m = \boxed{2}$ $\quad \exists\, k = \boxed{5}$

$\forall\, n \geqslant \boxed{5}$

$\log_2 n + 5 \leq 2 \cdot n$



START HERE

if $n \geqslant 5$

$\log_2 n + 5 \le \log_2 n + n$

I need

$\log_2 n \le n$?

$e^{\log_2 n}$

iff (~~to~~ raise b.s power 2.)

$n = 2^{\log_2 n} \le 2^n$

START AGAIN

$p(n)$: $n \le 2^n$

Prove by induction

$n = 1$ $p(1)$: $1 \le 2^1 = 2$ ✓

Assume $p(k)$ true

$p(k+1)$: $k+1 \le k+k$

$= 2k$

$\le 2 \cdot 2^k$ by Ind. hyp.

$= 2^{k+1}$.

by PMI true $\forall n \geq 1$.

$n \le 2^n$ — assume $n \geq 1$

$\log_2 n \le n$

$\log_2 n + 5 \le n + n$ (assume $n \geq 5$)

$\le 2n$

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}, n \geqslant k$.

Ex: who dominates who: $f(n) = \log_2 n + 5$ versus $g(n) = n$:

Here is my prepared solution in case my "live" one is too messy:

Claim 1: $n \leqslant 2^n$. Let $P(n)$ be the statement $n \leqslant 2^n$. $P(1)$ true. Assume $P(k)$ then $n + 1 \leqslant n + n = 2n \leqslant 2.2^n = 2^{n+1}$ so $P(k+1)$ is implies, so by PMI true for all $n \geqslant 1$.

Take $\log_2$ both sides gives $\log_2 n \leqslant n$, so

$$\log_2 n + 5 \leqslant n + 5 \leqslant n + n$$

(if $n \geqslant 5$)

$$= 2n$$

so $m = 2, k = 5$ gives the result.

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}, n \geqslant k$.

Some careful analysis of the two algorithms above shows that, up to some constants, power_fast takes about $\log_2 n$ +5 steps (because each step divides i by 2, roughly) and power_slow takes ~~n~~ steps.

Because of Big O, if you count steps slightly differently, and/or count the initial and final lines of the code, the time complexity function changes a bit but is **the same** up to Big O.

Reason for m:

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}, n \geqslant k$.

Recall the formula

$$\log_b y = \frac{\log_a y}{\boxed{\log_a b}}.$$ [1]

Constant.

$$y = a^{\log_a y}$$

$$\log_b y = \log_b a^{(\log_a y)}$$

$$= \log_a y \, \log_b a$$

[1] Proof: $y = b^{\log_b y}$, so sub this into the LHS.

RECALL

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}$, $n \geqslant k$.

Recall the formula

$$\log_b y = \frac{\log_a y}{\log_a b}.$$ [1]

If $b = 2$ and your calculator only has a button for $\log_{10}$ or $\log_e = \ln$ then you can use this formula:

$$\log_2 n = \frac{\log_e n}{\log_e 2}.$$

$$\boxed{\frac{1}{\log_e 2}}$$

just a constant.

[1] ~~Proof~~ Hint: $y = a^{\log_a y}$, so sub this into the LHS.

$\log_7 x \in O(\log_{17} x)$
$\log_{17} x \in O(\log_7 x)$

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}$, $n \geqslant k$.

Recall the formula

$$\log_b y = \frac{\log_a y}{\log_a b}.^{1}$$

If $b = 2$ and your calculator only has a button for $\log_{10}$ or $\log_e = \ln$ then you can use this formula:

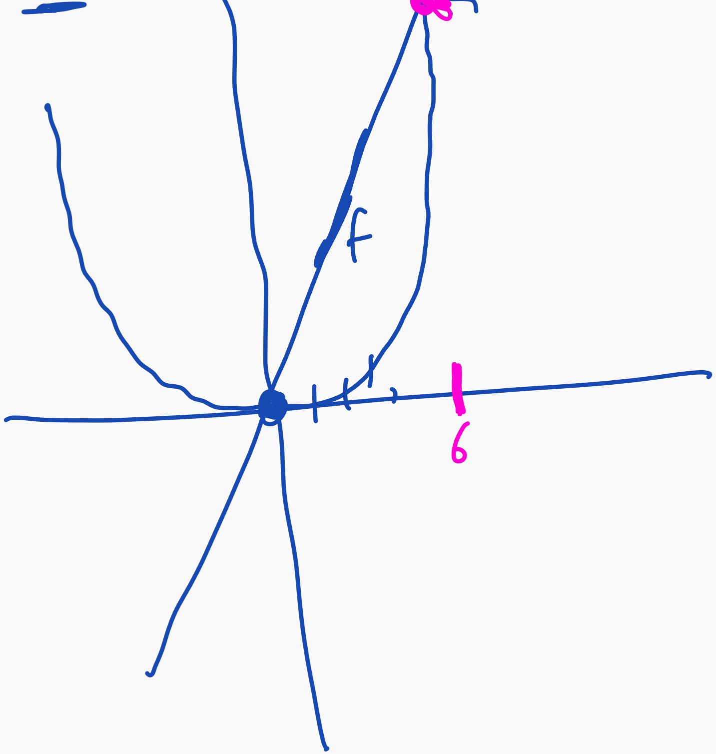
$$\log_2 n = \frac{\log_e n}{\boxed{\log_e 2}}.$$

Since $\log_e 2 \approx 4.6$ is a fixed number, it doesn't matter which log we use because of the m in the definition.

[1] Proof: $y = a^{\log_a y}$, so sub this into the LHS.

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}, n \geqslant k$.

Let $f(n) = 6n$ and $g(n) = n^2$. Show that g dominates f, that is, $6n \in O(n^2)$.



$\exists\ k = \boxed{6}$

$m = \boxed{1}$

$\forall\ n \geqslant 6$

$6 \cdot n \leq n \cdot n$ since $n \geqslant 6$

$= 1 \cdot n^2$

so $6n \in O(n^2)$ $\square$

Another proof

$\exists\ \underline{k = 1}\ ,\ \underline{m = 6}$

for all $n \geq \underbrace{1}_{k}$,

$6n = 6n \cdot 1 \leq 6n \cdot n$ since $n \geq 1$

$= \underbrace{6}_{m} n^2$

$\boxed{\text{So } 6n \in O(n^2).}$

EG

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}, n \geqslant k$.

Let $f(n) = 6n$ and $g(n) = n^2$. Show that g dominates f, that is, $6n \in O(n^2)$.

Here is my prepared solution. It might be different that the one I just did “live” – many choices for m, k can work.

There exist $m = 1, k = 6$ so that

$$6n \leqslant n.n = n^2$$

for all $n \geqslant 6$.

EG

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}$, $n \geqslant k$.

Show that $g(n) = n^2$ is not dominated by $f(n) = 6n$. That is, $n^2 \notin O(6n)$.

m

negation

EG

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}, n \geqslant k$.

Show that $g(n) = n^2$ is not dominated by $f(n) = 6n$. That is, $n^2 \notin O(6n)$.

We need the *negation* of the Big O definition.

$\neg (n \geq k) \vee (|f(n)| \leq m|g(n)|)$

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}, n \geqslant k$.

Show that $g(n) = n^2$ is not dominated by $f(n) = 6n$. That is, $n^2 \notin O(6n)$.

We need the *negation* of the Big O definition.

$$\forall m \in \mathbb{R}_+ \forall k \in \mathbb{N}_+ \exists n \in \mathbb{N}_+ [(n \geqslant k) \wedge (|f(n)| > m|g(n)|)].$$

$n^2 \notin O(6n)$

$\forall m \in \mathbb{R}_+ \quad \forall k \in \mathbb{N}_+$

$\exists n \qquad n \geq k, \quad \text{and} \quad n^2 > m \cdot 6n$

Choose ~~$n = \max\{k, m\}$.~~

Proof

$n^2 \notin O(6n)$.

since $\forall m \in \mathbb{R}_+$, $\forall k \in \mathbb{N}_+$

$\exists\ n = \boxed{\max\{k, 6m+1\}}$.

so that ① $n \geq k$

and ② $n^2 \geq (6m+1)n$ → since $n \geq 6m+1$

strictly $> 6m \cdot n$ $\qquad \max\{6m, k\}$

EG

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}$, $n \geqslant k$.

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We need the *negation* of the Big O definition.

$$\forall m \in \mathbb{R}_+ \forall k \in \mathbb{N}_+ \exists n \in \mathbb{N}_+ [(n \geqslant k) \wedge (|f(n)| > m|g(n)|)].$$

Here is my prepared solution.

Given m, k fixed numbers (positive), there is a number $n = \max\{6m + 1, k\}$ so that $n > 6m$ so

$$n^2 > m6n.$$

$f \in O(g) \wedge g \in O(f)$

$\Rightarrow f \neq g.$

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}$, $n \geqslant k$.

Let $f(n) = 6n^2 + 5n + 2$ and $g(n) = n^2$. Show that $f \in O(g)$ and $g \in O(f)$. So they are (up to Big O equivalence) the "same".

refl: $f \in O(f)$

$|f(n)| \leq |f(n)|$

$\forall n \geqslant 1$

$m = 1$

$k = 1$

Define

$f \cancel{R} g$

if $f \in O(g)$.

Then $\cancel{R}$ is not antisymmetric.

$n^2 \le 6n^2$ $\quad 1 \le 6$

$\le 6n^2 + 5n + 2$ $\quad 5n \ge 5$

$5n+2 \ge 7$

≥ 0

$m = 1 \quad c^2 = 1$

$6n^2 + 5n + 2 \le 6n^2 + n^2 + 2$

since

$n \ge 5$

$\le 6n^2 + n^2 + n$ $\quad n \ge 2$

$\le 6n^2 + n^2 + n^2$ $\quad n \ge 1$

$= 8n^2$

$k = 5$

$m = 8$

EG

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}$, $n \geqslant k$.

Show that $f(n) = n^3$ is dominated e^n.

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Show that $f(n) = n^3$ is dominated e^n.

My prepared solution:

Let $P(n)$ be the statement $n^3 \leqslant e^n$. Need to find a value k so that $P(k)$ is true, then

assume for $s \geqslant k$ that $P(s)$ is true, then $P(s+1)$:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1 \leqslant n^3 + n^3$$

(by a Lemma I will prove below)

$$= 2n^3 \leqslant 2e^n < e.e^n = e^{n+1}$$

since $e = 2.7... > 2$.

Lemma: $3n^2 + 3n + 1 \leqslant n^3$ for n big enough.

CHALLENGE QUESTION

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}, n \geqslant k$.

Show that $f(n) = n^c$ is dominated e^n where c is any positive integer.

So polynomials are "slower" than exponentials.

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So polynomials are “slower” than exponentials.

Your proof might use the Binomial Theorem:

$$(x+y)^c = x^c + \binom{c}{1}x^{c-1}y + \binom{c}{2}x^{c-2}y^2 + \cdots + \binom{c}{c-1}x^1y^{c-1} + y^c$$

CHALLENGE QUESTION

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$$(n+1)^c = n^c + \binom{c}{1}n^{c-1} + \binom{c}{2}n^{c-2} + \cdots + \binom{c}{c-1}n + 1$$

In your proof, c is a fixed number, so all the $\binom{c}{i}$ terms are bounded above by some fixed number.

Defn: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}$, $n \geqslant k$.

Which is faster (who dominates who?) – exponentials or factorials?

That is, show that one of $f(n) = c^n$, $g(n) = n!$ dominates the other (for any $c \in \mathbb{R}_+$ say).

If you do more computer science, you will use Big O. You count (roughly, since constant multiples don't matter) the number of steps an algorithm takes on inputs of size $n \in \mathbb{N}$.

The algorithm is "fast" if the number of steps is $O(n)$ or maybe $O(n \log n)$, and "bad" if it takes $O(c^n)$ or worse steps.

Issues: is it bad if it takes this many steps on *all* inputs, or some inputs, or most inputs.

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See Worksheet 6 for a table of standard functions and their names used in time complexity.

Exercise: prove that the worst case running time for the Euclidean algorithm is when the input is two of the Fibonacci numbers (worksheet 3)

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Standard exercise in comp sci: compare running times for algorithms which sort a list of numbers: bubble sort versus merge sort.

Recall from the Canvas "Welcome" page:
`https://www.youtube.com/watch?v=kVgy1GSDHG8`

And here is another video that mentions Big O:
`https://www.youtube.com/watch?v=xFFs9UgOAlE`

Next time:

- pigeonhole principle