37181 DISCRETE MATHEMATICS

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Lecture 13: Counting

PLAN

- multiplication
- addition
- inclusion-exclusion
- permutations
- permutations with repetition
- combinations

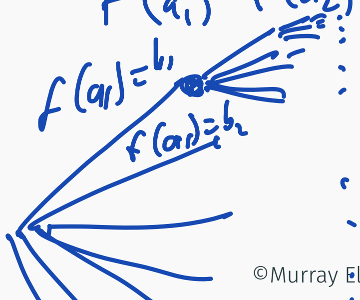
In this lecture, we will learn how to count.

Ex 1: let $|A| = 4$, $|B| = 7$. How many one-to-one functions are there from A to B?

$\leftarrow B$ b_1 $\vdots$ b_7

7	6	5	4
$f(a_1)$	$f(a_2)$	$f(a_3)$	$f(a_4)$

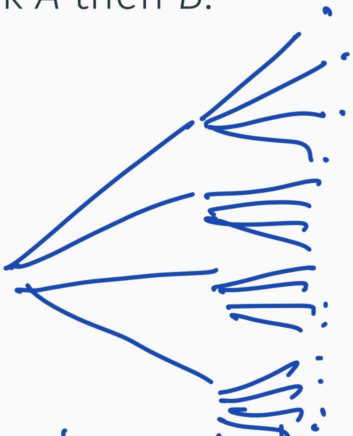
$f(a_1) \neq f(a_2)$



$7 \cdot 6 \cdot 5 \cdot 4$

$f(a_1) = b_7$

<u>Multiplication rule</u>: if there are a ways to do task A, and b ways to do task B, then there are ab ways to do task A then B.



task A then do task B.

Multiplication rule: if there are a ways to do task A, and b ways to do task B, then there are ab ways to do task A then B.

Ex 2: I have 3 pairs of shorts, 2 pairs of jeans, 5 t-shirts, 4 shirts, and 3 hats.

How many different outfits can I wear?

Task C →

Task B →

Task A →

$5 \cdot 9 \cdot 3$

Ex 3: I have 3 pairs of shorts, 2 pairs of jeans, 5 t-shirts and 4 shirts, and my outfit is always either shorts and t-shirt, or jeans and a shirt.

casual

not both

formal

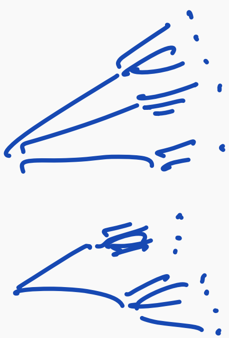
How many different outfits can I wear?

Casual

Formal

$3 \cdot 5 = 15$

$2 \cdot 4 = 8$



15

$15 + 8 = 23.$

8

Ex 3: I have 3 pairs of shorts, 2 pairs of jeans, 5 t-shirts and 4 shirts, and my outfit is always either shorts and t-shirt, or jeans and a shirt.

How many different outfits can I wear?

Addition rule: if there are a ways to do task A, and b ways to do task B, and you can only do A or B (not both), then there are $a + b$ ways to do A or B.

Alternate: $55 + 58 - 20$

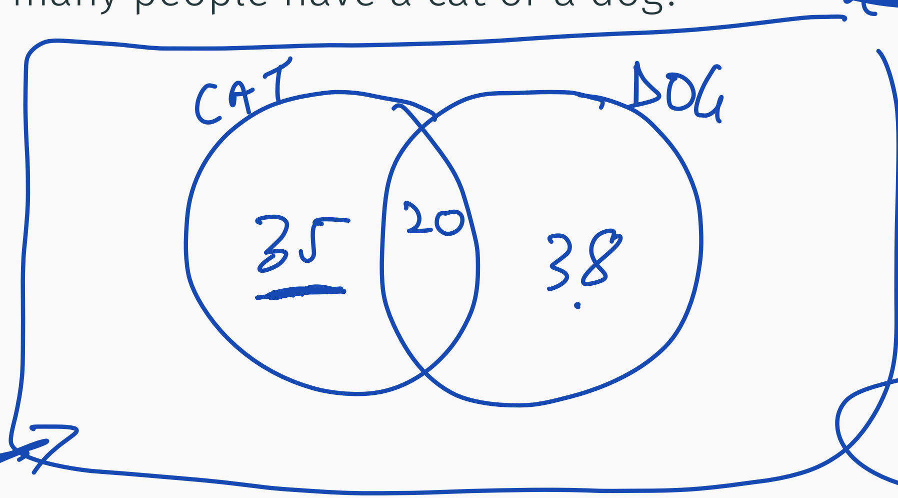
too many ??

dog + cat people once

Ex 4: Suppose a survey of 100 people asks if they have a cat or dog as a pet. The results are as follows:

55 answered yes for cat, 58 answered yes for dog and 20 people said yes for both cat and dog.

How many people have a cat or a dog?



$35 + 38 + 20$

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Ex 4: Suppose a survey of 100 people asks if they have a cat or dog as a pet. The results are as follows:

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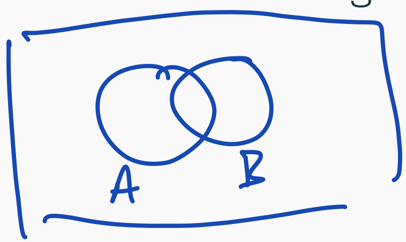
How many people have a cat or a dog?



because we overcounted $A \wedge B$

formula: $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-exclusion principle: if $A, B, C, \ldots$ are sets (whose elements might be some events like some tasks which may or may not be able to be performed simultaneously, or people having a cat or a dog or both . . .)

then

too many

subtract, BUT, we are taking away too many

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

so we add back

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C|$ −

$|A|+|B|-|A\cap B|$

$=50+33-16=67$

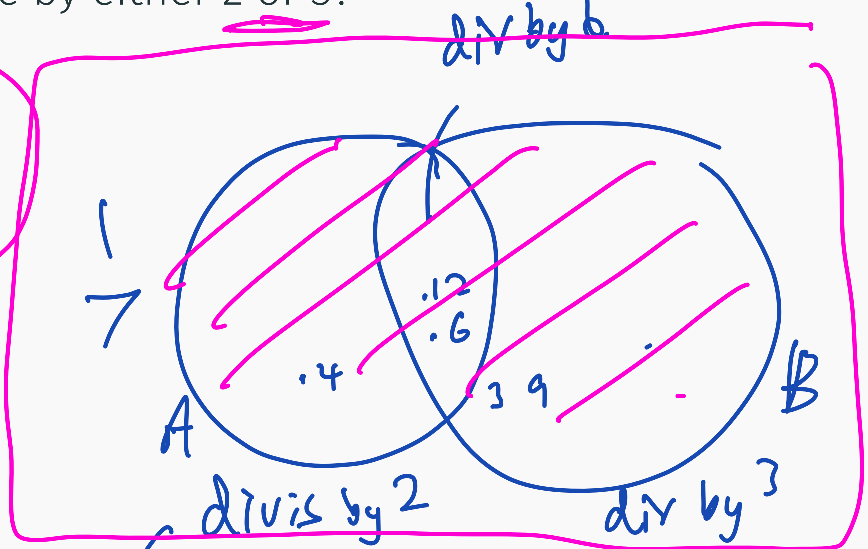
$$\begin{array}{r} 83 \\ -16 \\ \hline 67 \end{array}$$

include 1, 100.

Ex 5: How many numbers between 1 and 100 (inclusive) are not divisible by either 2 or 3?

$100-67$

$=33$



$|B|=33$

3, 6, 9, 12, 15, ..., 99

1 2 3 ... 33

$|A|=50$

2 4 6 ... 100

1 2 3 ... 50

$|A\cap B|$ — div by 6

6, 12, 18, ..., 96

$1 \quad 2 \quad 3 \ \ldots 16$

$|A \cap B \cap C| = 2$

$7 \quad 14 \quad 21 \ \ldots \ 98$

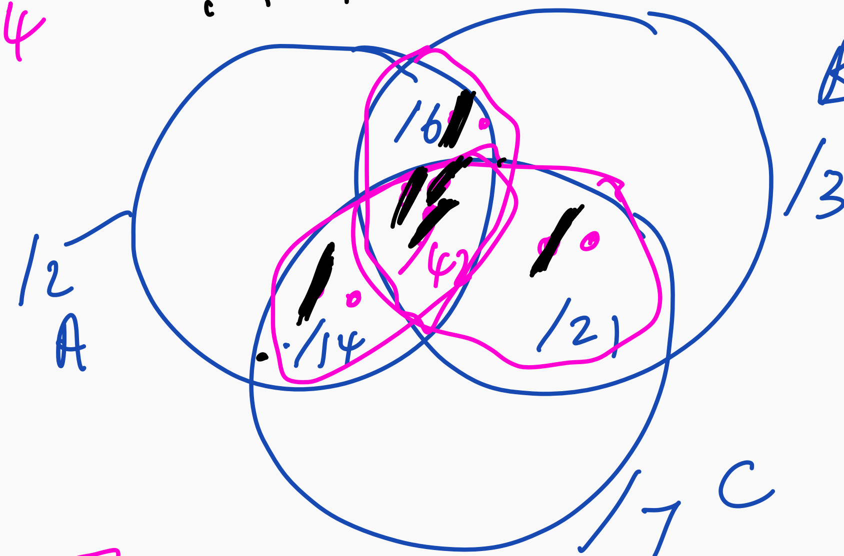
$70+28$

$7 \cdot 14$

Ex 6: How many numbers between 1 and 100 (inclusive) are not divisible by either 2, 3 or 7?

$|B \cap C| = 4$

$|C| = 14$



$|A \cap B| = 16$

$\dfrac{14 \quad 28 \ \ldots \ 98}{7}$

$|A| = 50$

$|B| = 33$

$|A \cap C| = 7$ → divisible by 2 AND 7

$100 - \underline{\qquad}$

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$$\underbrace{|A| + |B| + |C|}_{50 + 33 + 14} - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$|A \cap B|$: 16

$|A \cap C|$: 7

$|B \cap C|$: 4

$|A \cap B \cap C|$: 2

$$50 + 33 + 14 - 16 - 7 - 4 + 2 = \underbrace{72}$$

Answer: $100 - \underbrace{72}$

$= 28$

$\{m, e, a, t\}$

Ex 7: How many ways can you arrange the letters in the word *meat*?

emat

eamt

$\vdots$

4 3 2 1

$= 4! = 24.$

factorial

Ex 8: How many ways can you arrange the letters in the word *meet*?

Trick: $m\ e_1\ e_2\ t$

Count ways to arrange

2 elements

$e_1 m e_2 t$

$e_2 m e_1 t$

$e_1 e_2 m t$

$e_2 e_1 m t$

⋮

$4!$ "too many"

Step 1

Step 2 group "same" words together.

How many pink boxes?

$\frac{4!}{2}$

$m e_1 - - e_2 t$
$m e_2 - - - e_1 t$

$m e_1 t e_2 - -$
$m e_2 t e_1 - -$

$e_1 e_2 m t$
$e_2 e_1 m t$

$\vdots$

Total $4!$

but each pink box has 2 things in it

$$\frac{4!}{2!}$$

$$\frac{24}{2} = \underline{12}$$

Ex 9: How many ways can you arrange the letters in the word *coonabarabran*?

$$\frac{13!}{2!\,2!\,2!\,2!\,4!}$$

b_1, b_2

o_1, o_2

a_1, a_2, a_3, a_4

r_1, r_2

n_1, n_2

c

$\frac{13!}{2!}$

$o_1 b_1 o_2 a_1 b_2 \ldots$

$o_1 b_2 o_2 a_1 b_1 \ldots$

— group words which are identical except for $\boxed{b_1 b_2}$

$\left(\frac{13!}{2!}\right)\frac{1}{2!}$ ← o_1, o_2

$\left(\frac{13!}{2!2!2!2!}\right)\frac{1}{4!}$ →

coona_1ba_2r$a_3$bra_4n

coona_2ba_4r...

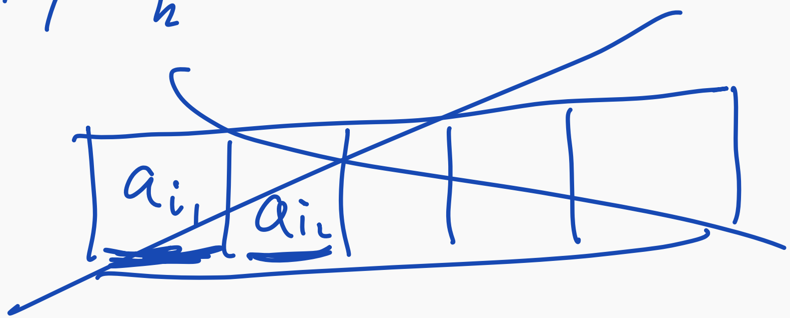
$4!$

"subset"

Ex 10: How many ways can you choose k elements from a set of size n?

$\{a_1, a_2 \ldots a_n\}$

$\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$



$a_{i_2}\, a_{i_1}$

Answer 1: $\binom{n}{k}$ "n choose k"

Notation — define $\{n \backslash choose\ k\}$ C^n_k

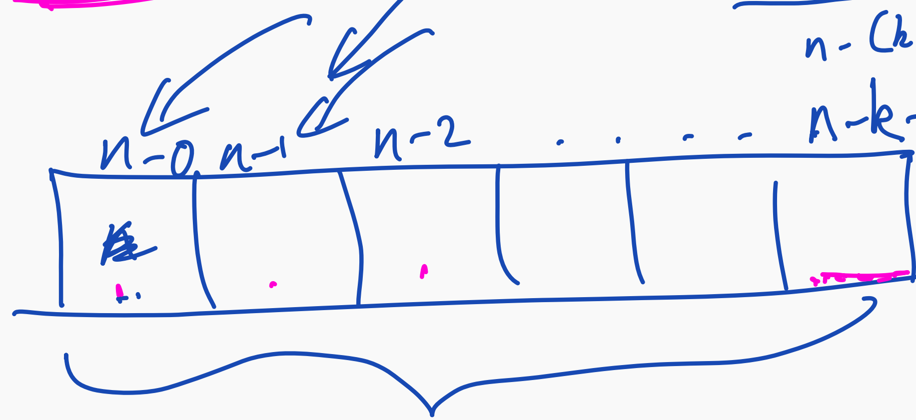
Ex 10: How many ways can you choose k elements from a set of size n?

$\{a_1, \ldots, a_n\}$

Answer 2:

Step 1: How many *ordered lists* of k elements from n? (overcount first)

$n-(k-1)$



$$\underbrace{n(n-1)(n-2)\cdots(n-k+1)}\cdot\frac{(n-k)(n-k-1)\cdots 2\cdot 1}{(n-k)(n-k-1)\cdots 2\cdot 1}$$

$= n!$

$(n-k)!$ ← fancy way to write it.

Step 2

$\binom{n}{k}$

$k!$
$a_1, a_2, a_7, a_9 \dots$
$a_2, a_1, a_7, a_9 \dots$
$a_1, a_7, a_2, a_9 \dots$

← all the ordered lists which contain the <u>same</u> elements

$k!$
$a_3\ a_1\ a_8\ a_6 \dots$
$\vdots$

$k!$
$\dots$

How many pink boxes?

$$\frac{n!}{(n-k)!\,k!}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Ex 10: How many ways can you choose k elements from a set of size n?

Step 1: How many *ordered lists* of k elements from n? (overcount first)

Step 2: deal with the overcount

Ex 10: How many ways can you choose k elements from a set of size n?

Step 1: How many *ordered lists* of k elements from n? (overcount first)

Step 2: deal with the overcount

Formula for $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

positive.

Notation: a *binary string* of length $n \in \mathbb{N}_+$ is an expression of the form $d_1 d_2 \dots d_n$ where $d_i \in \{0, 1\}$ for $1 \leqslant i \leqslant n$. For example, 11011 is a binary string of length 5. The set of all binary strings of length n is denoted $\{0, 1\}^n$.

1010

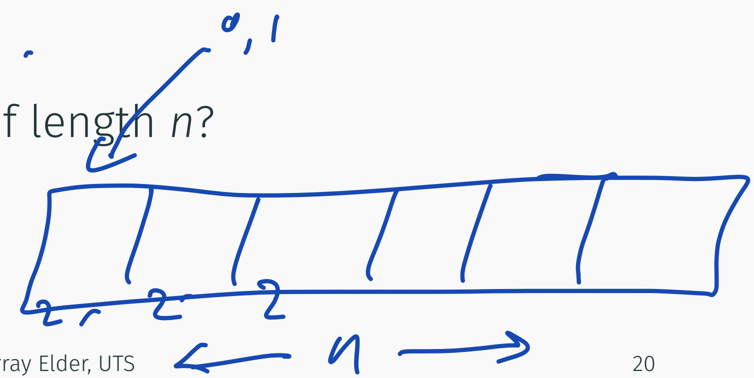
A finite length string is also called a *word*.

Whenever people talk about strings, it is helpful to include strings of length 0. For this subject, let us denote the (unique) string of length 0 by the symbol λ.

lambda

How many binary strings are there of length n?

2^n – includes $n = 0$.



A *square free word* is a word (a sequence of symbols) that does not contain any squares. A square is a word of the form *XX*, where *X* is not empty. `https://en.wikipedia.org/wiki/Square-free_word`

eg

$\underbrace{10}_{X}\underbrace{10}_{X}$

square

ll

$11\underbrace{10}\underbrace{10}1111$

as a factor

A *square free word* is a word (a sequence of symbols) that does not contain any squares. A square is a word of the form *XX*, where *X* is not empty. https://en.wikipedia.org/wiki/Square-free_word

How many binary strings of length 5 are square free? None.

λ

1, 0

10, 01

010, 101, 111

0 1 0 1 ...

1 0 1 0 ...

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3^5 total # ternary strings length 5.

A *square free word* is a word (a sequence of symbols) that does not contain any squares. A square is a word of the form XX, where X is not empty. https://en.wikipedia.org/wiki/Square-free_word

How many binary strings of length 5 are square free? None.

How many *ternary* strings ($d_i \in \{0, 1, 2\}$) of length 5 are square free?

k

Homework.

2 examples.

0 1 0 2 0
0 1 0 2 1
0 2 0 1 0
0 2 0 1 2
0 2 0 ...

0 1 2 0 1
0 1 2 0 2
0 2 1 ...
0 2 1 ...

lots.

- binomial theorem
- combinatorial proofs
- some famous counting sequences
- Catalan numbers

- Fibonacci numbers.

Early Student Feedback

finishes tonight.

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NEXT TIME

- binomial theorem
- combinatorial proofs
- some famous counting sequences
- Catalan numbers

(LPC6 tomorrow, then tutorials on counting, then StuVac next week)

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