37181 DISCRETE MATHEMATICS

Lecture 19: graph theory part 2

PLAN

- simple path
- graph isomorphism

shape

same

- Euler paths
- Hamiltonian circuits

q: $\{v_0 v_1\}$, $\{v_1, v_5\}$, $\{v_5 - v_6\}$, $\{v_6, v_7\}$, $\{v_7, v_8\}$, ~~$\{v_0 v_1 v_5 v_6 v_7 v_8\}$~~

- a path p *contains* a subpath q if p is the sequence

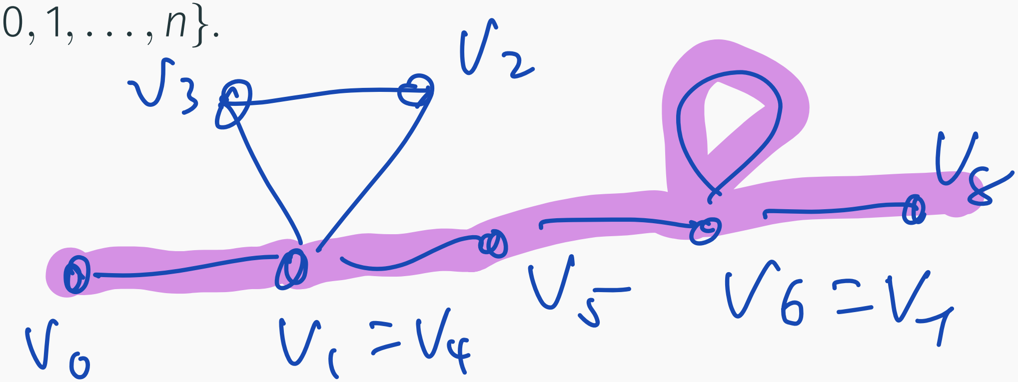
$$\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}$$

  and q is a path with the sequence

$$\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \ldots, \{v_{i_{j-1}}, v_{i_j}\}$$

  for $i_j \in \{0, 1, \ldots, n\}$.

p :



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for $i_j \in \{0, 1, \ldots, n\}$.

In other words, you can delete some edges from p and obtain another path q.

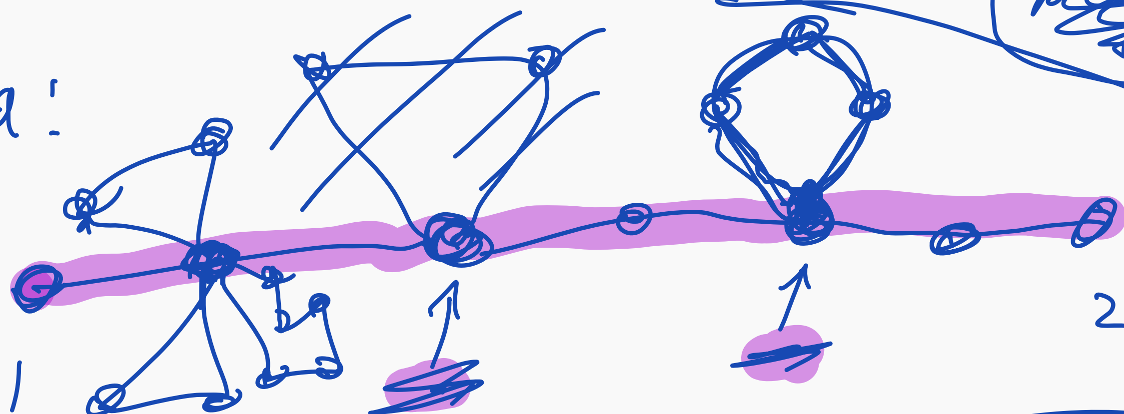
as a subpath

no repeated vertices. (except possibly first + last)

Theorem

Every path contains a simple path

Idea:



length $\underline{k+1}$

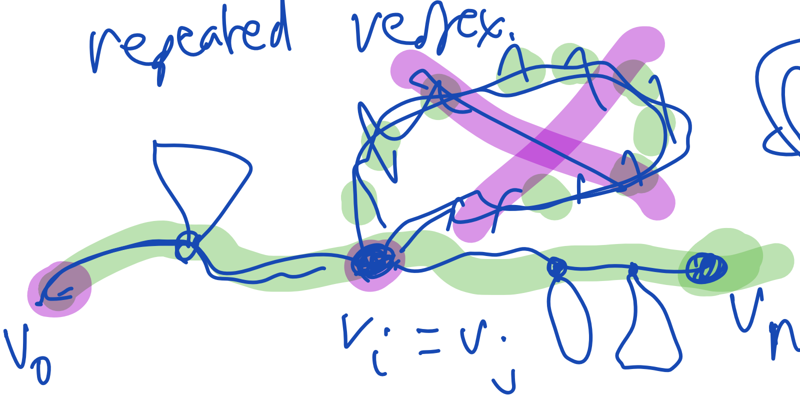
shorter path? ? ?

~~Proof~~ If p is simple

p contains p.

Else, p not simple.

Therefore, p has some "internal" repeated vertex.



SHORTEN

What can I do to this path?

$P(n)$: a path of length $n \in \mathbb{N}$ has a simple subpath.

Theorem

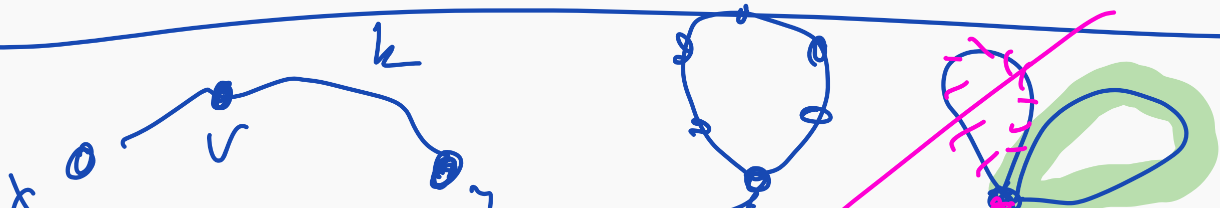
Every path contains a simple path

$n =$

Proof: by (strong) induction on length of the path.

True for path of length 0. Let p be a path of length k from x to y. If p is simple we are done. Else there is some vertex $v \neq y$ so that p visits v twice. Delete the subpath starting and ending at v, this shorter path contains a simple path by inductive assumption, and is contained in p so we are done.

$P(0)$ true: $\bullet$ already is a simple path.



GRAPH ISOMORPHISM

multi-edge

In the next definition, we assume our graphs don't have multi-edges, just to make the statements easier to say.

GRAPH ISOMORPHISM

$\{x, y\}$ $\{x, x\}$

degen

Definition

Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs that do not have multi-edges, and so we can assume $E_i \subseteq \mathscr{P}(V_i)$.

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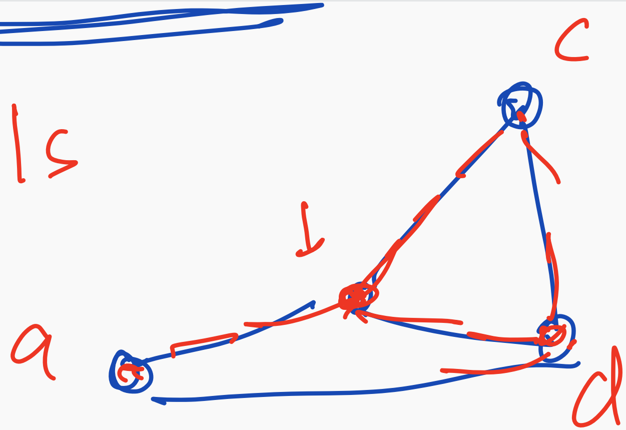
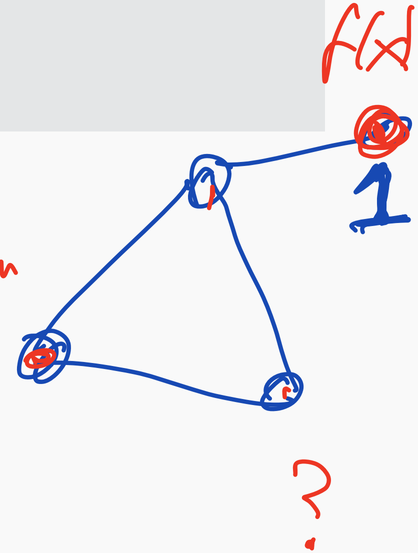
We say G, H are *isomorphic* if there is a bijection $f : V_1 \to V_2$ such that for all $x, y \in V$, we have $\{x, y\} \in E_1$ if and only if $\{f(x), f(y)\} \in E_2$.

Is



Isomorphism to



GRAPH ISOMORPHISM

Definition

Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs that do not have multi-edges, and so we can assume $E_i \subseteq \mathscr{P}(V_i)$.

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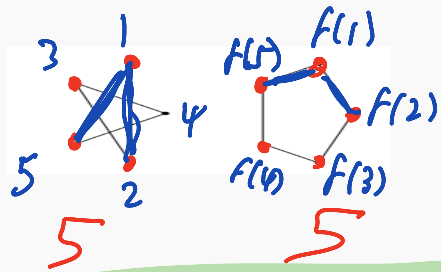
Ex: Decide if these two graphs are isomorphic.

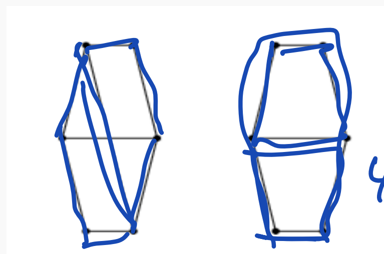


2, 3

Ex: Decide if these two graphs are isomorphic.

or • degree sequence

• # edge



4, 6

6   6

No, because • # verts of deg 2

or • # vert of deg 3

111122

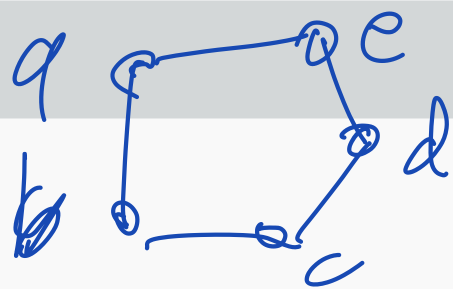
211211

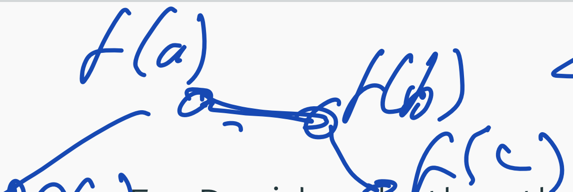
To show non-isomorphic, it is useful to have some *invariants*. For example, if the number of vertices is different, you can say No straight away.

"Invariant".

What other things might be preserved by an isomorphism?

- # edges
- # loops
- ~~adjacency matrix ← Murray.~~
- # verts of deg d.
- degree sequence (up to re-ordering)

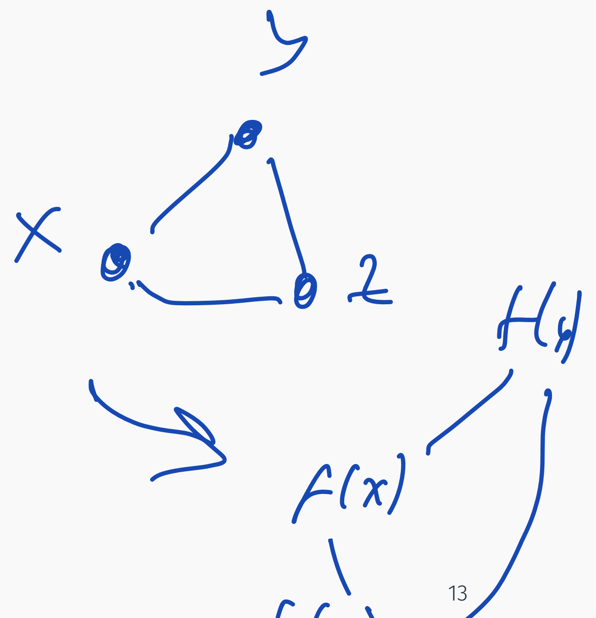




Ex: Decide whether these are invariants of a graph under isomorphism (that is, if you apply an isomorphism map f to G then $f(G)$ has the same number of these things as G does.)

f(d)

1. number of loops (at each vertex)
2. number of vertices of degree d
3. number of edges
4. number of cycles of length r



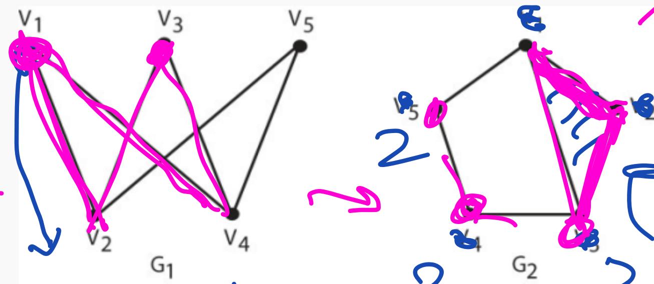
$\{v_1, v_2\}, \{v_2 v_3\}, \{v_3 v_4\}, \{v_4 v_1\}$

even.

Ex: Decide if these two graphs are isomorphic.

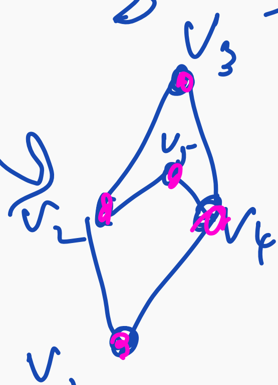
6, 5, 3, 2, 0

$K_{2,3}$



every cycle has even length.

cycle length 3



GRAPH ISOMORPHISM

There is some very interesting current research on the complexity of deciding if two graphs are isomorphic – can it be done in polynomial time?

https://en.wikipedia.org/wiki/Graph_isomorphism_problem

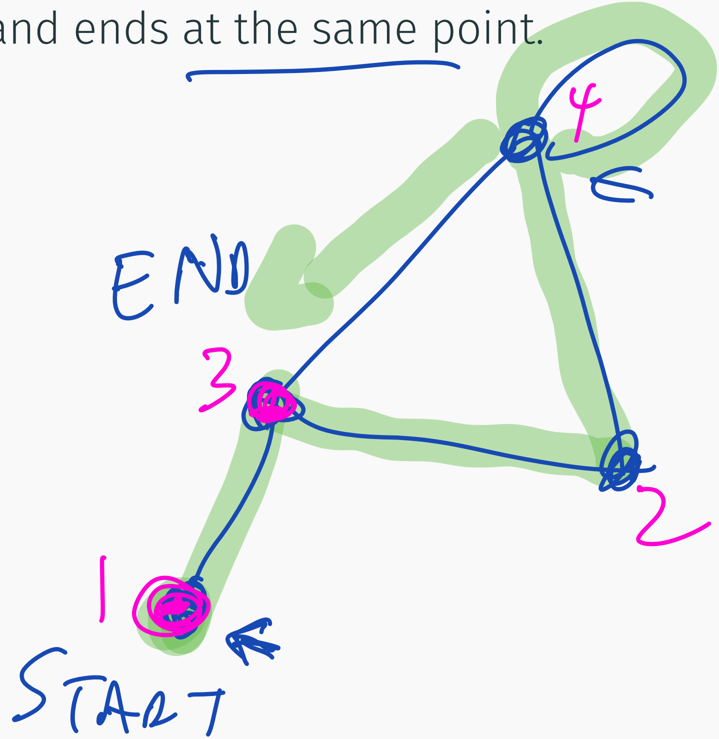
Same

Note: . . . computationally equivalent to the problem of computing the automorphism group of a graph – ~~Worksheet~~ 9 triangle question.

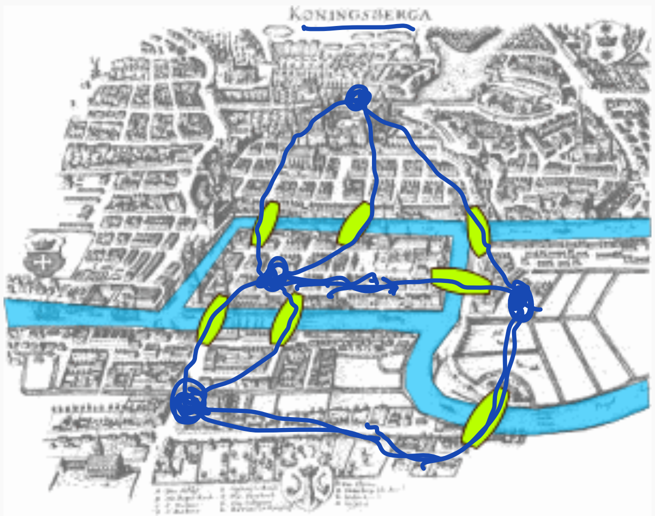
Homework sheet

shape

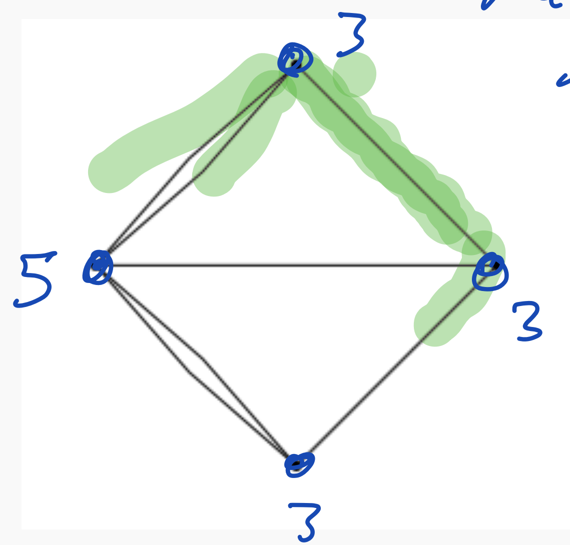
An Euler path in a graph is a path that traverses (uses) every edge exactly once. An Euler circuit in a graph is an Euler path that starts and ends at the same point.



Ex: Decide if there is a way to walk around this town crossing every bridge exactly once (and return to your starting point).



or

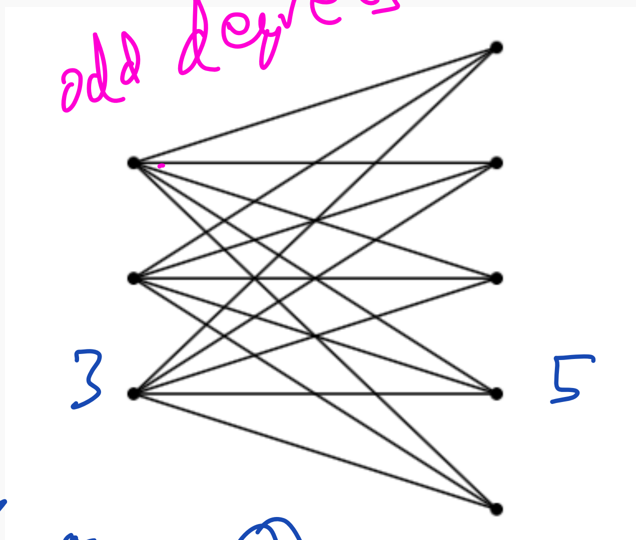


multi graph

7 bridges

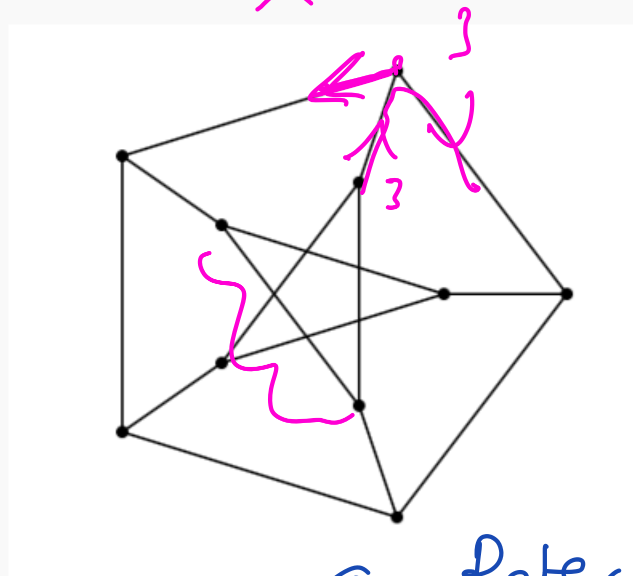
Use every edge exactly once

Ex: Decide if these graphs have Euler paths or circuits.



$K_{3,5}$

1



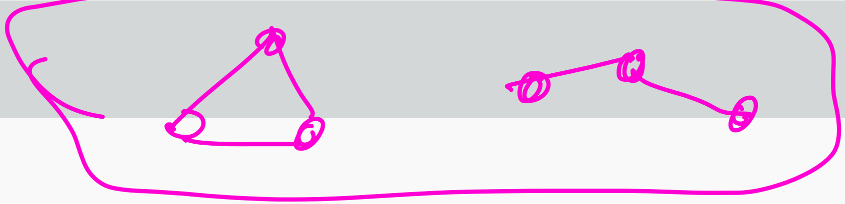
2

Petersen graph.

(Squid game)

It turns out it is easy to decide whether or not a graph has an Euler path or circuit: if a vertex has odd degree, then you cannot cross every edge of the graph without getting stuck at this vertex. This becomes the following theorem.

EULER PATHS AND CIRCUITS



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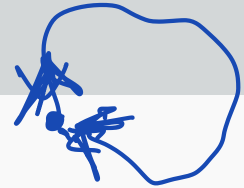
Theorem (Euler)

A connected graph $G = (V, E)$ has an Euler circuit if and only if every vertex has even degree. G has an Euler path if either it has exactly 0 or 2 vertices of odd degree.

EULER PATHS AND CIRCUITS



It turns out it is easy to decide whether or not a graph has an Euler path or circuit: if a vertex has odd degree, then you cannot cross every edge of the graph without getting stuck at this vertex. This becomes the following theorem.

Theorem 1A, 1B

1 *A connected graph $G = (V, E)$ has an Euler circuit if and only if every vertex has even degree. G has an Euler path if either it has exactly 0 or 2 vertices of odd degree.* 2

This gives us a polynomial time algorithm to decide if G has a circuit/path or not: just compute the degree of each vertex. Euler

Theorem

A connected graph $G = (V, E)$ has an Euler circuit if and only if every vertex has even degree. G has an Euler path if and only if either it has exactly 0 or 2 vertices of odd degree.

Proof:

NOT

① Euler circ. iff no odd degree → 1A ← 1B

② Euler path iff 0 or 2 odd deg. → 2A ← 2B

Proof: First suppose G has a vertex v of odd degree and has an Euler circuit. Assume without loss of generality the circuit starts and ends at v. The circuit leaves and returns to v via different edges since it uses each edge exactly once, and each return to v occurs after an even number of edges have been used to exit/enter. Since v has odd degree after it exits v the last time it can never return, contradiction.

1B

Contradiction

Next suppose G has an Euler path, which starts at x and ends at y, and we may assume $x \neq y$ (if $x = y$ then it is an Euler circuit and the previous argument shows it has 0 vertices of odd degree). If the path exits x n times, then it must enter x $n-1$ times, so the degree of x is odd. If the path enters y n times, then it must exit y $n-1$ times, so the degree of y is odd. Every other vertex is entered and exited the same number of times, so G has exactly two vertices of odd degree.

2A

Direct.

the rest

Now to prove ~~the converse~~ by induction.

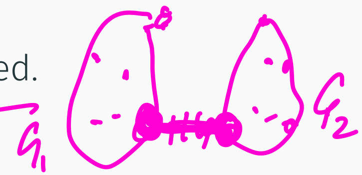
Let $P(n)$ be the statement about $n \in \mathbb{N}$ that if a connected graph G has n edges and 0 vertices of odd degree then it has an Euler circuit (which is also an Euler path), and if it has 2 vertices of odd degree then it has an Euler path which starts and ends at the two odd degree vertices.

$P(0)$: A connected graph with 0 edges is a single vertex. Degree is even (zero) and has an Euler circuit/path (empty path, sequence of zero edges).

EULER PATHS AND CIRCUITS

Assume $P(k)$ is true for $k \in \mathbb{N}$ and consider a connected graph G with $k + 1$ edges. $k+1$

Case 1: Suppose removing one edge from G makes it disconnected.



If G had 0 vertices of odd degree then the two graphs G_1, G_2 that result after deleting the edge will have one vertex of odd degree, and all other vertices even degree, which contradicts the formula that the sum of degrees of vertices in a graph should be even.

If G has two vertices of odd degree, say v_1, v_2, then each G_1, G_2 should contain exactly one of these odd degree vertices, or we contradict the formula again. Say G_1 contains v_1 and G_2 contains v_2. Either v_1 is incident to the removed edge, in which case all vertices in G_1 have even degree and G_1 contains an Euler circuit (without loss of generality starting and ending at v_1), or v_1 is not incident which means G_1 contains two vertices of odd degree, including one which is incident to the removed edge, so by inductive assumption is has an Euler path ending at the vertex incident to the removed edge. Same argument applies to G_2, and now you can form an Euler path by following the path/circuit ending at the vertex incident to the removed edge, crossing the deleted edge, then following the path/circuit starting at the vertex in G_2 incident to the removed edge.

EULER PATHS AND CIRCUITS

Case 2: Now we can assume after deleting an edge, the resulting graph is connected.

If the edge was a loop based at a single vertex, the smaller graph has an Euler circuit/path by inductive hypothesis, so G has a circuit/path by following the same circuit and going around the extra loop when you first visit that vertex.

Else suppose G has no loops. If G has 0 vertices of odd degree, deleting one edge gives two vertices x, y of odd degree and the rest remain even. By inductive hypothesis the smaller graph has an Euler path from x to y, so follow this path and then cross the deleted edge to give an Euler circuit in G.

If G has 2 vertices of odd degree, say x, y, then either $\{x, y\}$ is an edge or not. If it is an edge, removing it gives a graph with k edges and all vertices even degree, so by inductive hypothesis it has an Euler circuit. Create an Euler path for G starting at x, following the circuit back to x, then crossing the deleted edge y to finish.

If it is not an edge, x is joined to a vertex z of even degree. Deleting $\{x, z\}$ gives a graph with k edges and now y, z have odd degree and x even. By inductive hypothesis it has an Euler path from y to z, so create an Euler path for G by starting at y, following the path to z, then crossing the deleted edge x to finish. $\square$

Instead of visiting every edge, how about a path or circuit that visits every node exactly once. This would be useful if you were a salesperson who needed to visit a whole bunch of cities and didn't want to waste time/money visiting the same city twice.

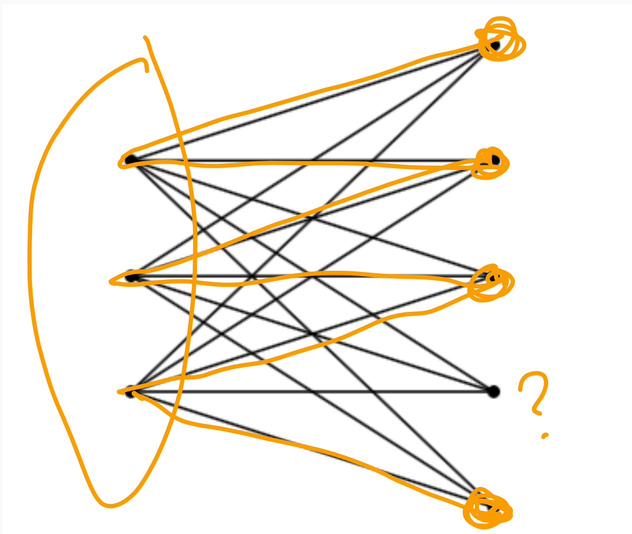
Definition

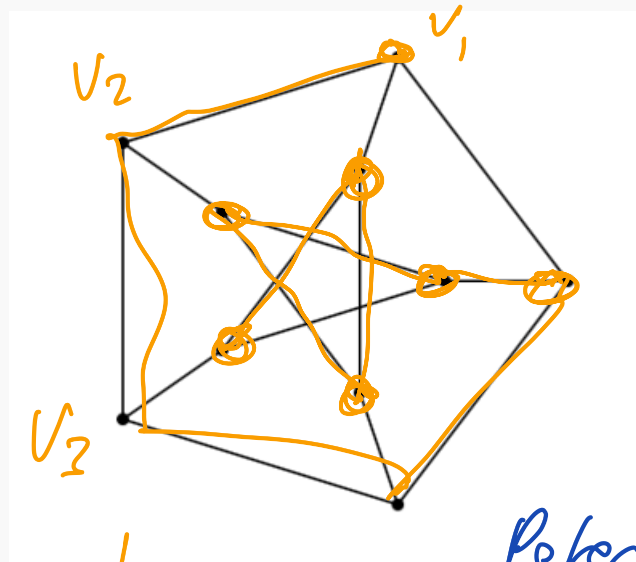
A Hamiltonian cycle is a circuit in G that visits every vertex exactly once. A Hamiltonian path is a path in G that visits every vertex exactly once.

HAMILTONIAN PATHS AND CIRCUITS

$p: \{v_1 v_2\}, \{v_2 v_3\}, --$

Ex: Decide if these graphs have Hamiltonian paths or cycles.





$K_{3,5}$

no Ham path

has Hamiltonian path.

BUT NO HAM CIRCUIT.

Petersen graph

This time, deciding if a graph has a Hamiltonian path is **NP-complete**.

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Recall: this means, in NP which means we can *verify* in polynomial time if an alleged solution is correct, AND that if you can solve this problem in polynomial time then you can solve 3-SAT, and also any other NP-complete problem.

$$(p_1 \wedge p_2 \wedge p_3) \vee (p_4 \wedge p_5 \wedge p$$

$$\vee \ldots$$

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`https://math.stackexchange.com/questions/38106/proof-hampath-is-np-complete`

Visit every vertex

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Euler! visit every edge.

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See the worksheet for this theorem: Suppose $G = (V, E)$ is a simple connected graph with $|V| = n \geqslant 3$. Suppose that for every pair of non-adjacent vertices v_1 and v_2 we have $\mathbf{deg}(v_1) + \mathbf{deg}(v_2) \geqslant n$. Then G has a Hamiltonian cycle.

"thick" graph.

NEXT

- Trees