37181 DISCRETE MATHEMATICS

Lecture 10: Ackermann's function, bijection, countable/uncountable

PLAN

- Ackermann's function
- bijection
- countable/uncountable

$\mathbb{N}^2 \times \mathbb{N}$

Define a function $A : \mathbb{N}^2 \to \mathbb{N}$ using the following *recursive* definition.

$$
\begin{aligned}
A(0, n) &= n + 1 && n \geqslant 0, \\
A(m, 0) &= A(m-1, 1) && m > 0, \\
A(m, n) &= A(m-1, A(m, n-1)) && m, n > 0.
\end{aligned}
$$

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(a) Compute $A(1,3)$. $= 5$

$A(1,3) = A(0, A(1,2)) = A(0,4) = 5.$

$A(1,2) = A(0, A(1,1)) = A(0,3) = 4$

$A(1,1) = A(0, A(1,0))$

$= A(0, A(0,1))$

$= A(0,2) = \cancel{4}\ 3.$

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(a) Compute $A(1,3)$.

(b) Compute $A(2,3)$. — Exercize.

Define a function $A : \mathbb{N}^2 \to \mathbb{N}$ using the following *recursive* definition.

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\end{array}
$$

(a) Compute $A(1,3)$.

(b) Compute $A(2,3)$.

(c) Prove that $A(1,n) = n+2$ for all $n \in \mathbb{N}$.

BIG HINT To use Induction.

Proof:

Let $P(n)$ be the statement that $A(1,n) = n+2$

We have $P(0)$ is true because

$$A(1,0) = A(0,1) = 2$$
$$= 0+2$$

Assume $\underline{P(k)}$ is true for some $k \geq 0$.

Then $P(k+1)$:

$$A(1,k+1) = A(0, A(1,k))$$
$$= A(0, k+2)$$
$$= k+3$$
$$= (k+1)+2$$

So $P(k+1)$ is true.

$\therefore$ by PMI, $P(n)$ is true for all $n \geq 0$.

Define a function $A : \mathbb{N}^2 \to \mathbb{N}$ using the following *recursive* definition.

$$\begin{array}{lcll} A(0,n) & = & n+1 & n \geqslant 0, \\ A(m,0) & = & A(m-1,1) & m > 0, \\ A(m,n) & = & A(m-1, A(m,n-1)) & m,n > 0. \end{array}$$

(a) Compute $A(1,3)$.

(b) Compute $A(2,3)$. ← $3+6=9$.

(c) Prove that $A(1,n) = n+2$ for all $n \in \mathbb{N}$.

(d) Prove that $A(2,n) = 3+2n$ for all $n \in \mathbb{N}$. — Induction Exercise.

Define a function $A : \mathbb{N}^2 \to \mathbb{N}$ using the following *recursive* definition.

$$\begin{array}{lcll} A(0,n) & = & n+1 & n \geqslant 0, \\ A(m,0) & = & A(m-1,1) & m > 0, \\ A(m,n) & = & A(m-1, A(m,n-1)) & m,n > 0. \end{array}$$

(a) Compute $A(1,3)$.

(b) Compute $A(2,3)$.

(c) Prove that $A(1,n) = n+2$ for all $n \in \mathbb{N}$.

$A(2,1) = 3+2$
$= 5$

(d) Prove that $A(2,n) = 3+2n$ for all $n \in \mathbb{N}$.

(e) Prove that $A(3,n) = 2^{n+3} - 3$ for all $n \in \mathbb{N}$. — Induction

Proof Let $P(n)$ be the statement that $A(3, n) = 2^{n+3} - 3$.

Then $P(0)$: $A(3, 0)$

$= A(2, 1) = 5 = 8-3$

$= 2^{0+3} - 3$

Assume $P(k)$ true $k \geq 0$

$P(k+1)$: $A(3, k+1)$

$= A(2, A(3, k))$

$= A(2, 2^{k+3} - 3)$

$= 3 + 2(2^{k+3} - 3)$

$= 3 + 2^{k+3+1} - 3 \cdot 2$

$= 2^{k+4} - 3$ ✓✓

So by PMI

$P(n)$ true for all $n \in \mathbb{N}$.

$A(4, n)$

Definition

A function $f : A \to B$ is a *bijection* if it is both 1-1 and onto.

Eg: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is a bijection.

1-1: $\forall a, b \in \mathbb{R}$ if $f(a) = f(b)$

then $5a + \cancel{3} = 5b + \cancel{3}$

$\rightarrow a = b$ ✓✓

onto: $\forall b \in \mathbb{R}$ $\exists a = \underbrace{\frac{b-3}{5}}$ $\in \mathbb{R}$

so that $f(a) = \cancel{5}\left(\frac{b-3}{\cancel{5}}\right) + 3 = b - 3 + 3 = b$ ✓✓

i.e. there exists some bijective map $f : A \to B$.

A, B

We think if two sets are in bijection, they are the same size (you can match them up by pairs of elements).

Claim: $\mathbb{N}$ and $2\mathbb{N}$ are in bijection.

0 1 2 3 4 5

0 ~~1~~ 2 ~~3~~ 4 ~~5~~ 6 8 ...

Proof: There is a matching

0	1	2	3	4	5	...
0	2	4	6	8	10	...

$f : \mathbb{N} \to 2\mathbb{N}$

$$\boxed{f(a) = 2a}$$

Check:

1-1: $\forall a, b \in \mathbb{N}$

if $f(a) = f(b)$

then $\cancel{2}a = \cancel{2}b$

so $a = b$

Onto: $\forall b \in 2\mathbb{N}$ (even integers)

$\exists\ a = \frac{b}{2} \in \mathbb{N}$

so that $f(a) = f\left(\frac{b}{2}\right) = 2 \cdot \frac{b}{2}$

$= b.$

So $\mathbb{N}$ and $2\mathbb{N}$ are in bijection

So we think of $\mathbb{N}$ and $2\mathbb{N}$ to be the same size.

We think if two sets are in bijection, they are the same size (you can match them up by pairs of elements).

Claim: $\mathbb{N}$ and $\mathbb{Z}$ are in bijection.

0	1	2	3	4	5	6	...
0	-1	1	-2	2	-3	3	...

1.5

"Send odd to negs".

ceiling.

$f : \mathbb{N} \to \mathbb{Z}$

$$f(a) = (-1)^a \left\lceil \frac{a}{2} \right\rceil$$

$\underline{\text{ex}}$ $f(a) = \begin{cases} \frac{a}{2} & a \text{ even} \\ -\left(\frac{a+1}{2}\right) & a \text{ odd} \end{cases}$

Show 1-1, onto:

$\forall a, b \in \mathbb{N}$ → Cases

① a, b both even
② a, b both odd
③ a even, b odd
④ a odd, b even

Case ①
if $f(a) = f(b)$
$\frac{a}{2} = \frac{b}{2} \rightarrow a = b$ ✓

Case ②
if $f(a) = f(b)$
$\cancel{-}\left(\frac{a+1}{2}\right) = \cancel{-}\left(\frac{b+1}{2}\right)$
-1 -1
$a = b$. ✓

Case ③
if $f(a) = f(b)$
$\frac{a}{2} = -\left(\frac{b+1}{2}\right)$
$a = -b - 1$

$f: \mathbb{N} \to \mathbb{Z}$ is 1-1 if
$\forall a, b \in \mathbb{N}$
$a \neq b \rightarrow f(a) \neq f(b)$

$\forall a, b \ (f(a) = f(b) \rightarrow a = b)$

$p \to q$
$\neg q \to \neg p$

If a even, b odd $f(a) = \frac{a}{2}$
$f(b)$ $b+1)$

$f(b) = -\frac{b+1}{2}$

$a+b+1 > 0$ since $a, b \in \mathbb{N}$

so $a+b+1 \neq 0$

so $a \neq -(b+1)$

so $\frac{a}{2} \neq -\frac{(b+1)}{2}$ so $f(a) \neq f(b)$

If a odd, b even, SAME ARGUMENT.

If both even

$a \neq b \to \frac{a}{2} \neq \frac{b}{2}$

so $f(a) \neq f(b)$.

If both odd

$a \neq b \to a+1 \neq b+1$

$\to -(a+1) \neq -(b+1)$

$\to -\frac{a+1}{2} \neq -\frac{b+1}{2}$

$\to f(a) \neq f(b)$

Onto

$\forall\ b \in \mathbb{Z}$

if $b < 0$ then $\exists\ a = -2b-1 \in \mathbb{N}$

if $b \geq 0$ then $\exists\ a = 2b$

so that $f(a) = b$

$$f(a) = -\frac{(a+1)}{2} = b$$

$$-(a+1) = 2b$$

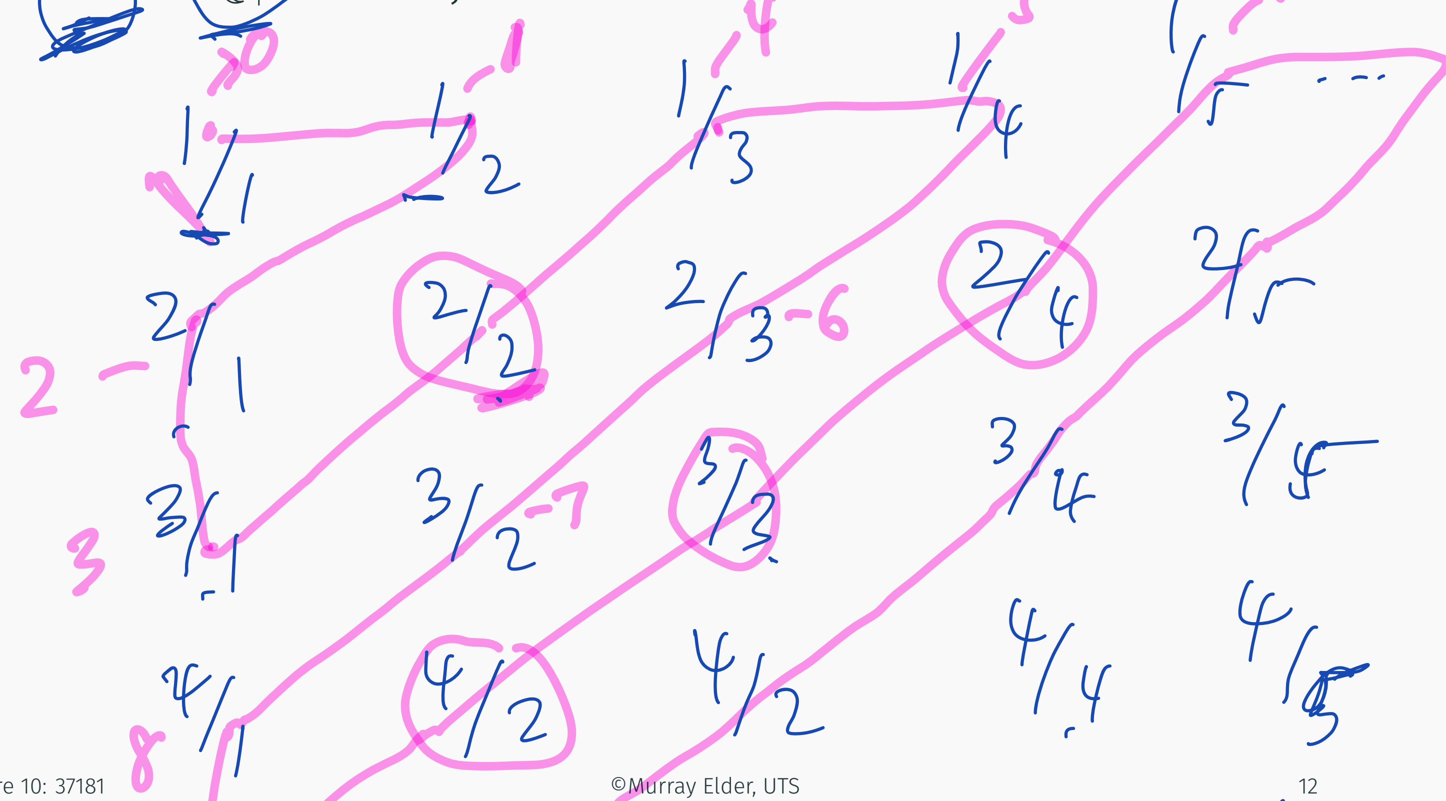
$$a+1 = -2b$$

$$a = -1 - 2b \in \mathbb{N} \quad > 0$$

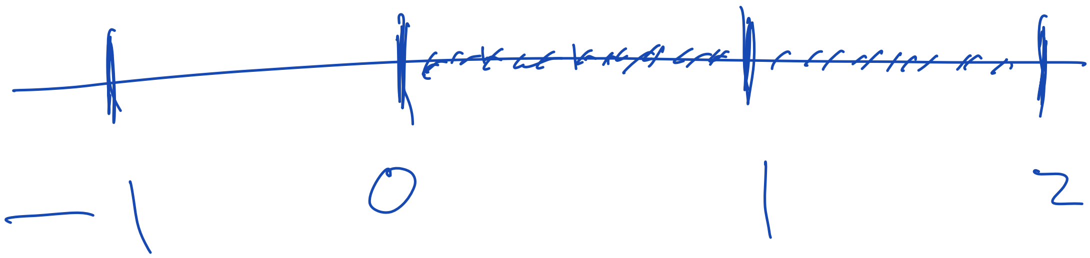
We think if two sets are in bijection, they are the same size (you can match them up by pairs of elements).

$f: \mathbb{Q}_+ \to \mathbb{N}$

Claim: $\mathbb{N}$ and $\mathbb{Q}_+$ are in bijection.



$3/5$



COUNTABLE

Definition: if a set X is in bijection with a finite set or $\mathbb{N}$ then we say it is *countable*.

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So, $\{1, 2, 3, 4\}$, $\mathbb{Z}$, $\mathbb{Q}_+$, $2\mathbb{N}$ are countable.

COUNTABLE

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0 1

Definition: if a set X is in bijection with $\mathbb{N}$ then we also say it is *countably infinite*.

in biject with $\mathbb{N}$.

$\mathbb{Q}_+$

$\mathbb{Q}$

Is there any set that is "bigger" than $\mathbb{N}$?

BIJECTION

Claim: $\mathbb{N}$ and $\mathbb{R}$ are not in bijection. That is, $\mathbb{R}$ is "strictly bigger" than $\mathbb{N}$.

Claim: $\mathbb{N}$ and $\mathbb{R}$ are not in bijection. That is, $\mathbb{R}$ is "strictly bigger" than $\mathbb{N}$.

(If an infinite set is not in bijection with $\mathbb{N}$, we call it *uncountable*.)

Proof: Due to Cantor. "diagonalisation".

Proof by contradiction.

Suppose there is a bijection

$f : \mathbb{N} \to \mathbb{R}$.

So write it down:

$a_i \in \mathbb{Z}$

$a_i \in$

$f(0) = a_0 . a_{01}\ a_{02}\ a_{03} \ldots$ $\to \{0,1,2,\ldots 9\}$

$f(1) = a_1 . a_{11}\ a_{12}\ a_{13} \ldots$

$f(2) = a_2 . a_{21}\ a_{22}\ a_{23} \ldots$ ← decimal numbers

$f(3) = a_3 . a_{31}\ a_{32}\ a_{33}\ a_{34} \ldots$

$f(4) = a_4 . a_{41}\ a_{42}\ a_{43}\ a_{44}\ a_{45}$

$0 . \hat{a}_{01}\ \hat{a}_{12}\ \hat{a}_{23}\ \hat{a}_{34} \ldots$

Now I will show you a number in $\mathbb{R}$ that is not mapped to

$\hat{x}$ hat means pick something other than x.

(so f was not onto

→ contradiction that f is a bijection)

eg $\hat{x} = x+1 \mod 10.$

$0 \to 1$

$1 \to 2$

$\vdots$

$8 \to 9$

$9 \to 0$

... does not

PROOF

Claim $0.a_{01}a_{12}a_{23}\ldots$ appear anywhere on $\mathbb{R}$ list.

Suppose (for contradiction) that $\mathbb{R}$ is the same size as $\mathbb{N}$.

This means that there is some bijection from one set to the other.

Let's suppose this bijection is $f : \mathbb{N} \to \mathbb{R}$, and write $f(0), f(1), f(2), \ldots$

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for example

$$\begin{array}{rcl} f(0) & = & 376.72333\ldots \\ f(1) & = & -0.1111111\ldots \\ f(2) & = & -0.5432100\ldots \\ f(3) & = & 17.0000000\ldots \\ & \vdots & \end{array}$$

$\neq$

0.8241 . . .

Now I will tell you a real number that f has missed. So f is not onto. (Contradiction).

Here is my real number. It is the decimal number $0.x_0x_1x_2x_3x_4\ldots$ where I have to tell you what each x_i is.

For each $i \in \mathbb{N}$, I choose x_i to be a digit that is *not* the i-th digit in $f(i)$. (Say add 1 to it and reduce mod 10).

Now, tell me where my number is on the list?

Nowhere

$\therefore$ not onto.

$\mathbb{R}$ IS BIGGER THAN $\mathbb{N}$

This famous proof (due to Cantor) is known as a *diagonalisation argument*.

The same idea is used to prove that the Halting Problem is undecidable (see 41080 Theory of Computing Science).

So we have $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are all countably infinite, and $\mathbb{R}$ is uncountable.

Question: is there any set of size strictly between these?

P(A) IS BIGGER THAN A

A set.

Claim: For any set A (think infinite), you will prove on the team assignment that A is *not* in bijection with $\mathscr{P}(A)$.

Note, it is possible to think of $\mathbb{R}$ as (in bijection with) $\mathscr{P}(\mathbb{N})$: idea:

$[0, 1)$

0.0 1 0 1 1 1 0 1 1 1 1 1 0 ...

0 is out

1 is in

subset of $\mathbb{N}$

$\{1, 3, 4, 5, 7, 8, 9, 10, 11$

P(A) IS BIGGER THAN A

Claim: For any set A (think infinite), you will prove on the team assignment that A is *not* in bijection with $\mathscr{P}(A)$.

Note, it is possible to think of $\mathbb{R}$ as (in bijection with) $\mathscr{P}(\mathbb{N})$: idea:

So what your assignment question will imply is quite amazing: there are many different sizes of infinity.

$\mathbb{N}$ $\quad \mathbb{R}$ $\quad \mathscr{P}(\mathbb{R})$ $\quad \mathscr{P}(\mathscr{P}(\mathbb{R}))$

$\mathscr{P}(\mathbb{N})$

PUT THESE ON YOUR FORMULA SHEET

- ordered pair
- relation
- reflexive
- symmetric
- antisymmetric
- transitive
- equivalence relation
- partial order
- Hasse diagram
- function
- one to one
- onto
- bijection
- Ackermann's function
- countable
- countably infinite
- uncountable

NEXT

Next lecture:

- Big O
- comparing speed of algorithms