37181 DISCRETE MATHEMATICS

Lecture 10: Ackermann's function, bijection, countable/uncountable

PLAN

- Ackermann's function
- bijection
- countable/uncountable

Define a function $A : \mathbb{N}^2 \to \mathbb{N}$ using the following *recursive* definition.

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(e) Prove that $A(3,n) = 2^{n+3} - 3$ for all $n \in \mathbb{N}$.

Definition

A function $f : A \to B$ is a *bijection* if it is both 1-1 and onto.

Eg: $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 3$ is a bijection.

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Definition: if a set X is in bijection with $\mathbb{N}$ then we also say it is *countably infinite*.

Is there any set that is "bigger" than $\mathbb{N}$?

Claim: $\mathbb{N}$ and $\mathbb{R}$ are not in bijection. That is, $\mathbb{R}$ is "strictly bigger" than $\mathbb{N}$.

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(If an infinite set is not in bijection with $\mathbb{N}$, we call it *uncountable*.)

Proof:

Suppose (for contradiction) that $\mathbb{R}$ is the same size as $\mathbb{N}$.

This means that there is some bijection from one set to the other.

Let's suppose this bijection is $f : \mathbb{N} \to \mathbb{R}$, and write $f(0), f(1), f(2), \ldots$

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for example

$$\begin{array}{rcl} f(0) & = & 376.72333\ldots \\ f(1) & = & -0.1111111\ldots \\ f(2) & = & -0.5432100\ldots \\ f(3) & = & 17.0000000\ldots \\ & \vdots & \end{array}$$

PROOF

Now I will tell you a real number that f has missed. So f is not onto. (Contradiction).

Here is my real number. It is the decimal number $0.x_0x_1x_2x_3x_4\ldots$ where I have to tell you what each x_i is.

For each $i \in \mathbb{N}$, I choose x_i to be a digit that is *not* the i-th digit in $f(i)$. (Say add 1 to it and reduce mod 10).

Now, tell me where my number is on the list?

This famous proof (due to Cantor) is known as a *diagonalisation argument*.

The same idea is used to prove that the Halting Problem is undecidable (see 41080 Theory of Computing Science).

So we have $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are all countably infinite, and $\mathbb{R}$ is uncountable.

Question: is there any set of size strictly between these?

Claim: For any set A (think infinite), you will prove on the team assignment that A is *not* in bijection with $\mathscr{P}(A)$.

Note, it is possible to think of $\mathbb{R}$ as (in bijection with) $\mathscr{P}(\mathbb{N})$: idea:

P(A) IS BIGGER THAN A

Claim: For any set A (think infinite), you will prove on the team assignment that A is *not* in bijection with $\mathscr{P}(A)$.

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So what your assignment question will imply is quite amazing: there are many different sizes of infinity.

PUT THESE ON YOUR FORMULA SHEET

- ordered pair
- relation
- reflexive
- symmetric
- antisymmetric
- transitive
- equivalence relation
- partial order
- Hasse diagram

- function
- one to one
- onto
- bijection
- Ackermann's function
- countable
- countably infinite
- uncountable

NEXT

Next lecture:

- Big O
- comparing speed of algorithms