37181 DISCRETE MATHEMATICS

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Lecture 11: Big O, complexity of algorithms

PLAN

- Big O
- comparing speed of algorithms

COMPARING SPEEDS OF ALGORITHMS

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for i = 1 to n do
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    i = floor(i/2)
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COMPARING SPEEDS OF ALGORITHMS

How can we *formally* capture the idea that the second algorithm is *faster*? Can we estimate that on input of size n, the algorithm will take roughly some function of n steps?

Look back at the two algorithms and try to roughly guess a function for each one.

Luckily, someone already thought of how to capture the idea of algorithm speed/complexity, using this definition.

Definition

Let $f, g : \mathbb{N}_+ \to \mathbb{R}$. We say that g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that

$$|f(n)| \leqslant m|g(n)|$$

for all $n \in \mathbb{N}, n \geqslant k$.

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$O(g)$ is the set of all functions from $\mathbb{N}_+$ to $\mathbb{R}$ which are dominated by g.

EG

Defn: *g dominates f* if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}$, $n \geqslant k$.

Ex: who dominates who: $f(n) = \log_2 n + 5$ versus $g(n) = n$:

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Ex: who dominates who: $f(n) = \log_2 n + 5$ versus $g(n) = n$:

Here is my prepared solution in case my "live" one is too messy:

Claim 1: $n \leqslant 2^n$. Let $P(n)$ be the statement $n \leqslant 2^n$. $P(1)$ true. Assume $P(k)$ then $n + 1 \leqslant n + n = 2n \leqslant 2.2^n = 2^{n+1}$ so $P(k + 1)$ is implies, so by PMI true for all $n \geqslant 1$.

Take $\log_2$ both sides gives $\log_2 n \leqslant n$, so

$$\log_2 n + 5 \leqslant n + 5 \leqslant n + n$$

(if $n \geqslant 5$)

$$= 2n$$

so $m = 2, k = 5$ gives the result.

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Some careful analysis of the two algorithms above shows that, up to some constants, power_fast takes about $\log_2 n$ steps (because each step divides i by 2, roughly) and power_slow takes n steps.

Because of Big O, if you count steps slightly differently, and/or count the initial and final lines of the code, the time complexity function changes a bit but is **the same** up to Big O.

RECALL

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Recall the formula

$$\log_b y = \frac{\log_a y}{\log_a b}. \quad [1]$$

[1] Proof: $y = a^{\log_a y}$, so sub this into the LHS.

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Since $\log_e 2 \approx 4.6$ is a fixed number, it doesn't matter which log we use because of the m in the definition.

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Let $f(n) = 6n$ and $g(n) = n^2$. Show that g dominates f, that is, $6n \in O(n^2)$.

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Here is my prepared solution. It might be different that the one I just did "live" – many choices for m, k can work.

There exist $m = 1, k = 6$ so that

$$6n \leqslant n.n = n^2$$

for all $n \geqslant 6$.

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Show that $g(n) = n^2$ is not dominated by $f(n) = 6n$. That is, $n^2 \notin O(6n)$.

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Given m, k fixed numbers (positive), there is a number $n = \mathbf{max}\{6m + 1, k\}$ so that $n > 6m$ so

$$n^2 > m6n.$$

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Let $f(n) = 6n^2 + 5n + 2$ and $g(n) = n^2$. Show that $f \in O(g)$ and $g \in O(f)$. So they are (up to Big O equivalence) the "same".

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Show that $f(n) = n^3$ is dominated e^n.

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My prepared solution:

Let $P(n)$ be the statement $n^3 \leqslant e^n$. Need to find a value k so that $P(k)$ is true, then

assume for $s \geqslant k$ that $P(s)$ is true, then $P(s+1)$:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1 \leqslant n^3 + n^3$$

(by a Lemma I will prove below)

$$= 2n^3 \leqslant 2e^n < e.e^n = e^{n+1}$$

since $e = 2.7... > 2$.

Lemma: $3n^2 + 3n + 1 \leqslant n^3$ for n big enough.

CHALLENGE QUESTION

Defn: *g dominates f* if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}_+$ such that $|f(n)| \leqslant m|g(n)|$ for all $n \in \mathbb{N}$, $n \geqslant k$.

Show that $f(n) = n^c$ is dominated e^n where c is any positive integer.

So polynomials are "slower" than exponentials.

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Your proof might use the Binomial Theorem:

$$(x+y)^c = x^c + \binom{c}{1}x^{c-1}y + \binom{c}{2}x^{c-2}y^2 + \cdots + \binom{c}{c-1}x^1y^{c-1} + y^c$$

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$$(n+1)^c = n^c + \binom{c}{1}n^{c-1} + \binom{c}{2}n^{c-2} + \cdots + \binom{c}{c-1}n + 1$$

In your proof, c is a fixed number, so all the $\binom{c}{i}$ terms are bounded above by some fixed number.

EG

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Which is faster (who dominates who?) – exponentials or factorials?

That is, show that one of $f(n) = c^n$, $g(n) = n!$ dominates the other (for any $c \in \mathbb{R}_+$ say).

COMPLEXITY OF ALGORITHMS

If you do more computer science, you will use Big O. You count (roughly, since constant multiples don't matter) the number of steps an algorithm takes on inputs of size $n \in \mathbb{N}$.

The algorithm is "fast" if the number of steps is $O(n)$ or maybe $O(n \log n)$, and "bad" if it takes $O(c^n)$ or worse steps.

Issues: is it bad if it takes this many steps on *all* inputs, or some inputs, or most inputs.

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See Worksheet 6 for a table of standard functions and their names used in time complexity.

Exercise: prove that the worst case running time for the Euclidean algorithm is when the input is two of the Fibonacci numbers (worksheet 3)

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Standard exercise in comp sci: compare running times for algorithms which sort a list of numbers: bubble sort versus merge sort.

Recall from the Canvas “Welcome” page:
`https://www.youtube.com/watch?v=kVgy1GSDHG8`

And here is another video that mentions Big O:
`https://www.youtube.com/watch?v=xFFs9UgOAlE`

NEXT

Next time:

- pigeonhole principle