37181 DISCRETE MATHEMATICS

Lecture 14: Binomial theorem; some famous counting sequences; some extra PHP problem applications

PLAN

- binomial theorem
- combinatorial proofs
- some famous counting sequences
- Catalan numbers
- more applications of PHP

Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_+$. Then

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \dots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Eg: $(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3$

$= x^3 + 3x^2y + 3xy^2 + y^3$

$$\binom{n}{k} = \binom{n}{n-k}$$

Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_+$. Then

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Proof:

$(x+y)(x+y) \ldots (x+y)$

when you expand, the terms will look like

$x^{n-i} y^{i}$

$1 \cdot x^n$

$\binom{n}{1} x^{n-1} y$

y^n

How many times will the product give $x^{n-i} y^{i}$?

↳ Same as choosing i of the bracket terms $(x+y)$ to give the "y".

$\rightarrow \binom{n}{i}$

$$\therefore (x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^{i}.$$

$\square$

Prove:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Lecture 13.

$\overbrace{a_1 \ldots a_{n+1}}$ each choice either contains a_{n+1} or not.

Contain a_{n+1} Don't

$$\binom{n}{k-1} + \binom{n}{k}$$

~~$(x+y)^{10}$~~

Prove:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Application: Pascal's triangle

$$\binom{3}{2} = \binom{2}{1} + \binom{2}{2}$$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} + \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Ex 11: let f_n = the number of strings of 1, 2 whose digits add up to $n-1$.

empty
1 string.

$f_0 = 0$

add up to 0 $\{\lambda\}$

$f_1 = 1$

$f_2 = 1$ → $\{1\}$

$f_3 = 2$ $\{11, 2\}$

$f_4 = 3$ → $\{111, 12, 21\}$

$f_5 = 5$ $\{1111, 121, 112, 211, 22\}$

adds up to 4

This sequence is famous, called Fibonacci sequence.

Ex 11: let f_n = the number of strings of 1, 2 whose digits add up to $n - 1$.

General formula: think recursively

f_{n+1} :

strings start with 1 — and adds up to n

1 ______ adds up to $n-1$

strings start with 2.

2 ______ adds up to $n-2$

$$\boxed{f_{n+1} = f_n + f_{n-1}}$$

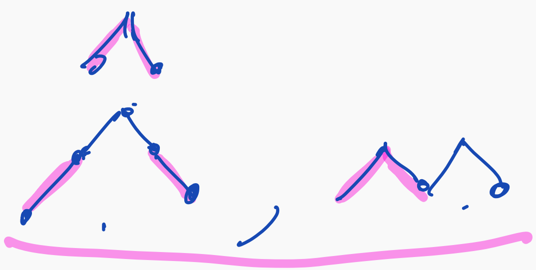
start with 1 — start with 2

Ex 12: let c_n = the number of paths in the xy-plane consisting of $2n$ diagonal lines of length $\sqrt{2}$ and slope ± 1, starting at $(0,0)$ and ending at $(2n, 0)$, and never going below the x-axis.

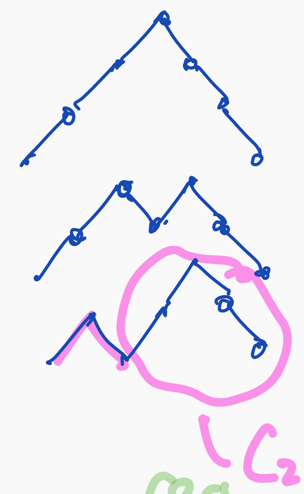
$c_0 = 1$

$c_1 = 1$

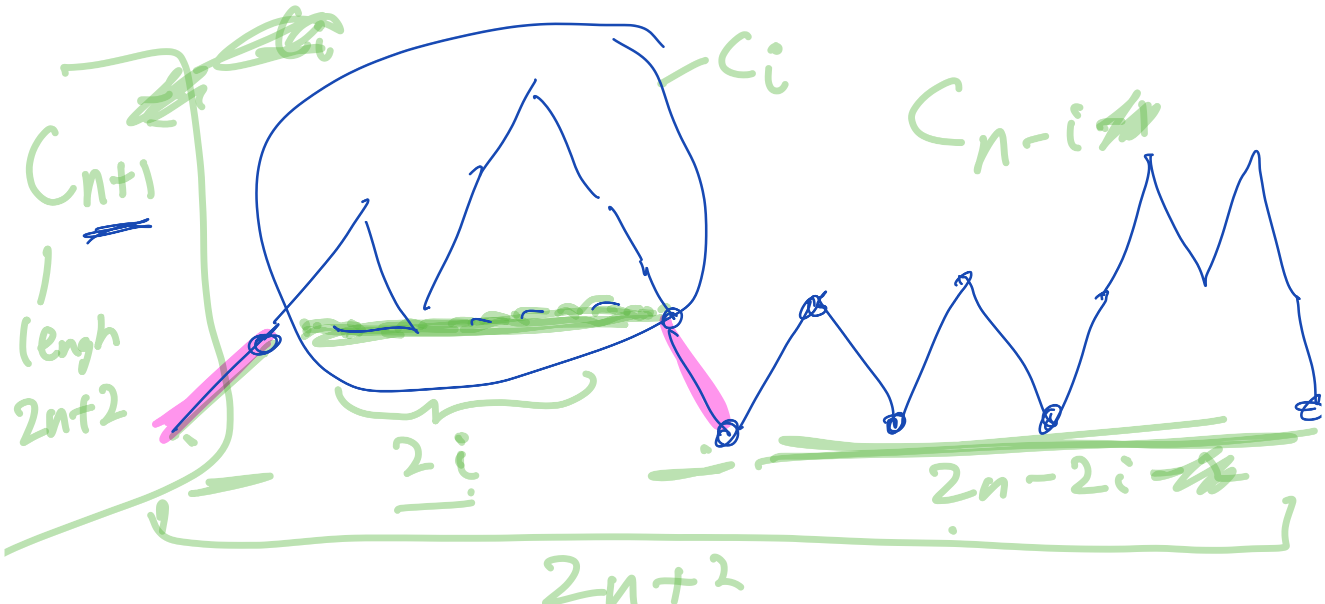
$c_2 = 2$



$C_3 =$



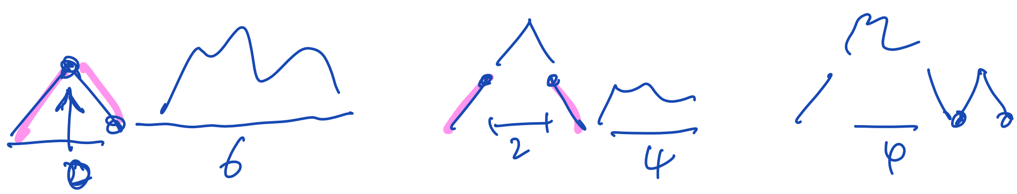
General formula: think recursively

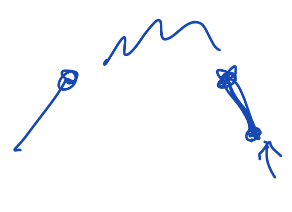


$$\boxed{C_{n+1} = \sum_{i=0}^{n} C_i \cdot C_{n-i}}$$

$$C_4 = C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0$$

$n=3$





$$= 1\cdot 5 + 1\cdot 2 + 2\cdot 1 + 5\cdot 1$$
$$= 14$$

$$
\begin{aligned}
C_3 &= C_0 C_2 + C_1 C_1 + C_2 C_0 \\
&= 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 \\
&= 5
\end{aligned}
$$

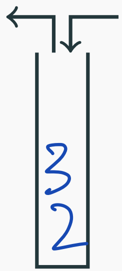
Famous sequence
Catalan numbers

1, 1, 2, 5, 14, ...

$$
\begin{aligned}
C_5 &= C_0 C_4 + C_1 C_3 + C_2 C_2 + C_3 C_1 + C_4 C_0 \\
&= 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 \\
&= 28 + 14 = \underline{42}
\end{aligned}
$$

(oeis.org)

Sort out-of-order data (permutations) with a stack right-to-left.



32451 — can't sort. 241 problem / not adjacent.

312 ✓✓ can sort

231

single stack

Which permutations can't be sorted?

Knuth: problem is when permutation "contains middle, Big, small" somewhere in the

 Sequence.

Theorem (Knuth 1968)

A permutation can be sorted by passing it -right-to-left through an infinite stack if and only if it avoids 231.

And, they are enumerated by the Catalan numbers. 20, 100, 5

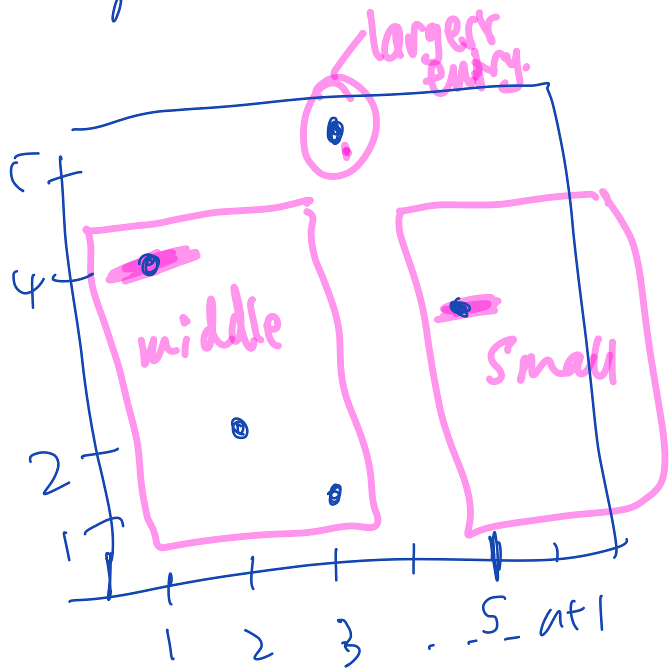
counted

Proof:

C_n = # permutations sorted by single stack.

If a perm (length $n+1$) avoids mid Big small
(2 3 1)

plot:



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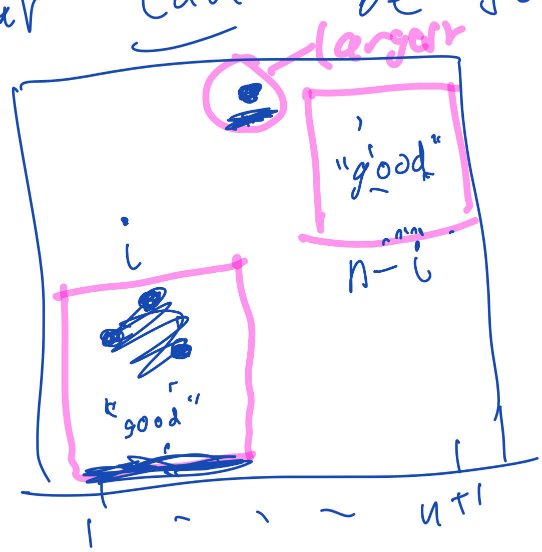
Plot of a perm of length $\underline{n+1}$ that $\underline{\text{can}}$ be sorted:

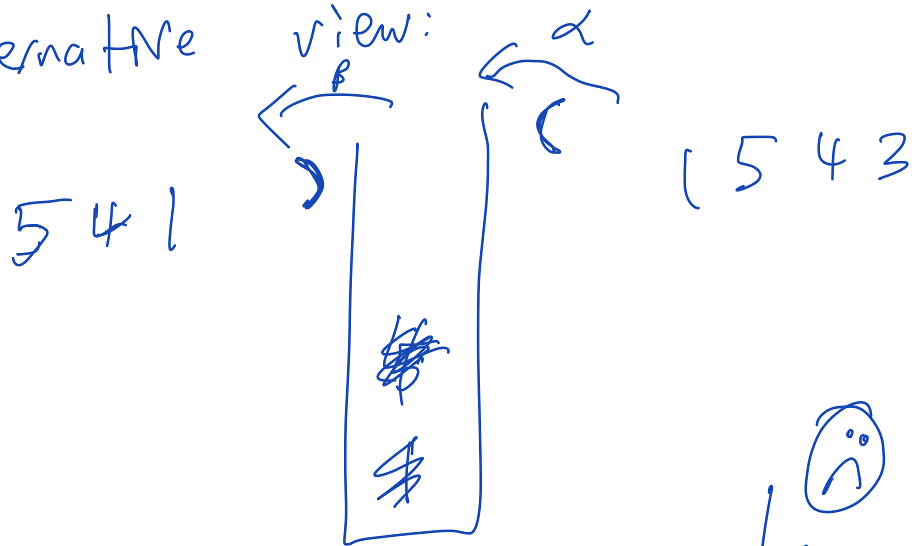


$C_0 = 1$

$C_1 = 1$

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

Alternative view:



Codeword: $\alpha\ \alpha\ \beta\ \alpha\ \beta\ \beta \mid \beta\ \alpha$

instruct to push + pop numbers in + out of stack

(( ) ( ))) (

PHP

distinct

We say a sequence of numbers $\cancel{3}, \cancel{9}, 2, \cancel{7}, 6, \cancel{1}, 4, \cancel{10}, \cancel{5}, 8$ *contains a subsequence* if you can remove a few numbers to obtain the subsequence, for example the above contains $2, 6, 4, 8$. It contains an increasing sequence $3, 6, 10$ and a decreasing sequence $9, 7, 6, 4$.

$n^2+1 = 10 = 3^2+1$ $n+1 = 4$

We say a sequence of numbers $3, 9, 2, 7, 6, 1, 4, 10, 5, 8$ *contains a subsequence* if you can remove a few numbers to obtain the subsequence, for example the above contains $2, 6, 4, 8$. It contains an increasing sequence $3, 6, 10$ and a decreasing sequence $9, 7, 6, 4$.

Theorem

~~Lemma~~ (Erdős and Szekeres 1935)

For each $n \in \mathbb{N}_+$, any sequence of $n^2 + 1$ distinct real numbers contains a decreasing or increasing subsequence of length $n + 1$.

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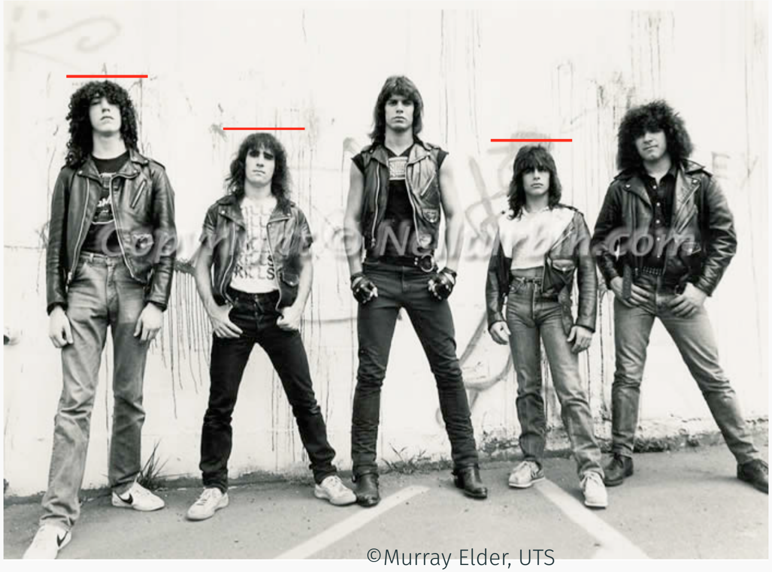

n^2+1
$=5$

Lemma (Erdős and Szekeres 1935)

For each $n \in \mathbb{N}_+$, any sequence of n^2+1 distinct real numbers contains a decreasing or increasing subsequence of length $n+1$.

Proof: See Grimaldi – Discrete Math.

... of them must be equal, and we conclude that there exist $1 \le j < i \le 28$ with $x_i = x_j + 15$. Hence, from the start of day $j+1$ to the end of day i, Herbert will play exactly 15 sets of tennis.

Our last example for this section deals with a classic result that was first discovered in 1935 by Paul Erdös and George Szekeres.

LE 5.49

Let us start by considering two particular examples:

1) Note how the sequence 6, 5, 8, 3, 7 (of length 5) contains the decreasing subsequence 6, 5, 3 (of length 3).
2) Now note how the sequence 11, 8, 7, 1, 9, 6, 5, 10, 3, 12 (of length 10) contains the increasing subsequence 8, 9, 10, 12 (of length 4).

These two instances demonstrate the general result: For each $n \in \mathbf{Z}^+$, a sequence of n^2+1 distinct real numbers contains a decreasing or increasing subsequence of length $n+1$.

To verify this claim let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of n^2+1 distinct real numbers. For $1 \le k \le n^2+1$, let

x_k = the maximum length of a decreasing subsequence that ends with a_k, and

y_k = the maximum length of an increasing subsequence that ends with a_k.

For instance, our second particular example would provide

k	1	2	3	4	5	6	7	8	9	10
a_k	11	8	7	1	9	6	5	10	3	12
x_k	1	2	3	4	2	4	5	2	6	1
y_k	1	1	1	1	2	2	2	3	2	4

If, in general, there is no decreasing or increasing subsequence of length $n+1$, then $1 \le x_k \le n$ and $1 \le y_k \le n$ for all $1 \le k \le n^2+1$. Consequently, there are at most n^2 *distinct* ordered pairs (x_k, y_k). But we have n^2+1 ordered pairs (x_k, y_k), since $1 \le k \le n^2+1$. So the pigeonhole principle implies that there are two identical ordered pairs (x_i, y_i), (x_j, y_j), where $i \ne j$ — say $i < j$. Now the real numbers $a_1, a_2, \ldots, a_{n^2+1}$ are distinct, so if $a_i < a_j$ then $y_i < y_j$, while if $a_j < a_i$ then $x_j > x_i$. In either case we no longer have $(x_i, y_i) = (x_j, y_j)$. This contradiction tells us that $x_k = n+1$ or $y_k = n+1$ for some $n+1 \le k \le n^2+1$; the result then follows.

For an interesting application of this result, consider n^2+1 sumo wrestlers facing forward and standing shoulder to shoulder. (Here no two wrestlers have the same weight.) We can select $n+1$ of these wrestlers to take one step forward so that, as they are scanned from left to right, their successive weights either decrease or increase.

NEXT TIME

- Number theory

RSA