37181 DISCRETE MATHEMATICS

Lecture 14: Binomial theorem; some famous counting sequences; some extra PHP problem applications

PLAN

- binomial theorem
- combinatorial proofs
- some famous counting sequences
- Catalan numbers
- more applications of PHP

Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_+$. Then

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Eg: $(x+y)^2$

$(x+y)^3$

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Proof:

Prove:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

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Application: Pascal's triangle

Ex 11: let $f_n =$ the number of strings of $1, 2$ whose digits add up to $n - 1$.

$f_0 =$

$f_1 =$

$f_2 =$

$f_3 =$

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General formula: think recursively

Ex 12: let $c_n =$ the number of paths in the xy-plane consisting of $2n$ diagonal lines of length $\sqrt{2}$ and slope ± 1, starting at $(0,0)$ and ending at $(2n, 0)$, and never going below the x-axis.

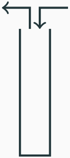
$c_0 =$

$c_1 =$

$c_2 =$

General formula: think recursively

Sort out-of-order data (permutations) with a stack right-to-left.



32451

312

231

Which permutations can't be sorted?

Theorem (Knuth 1968)

A permutation can be sorted by passing it -right-to-left through an infinite stack if and only if it avoids 231.

And, they are enumerated by the Catalan numbers.

Proof:

PHP

We say a sequence of numbers $3, 9, 2, 7, 6, 1, 4, 10, 5, 8$ *contains a subsequence* if you can remove a few numbers to obtain the subsequence, for example the above contains $2, 6, 4, 8$. It contains an increasing sequence $3, 6, 10$ and a decreasing sequence $9, 7, 6, 4$.

PHP

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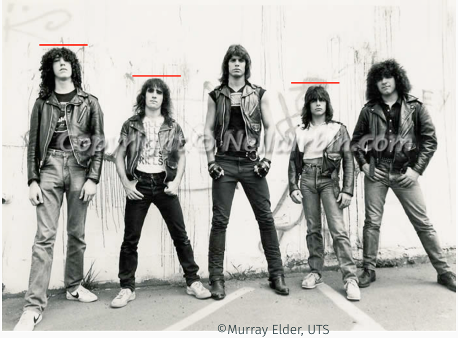

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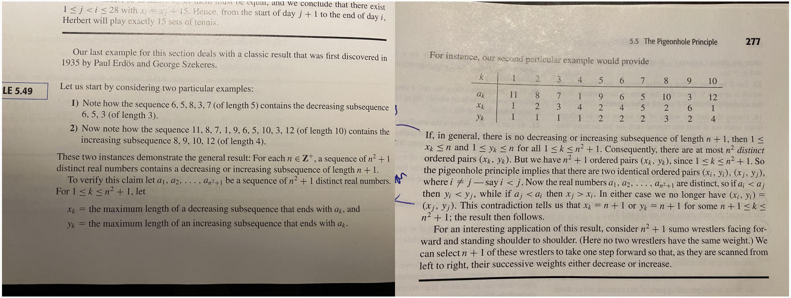
Proof: See Grimaldi – Discrete Math.



... and we conclude that there exist $1 \le j < i \le 28$ with $x_i = x_j + 15$. Hence, from the start of day $j+1$ to the end of day i, Herbert will play exactly 15 sets of tennis.

Our last example for this section deals with a classic result that was first discovered in 1935 by Paul Erdös and George Szekeres.

LE 5.49

Let us start by considering two particular examples:

1) Note how the sequence 6, 5, 8, 3, 7 (of length 5) contains the decreasing subsequence 6, 5, 3 (of length 3).
2) Now note how the sequence 11, 8, 7, 1, 9, 6, 5, 10, 3, 12 (of length 10) contains the increasing subsequence 8, 9, 10, 12 (of length 4).

These two instances demonstrate the general result: For each $n \in \mathbf{Z}^+$, a sequence of n^2+1 distinct real numbers contains a decreasing or increasing subsequence of length $n+1$.

To verify this claim let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of n^2+1 distinct real numbers. For $1 \le k \le n^2+1$, let

x_k = the maximum length of a decreasing subsequence that ends with a_k, and

y_k = the maximum length of an increasing subsequence that ends with a_k.

5.5 The Pigeonhole Principle 277

For instance, our second particular example would provide

k	1	2	3	4	5	6	7	8	9	10
a_k	11	8	7	1	9	6	5	10	3	12
x_k	1	2	3	4	2	4	5	2	6	1
y_k	1	1	1	1	2	2	2	3	2	4

If, in general, there is no decreasing or increasing subsequence of length $n+1$, then $1 \le x_k \le n$ and $1 \le y_k \le n$ for all $1 \le k \le n^2+1$. Consequently, there are at most n^2 *distinct* ordered pairs (x_k, y_k). But we have n^2+1 ordered pairs (x_k, y_k), since $1 \le k \le n^2+1$. So the pigeonhole principle implies that there are two identical ordered pairs (x_i, y_i), (x_j, y_j), where $i \ne j$ — say $i < j$. Now the real numbers $a_1, a_2, \ldots, a_{n^2+1}$ are distinct, so if $a_i < a_j$ then $y_i < y_j$, while if $a_j < a_i$ then $x_j > x_i$. In either case we no longer have $(x_i, y_i) = (x_j, y_j)$. This contradiction tells us that $x_k = n+1$ or $y_k = n+1$ for some $n+1 \le k \le n^2+1$; the result then follows.

For an interesting application of this result, consider n^2+1 sumo wrestlers facing forward and standing shoulder to shoulder. (Here no two wrestlers have the same weight.) We can select $n+1$ of these wrestlers to take one step forward so that, as they are scanned from left to right, their successive weights either decrease or increase.

NEXT TIME

- Number theory