37181 DISCRETE MATHEMATICS

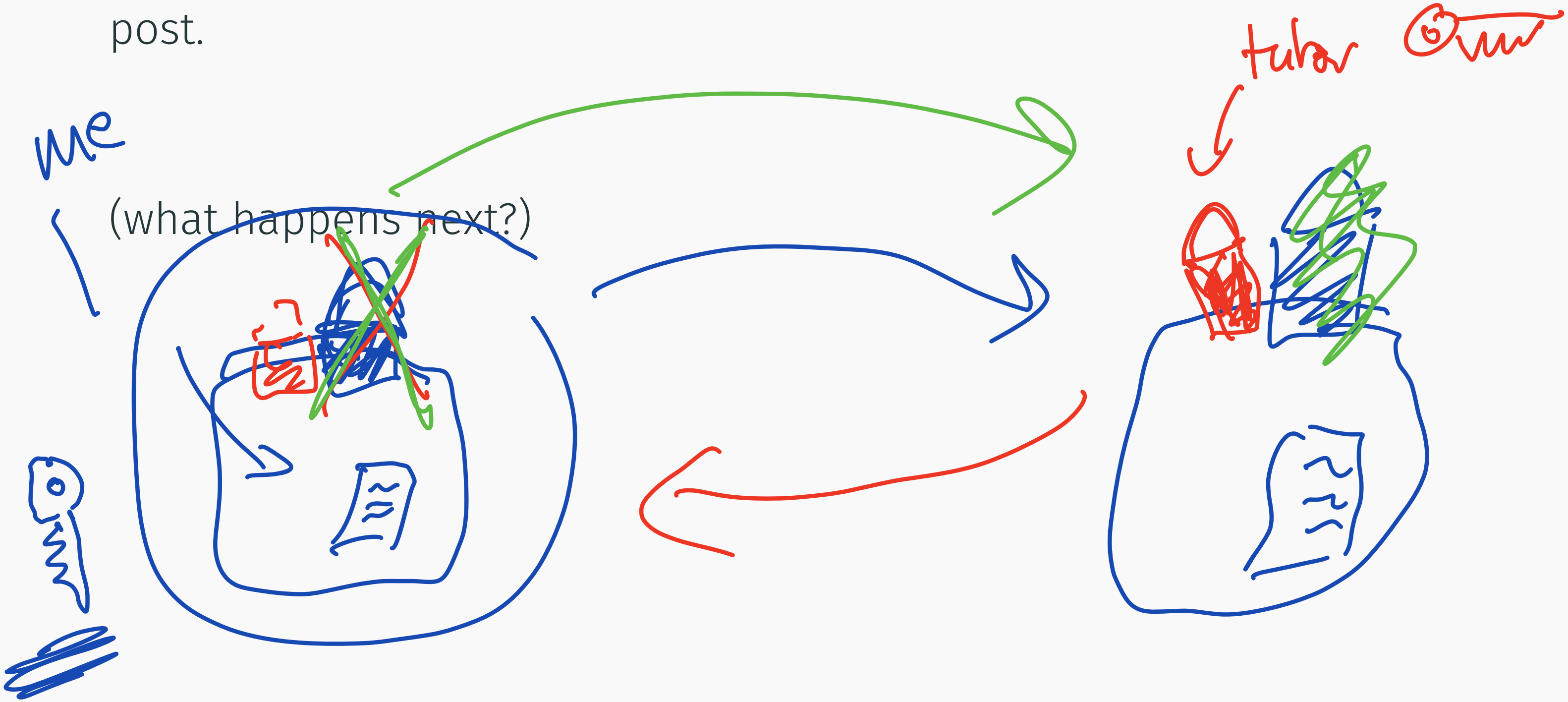
Lecture 17: the RSA cryptosystem

PLAN

- RSA

I have a box, and a padlock. I put a secret message in the box (not a pigeon), put my padlock on it, keep the key, and send the box in the post.

(what happens next?)



$\exists p \exists q$

$$\boxed{n = p \cdot q}$$

Security assumption:

if you are given $n \in \mathbb{N}$ and told it is the product of two primes, how hard is it to find the two primes?

"very hard"

(we hope)

[1] you can stop when you hit $\lfloor\sqrt{n}\rfloor$. Why?

$\varphi(n) = \varphi(p \cdot q)$

$=$

Security assumption:

if you are given $n \in \mathbb{N}$ and told it is the product of two distinct primes, how hard is it to find the two primes?

≥ 100 digits each

- compute $\varphi(n)$ (recall end of Lectures 16 - how hard is it to compute $\varphi(n)$?)
- divide by 2, 3, 5, 7, 11, 13, ...[1]

$\sqrt{n} \cdot \sqrt{n} = n$

$(\lfloor \sqrt{n} \rfloor + 1)(\ldots$

[1]you can stop when you hit $\lfloor \sqrt{n} \rfloor$. Why?

Security assumption:

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- compute $\varphi(n)$ (recall end of Lectures 16 - how hard is it to compute $\varphi(n)$?)
- divide by $2, 3, 5, 7, \ldots$ [1]
- Quantum computer?
  https://en.wikipedia.org/wiki/Shors_algorithm
  https://en.wikipedia.org/wiki/Integer_factorization_records

[1] you can stop when you hit $\lfloor \sqrt{n} \rfloor$. Why?

RSA

Idea: Alice (pronouns she/her) and Bob (pronouns he/him) want to communicate over open channels (the internet)

so that at the end, they have a *shared secret* nobody else knows

even though everybody can see their communication.

Alice:

- chooses p, q two large distinct (and keeps them secret).
- She computes $n = pq$.
- She publishes n on her webpage.
- She secretly computes $\varphi(n) = (p-1)(q-1)$ (that's easy for her)

"multiplication is easy"

Public : n

Secret: $\varphi(n)$ p, q

RSA

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- Then she chooses (any) $e \in \mathbb{Z}^*_{\varphi(n)}$. of e.
- She computes the multiplicative inverse mod $\varphi(n)$: that is, she computes d so that $de \equiv 1 \bmod \varphi(n)$.

RSA

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- Since Alice worked hard on the worksheets and lectures 15 and 16, all of that is easy for her to do.

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- Since Alice worked hard on the worksheets and lectures 15 and 16, all of that is easy for her to do.
- Finally, she publishes e on her webpage as well.

Public: n, e

Secret: p, q, d, $\varphi(n)$

RSA

Bob:

- wants to send Alice a secret message which will be a number in $\mathbb{Z}_n$. He knows n since it is public.
- He picks m relatively prime to n.

$\gcd(m, n) = 1$

n, e Public

Bob:

- wants to send Alice a secret message which will be a number in $\mathbb{Z}_n$. He knows n since it is public.
- He picks m relatively prime to n.
- He computes $[m^e]_n$ (he might need repeated squaring). Remember e, n are both public.
- He sends the number $c = [m^e]_n$ to Alice over the internet. Assuming raising to a really high power e then reducing mod n "mixes up" the number m, it should not be obvious what m is when anyone else sees c in the open channels.

Public: $c(= [m^e]_n)$ Secret: m.

c, n, e

$a^{\varphi(n)} \equiv 1 \bmod n$

Alice:

- She knows d, p, q so she takes the number c from Bob and computes
$$c^d \equiv (m^e)^d = m^{ed} \bmod n$$
some $s \in \mathbb{Z}$
- Remember $ed \equiv 1 \bmod \varphi(n)$ so $ed = 1 + s\varphi(n)$
- So $m^{ed} = m^{1+s\varphi(n)} = m.(m^{\varphi(n)})^s$ but by Euler's theorem $m^{\varphi(n)} \equiv 1 \bmod n$.
- So Alice just computed $m = [m^{ed}]_n$.

prime

$2 \cdot 37$

Practice:

(You will need a copy of the RSA procedure in front of you to follow this.)

1. Alice constructs an RSA system by choosing $n = 74$ and $e = 7$. What is her corresponding value for d?

inverse of e

$\varphi(n) = \varphi(2)\varphi(37)$

$= 1 \cdot 36$

$\boxed{3^2 \cdot 2^2}$

$36 = 5 \cdot 7 + 1$

$1 = 36 - \underbrace{5}_{d} \cdot 7$

$d \equiv -5 \equiv 31$

$\boxed{d = 31}$

Practice:

(You will need a copy of the RSA procedure in front of you to follow this.)

1. Alice constructs an RSA system by choosing $n = 74$ and $e = 7$. What is her corresponding value for d?

2. Then Bob wants to send $m = 21$. What c does he send?

$$c = \left[21^7\right]_{74}$$

$$7 = 4 + 2 + 1$$

$$21 = $$

$$21^2 = 441 = -3 \quad \text{mod } 74$$

$$21^4 = (-3)^2 = 9$$

$$21^7 = 21^4 \cdot 21^2 \cdot 21^1$$

$c = 25$

$$= 9\cdot(-3)\cdot 21$$
$$= 9\cdot(-63)$$
$$= 9\cdot 11$$
$$= 99 = 25$$

Practice:

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1. Alice constructs an RSA system by choosing $n = 74$ and $e = 7$. What is her corresponding value for d?

2. Then Bob wants to send $m = 21$. What c does he send?

3. Then do the steps Alice would do to decode c.

knows ~~(n)~~, d

$$[25^{31}]_{74}$$

$$31 = 16+8+4+2+1$$

$$25^{31} = 25^{16}\cdot 25^{8}\cdot 25^{4}\cdot 25^{2}\cdot 25$$

$$= 9(-3)(53)(33)(25)$$

$\equiv 21$

Practice:

(You will need a copy of the RSA procedure in front of you to follow this.)

1. Alice constructs an RSA system by choosing $n = 74$ and $e = 7$. What is her corresponding value for d?

2. Then Bob wants to send $m = 21$. What c does he send?

3. Then do the steps Alice would do to decode c.

$\varphi(74) = \varphi(2)\varphi(37) = 36, 36 = 5.7 + 1, 1 = 36 - 5.7$ so $d = -5 = 31$.
$c = 21^4 21^2 21^1 = 9(-3)21 = 9.(-63) = 9.11 = 99 = 25$. Alice computes
$25^d = 25^{31} = 25^{16} 25^8 25^4 25^2 25 = 21$

$23^{43} = 23^{7} \mod 77$

$7 \cdot 11$ $\varphi(77) = 6 \cdot 10 = 60$

Alice constructs an RSA system and publishes $n = 77$ and $e = 43$. Bob then sends Alice the encoded message $c = 23$. What was Bob's intended message?

$$m = [c^{d}]_{n} = [23^{?}]_{77}$$

$$\varphi(77) = 60$$

$$60 = 1 \cdot 43 + 17$$
$$43 = 2 \cdot 17 + 9$$
$$17 = 1 \cdot 9 + 8$$
$$9 = 1 \cdot 8 + 1$$

$$\begin{aligned}
1 &= 9 - 8 \\
&= 9 - (17 - 9) \\
&= 2 \cdot 9 - 17 \\
&= 2(43 - 2 \cdot 17) - 17 \\
&= 2 \cdot 43 - 5 \cdot 17 \\
&= 2 \cdot 43 - 5(60 - 43) \\
&= 2 \cdot 43 - 5 \cdot 60 + 5 \cdot 43
\end{aligned}$$

$\boxed{d = 7}$

$$[23^7]_{77} \equiv 23 \quad !!$$

$$23^4 \cdot 23^2 \cdot 23^1 \equiv \cancel{100} \cdot \underline{23} \cdot (-10) \cdot \underline{23}$$

$$\equiv -10 \cdot -10$$

$$= 100$$

$$= 23$$

$$23^2 = 529 \equiv 67 \mod 77$$

$$\equiv -10$$

$$23^4 = (-10)^2 = 100 \equiv \underline{23}$$

Alice finds $m = 23$

Check

WAYS TO CHEAT

https://www.mtholyoke.edu/courses/quenell/s2003/ma139/js/powermod.html

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MATH 139

PowerMod Calculator

Computes (base)$^{\text{(exponent)}}$ mod (modulus) in log(exponent) time.

Base: 21	Exponent: 7	Modulus: 74
Compute	b^e MOD m =	25

The program is written in JavaScript, and runs on the client computer. Most implementations seem to handle numbers of up to 16 digits correctly.

mtholyoke.edu/courses/quenell/s2003/ma139/js/powermod.html

MATH 139

PowerMod Calculator

Computes (base)$^{\text{(exponent)}}$ mod (modulus) in log(exponent) time.

Base: 25	Exponent: 31	Modulus: 74
Compute	b^e MOD m =	21

The program is written in JavaScript, and runs on the client computer. Most implementations seem to handle numbers of up to 16 digits correctly.

LPC7: HASSE DIAGRAM

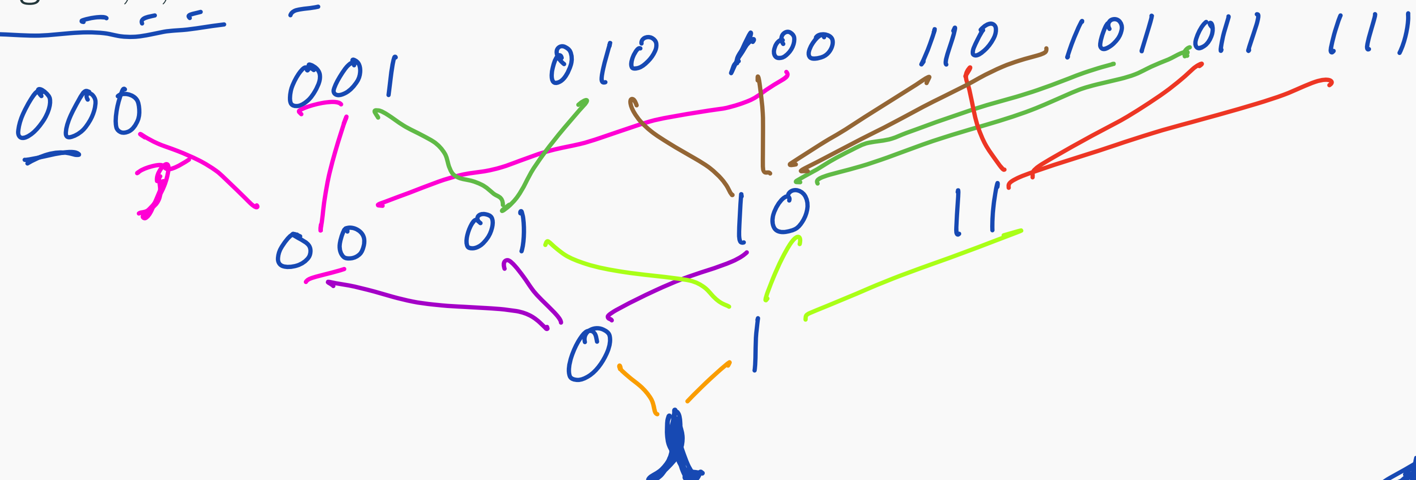
partial order

Define a relation $\mathcal{T}$ on the set of all finite length (including 0) binary strings by

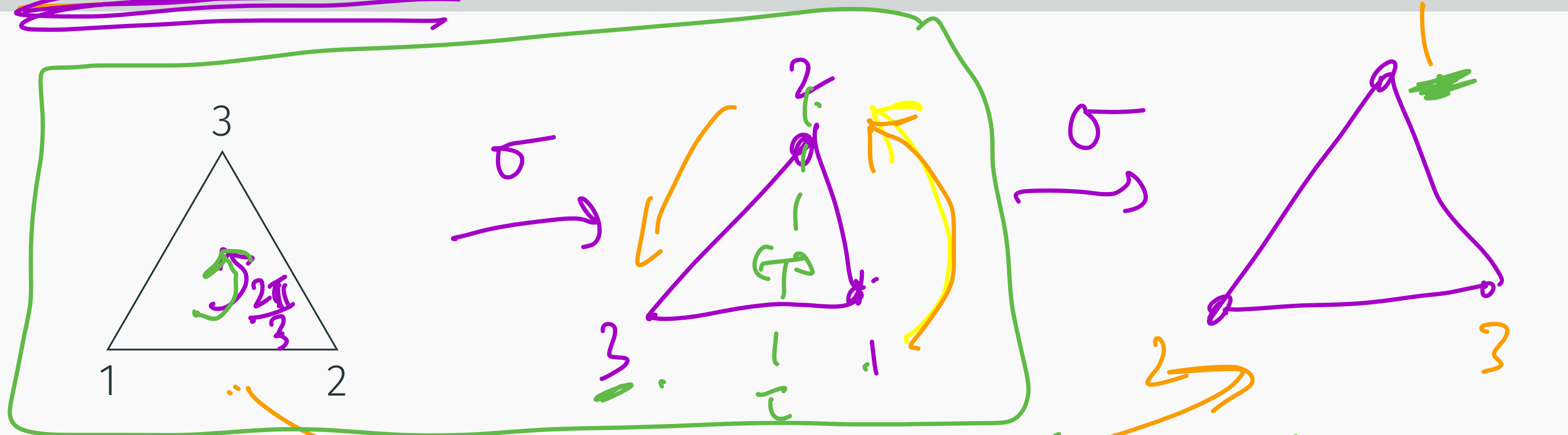
$$a \mathcal{T} b \quad \text{if} \quad a \text{ is a factor of } b.$$

reflexive ✓
antisymm ✓
transitive ✓

(b) Draw the *Hasse diagram*[2] for $\mathcal{T}$ on the set of all binary strings of length 0, 1, 2 and 3.



[2] this was in Lecture 9.



$*(*(\sigma,\sigma),\sigma) = e$

(a) Is $*(\sigma,\tau)$ the same motion as $*(\tau,\sigma)$?

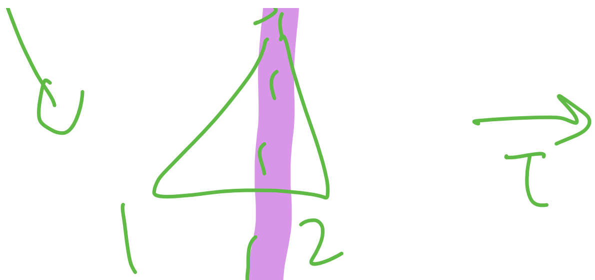
(b) What is $*(\tau,\tau)$?

(c') What is $*(\sigma,\sigma)$?

rotate by
120 + 120
= 240°

Also, same as

(d') Prove that τ, σ both have inverses.



$*: G \times G \to G$

$\tau * \tau = e$

$$\boxed{*(\tau, \tau) = e}$$

Worksheet 9

$\mathbb{Z}_n^*$, mult mod n ← 1

Qn1 old woman.

$n \equiv 1 \mod 2$: ×

$n \equiv 1 \mod 3$..

$n \equiv 1 \mod 4$.

$n \equiv 1 \mod 5$

$n \equiv 1 \mod 6$

$n \equiv 0 \mod 7$

$\times 4$

21

- Graph theory

$n = \cancel{2p+1}$

$3q+1$

$n = \cancel{2 \cdot 3 \cdot 2^4 \cdot 5s} + 1$

$= 60s + 1$

61

121

181

241

301

301