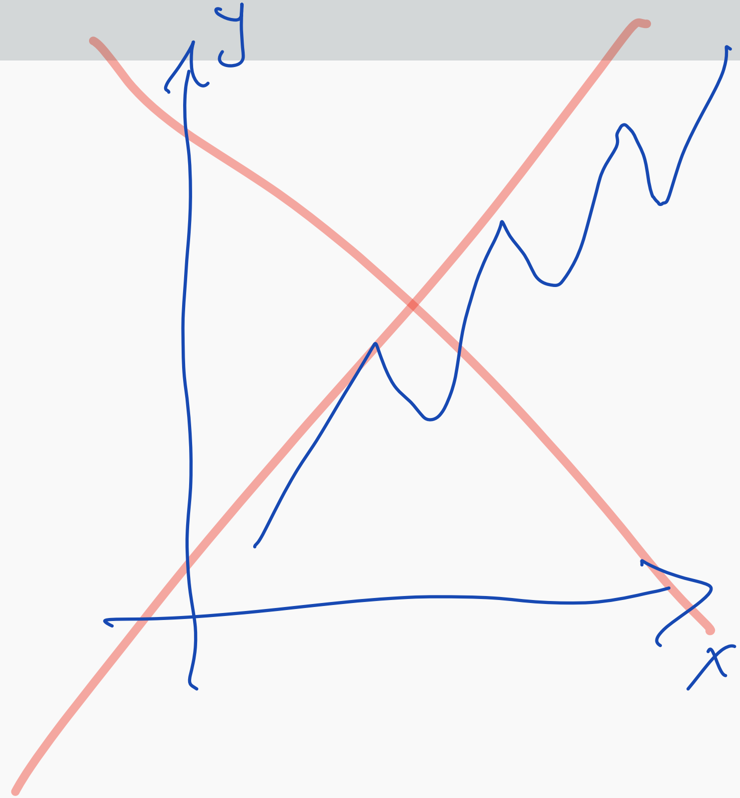
37181 DISCRETE MATHEMATICS

Lecture 18: graph theory

PLAN

- Definition: graph
- Applications, theorems




INDIAN OCEAN
PACIFIC OCEAN
TASMAN SEA
CORAL SEA
TIMOR SEA
Great Australian Bight
WESTERN AUSTRALIA
NORTHERN TERRITORY
QUEENSLAND
SOUTH AUSTRALIA
NEW SOUTH WALES
VICTORIA
TASMANIA
DARWIN
BROOME
PORT HEDLAND
KARRATHA
LEARMONTH
PARABURDOO
NEWMAN
GERALDTON
KALGOORLIE
PERTH
KUNUNURRA
ULURU (AYERS ROCK)
ALICE SPRINGS
MT ISA
CAIRNS
TOWNSVILLE
HAMILTON ISLAND
MACKAY
ROCKHAMPTON
GLADSTONE
BUNDABERG
HERVEY BAY
MAROOCHYDORE (SUNSHINE COAST)
BRISBANE
GOLD COAST
BALLINA BYRON
COFFS HARBOUR
ARMIDALE
TAMWORTH
PORT MACQUARIE
LORD HOWE ISLAND
DUBBO
NEWCASTLE
SYDNEY
CANBERRA
WAGGA WAGGA
ALBURY
MILDURA
ADELAIDE
PORT LINCOLN
KINGSCOTE
MELBOURNE
MELBOURNE (AVALON)
DEVONPORT
LAUNCESTON
HOBART
08:00
10:00
ROUTE KEY
Qantas routes
Jetstar routes
QantasLink routes
Routes operated by other airlines for Qantas
Qantas dedicated freight route
Qantas Club and Qantas regional lounge locations
Qantas Group international gateway port
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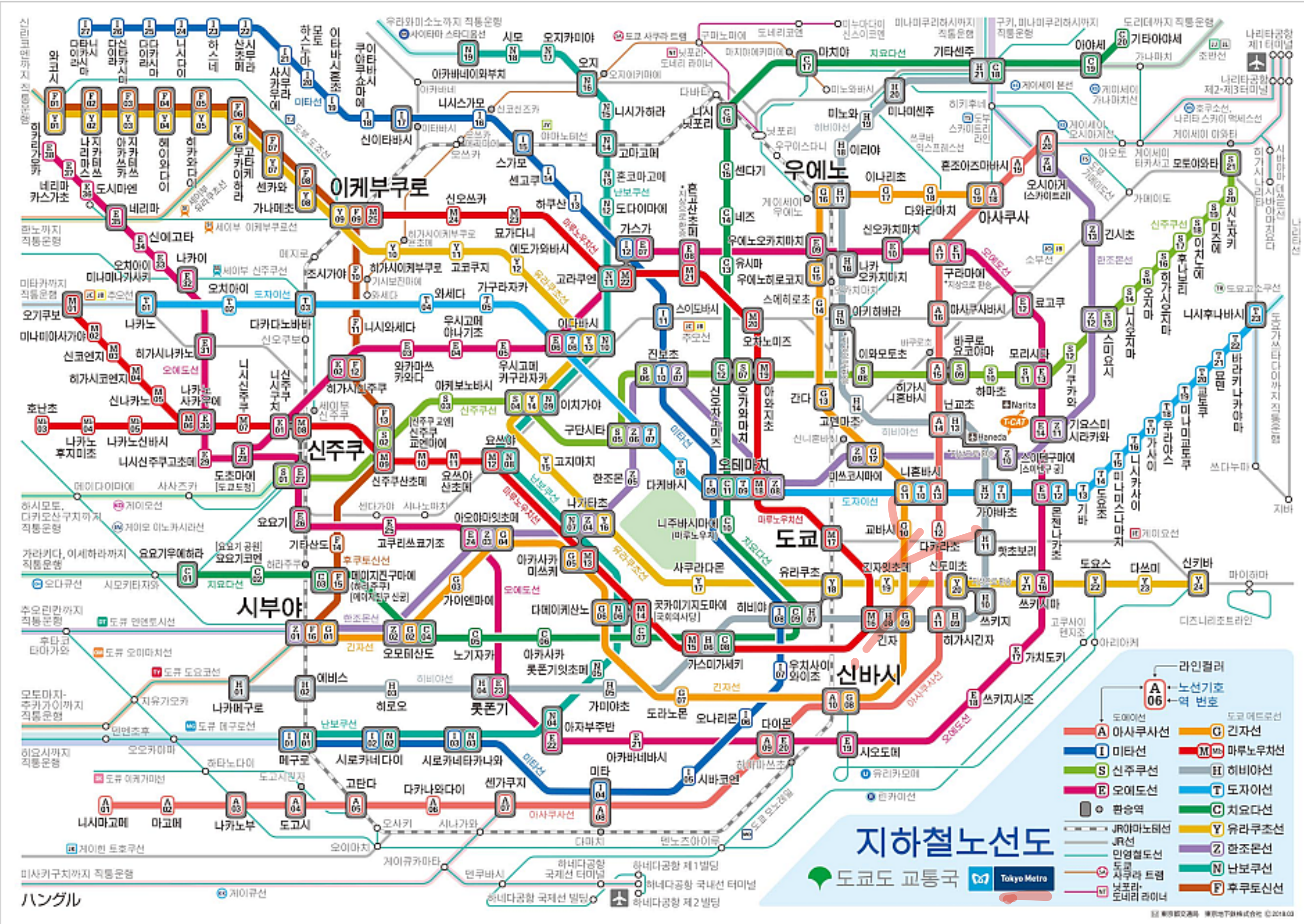
Qantas
Jetstar
Airnorth



지하철노선도
도쿄도 교통국
Tokyo Metro
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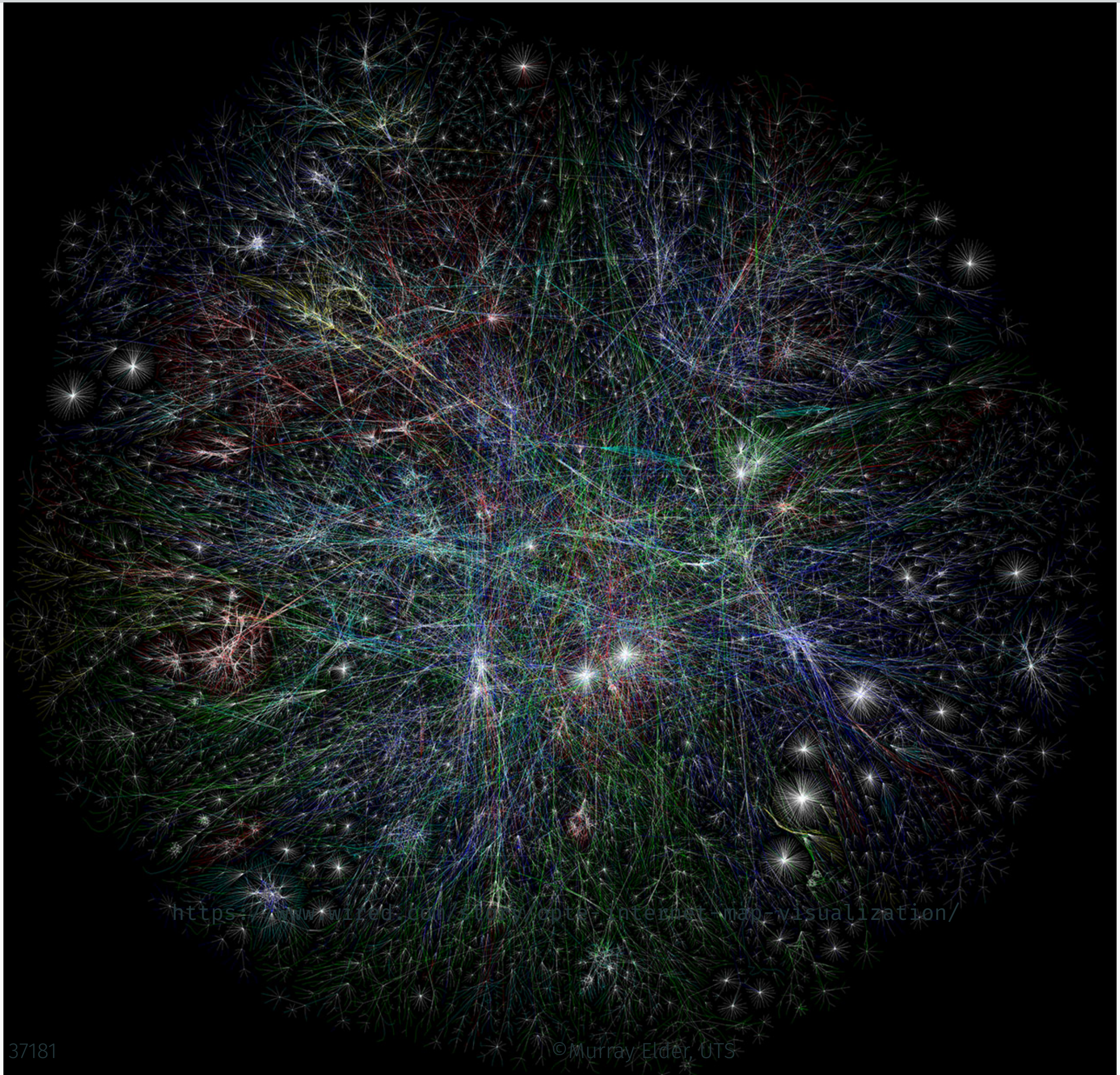
이케부쿠로
우에노
신주쿠
도쿄
시부야
신바시
라인컬러
노선기호
역 번호
아사쿠사선
미타선
신주쿠선
오에도선
환승역
긴자선
마루노우치선
히비야선
도자이선
치요다선
유라쿠조선
한조몬선
난보쿠선
후쿠토신선


https://www.wired.com/story/opte-internet-map-visualization/

GRAPH THEORY

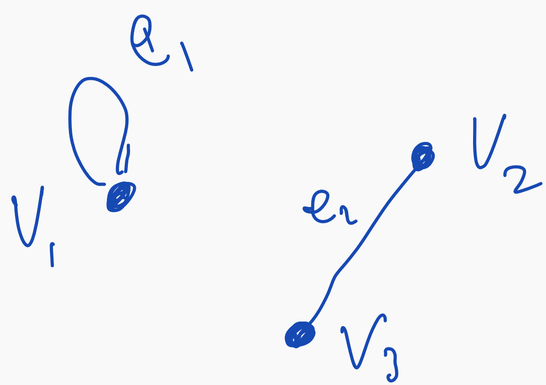
$V = \{v_1, v_2, v_3\}$

$E = e_1 \rightsquigarrow \{v_1\}$, $e_2 \rightsquigarrow \{v_2, v_3\}$

Definition (Graph)

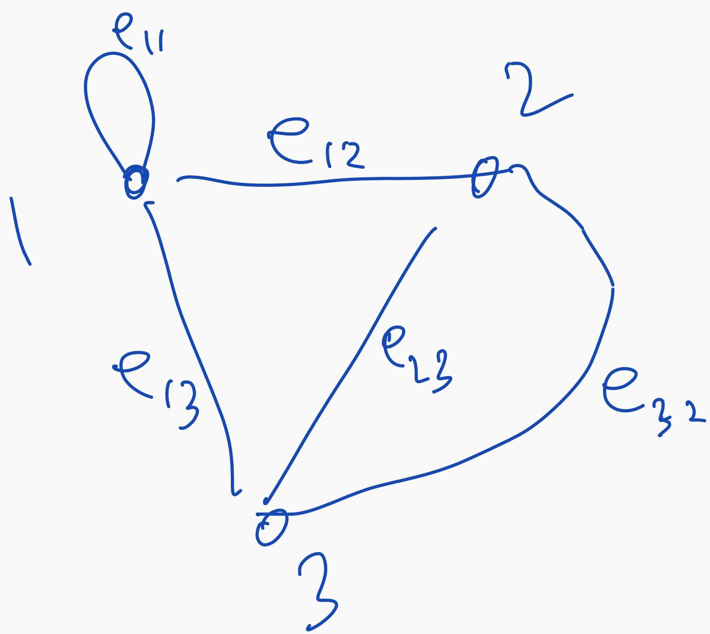
A *graph* $G = (V, E)$ *is a pair of sets* V, E *such that* each $e \in E$ is associated to some subset $\{v_1, v_2\} \subseteq V$ of size 1 or 2.

The elements of V are called [plural: *vertices*, singular: *vertex*] or *nodes*, and the elements of E are called *edges* or *arcs*.

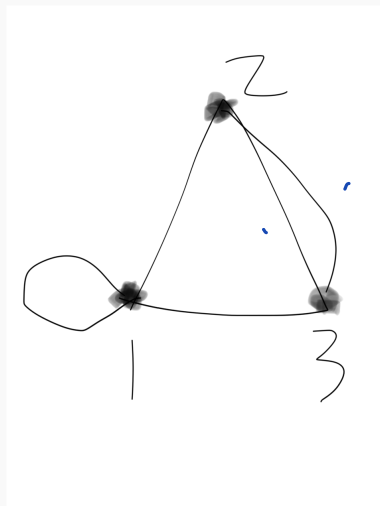


not connected.

Eg: If $V = \{1, 2, 3\}$, $E = \{e_{11}, e_{12}, e_{13}, e_{23}, e_{32}\}$ where e_{ij} is associated to $\{i, j\} \subseteq V$ then $G = (V, E)$ is a graph. We can *visualise* G as a picture, with a dot for each element of V and a line between v_1 and v_2 if $\{v_1, v_2\}$ is associated to some $e \in E$, so in this case we get



Eg: If $V = \{1, 2, 3\}$, $E = \{e_{11}, e_{12}, e_{13}, e_{23}, e_{32}\}$ where e_{ij} is associated to $\{i, j\} \subseteq V$ then $G = (V, E)$ is a graph. We can *visualise* G as a picture, with a dot for each element of V and a line between v_1 and v_2 if $\{v_1, v_2\}$ is associated to some $e \in E$, so in this case we get



If $\{v_1, v_2\}$ is associated to more than one edge in some graph G, these edges are called *multiple edges* or *multi-edges* and G is sometimes called a *multi-graph*.

When each subset $\{v_1, v_2\} \subseteq V$ of size 1 or 2 is associated to at most one element of E, we can choose to label each edge by a subset of V of size 1 or 2, and write $E \subseteq \mathscr{P}(V)$.

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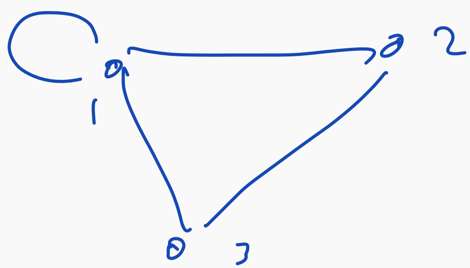
Ex: draw the graph with edge set $\{\{1, 2\}, \{1, 1\}, \{2, 3\}, \{1, 3\}\}$.

$\boxed{V = \{1, 2, 3\}}$



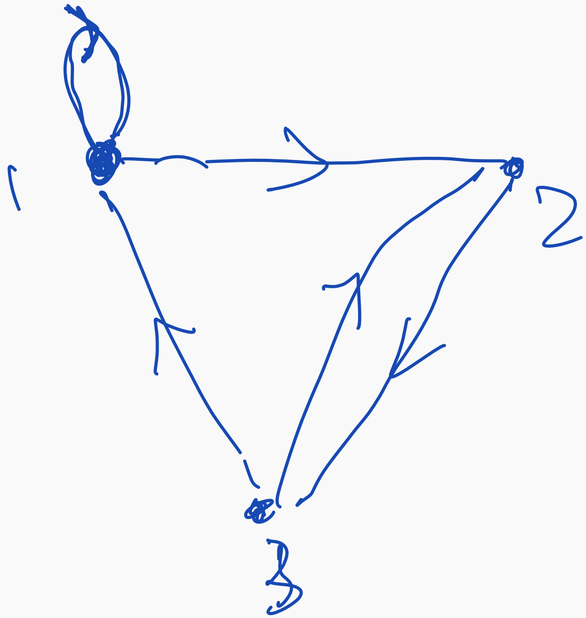
Definition (Directed graph)

A *directed* graph $G = (V, E)$ is a pair of sets V, E such that each $e \in E$ is associated to some ordered pair $(v_1, v_2) \in V \times V$. If $\cancel{(u, v) \in E}$ we call $u \in V$ the *source* vertex and v the *terminal* vertex.

In this case when we visualise G as a picture, we draw arrows on the edges to indicate their *direction*, from source to terminal.

if $e \in E$ is associated to (u, v)

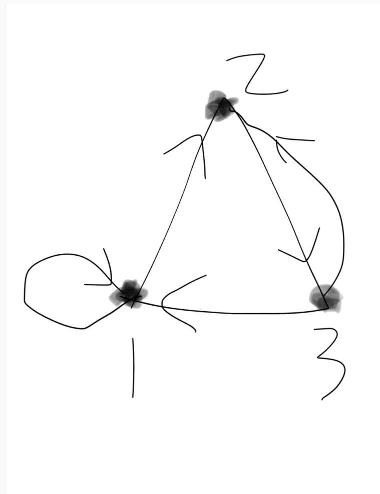
Ex: draw the directed graph with
$V = \{1, 2, 3\}, E = \{e_{11}, e_{12}, e_{31}, e_{23}, e_{32}\}$ where e_{ij} is associated to $(i, j) \in V^2$.



GRAPH THEORY – BASIC DEFINITIONS

$(2,3)$

Ex: draw the directed graph with
$V = \{1, 2, 3\}, E = \{e_{11}, e_{12}, e_{31}, e_{23}, e_{32}\}$ where e_{ij} is associated to $(i, j) \in V^2$.



~~multi-digraph~~

Again if more than one edge is associated to the same ordered pair, we call them *multi-edges*. If G is directed with no multi-edges we can choose to label E by ordered pairs $V \times V = V^2$, and we write $E \subseteq V^2$.
In our example there are no multi-edges, since $(2, 3)$ and $(3, 2)$ are different elements of V^2.

1. loop

GRAPH THEORY – BASIC DEFINITIONS

1. loop: an edge associated to a singleton set $\{x\}$

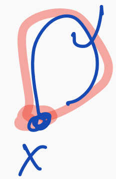
2. multiple edge/multi-edge

GRAPH THEORY – BASIC DEFINITIONS

1. loop: an edge associated to a singleton set $\{x\}$
2. multiple edge/multi-edge: when more than one edge is associated to the same set $\{x, y\}$
3. simple graph

Undirected graphs

1. loop: an edge associated to a singleton set $\{x\}$



2. multiple edge/multi-edge: when more than one edge is associated to the same set $\{x, y\}$

3. simple graph: a graph with no loops and no multi-edges

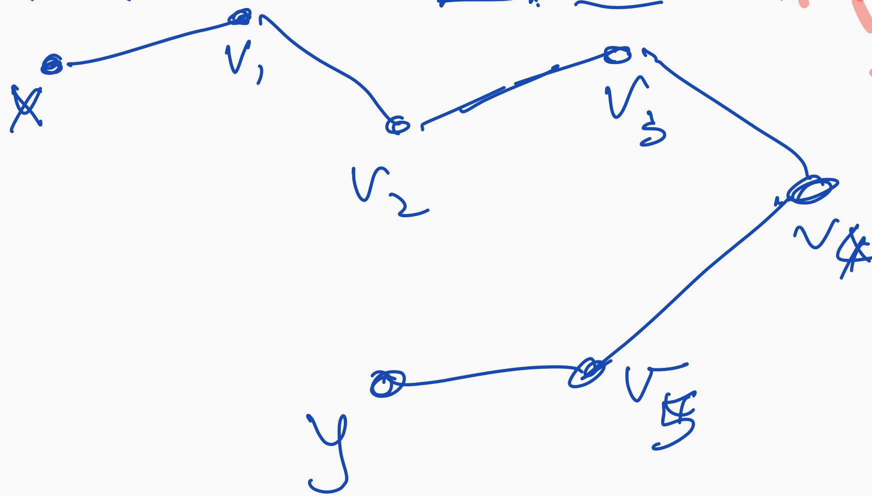
4. path

~~$(x, v_1)\ (v_1, v_2)\ (v_3, v_2), (v_2, y)$~~

$\{v_2, v_3\} \ldots$

4. path : a sequence of edges associated to sets $\{x, v_1\}, \{v_1, v_2\}, \ldots, \{v_n, y\}$

$n+1$



5. length of a path

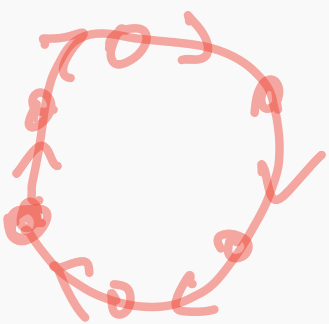
\# edges in the sequence defining the path.

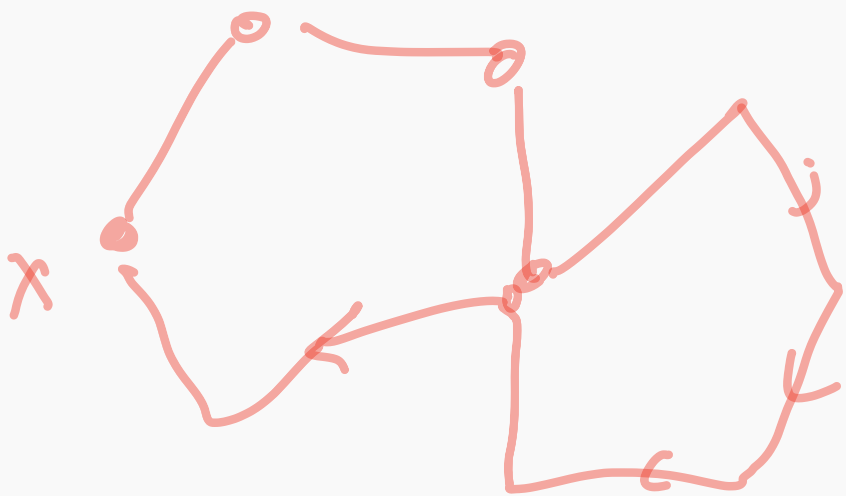
$\{x, x\}$.

4. path : a sequence of edges associated to sets $\{x, v_1\}, \{v_1, v_2\}, \ldots, \{v_n, y\}$

5. length of a path : the number of edges in the path

6. circuit $x = y$





4. path : a sequence of edges associated to sets $\{x, v_1\}, \{v_1, v_2\}, \ldots, \{v_n, y\}$

5. length of a path : the number of edges in the path

6. circuit : a path where $x = y$

eg $\{x, x\}$.

loop $\Rightarrow$ circuit.

$\exists\, x, y \in V$
$\forall$ path

7. connected

$\forall\, x, y \in V$

$\exists$ path P
of the form
$\{x, v_1\} \ldots \{v_n, y\}$.

whole graph

7. connected : if every two vertices x, y are joined by a path

8. disconnected

7. connected : if every two vertices x, y are joined by a path

8. disconnected : not connected

9. simple path

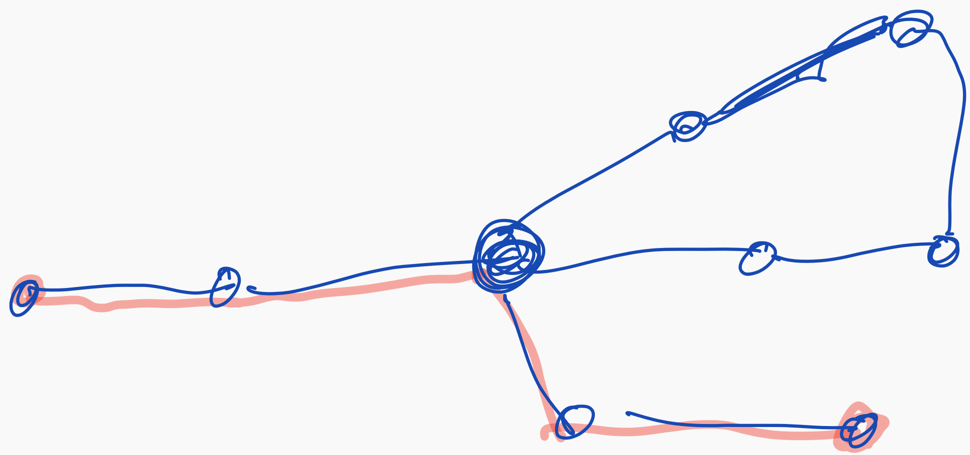
9. simple path : a path $\{x, v_1\}, \{v_1, v_2\}, \ldots, \{v_n, y\}$ where each v_i is different from all other v_j (note x, y are allowed to be the same vertex).

and x and y

$\{x, y\}$

$\{x, x\}$

10. endpoint(s) of an edge



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10. endpoint(s) of an edge : if e is associated with $\{x, y\}$ then x, y are called the endpoints of e

11. edge incident to a vertex

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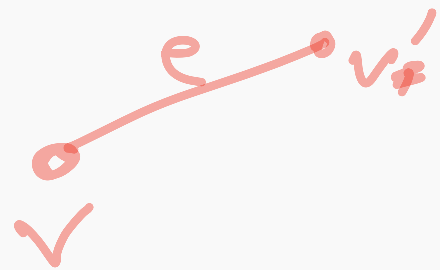
10. endpoint(s) of an edge : if e is associated with $\{x, y\}$ then x, y are called the endpoints

"next to"

11. edge incident to a vertex : e is incident to v if e is associated to $\{v, v'\}$

12. adjacent vertices



9. simple path : a path $\{x, v_1\}, \{v_1, v_2\}, \ldots, \{v_n, y\}$ where each v_i is different from all other v_j (note x, y are allowed to be the same vertex).

10. endpoint(s) of an edge : if e is associated with $\{x, y\}$ then x, y are called the endpoints

11. edge incident to a vertex : e is incident to v if e is associated to $\{v, v'\}$

12. adjacent vertices : v, v' are adjacent if $\{v, v'\}$ is associated to an edge

13. degree of a vertex (for undirected graphs), notation $\deg(v)$

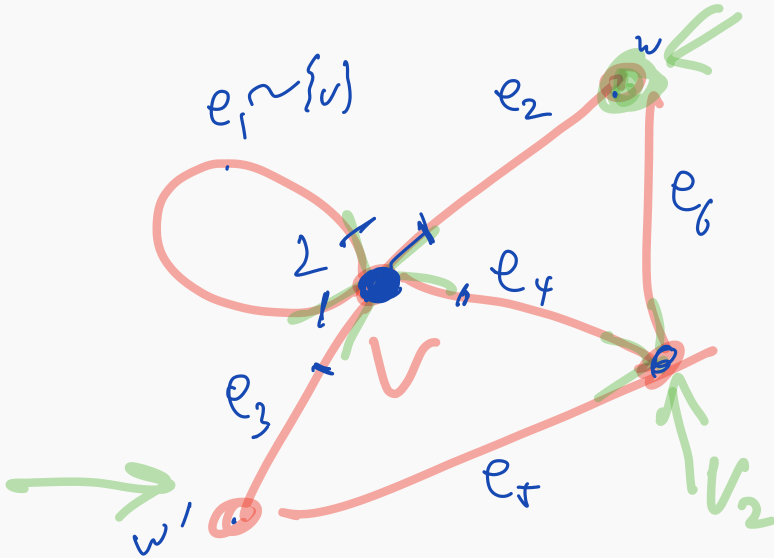
associated to

13. degree of a vertex (for undirected graphs), notation **deg**(v) :

$$\mathbf{deg}(v) = \sum_{e \in E} p(e) \text{ where } p(e) = \begin{cases} 2 & e \sim \{v\} \\ 1 & e \sim \{v, w\}, w \neq v \\ 0 & e \sim \{u, w\}, u, w \neq v \end{cases}$$

$u \neq v$ and $w \neq v$

14. in-degree and out-degree (for directed graphs)



$p(e_1) + p(e_2) + p(e_3) + p(e_4)$

$= 2 + 1 + 1 + 1 + 0 + 0$

13. degree of a vertex (for undirected graphs), notation $\mathbf{deg}(v)$:

$$\mathbf{deg}(v) = \sum_{e \in E} p(e) \text{ where } p(e) = \begin{cases} 2 & e \sim \{v\} \\ 1 & e \sim \{v, w\}, w \neq v \\ 0 & e \sim \{u, w\}, u, w \neq v \end{cases}$$

14. in-degree and out-degree (for directed graphs) : exercise

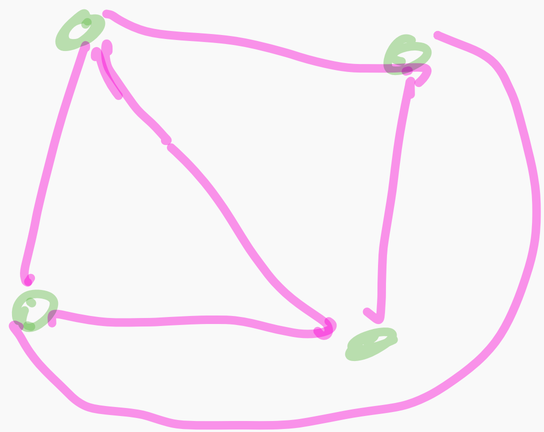
15. complete graph K_n on n vertices

(simple) and no multi edges

no loops

$\{x\} = \{x, x\}$

$n = 4$



13. degree of a vertex (for undirected graphs), notation **deg**(v) :

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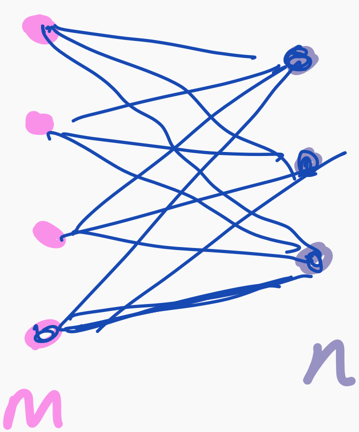
14. in-degree and out-degree (for directed graphs) :

15. complete graph K_n on n vertices :

16. complete bipartite graph on $m + n$ vertices

eg: $K_{3,4}$



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$$\deg(v) = \sum_{e \in E} p(e) \text{ where } p(e) = \begin{cases} 2 & e \sim \{v\} \\ 1 & e \sim \{v, w\}, w \neq v \\ 0 & e \sim \{u, w\}, u, w \neq v \end{cases}$$

14. in-degree and out-degree (for directed graphs) :

15. complete graph K_n on n vertices :

16. complete bipartite graph on $m + n$ vertices :

Note that all our definitions are given (precisely) in terms of set theory, rather than some picture-description.

You should know now that this is important for when it comes time to *proving* facts about graphs. If we have imprecise definitions, we will have trouble in our proofs.

I often use the word *node* instead of *vertex* to make the English simpler. Remember plural: *vertices*, singular: *vertex*.

finite

Definition (Adjacency matrix)

Let $G = (V, E)$ be a graph (directed or undirected). Assume $|V| = n$ and $V = \{1, 2, \ldots, n\}$.

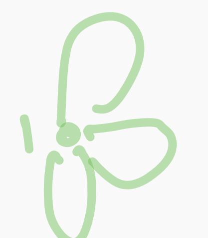
between (undirected)

The *adjacency matrix* for G is a $n \times n$ matrix $A = (a_{ij})$ where a_{ij} is the number of edges from vertex i to vertex j.

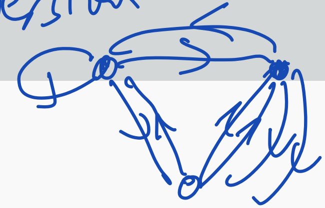
$$
\begin{array}{c|cccc|}
 & 1 & 2 & & n \\
\hline
1 & a_{11} & a_{12} & \cdots & a_{1n} \\
2 & a_{21} & a_{22} & & \\
3 & & & & \\
\vdots & \vdots & & & \\
n & a_{n1} & a_{n2} & & a_{nn} \\
\hline
\end{array}
$$

$a_{ij} = a_{ji}$

a_{11}



directed version



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Let $G = (V, E)$ be a graph (directed or undirected). Assume $|V| = n$ and $V = \{1, 2, \ldots, n\}$.

The *adjacency matrix* for G is a $n \times n$ matrix $A = (a_{ij})$ where a_{ij} is the number of edges from vertex i to vertex j.

Eg:

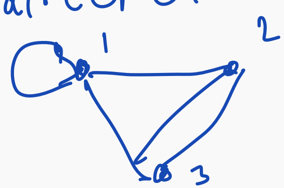
natural numbers

~~integer~~

$\mathbb{N}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

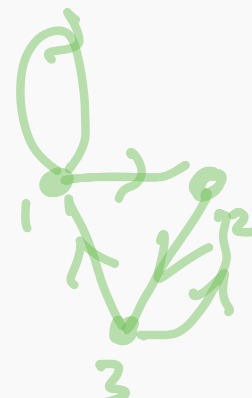
Undirected:

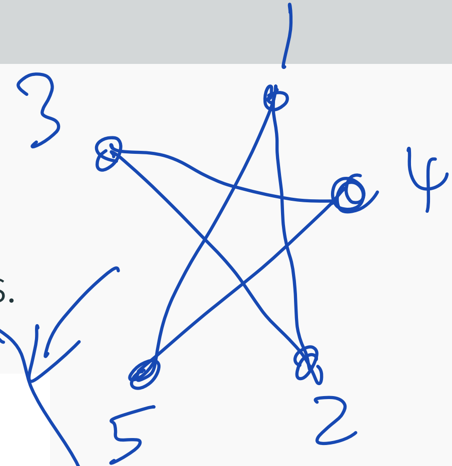


$$\begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

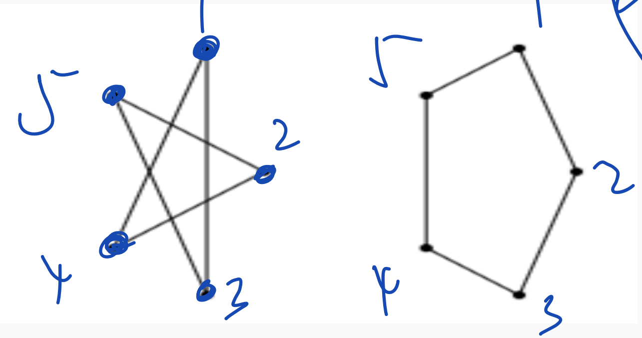
1,3

directed





Ex: Give the adjacency matrices for these graphs.



(In set notation: $V_1 = \{1, 2, 3, 4, 5\}$, $E_1 = \{\{1, 3\}, \{3, 5\}, \{5, 2\}, \{2, 4\}, \{4, 1\}\}$ and $V_2 = \{1, 2, 3, 4, 5\}$, $E_2 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$.)

$$\begin{matrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{matrix}$$

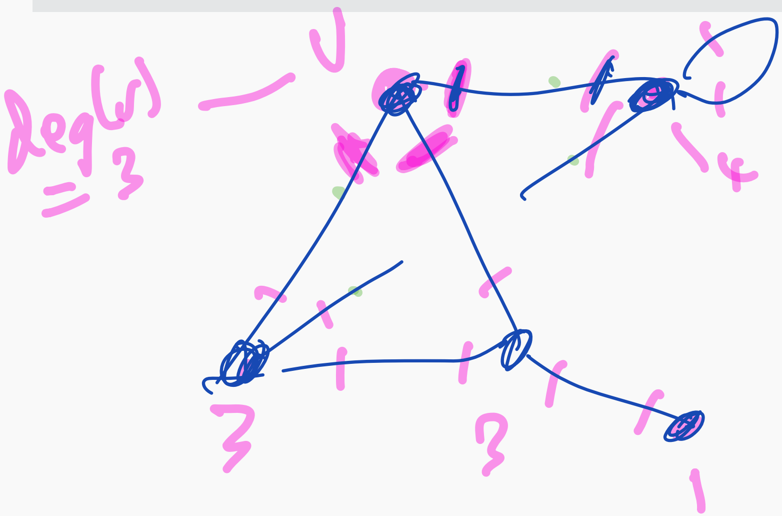
Now that we have all these definitions, we can prove some theorems.

Theorem

If $G = (V, E)$ is a graph (undirected) and $|E| = n$ then

$$\sum_{v \in V} \deg(v) = 2n.$$

V finite



LHS counts # pink dashes.

~~RHS is~~

Every edge has 2 pink dashes on it.

Now that we have all these definitions, we can prove some theorems.

Theorem

If $G = (V, E)$ is a graph (undirected) and $|E| = n$ then

$$\sum_{v \in V} \deg(v) = 2n.$$

Do we have our definitions correct? How does degree work with loops – does a loop count 1 or 2 towards the degree?

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Do we have our definitions correct? How does degree work with loops – does a loop count 1 or 2 towards the degree?

Proof.

Imagine drawing a mark on your picture of G for each pair (v, e) where v is an endpoint of e (and draw it close to v). Then the number of marks you have drawn is $\sum_{v \in V} \deg(v)$. Now if each edge has exactly two marks drawn on it, we have our theorem. □

Theorem

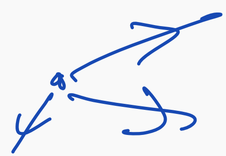
If $G = (V, E)$ is a graph (undirected) and $|E| = n$ then

$$\sum_{v \in V} \deg(v) = 2n.$$

Ex: What would be the corresponding statement for directed graphs? Can you prove it?

in-deg

out-degree.

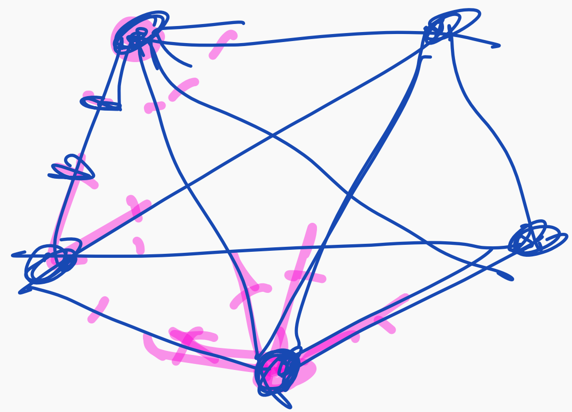


Eg: ~~…~~ draw graph with degree sequence ~~…~~ 1, 3, 5, 6

Ex: How many edges does the complete graph on *n* vertices have?

K_n

$$\sum_{v \in K_n} \deg(v) = (n-1)\cdot n \,/\, 2$$



RECALL: MATRIX MULTIPLICATION

LAS

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} =$$

2 2 1
3
2

undirected

Theorem

If $G = (V, E)$ is a graph with $V = \{1, 2, \ldots, s\}$ and $s \times s$ adjacency matrix A, then the number of paths starting at i and ending at j of length $n \geqslant 1$ is the ij-th entry in A^n.

$$\begin{matrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix} = A^1$$

paths of length 1

a_{12}

True for $n = 1$.

by definition of adjacency matrix.

Assume statement is true for $n = k$

A^{k+1}

COUNTING PATHS IN A GRAPH

$= \underline{A} \cdot A^k \rightarrow$ i, j-th entry of

Theorem

If $G = (V, E)$ is a graph with $V = \{1, 2, \dots, s\}$ and $s \times s$ adjacency matrix A, then the number of paths starting at i and ending at j of length $n \geqslant 1$ is the ij-th entry in A^n.

First of all, check its true (do we have our definitions of paths, and length of path, correct?)

Theorem

If $G = (V, E)$ is a graph with $V = \{1, 2, \ldots, s\}$ and $s \times s$ adjacency matrix A, then the number of paths starting at i and ending at j of length $n \geqslant 1$ is the ij-th entry in A^n.

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This is a statement about paths of length $n \in \mathbb{N}_+$ so maybe *induction* is appropriate.

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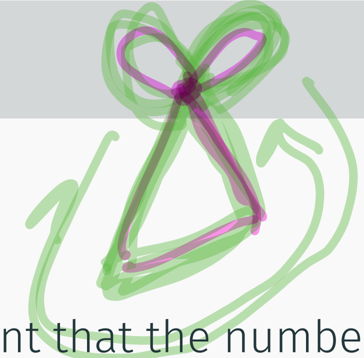

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

a_{12}

$a_{11} = 2$

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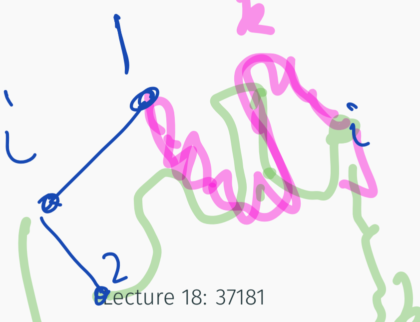
Assume $P(k)$ is true and consider $A^{k+1} = AA^k$. If we write $A^k = (b_{ij})_{1 \leqslant i,j \leqslant s}$ then by assumption b_{ij} is the number of paths from i to j of length k.



A: $\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{is} \end{pmatrix}$ edges

A^k: j-th column $\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{sj} \end{pmatrix}$

$a_{i1} b_{1j}$
$+ a_{i2} b_{2j}$
$+ a_{i3} b_{3j}$

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$A = PDP^{-1}$

$A^{1000000} = PD^{1000000}P^{-1}$

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By the definition of matrix multiplication the ij-th entry of $A^{k+1} = AA^k$ is

via1 via2 vian

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{t=1}^{n} a_{it}b_{tj}$$

which counts paths whose first step goes via some vertex t, so counts all the paths that start at i and make one step to be at vertex t and then follow a path from t to j.

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Since all of these paths are different, we get the correct count. $\square$

Ex: What would be the corresponding statement for directed graphs?

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Application: `https://math.stackexchange.com/questions/92555/counting-the-number-of-paths-on-a-graph`
How many different ways are there to unlock an android phone?

NEXT TIME

- graph isomorphism
- Eular paths
- Hamiltonian circuits

— Trees