37181 DISCRETE MATHEMATICS

Lecture 18: graph theory

PLAN

- Definition: graph
- Applications, theorems


DARWIN
CAIRNS
TOWNSVILLE
BROOME
PORT HEDLAND
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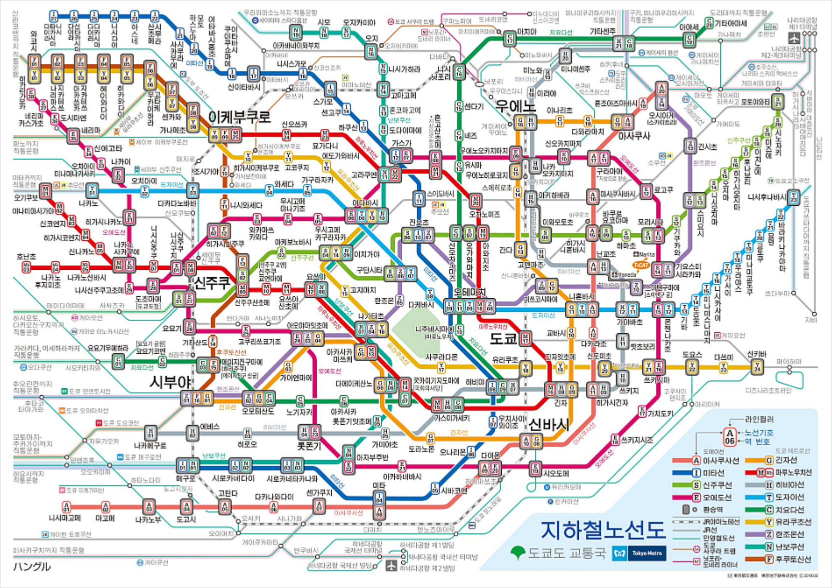
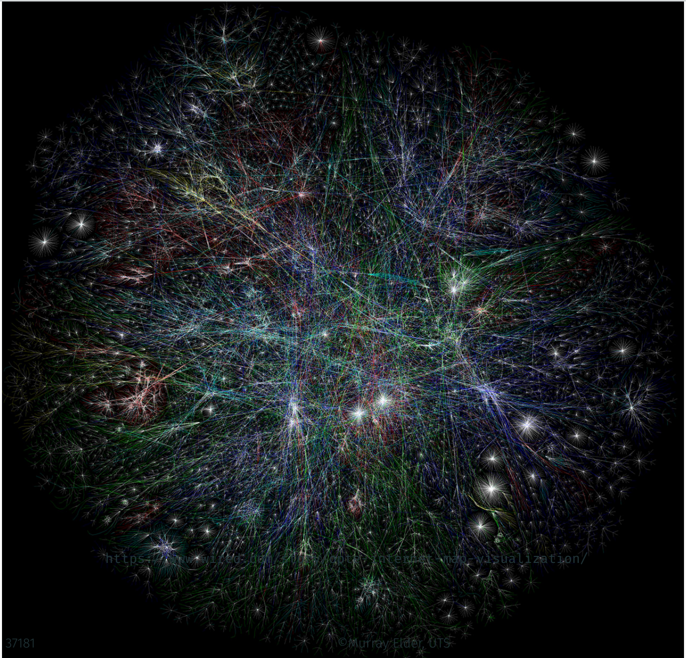
ALICE SPRINGS
ULURU (AYERS ROCK)
KALGOORLIE
PERTH
ADELAIDE
MELBOURNE
SYDNEY
CANBERRA
BRISBANE
HOBART
LAUNCESTON
DEVONPORT
MILDURA
NEWCASTLE
INDIAN OCEAN
PACIFIC OCEAN
TASMAN SEA
CORAL SEA
TIMOR SEA
NORTHERN TERRITORY
QUEENSLAND
WESTERN AUSTRALIA
SOUTH AUSTRALIA
NEW SOUTH WALES
VICTORIA
TASMANIA
Great Australian Bight
Gulf of Carpentaria
ROUTE KEY
Qantas routes
Jetstar routes
QantasLink routes
Routes operated by other airlines for Qantas
Qantas dedicated freight route
Qantas Club and Qantas regional lounge locations
Qantas Group international gateway port
National capital
Qantas Frequent Flyer domestic partners and codeshare airlines
Qantas
Jetstar
Airnorth

GRAPH THEORY





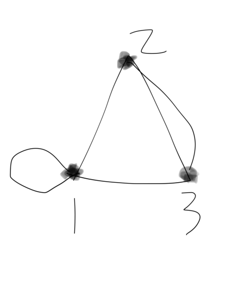
https://www.wired.com/story/opte-internet-map-visualization/

Definition (Graph)

A *graph* $G = (V, E)$ *is a pair of sets* V, E *such that* each $e \in E$ is associated to some subset $\{v_1, v_2\} \subseteq V$ of size 1 or 2.

The elements of V are called [plural: *vertices*, singular: *vertex*] or *nodes*, and the elements of E are called *edges* or *arcs*.

Eg: If $V = \{1,2,3\}$, $E = \{e_{11}, e_{12}, e_{13}, e_{23}, e_{32}\}$ where e_{ij} is associated to $\{i,j\} \subseteq V$ then $G = (V,E)$ is a graph. We can *visualise* G as a picture, with a dot for each element of V and a line between v_1 and v_2 if $\{v_1, v_2\}$ is associated to some $e \in E$, so in this case we get



If $\{v_1, v_2\}$ is associated to more than one edge in some graph G, these edges are called *multiple edges* or *multi-edges* and G is sometimes called a *multi-graph*.

When each subset $\{v_1, v_2\} \subseteq V$ of size 1 or 2 is associated to at most one element of E, we can choose to label each edge by a subset of V of size 1 or 2, and write $E \subseteq \mathscr{P}(V)$.

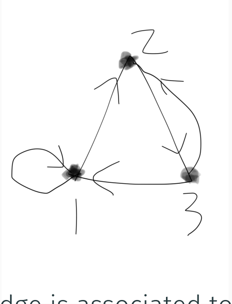
Ex: draw the graph with edge set $\{\{1,2\}, \{1,1\}, \{2,3\}, \{1,3\}\}$.

Definition (Directed graph)

A *directed* graph $G = (V, E)$ is a pair of sets V, E such that each $e \in E$ is associated to some ordered pair $(v_1, v_2) \in V \times V$. If $(u, v) \in E$ we call $u \in V$ the *source* vertex and v the *terminal* vertex.

In this case when we visualise G as a picture, we draw arrows on the edges to indicate their *direction*, from source to terminal.

Ex: draw the directed graph with
$V = \{1, 2, 3\}, E = \{e_{11}, e_{12}, e_{31}, e_{23}, e_{32}\}$ where e_{ij} is associated to $(i, j) \in V^2$.



Again if more than one edge is associated to the same ordered pair, we call them *multi-edges*. If G is directed with no multi-edges we can choose to label E by ordered pairs $V \times V = V^2$, and we write $E \subseteq V^2$. In our example there are no multi-edges, since $(2, 3)$ and $(3, 2)$ are different elements of V^2.

1. loop: an edge associated to a singleton set $\{x\}$
2. multiple edge/multi-edge: when more than one edge is associated to the same set $\{x, y\}$
3. simple graph: a graph with no loops and no multi-edges

4. path : a sequence of edges associated to sets $\{x, v_1\}, \{v_1, v_2\}, \ldots, \{v_n, y\}$

5. length of a path : the number of edges in the path

6. circuit : a path where $x = y$

7. connected : if every two vertices x, y are joined by a path

8. disconnected : not connected

9. simple path : a path $\{x, v_1\}, \{v_1, v_2\}, \ldots, \{v_n, y\}$ where each v_i is different from all other v_j (note x, y are allowed to be the same vertex).

10. endpoint(s) of an edge : if e is associated with $\{x, y\}$ then x, y are called the endpoints

11. edge incident to a vertex : e is incident to v if e is associated to $\{v, v'\}$

12. adjacent vertices : v, v' are adjacent if $\{v, v'\}$ is associated to an edge

13. degree of a vertex (for undirected graphs), notation $\deg(v)$:

$$\deg(v) = \sum_{e \in E} p(e) \text{ where } p(e) = \begin{cases} 2 & e \sim \{v\} \\ 1 & e \sim \{v, w\}, w \neq v \\ 0 & e \sim \{u, w\}, u, w \neq v \end{cases}$$

14. in-degree and out-degree (for directed graphs) :

15. complete graph K_n on n vertices :

16. complete bipartite graph on $m + n$ vertices :

Note that all our definitions are given (precisely) in terms of set theory, rather than some picture-description.

You should know now that this is important for when it comes time to *proving* facts about graphs. If we have imprecise definitions, we will have trouble in our proofs.

I often use the word *node* instead of *vertex* to make the English simpler. Remember plural: *vertices*, singular: *vertex*.

Definition (Adjacency matrix)

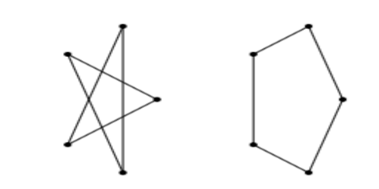
Let $G = (V, E)$ be a graph (directed or undirected). Assume $|V| = n$ and $V = \{1, 2, \ldots, n\}$.

The *adjacency matrix* for G is a $n \times n$ matrix $A = (a_{ij})$ where a_{ij} is the number of edges from vertex i to vertex j.

Eg:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Ex: Give the adjacency matrices for these graphs.



(In set notation: $V_1 = \{1, 2, 3, 4, 5\}$, $E_1 = \{\{1, 3\}, \{3, 5\}, \{5, 2\}, \{2, 4\}, \{4, 1\}\}$ and $V_2 = \{1, 2, 3, 4, 5\}$, $E_2 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$.)

Now that we have all these definitions, we can prove some theorems.

Theorem

If $G = (V, E)$ is a graph (undirected) and $|E| = n$ then

$$\sum_{v \in V} \deg(v) = 2n.$$

Do we have our definitions correct? How does degree work with loops – does a loop count 1 or 2 towards the degree?

Proof.

Imagine drawing a mark on your picture of G for each pair (v, e) where v is an endpoint of e (and draw it close to v). Then the number of marks you have drawn is $\sum_{v \in V} \deg(v)$. Now if each edge has exactly two marks drawn on it, we have our theorem. $\square$

Theorem

If $G = (V, E)$ is a graph (undirected) and $|E| = n$ then

$$\sum_{v \in V} \deg(v) = 2n.$$

Ex: What would be the corresponding statement for directed graphs? Can you prove it?

Ex: How many edges does the complete graph on n vertices have?

RECALL: MATRIX MULTIPLICATION

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} =$$

Theorem

If $G = (V, E)$ is a graph with $V = \{1, 2, \ldots, s\}$ and $s \times s$ adjacency matrix A, then the number of paths starting at i and ending at j of length $n \geqslant 1$ is the ij-th entry in A^n.

First of all, check its true (do we have our definitions of paths, and length of path, correct?)

What technique to prove this?

This is a statement about paths of length $n \in \mathbb{N}_+$ so maybe *induction* is appropriate.

Proof. Let $P(n)$ be the statement that the number of paths starting at i and ending at j of length n is the ij-th entry in A^n.

Then $P(1)$ is exactly the definition of adjacency matrix: the number of paths of length 1 from i to j is given by a_{ij}.

Assume $P(k)$ is true and consider $A^{k+1} = AA^k$. If we write $A^k = (b_{ij})_{1 \leqslant i,j \leqslant s}$ then by assumption b_{ij} is the number of paths from i to j of length k.

Assume $P(k)$ is true and consider $A^{k+1} = AA^k$. If we write $A^k = (b_{ij})_{1 \leqslant i,j \leqslant s}$ then by assumption b_{ij} is the number of paths from i to j of length k.

By the definition of matrix multiplication the ij-th entry of $A^{k+1} = AA^k$ is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{t=1}^{n} a_{it}b_{tj}$$

which counts paths whose first step goes via some vertex t, so counts all the paths that start at i and make one step to be at vertex t and then follow a path from t to j.

Since all of these paths are different, we get the correct count. $\square$

Ex: What would be the corresponding statement for directed graphs?

Application: `https://math.stackexchange.com/questions/92555/counting-the-number-of-paths-on-a-graph`
How many different ways are there to unlock an android phone?

NEXT TIME

- graph isomorphism
- Eular paths
- Hamiltonian circuits