# 37181 DISCRETE MATHEMATICS

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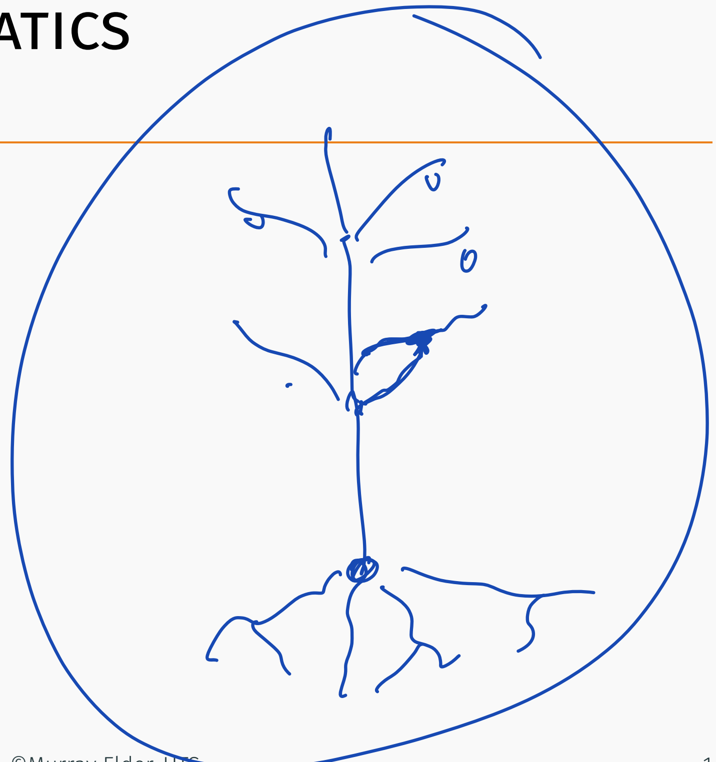
Lecture 20: Trees



# PLAN

- Defn: tree
- Spanning tree
- Rooted trees, bracket-free expressions (pre-post-in orders)

circuit = cycle

Recall that a circuit is a path $(x, v_1), \ldots, (v_n, x)$. We will also call a circuit a *cycle*.

non-trivial

A *tree* is an undirected graph $G$ which is connected and has no cycles.

1.

2.

# nodes :

1 2 3 4

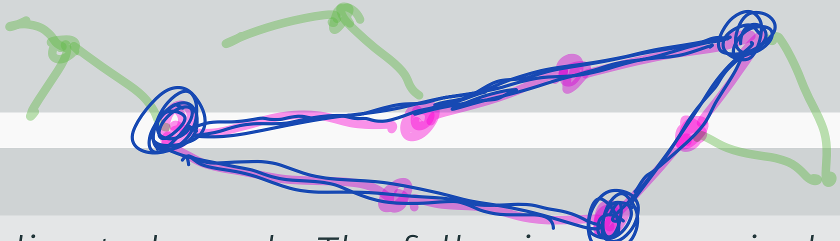
no edges

# DEFN: TREE

Recall that a circuit is a path $(x, v_1), \ldots, (v_n, x)$. We will also call a circuit a *cycle*.

A *tree* is an undirected graph $G$ which is connected and has no cycles.

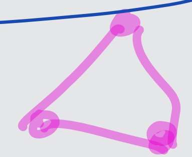
By convention, we don't allow the empty graph to be a tree.



## Theorem

*Let G be an undirected graph. The following are equivalent (TFAE):*

- *G is a tree.* 1. connected 2. no cycles
- *G has no cycles, and a simple cycle is formed if any edge is added to G.*
- *G is connected, but would become disconnected if any single edge is removed from G.*
- *G is connected and $K_3$ is not a minor of G.*



Subgraph plus smooth over degree 2 verts

- *Any two vertices in G can be connected by a unique simple path.*

We will come back and prove just one or two of these are equivalent, leaving the rest for exercises.

# PLAY AROUND WITH DEFN. HOW MANY?

$|V| = 1, 2, 3, 4, 5, \ldots.$

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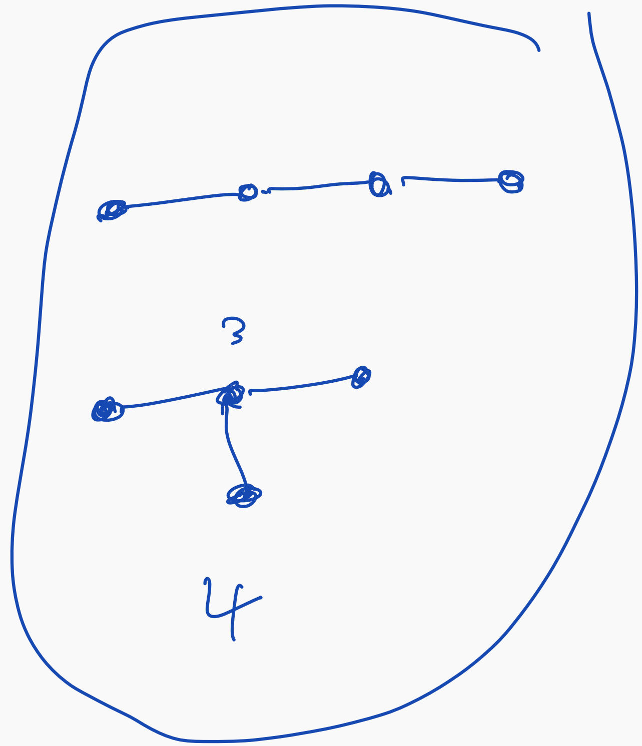
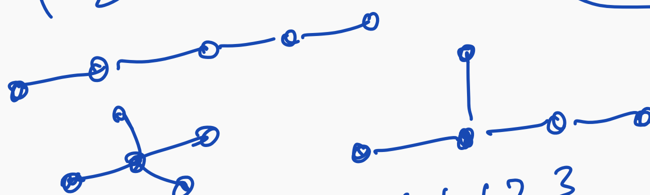
remember: sum degrees

1 2 3




5

1 2 2 2 1



1 1 1 1 4

1 1 1 2 3

# PLAY AROUND WITH DEFN. HOW MANY?

$|V| = 1, 2, 3, 4, 5, \ldots.$

remember: sum degrees

`https://oeis.org/A000055`

# PLAY AROUND WITH DEFN. HOW MANY?

$|V| = 1, 2, 3, 4, 5, \ldots.$

remember: sum degrees

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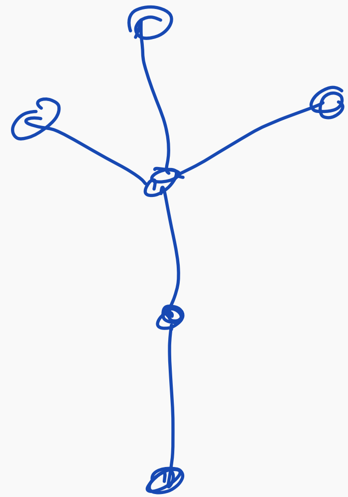
Guess: what is a leaf?

deg 1. vertex.



# PLAY AROUND WITH DEFN. HOW MANY?

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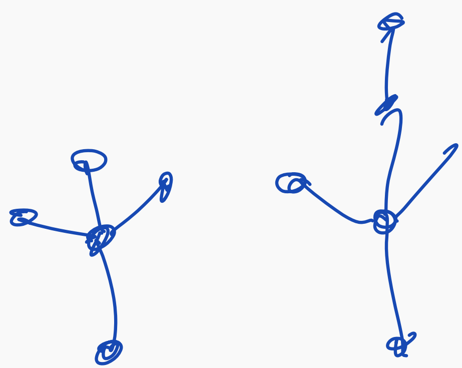
Guess: what is a leaf?



a forest?

disjoint union of trees

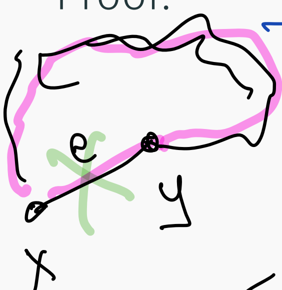
1.

Recall: G is a tree if it is connected and has no cycles.

**Theorem**

*G is a tree if and only if G is connected, but would become disconnected if any single edge is removed from G.*

Proof:




Assume G is a tree, so connected + no cycles.
Then G is connected. Suppose G is not disconnected if I remove a single edge → e.
since G is a tree, the edge was not a loop.
so it is x•—•y $x \neq y$.

$G \setminus \{(x,y)\} = G'$ is connected. Thus there is a path inside $G'$ from x to y called p. Then pe is a cycle inside G. Contradiction.

Assume G is connected but becomes disconnected if any single edge is removed.
Then G is connected. Also

Recall: $G$ is a tree if it is connected and has no cycles.

**Theorem**

*$G$ is a tree if and only if $G$ is connected, but would become disconnected if any single edge is removed from $G$.*

Proof: Two directions.

1. 2.

Recall: $G$ is a tree if it is connected and has no cycles.

**Theorem**

*$G$ is a tree if and only if $G$ is connected, but would become disconnected if any single edge is removed from $G$.*

Proof: Two directions.

Assume $G$ is connected and has no cycles. Then $G$ is connected. Suppose (for contradiction) some single edge $e$ is removed (keeping its endpoints $x, y$), and $G$ is still connected. Then there is a path $p$ in $G'$ from $x$ to $y$, so in $G$ there is a cycle $pe$. Contradiction.

$x \neq y$ since $G$ tree can't have a loop.

$= G \setminus \{e\}$

Assume $G$ is a connected graph with the property that removing a single edge always disconnects it. Then $G$ is connected. Suppose $G$ has a cycle. Then removing an edge on that cycle does not disconnect, contradiction. So $G$ doesn't have any cycles. □

$\sum_{v \in V} \deg(v) = 2|E|$

**Theorem**

*A tree with $n \in \mathbb{N}_+$ vertices has $n-1$ edges*

Proof: ~~Induction~~ ← need Strong.

Let $P(n)$ be statement that a tree with $n$ verts has $n-1$ edges.

$P(1)$ : the only tree with one vertex is • and has 0 edges, so $P(1)$ true.

Assume that $P(k)$ true for $k \geq 1$.

Consider a tree $T$ with $k+1$ vertices.


disconnect $\rightarrow$ 2

Remove edge and keep its endpoints.

Two trees $T_1$, $T_2$

\# vertices of $T$ was $k+1$

$= \underbrace{\#\text{ verts of } T_1}_{\geq 1,\ i,\ \leq k} + \underbrace{\#\text{ verts of } T_2}_{\geq 1,\ \leq k}$

$i$ verts

$k+1-i$ verts

**Theorem**

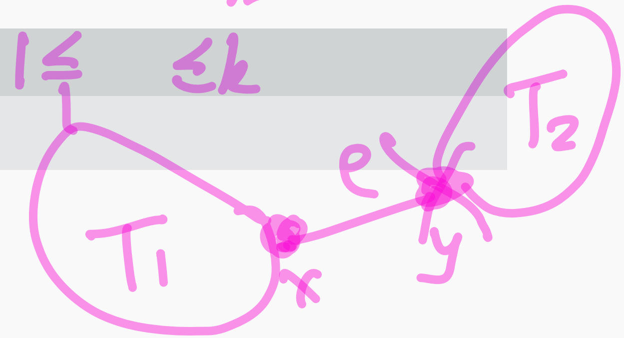
*A tree with $n \in \mathbb{N}_+$ vertices has $n-1$ edges*

Proof: induction, remove an edge.

~~k+1~~

**Theorem**

*A tree with $n \in \mathbb{N}_+$ vertices has $n-1$ edges*




idea

Proof: induction, remove an edge.

Proof: (strong induction) Let $P(n)$ be the statement that a tree with $n$ vertices has $n-1$ edges.

True for $n=1$ by inspection.

1 vertex, 0 edges

Assume true for all $1 \leqslant i \leqslant k$ and consider $T$ with $k+1$ vertices. If $T$ has no edges then it is disconnected ($k+1 \geqslant 2$), so $T$ has an edge. Choose one (finitely many) and erase it, leaving its endpoints, to get two trees (not connected after removing an edge by previous). Call the two connected components $T_1, T_2$ and say $T_1$ has $1 \leqslant j \leqslant k$ vertices, so $T_2$ has $1 \leqslant k+1-j \leqslant k$ vertices. By strong induction, we know $T_1$ has $j-1$ edges and $T_2$ has $k+1-j-1 = k-j$ edges, so the original $T$ had $\underbrace{(j-1)}_{T_1} + \underbrace{(k-j)}_{T_2} + \underbrace{1}_{\text{removed edge}} = k$ edges (those from $T_1, T_2$ plus the edge we removed).

Thus by (strong) PMI the statement is true for all $n \in \mathbb{N}_+$. □

We mostly restrict to finite graphs and trees in First Year Discrete Maths.

But infinite graphs and trees lead to very interesting mathematics.

$|V|$ infinite

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But infinite graphs and trees lead to very interesting mathematics.

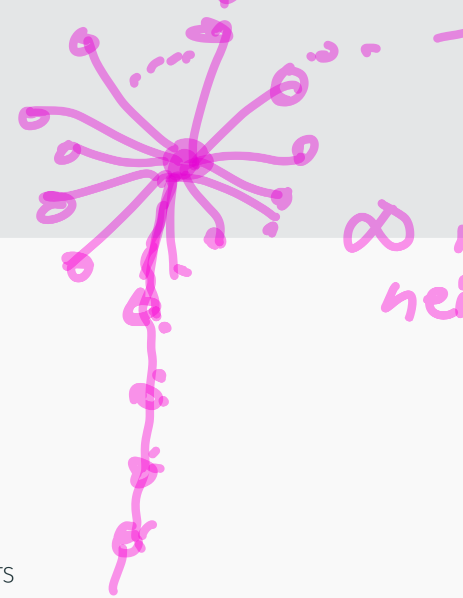
connected.

**Theorem (König's lemma)**

*Every infinite tree contains either*

- *a vertex of infinite degree, or*
- *an infinite simple path.*

0.0001



$\infty$ many neighbours

Countable
sdr
$\mathbb{N}$.

We mostly restrict to finite graphs and trees in First Year Discrete Maths.

But infinite graphs and trees lead to very interesting mathematics.

**Theorem (König's lemma)**

*Every infinite tree contains either*

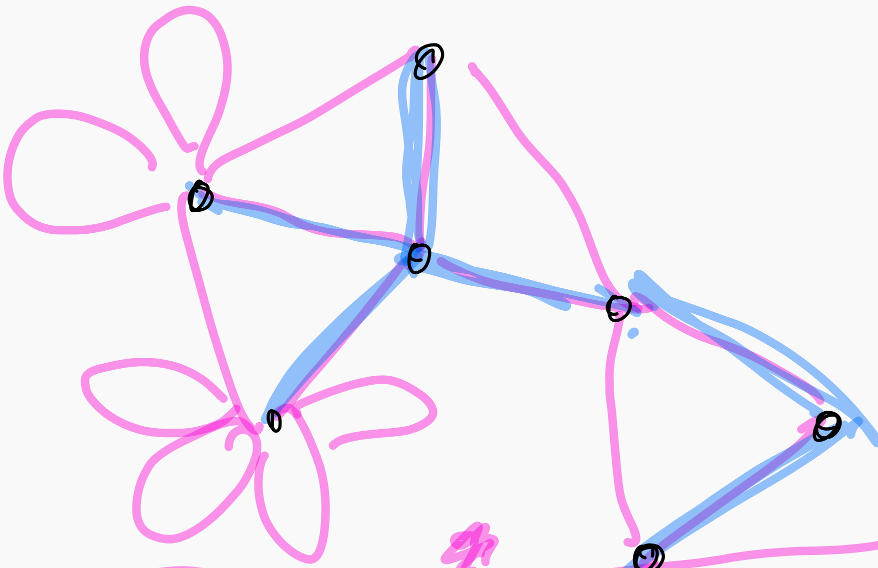
- *a vertex of infinite degree, or*
- *an infinite simple path.*

Wikip: This proof is not generally considered to be *constructive*, because at each step it uses a proof by contradiction to establish that there exists an adjacent vertex from which infinitely many other vertices can be reached, and because of the reliance on a weak form of the *Axiom of Choice*.

**Definition**

A spanning tree of a graph $G = (V, E)$ is a tree $H = (V, E')$ with $E' \subseteq E$.

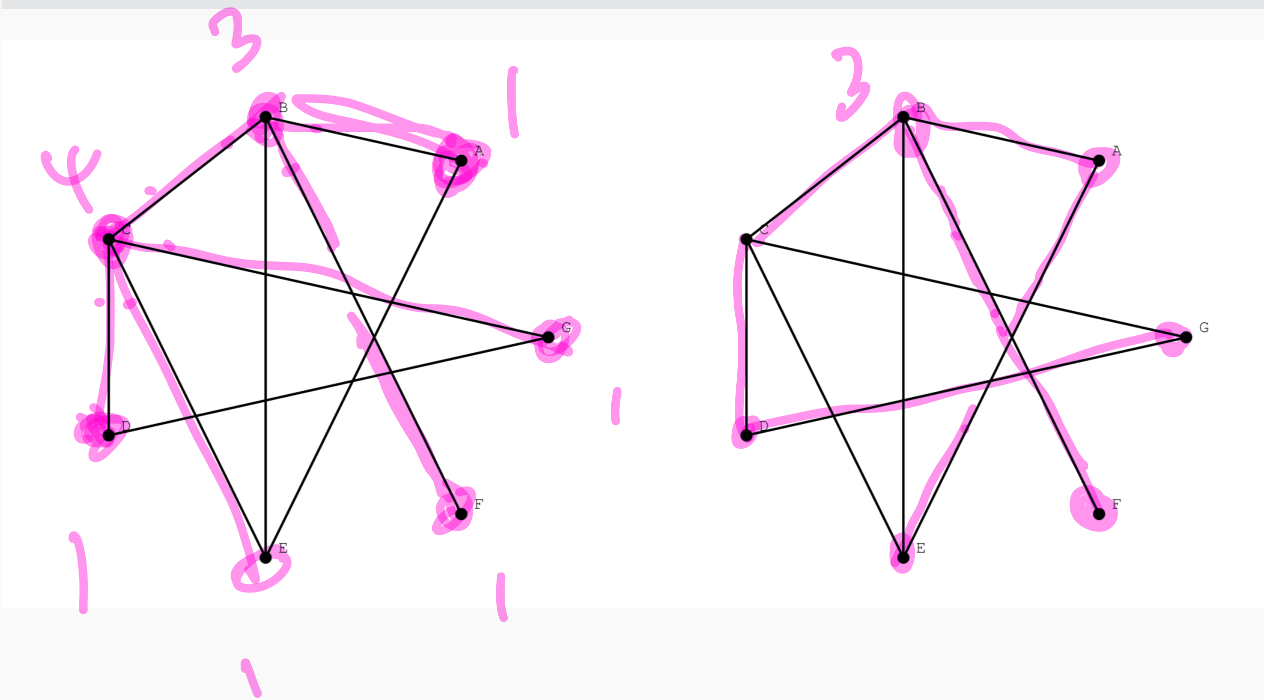
same V



If graph disconnected. → No Spanning graph

## Definition

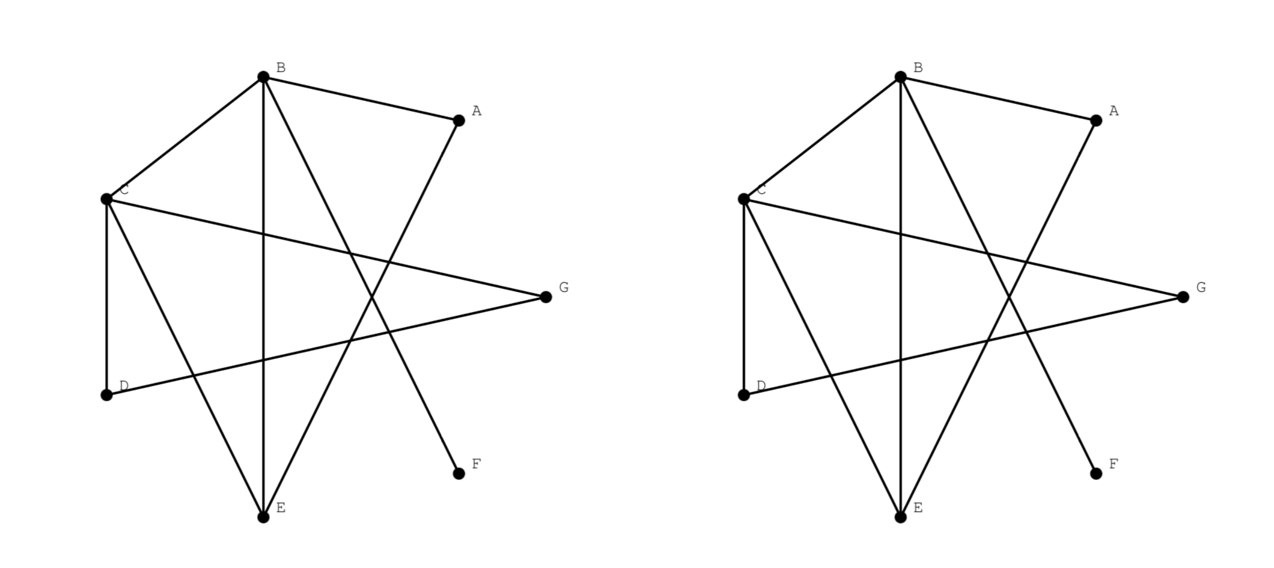
A spanning tree of a graph $G = (V, E)$ is a tree $H = (V, E')$ with $E' \subseteq E$.



## Definition

A spanning tree of a graph $G = (V, E)$ is a tree $H = (V, E')$ with $E' \subseteq E$.




c.f. Wikip: A *spanning tree* $T$ of an undirected graph $G$ is a subgraph that is a tree which includes all of the vertices of G, with minimum possible number of edges.

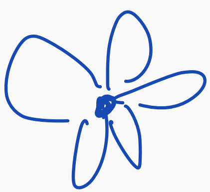
n vertices, $n \in \mathbb{N}_+$

**Theorem**

*Every connected non-empty finite graph contains a spanning tree*

Proof: Induction.

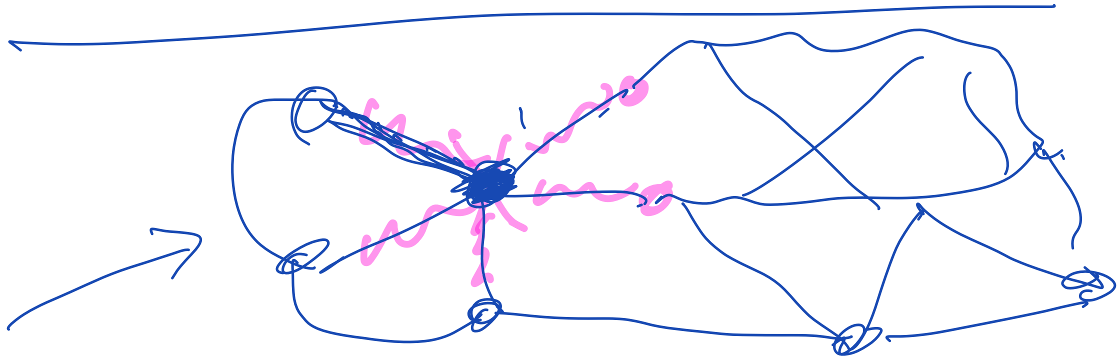
$P(n)$ statement that a connected graph with $n$ vertices contains a spanning tree.

$P(1)$ :  some number of loops, Spanning tree is a single dot.

Assume $P(k)$ true $k \geq 1$.

Consider graph $G$ with $k+1$ vertices

Remove a vertex plus all ~~adjacent~~ ~~ed~~ incident edges



Either resulting graph $G' = G \setminus \{$vertex plus incident edges$\}$

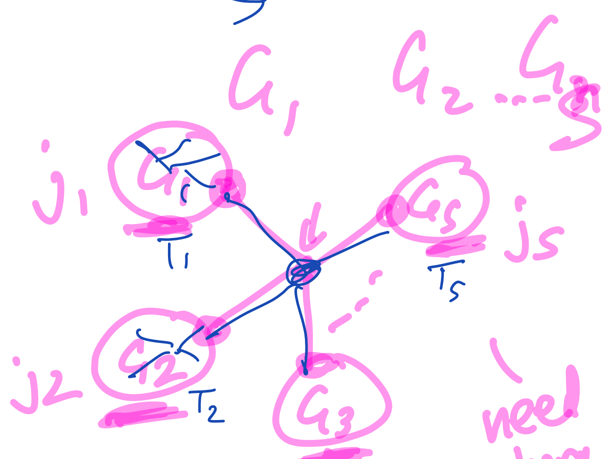
is connected or disconnected.

$G'$ connected, has $k$ vertices.

by inductive assump

$G'$ has a spanning tree $\to T'$

~~add to T~~




$T = T'$ plus one edge to the removed vertex.

$1 \leq j_{k_n} \leq k$

**Theorem**

*Every connected* *non-empty* *finite graph contains a spanning tree*

Proof:

Induction (strong). Let $P(n)$ be the statement that:

existence proof.

**Theorem**

*Every connected non-empty finite graph contains a spanning tree*

Proof:

Induction (strong). Let $P(n)$ be the statement that:

Note the statement is also true for infinite graphs, but requires more interesting logic and techniques.

In later optimisation courses you will study *efficient* algorithms to construct spanning trees. Use Big O to make precise how efficient.

# NEXT TIME (LAST LECTURE!)

- rooted trees
- bracket-free expressions.

- planar graph
- Euler's formula