37181 DISCRETE MATHEMATICS

Lecture 21: Rooted trees; bracket-free expressions; planar graphs; Euler's formula

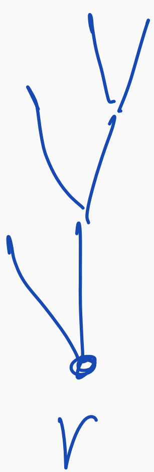
PLAN

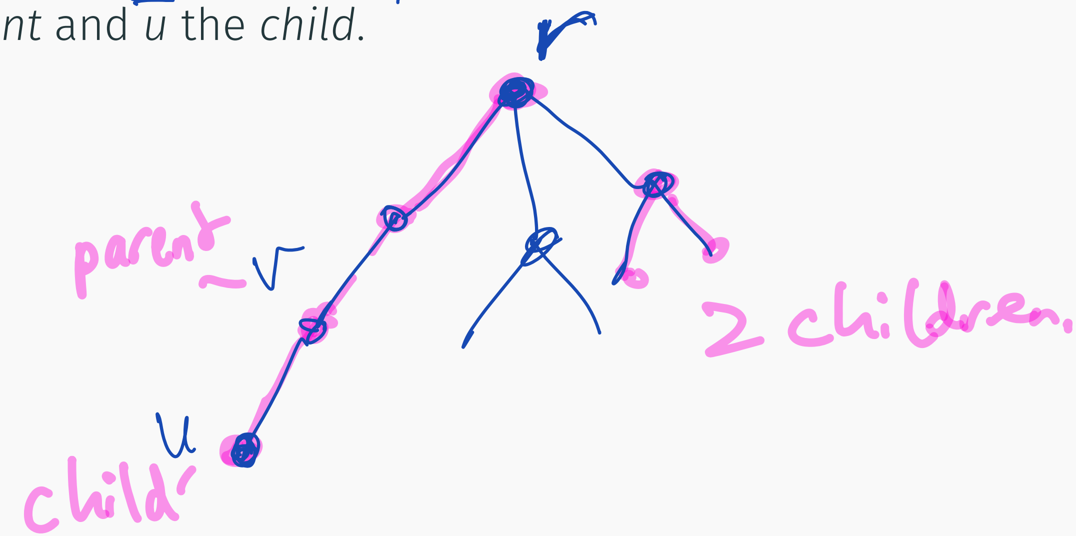
- Rooted trees, bracket-free expressions (pre-post-in orders)
- planar graphs
- Euler's formula

ROOTED TREES

A *rooted tree* is a tree which has a special node r called the *root*.

In a rooted tree, if v is a vertex and u is connected by an edge to v, such that the path from u to r passes v (a picture would help here), we call v the *parent* and u the *child*.





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Rooted trees are very useful as data structures, with efficient search algorithms. There are many other applications of rooted trees. Here we consider just one.

BRACKET-FREE EXPRESSIONS

$$3 + 4(x + y)$$

$$\frac{2+4}{x+2y}$$

$$(2+4) \div (x+2y)$$

BRACKET-FREE EXPRESSIONS

$3 + 4(x + y)$

$$\frac{2+4}{x+2y}$$

$3 * 2^4 + (1 + 3)$

^ 4

2

Inorder, preorder, postorder are three different conventions on how to write the nodes of a rooted binary tree as a string.

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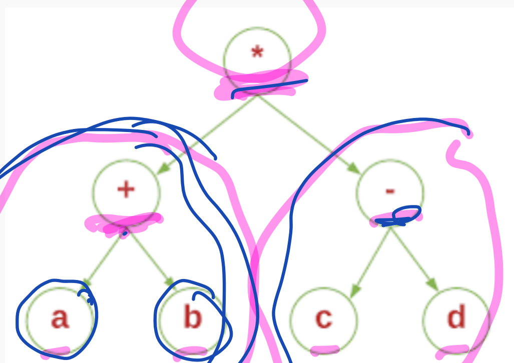
useful in computing, efficient to represent (old calculators would use this for display, input); recursively defined:



left child

right child

pre: parent, left, right

pre-order

* + a b - c d

binary operators.

a, b, c, d

"numbers"

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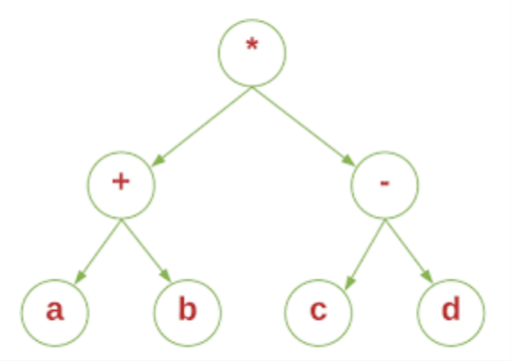

pre: parent, left, right

in: left, parent, right

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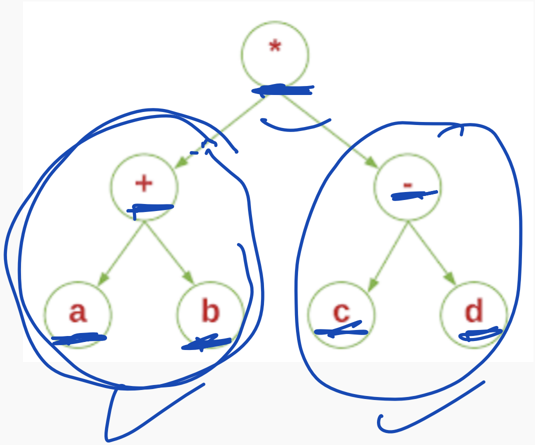


pre: parent, left, right

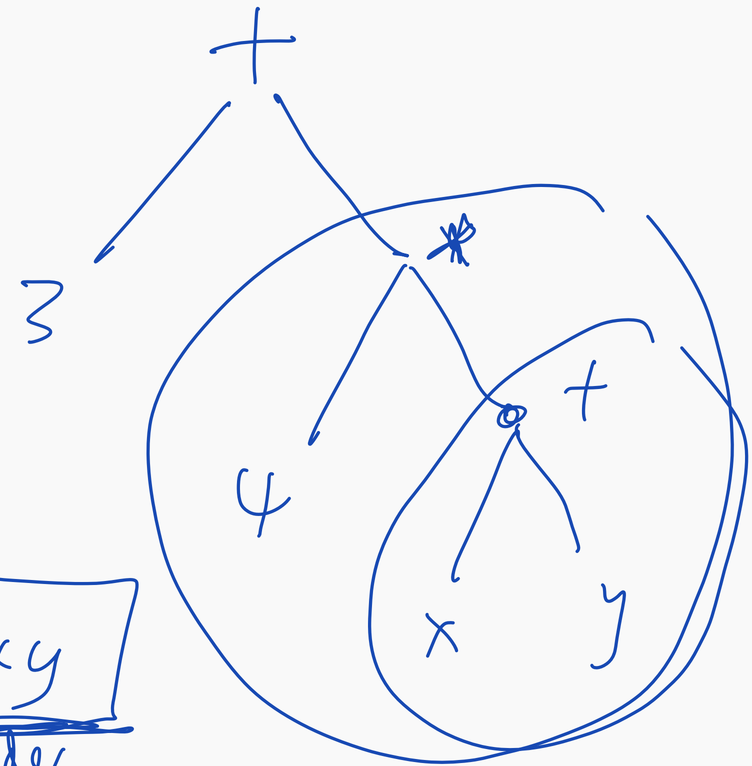
in: left, parent, right

post: left, right, parent

Recall:

- pre: parent, left, right
- in: left, parent, right
- post: left, right, parent

$3 + 4(x + y)$



$\boxed{+\ 3\ *\ 4\ +\ x\ y}$

pre-order

BRACKET-FREE EXPRESSIONS

Recall:

- pre: parent, left, right
- in: left, parent, right
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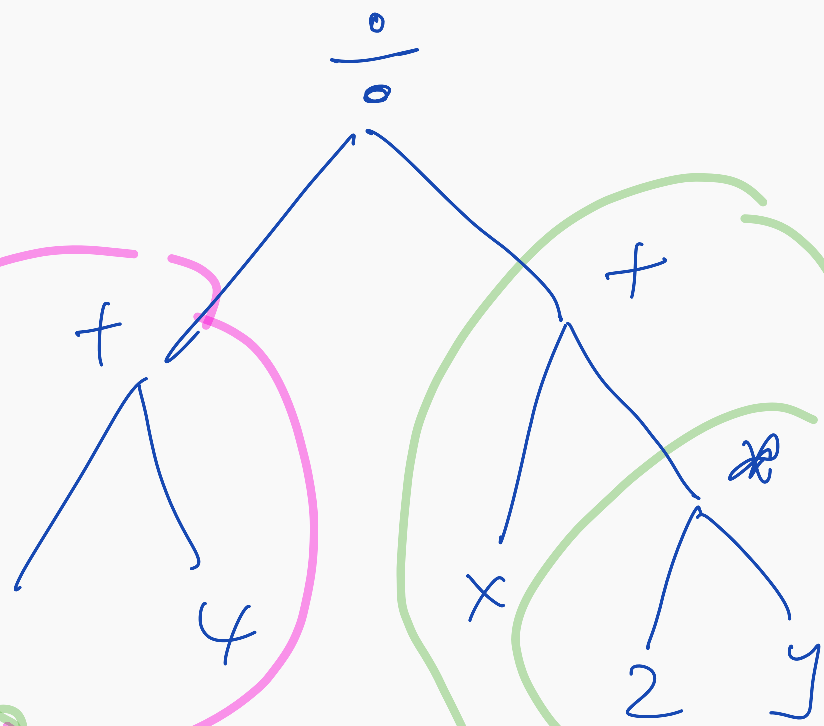
$3 + 4(x + y)$

$$\frac{2+4}{x+2y}$$



$2+4 \div x+2*y$

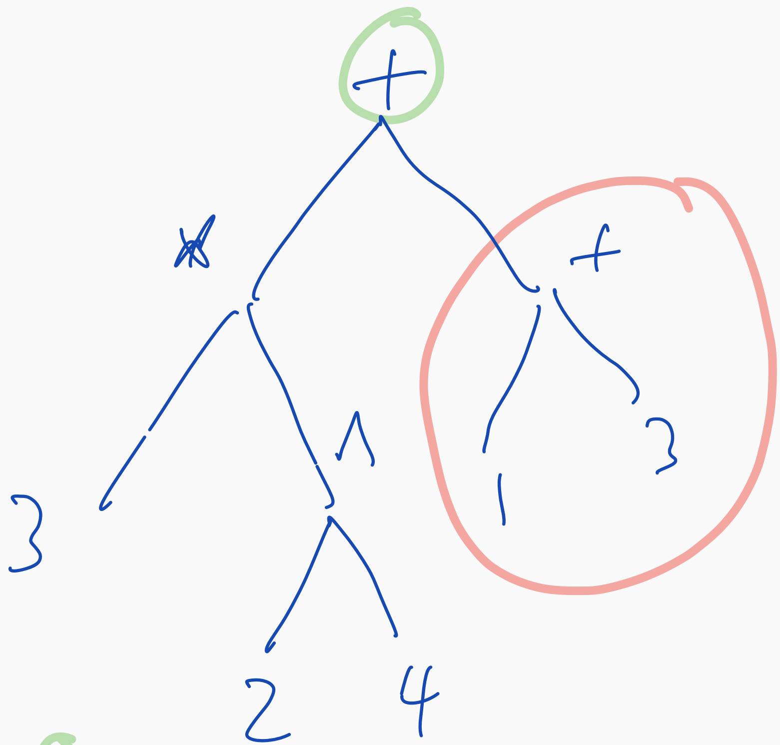
Recall:

- pre: parent, left, right
- in: left, parent, right
- post: left, right, parent

$3 + 4(x + y)$

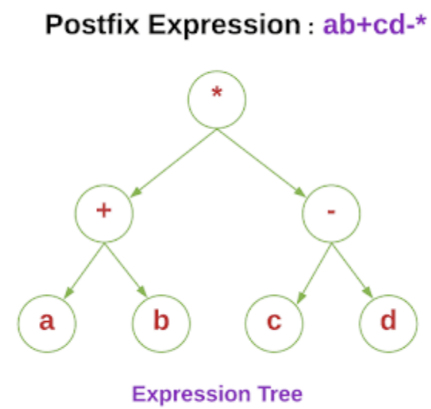
$$\frac{2+4}{x+2y}$$

$3 * 2^4 + (1 + 3)$



3 2 4 ∧ * 1 3 + +

INORDER, PREORDER, POSTORDER TRAVERSAL



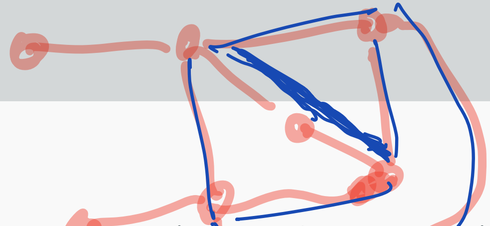
Preorder Traversal	:	***+ab-cd**
Inorder Traversal	:	**a+b * c-d**
Postorder Traversal	:	**ab+cd-***

PLANAR GRAPHS

A graph G is *planar* if it can be drawn on a piece of paper without any edges crossing.

This is an example of mathematics being purely visual.

PLANAR GRAPHS



— K_4 minor.

A graph G is *planar* if it can be drawn on a piece of paper without any edges crossing.

This is an example of mathematics being purely visual.

A very cool theorem is that G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a *minor*.

— subgraph with degree 2 vertices "smoothed over"

See `https://en.wikipedia.org/wiki/Wagner%27s_theorem`

We will prove one direction only: that K_5 ~~or~~ and $K_{3,3}$ are not planar, so if you could draw G without crossing on a piece of paper, then you can draw all its minors too, so if G is planar it cannot have K_5 or $K_{3,3}$ as a minor.

If G is planar, we can draw it on the surface of a balloon without any edges crossing.

Define a *face* to be a region bounded by edges of the graph. You might think at first the number of faces will depend on how we choose to draw G.

draw on sphere without crossing edge

Theorem

If G is planar, finite, connected then $|V| - |E| + |F| = 2$ *(for any representation/drawing of G on the plane without edge crossings.)*

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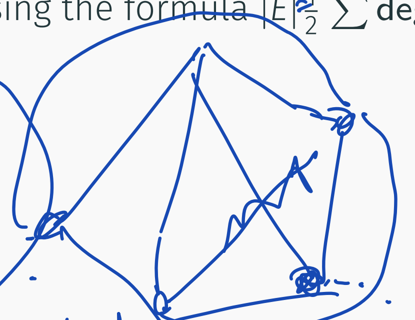
If G is planar, finite, connected then $|V| - |E| + |F| = 2$ (for any representation/drawing of G on the plane without edge crossings.)

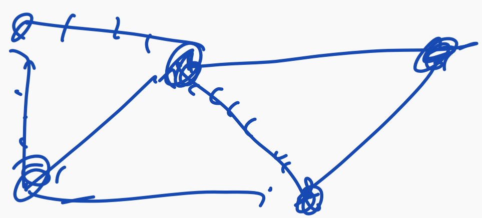
Test it out: balloon

Suppose it was.

Application: K_5 cannot be planar. Proof: add up $2 = |V| - |E| + |F| = 5 - 10 + |F|$ so $|F| = 7$. (Using the formula $|E| = \frac{1}{2}\sum \mathbf{deg}(v_i)$.) But each face is a triangle, so ...

each face contributes 3 edges





21 edges counted twice

Contradiction since 21 is odd.

Theorem

If G is planar, finite, connected then $|V| - |E| + |F| = 2$ (for any representation/drawing of G on the plane without edge crossings.)

Test it out: balloon

Application: K_5 cannot be planar. Proof: add up $2 = |V| - |E| + |F| = 5 - 10 + F$ so $|F| = 7$. (Using the formula $|E| \frac{1}{2} \sum \mathbf{deg}(v_i)$.) But each face is a triangle, so ...

Suppose it is.

Also, $K_{3,3}$ cannot be planar. Proof: add up $2 = |V| - |E| + |F| = 6 - 9 + |F|$ so $|F| = 5$. But each face is a square, so ...

Exercize ... lead to contradiction.

EULER'S FORMULA

Theorem

If G is planar, finite, connected then $|V| - |E| + |F| = 2$ (for any representation/drawing of G on the plane without edge crossings.)

Theorem

If G is planar, finite, connected then $|V| - |E| + |F| = 2$ (for any representation/drawing of G on the plane without edge crossings.)

Proof: induction on number of edges.

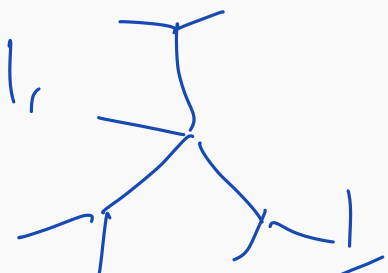
Trick: G has a cycle, or not.

tree

life easy

$|F| = 1,$

$|V| = |E| + 1$



$|V| - |E| + |F|$

$= |E| + 1 - |E| + 1 = 2$

Let $P(n)$ statement that
a planar, finite connected nonempty graph with n edges satisfies
$$|V| - |E| + |F| = 2$$

$P(0)$: , $|V| = 1$ $|E| = 0$
$|F| = 1$
$$|V| - |E| + |F| = 1 + 1 = 2$$ ✓

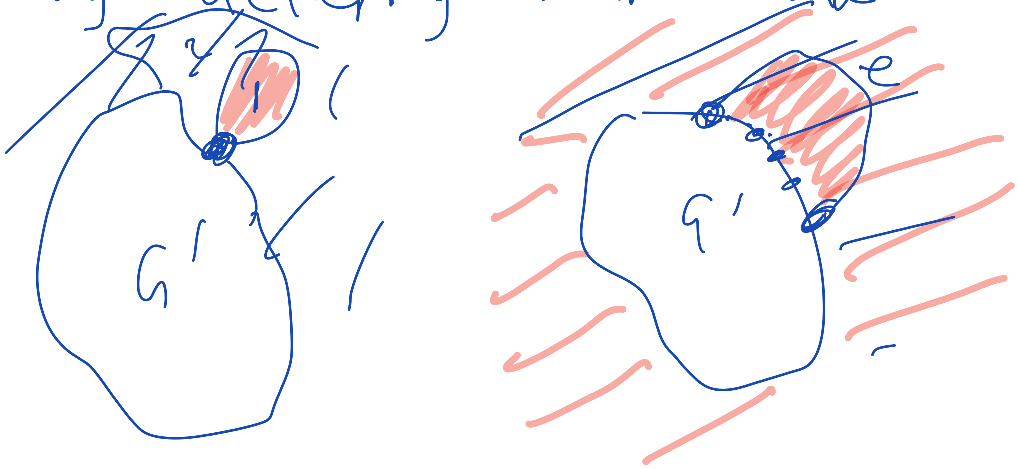
Assume $P(k)$ is true for $k \geq 0$.
Consider a graph G with $k+1$ edges.
G planar, finite, connected.

Either G has a cycle, or not.

If not, G is a tree, and $|F| = 1$
and $|V| = |E| + 1$ so
$$|V| - |E| + |F| = |E| + 1 - |E| + 1 = 2$$ ✓

Else, G has a cycle.
Choose an edge in this cycle

and let G' be the graph obtained by deleting that edge



G' connected, planar, finite so by inductive hypothesis — k edges

$$|V'| - |E'| + |F'| = 2$$

so
$$|V| = |V'|$$
$$|E| = |E'| + 1$$
$$|F| = |F'| + 1$$

so
$$|V| - |E| + |F| = |V'| - |E'| \cancel{-1} + |F'| \cancel{+1} = 2.$$

$\square$

Theorem

If G is planar, finite, connected then $|V| - |E| + |F| = 2$ (for any representation/drawing of G on the plane without edge crossings.)

Proof: induction on number of edges. Trick: either G has a cycle, or it doesn't.

If $|E| = 0$ then $|V| = 1, |F| = 1$ so true (single vertex, outside space is the single face).

Assume true for $|E| = k \geqslant 1$ and consider G planar connected with $k + 1$ edges.

If G has a cycle, deleting one edge from this cycle gives a connected graph G' with k edges, and is planar since it is a subgraph of G. Let V', E', F' be the vertices, edges and faces of G'. Then by inductive assumption $|V'| - |E'| + |F'| = 2$. Now $V = V'$ since we only deleted an edge and kept the adjacent vertices. $|E| = |E'| + 1$ and $|F| = F'| + 1$ since when we add the edge back in, we divide one face up into two. Thus $|V| - |E| + |F| = |V'| - |E'| - 1 + |F'| + 1 = 2$.

Otherwise, if G has no cycles, since G is connected, G is a tree. Then $|V| = |E| + 1$ and $|F| = 1$ (just the outside).

So $|V| - |E| + |F| = |E| + 1 - |E| + 1 = 2$. $\square$

Thanks everyone. Tutorial Wed/Thu/Fri this week, then that's it.

Final exam covers *all topics*. Please review all content over StuVac, and make yourself a summary/formula sheets to be able to quickly recall definitions and facts during the online exam.

3 pages