37181 DISCRETE MATHEMATICS

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Lecture 21: Rooted trees; bracket-free expressions; planar graphs; Euler's formula

PLAN

- Rooted trees, bracket-free expressions (pre-post-in orders)
- planar graphs
- Euler's formula

A *rooted tree* is a tree which has a special node r called the *root*.

In a rooted tree, if v is a vertex and u is connected by an edge to v, such that the path from u to r passes v (a picture would help here), we call v the *parent* and u the *child*.

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Rooted trees are very useful as data structures, with efficient search algorithms. There are many other applications of rooted trees. Here we consider just one.

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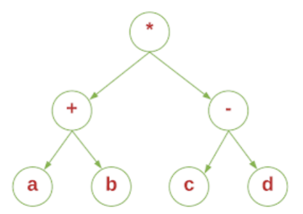
useful in computing, efficient to represent (old calculators would use this for display, input); recursively defined:



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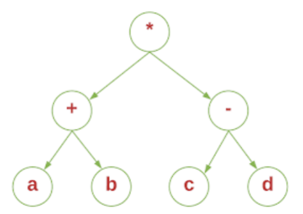

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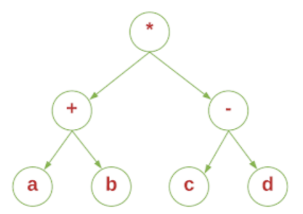

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post: left, right, parent

Recall:

- pre: parent, left, right
- in: left, parent, right
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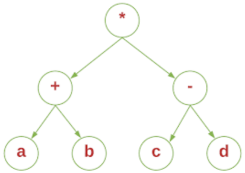
$3 * 2^4 + (1 + 3)$

INORDER, PREORDER, POSTORDER TRAVERSAL

Postfix Expression : ab+cd-*



Expression Tree

Preorder Traversal	:	***+ab-cd**
Inorder Traversal	:	**a+b * c-d**
Postorder Traversal	:	**ab+cd-***

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PLANAR GRAPHS

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A very cool theorem is that G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a *minor*.

See `https://en.wikipedia.org/wiki/Wagner%27s_theorem`

We will prove one direction only: that K_5 and $K_{3,3}$ are not planar, so if you could draw G without crossing on a piece of paper, then you can draw all its minors too, so if G is planar it cannot have K_5 or $K_{3,3}$ as a minor.

If G is planar, we can draw it on the surface of a balloon without any edges crossing.

Define a *face* to be a region bounded by edges of the graph. You might think at first the number of faces will depend on how we choose to draw G.

EULER'S FORMULA

Theorem

If G is planar, finite, non-empty, connected then $|V| - |E| + |F| = 2$ *(for any representation/drawing of G on the plane without edge crossings.)*

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Application: K_5 cannot be planar. Proof: add up $2 = |V| - |E| + |F| = 5 - 10 + F$ so $|F| = 7$. (Using the formula $|E|\frac{1}{2}\sum \mathbf{deg}(v_i)$.) But each face is a triangle, so ...

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Also, $K_{3,3}$ cannot be planar. Proof: add up $2 = |V| - |E| + |F| = 6 - 9 + |F|$ so $|F| = 5$. But each face is a square, so ...

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Proof: induction on number of edges. Trick: either G has a cycle, or it doesn't.

If $|E| = 0$ then $|V| = 1, |F| = 1$ so true (single vertex, outside space is the single face).

Assume true for $|E| = k \geqslant 1$ and consider G planar connected with $k+1$ edges.

If G has a cycle, deleting one edge from this cycle gives a connected graph G' with k edges, and is planar since it is a subgraph of G. Let V', E', F' be the vertices, edges and faces of G'. Then by inductive assumption $|V'| - |E'| + |F'| = 2$. Now $V = V'$ since we only deleted an edge and kept the adjacent vertices. $|E| = |E'| + 1$ and $|F| = F'| + 1$ since when we add the edge back in, we divide one face up into two. Thus $|V| - |E| + |F| = |V'| - |E'| - 1 + |F'| + 1 = 2$.

Otherwise, if G has no cycles, since G is connected, G is a tree. Then $|V| = |E| + 1$ and $|F| = 1$ (just the outside).

So $|V| - |E| + |F| = |E| + 1 - |E| + 1 = 2$. $\square$

Thanks everyone. Tutorial Wed/Thu/Fri this week, then that's it.

Final exam covers *all topics*. Please review all content over StuVac, and make yourself a summary/formula sheets to be able to quickly recall definitions and facts during the online exam.