37181 DISCRETE MATHEMATICS

Lecture 4: rational numbers, well ordering principle

PLAN

- rational and irrational numbers
- first element
- well ordering principle

Definition:
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Lemma

$\sqrt{2}$ *is irrational.*

$\sqrt{2}$ IS IRRATIONAL $\quad (\neg p \rightarrow \mathsf{F}) \rightarrow p$

Proof.

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- square both sides
- multiply both sides by b^2
- then a^2 is even
- so by our Lemma, a is even
- do some more manipulating
- now b^2 is even

$\square$

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Let the universe of discourse be $\mathbb{Q}$.

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Construct a number in between them:

$p = \frac{ad}{bd}$ and $q = \frac{cb}{db}$, and we know $ad < cb$ and they are both integers. What if they were just 1 apart?

□

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I will make some conjectures, based on the previous lemma, and you decide whether or not you think they are true.

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Between any two distinct real numbers you can find both a rational and an irrational number.

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Then $s = t$ so there was only one. □

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Eg: $\{5, 4, 6, 7\}$ has a first element, 4.

Next lecture:

- Set theory notation
- power set