37181 DISCRETE MATHEMATICS

©Murray Elder, UTS

Lecture 7: induction

PLAN

- principle of mathematical induction

Lemma

For all $n \in \mathbb{N}$, $11^n - 4^n$ is divisible by 7.

?

Lemma

If A is a set of size $n \in \mathbb{N}$, then $\mathscr{P}(A)$ has size 2^n.

?

Axiom (Principle of mathematical induction)

Let $P(n)$ be a statement about natural numbers. Let $s \in \mathbb{N}$, eg. $s = 0, 1$

Axiom (Principle of mathematical induction)

Let $P(n)$ be a statement about natural numbers. Let $s \in \mathbb{N}$, eg. $s = 0, 1$

If

1. $P(s)$ is true
2. $P(k) \rightarrow P(k+1)$ is true

Axiom (Principle of mathematical induction)

Let $P(n)$ be a statement about natural numbers. Let $s \in \mathbb{N}$, eg. $s = 0, 1$

If

1. $P(s)$ is true
2. $P(k) \to P(k+1)$ is true

then $P(n)$ is true for all $n \geqslant s$.

(domino picture)

APPLICATION

Lemma

For all $n \in \mathbb{N}, n \geqslant 1$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

©Murray Elder, UTS

APPLICATION

Lemma

For all $n \in \mathbb{N}, n \geqslant 1$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof.

Let $P(n)$ be the statement that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

©Murray Elder, UTS

Lemma

For all $n \in \mathbb{N}, n \geqslant 1$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof.

Let $P(n)$ be the statement that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Thus by PMI $P(n)$ is true for all $n \geqslant 1$. $\square$

Lemma

For all $n \in \mathbb{N}, n \geqslant 1$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Lemma

For all $n \in \mathbb{N}, n \geqslant 1$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof.

Let $P(n)$ be the statement that

Lemma

For all $n \in \mathbb{N}, n \geqslant 1$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof.

Let $P(n)$ be the statement that

Thus by PMI $P(n)$ is true for all $n \geqslant 1$. $\square$

Lemma

For all $n \in \mathbb{N}$, $11^n - 4^n$ is divisible by 7.

Lemma

For all $n \in \mathbb{N}$, $11^n - 4^n$ is divisible by 7.

Proof.

Let $P(n)$ be the statement that

Lemma

For all $n \in \mathbb{N}$, $11^n - 4^n$ is divisible by 7.

Proof.

Let $P(n)$ be the statement that

Thus by PMI $P(n)$ is true for all $n \geqslant 0$. $\square$

EG

Lemma

If A is a set of size $n \in \mathbb{N}$, then $\mathscr{P}(A)$ has size 2^n.

Proof.

Let $P(n)$ be the statement that

Thus by PMI $P(n)$ is true for all $n \geqslant 0$. $\square$

©Murray Elder, UTS

Lemma

For all $n \in \mathbb{N}$, if $n \geqslant \square$ then (some statement).

Proof.

Let $P(n)$ be the statement $\square$

Then $P(\square)$ is true since $\square$

Assume $P(k)$ for $k \geqslant \square$. Then

$\square$

Thus by PMI $P(n)$ is true for all $n \geqslant \square$. $\square$

PMI is equivalent to the following: Let $s \in \mathbb{N}$.

If

- $P(s)$ is true and
- if *for all* $s \leqslant i \leqslant n$ $P(i)$ is true, then $P(n+1)$ is true,

PMI is equivalent to the following: Let $s \in \mathbb{N}$.

If

- $P(s)$ is true and
- if *for all* $s \leqslant i \leqslant n$ $P(i)$ is true, then $P(n+1)$ is true,

then $P(n)$ is true for all $n \in \mathbb{Z}, n \geqslant s$.

Lemma

For all $n \in \mathbb{N}, n > 1$ if n is not prime then some prime number p divides n.

Proof.

Lemma

For all $n \in \mathbb{N}, n > 1$ *if* n *is not prime then some prime number* p *divides* n*.*

Proof.

Let $P(n)$ be the statement that either n is prime or some prime divides n.

$\square$

Lemma

For all $n \in \mathbb{N}$, $n! \geqslant 2^{n-1}$

Proof.

Let $P(n)$ be the statement that

$\square$

(start at 0)

EG

Consider the statement: For all $n \in \mathbb{N}$, $n! \geqslant 3^{n-1}$.

Is this true? Is it true for all $n \geqslant$??

Proof.

Let $P(n)$ be the statement that

$\square$

(start at ??)

EG

Lemma

All horses are black.

Lemma

All horses are black.

Proof.

Let $P(n)$ be the statement that

$\square$

Next lecture:

- application of induction: correctness of computer code/algorithms
- WOP and PMI

Important to gets lots of practice doing proofs by induction.